Gravity

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I. RIEMANN GEOMETRY

A. Connection

For a general coordinate system, we can choose a coordinate basis $\{e_\mu \equiv \frac{\partial}{\partial x^\mu}\}$ and define the connection:

$$\nabla_\mu e_\nu = e_\lambda \Gamma^\lambda{}_{\mu\nu}. \tag{1}$$

We immediately know the covariant derivative for vector field:

$$\nabla_\mu (W^\nu e_\nu) = \frac{\partial W^\nu}{\partial x^\mu} e_\nu + W^\nu e_\lambda \Gamma^\lambda{}_{\mu\nu} = \left(\frac{\partial W^\lambda}{\partial x^\mu} + \Gamma^\lambda{}_{\mu\nu} W^\nu \right) e_\lambda, \tag{2}$$

For the dual vector, consider the equation:

$$\nabla_\mu (W^\nu V_\nu) = (\partial_\mu W^\nu) V_\nu + W^\nu (\partial_\mu V_\nu) = \left(\frac{\partial W^\lambda}{\partial x^\mu} + \Gamma^\lambda{}_{\mu\nu} W^\nu \right) V_\lambda + W^\nu (\nabla_\mu V)_\nu, \tag{3}$$

which leads to $\nabla_\mu V_\nu = \partial_\mu V_\nu - V_\lambda \Gamma^\lambda{}_{\mu\nu}$. The covariant derivative is used to defined the *parallel transport*: let X be a vector field defined along curve $c(t)$. X is said to be parallel transported if

$$\nabla_V X = 0 \quad \Longrightarrow \quad \frac{dx^\mu}{dt} \left(\frac{\partial X^\lambda}{\partial x^\mu} + \Gamma^\lambda{}_{\mu\nu} X^\nu \right) = \frac{dX^\lambda}{dt} + \Gamma^\lambda{}_{\mu\nu} \frac{dx^\mu}{dt} X^\nu = 0. \tag{4}$$

where $V^\mu = \frac{d}{dt} x^\mu |_{c(t)}$. Further, a curve $c(t)$ is a geodesic if

$$\nabla_V V = 0 \quad \Longrightarrow \quad \frac{d^2 x^\mu}{dt^2} + \Gamma^\mu{}_{\nu\lambda} \frac{dx^\nu}{dt} \frac{dx^\lambda}{dt} = 0. \tag{5}$$

Note that the connection is not a tensor: for a coordinate transformation $x \to \tilde{x}$, the basis transforms as $\tilde{e}_\nu = (\partial x^\mu / \partial \tilde{x}^\nu) e_\mu$. According to (1), the new connection is

$$\nabla_{\tilde{e}_\mu} \tilde{e}_\nu = \tilde{e}_\lambda \tilde{\Gamma}^\lambda_{\mu\nu} = \frac{\partial x^\sigma}{\partial \tilde{x}^\mu} \nabla_\sigma \left(\frac{\partial x^\tau}{\partial \tilde{x}^\nu} e_\tau \right) = \frac{\partial x^\sigma}{\partial \tilde{x}^\mu} \left(\frac{\partial}{\partial x^\sigma} \frac{\partial x^\tau}{\partial \tilde{x}^\nu} e_\tau + \Gamma^\rho{}_{\sigma\tau} \frac{\partial x^\tau}{\partial \tilde{x}^\nu} e_\rho \right) \tag{6}$$

Therefore, the connection transforms as:

$$\tilde{\Gamma}^\lambda_{\mu\nu} = \Gamma^\rho{}_{\sigma\tau} \frac{\partial \tilde{x}^\lambda}{\partial x^\rho} \frac{\partial x^\sigma}{\partial \tilde{x}^\mu} \frac{\partial x^\tau}{\partial \tilde{x}^\nu} + \frac{\partial \tilde{x}^\lambda}{\partial x^\tau} \frac{\partial^2 x^\tau}{\partial \tilde{x}^\mu \partial \tilde{x}^\nu}. \tag{7}$$

B. Curvature

C. Vielbeins

II. THE EINSTEIN EQUATIONS

A. The Einstein-Hilbert Action

Given a Ricci scalar R, the action for gravitational field is

$$S = \frac{M_{\rm pl}^2}{2} \int d^4x \sqrt{-g} R. \tag{8}$$

B. Schwarzschild Spacetime

C. de Sitter Space