Notes on Quantum Field Theory

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Chapter 1

Relativistic Free Field Theories

1.1 The Lorentz Invariance

The metric for $(3+1)$-dimensional flat space-time is chosen to be

$$g_{\mu\nu} = g^{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1). \tag{1.1}$$

The Lorentz transformation $\Lambda^{\mu}{}_{\nu}$ satisfies

$$\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}g^{\alpha\beta} = g^{\mu\nu}. \tag{1.2}$$

From this we have

$$g^{\gamma\alpha}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}g_{\mu\nu} = g^{\gamma\alpha}g_{\alpha\beta} \quad \Longrightarrow \quad \Lambda_{\nu}{}^{\gamma}\Lambda^{\nu}{}_{\beta} = \delta^{\gamma}{}_{\beta}, \tag{1.3}$$

The inverse Lorentz transformation satisfies:

$$(\Lambda^{-1})^{\mu}{}_{\nu} = \Lambda_{\nu}{}^{\mu}. \tag{1.4}$$

The infinitesimal transformation is denoted as

$$\begin{aligned} \Lambda^{\mu}{}_{\nu} &= \delta^{\mu}_{\nu} + \delta\omega^{\mu}{}_{\nu}, \\ (\Lambda^{-1})^{\mu}{}_{\nu} &= \delta^{\mu}_{\nu} - \delta\omega^{\mu}{}_{\nu}. \end{aligned} \tag{1.5}$$

which means $\delta\omega^{\mu}{}_{\nu} = -\delta\omega_{\nu}{}^{\mu}$. We can further use the metric tensor $g_{\mu\nu}$ to lower the indices and get $\delta\omega_{\alpha\beta} = -\delta\omega_{\beta\alpha}$, i.e., the infinitesimal parameter $\delta\omega_{\mu\nu}$ is anti-symmetric for $(\mu \leftrightarrow \nu)$.

In general, a representation of Lorentz group $U_R(\Lambda)$ can be parametrized as:

$$U_R(\Lambda) = \exp\left(\frac{i}{2}\omega_{\mu\nu}M_R^{\mu\nu}\right). \tag{1.6}$$

Another useful parametrization is

$$\theta_i \equiv \frac{1}{2}\varepsilon_{ijk}\omega_{jk}, \quad \beta_i \equiv \omega_{i0}. \tag{1.7}$$

A new set of generators are:

$$J_i \equiv \frac{1}{2}\varepsilon_{ijk}M^{jk}, \quad K_i \equiv M^{i0}, \tag{1.8}$$

where J_i's are the generators of the spatial rotations, and K_i's are the generators of Lorentz boosts.

In the fundamental representation, the generators are represented by

$$\begin{aligned}
J_1 &= \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 0 & -i \\ & & i & 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & & & \\ & 0 & & i \\ & & 0 & \\ & -i & & 0 \end{bmatrix}, \quad J_3 = \begin{bmatrix} 0 & & & \\ & 0 & -i & \\ & i & 0 & \\ & & & 0 \end{bmatrix}, \\
K_1 &= \begin{bmatrix} 0 & -i & & \\ -i & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0 & & -i & \\ & 0 & & \\ -i & & 0 & \\ & & & 0 \end{bmatrix}, \quad K_3 = \begin{bmatrix} 0 & & & -i \\ & 0 & & \\ & & 0 & \\ -i & & & 0 \end{bmatrix}.
\end{aligned} \tag{1.9}$$

The Lie algebra of the Lorentz algebra can be explicitly done using the fundamental representation. The result is

$$\begin{aligned}
[J_i, J_j] &= i\varepsilon_{ijk}J_k, \\
[J_i, K_j] &= i\varepsilon_{ijk}K_k, \\
[K_i, K_j] &= -i\varepsilon_{ijk}J_k.
\end{aligned} \tag{1.10}$$

1.1.1 Left and Right Spinors

We introduce a new set of generators:

$$N_i^L \equiv \frac{J_i - iK_i}{2}, \quad N_i^R \equiv \frac{J_i + iK_i}{2}. \tag{1.11}$$

They satisfies two independent $\mathfrak{su}(2)$ algebra:

$$\begin{aligned}
\left[N_i^L, N_j^L\right] &= i\varepsilon_{ijk}N_k^L, \\
\left[N_i^R, N_j^R\right] &= i\varepsilon_{ijk}N_k^R, \\
\left[N_i^L, N_j^R\right] &= 0.
\end{aligned} \tag{1.12}$$

That is, the Lorentz algebra is isomorphic to two $\mathfrak{su}(2)$ algebra,

$$\mathfrak{so}(3,1) \approx \mathfrak{su}_L(2) \oplus \mathfrak{su}_R(2). \tag{1.13}$$

From Eq. (1.13), we know that the representation of the Lorentz algebra can be labelled by j_L and j_R. Note that the fundamental representation correspond to

$$\left(j_L = \frac{1}{2}, j_R = \frac{1}{2}\right).$$

The specific form of the group is

$$\Lambda(\vec{\theta}, \vec{\beta}) = \exp\left[i(\vec{\theta} + i\vec{\beta}) \cdot \vec{N}^L + i(\vec{\theta} - i\vec{\beta}) \cdot \vec{N}^R\right]. \tag{1.14}$$

The spinor representations are those with $j_L = 1/2$ or $j_R = 1/2$. Specifically, we define the left-hand spinor ψ_L and right-hand spinor ψ_R that transform as:

$$\begin{aligned} \Lambda_L(\vec{\theta},\vec{\beta})\psi_L &= \exp\left(\frac{i}{2}\vec{\theta}\cdot\vec{\sigma} - \frac{1}{2}\vec{\beta}\cdot\vec{\sigma}\right)\psi_L, \\ \Lambda_R(\vec{\theta},\vec{\beta})\psi_R &= \exp\left(\frac{i}{2}\vec{\theta}\cdot\vec{\sigma} + \frac{1}{2}\vec{\beta}\cdot\vec{\sigma}\right)\psi_R. \end{aligned} \tag{1.15}$$

