Quantum Electrodynamics

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The quantum electrodynamics is the field theory for the interaction of charged Dirac field with U(1) gauge field. The Lagrangian is obtained by minimal coupling:

$$\mathcal{L}_{\text{QED}} = \bar{\psi}\left(i\not{D} - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{1}$$

where the ordinary derivative is replaced by the covariant derivative:

$$\not{D} = \gamma^\mu D_\mu = \gamma^\mu[\partial_\mu - iqA_\mu(x)] = \not{\partial} - iq\not{A}. \tag{2}$$

The Lagrangian is invariant under the gauge transformation:

$$\psi(x) \to e^{iq\alpha(x)}\psi(x), \quad A^\mu(x) \to A^\mu(x) + \partial^\mu\alpha(x). \tag{3}$$

In the chapter, we first canonically quantize QED, and then relate the correlation function with the cross section. We then discuss some important elementary scattering processes. Finally we study the renormalization of the QED.

Contents

I. SCATTERING

A. $e^+e^- \to \mu^+\mu^-$

Consider the scattering process $(e_{p_1} + \bar{e}_{p_2} \to \mu_{p_3} + \bar{\mu}_{p_4})$. To the first order, the amplitude correspond to the simplest tree level diagram. Using the Feynman rule, this process gives the amplitude

$$i\mathcal{M} = (-ie)^2\bar{u}(p_3)\gamma^\mu v(p_4)\frac{-ig_{\mu\nu}}{(p_1+p_2)^2}\bar{v}(p_2)\gamma^\nu u(p_1). \tag{4}$$

Use the Mandelstam variables $s = p_1 + p_2$, the amplitude is

$$\mathcal{M} = \frac{e^2}{s}\left[\bar{u}(p_3)\gamma^\mu v(p_4)\right]\left[\bar{v}(p_2)\gamma_\mu u(p_1)\right]. \tag{5}$$

The complex conjugate of the amplitude is

$$\mathcal{M}^\dagger = \frac{e^2}{s} \left[\bar{u}(p_1)\gamma_\mu v(p_2)\right] \left[\bar{v}(p_4)\gamma^\mu u(p_3)\right]. \tag{6}$$

Note that we have use the relation

$$(\bar{u}\gamma^\mu v)^\dagger = v^\dagger \gamma^{\mu\dagger}\gamma^0 u = v^\dagger \gamma^0 \gamma^\mu u = \bar{v}\gamma^\mu u. \tag{7}$$

Therefore,

$$|\mathcal{M}|^2 = \frac{e^4}{s^2} \left[\bar{v}(p_4)\gamma^\mu u(p_3)\right] \left[\bar{u}(p_3)\gamma^\nu v(p_4)\right] \left[\bar{v}(p_2)\gamma_\mu u(p_1)\right] \left[\bar{u}(p_1)\gamma_\nu v(p_2)\right]. \tag{8}$$

If in the scattering experiment the spins are nor measured, the spin averaged cross section is just the sum of spin indices. The spin sum can actually simplifies the expression, as it gives the orthogonal relations

$$\sum_s u(p)_s \bar{u}_s(p) = \not{p} + m, \quad \sum_s v(p)_s \bar{v}_s(p) = \not{p} - m. \tag{9}$$

The spin averaged amplitude is

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{e^2}{4s^2} \operatorname{Tr}\left[(\not{p}_4 - m_\mu)\gamma^\mu(\not{p}_3 + m_\mu)\gamma^\nu\right] \operatorname{Tr}\left[(\not{p}_2 - m_e)\gamma_\mu(\not{p}_1 + m_e)\gamma_\nu\right]. \tag{10}$$

Now we evaluate the gamma traces, using the facts $\operatorname{Tr}[\gamma^\mu\gamma^\nu] = 4g^{\mu\nu}$ and $\operatorname{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = 4g^{\mu\nu}g^{\rho\sigma} - 4g^{\mu\rho}g^{\nu\sigma} + 4g^{\mu\sigma}g^{\nu\rho}$. The first trace is

$$\begin{aligned}\operatorname{Tr}\left[(\not{p}_4 - m_\mu)\gamma^\mu(\not{p}_3 + m_\mu)\gamma^\nu\right] &= \operatorname{Tr}\left[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu\right] p_{4\alpha}p_{3\beta} - m_\mu^2 \operatorname{Tr}[\gamma^\mu\gamma^\nu] \\ &= 4(p_3^\mu p_4^\nu + p_3^\nu p_4^\mu) - 4g^{\mu\nu}(p_3\cdot p_4 + m_\mu^2)\end{aligned} \tag{11}$$

The second is of the same form as the first, we can similarly get

$$\operatorname{Tr}\left[(\not{p}_2 - m_e)\gamma_\mu(\not{p}_1 + m_e)\gamma_\nu\right] = 4(p_{1\mu}p_{2\nu} + p_{1\nu}p_{2\mu}) - 4g_{\mu\nu}(p_1\cdot p_2 + m_e^2). \tag{12}$$

The final result of the spin average amplitude is

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{8e^4}{s^2}\left(p_{13}p_{24} + p_{14}p_{23} + m_\mu^2 p_{12} + m_e^2 p_{34} + 2m_e^2 m_\mu^2\right), \tag{13}$$

where $p_{ij} \equiv p_i \cdot p_j$. The amplitude is in a better form with the Mandelstam variables s, t and u:

$$\begin{aligned} s &= (p_1+p_2)^2 = (p_3+p_4)^2 = 2m_e^2 + 2p_{12} = 2m_\mu^2 + 2p_{34}, \\ t &= (p_1-p_3)^2 = (p_2-p_4)^2 = m_e^2 + m_\mu^2 - 2p_{13} = m_e^2 + m_\mu^2 - 2p_{24}, \\ u &= (p_1-p_4)^2 = (p_2-p_3)^2 = m_e^2 + m_\mu^2 - 2p_{14} = m_e^2 + m_\mu^2 - 2p_{23}. \end{aligned} \tag{14}$$

