Notes on Quantum Field Theory

Jie Ren

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Chapter 1

Real Klein-Gordon Field Theory

In this note, we use the $(+,-,-,-)$ metric, where the inner product of two 4-momentum and 4-coordinate is

$$k \cdot x = \omega t - \vec{k} \cdot \vec{x}. \tag{1.1}$$

In this chapter, we consider the real Klein-Gordon field. The free field Lagrangian density:

$$\mathcal{L}_0 = \frac{1}{2}\partial^\mu \phi \partial_\mu \phi - \frac{m^2}{2}\phi^2 \simeq -\frac{1}{2}\phi(\partial^2 + m^2)\phi. \tag{1.2}$$

The action for free field with source is

$$S_0[\phi, J] = \int d^d x \left(\frac{1}{2}\partial^\mu \phi \partial_\mu \phi - \frac{m^2}{2}\phi^2 + J \cdot \phi \right). \tag{1.3}$$

The space-time Fourier transformation is defined as

$$\begin{aligned}
\tilde{\phi}(k) &= \int d^d x e^{ik\cdot x} \phi(x), \\
\phi(x) &= \int \frac{d^d k}{(2\pi)^d} e^{-ik\cdot x} \tilde{\phi}(k).
\end{aligned} \tag{1.4}$$

The Lagrangian in momentum space is

$$\tilde{\mathcal{L}}_0[\phi_k, J] = \tilde{\phi}(k)(k^2 - m^2)\tilde{\phi}(-k) + \tilde{J}(k) \cdot \tilde{\phi}(-k) + \tilde{\phi}(k) \cdot \tilde{J}(-k).$$

1.1 Quantization of Free Field

1.1.1 Path Integral Formalism

In the path integral formula,

$$Z_0[J] = \int D[\phi] \exp(iS_0[\phi, J]). \tag{1.5}$$

The partition function for free field:

$$\begin{aligned}\frac{Z_0[J]}{Z_0[0]} &= \exp\left(-\frac{i}{2}\int \frac{d^dk}{(2\pi)^d}\frac{\tilde{J}(k)\tilde{J}(-k)}{k^2-m^2+i\epsilon}\right) \\ &= \exp\left(-\frac{i}{2}\int d^dx_1 d^dx_2 J(x_1)\Delta_0(x_1-x_2)J(x_2)\right).\end{aligned}$$

where the propagator is[1]

$$\Delta_0(x_1-x_2) = \int \frac{d^dk}{(2\pi)^d}\frac{e^{-ik\cdot x}}{k^2-m^2+i\epsilon}. \tag{1.6}$$

Remark 1. Gaussian Integral for Real Scalar Field

The real Gaussian integral formula is

$$\int d\boldsymbol{v}\exp\left(-\frac{1}{2}\boldsymbol{v}^T\cdot A\cdot\boldsymbol{v}+\boldsymbol{b}^T\cdot\boldsymbol{v}\right) = \sqrt{\frac{(2\pi)^N}{\det A}}\exp\left(\frac{1}{2}\boldsymbol{b}^T\cdot A^{-1}\cdot\boldsymbol{b}\right), \tag{1.7}$$

where $\boldsymbol{v},\boldsymbol{b}$ are two N-dimensional vector, and A is an $N\times N$ matrix. For the field integral, we absorbed the $(2\pi)^{N/2}$ term into the measure, and express the path integral for the Gaussian field as:

$$\begin{aligned}Z[J] &= \int D[\phi]\exp\left(\frac{i}{2}\int d^dx\phi\hat{A}\phi + i\int d^dxJ\phi\right) \\ &= Z[0]\exp\left[-\frac{i}{2}\int d^dx_1d^dx_2J(x_1)A^{-1}(x_1-x_2)J(x_2)\right].\end{aligned}$$

We make use of (1.7) by making the identification

$$A = \bigoplus_{|k|}\begin{pmatrix} 0 & k^2-m^2 \\ k^2-m^2 & 0\end{pmatrix},\ b = \bigoplus_{|k|}\begin{pmatrix}\tilde{J}(k) \\ \tilde{J}(-k)\end{pmatrix}.$$

This gives the propagator in the momentum space:

$$\tilde{\Delta}_0(k) = \frac{1}{k^2-m^2}.$$

Note that $\Delta_0(x_1-x_2)$ is related to the correlation function:

$$\begin{aligned}\langle 0|T\phi(x_1)\phi(x_2)|0\rangle &= \frac{\delta}{i\delta J(x_1)}\frac{\delta}{i\delta J(x_2)}Z_0[J] \\ &= i\Delta(x_1-x_2).\end{aligned} \tag{1.8}$$

For interaction theory, the partition function is

$$Z[J] = \exp\left(i\int d^dx\mathcal{L}_{\text{int}}\left[\frac{\delta}{i\delta J(x)}\right]\right)Z_0[J]. \tag{1.9}$$

[1] The extra $i\epsilon$ term is use to bring the singularities infinitesimally below the real axis. This infinitesimal value can be absorbed into the mass term, by regarding the mass term m^2 as $m^2-i\epsilon$.

The expectation values for a generic operator of the form $O(\phi)$ can be evaluated by the true partition function

$$\langle O(\phi)\rangle = \frac{1}{Z[0]}\, O\left[\frac{\delta}{i\delta J(x)}\right] Z[J]\bigg|_{J=0}. \tag{1.10}$$

Remark 2. Connected Diagrams

The expression (1.10) can be expanded order by order using the Feynman diagram. Since the unconnected diagram can be absorbed into $Z[0]$, we only need to calculate the connected diagram.

The procedure of perturbative expansion with only connected diagrams can be formally represented by introducing the quantity

$$Z[J] = \exp\left(iW[J]\right). \tag{1.11}$$

The perturbative expansion of $W[J]$ contain only the connected diagrams. Eq. (1.10) can then be rephrased as

$$\langle O(\phi)\rangle = O\left[\frac{\delta}{\delta J(x)}\right] W[J]\bigg|_{J=0}. \tag{1.12}$$

For example, the two-point connected correlation (propagator) is

$$\begin{aligned}
\langle \mathcal{T}\phi(x_1)\phi(x_2)\rangle_c =& \frac{1}{i}\,\frac{\delta^2 W[J]}{\delta J(x_1)\delta J(x_2)}\bigg|_{J=0} \\
=& -\frac{\delta^2}{\delta J(x_1)\delta J(x_2)} \ln Z[J]\bigg|_{J=0} \\
=& \frac{1}{Z[0]}\,\frac{\delta^2 Z[J]}{i\delta J(x_1) i\delta J(x_2)}\bigg|_{J=0}.
\end{aligned} \tag{1.13}$$

1.1.2 Canonical Quantization

The classical equation of motion for the real Klein-Gordon field is

$$(-\partial_t^2 + \nabla^2 - m^2)\phi(\vec{x}, t) = 0. \tag{1.14}$$

The solution to Eq. (1.14) is proportional to the plane wave:

$$\phi(\vec{x}, t) \propto e^{-i\omega_{\boldsymbol{k}} t + i\vec{p}\cdot\vec{x}} + e^{i\omega_{\vec{k}} t - i\boldsymbol{p}\cdot\vec{x}}, \tag{1.15}$$

where the energy is $\omega_{\boldsymbol{k}} = \boldsymbol{k}^2 + m^2$ and $\vec{k}$ is the momentum as the conserved quantity. The general solution to the EOM is

$$\phi(\vec{x}, t) \propto \int \frac{d^d k}{(2\pi)^d}\left(a_k e^{-i\omega_{\boldsymbol{k}} t + i\vec{k}\cdot\vec{x}} + a_k^* e^{i\omega_{\boldsymbol{k}} t - i\vec{k}\cdot\vec{x}}\right). \tag{1.16}$$

The canonical quantization promote the coefficient a_k/a_k^* to the particle annihilation/creation operator $a_k/a_k^\dagger$, with the commutation relation

$$[a_k, a_p^\dagger] = (2\pi)^d \delta^d(\vec{k} - \vec{p}). \tag{1.17}$$

The single-particle state with momentum $\vec{k}$ is created by $a_k^\dagger$ operators acting on the vacuum:

$$a_k^\dagger|0\rangle = \frac{1}{\sqrt{2\omega_{\boldsymbol{k}}}}|\vec{k}\rangle, \tag{1.18}$$

where $|\vec{k}\rangle$ is a state with a single particle of momentum $\vec{k}$.