Using the fact $\sigma^2 \cdot \vec{\sigma}^* \cdot \sigma^2 = -\vec{\sigma}$, the left-hand and the right-hand representations are related by:

$$\begin{aligned} \sigma^2\Lambda_L^*\sigma^2 &= \Lambda_R, \quad \sigma^2\Lambda_L^T\sigma^2 = \Lambda_L^{-1}, \\ \sigma^2\Lambda_R^*\sigma^2 &= \Lambda_L, \quad \sigma^2\Lambda_R^T\sigma^2 = \Lambda_R^{-1}. \end{aligned} \tag{1.16}$$

For this reason, the left-hand and right-hand spinor can be interchanged by

$$\begin{aligned} \sigma^2\psi_L^* &\sim \chi_R, \quad \psi_L^\dagger\sigma^2 \sim \chi_R^\dagger, \\ \sigma^2\psi_R^* &\sim \chi_L, \quad \psi_R^\dagger\sigma^2 \sim \chi_L^\dagger. \end{aligned} \tag{1.17}$$

1.1.2 The Invariant Symbols

The invariant symbols can be thought as the Clebsch-Gordan coefficients that help to form singlets. The first singlet comes from the decomposition

$$\frac{1}{2}\otimes\frac{1}{2} \approx 0 \oplus 1.$$

Correspondingly, we can check that for each-hand-side spinor, the quadratic forms

$$\psi_L^T\sigma^2\chi_L \quad \text{or} \quad \psi_R^T\sigma^2\chi_R \tag{1.18}$$

are singlets. We can define the first invariant symbol as[1]

$$\varepsilon^{ab} = \varepsilon^{\dot{a}\dot{b}} = i(\sigma^2)_{ab}, \quad \varepsilon_{ab} = \varepsilon_{\dot{a}\dot{b}} = -i(\sigma^2)_{ab}. \tag{1.19}$$

The symbol ε^{ab} or ε_{ab} also serve as the index raising/lowering symbol, i.e.,

$$\varepsilon^{ab}\psi_b = \psi^a, \ \varepsilon_{ab}\psi^b = \psi_a. \tag{1.20}$$

The singlet (1.18) is then defined as the inner product of two spinors:

$$\psi\cdot\chi \equiv \varepsilon_{ab}\psi^a\chi^b = \psi^a\chi_a = -\varepsilon_{ba}\psi^a\chi^b = -\psi_b\chi^b. \tag{1.21}$$

In addition, because of (1.17), the expressions

$$\psi_L^\dagger\chi_R \quad \text{and} \quad \psi_R^\dagger\chi_L$$

are also singlets.

Besides, we know there should be another invariant symbol from the decomposition

$$\left(\frac{1}{2},0\right)\otimes\left(0,\frac{1}{2}\right) \approx (0,0)\oplus\cdots.$$

[1]We use the dotted symbol to denote the right-hand spinor indices.

For this reason, we are searching for the symbol M that the expression

$$M^{\mu}_{a\dot{b}}\psi^{a}_{L}\chi^{\dot{b}}_{R}$$

transforms as the Lorentz vector. The matrix M^{μ} should transform as

$$M^{\mu} \longrightarrow \Lambda_L^T \cdot M^{\mu} \cdot \Lambda_R = \Lambda^{\mu}{}_{\nu} M^{\nu}.$$

Use the fact that $\sigma^2 \cdot \Lambda_L^T \cdot \sigma^2 = \Lambda_L^{-1}$, the above equation transforms to

$$\left(\sigma^2 M^{\mu}\right) \longrightarrow \Lambda_L^{-1} \cdot \left(\sigma^2 M^{\mu}\right) \cdot \Lambda_R.$$

We then show the matrices $\sigma^{\mu} = (\sigma^0, \vec{\sigma})$ satisfies the requirement. Firstly, for the spatial rotation,

$$\Lambda_L(\vec{\theta}, \vec{0}) = \Lambda_R(\theta, \vec{0}) = \exp\left(i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) \tag{1.22}$$

The Pauli matrix transform as

$$\left(1 - i\delta\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) \sigma^j \left(1 + i\delta\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) = \sigma^j + i\delta\theta_i \left(-i\varepsilon_{ijk}\sigma^k\right)$$

Secondly, for the boosts,

$$\Lambda_{L/R}(\vec{0}, \vec{\beta}) = \exp\left(\mp\vec{\beta} \cdot \frac{\vec{\sigma}}{2}\right). \tag{1.23}$$

The Pauli matrix transform as

$$\left(1 + \delta\vec{\beta} \cdot \frac{\vec{\sigma}}{2}\right) \sigma^{\mu} \left(1 + \delta\vec{\beta} \cdot \frac{\vec{\sigma}}{2}\right) = \begin{cases} \sigma^0 + i\delta\beta_i \cdot (-i\sigma^i), & \mu = 0 \\ \sigma^j + i\delta\beta_j(-i\sigma^0), & \mu = j \end{cases}.$$

We thus have shown indeed that

$$\psi_L^T \sigma^2 \sigma^{\mu} \chi_R \tag{1.24}$$

is a Lorentz vector. Further more, from (1.17), we know that

$$\eta_R^{\dagger} \sigma^{\mu} \chi_R \tag{1.25}$$

is also a Lorentz vector. Similarly, consider the Lorentz vector

$$N^{\mu}_{\dot{a}b}\psi^{\dot{a}}_{R}\chi^{b}_{R},$$

which together with σ^2 should transforms as

$$\left(\sigma^2 N^{\mu}\right) \longrightarrow \Lambda_R^{-1} \cdot \left(\sigma^2 N^{\mu}\right) \cdot \Lambda_L.$$

We can check that $\bar{\sigma}^{\mu} = (\sigma^0, -\vec{\sigma})$ satisfies the requirement, and thus

$$\eta_L^{\dagger} \bar{\sigma}^{\mu} \chi_L \tag{1.26}$$

is also a Lorentz vector.

1.1.3 Lorentz-invariant Lagrangian

In relativistic quantum field theory, the Lagrangian should be a singlet under Lorentz transformation. Different free fields correspond to different representation of the Lorentz algebra. The symmetry under Lorentz transformation also restrict the possible terms that can appear in the Lagrangian.