Note that $s + t + u = 2m_e^2 + 2m_\mu^2$. After some algebra, we get

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{2e^4}{s^2}\left[t^2 + u^2 + 4s\left(m_e^2 + m_\mu^2\right) - 2\left(m_e^2 + m_\mu^2\right)^2\right]. \tag{15}$$

B. $e^- p^+ \to e^- p^+$

Next we consider the Rutherford scattering ($e_{p_1} + p_{p_2} \to e_{p_3} + p_{p_4}$). This correspond to the same tree-level diagram as ($e_{p_1} + \bar{e}_{p_2} \to \mu_{p_3} + \bar{\mu}_{p_4}$) except a rotation. We immediately get the amplitude:

$$|\mathcal{M}|^2 = -\frac{e^4}{t^2}\left[\bar{u}(p_3)\gamma^\mu u(p_1)\right]\left[\bar{u}(p_1)\gamma^\nu u(p_3)\right]\left[\bar{u}(p_4)\gamma_\mu u(p_2)\right]\left[\bar{u}(p_2)\gamma_\nu u(p_4)\right]. \tag{16}$$

Note that the minus sign comes from the positive charge of proton. The spin-averaged amplitude is

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{e^2}{4t^2}\operatorname{Tr}\left[(\not{p}_3 + m_e)\gamma^\mu(\not{p}_1 + m_e)\gamma^\nu\right]\operatorname{Tr}\left[(\not{p}_4 + m_p)\gamma_\mu(\not{p}_2 + m_p)\gamma_\nu\right]. \tag{17}$$

The trace is evaluated similarly:

$$\begin{aligned}\operatorname{Tr}\left[(\not{p}_3 + m_e)\gamma^\mu(\not{p}_1 + m_e)\gamma^\nu\right] &= 4(p_1^\mu p_3^\nu + p_1^\nu p_3^\mu) - 4g^{\mu\nu}(p_{13} - m_e^2),\\ \operatorname{Tr}\left[(\not{p}_4 + m_p)\gamma_\mu(\not{p}_2 + m_p)\gamma_\nu\right] &= 4(p_{2\mu}p_{4\nu} + p_{2\nu}p_{4\mu}) - 4g_{\mu\nu}(p_{24} - m_p^2).\end{aligned} \tag{18}$$

Therefore,

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{8e^4}{t^2}\left(p_{12}p_{34} + p_{14}p_{23} - m_p^2 p_{13} - m_e^2 p_{24} + 2m_e^2 m_p^2\right). \tag{19}$$

The Mandelstam variables are

$$\begin{aligned}s &= m_e^2 + m_p^2 + 2p_{12} = m_e^2 + m_p^2 + 2p_{34},\\ t &= 2m_e^2 - 2p_{13} = 2m_p^2 - 2p_{24},\\ u &= m_e^2 + m_p^2 - 2p_{14} = m_e^2 + m_p^2 - 2p_{23}.\end{aligned} \tag{20}$$

After some algebra we get

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{2e^4}{t^2}\left[u^2 + s^2 + 4t\left(m_e^2 + m_p^2\right) - 2\left(m_e^2 + m_p^2\right)^2\right]. \tag{21}$$

C. $\gamma e^- \to \gamma e^-$

Next we consider the Compton scattering ($\gamma_k + e_p \to \gamma_{k'} + e_{p'}$). There are two corresponding tree-level diagrams: one is an s-process, the other is a u-process. The amplitude for s-process is

$$\mathcal{M}_s = -\frac{e^2}{s - m_e^2}\epsilon_\mu(k')\bar{u}(p')\gamma^\mu(\not{p} + \not{k} + m_e)\gamma^\nu u(p)\epsilon_\nu(k), \tag{22}$$

and the amplitude for u-process is

$$\mathcal{M}_u = -\frac{e^2}{u - m_e^2}\epsilon_\mu(k)\bar{u}(p')\gamma^\mu(\not{p} - \not{k}' + m_e)\gamma^\nu u(p)\epsilon_\nu(k'). \tag{23}$$

Together, the coherent amplitude is

$$\mathcal{M} = e^2\epsilon_\mu(k')\bar{u}(p')\left[\frac{\gamma^\mu(\not{p} + \not{k} + m_e)\gamma^\nu}{s - m_e^2} + \frac{\gamma^\nu(\not{p} - \not{k}' + m_e)\gamma^\mu}{u - m_e^2}\right]u(p)\epsilon_\nu(k). \tag{24}$$

The conjugate amplitude is

$$\mathcal{M}^\dagger = e^2\epsilon_\alpha(k)\bar{u}(p)\left[\frac{\gamma^\alpha(\not{p} + \not{k} + m_e)\gamma^\beta}{s - m_e^2} + \frac{\gamma^\beta(\not{p} - \not{k}' + m_e)\gamma^\alpha}{u - m_e^2}\right]u(p')\epsilon_\beta(k'). \tag{25}$$