Remark 3. Lorentz Invariance of Single-particle State

The factor of $\sqrt{2\omega_{\boldsymbol{k}}}$ in Eq. (1.18) is just a convention, but it will make some calculations easier. To compute the normalization of one-particle states, we start with

$$\langle 0|0\rangle = 1, \tag{1.19}$$

which leads to

$$\langle \vec{p}|\vec{k}\rangle = 2\sqrt{\omega_{\boldsymbol{p}}\omega_{\boldsymbol{k}}} \left\langle 0 \left| a_p a_k^\dagger \right| 0 \right\rangle = 2\omega_{\boldsymbol{p}}(2\pi)^d \delta^d(\vec{p} - \vec{k}). \tag{1.20}$$

The identity operator for one-particle states is

$$1 = \int \frac{d^d p}{(2\pi)^d} \frac{1}{2\omega_{\boldsymbol{p}}} |\vec{p}\rangle\langle\vec{p}|, \tag{1.21}$$

which we can check with

$$|\vec{k}\rangle = \int \frac{d^d p}{(2\pi)^d} \frac{1}{2\omega_{\boldsymbol{p}}} |\vec{p}\rangle\langle\vec{p}|\vec{k}\rangle = \int \frac{d^d p}{(2\pi)^d} \frac{1}{2\omega_{\boldsymbol{p}}} 2\omega_{\boldsymbol{p}}(2\pi)^d \delta^d(\vec{p} - \vec{k})|\vec{p}\rangle = |\vec{k}\rangle.$$

The identity operator Eq. (1.21) is Lorentz invariant since it can be expressed as

$$1 = \int \frac{d^d p}{(2\pi)^d} \int \frac{d\omega}{2\pi} 2\pi\delta(\omega^2 - \boldsymbol{p}^2 - m^2). \tag{1.22}$$

By requiring $\langle\vec{k}|\phi|0\rangle = 1$, the normalization Eq. (1.18) determines the normalization for the quantized field operator:

$$\phi(\vec{x}, t) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{\sqrt{2\omega_{\boldsymbol{k}}}} \left(a_k e^{-ik\cdot x} + a_k^\dagger e^{ik\cdot x} \right). \tag{1.23}$$

1.1.3 Scattering Amplitude

Consider the scattering process in the interaction picture,

$$
\begin{aligned}
\langle f|e^{-iHt}|i\rangle &= \langle f|T\exp\left(-i\int dt V_{\text{int}}(t)\right)|i\rangle \\
&= \langle f|T\exp\left(i\int d^dx\mathcal{L}_{\text{int}}(t)\right)|i\rangle.
\end{aligned}
\tag{1.24}
$$

The S-matrix is defined as

$$
S = \mathcal{T}\exp\left(i\int d^dx\mathcal{L}_{\text{int}}(t)\right) = 1 + i\mathcal{T}. \tag{1.25}
$$

Because of the additional momentum conservation,

$$
\mathcal{T} = (2\pi)^d\delta^d\left(\sum p\right)\mathcal{M}. \tag{1.26}
$$

1.2 ϕ^3 Theory in $(d = 6 - \epsilon)$ Space-time

Now consider the interaction theory with additional Lagrangian

$$
\mathcal{L}_{\text{int}}[\phi] = \frac{g}{3!}\phi^3. \tag{1.27}
$$

Note that the field ϕ has the mass dimension $[\frac{d-2}{2}]$. When $d = 6$, the coupling constant g is dimensionless. For interaction theory, the renormalized Lagrangian has the form:

$$
\begin{aligned}
\mathcal{L} &= Z_\phi\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - Z_m\frac{m^2}{2}\phi^2 + Z_g\frac{g}{3!}\phi^3 \\
&= \mathcal{L}_0 + \mathcal{L}_{\text{int}} + \mathcal{L}_{\text{ct}},
\end{aligned}
\tag{1.28}
$$

where the counter terms are:

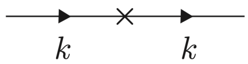
$$
\begin{aligned}
\mathcal{L}_{\text{ct}}[\phi] &= \frac{A}{2}\partial^\mu\phi\partial_\mu\phi - \frac{B}{2}m^2\phi^2 + \frac{C}{3!}g\phi^3 \\
&\simeq -\frac{A}{2}\phi\partial^2\phi - \frac{B}{2}m^2\phi^2 + \frac{C}{3!}g\phi^3,
\end{aligned}
\tag{1.29}
$$

where

$$
A = Z_\phi - 1, B = Z_m - 1, C = Z_g - 1.
$$

The counter term for the the free field gives additional corrrection

$$
\begin{aligned}
i\tilde{\Delta}^{(\text{ct})}(k) &= i\tilde{\Delta}_0(k)(Ak^2 - Bm^2)i\tilde{\Delta}_0(k) \\
&=
\end{aligned}
\tag{1.30}
$$



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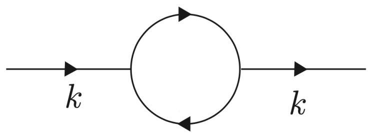
1.2.1 Self Energy Correction

To second order, we consider the one-loop correction to the propagator with the diagram:



This correspond to

$$i\tilde{\Delta}^{(2)}(k) = i\tilde{\Delta}_0(k)\left[i\Sigma^{(2)}(k^2)\right] i\tilde{\Delta}_0(k), \tag{1.31}$$

where the self energy term to the second order $i\Pi^{(2)}(k)$ is defined as:

$$i\Sigma^{(2)}(k^2) \equiv \frac{g^2}{2}\int \frac{d^dq}{(2\pi)^d}\tilde{\Delta}_0(q)\tilde{\Delta}_0(k-q) + (Ak^2 - Bm^2). \tag{1.32}$$

Remark 4. Symmetry Factor

The coefficient $g^2/2$ comes from the symmetry factor in the diagram. We can also check the coefficient explicitly, by considering the expansion to the second order (we denote $\delta/\delta J(x_i)$ as δ_i):

$$\delta_1\delta_2\frac{1}{2!4!}\left[\frac{ig}{3!}\int d^dy\left(\frac{\delta}{\delta J(y)}\right)^3\right]^2\left[-\frac{i}{2}\int d^dy_1d^dy_2J(y_1)\Delta(y_1-y_2)J(y_2)\right]^4.$$

The expansion gives the coefficient

$$\left(\frac{ig}{6}\right)^2 \times \frac{1}{2!\times 4!\times 2^4}.$$

Now consider the combinatorial factor, which comes from the exchange of $\phi(x_i)$ in the propagator, the exchange of $\phi(x_i)$ in the vertex, the exchange of propagator in the diagram, and the change of vertices in the diagram:

$$(2!)^4\times(3!)^2\times(4\times 3)\times 2.$$

Those two factors produce a $-g^2/2$ coefficient. Note that in the self energy expression (1.31), we put a i factor in front of each propagator, which absorbs the minus sign.

Once we obtain the self energy, the one-loop corrected propagator has the form:

$$\begin{aligned} i\tilde{\Delta}(k) &= i\tilde{\Delta}_0(k) + i\tilde{\Delta}_0(k)\left[\sum_{n=1}^{\infty} i\Sigma(k^2)\right] i\tilde{\Delta}_0(k) \\ &= \frac{i}{\tilde{\Delta}_0^{-1}(k) - \Sigma(k^2)} \\ &= \frac{i}{k^2 - m^2 - \Sigma(k^2)}. \end{aligned} \tag{1.33}$$

Now we are going to evaluate the divergent integral in the self energy expression, using the Feynman parameters:

$$\begin{aligned}&\int \frac{d^dq}{(2\pi)^d}\frac{1}{q^2-m^2}\frac{1}{(k-q)^2-m^2}\\ &=\int \frac{d^dq}{(2\pi)^d}\int_0^1 dx \frac{1}{\left[q^2-m^2+x\left((q-k)^2-q^2\right)\right]^2}\\ &=\int_0^1 dx\int \frac{d^dq}{(2\pi)^d}\frac{1}{\left[(q-kx)^2-D\right]^2},\end{aligned}$$

where $D=m^2-k^2x(1-x)$. Then we can shift $q\to q+kx$ leaving an integral that only depends on q^2. In this way,

$$\Sigma(k^2)=\int_0^1 I(x)dx.$$

To evaluate the self-energy, it suffices to obtain the integral

$$I(x)=\frac{g^2}{2i}\int \frac{d^dq}{(2\pi)^d}\frac{1}{\left[q^2-D\right]^2}.$$

Remark 5. Feyman Parameters

We use Feynman's formula to combine denominators,

$$\frac{1}{A_1\dots A_n}=\int dF_n\left(x_1A_1+\dots+x_nA_n\right)^{-n}, \tag{1.34}$$

where the integration measure over the Feynman parameters x_i is

$$\int dF_n=(n-1)!\int_0^1 dx_1\dots dx_n\delta\left(x_1+\dots+x_n-1\right). \tag{1.35}$$

This measure is normalized so that $\int dF_n=1$. The simplest case is

$$\frac{1}{AB}=\int_0^1\frac{dx}{[A+(B-A)x]^2}=\int_0^1\frac{\delta(x+y-1)}{[xA+yB]^2}dxdy. \tag{1.36}$$

Other useful identities are

$$\begin{aligned}\frac{1}{AB^n}&=\int_0^1 dxdy\frac{\delta(x+y-1)ny^{n-1}}{[xA+yB]^{n+1}},\\ \frac{1}{ABC}&=\int_0^1 dxdydz\frac{2\delta(x+y+z-1)}{[xA+yB+zC]^3}.\end{aligned} \tag{1.37}$$

By making the Wick rotation $q^0\to iq_E^0$, the integral becomes:[2]

$$I(x)=\frac{g}{2}\int\frac{d^dq_E}{(2\pi)^d}\frac{1}{\left(q_E^2+D\right)^2}=\frac{g\Omega_d}{2(2\pi)^d}\int dq\frac{q^{d-1}}{\left(q^2+D\right)^2}.$$

[2] The d-dimensional solid angle is

$$\Omega_d=\frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}, \tag{1.38}$$

Dimensional Regularization

We set the dimension to $d = 6 - \epsilon$, and rewrite the Lagrangian as

$$\mathcal{L} = Z_\phi \frac{1}{2}\partial^\mu \phi \partial_\mu \phi - Z_m \frac{m^2}{2}\phi^2 + Z_g \frac{g\tilde{\mu}^{\epsilon/2}}{3!}\phi^3. \tag{1.40}$$

Note that the coupling constant should be changed to $g \to g\tilde{\mu}^{\epsilon/2}$ where μ is of mass dimension [1] in order to get the correct dimensionality. We then expand the expression to zeroth order of ϵ. A useful identity is:

$$\int dk \frac{k^a}{(k^2+D)^b} = D^{\frac{a+1}{2}-b}\frac{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(b-\frac{a+1}{2}\right)}{2\Gamma(b)}. \tag{1.41}$$

Actually, we can compute the integral and series expansion in `Mathematica` all together:

```
omg=(2*Pi^(d/2))/(Gamma[d/2]);
cof=g^2*\[Mu]^(6-d)/2*omg/(2*Pi)^d;
int=cof*Integrate[q^(d-1)/(q^2+D)^2,{q,0,Infinity}][[1]];
map={g^2->\[Alpha]*(4*Pi)^3,EulerGamma->Subscript[\[Gamma],E]};
ans=Series[int/.{d->6-\[Epsilon]},{\[Epsilon],0,0}];
ans/.map//Simplify
```

The result is (where $\alpha \equiv g^2/(4\pi)^3$)

$$I(x) = \frac{\alpha D}{2}\left[\ln\left(\frac{De^{\gamma_E}}{4\pi\tilde{\mu}^2}\right) - \left(\frac{2}{\epsilon}+1\right)\right] + O(\epsilon).$$

Now insert $D = m^2 - k^2 x(1-x)$. Note that

$$\int_0^1 dx D = m^2 - \frac{k^2}{6}.$$

This simplifies the result to

$$\Sigma^{(2)}(k^2) = \frac{\alpha}{2}\left(\frac{2}{\epsilon}+1\right)\left(\frac{k^2}{2}-m^2\right) + \frac{\alpha}{2}\int_0^1 dx D(x)\ln\left(\frac{D(x)}{\mu^2}\right), \tag{1.42}$$

where we have replace $\tilde{\mu}$ with

$$\mu \equiv \sqrt{\frac{4\pi}{e^{\gamma_E}}}\tilde{\mu}. \tag{1.43}$$

where $\Gamma(x)$ is the gamma function, satisfing

$$\Gamma(1+x) = x\Gamma(x),\ \Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + O(\epsilon). \tag{1.39}$$

In particular, $\Gamma(n+1) = n!$.