Scalar Field

The simplest case is when $j_L = j_R = 0$, corresponding to the scalar field, which we denote as $\phi(x)$. Since the field it self is singlet, any polynomial of the field in principle can appear in the theory. When considering the free theory, we restrict our attention to the quadratic terms. We require the field theory to have a dynamical term, which contains derivative the the field. The derivative operator ∂^μ transforms as the fundamental representation. To be Lorentz invariant, the allowed free theory can only be

$$\mathcal{L}_{\text{KG}} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{m^2}{2}\phi^2 \simeq -\frac{1}{2}\phi(\partial^2 + m^2)\phi. \tag{1.27}$$

For general discussion, we consider the field theory on d-dimensional space-time. Note that the space-time Fourier transformation is defined as

$$\begin{aligned} \tilde{\phi}(k) &= \int d^d x e^{ik\cdot x}\phi(x), \\ \phi(x) &= \int \frac{d^d k}{(2\pi)^d} e^{-ik\cdot x}\tilde{\phi}(k), \end{aligned} \tag{1.28}$$

where the inner product of two d-momentum and d-coordinate is

$$k \cdot x \equiv \omega t - \vec{k}\cdot\vec{x}. \tag{1.29}$$

The action can be expressed as

$$S_{\text{KG}} = \int \frac{d^d k}{(2\pi)^d}\frac{1}{2}\tilde{\phi}^*(k)(k^2 - m^2)\tilde{\phi}(k). \tag{1.30}$$

Vector Field

If we can choose $j_L = j_R = 1/2$, the field is transformed as Lorentz vector. We denote the field as $A^\mu(x)$. Some possible quadratic forms for the vector field that forms singlets are

$$A^\mu A_\mu,\ (\partial_\mu A^\mu)^2,\ A^\nu\partial^2 A_\nu,\ \varepsilon_{\mu\nu\rho\lambda}\partial^\mu A^\nu \partial^\rho A^\lambda. \tag{1.31}$$

For the field theory describe the electromagnetic field, we require the theory to further have gauge symmetry, i.e., invariant under

$$A^\mu(x) \to A^\mu(x) + \partial^\mu\alpha(x). \tag{1.32}$$

The gauge invariant forbids the first term, and forces the second and third term to combine as

$$(\partial_\mu A^\mu)^2 - A^\nu\partial^2 A_\nu \sim \frac{1}{2}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A^\nu - \partial_\nu A_\mu) \equiv \frac{1}{2}F^{\mu\nu}F_{\mu\nu}.$$

where we have define a field-strength tensor

$$F^{\mu\nu} \equiv (\partial^\mu A^\nu - \partial^\nu A^\mu) = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{bmatrix}, \tag{1.33}$$

where we notice that from Maxwell equations:

$$E^i = \partial_t \vec{A} = -\vec{\nabla} A^0, \quad B^i = \nabla \times \vec{A}. \tag{1.34}$$

Note that the fourth term is called the *theta term*, which can be written as a boundary term

$$\varepsilon_{\mu\nu\rho\lambda}\partial^\mu A^\nu \partial^\rho A^\lambda = \partial^\mu(\varepsilon_{\mu\nu\rho\lambda} A^\nu \partial^\rho A^\lambda). \tag{1.35}$$

The Lagrangian describing the electromagnetic field is given by

$$\mathcal{L}_{\text{Maxwell}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{1.36}$$

Spinor Field

Based on previous discussion, the Lagrangian for spinor field can have

$$\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L,\ \psi_R^\dagger \sigma^\mu \partial_\mu \psi_R,\ \psi_L^\dagger \psi_R,\ \psi_R^\dagger \psi_L,\ \psi_L \cdot \psi_L,\ \psi_R \cdot \psi_R. \tag{1.37}$$

The Dirac field describe the theory with both left-hand and right-hand spinors. The Lagrangian is

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi, \tag{1.38}$$

where

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix},\ \bar{\psi} = \begin{pmatrix} \psi_R^\dagger & \psi_L^\dagger \end{pmatrix},\ \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}. \tag{1.39}$$

In addition, we could consider using the last two terms as the mass, the result theory is the *Majorana field theory*:

$$\begin{aligned} \mathcal{L}^L_{\text{Majorana}} &= \psi_L^\dagger \left(i\bar{\sigma}^\mu \partial_\mu - m\sigma^2\right)\psi_L, \\ \mathcal{L}^R_{\text{Majorana}} &= \psi_R^\dagger \left(i\sigma^\mu \partial_\mu - m\sigma^2\right)\psi_R. \end{aligned} \tag{1.40}$$

For the spinor basis, the Dirac Algebra is generated by

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]. \tag{1.41}$$

The Lorentz group is represented by

$$\Lambda_{\frac{1}{2}} = \exp\left(\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right). \tag{1.42}$$

Using the familiar parametrization,

$$S^{i0} = \frac{i}{2}\begin{bmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{bmatrix}, \quad S^{ij} = \frac{1}{2}\epsilon^{ijk}\begin{bmatrix} \sigma^k & 0 \\ 0 & -\sigma^k \end{bmatrix}, \tag{1.43}$$

which agree with the transformation property (1.15).

1.2 Canonical Quantization

1.2.1 Scalar Field

For the Klein-Gordon Lagrangian

$$\mathcal{L} = -\frac{1}{2}\phi(x)(\partial^2 + m^2)\phi(x), \tag{1.44}$$

the equation of motion is:

$$\begin{aligned} \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right] - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \\ \Rightarrow (\partial_t^2 - \nabla^2 + m^2)\phi(\boldsymbol{x}, t) = 0. \end{aligned} \tag{1.45}$$

The (classical) solution to Eq. (1.45) is proportional to the plane wave:

$$\phi_{\boldsymbol{k}}(\boldsymbol{x}, t) \propto e^{-i\omega_{\boldsymbol{k}} t + i\boldsymbol{k}\cdot\boldsymbol{x}} + e^{i\omega_{\boldsymbol{k}} t - i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{1.46}$$

where the energy is $\omega_{\boldsymbol{k}} = \boldsymbol{k}^2 + m^2$ and $\boldsymbol{k}$ is the momentum as the conserved quantity. The general solution to Eq. (1.45) is

$$\phi(\boldsymbol{x}, t) \propto \int \frac{d^3k}{(2\pi)^3} \left(a_k e^{-i\omega_{\boldsymbol{k}} t + i\boldsymbol{k}\cdot\boldsymbol{x}} + a_k^* e^{i\omega_{\boldsymbol{k}} t - i\boldsymbol{k}\cdot\boldsymbol{x}}\right), \tag{1.47}$$

where a_k's are arbitrary c-numbers.