If we do note measure the photon polarization, we can use the polarization averaged result. The photon polarization sum is

$$\sum_i \epsilon_\mu^i(k)\epsilon_\nu^i(k) = -g_{\mu\nu} + \frac{1}{2E^2}(p_\mu\bar{p}_\nu + \bar{p}_\mu p_\nu). \tag{26}$$

The second term does not contribute to the amplitude due to the Ward identity. We can then simply replace the polarization sum with the metric $-g_{\mu\nu}$. In this way,

$$\frac{1}{4}\sum_{\text{s,p}}|\mathcal{M}_s|^2 = \frac{e^4}{(s - m_e^2)^2}\operatorname{Tr}\left[(\not{p}_2 - m_e)\gamma_\nu(\not{p}_1 + \not{p}_2 + m_e)\gamma_\mu(\not{p}_4 + m_e)\gamma^\mu(\not{p}_1 + \not{p}_2 + m_e)\gamma^\nu\right]. \tag{27}$$

II. PERTURBATIVE RENORMALIZATION

As with the scalar field, the partition function with source is defined as

$$Z[\bar{\eta},\eta,J]=\exp\left\{i\int d^dx\mathcal{L}_{\text{int}}\left[\frac{\delta}{i\delta J},\frac{\delta}{i\delta\eta},\frac{i\delta}{\delta\bar{\eta}}\right]\right\}Z_0[\bar{\eta},\eta,J]. \tag{28}$$

In the following, we will use the dimensional regularization by default. Note that the mass dimensionality for the fermion and gauge field is

$$[\psi]=\left[\frac{d-1}{2}\right],\quad [A]=\left[\frac{d}{2}-1\right],$$

which lead to the coupling constant e to have the dimension

$$[e]=\left[2-\frac{d}{2}\right].$$

When $d=4-\epsilon$, we make the replacement

$$e\to e\tilde{\mu}^{\epsilon/2},$$

so that to make the coupling constant e dimensionless.

The interaction will renormalize the field and coefficients to

$$\begin{aligned}\psi&=\sqrt{Z_\psi}\psi_R,\quad m=Z_m m_R,\\ A&=\sqrt{Z_A}A_R,\quad\ \ e=Z_e e_R.\end{aligned} \tag{29}$$

The renormalized Lagrangian is

$$\begin{aligned}\mathcal{L}&=Z_\psi\bar{\psi}_R(i\gamma^\mu\partial_\mu)\psi_R-Z_mZ_\psi m_R\bar{\psi}_R\psi_R+Z_A\frac{1}{4}F_{R,\mu\nu}F_R^{\mu\nu}-Z_eZ_\psi\sqrt{Z_A}e_RA_R^\mu\bar{\psi}_R\gamma^\mu\psi_R\\ &=\mathcal{L}_0+\mathcal{L}_{\text{int}}+\mathcal{L}_{\text{ct}}.\end{aligned} \tag{30}$$

The we define the coefficients

$$\delta_\psi=Z_\phi-1,\quad \delta_m=Z_m-1,\quad \delta_A=Z_A-1,\quad \delta_1=Z_eZ_\psi\sqrt{Z_A}-1. \tag{31}$$

The counter term also contribute to the perturbative expansion like the interactions. The counter terms for the fermion propagator come from the diagram expression:

$$iG_F^{(\text{ct})}(p)=$$

$$=G_F^{(0)}(p)\left[\delta_\psi\not{p}-(\delta_m+\delta_\psi)m_R\right]G_F^{(0)}(p). \tag{32}$$

The contribution to the electron self energy is

$$i\Sigma^{(\text{ct})}(p)\equiv\delta_\psi\not{p}-(\delta_m+\delta_\psi)m_R. \tag{33}$$

Similarly, the counter term contribution to the photon self energy is

$$i\Pi_{\mu\nu}^{(\text{ct})}(k)=\ \mu$$

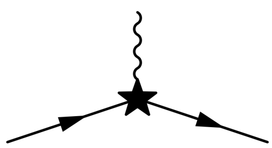
$$\nu$$

$$=\delta_A[-p^2g_{\mu\nu}+(1-\xi)p_\mu p_\nu]. \tag{34}$$

In Landau gauge, $\xi=1$ and we can ignore the $p_\mu p_\nu$ terms, and the photon self-energy is always proportional to $g_{\mu\nu}$. Whenever other gauge choices are concerned, we can simply replace $g_{\mu\nu}$ with $g_{\mu\nu}-(1-\xi)\frac{p_\mu p_\nu}{p^2}$.

The counter term contribution to the QED vertex is

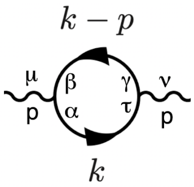
$$\Gamma_{\alpha\beta}^{(\text{ct})\mu}=$$



$$=-\delta_1\gamma_{\alpha\beta}^\mu. \tag{35}$$

A. Vacuum Polarization

Consider the one-loop correction to the photon propagator:



$$\begin{aligned} &\simeq (-ie_R)^2 A_\mu \bar{\psi}_\alpha \gamma^\mu_{\alpha\beta} \psi_\beta A_\nu \bar{\psi}_\gamma \gamma^\nu_{\gamma\tau} \psi_\tau \\ &\equiv i A_\mu \Pi^{\mu\nu}(p) A_\nu. \end{aligned} \tag{36}$$