Renormalization

The counter terms also contribute to the perturbative correction,

$$\begin{aligned}\Sigma^{(2)}\left(k^2\right) =& \frac{\alpha}{2}\int_0^1 dx D \ln\left(\frac{D}{m^2}\right) + \left\{\frac{\alpha}{6}\left[\frac{1}{\varepsilon} + \ln\left(\frac{\mu}{m}\right) + \frac{1}{2}\right] + A\right\} k^2 \\ & - \left\{\alpha\left[\frac{1}{\varepsilon} + \ln\left(\frac{\mu}{m}\right) + \frac{1}{2}\right] + B\right\} m^2 + O\left(\alpha^2\right).\end{aligned}$$

Consider the on-shell condition for the subtraction:

$$\Sigma(m^2) = \Sigma'(m^2) = 0. \tag{1.44}$$

Set $D_0 \equiv D(x)|_{k^2=m^2} = m^2(1-x+x^2)$, the self energy has the form:

$$\Sigma^{(2)}(k^2) = \frac{\alpha}{2}\int_0^1 dx D(x) \ln\left(\frac{D(x)}{D_0(x)}\right) + C_k k^2 + C_m m^2. \tag{1.45}$$

The condition $\Pi(m^2) = 0$ requires

$$\Sigma^{(2)}(k^2) = \frac{\alpha}{2}\int_0^1 dx D(x) \ln\left(\frac{D(x)}{D_0(x)}\right) + C_k(k^2 - m^2).$$

The condition $\Pi'(m^2) = 0$ requires

$$\begin{aligned}\left.\frac{d\Sigma^{(2)}(k^2)}{dk^2}\right|_{k^2=m^2} &= \frac{\alpha}{2}\int_0^1 dx \left.\left[\frac{D(x)}{dk^2}\ln\left(\frac{D(x)}{D_0(x)}\right) + D_0(x)\right]\right|_{q^2=m^2} + C_k \\ &= \frac{\alpha}{2}\int_0^1 dx(x^2 - x) + C_k \\ &= C_k - \frac{\alpha}{12} = 0.\end{aligned}$$

In this way, we obtained the renormalized self-energy:

$$\Sigma^{(2)}(k^2) = \frac{\alpha}{2}\int_0^1 dx D(x) \ln\left(\frac{D(x)}{D_0(x)}\right) + \frac{\alpha}{12}(k^2 - m^2). \tag{1.46}$$

On the other hand, we chan choose the $\overline{\text{MS}}$ subtraction scheme, i.e.,

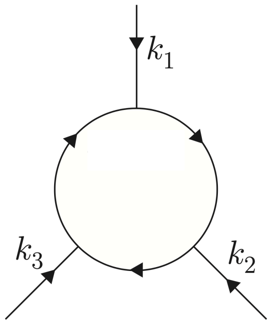
$$A = -\frac{\alpha}{6\epsilon},\ B = -\frac{\alpha}{\epsilon}. \tag{1.47}$$

The self energy under $\overline{\text{MS}}$ scheme will depend on the the mass scale μ we choose:

$$\Sigma^{(2)}\left(k^2\right) = \frac{\alpha}{2}\int_0^1 dx D \ln\left(\frac{D}{m^2}\right) + \alpha\left[\ln\left(\frac{\mu}{m}\right) + \frac{1}{2}\right]\left(\frac{k^2}{6} - m^2\right). \tag{1.48}$$

1.2.2 Vertex Correction

Now consider the simplest one-loop correction to the vertex function from the diagram:



The vertex function corresponding to such correction, together with the counter term, can be expressed as:

$$iV_3^{(3)}(k_1, k_2, k_3) = (ig)^3 i^3 \int \frac{d^a q}{(2\pi)^d} \tilde{\Delta}(q - k_1)\tilde{\Delta}(q + k_2)\tilde{\Delta}(q) + iCg, \tag{1.49}$$

Using the Feynman parameter, the integrant is

$$\tilde{\Delta}(q - k_1)\tilde{\Delta}(q + k_2)\tilde{\Delta}(q) = \int dF_3 \frac{1}{(q^2 - D)^3}$$

where we have shift the value of q, and D can be evaluate by the following code:

```
A1=(l-k1)^2-m^2;
A2=(l+k2)^2-m^2;
A3=(l)^2-m^2;
{c,b,a}=CoefficientList[x1*A1+x2*A2+(1-x1-x2)*A3,{l}];
-c+b^2/(4*a)//Expand
```

The result is

$$D = m^2 - k_1^2 x_1(1 - x_1) - k_2^2 x_2(1 - x_2) - 2k_1 k_2 x_1 x_2.$$

The same procedure gives:

$$V_3^{(3)}/g = \int dF_3 I(x_1, x_2, x_3) + C, \tag{1.50}$$

where

$$I(x_1, x_2, x_3) = \frac{g^2 \Omega_d}{(2\pi)^d} \int dq \frac{q^{d-1}}{(q^2 + D)^3}.$$

The same regularization procedure in `Mathematica`:

```
omg=(2*Pi^(d/2))/(Gamma[d/2]);
cof=g^2*\[Mu]^(6-d)*omg/(2*Pi)^d;
int=cof*Integrate[q^(d-1)/(q^2+D)^3,{q,0,Infinity}][[1]];
map={g^2->\[Alpha]*(4*Pi)^3,EulerGamma->Subscript[\[Gamma],E]};
ans=Series[int/.{d->6-\[Epsilon]},{\[Epsilon], 0, 0}];
ans/.map//Simplify
```

The result is

$$
\begin{aligned}
V_3^{(3)}/g &= \frac{\alpha}{\epsilon} + \frac{\alpha}{2}\int dF_3 \ln\left(\frac{4\pi\tilde{\mu}^2 e^{-\gamma_E}}{D}\right) + C + O(\epsilon) \\
&= \frac{\alpha}{\epsilon} + \alpha \ln\left(\frac{\mu}{m}\right) - \frac{\alpha}{2}\int dF_3 \ln\left(\frac{D}{m}\right) + C.
\end{aligned}
\tag{1.51}
$$

The on-shell subtraction requires

$$
V_3(0,0,0) = g, \tag{1.52}
$$

which gives

$$
C = -\frac{\alpha}{\epsilon} - \alpha \ln\left(\frac{\mu}{m}\right). \tag{1.53}
$$

So the vertex function to the third order is

$$
V_3(k_1,k_2,k_3) = g\left\{1 - \frac{\alpha}{2}\int dF_3 \ln\left[\frac{D(x_1,x_2,x_3)}{m}\right]\right\}. \tag{1.54}
$$

The $\overline{\text{MS}}$ scheme, on the other hand, sets

$$
C = -\frac{\alpha}{\epsilon}. \tag{1.55}
$$

1.2.3 Renormalization Group

We fist summarize the normalization factor obtained on the one-loop level (with $\overline{\text{MS}}$ subtraction scheme):

$$
\begin{aligned}
Z_\phi &= 1 - \frac{\alpha}{6\epsilon} + O(\alpha^2), \\
Z_m &= 1 - \frac{\alpha}{\epsilon} + O(\alpha^2), \\
Z_g &= 1 - \frac{\alpha}{\epsilon} + O(\alpha^2).
\end{aligned}
\tag{1.56}
$$

For the renormalized Lagrangian in $(6-\epsilon)$-dimension

$$
\mathcal{L} = Z_\phi \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - Z_m\frac{m^2}{2}\phi^2 + Z_g\frac{g\tilde{\mu}^{\epsilon/2}}{3!}\phi^3, \tag{1.57}
$$

the factors relate the original field and bare coefficients

$$
\phi_0 = Z_\phi^{1/2}\phi,\ m_0 = Z_m^{1/2}Z_\phi^{-1/2}m,\ g_0 = Z_g Z_\phi^{-3/2}\tilde{\mu}^{\epsilon/2}g. \tag{1.58}
$$

The renormalization group requires that the bare parameter is independent of the mass scale μ we choose, that is:

$$
\frac{d\phi_0}{d\ln\mu} = \frac{dm_0}{d\ln\mu} = \frac{dg_0}{d\ln\mu} = 0. \tag{1.59}
$$

Beta Function

Star with g_0, it is more convenient to use

$$\alpha_0 \equiv \frac{g_0^2}{4\pi} = Z_g^2 Z_\phi^{-3} \tilde{\mu}^\epsilon \alpha. \tag{1.60}$$