The canonical quantization promote the coefficient a_k/a_k^* to the particle annihilation/creation operator $a_k/a_k^\dagger$, with the commutation relation

$$[a_k, a_p^\dagger] = (2\pi)^3\delta^3(\boldsymbol{k} - \boldsymbol{p}). \tag{1.48}$$

Single-particle States

The single-particle state with momentum $\boldsymbol{k}$ is created by $a_k^\dagger$ operators acting on the vacuum:

$$|\boldsymbol{k}\rangle \equiv \sqrt{2\omega_{\boldsymbol{k}}}\, a_k^\dagger |0\rangle, \tag{1.49}$$

where $|\boldsymbol{k}\rangle$ is a state with a single particle of momentum $\boldsymbol{k}$. The factor of $\sqrt{2\omega_{\boldsymbol{k}}}$ in (1.49) is a convention to ensure Lorenz invariant. To compute the normalization of one-particle states, we start by requiring the vacuum state to be of unit norm:

$$\langle 0|0\rangle = 1, \tag{1.50}$$

which, together with the canonical commutation relation of particle annihilation and creation operators leads to

$$\langle \boldsymbol{p}|\boldsymbol{k}\rangle = 2\sqrt{\omega_{\boldsymbol{p}}\omega_{\boldsymbol{k}}} \left\langle 0 \left| a_p a_k^\dagger \right| 0 \right\rangle = 2\omega_{\boldsymbol{p}}(2\pi)^3\delta^3(\boldsymbol{p} - \boldsymbol{k}). \tag{1.51}$$

The identity operator for one-particle states under such norm is

$$1 = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\boldsymbol{p}}} |\boldsymbol{p}\rangle\langle \boldsymbol{p}|, \tag{1.52}$$

which we can check with

$$|\boldsymbol{k}\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\boldsymbol{p}}}|\boldsymbol{p}\rangle\langle\boldsymbol{p}|\boldsymbol{k}\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\boldsymbol{p}}}2\omega_{\boldsymbol{p}}(2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{k})|\boldsymbol{p}\rangle = |\boldsymbol{k}\rangle.$$

We see that the identity operator (1.52) under such convention is Lorentz invariant, since it can be expressed as

$$1 = 2\pi\int \frac{d^3p\,d\omega}{(2\pi)^4}\delta(\omega^2-\boldsymbol{p}^2-m^2)|\boldsymbol{p}\rangle\langle\boldsymbol{p}|. \tag{1.53}$$

The single-particle defined above can be used to fix the normalization:

$$\langle\boldsymbol{k}|\phi(\boldsymbol{x},0)|0\rangle = e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{1.54}$$

leading to the field expansion

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{k}}}}\left(a_k e^{-ik\cdot x} + a_k^\dagger e^{ik\cdot x}\right). \tag{1.55}$$

Hamiltonian

We can obtain the Hamiltonian for the Klein-Gordon field using the Legendre transformation:

$$\begin{aligned} H &= \int d^4x\ \left[\pi(x)\dot{\phi}(x) - \mathcal{L}(x)\right] \\ &= \int d^4x\ \frac{1}{2}\left[\pi^2 + (\nabla\phi)^2 + m^2\phi^2\right] \end{aligned} \tag{1.56}$$

where the canonical momentum is defined as

$$\begin{aligned} \pi(x) &= \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}(x) \\ &= -i\int \frac{d^3k}{(2\pi)^3}\sqrt{\frac{\omega_k}{2}}\left(a_k e^{-ik\cdot x} - a_k^\dagger e^{ik\cdot x}\right) \end{aligned} \tag{1.57}$$

The π^2 term expands as

$$\pi^2(x) = \int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\frac{\sqrt{\omega_{k_1}\omega_{k_2}}}{2}\left(a_{k_1}^\dagger a_{k_2}e^{i(k_1-k_2)x} - a_{k_1}^\dagger a_{k_2}^\dagger e^{i(k_1+k_2)x} + h.c.\right). \tag{1.58}$$

We note that after integrate over x, the phase factor $e^{i(k_1-k_2)x}$ produce a delta function for k_1 and k_2. The $a_{k_1}^\dagger a_{k_2}^\dagger$ terms will finally be cancelled by other terms. We temporally ignore such term. The contribution from the first term is then

$$\int d^4x\ \pi^2(x) = \int \frac{d^3k}{(2\pi)^3}\frac{\omega_k}{2}a_k^\dagger a_k + h.c. \tag{1.59}$$

The second term is

$$(\nabla\phi)^2 = \int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\frac{\boldsymbol{k}_1\boldsymbol{k}_2}{2\sqrt{\omega_{k_1}\omega_{k_2}}}\left(a_{k_1}^\dagger a_{k_2}e^{i(k_1-k_2)x} - a_{k_1}^\dagger a_{k_2}^\dagger e^{i(k_1+k_2)x} + h.c.\right). \tag{1.60}$$

The third term is

$$m^2\phi^2 = \int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\frac{m^2}{2\sqrt{\omega_{k_1}\omega_{k_2}}}\left(a^\dagger_{k_1}a_{k_2}e^{i(k_1-k_2)x} + a^\dagger_{k_1}a^\dagger_{k_2}e^{i(k_1+k_2)x} + h.c.\right). \tag{1.61}$$

All three contributions sum up as

$$\begin{aligned} H &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2}\left(\frac{\omega_k}{2} + \frac{\boldsymbol{k}^2+m^2}{2\omega_k}\right)\left(a^\dagger_k a_k + h.c.\right) \\ &= \int \frac{d^3k}{(2\pi)^3}\omega_k\left(a^\dagger_k a_k + \frac{1}{2}\right). \end{aligned} \tag{1.62}$$