The self energy is:

$$\begin{aligned} i\Pi^{\mu\nu}(p) &= -e_R^2 \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left[\gamma^\mu G_F^{(0)}(k-p)\gamma^\nu G_F^{(0)}(k)\right] \\ &= -e_R^2 \int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{Tr}\left[\gamma^\mu(\not{k} - \not{p} + m_R)\gamma^\nu(\not{k} + m)\right]}{(k^2 - m_R^2)[(p-k)^2 - m_R^2]}. \end{aligned} \tag{37}$$

The trace of the Dirac matrices can be evaluated in `Mathematic` using the `FeynCalc` package:

```
(*Dirac trace using FeynCalc`*)
res=DiracTrace[GA[\[Mu]].(GS[k−p]+m).GA[\[Nu]].(GS[k]+m)];
DiracSimplify[res]
```

The Dirac trace is:

$$\begin{aligned} &\mathrm{Tr}\left[\gamma^\mu(\not{k} - \not{p} + m_R)\gamma^\nu(\not{k} + m_R)\right] \\ &= 4\left[g^{\mu\nu}\left(k\cdot p - k^2 + m_R^2\right) + 2k^\mu k^\nu - k^\mu p^\nu - p^\mu k^\nu\right]. \end{aligned} \tag{38}$$

Using the Feynman parameters, the denominator is:

$$\begin{aligned} \frac{1}{(k^2 - m_R^2)[(p-k)^2 - m_R^2]} &= \frac{1}{\left\{[k - p(1-x)]^2 - [m_R^2 + p^2 x(x-1)]\right\}^2} \\ &\equiv \frac{1}{\left\{[k - p(1-x)]^2 - D_x\right\}^2}. \end{aligned} \tag{39}$$

Since the Ward identity requires that the p^μ term in the propagator do not contribute to any scattering process, we then shift $k \to k + p(1-x)$ and drop all p^μ linear term. The final result is simplified to:

$$\begin{aligned} i\Pi^{\mu\nu}(p) &= -4e_R^2 \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{2k^\mu k^\nu - g^{\mu\nu}\left[k^2 - x(1-x)p^2 - m_R^2\right]}{\left[k^2 - D_x\right]^2} \\ &\simeq 4e_R^2 g^{\mu\nu} \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{\frac{1}{2}k^2 - x(1-x)p^2 - m_R^2}{\left[k^2 - D_x\right]^2} \\ &\simeq -ie_R^2 g^{\mu\nu} \int_0^1 dx\, \frac{\Omega_d \tilde{\mu}^\varepsilon}{(2\pi)^d} \int_0^\infty dk\ k^{d-1} \frac{\left(4 - \frac{2}{d}\right)k^2 + 4x(1-x)p^2 + 4m_R^2}{\left[k^2 + D_x\right]^2}. \end{aligned} \tag{40}$$

where we have made the Wick rotation, shifted the dimensionality to ($d = 4 - \varepsilon$), and made the substitution (since the self-energy $i\Pi^{\mu\nu} \propto g^{\mu\nu}$):

$$k^\mu k^\nu \to \frac{1}{d} k^2 g^{\mu\nu}. \tag{41}$$

The remaining problem is to regularize and renormalize the divergent integral

$$I_\varepsilon(x) \equiv \frac{\Omega_{4-\varepsilon}\tilde{\mu}^\varepsilon}{(2\pi)^{4-\varepsilon}} \int_0^\infty dk\ k^{3-\varepsilon} \frac{\left(4 - \frac{8}{4-\varepsilon}\right)k^2 + 4x(1-x)p^2 + 4m_R^2}{\left[k^2 + D_x\right]^2}. \tag{42}$$

1. Regularization and Renormalization

In $(4-\varepsilon)$-dimensional Euclidean space, the integral is convergent. The ε-expansion is carried out in Mathematica using the following code:

```
omg = (2*Pi^(d/2))/(Gamma[d/2]);
cof = \[Mu]^(4-d)*omg/(2*Pi)^d;
nom = k^(d-1)*((4-8/(4-\[Epsilon]))k^2+4x*(1-x)p^2+4m^2);
int = cof*Integrate[nom/(k^2+D)^2,{k,0,Infinity}][[1]];
map = D->m^2-p^2*x*(1-x);
ans = Series[int/.{d->4-\[Epsilon]},{\[Epsilon],0,0}];
ans /. map // Simplify
```

The result is

$$I_\varepsilon(x) = \frac{p^2 x(1-x)}{2\pi^2}\left[\frac{2}{\varepsilon} + \ln\left(\frac{4\pi e^{-\gamma_E}\tilde{\mu}^2}{m_R^2 - p^2 x(1-x)}\right)\right] \tag{43}$$

So the photon self-energy is (also denote $\mu^2 = 4\pi e^{-\gamma_E}\tilde{\mu}^2$):

$$\Pi^{\mu\nu}(p) = -\frac{e_R^2 p^2 g^{\mu\nu}}{6\pi^2\varepsilon} - \frac{e_R^2 p^2 g^{\mu\nu}}{2\pi^2}\int_0^1 dx\; x(1-x)\ln\left[\frac{\mu^2}{m_R^2 - p^2 x(1-x)}\right] \tag{44}$$

The counter term coefficient can be chosen as

$$\delta_A = -\frac{e_R^2}{6\pi^2\epsilon}. \tag{45}$$

The renormalized photon self-energy is then

$$\begin{aligned}\Pi^{\mu\nu}(p) &= -\frac{e_R^2 p^2 g^{\mu\nu}}{2\pi^2}\int_0^1 dx\; x(1-x)\ln\left[\frac{\mu^2}{m_R^2 - p^2 x(1-x)}\right] \\ &= \frac{e_R^2 p^2 g^{\mu\nu}}{2\pi^2}\left\{\frac{1}{3}\ln\left(\frac{m_R}{\mu}\right) + \int_0^1 dx\; x(1-x)\ln\left[1 - \frac{p^2 x(1-x)}{m_R^2}\right]\right\}.\end{aligned} \tag{46}$$