Take logarithm on both side:

$$\ln \alpha_0 = \ln(Z_g^2 Z_\phi^{-3}) + \ln \alpha + \epsilon \ln \tilde{\mu}. \tag{1.61}$$

The RG equation is

$$\frac{d \ln \alpha_0}{d \ln \mu} = \frac{d \ln(Z_g^2 Z_\phi^{-3})}{d\alpha} \frac{d\alpha}{d \ln \mu} + \frac{1}{\alpha} \frac{d\alpha}{d \ln \mu} + \epsilon = 0. \tag{1.62}$$

To the first order of α:

$$\frac{d \ln(Z_g^2 Z_\phi^{-3})}{d\alpha} = \frac{d}{d\alpha} \left(-\frac{2\alpha}{\epsilon} + \frac{\alpha}{2\epsilon} \right) = -\frac{3}{2\epsilon}, \tag{1.63}$$

which leads to

$$\frac{d\alpha}{d \ln \mu} \left(1 - \frac{3\alpha}{2\epsilon} + O(\alpha^2) \right) + \epsilon\alpha = 0. \tag{1.64}$$

The beta function is defined as

$$\beta(\alpha) = \frac{d\alpha}{d \ln \mu} = \beta_1 \alpha + \beta_2 \alpha^2 + O(\alpha^3). \tag{1.65}$$

Insert such definition into the original expression, and keep track of the order of α, we get

$$(\beta_1 + \epsilon)\alpha + \left(\beta_2 - \frac{3\beta_1}{2\epsilon} \right) \alpha^2 + O(\alpha^3) = 0. \tag{1.66}$$

The beta function is

$$\beta(\alpha) = -\epsilon\alpha - \frac{3}{2}\alpha^2 + O(\alpha^3). \tag{1.67}$$

Anomalous Dimension

Consider the RG equation with bare mass:

$$\begin{aligned} \frac{d \ln m_0}{d \ln \mu} &= \frac{1}{2} \frac{d(\ln Z_m - \ln Z_\phi)}{d\alpha} \frac{d\alpha}{d \ln \mu} + \frac{1}{m} \frac{dm}{d \ln \mu} \\ &= \frac{5\alpha}{12} + \frac{1}{m} \frac{dm}{d \ln \mu} + O(\alpha^2) = 0. \end{aligned} \tag{1.68}$$

We get the anomalous dimension of the mass:

$$\gamma_m(\alpha) \equiv \frac{1}{m} \frac{dm}{d \ln \mu} = -\frac{5\alpha}{12} + O(\alpha^2). \tag{1.69}$$

Also, for the bare field

$$\frac{d \ln \phi_0}{d \ln \mu} = \frac{1}{2} \frac{d \ln Z_\phi}{d \ln \mu} + \frac{d \ln \phi}{d \ln \mu} = 0. \tag{1.70}$$

We can define the anomalous dimension of the field as

$$\gamma_\phi \equiv \frac{1}{2} \frac{d \ln Z_\phi}{d \ln \mu} = \frac{1}{2} \frac{d \ln Z_\phi}{d\alpha} \frac{d\alpha}{d \ln \mu} = \frac{\alpha}{12} + O(\alpha^2). \tag{1.71}$$

Callan-Symanzik Equation

Consider the bare propagator:

$$\tilde{\Delta}_0(k) = Z_\phi \tilde{\Delta}(k) \tag{1.72}$$

The RG condition for the bare propagator gives:

$$\frac{d\ln\tilde{\Delta}_0(k)}{d\ln\mu} = \frac{d\ln Z_\phi}{d\ln\mu} + \frac{1}{\tilde{\Delta}(k)}\left(\frac{\partial}{\partial\ln\mu} + \frac{d\alpha}{d\ln\mu}\frac{\partial}{\partial\alpha} + \frac{dm}{d\ln\mu}\frac{\partial}{\partial m}\right)\tilde{\Delta}(k) = 0.$$

The Callan-Symanzik equation is

$$\left(2\gamma_\phi + \frac{\partial}{\partial\ln\mu} + \beta(\alpha)\frac{\partial}{\partial\alpha} + \gamma_m(\alpha)m\frac{\partial}{\partial m}\right)\tilde{\Delta}(k) = 0. \tag{1.73}$$

1.3 ϕ^4 Theory in $(d = 4 - \epsilon)$ Space-time

In this section, we consider the real Klein-Gordon field with ϕ^4 interaction in $(4-\epsilon)$-dimension space-time:

$$\mathcal{L} = Z_\phi\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - Z_m\frac{m^2}{2}\phi^2 - Z_g\frac{g\tilde{\mu}^\epsilon}{4!}\phi^4. \tag{1.74}$$

Note that the field ϕ has mass dimension $[\frac{d-2}{2}] = [1]$, so the original coupling constant g is dimensionless.

As the ϕ^3 theory, we can rewrite the Lagrangian as:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}} + \mathcal{L}_{\text{ct}}. \tag{1.75}$$

In the following we investigate the loop correction to the mass and the coupling constant.

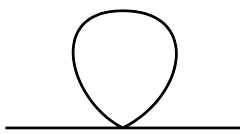
1.3.1 One-loop Correction

Self-energy

Following the same procedure, the one-loop self-energy correction is

$$i\Sigma(k^2) = -\frac{g\tilde{\mu}^\epsilon}{2}\int\frac{d^dq}{(2\pi)^d}\frac{1}{q^2-m^2} + i(Ak^2 - Bm^2). \tag{1.76}$$

The first term comes from the diagram



and the second term comes from the counter terms. After the Wick rotation,

$$\Sigma(k^2) = -\frac{g\tilde{\mu}^\epsilon}{2}\frac{\Omega_d}{(2\pi)^d}\int\frac{q^{d-1}dq}{q^2+m^2} + (Ak^2 - Bm^2). \tag{1.77}$$

The dimensional regulation is carried out using the following code:

```
omg=(2*Pi^(d/2))/(Gamma[d/2]);
cof=g*\[Mu]^(4-d)/2*omg/(2*Pi)^d;
int=cof*Integrate[q^(d-1)/(q^2+m^2),{q,0,Infinity}][[1]];
map={EulerGamma->Subscript[\[Gamma],E]};
ans=Series[int/.{d->4-\[Epsilon]},{\[Epsilon],0,0}];
ans/.map//Simplify
```

The result is

$$\Sigma(k^2) = \frac{gm^2}{32\pi^2}\left[\frac{2}{\epsilon} + 1 + \log\left(\frac{4\pi\tilde{\mu}^2 e^{-\gamma_E}}{m^2}\right)\right] + (Ak^2 - Bm^2) + O(\epsilon). \tag{1.78}$$

Using the $\overline{\mathrm{MS}}$ renormalization scheme, we set

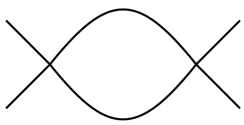
$$A = 0,\ \ B = \frac{g}{16\pi^2\epsilon}. \tag{1.79}$$

The result is

$$\Sigma(k^2) = \frac{gm^2}{16\pi^2}\log\left(\frac{\mu}{m}\right) + \frac{gm^2}{32\pi^2} + O(\epsilon). \tag{1.80}$$

Vertex Correction

Now consider the vertex correction. To the lowest order the diagram is



Together with the counter term, the vertex function is

$$iV_4^{(2)}(k_1,k_2,k_3,k_4) = \frac{g^2}{2}\left[iF(s) + iF(t) + iF(u)\right] - iCg, \tag{1.81}$$

where

$$s = (k_1+k_2)^2,\ \ t = (k_1+k_3)^2,\ \ u = (k_1+k_4)^2, \tag{1.82}$$

and

$$iF(k^2) = \tilde{\mu}^\epsilon \int \frac{d^dq}{(2\pi)^d}\tilde{\Delta}_0(q)\tilde{\Delta}_0(q+k) \tag{1.83}$$

$$= \frac{i\tilde{\mu}^\epsilon\Omega_d}{(2\pi)^d}\int_0^1 dx \int \frac{q^{d-1}dq}{[q^2+m^2+x(1-x)k^2]^2}. \tag{1.84}$$

Then we carry out the calculation (set $D(k^2,x) = m^2 + x(1-x)k^2$)

```
omg=(2*Pi^(d/2))/(Gamma[d/2]);
cof=g^2*\[Mu]^(4-d)/2*omg/(2*Pi)^d;
int=cof*Integrate[q^(d-1)/(q^2+D)^2,{q,0,Infinity}][[1]];
map={EulerGamma->Subscript[\[Gamma],E]};
ans=Series[int/.{d->4-\[Epsilon]},{\[Epsilon],0,0}];
ans/.map//Simplify
```

The result is:

$$F(s) = \frac{1}{8\pi^2\epsilon} + \frac{1}{16\pi^2}\int_0^1 dx \ln\left(\frac{4\pi\tilde{\mu}^2 e^{-\gamma_E}}{D}\right)$$
$$= \frac{1}{8\pi^2\epsilon} + \frac{1}{8\pi^2}\ln\left(\frac{\mu}{m}\right) - \frac{1}{16\pi^2}\int_0^1 dx \ln\left(\frac{D(s,x)}{m^2}\right). \tag{1.85}$$

The $\overline{\text{MS}}$ scheme absorbs the $\frac{1}{8\pi^2\epsilon}$ term, i.e.,

$$C = \frac{3g}{16\pi^2}. \tag{1.86}$$

The result is:

$$V_4(k_1,k_2,k_3,k_4) = -g + \frac{g^2}{32\pi^2}\int_0^1 dx \ln\left(\frac{\mu^6}{D(s,x)D(t,x)D(u,x)}\right). \tag{1.87}$$

To summarize, the normalization is:

$$Z_\phi = 1, \tag{1.88}$$
$$Z_m = 1 + \frac{g}{16\pi^2\epsilon}, \tag{1.89}$$
$$Z_g = 1 + \frac{3g}{16\pi^2\epsilon}. \tag{1.90}$$