We can now check that the $a^\dagger a^\dagger$ terms indeed have no contributions, as the total contribution for each momentum k is

$$-\frac{\omega_k}{2} + \frac{\boldsymbol{k}^2}{2\omega_k} + \frac{m^2}{2\omega_k} = 0. \tag{1.63}$$

The Hamiltonian in the operator form also make it manifest that

$$H|\boldsymbol{k}\rangle = \omega_k|\boldsymbol{k}\rangle. \tag{1.64}$$

Correlation Function

Consider the two-point correlation (propagator):

$$\begin{aligned} i\Delta(x_1-x_2) &\equiv \langle 0|T\phi(x_1)\phi(x_2)|0\rangle \\ &= \theta(t_1-t_2)\langle 0|\phi(x_1)\phi(x_2)|0\rangle + \theta(t_2-t_1)\langle 0|\phi(x_2)\phi(x_1)|0\rangle. \end{aligned} \tag{1.65}$$

Note that

$$\langle 0|\phi(x_1)\phi(x_2)|0\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}e^{i\boldsymbol{k}\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)-i\omega_{\boldsymbol{k}}\tau}, \tag{1.66}$$

where $\tau = t_1 - t_2$. The propagator can be written as

$$\begin{aligned} i\Delta(x_1-x_2) &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}e^{i\boldsymbol{k}\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)}\left[e^{-i\omega_{\boldsymbol{k}}\tau}\theta(\tau) + e^{i\omega_{\boldsymbol{k}}\tau}\theta(-\tau)\right] \\ &= \int \frac{d^3k}{(2\pi)^3}e^{i\boldsymbol{k}\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)}\int\frac{d\omega}{2\pi i}\frac{-e^{i\omega\tau}}{\omega^2-\omega_k^2+i\epsilon} \\ &= \int \frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x_1-x_2)}\frac{i}{k^2-m^2+i\epsilon}. \end{aligned} \tag{1.67}$$

We have used the identity

$$\frac{1}{2\omega_k}\left[e^{-i\omega_{\boldsymbol{k}}\tau}\theta(\tau) + e^{i\omega_{\boldsymbol{k}}\tau}\theta(-\tau)\right] = \int\frac{d\omega}{2\pi i}\frac{-e^{i\omega\tau}}{\omega^2-\omega_k^2+i\epsilon},$$

where an infinitesimal number ϵ is included to move the singularities away from the real axis. Any final result shall take the $(\epsilon\to 0^+)$ limit. Sometimes the infinitesimal ϵ will be absorbed into the mass, i.e., $m^2 \to m^2 - i\epsilon$.

1.2.2 Vector Field

Although forbid by gauge invariance, we consider a vector field with nonzero mass term. Actually the vector field can obtain mass from the spontaneous symmetry breaking. For example the W and Z boson in the week interaction have nonzero mass. The action with mass term is:

$$S = \int \frac{d^4k}{(2\pi)^4} \tilde{A}^{\mu *}(k) \left(-k^2 g_{\mu\nu} + k_\mu k_\nu + m^2 \right) \tilde{A}^\nu(k). \tag{1.68}$$

The equation of motion for the action is

$$\frac{\delta S}{\delta \tilde{A}^{\mu *}(k)} = 0 \quad \Longrightarrow \quad \left(-k^2 g_{\mu\nu} + k_\mu k_\nu + m^2 \right) \tilde{A}^\nu(k) = 0. \tag{1.69}$$

Such equation is sometimes called the *Proca equation*. Note that the Proca equation implies

$$\partial_\mu A^\mu = k_\nu \tilde{A}^\nu = 0. \tag{1.70}$$

So the Proca equation becomes

$$\left(-k^2 g_{\mu\nu} + m^2 \right) \tilde{A}^\nu(k) = 0, \tag{1.71}$$

which is similar to the Klein-Gordon field with multiple components.

Polarization Vectors

Since we are now dealing with a field with space-time indices, it is helpful to introduce a set of basis vectors. The general solution to the Proca equation is also a plane wave labelled by momentum $\boldsymbol{k}$. For each momentum, we introduce a *longitudinal polarization vector*

$$\epsilon(\boldsymbol{k}, 3) \equiv \left(\frac{|\boldsymbol{k}|}{m}, \frac{\boldsymbol{k}}{|\boldsymbol{k}|} \frac{k_0}{m} \right) \tag{1.72}$$

and two *transverse polarization vectors*

$$\epsilon(\boldsymbol{k}, 1) \equiv (0, \boldsymbol{\epsilon}(\boldsymbol{k}, 1)), \quad \epsilon(\boldsymbol{k}, 2) \equiv (0, \boldsymbol{\epsilon}(\boldsymbol{k}, 2)), \tag{1.73}$$

satisfying the orthogonal relation

$$\boldsymbol{\epsilon}(\boldsymbol{k}, 1) \cdot \boldsymbol{k} = \boldsymbol{\epsilon}(\boldsymbol{k}, 2) \cdot \boldsymbol{k} = k^\mu \epsilon_\mu(\boldsymbol{k}, 3) = 0. \tag{1.74}$$

So these three polarization vector together with 4-momentum k form a basis for the space-time. For the notational convenience, we define

$$\epsilon(\boldsymbol{k}, 0) \equiv \frac{k}{m}. \tag{1.75}$$

The vector field can be

$$A_\mu(\boldsymbol{k}, \lambda; x) \propto e^{-i\omega_k t + \boldsymbol{k}\cdot\boldsymbol{x}} \epsilon_\mu(\boldsymbol{k}, \lambda) \tag{1.76}$$

The condition $k \cdot \tilde{A} = 0$ is satisfied if we require no mode in the $\epsilon(\boldsymbol{k}, 0)$ polarization. Then the vector field is basically three independent scalar field, leading to the field expansion:

$$A^\mu(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \sum_{j=1}^{3} \left[\epsilon^\mu(\boldsymbol{k}, j) a_{k,j} e^{-ik\cdot x} + \epsilon^\mu(\boldsymbol{k}, j) a^\dagger_{k,j} e^{ik\cdot x} \right]. \tag{1.77}$$

A single-particle state with polarization vector $\epsilon(\boldsymbol{k}, j)$ is defined as

$$|k, \epsilon_j\rangle = \epsilon(\boldsymbol{k}, j) \sqrt{2\omega_k} a^\dagger_j |0\rangle. \tag{1.78}$$

Massless Polarization Vectors

For the massless vector field, the polarization $\epsilon(\boldsymbol{k},3)$ is not well-defined. We modify the definition to

$$\begin{aligned}\epsilon(\boldsymbol{k},0) &\equiv (1,0,0,0),\\ \epsilon(\boldsymbol{k},3) &\equiv (0,0,0,1).\end{aligned} \tag{1.79}$$

Note that we have choose the spatial direction so that the momentum point to the z-direction.