2. Physical Observable

The photon self-energy has the form

$$\Pi^{\mu\nu}(p) = -e_R^2\left[g^{\mu\nu} - (1-\xi)p^\mu p^\nu\right] g^{\mu\nu}\Pi_2(p), \tag{47}$$

where

$$\Pi_2(p) = \frac{1}{2\pi^2}\int_0^1 dx\; x(1-x)\ln\left[\frac{\mu^2}{m_R^2 - p^2 x(1-x)}\right] \tag{48}$$

The one-loop correction to photon propagator is

$$\begin{aligned} iG_\gamma^{\mu\nu}(p) &= -i\frac{g^{\mu\nu}}{p^2}\left(1 + \sum_{n=1}^{\infty}(-e_R^2)^n \Pi_2^n(p)\right) \\ &= -i\frac{g^{\mu\nu}}{p^2\left[1 + e_R^2\Pi_2(p)\right]}.\end{aligned} \tag{49}$$

We can choose the on-shell condition that the photon has no rest mass:

$$\Pi_2(0) = 0 \quad\Longrightarrow\quad \mu = m_R. \tag{50}$$

Note that the propagator is related to the Coulomb potential.[1] To the second order,

$$\begin{aligned} V(p) &= e_R^2 \frac{1 - e_R^2 \Pi_2(p)}{p^2} + O(e_R^6) \\ &= \frac{e_R^2}{p^2} \left\{ 1 + \frac{e_R^2}{2\pi^2} \int_0^1 dx\; x(1-x) \ln \left[1 - \frac{p^2 x(1-x)}{m_R^2} \right] + O(e_R^4) \right\}. \end{aligned} \tag{51}$$

Consider the small momentum limit, where the integral in approximated by

$$\int_0^1 dx\; x(1-x) \ln \left[1 - \frac{p^2 x(1-x)}{m_R^2} \right] \approx -\frac{p^2}{m_R^2} \int_0^1 dx\; x^2 (1-x)^2 = -\frac{p^2}{30 m_R^2}. \tag{52}$$

This implies

$$V(p) = \frac{e_R^2}{p^2} - \frac{e_R^4}{60\pi^2 m_R^2}. \tag{53}$$

The Fourier transformation gives

$$V(r) = -\frac{e_R^2}{4\pi r} - \frac{e_R^4}{60\pi^2 m_R^2} \delta^{(3)}(r). \tag{54}$$

For atomic orbit, since only the ($L = 0$)-orbit have support at $r = 0$, this extra potential will shift the spectrum. This effect is called the *Lamb shift*.

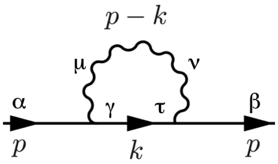
On the other hand, the the large momentum limit,

$$V(p) \approx \frac{e_R^2}{p^2} \left[1 + \frac{e_R^2}{12\pi^2} \ln \frac{-p^2}{m_R^2} \right], \tag{55}$$

which predicts a *Landau pole* beyond which perturbation theory breaks down.

B. One-loop Correction to Electron Propagator

Consider the one-loop correction to the particle propagator:



$$\simeq (-ie_R)^2 A_\mu \bar{\psi}_\alpha \gamma^\mu_{\alpha\gamma} \psi_\gamma A_\nu \bar{\psi}_\tau \gamma^\nu_{\tau\beta} \psi_\beta \equiv i\bar{\psi}_\alpha \Sigma^{\alpha\beta}(p) \psi_\beta. \tag{56}$$

The self energy is

$$\begin{aligned} i\Sigma_{\alpha\beta}(p) &= e_R^2 \int \frac{d^4 k}{(2\pi)^4} G_\gamma^{\mu\nu}(p-k) \left[\gamma_\mu G_F(k) \gamma_\nu \right]_{\alpha\beta} \\ &= -e_R^2 \int \frac{d^d k}{(2\pi)^d} \frac{\gamma^\mu (\not{k} + m_R) \gamma_\mu}{(p-k)^2 (k^2 - m_R^2)}. \end{aligned} \tag{57}$$

The seconded equality comes from the contraction:

$$(-ie)^2 A_\mu \bar{\psi}_\alpha \gamma^\mu_{\alpha\gamma} \psi_\gamma A_\nu \bar{\psi}_\tau \gamma^\nu_{\tau\beta} \psi_\beta \tag{58}$$

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[1] The Coulomb potential arises just like we derive the force (**??**), but the sources have additional charge e_R, and the photon is mass less, so $V(p) = \frac{e_R^2}{p^2}$ for free field.

The nominator can be simplified using the Dirac matrix identities:

$$\gamma^\mu \gamma_\mu = d, \quad \gamma^\mu \gamma^\nu \gamma_\mu = (2-d)\gamma^\nu \quad \Longrightarrow \quad \gamma^\mu (\not{k} + m_R)\gamma_\mu = d m_R + (2-d)\not{k}. \tag{59}$$

The denominator can be simplify using the Feynman parameter:

$$\begin{aligned} \frac{1}{(p-k)^2(k^2-m_R^2)} &= \int_0^1 \frac{dx}{\left[(k-px)^2 - (1-x)(m_R^2 - p^2 x)\right]^2} \\ &\to \int_0^1 \frac{dx}{\left(k^2 - D_x\right)^2} \end{aligned} \tag{60}$$

where we have shifted $k \to k + px$ (note this shift also change the numerator).