1.3.2 Renormalization Group

Now consider the RG equation for the one-loop correction. The bare parameters are:

$$g_0 = Z_g g\tilde{\mu}^\epsilon, \ m_0 = Z_m^{1/2} m, \tag{1.91}$$

The RG conditions are:

$$\frac{dg_0}{d\ln\mu} = \left(\frac{3}{16\pi^2\epsilon} + \frac{1}{g}\right)\frac{dg}{d\ln\mu} + \epsilon = 0, \tag{1.92}$$
$$\frac{dm_0}{d\ln\mu} = \frac{1}{32\pi^2\epsilon}\frac{dg}{d\ln\mu} + \frac{1}{m}\frac{dm}{d\ln\mu} = 0. \tag{1.93}$$

Consider the series expansion of beta function:

$$\beta(g) = \frac{dg}{d\ln\mu} = \beta_1 g + \beta_2 g^2 + O(g^3). \tag{1.94}$$

The beta function is

$$\beta(g) = -\epsilon g + \frac{3g^2}{16\pi^2} + O(g^3). \tag{1.95}$$

The anomalous dimension of mass is

$$\gamma_m = \frac{1}{m}\frac{dm}{d\ln\mu} = \frac{g}{32\pi^2} + O(g^2) \tag{1.96}$$

Chapter 2

Quantum Electrodynamics

The Lagrangian for quantum electrodynamics is

$$\begin{aligned}\mathcal{L}_{\text{QED}} &= \bar{\psi}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - eA_{\mu}\bar{\psi}\gamma^{\mu}\psi \\ &= \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{Maxwell}} + \mathcal{L}_{\text{int}},\end{aligned} \tag{2.1}$$

where

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\nu} = (dA)_{\mu\nu}. \tag{2.2}$$

The Lagrangian is invariant under the gauge transformation:

$$\begin{aligned}\psi(x) &\to e^{-ie\alpha(x)}\psi(x), \\ A^{\mu}(x) &\to A^{\mu}(x) + \partial^{\mu}\alpha(x).\end{aligned} \tag{2.3}$$

It is convenient to rewrite Lagrangian as

$$\mathcal{L}_{\text{QED}} = \bar{\psi}\left(i\not{D} - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{2.4}$$

where we have define the covariant derivative as:

$$\not{D} = \gamma^{\mu}D_{\mu} = \gamma^{\mu}[\partial_{\mu} + ieA_{\mu}(x)] = \not{\partial} + ie\not{A}. \tag{2.5}$$

2.1 Free Field Theory

2.1.1 Dirac Field

The free Dirac field is

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi \tag{2.6}$$

Consider the partition function with source

$$\begin{aligned}Z_{\text{Dirac}}[J] &= \int D[\bar{\psi},\psi]\exp\left[i\int d^{d}x\left(\mathcal{L}_{\text{Dirac}} + \bar{\eta}\psi + \bar{\psi}\eta\right)\right] \\ &= Z_{\text{Dirac}}[0]\exp\left[-i\int d^{d}x_{1}d^{d}x_{2}\bar{\eta}(x_{1})\cdot D_{F}(x_{1}-x_{2})\cdot\eta(x_{2})\right].\end{aligned} \tag{2.7}$$

where

$$D_F(x_1-x_2)=\int\frac{d^dk}{(2\pi)^d}\frac{e^{-ik\cdot(x_1-x_2)}}{\not{k}-m}=\int\frac{d^dk}{(2\pi)^d}e^{-ik\cdot(x_1-x_2)}\frac{\not{k}+m}{k^2-m^2}. \tag{2.8}$$

Note that the propagator is[1]

$$\begin{aligned}\langle 0|T\psi(x_1)\bar{\psi}(x_2)|0\rangle &= \frac{1}{Z_{\text{Dirac}}[0]}\frac{\delta}{i\delta\bar{\eta}(x_1))}\frac{i\delta}{\delta(\eta(x_2))}Z_{\text{Dirac}}[J]\\ &= iD_F(x_1-x_2).\end{aligned} \tag{2.9}$$

2.1.2 Electromagnetic Field

The Lagrangian for the classical electromagnetic field is

$$\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+A_\mu J^\mu. \tag{2.10}$$

Consider the partition function

$$\frac{Z_{\text{maxwell}}[J]}{Z_{\text{maxwell}}[0]}=\exp\left[-\frac{i}{2}\int dx_1dx_2J_\mu(x_1)\Pi^{\mu\nu}(x_1-x_2)J_\nu(x_2)\right]. \tag{2.11}$$

The propagator is

$$\Pi^{\mu\nu}(x_1-x_2)=\int\frac{d^dk}{(2\pi)^d}e^{-ik\cdot(x_1-x_2)}\frac{-g^{\mu\nu}+(1-\xi)k^\mu k^\nu}{k^2}. \tag{2.12}$$

2.1.3 Perturbation Theory

As with the scalar field,

$$Z[\bar{\eta},\eta,J]=\exp\left\{i\int d^dx\mathcal{L}_{\text{int}}\left[\frac{\delta}{i\delta J},\frac{\delta}{i\delta\eta},\frac{i\delta}{\delta\bar{\eta}}\right]\right\}Z_0[\bar{\eta},\eta,J]. \tag{2.13}$$

We use the dimensional regularization by default. Note that ψ has the mass dimension $[\frac{d-1}{2}]$, A^μ had the mass dimension $[\frac{d}{2}-1]$, and e has the mass dimension $[2-\frac{d}{2}]$. When $d=4-\epsilon$, we replace e with $e\tilde{\mu}^{\epsilon/2}$, so that to make the coupling constant e dimensionless.

The renormalized Lagrangian is

$$\begin{aligned}\mathcal{L}&=Z_\psi\bar{\psi}_R(i\gamma^\mu\partial_\mu)\psi_R-Z_m m\bar{\psi}_R\psi_R+\frac{1}{4}Z_AF_{R,\mu\nu}F_R^{\mu\nu}-Z_ee_RA_R^\mu\bar{\psi}_R\gamma^\mu\psi_R\\ &=\mathcal{L}_0+\mathcal{L}_{\text{int}}+\mathcal{L}_{\text{ct}}.\end{aligned} \tag{2.14}$$

The we define the coefficients

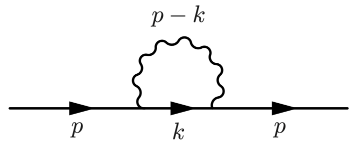
$$\delta_\psi=Z_\phi-1,\ \delta_m=Z_m-1,\ \delta_Z=Z_A-1,\ \delta_e=Z_e-1. \tag{2.15}$$

The counter term also contribute to the perturbative expansion like the interactions.

[1]The sign in the variational derivative comes from the anti-commutation relation of the fermionic fields.

Example 1. One-loop Correction to Electron Propagator

Consider the diagram



This contains 3 electron propagator, 1 photon propagator, and 2 vertices. The coefficient is (omit all the integration and summation for simplicity):

$$iD_F^{(2)}(p) \sim \frac{\delta^2}{\delta\bar{\eta}\delta\eta}\frac{1}{2!}\left(\frac{-ie\gamma^{\mu}_{\alpha\beta}\delta^3}{i\delta J^{\mu}\delta\eta_{\alpha}\delta\bar{\eta}_{\beta}}\right)^2\frac{1}{3!}\left(-i\bar{\eta}_{\alpha}D_F^{\alpha\beta}\eta_{\beta}\right)^3\left(-\frac{i}{2}J^{\mu}\Pi_{\mu\nu}J^{\nu}\right).$$

First consider the scalar coefficient. Since there is no additional symmetry, the abstract value is e^2. There is an additional sign factor by the proper order of the fermion operators:

$$\frac{\delta^2}{\delta\bar{\eta}_f\delta\eta_i}\frac{\delta^2}{\delta\eta_1\delta\bar{\eta}_1}\frac{\delta^2}{\delta\eta_2\delta\bar{\eta}_2} = -\frac{\delta}{\delta\eta_i}\frac{\delta^2}{\delta\bar{\eta}_1\delta\eta_1}\frac{\delta^2}{\delta\bar{\eta}_2\delta\eta_2}\frac{\delta}{\delta\bar{\eta}_f}.$$

Then consider the tensor contraction,

$$\Pi_{\mu\nu}D_F^{\alpha\lambda}\gamma^{\mu}_{\lambda\rho}D_F^{\rho\tau}\gamma^{\nu}_{\tau\sigma}D_F^{\sigma\beta}.$$

The total amplitude is

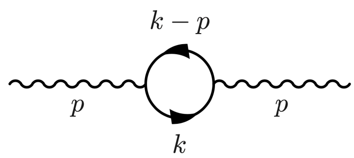
$$\begin{aligned} iD_F^{(2)}(p) &= -e^2\int\frac{d^4k}{(2\pi)^4}\Pi_{\mu\nu}(p-k)\left[D_F(p)\gamma^{\mu}D_F(k)\gamma^{\nu}D_F(p)\right]_{\alpha\beta} \\ &= iD_F(p)i\Sigma(p^2)iD_F(p), \end{aligned}$$

where $i\Sigma(p^2)$ is the self energy:

$$i\Sigma(p^2) = e^2\int\frac{d^4k}{(2\pi)^4}\Pi_{\mu\nu}(p-k)\gamma^{\mu}D_F(k)\gamma^{\nu}, \tag{2.16}$$

Example 2. One-loop Correction to Photon Propagator

Consider the diagram



There is 2 electron propagator, 2 photon propagator, and 2 vertices. Consider the perturbative expansion:

$$i\Pi^{(2)}(p) \sim \frac{\delta^2}{i\delta J i\delta J}\frac{1}{2!}\left(\frac{-e\gamma^{\mu}_{\alpha\beta}\delta^3}{\delta J^{\mu}\delta\eta_{\alpha}\delta\bar{\eta}_{\beta}}\right)^2 \frac{1}{2!}\left(-i\bar{\eta}_{\alpha}D_F^{\alpha\beta}\eta_{\beta}\right)^2 \frac{1}{2!}\left(\frac{i}{2}J^{\mu}\Pi_{\mu\nu}J^{\nu}\right)^2.$$

The diagram has no symmetry factor, but with a -1 sign, which is canceled out by the operator reordering:

$$\bar{\eta}_{\beta}D_F^{\beta\tau}\eta_{\tau}\bar{\eta}_{\sigma}D_F^{\sigma\alpha}\eta_{\alpha} = -\eta_{\alpha}\bar{\eta}_{\beta}D_F^{\beta\tau}\eta_{\tau}\bar{\eta}_{\sigma}D_F^{\sigma\alpha}. \tag{2.17}$$

The overall value is e^2.