We can add two types of virtual particles generated by $a_{k,0}^\dagger$ and $a_{k,3}^\dagger$ respectively, which are usually called the *scalar photons* and *longitudinal photons*. However, the gauge fixing condition requires

$$\partial_\mu A^\mu(x)|\psi\rangle = 0 \quad \Longrightarrow \quad (a_{k,0} - a_{k,3})|\psi\rangle = 0 \tag{1.80}$$

for all state $|\psi\rangle$ in the gauge-fixed Hilbert space.

We can show that the scalar and longitudinal modes are just the result of gauge transformation. The physical polarization are the transverse polarization modes, and the field expansion is

$$A^\mu(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\sum_{j=1}^{2}\left[\epsilon^\mu(\boldsymbol{k},j)a_{k,j}e^{-ik\cdot x} + \epsilon^\mu(\boldsymbol{k},j)a_{k,j}^\dagger e^{ik\cdot x}\right]. \tag{1.81}$$

Correlation Function

To obtain the correlation for the massless vector field, we consider a modified Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\xi}{2}(\partial_\mu A^\mu)^2. \tag{1.82}$$

In the momentum space:

$$\tilde{\mathcal{L}}_k = \tilde{A}^\mu(-k)\left(-k^2 g_{\mu\nu} + (1-\xi)k_\mu k_\nu\right)\tilde{A}^\nu(k) \tag{1.83}$$

To construct the inverse matrix we make a general symmetric ansatz

$$(G_\gamma^{-1})^{\mu\nu}(k) = A\left(k^2\right)g^{\mu\nu} + B\left(k^2\right)k^\mu k^\nu. \tag{1.84}$$

Requiring that

$$(G_\gamma)_{\mu\sigma}(k)(G_\gamma^{-1})^{\sigma\nu}(k) = \delta_\mu^\nu, \tag{1.85}$$

and comparing the coefficients, we get the conditions

$$\begin{aligned}-k^2 A\left(k^2\right) &= 1,\\ \xi k^2 B\left(k^2\right) &= (\xi-1)A\left(k^2\right).\end{aligned} \tag{1.86}$$

In the case $\xi = 0$ these equations are not compatible. Without the gauge-fixing term the matrix $(G_\gamma)_{\mu\nu}$ cannot be inverted (since the determinant vanishes) and the Feynman propagator cannot be constructed. If $\xi \neq 0$, however, no problems arise and the system of equations (1.86) is solved by

$$A\left(k^2\right) = -\frac{1}{k^2}, \quad B\left(k^2\right) = \frac{\xi - 1}{\xi}\frac{1}{\left(k^2\right)^2}, \tag{1.87}$$

which leads to

$$G_\gamma(k) = \frac{-g^{\mu\nu} + (1-\xi)k^\mu k^\nu}{k^2}. \tag{1.88}$$

Different choice of ξ correspond to different gauge fixing. The Landau gauge choose $\xi = 1$, and the propagator has the simplest form

$$G_\gamma(k) = \frac{-g_{\mu\nu}}{k^2}. \tag{1.89}$$

1.2.3 Spinor Field

The equation of motion for Dirac field is

$$\begin{aligned} \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} - \frac{\partial \mathcal{L}}{\partial \psi} = 0 \quad &\Longrightarrow \quad \bar{\psi}(i\overleftarrow{\not{\partial}} - m) = 0, \\ \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} - \frac{\partial \mathcal{L}}{\partial \bar{\psi}} = 0 \quad &\Longrightarrow \quad (i\overrightarrow{\not{\partial}} - m)\psi = 0. \end{aligned} \tag{1.90}$$

This EOM is a matrix equation. The general solution of the Dirac equation can be written as a linear combination of plane waves (with positive and negative energy):[2]

$$\psi_p(x) = \begin{cases} u(p)e^{-ip\cdot x} & p^0 > 0 \\ v(p)e^{+ip\cdot x} & p^0 < 0 \end{cases}, \quad p^2 = m^2. \tag{1.91}$$

In momentum space, $u(p)$ and $v(p)$ satisfies:

$$\begin{bmatrix} -m & p\cdot\sigma \\ p\cdot\bar{\sigma} & -m \end{bmatrix} u_s(p) = \begin{bmatrix} -m & -p\cdot\sigma \\ -p\cdot\bar{\sigma} & -m \end{bmatrix} v_s(p) = 0 \tag{1.92}$$

For massive Dirac field, we can choose the rest frame where $p = (m, 0, 0, 0)$, the matrix equation is[3]

$$\begin{aligned} \begin{bmatrix} -m & m \\ m & -m \end{bmatrix} u_s = 0 \quad &\Longrightarrow \quad u_s = \sqrt{m}\begin{bmatrix} \xi_s \\ \xi_s \end{bmatrix}, \\ \begin{bmatrix} m & m \\ m & m \end{bmatrix} v_s = 0 \quad &\Longrightarrow \quad v_s = \sqrt{m}\begin{bmatrix} \eta_s \\ -\eta_s \end{bmatrix}, \end{aligned} \tag{1.93}$$

where ξ and η has two independent solutions. For example, four linearly independent solutions are

$$u_\uparrow = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad u_\downarrow = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad v_\uparrow = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_\downarrow = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}. \tag{1.94}$$

[2] Note that we have chosen to put the + sign into the exponential, rather than having $p^0 < 0$.