The self energy becomes (including a $\tilde{\mu}$ mass scale):

$$\begin{aligned} i\Sigma(p) &= e_R^2 \tilde{\mu}^\varepsilon \int_0^1 [(2-\varepsilon)x\not{p} - (4-\varepsilon)m_R] dx \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - D_x)^2} \\ &= i e_R^2 \int_0^1 dx \; [(2-\varepsilon)x\not{p} - (4-\varepsilon)m_R] \, \frac{\tilde{\mu}^\varepsilon \Omega_d}{(2\pi)^d} \int \frac{k^{d-1} dk}{(k^2 + D_x)^2}. \end{aligned} \tag{61}$$

1. Regularization and Renormalization

The regularization procedure is carried out by the following code:

```
omg=(2*Pi^(d/2))/(Gamma[d/2]);
cof=\[Mu]^(4-d)*omg/(2*Pi)^d;
int=cof*Integrate[q^(d-1)/(q^2+D)^2,{q,0,Infinity}][[1]];
map={e->Sqrt[4*Pi*\[Alpha]],EulerGamma->Subscript[\[Gamma],E]};
ans=Series[int/.{d->4-\[Epsilon]},{\[Epsilon],0,0}];
ans/.map//Simplify
```

The result is ($\mu^2 = 4\pi e^{-\gamma_E} \tilde{\mu}^2$)

$$\begin{aligned} \Sigma(p) &= \frac{e_R^2}{16\pi^2} \int_0^1 dx \, [(2-\varepsilon)x\not{p} - (4-\varepsilon)m_R] \left[\frac{2}{\varepsilon} + \ln \frac{\mu^2}{(1-x)(m_R^2 - p^2 x)}\right] \\ &= \frac{e_R^2}{16\pi^2} \left\{ \int_0^1 dx \, [2x\not{p} - 4m_R] \left[\frac{2}{\varepsilon} + \ln \frac{\mu^2}{(1-x)(m_R^2 - p^2 x)}\right] - \not{p} + 2m_R \right\}. \end{aligned} \tag{62}$$

The infinity comes from

$$\frac{e_R^2}{4\pi^2 \varepsilon} \int_0^1 dx (x\not{p} - 2m_R) = \frac{e_R^2}{8\pi^2 \varepsilon} \not{p} - \frac{e_R^2}{2\pi^2 \varepsilon} m_R. \tag{63}$$

Using the $\overline{\text{MS}}$ subtraction scheme, we choose

$$\delta_\psi = -\frac{e_R^2}{8\pi^2 \varepsilon}, \quad \delta_m = -\frac{3 e_R^2}{8\pi^2 \varepsilon}, \tag{64}$$

and the self energy is

$$\Sigma(p) = \frac{e_R^2}{16\pi^2} \left\{ \int_0^1 dx (2x\not{p} - 4m_R) \ln \left[\frac{\mu^2}{(1-x)(m_R^2 - p^2 x)}\right] - \not{p} + 2m_R \right\}. \tag{65}$$

2. Physical Observables

The Dyson series gives:

$$iG_F(p) = \frac{i}{\not{p} - m_R + \Sigma(p)} \tag{66}$$

Experimentally, for a given The on-shell subtraction requires that the m_R equals to the physical mass:

$$\Sigma(\not p)\big|_{\not p=m_R}=0,\qquad \frac{d}{d\not p}\Sigma(\not p)\bigg|_{\not p=m_R}=0. \tag{67}$$

To implement the on-shell condition, we have to modify the subtraction scheme to

$$\begin{aligned}\delta_\psi &= -\frac{e_R^2}{8\pi^2}\left(\frac{1}{\varepsilon}+\ln\frac{\mu}{m_R}+A\right),\\ \delta_m &= -\frac{e_R^2}{8\pi^2}\left(\frac{3}{\varepsilon}+3\ln\frac{\mu}{m_R}+B\right),\end{aligned} \tag{68}$$

and the self energy is

$$\begin{aligned}\Sigma(\not p) = &-\frac{e_R^2}{8\pi^2}\int_0^1 dx(x\not p-2m_R)\ln\left[(1-x)\left(1-\frac{p^2}{m_R^2}x\right)\right]\\ &-\frac{e_R^2}{8\pi^2}\left[\left(A+\frac{1}{2}\right)\not p-(A+B+1)m_R\right].\end{aligned} \tag{69}$$

The first condition

$$\begin{aligned}\Sigma(\not p)\big|_{\not p=m_R} &= -\frac{e_R^2}{8\pi^2}m_R\left[\int_0^1 dx\ (x-2)\ln(1-x)^2-B-\frac{1}{2}\right]\\ &= -\frac{e_R^2}{8\pi^2}m_R\,(2-B)=0\end{aligned} \tag{70}$$

gives the mass renormalization coefficient

$$\delta_m = -\frac{e_R^2}{8\pi^2}\left(\frac{3}{\varepsilon}+3\ln\frac{\mu}{m_R}+2\right). \tag{71}$$