Then consider the tensor contraction,

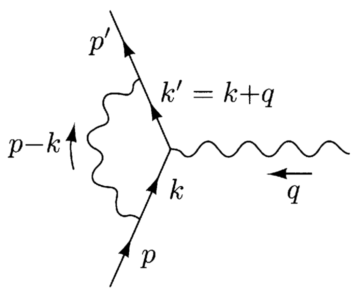
$$-i\Pi^{\mu\nu}_{(2)} \sim e^2\Pi_{\mu\rho}\gamma^{\rho}_{\alpha\beta}D_F^{\beta\tau}\gamma^{\eta}_{\tau\sigma}D_F^{\sigma\alpha}\Pi_{\eta\nu} \quad \sim i\Pi_{\mu\rho}i\Sigma^{\rho\sigma}i\Pi_{\sigma\nu}.$$

The photon self-energy is

$$i\Sigma^{\mu\nu}(p^2) = -e^2\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left[\gamma^{\mu}D_F(k-p)\gamma^{\nu}D_F(k)\right]. \tag{2.18}$$

Example 3. One-loop Correction to Vertex

Consider the diagram



There is 4 electron propagator, 2 photon propagator, and 3 vertices. Consider the perturbative expansion:

$$\frac{\delta^3}{i\delta J\delta\bar{\eta}\delta\eta}\frac{1}{2!}\left(\frac{-e\gamma^{\mu}_{\alpha\beta}\delta^3}{\delta J^{\mu}\delta\eta_{\alpha}\delta\bar{\eta}_{\beta}}\right)^3 \frac{1}{2!}\left(-i\bar{\eta}_{\alpha}D_F^{\alpha\beta}\eta_{\beta}\right)^4 \frac{1}{2!}\left(-\frac{i}{2}J^{\mu}\Pi_{\mu\nu}J^{\nu}\right)^2.$$

There is not symmetry factor, and an additional $-i$ factor. The total coefficient is $-ie^3$.

Then consider the tensor contraction

$$D_F^{\alpha\gamma}\gamma^{\nu}_{\gamma\rho}D_F^{\rho\sigma}\gamma^{\zeta}_{\sigma\tau}D_F^{\tau\eta}\gamma^{\lambda}_{\eta\xi}D_F^{\xi\beta}\Pi_{\nu\lambda}\Pi_{\mu\zeta}.$$

The vertex correction is:

$$iV_3(q,p,p') = [iD_F(p)][iD_F(p')][i\Pi^{\mu\nu}(q)][-ie\Gamma^{\nu}(q,p,p')]$$

where

$$i\Gamma^\mu(q,p,p') = -e^2 \int \frac{d^4k}{(2\pi)^4} \Pi_{\nu\lambda}(p-k)\gamma^\nu D_F(k')\gamma^\mu D_F(k)\gamma^\lambda. \tag{2.19}$$

Example 4. Counter Terms

Consider the counter term in the diagram

The perturbative expansion is

$$iD_F^{(\mathrm{ct})} \sim \frac{\delta^2}{\delta\bar{\eta}\delta\eta} i(\delta_\psi \gamma^\mu_{\alpha\beta} k_\mu - \delta_m \mathbb{I}_{\alpha\beta}) \frac{\delta^2}{\delta\eta_\alpha \delta\bar{\eta}_\beta} \frac{1}{2!} \left(-i\bar{\eta}_\alpha D_F^{\alpha\beta} \eta_\beta\right)^2.$$

The contribution to the electron self energy is

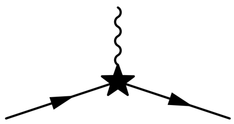
$$\delta_\psi \not{k} - \delta_m m_R.$$

Similarly, the digram

contribute to the photon self energy with term

$$\delta_A[-p^2 g^{\mu\nu} + (1-\xi)p^\mu p^\nu],$$

and the diagram



contribute to the vertex with term

$$\delta_e \gamma^\mu.$$

2.2 Loop Correction

In this section, we consider the QED in $(d = 4 - \epsilon)$ dimensional space-time.

2.2.1 Electron Propagator

Consider the one-loop correction to the electron propagator, where the self energy (2.16) is

$$\begin{aligned} i\Sigma(p^2) &= e^2 \tilde{\mu}^\epsilon \int \frac{d^dk}{(2\pi)^d} \Pi_{\mu\nu}(p-k) \left[\gamma^\mu D_F(k) \gamma^\nu\right]_{\alpha\beta} \\ &= -e^2 \tilde{\mu}^\epsilon \int \frac{d^dk}{(2\pi)^d} \frac{\gamma^\mu(\not{k}+m)\gamma_\mu}{(p-k)^2(k^2-m^2)}. \end{aligned} \tag{2.20}$$

The nominator can be simplified using the `FeynCalc` Package:

```
(*load FeynCalc Package*)
<< FeynCalc`

(*simplify the gamma expression*)
Contract[GA[\[Mu]].(GS[k]+m).GA[\[Mu]]]//DiracSimplify
```

The result is

$$4m - 2\not{k}.$$

The denominator can be simplify using the Feynman parameter:

$$\frac{1}{(p-k)^2(k^2-m^2)} = \int_0^1 \frac{dx}{[(k-b)^2 - D]^2}$$

where b and D can be calculated by

```
A1=(k-p)^2;
A2=k^2-m^2;
{c,b,a}=CoefficientList[x*A1+(1-x)*A2,{k}];
-b/2
-c+b^2/(4*a)//Simplify
```

The result is

$$b = px,\ D = (1-x)(m^2 - p^2 x).$$

Shift $k \to k + px$, the self energy becomes:

$$\begin{aligned} i\Sigma(p^2) &= 2e^2\tilde{\mu}^\epsilon \int_0^1 (x\not{p} - 2m)dx \int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2-D)^2} \\ &= i\frac{2e^2\tilde{\mu}^\epsilon\Omega_d}{(2\pi)^d}\int_0^1 (x\not{p} - 2m)dx \int \frac{k^{d-1}dk}{(k^2+D)^2}. \end{aligned} \tag{2.21}$$

The regularization procedure

```
omg=(2*Pi^(d/2))/(Gamma[d/2]);
cof=2*e^2*\[Mu]^(4-d)*omg/(2*Pi)^d;
int=cof*Integrate[q^(d-1)/(q^2+D)^2,{q,0,Infinity}][[1]];
map={e->Sqrt[4*Pi*\[Alpha]],EulerGamma->Subscript[\[Gamma],E]};
ans=Series[int/.{d->4-\[Epsilon]},{\[Epsilon],0,0}];
ans/.map//Simplify
```

The result is ($\mu^2 = 4\pi\tilde{\mu}^2 e^{-\gamma_E}$)

$$\Sigma(p^2) = \frac{e_R^2}{8\pi^2}\int_0^1 dx(x\not{p} - 2m_R)\left[\frac{2}{\epsilon} + \ln\left(\frac{\mu^2}{D}\right)\right]. \tag{2.22}$$

The infinity comes from

$$\frac{e_R^2}{4\pi^2\epsilon}\int_0^1 dx(x\not{p} - 2m_R) = \frac{e_R^2}{8\pi^2\epsilon}\not{p} - \frac{e_R^2}{2\pi^2\epsilon}m_R.$$

Using the $\overline{\text{MS}}$ subtraction scheme, we choose

$$\delta_\psi = -\frac{e_R^2}{8\pi^2\epsilon},\ \delta_m = -\frac{e_R^2}{2\pi^2\epsilon}, \tag{2.23}$$

and the self energy is

$$\begin{aligned}\Sigma(p^2) &= \frac{e_R^2}{8\pi^2}\int_0^1 dx(x\not{p} - 2m_R)\ln\left[\frac{\mu^2}{(1-x)(m_R^2 - p^2x)}\right] \\ &= \frac{e_R^2}{8\pi^2}(\not{p} - 4m_R)\ln\left(\frac{\mu}{m_R}\right) - \int_0^1 dx \ln\left[(1-x)\left(1 - \frac{p^2x}{m_R^2}\right)\right].\end{aligned} \tag{2.24}$$

2.2.2 Photon Self-energy

Consider the one-loop correction to the photon propagator, where the self energy (2.18) is

$$i\Sigma^{\mu\nu} = -e^2\tilde{\mu}^\epsilon \int \frac{d^dk}{(2\pi)^d}\frac{\text{Tr}\left[\gamma^\mu D_F(k-p)\gamma^\nu D_F(k)\right]}{(k^2-m^2)[(p-k)^2-m^2]}. \tag{2.25}$$

The Dirac trace and Feynman parameter is calculated by

```
(*Dirac trace*)
DiracTrace[GA[\[Mu]].(GS[k-p]+m).GA[\[Nu]].(GS[k]+m)]//DiracSimplify

(*Feynman paramater*)
A1=k^2-m^2;
A2=(k-p)^2-m^2;
{c,b,a}=CoefficientList[x*A1+(1-x)*A2,{k}];
-b/2
-c+b^2/(4*a)//Simplify
```

The nominator is

$$4\left[g^{\mu\nu}\left(k\cdot p - k^2 + m^2\right) + 2k^\mu k^\nu - k^\mu p^\nu - p^\mu k^\nu\right]$$

The denominator is:

$$\frac{1}{(k^2-m^2)[(p-k)^2-m^2]} = \frac{1}{\left\{[k-p(1-x)]^2 - [m^2 + p^2x(x-1)]\right\}^2}$$

Let $D = m^2 - p^2x(1-x)$, shift $k \to k + p(1-x)$, and drop all p^μ linear term,[2] the result is

$$i\Sigma^{\mu\nu} = -4e^2\tilde{\mu}^\epsilon\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\frac{2k^\mu k^\nu - g^{\mu\nu}\left[k^2 - x(1-x)p^2 - m^2\right]}{\left[k^2 - D\right]^2} \tag{2.26}$$

The self-energy $i\Sigma^\mu \propto g^{\mu\nu}$, we can make the substitution

$$k^\mu k^\nu \to \frac{1}{d}k^2 g^{\mu\nu}.$$

[2] The Ward identity requires that the p^μ term in the propagator do not contribute to any scattering process.