[3] We first consider the case where there is only one spatial dimension. It correspond to the choice of coordinate such that the momentum point to the z direction.

The Dirac spinor is a complex four-component object, with eight real degrees of freedom. The equations of motion reduce it to four degrees of freedom, which, as we will see, can be interpreted as spin up and spin down for particle and antiparticle.

Solution in General Frame

To derive a more general expression, we can solve the equations again in the boosted frame and match the normalization. If $p = (E, 0, 0, p_z)$ then

$$p \cdot \sigma = \begin{bmatrix} E - p_z & 0 \\ 0 & E + p_z \end{bmatrix}, \quad p \cdot \bar{\sigma} = \begin{bmatrix} E + p_z & 0 \\ 0 & E - p_z \end{bmatrix}. \tag{1.95}$$

Let $a = \sqrt{E - p_z}$ and $b = \sqrt{E + p_z}$, then $m^2 = (E - p_z)(E + p_z) = a^2 b^2$ and Dirac equation becomes

$$\begin{bmatrix} -ab & 0 & a^2 & 0 \\ 0 & -ab & 0 & b^2 \\ b^2 & 0 & -ab & 0 \\ 0 & a^2 & 0 & -ab \end{bmatrix} u_s(p) = \begin{bmatrix} ab & 0 & a^2 & 0 \\ 0 & ab & 0 & b^2 \\ b^2 & 0 & ab & 0 \\ 0 & a^2 & 0 & ab \end{bmatrix} v_s(p) = 0. \tag{1.96}$$

The solutions are

$$u_s = \begin{pmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \xi_s \\ \begin{bmatrix} b & 0 \\ 0 & a \end{bmatrix} \xi_s \end{pmatrix}, \quad v_s = \begin{pmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \eta_s \\ -\begin{bmatrix} b & 0 \\ 0 & a \end{bmatrix} \eta_s \end{pmatrix}. \tag{1.97}$$

Using

$$\sqrt{p \cdot \sigma} = \begin{bmatrix} \sqrt{E - p_z} & 0 \\ 0 & \sqrt{E + p_z} \end{bmatrix}, \quad \sqrt{p \cdot \bar{\sigma}} = \begin{bmatrix} \sqrt{E + p_z} & 0 \\ 0 & \sqrt{E - p_z} \end{bmatrix}, \tag{1.98}$$

we can write more generally

$$u_s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi_s \\ \sqrt{p \cdot \bar{\sigma}} \xi_s \end{pmatrix}, \quad v_s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \eta_s \\ -\sqrt{p \cdot \bar{\sigma}} \eta_s \end{pmatrix}, \tag{1.99}$$

where the square root of a matrix can be defined by changing to the diagonal basis, taking the square root of the eigenvalues, then changing back to the original basis. In practice, we will usually pick p along the z axis, so we do not need to know how to make sense of $\sqrt{p \cdot \sigma}$. Then the four solutions are

$$\begin{aligned} u^1(p) &= \begin{pmatrix} \sqrt{E - p_z} \\ 0 \\ \sqrt{E + p_z} \\ 0 \end{pmatrix}, \quad u^2(p) = \begin{pmatrix} 0 \\ \sqrt{E - p_z} \\ 0 \\ \sqrt{E + p_z} \end{pmatrix}, \\ v^1(p) &= \begin{pmatrix} \sqrt{E - p_z} \\ 0 \\ -\sqrt{E + p_z} \\ 0 \end{pmatrix}, \quad v^2(p) = \begin{pmatrix} 0 \\ \sqrt{E - p_z} \\ 0 \\ -\sqrt{E + p_z} \end{pmatrix}. \end{aligned} \tag{1.100}$$

In any frame u^s are the positive frequency electrons, and the v^s are negative frequency electrons, or equivalently, positive frequency positrons.

For massless spinors, $p_z = \pm E$ and the explicit solutions in Eq. (1.100) are 4-vectors with one non-zero component describing spinors with fixed helicity. The spinor solutions for massless electrons are sometimes called polarizations, and are useful for computing polarized electron scattering amplitudes.

For Weyl spinors, there are only four real degrees of freedom off-shell and two real degrees of freedom on-shell. Recalling that the Dirac equation splits up into separate equations for ψ_L and ψ_R, the Dirac spinors with zeros in the bottom two rows will be ψ_L and those with zeros in the top two rows will be ψ_R. Since ψ_L and ψ_R have two degrees of freedom each, these must be particle and antiparticle for the same helicity. The embedding of Weyl spinors into fields this way induces irreducible unitary representations of the Poincare group for $m = 0$.

Normalization and Spin Sum

The normalization chosen this way gives the orthogonal relation:

$$\begin{aligned} \bar{u}^r(p)u^s(p) &= +2m\delta^{rs}, \\ \bar{v}^r(p)v^s(p) &= -2m\delta^{rs}. \end{aligned} \tag{1.101}$$

This is the (conventional) normalization for the spinor inner product for massive Dirac spinors. It is also easy to check that

$$\bar{u}_s(p)v_{s'}(p) = \bar{v}_s(p)u_{s'}(p) = 0. \tag{1.102}$$

We can further check that an additional orthogonal relation hold

$$\begin{aligned} u^{r\dagger}(p)u^s(p) &= -2\omega_{\boldsymbol{p}}\delta^{rs}, \\ v^{r\dagger}(p)v^s(p) &= +2\omega_{\boldsymbol{p}}\delta^{rs}. \end{aligned} \tag{1.103}$$

And if we define $\bar{p} \equiv (E, -\vec{p})$, there is another set of orthogonal relation:

$$u^{r\dagger}(p)v^s(\bar{p}) = v^{r\dagger}(p)u^s(\bar{p}) = 0. \tag{1.104}$$

A useful identity is the spin sum identity:

$$\begin{aligned} \sum_s u^s(p)\bar{u}^s(p) &= \not{p} + m, \\ \sum_s v^s(p)\bar{v}^s(p) &= \not{p} - m. \end{aligned} \tag{1.105}$$