While in the derivative of the self-energy:

$$\frac{d}{d\not p}\Sigma(m_R) = -\frac{e_R^2}{8\pi^2}\left\{\int_0^1 dx\ \left[x\ln(1-x)^2-\frac{2x(x-2)}{1-x}\ln(1-x)\right]+A+\frac{1}{2}\right\}, \tag{72}$$

there is a divergent integral:

$$\int_0^1 dx\ \frac{2x(x-2)}{1-x}\ln(1-x), \tag{73}$$

indicating an IR divergence. We can never the less get rid of it by introducing a small mass m_γ for photon (which will be set to zero). This mass term change the denominator in the loop integral:

$$\frac{1}{\left[(p-k)^2-m_\gamma^2\right](k^2-m_R^2)}=\int_0^1\frac{dx}{\left[(k-px)^2-D_x-xm_\gamma^2\right]^2}. \tag{74}$$

Most derivation remains the same, we just need to make a substitution in the finial result:

$$D_x\to D_x+xm_\gamma^2. \tag{75}$$

Especially, the introducing of the photon mass will not change the result of the mass renormalization factor we have computed.

The modified self-energy is then

$$\begin{aligned}\Sigma(\not p) = &-\frac{e_R^2}{8\pi^2}\int_0^1 dx(x\not p-2m_R)\ln\left[(1-x)\left(1-\frac{p^2}{m_R^2}x\right)+x\frac{m_\gamma^2}{m_R^2}\right]\\ &-\frac{e_R^2}{8\pi^2}\left[\left(A+\frac{1}{2}\right)\not p-(A+B+1)m_R\right].\end{aligned} \tag{76}$$

The derivative is now

$$\frac{d}{d\not{p}}\Sigma(m_R) = -\frac{e_R^2}{8\pi^2}\left\{\int_0^1 dx\ \left[x\ln(1-x)^2 + \frac{2x(2-x)(1-x)}{(1-x)^2 + x\frac{m_\gamma^2}{m_R^2}}\right] + A + \frac{1}{2}\right\}, \tag{77}$$

Note that in the $(m_\gamma \to 0)$ limit, the asymptotic behavior of the originally divergent integral is

$$\lim_{m_\gamma\to 0}\int_0^1 dx\ \frac{2x(2-x)(1-x)}{(1-x)^2 + x\frac{m_\gamma^2}{m_R^2}} = -1 - 2\ln\frac{m_\gamma}{m_R}. \tag{78}$$

So the second subtraction condition is:

$$\frac{d}{d\not{p}}\Sigma(m_R) = -\frac{e_R^2}{8\pi^2}\left(A - 2 - 2\frac{m_\gamma}{m_R}\right) = 0. \tag{79}$$

The field strength renormalization is

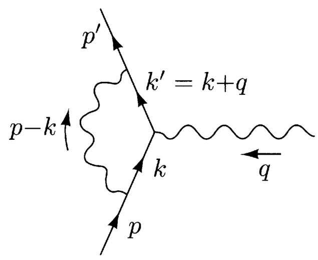
$$\delta_\psi = -\frac{e_R^2}{8\pi^2}\left(\frac{1}{\varepsilon} + \ln\frac{\mu}{m_R} + 2 + 2\ln\frac{m_\gamma}{m_R}\right). \tag{80}$$

The final self energy is (shall take the $m_\gamma \to 0$ limit):

$$\begin{aligned}\Sigma(\not{p}) = &-\frac{e_R^2}{16\pi^2}\int_0^1 dx(2x\not{p} - 4m_R)\ln\left[(1-x)\left(1 - \frac{p^2}{m_R^2}x\right) + 2x\ln\frac{m_\gamma}{m_R}\right] \\ &-\frac{e_R^2}{16\pi^2}\left[\left(5 + 4\ln\frac{m_\gamma}{m_R}\right)\not{p} - \left(10 + 4\ln\frac{m_\gamma}{m_R}\right)m_R\right].\end{aligned} \tag{81}$$

C. One-loop Correction to Vertex

Consider the one-loop correction to interaction:



$$\simeq (-ie_R)^3 A_\nu\bar{\psi}_\alpha\gamma^\nu_{\alpha\beta}\psi_\beta A_\mu\bar{\psi}_\lambda\gamma^\mu_{\lambda\tau}\psi_\tau A_\xi\bar{\psi}_\rho\gamma^\nu_{\rho\sigma}\psi_\sigma \tag{82}$$

$$\equiv -ieA_\mu\Gamma^\mu_{\alpha\beta}(q,p,p')\bar{\psi}_\alpha\psi_\beta.$$

The vertex function is:

$$\begin{aligned}i\Gamma^\mu_{\alpha\beta}(q,p,p') &= -e_R^2\int\frac{d^4k}{(2\pi)^4}G_\gamma^{\nu\lambda}(p-k)[\gamma_\nu G_F(k')\gamma^\mu G_F(k)\gamma_\lambda]_{\alpha\beta} \\ &= e_R^2\int\frac{d^4k}{(2\pi)^4}\frac{[\gamma^\nu(\not{k}' + m_R)\gamma^\mu(\not{k} + m_R)\gamma_\nu]_{\alpha\beta}}{(k^2 - m_R^2)(k'^2 - m_R^2)(p-k)^2}\end{aligned} \tag{83}$$

Using the following code

```
(*numerator*)
den=Contract[GA[\[Nu]].(GS[kp]+m).GA[\[Mu]].(GS[k]+m).GA[\[Nu]]];
DiracSimplify[den]

(*Feynman parameter*)
A1=k^2-m^2;
```

```
A2=(k+q)^2-m^2;
A3=(p-k)^2;
{c,b,a}=CoefficientList[x*A1+y*A2+z*A3,{k}];
-b/2//Simplify
-c+b^2/4//Simplify
```