We then need to consider the integral

$$
\begin{aligned}
iI(x) &= 4e^2\tilde{\mu}^\epsilon \int \frac{d^dk}{(2\pi)^d}\frac{(1-\frac{2}{d})k^2 - x(1-x)p^2 - m^2}{[k^2-D]^2}, \\
I(x) &= -\frac{4e^2\tilde{\mu}^\epsilon\Omega_d}{(2\pi)^d}\int k^{d-1}dk\frac{(1-\frac{2}{d})k^2 + x(1-x)p^2 + m^2}{[k^2+D]^2}.
\end{aligned}
$$

The regulation is carried out by the following code:

```
omg=(2*Pi^(d/2))/(Gamma[d/2]);
cof=-4*e^2*\[Mu]^(4-d)*omg/(2*Pi)^d;
den=q^(d-1)*((1-2/d)q^2+x*(1-x)p^2+m^2);
int=cof*Integrate[den/(q^2+D)^2,{q,0,Infinity}][[1]];
map={EulerGamma->Subscript[\[Gamma],E],D->m^2-p^2*x*(1-x)};
ans=Series[int/.{d->4-\[Epsilon]},{\[Epsilon], 0, 0}];
ans/.map//Simplify
```

The result is

$$
\Sigma^{\mu\nu}(p^2) = -\frac{e_R^2p^2g^{\mu\nu}}{2\pi^2}\int_0^1 dx\ x(1-x)\left[\frac{2}{\epsilon} + \ln\left(\frac{\mu^2}{m_R^2 - p^2x(1-x)}\right)\right] \tag{2.27}
$$

The divergent part is

$$
-\frac{e_R^2p^2g^{\mu\nu}}{\pi^2\epsilon}\int_0^1 dx\ x(1-x) = -\frac{e_R^2p^2g^{\mu\nu}}{6\pi^2\epsilon}.
$$

The counter term coefficient is

$$
\delta_A = -\frac{e_R^2}{6\pi^2\epsilon}. \tag{2.28}
$$

The photon self-energy is then

$$
\begin{aligned}
\Sigma^{\mu\nu}(p^2) &= -\frac{e_R^2p^2g^{\mu\nu}}{2\pi^2}\int_0^1 dx\ x(1-x)\ln\left[\frac{\mu^2}{m_R^2-p^2x(1-x)}\right] \\
&= -\frac{e_R^2p^2g^{\mu\nu}}{12\pi^2}\ln\left(\frac{\mu}{m}\right) + \frac{e_R^2p^2g^{\mu\nu}}{2\pi^2}\int_0^1 dx\ x(1-x)\ln\left[1-\frac{p^2}{m_R^2}x(1-x)\right].
\end{aligned} \tag{2.29}
$$

2.2.3 Vertex Correction

Consider the loop correction (2.19):

$$
i\Gamma^\mu(p,q_1,q_2) = e^2\tilde{\mu}^\epsilon\int\frac{d^4k}{(2\pi)^4}\frac{\gamma^\nu(\not{k}'+m)\gamma^\mu(\not{k}+m)\gamma_\nu}{(k^2-m^2)(k'^2-m^2)(p-k)^2}. \tag{2.30}
$$

Using the following code

```
(*numerator*)
den=Contract[GA[\[Nu]].(GS[kp]+m).GA[\[Mu]].(GS[k]+m).GA[\[Nu]]];
DiracSimplify[den]

(*Feynman parameter*)
```

```
A1=k^2-m^2;
A2=(k+q)^2-m^2;
A3=(p-k)^2;
{c,b,a}=CoefficientList[x*A1+y*A2+z*A3,{k}];
-b/2//Simplify
-c+b^2/4//Simplify
```

The numerator is

$$-2\not{k}\gamma^\mu \not{k}' - 2m^2\gamma^\mu + 4m(k+k')^\mu.$$

The denominator is

$$\int \frac{dF_3}{[(k+yq-zp)^2-D]^3},$$

where

$$\begin{aligned} D &= (x+y)m^2 - z(1-z)p^2 - y(1-y)q^2 - 2yzpq \\ &= (x+y)m^2 - xyq^2 - yzp'^2 - xzp^2. \end{aligned}$$

Shift $k^\mu \to k^\mu + zq_1^\mu - yp^\mu$, throw away all terms with linear k^μ, and replace $k^\mu k^\nu$ with $\frac{1}{d}k^2 g^{\mu\nu}$, the result is

$$\frac{4}{d}k^2\gamma^\mu - 2(-y\not{q} + z\not{p})\gamma^\mu[(1-y)\not{q} + z\not{p}] + 4m^2\gamma^\mu - 2m\left[(1-2y)q^\mu + 2zp^\mu\right].$$

Note that only the quadratic term is divergent.

$$\Gamma^\mu(p, q_1, q_2) = -i\frac{4e^2\tilde{\mu}^\epsilon\gamma^\mu}{d}\int dF_3 \int \frac{d^dk}{(2\pi)^d}\frac{k^2}{(k^2-D)^3} + \delta\Gamma^\mu(p, q_1, q_2).$$

where $\delta\Gamma^\mu$ stores all the finite part

$$\begin{aligned} &\delta\Gamma^\mu(p, q_1, q_2) \\ &= \int \frac{e^2k^3dkdF_3}{(2\pi)^2(k^2+D)^3}\left\{(-y\not{q} + z\not{p})\gamma^\mu[(1-y)\not{q} + z\not{p}] - 2m^2\gamma^\mu + m\left[(1-2y)q^\mu + 2zp^\mu\right]\right\}. \end{aligned}$$

The divergent part is

$$\frac{4e^2\tilde{\mu}^\epsilon\Omega_d\gamma^\mu}{d(2\pi)^d}\int dF_3 \int \frac{k^{d+1}dk}{(k^2+D)^3} = \frac{e_R^2}{16\pi^2}\gamma^\mu \int dF_3 \left[\frac{2}{\epsilon} + \ln\left(\frac{\mu^2}{D}\right)\right].$$

Using the $\overline{\text{MS}}$ scheme, the coefficient of the counter term is

$$\delta_e = -\frac{e_R^2}{8\pi^2\epsilon}. \tag{2.31}$$

2.2.4 Renormalization Group

In summery, the renormalization factors are

$$\begin{aligned} Z_\psi &= 1 - \frac{e_R^2}{8\pi^2\epsilon} + O(e_R^3), \\ Z_A &= 1 - \frac{e_R^2}{6\pi^2\epsilon} + O(e_R^3), \\ Z_m &= 1 - \frac{e_R^2}{2\pi^2\epsilon} + O(e_R^3), \\ Z_e &= 1 - \frac{e_R^2}{8\pi^2\epsilon} + O(e_R^3), \end{aligned} \tag{2.32}$$

which means

$$\begin{aligned}
\frac{d\ln Z_\phi}{de_R} &= -\frac{e_R}{4\pi^2\epsilon} + O(e_R^2), \\
\frac{d\ln Z_A}{de_R} &= -\frac{e_R}{3\pi^2\epsilon} + O(e_R^2), \\
\frac{d\ln Z_m}{de_R} &= -\frac{e_R}{\pi^2\epsilon} + O(e_R^2), \\
\frac{d\ln Z_e}{de_R} &= -\frac{e_R}{4\pi^2\epsilon} + O(e_R^2).
\end{aligned} \tag{2.33}$$

The bare parameters are

$$\begin{aligned}
\psi_0 &= Z_\psi^{1/2}\psi_R, \\
A_0 &= Z_A^{1/2}A_R, \\
m_0 &= Z_m Z_\psi^{-1} m_R, \\
e_0 &= Z_e Z_\psi^{-1} Z_A^{-1/2} e_R \tilde{\mu}^{\epsilon/2}.
\end{aligned} \tag{2.34}$$

The RG equation for e_0 is

$$\frac{d\ln e_0}{d\ln\mu} = \left(\frac{\ln Z_e}{de_R} - \frac{\ln Z_\psi}{de_R} - \frac{1}{2}\frac{\ln Z_A}{de_R} + \frac{1}{e_R}\right)\frac{de_R}{d\ln\mu} + \frac{\epsilon}{2} = 0. \tag{2.35}$$

The beta function is

$$\beta(e_R) = \frac{de_R}{d\ln\mu} = -\frac{\epsilon}{2}e_R + \frac{e_R^3}{12\pi^2} + O(e_R^4). \tag{2.36}$$

The RG equation for m_0 is

$$\frac{d\ln m_0}{d\ln\mu} = \left(\frac{\ln Z_m}{de_R} - \frac{\ln Z_\psi}{de_R}\right)\frac{de_R}{d\ln\mu} + + \frac{1}{m_R}\frac{dm_R}{d\ln\mu} = 0. \tag{2.37}$$

The anomalous dimension of mass is

$$\gamma_m = \frac{d\ln m_R}{d\ln\mu} = -\frac{3e_R^2}{8\pi^2} + O(e_R^3). \tag{2.38}$$

Chapter 3

Non-relativistic QFT

A general non-relativistic field has the Lagrangian[1]

$$\mathcal{L} = \bar{\psi}_a(x)(i\delta_{ab}\partial_t - \hat{H}_{ab})\psi_b(x) + \mathcal{V}_{\text{int}} \tag{3.1}$$

where the field operator ψ can be bosonic or fermionic, which is denoted by a number $\zeta = \pm 1$, and $\mathcal{V}_{\text{int}}$ is the interaction Lagrangian. A general interaction has the form

$$\mathcal{V}_{\text{int}} = \bar{\psi}_a(x_1)\bar{\psi}_b(x_2)V_{ab}(x_1, x_2)\psi_b(x_2)\psi_a(x_1). \tag{3.2}$$

Note that the classical equation of motion for the free field is

$$\begin{aligned} 0 &= \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi}_a(x))} - \frac{\partial \mathcal{L}}{\partial \bar{\psi}_a(x)} \\ &= -i\partial_t \phi_a(x) + \hat{H}_{ab}\phi_b(x), \end{aligned} \tag{3.3}$$

which satisfies the Schrödinger equation.

