Master Equation

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1 General System-Environment Coupling

In this section, we consider a general system-bath coupling:[1]

$$H_T = H + H_B + V, \quad V = \sum_k A_k \otimes B_k. \tag{1}$$

Under certain conditions, we will show that the dynamics of the system are well approximated by the Lindblad equation. We first assume that initially, the total system is a product state

$$\rho_T(0) = \rho(0) \otimes \rho_B.$$

In the following, we will adopt the interacting picture, where the density operator evolves as

$$\partial_t \rho_T(t) = -i[V(t), \rho_T(t)] \equiv -i\mathcal{V}(t)|\rho_T(t)\rangle.$$

In the last equality, ρ_T is expressed as a ket in the Hilbert space of linear operator, and the commutator with V is expressed as a superoperator $\mathcal{V}$. This notation can simplify the expression. For example, the inner product in the operator space is the trace so that the partial trace operation can be denoted as $|\rho\rangle = \langle \mathbb{I}_B|\rho_T\rangle$. The evolution of the system is then

$$\begin{aligned}\partial_t|\rho(t)\rangle &= -i\langle \mathbb{I}_B|\mathcal{V}(t)|\rho_T(t)\rangle = -i\langle \mathbb{I}_B|\mathcal{V}(t)|\rho_T(0)\rangle - \int_0^t \langle \mathbb{I}_B|\mathcal{V}(t)\mathcal{V}(\tau)|\rho_T(\tau)\rangle d\tau \\ &= -\int_0^t \langle \mathbb{I}_B|\mathcal{V}(t)\mathcal{V}(\tau)|\rho_T(\tau)\rangle d\tau.\end{aligned} \tag{2}$$

Now we are taking the **Born approximation**, which states when the coupling is weak enough compared with the energy scale of the system and the bath, the total density matrix is approximated by the product state $|\rho_T(t)\rangle \approx |\rho(t)\rangle \otimes |\rho_B\rangle$. The evolution is now

$$\begin{aligned}\frac{d}{dt}\rho(t) &\approx \int_0^t \mathrm{Tr}_B\left[V(t)\rho_T(\tau)V(\tau) - \rho_T(\tau)V(\tau)V(t)\right] d\tau + h.c. \\ &= \sum_{kl}\int_0^t d\tau\ \left[C_{lk}(\tau - t)A_k(t)\rho(\tau)A_l(\tau) - C_{lk}(\tau - t)\rho(\tau)A_l(\tau)A_k(t) + h.c.\right],\end{aligned} \tag{3}$$

where $C_{kl}(t) \equiv \mathrm{Tr}_B[\rho_B B_k(t) B_l]$ is the correlation function of B_k's. We then take the **Markovian approximation**, which assumes that the correlations of the bath decay fast in time. We can thus make the substitution $\rho(\tau) \to \rho(t)$, the result equation of motion is Markovian:

$$\begin{aligned}\frac{d}{dt}\rho(t) &\approx \sum_{kl}\int_0^t dt'\left[C_{lk}(-t')A_k(t)\rho(t)A_l(t-t') - C_{lk}(-t')\rho(t)A_l(t-t')A_k(t) + h.c.\right] \\ &= \sum_k \int_0^t dt\left[A_k\rho B_k - \rho B_k A_k + h.c.\right],\end{aligned} \tag{4}$$

[1] Without loss of generality, we can also assume $\|A_k\| = 1$, $\mathrm{Tr}[\rho_B B_k] = 0$.

where we have defined $B_k(t) = \sum_l \int_0^\infty dt' A_l(t-t') C_{lk}(-t')$. Now, we switch to the frequency domain,

$$A_k(t) = \sum_\omega A_k(\omega) e^{-i\omega t}, \quad B_k(t) = \sum_{l,\omega} e^{-i\omega t} A_l(\omega) \Gamma_{lk}(\omega), \quad \Gamma_{kl}(\omega) = \int_0^\infty dt\, e^{i\omega t} C_{kl}(t).$$

We then take the **rotating wave approximation**, where we only keep the contributions from canceling frequency of operator A and B,

$$\begin{aligned} \frac{d}{dt}\rho(t) &= \sum_\omega [\Gamma_{lk}(\omega) A_k(\omega) \rho A_l(\omega) - \Gamma_{lk}(\omega) \rho A_l(\omega) A_k(\omega) + h.c.] \\ &= \sum_\omega \gamma_{kl}(\omega)(A_{l,\omega} \rho A^\dagger_{k,\omega} - \frac{1}{2}\{\rho, A^\dagger_{k,\omega} A_{l,\omega}\}) - i\left[\sum_\omega S_{kl}(\omega) A^\dagger_{k,\omega} A_{l,\omega}, \rho\right], \end{aligned} \tag{5}$$

where we defined

$$\gamma_{kl}(\omega) = \Gamma_{kl}(\omega) + \Gamma^*_{lk}(\omega), \quad S_{kl}(\omega) = \frac{1}{2i}[\Gamma_{kl}(\omega) - \Gamma^*_{lk}(\omega)]. \tag{6}$$

The matrices $\gamma(\omega)$ are positive; we can then take the square root of them. The jump operator is then

$$L_{i,\omega} = \sum_j \sqrt{\gamma_{ij}(\omega)} A_{j,\omega}.$$

The evolution is then in the Lindblad form.