The numerator is

$$-2\not{k}\gamma^\mu\not{k}' - 2m_R^2\gamma^\mu + 4m_R(k+k')^\mu. \tag{84}$$

The denominator is

$$\int \frac{dF_3}{[(k+yq-zp)^2 - D_{xyz}]^3}, \tag{85}$$

where

$$\begin{aligned} D_{xyz} &= (x+y)m^2 - z(1-z)p^2 - y(1-y)q^2 - 2yzpq \\ &= (x+y)m_R^2 - xyq^2 - yzp'^2 - xzp^2. \end{aligned}$$

Shift $k^\mu \to k^\mu + zq_1^\mu - yp^\mu$, throw away all terms with linear k^μ, and replace $k^\mu k^\nu$ with $\frac{1}{d}k^2 g^{\mu\nu}$, the result is

$$\frac{4}{d}k^2\gamma^\mu - 2(-y\not{q} + z\not{p})\gamma^\mu[(1-y)\not{q} + z\not{p}] + 4m_R^2\gamma^\mu - 2m_R\left[(1-2y)q^\mu + 2zp^\mu\right].$$

Note that only the quadratic term is divergent.

$$\Gamma^\mu(p,q_1,q_2) = -i\frac{4e^2\tilde{\mu}^\epsilon\gamma^\mu}{d}\int dF_3\int\frac{d^dk}{(2\pi)^d}\frac{k^2}{(k^2-D)^3} + \delta\Gamma^\mu(p,q_1,q_2).$$

where $\delta\Gamma^\mu$ stores all the finite part

$$\begin{aligned} &\delta\Gamma^\mu(p,q_1,q_2) \\ &= \int\frac{e^2k^3dkdF_3}{(2\pi)^2(k^2+D)^3}\left\{(-y\not{q}+z\not{p})\gamma^\mu[(1-y)\not{q}+z\not{p}] - 2m_R^2\gamma^\mu + m_R\left[(1-2y)q^\mu + 2zp^\mu\right]\right\}. \end{aligned}$$

The divergent part is

$$\frac{4e^2\tilde{\mu}^\epsilon\Omega_d\gamma^\mu}{d(2\pi)^d}\int dF_3\int\frac{k^{d+1}dk}{(k^2+D)^3} = \frac{e_R^2}{16\pi^2}\gamma^\mu\int dF_3\left(\frac{2}{\epsilon} + \ln\frac{\mu^2}{D_{xyz}}\right). \tag{86}$$

Using the $\overline{\text{MS}}$ scheme, the coefficient of the counter term is

$$\delta_e = -\frac{e_R^2}{8\pi^2\varepsilon}. \tag{87}$$

III. SYSTEMATIC RENORMALIZATION

A. Renormalization Group

In summery, the renormalization factors are

$$\begin{aligned} Z_\psi &= 1 - \frac{e_R^2}{8\pi^2\epsilon} + O(e_R^3), \\ Z_A &= 1 - \frac{e_R^2}{6\pi^2\epsilon} + O(e_R^3), \\ Z_m &= 1 - \frac{e_R^2}{2\pi^2\epsilon} + O(e_R^3), \\ Z_e &= 1 - \frac{e_R^2}{8\pi^2\epsilon} + O(e_R^3), \end{aligned} \tag{88}$$

which means

$$\begin{aligned}\frac{d\ln Z_\phi}{de_R} &= -\frac{e_R}{4\pi^2\epsilon} + O(e_R^2),\\ \frac{d\ln Z_A}{de_R} &= -\frac{e_R}{3\pi^2\epsilon} + O(e_R^2),\\ \frac{d\ln Z_m}{de_R} &= -\frac{e_R}{\pi^2\epsilon} + O(e_R^2),\\ \frac{d\ln Z_e}{de_R} &= -\frac{e_R}{4\pi^2\epsilon} + O(e_R^2).\end{aligned} \tag{89}$$

The bare parameters are

$$\begin{aligned}\psi_0 &= Z_\psi^{1/2}\psi_R,\\ A_0 &= Z_A^{1/2}A_R,\\ m_0 &= Z_m Z_\psi^{-1} m_R,\\ e_0 &= Z_e Z_\psi^{-1} Z_A^{-1/2} e_R \tilde{\mu}^{\epsilon/2}.\end{aligned} \tag{90}$$

The RG equation for e_0 is

$$\frac{d\ln e_0}{d\ln\mu} = \left(\frac{\ln Z_e}{de_R} - \frac{\ln Z_\psi}{de_R} - \frac{1}{2}\frac{\ln Z_A}{de_R} + \frac{1}{e_R}\right)\frac{de_R}{d\ln\mu} + \frac{\epsilon}{2} = 0. \tag{91}$$

The beta function is

$$\beta(e_R) = \frac{de_R}{d\ln\mu} = -\frac{\epsilon}{2}e_R + \frac{e_R^3}{12\pi^2} + O(e_R^4). \tag{92}$$

The RG equation for m_0 is

$$\frac{d\ln m_0}{d\ln\mu} = \left(\frac{\ln Z_m}{de_R} - \frac{\ln Z_\psi}{de_R}\right)\frac{de_R}{d\ln\mu} + + \frac{1}{m_R}\frac{dm_R}{d\ln\mu} = 0. \tag{93}$$

The anomalous dimension of mass is

$$\gamma_m = \frac{d\ln m_R}{d\ln\mu} = -\frac{3e_R^2}{8\pi^2} + O(e_R^3). \tag{94}$$