We are mostly work with finite system size L^d with UV cutoff $\Lambda = \frac{2\pi}{a}$,[2] in which case the spatial Fourier transformation is

$$\tilde{\psi}_a(k) = \int_{L^d} d^d x e^{-ik\cdot x}\psi(k), \tag{3.4}$$

$$\psi_a(x) = \frac{1}{L^d}\sum_k e^{ik\cdot x}\tilde{\psi}(k). \tag{3.5}$$

Note that for finite size, the momentum is discretized:

$$k_i = \frac{2\pi}{L}n_i, \;\; n_i = -N, \cdots, N. \tag{3.6}$$

By default, we take the thermodynamic limit. The summation is approximated by the integration:

$$\frac{1}{L^d}\sum_k \longrightarrow \int_{|k|<\Lambda} \frac{d^d k}{(2\pi)^d}. \tag{3.7}$$

[1] The repeated indices are automatically summed.

[2] We can regard a as the lattice spacing, and assume $L = Na$.

3.1 Finite Temperature Field Theory

The original real-time partition function is defined as[3]

$$Z[J] = \int D[\bar{\psi}, \psi] \exp\left\{ i \int dt \int d^d x \left[\mathcal{L} + \bar{J}_a(x)\psi_a(x) + \bar{\psi}_a(x) J_a(x) \right] \right\}. \tag{3.8}$$

For finite-temperature field theory, after making the wick rotation $t \to -i\tau$, the partition function for a generic non-relativistic lattice theory is:

$$Z[J] = \int D[\bar{\psi}, \psi] e^{-S[\bar{\psi},\psi] + \bar{J}\cdot\psi + \bar{\psi}\cdot J}, \tag{3.9}$$

where the action is

$$S = \int_0^\beta d\tau \int d^d x \left[\bar{\psi}_a(x)(\delta_{ab}\partial_\tau + \hat{H}_{ab})\psi_b(x) + \mathcal{V}_{\text{int}} \right]. \tag{3.10}$$

Remark 6. Temporal Fourier Transformation

The Fourier transformation on the imaginary time domain is defined as:

$$\tilde{\psi}(\omega_n) = \int_0^\beta d\tau e^{i\omega_n \tau} \psi(\tau), \tag{3.11}$$

$$\psi(\tau) = \frac{1}{\beta} \sum_{\omega_n} e^{-i\omega_n \tau} \tilde{\psi}(\omega_n). \tag{3.12}$$

Under such convention, in the thermodynamic limit and zero-temperature limit, the spatial-temporal Fourier transformation agrees with the relativistic case (up to a Wick rotation).

3.1.1 Free Field Theory

We first consider the action of free field

$$S_0 = \int_0^\beta d\tau \int d^d x \; \bar{\psi}_a(x)(\delta_{ab}\partial_\tau + \hat{H}_{ab})\psi_b(x). \tag{3.13}$$

The Fourier transformation

$$S_0 = \frac{1}{\beta} \sum_{\omega_n} \int_\Lambda \frac{d^d k}{(2\pi)^d} \tilde{\bar{\psi}}_a(k, \omega_n) \left[-i\omega_n + \tilde{H}_{ab}(k) \right] \tilde{\psi}_b(k, \omega_n). \tag{3.14}$$

The partition function with source is

$$\frac{Z_0[J]}{Z_0[0]} = \exp\left[-\frac{1}{\beta} \sum_{\omega_n} \int_\Lambda \frac{d^d k}{(2\pi)^d} \tilde{\bar{J}}_a(k, \omega_n) \tilde{G}_{ab}(k, \omega_n) \tilde{J}_b(k, \omega_n) \right], \tag{3.15}$$

[3] As with the relativistic case, we introduce an auxiliary source J, which is bosonic/fermionic if the field ψ is bosonic/fermionic.

where the Green's function is

$$\tilde{G}_{ab}(k,\omega_n) = \left[\frac{1}{i\omega_n - \tilde{H}(k)}\right]_{ab}. \tag{3.16}$$

Remark 7. Obtaining the Partition Function

Unlike the relativistic case, the value of the value of partition function without source $Z_0[0]$ is related to the free energy. We can express it formally as

$$Z_0[0] = \left[\det(-G_{ab})^{-1}\right]^{-\zeta}.$$

To get the correct dimensionality, we set the determinant as

$$Z_0[0] \equiv \prod_{k,\omega_n}\left\{\beta\det\left[-i\omega_n + \tilde{H}(k)\right]\right\}^{-\zeta}.$$

Thus the free energy is

$$F = -\frac{1}{\beta}\ln Z_0 = \zeta\sum_{k,\omega_n}\ln\left\{\beta\det\left[-i\omega_n + \tilde{H}(k)\right]\right\}. \tag{3.17}$$

Remark 8. Matsubara Summation

Now consider the summation on Matsubara frequency:

$$\sum_{\omega_n} f(\omega_n) = \begin{cases}\sum_n f(\frac{2n\pi}{\beta}) & \text{bosonic}\\ \sum_n f(\frac{(2n+1)\pi}{\beta}) & \text{fermionic}\end{cases}. \tag{3.18}$$

The frequency is capture by the singularities of the density function of the states:

$$\rho(z) = \begin{cases}\frac{1}{\exp(\beta z)-1} & \text{bosonic}\\ \frac{1}{\exp(\beta z)+1} & \text{fermionic}\end{cases}. \tag{3.19}$$

The residue on imaginary frequency $i\omega_n$ is alway $\frac{1}{\beta}$. In this way, the summation is:

$$\frac{1}{\beta}\sum_{\omega_n} f(i\omega_n) = \frac{1}{2\pi i}\oint \rho(z)f(z) = -\sum_k \operatorname{Res}\,\rho(z)f(z)|_{z=z_k}. \tag{3.20}$$

Example 5. Summation of Green's function

Consider the frequency summation for the correlation function:

$$\frac{1}{\beta}\sum_{\omega_n}\tilde{G}_0(k) = \frac{1}{\beta}\sum_{\omega_n}\frac{1}{i\omega_n - E_p} = -\operatorname{Res}\left.\frac{\rho(z)}{z-E_p}\right|_{z=E_p} = \rho(E_p).$$

Example 6. Summation of Green's function

Consider the frequency summation for the correlation function:

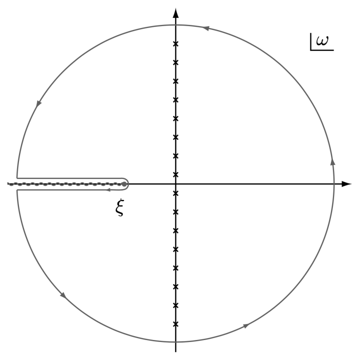
$$\sum_{\omega_n} \langle \bar{\psi}_{\vec{p},\omega_n} \psi_{\vec{p},\omega_n} \rangle = \frac{1}{\beta} \sum_{\omega_n} \frac{1}{-i\omega_n + \epsilon_{\vec{p}}} = \text{Res} \left. \frac{\rho(z)}{z - \epsilon_{\vec{p}}} \right|_{z - \epsilon_{\vec{p}}} = \rho(\epsilon_{\vec{p}}).$$

Example 7. Free Energy Summation

Consider the free energy

$$F = -\frac{1}{\beta} \ln Z = -\frac{1}{\beta} \sum_{\omega_n} \ln[\beta(-i\omega_n + E_{\vec{p}})] = \frac{1}{2\pi i} \oint dz \rho(z) \ln[\beta(\xi - z)].$$

To calculate the summation, we consider the line integral along the loop:



The free energy is

$$\begin{aligned} F &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} dx \rho(x) \ln \left(\frac{\xi - x - i\epsilon}{\xi - x + i\epsilon} \right) \\ &= \frac{-\zeta}{2\pi i \beta} \int_{-\infty}^{\infty} dx \ln(1 - \zeta e^{-\beta z}) \left(\frac{1}{x + i\epsilon - \xi} - \frac{1}{x - i\epsilon - \xi} \right), \end{aligned}$$

where we integrate the expression by part, noticing that

$$\frac{d}{dz} \frac{\zeta}{\beta} \ln(1 - \zeta e^{-\beta z}) = \frac{1}{e^{\beta z} - \zeta} = \rho(z) \tag{3.21}$$

Using the identity

$$\lim_{\epsilon \to 0^+} \frac{1}{x + i\epsilon} = -i\pi\delta(x) + \mathcal{P}\frac{1}{x},$$

the above expression can be simplified to

$$F = \frac{\zeta}{\beta} \ln(1 - \zeta e^{-\beta\zeta}). \tag{3.22}$$