Field Expansion and Correlation

The Dirac field expansion is

$$\begin{aligned} \psi(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\sum_s \left(a^s_{\mathbf{p}}u^s(p)e^{-ip\cdot x} + b^{s\dagger}_{\mathbf{p}}v^s(p)e^{ip\cdot x}\right), \\ \bar{\psi}(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\sum_s \left(b^s_{\mathbf{p}}\bar{v}^s(p)e^{-ip\cdot x} + a^{s\dagger}_{\mathbf{p}}\bar{u}^s(p)e^{ip\cdot x}\right). \end{aligned} \tag{1.106}$$

Now let us investigate the propagator

$$\begin{aligned} iD_{F,\alpha\beta}(x_1 - x_2) &= \langle 0|T\psi_\alpha(x_1)\bar{\psi}_\beta(x_2)|0\rangle \\ &= \theta(\tau)\langle 0|\psi_\alpha(x_1)\bar{\psi}_\beta(x_2)|0\rangle - \theta(-\tau)\langle 0|\bar{\psi}_\beta(x_2)\psi_\alpha(x_1)|0\rangle. \end{aligned} \tag{1.107}$$

On the RHS, the first term is

$$\begin{aligned} \langle 0|\psi_\alpha(x_1)\bar{\psi}_\beta(x_2)|0\rangle &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left[\sum_s u^s_\alpha(p)\bar{u}^s_\beta(p)\right] e^{-ip\cdot(x_1-x_2)} \\ &= (i\not{\partial} + m)_{\alpha\beta}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}e^{-ip\cdot(x_1-x_2)}. \end{aligned}$$

For the second term:

$$\begin{aligned} \langle 0|\bar{\psi}_\beta(x_2)\psi_\alpha(x_1)|0\rangle &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left[\sum_s \bar{v}^s_\beta(p)v^s_\alpha(p)\right] e^{ip\cdot(x_1-x_2)} \\ &= -(i\not{\partial} + m)_{\alpha\beta}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}e^{ip\cdot(x_1-x_2)}. \end{aligned}$$

Together, the Dirac propagator is:

$$\begin{aligned} iD_F(x_1 - x_2) &= (i\not{\partial} + m)i\Delta(x_1 - x_2) \\ &= \int \frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x_1-x_2)}\frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}. \end{aligned} \tag{1.108}$$

1.3 Path-integral Quantization

1.3.1 Scalar Field

Consider the action for free field with source

$$S_0[\phi, J] = \int d^dx\left[\mathcal{L}_0 + J(x)\cdot\phi(x)\right]. \tag{1.109}$$

In the path integral formalism, we consider the partition function

$$Z_0[J] = \int D[\phi]\exp(iS_0[\phi, J]) \equiv Z[0]\exp(iW_0[J]). \tag{1.110}$$

where we have introduced a new quantity

$$W_0[J] = -\frac{1}{2}\int d^dx_1d^dx_2J(x_1)\Delta_0(x_1 - x_2)J(x_2). \tag{1.111}$$

For free field, the free propagator $\Delta_0(x_1 - x_2)$ is:

$$i\Delta_0(x_1 - x_2) = \langle 0|T\phi(x_1)\phi(x_2)|0\rangle = \frac{\delta}{i\delta J(x_1)}\frac{\delta}{i\delta J(x_2)}iW_0[J]. \tag{1.112}$$

Now we evaluate the propagator in the path-integral formalism. In momentum space, the free action (with source) is

$$\frac{1}{V}\sum_k\left[\frac{1}{2}\tilde{\phi}^*(k)(k^2-m^2)\tilde{\phi}(k)+\tilde{J}^*(k)\cdot\tilde{\phi}(k)+\tilde{\phi}^*(k)\cdot\tilde{J}(k)\right].$$

For real field, $\tilde{\phi}^*(k)=\tilde{\phi}(-k)$. For our convenience, we have expressed the momentum integral as summation. Actually, consider the d-dimensional box of size L^d, the momentum along each axis is multiple of $2\pi/L$, so when $L\to\infty$, the summation approaches in integral,

$$\frac{1}{V}\sum_k\to\int\frac{d^dk}{(2\pi)^d}.$$

Let us omit the $1/V$ factor, the summation can be formally expressed as

$$\frac{1}{4}\mathbf{v}^T\cdot\mathbf{M}\cdot\mathbf{v}+\frac{1}{2}\mathbf{j}^T\cdot\mathbf{v} \tag{1.113}$$

where

$$\mathbf{v}=\bigoplus_{|\mathbf{k}|}\begin{bmatrix}\tilde{\phi}(k)\\ \tilde{\phi}^*(k)\end{bmatrix},\ \mathbf{M}=\bigoplus_{|\mathbf{k}|}\begin{bmatrix}0 & k^2-m^2\\ k^2-m^2 & 0\end{bmatrix},\ \mathbf{j}=\bigoplus_{|\mathbf{k}|}\begin{bmatrix}\tilde{J}^*(k)\\ \tilde{J}(k)\end{bmatrix}.$$

Note that in the above expression, we have made an infinitesimal shift of mass ($m^2\to m^2-i\epsilon$) to ensure the convergence of the Gaussian integral. The integrated variables v_i is not real. To use the real Gaussian integral formula, we make use of a unitary transformation:

$$\mathbf{U}=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ -i & i\end{bmatrix},\quad \mathbf{U}\cdot\begin{bmatrix}\tilde{\phi}(k)\\ \tilde{\phi}^*(k)\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}\tilde{\phi}(k)+\tilde{\phi}^*(k)\\ -i\tilde{\phi}(k)+i\tilde{\phi}^*(k)\end{bmatrix}\equiv\begin{bmatrix}\tilde{\phi}_1(k)\\ \tilde{\phi}_2(k)\end{bmatrix}$$

The path integral then becomes a real field integral. Recall the real Gaussian integral formula:

$$\int d\mathbf{x}\exp\left(-\frac{1}{2}\mathbf{x}^T\cdot\mathbf{A}\cdot\mathbf{x}+\mathbf{B}^T\cdot\mathbf{x}\right)=\sqrt{\frac{(2\pi)^N}{\det\mathbf{A}}}\exp\left(\frac{1}{2}\mathbf{B}^T\cdot\mathbf{A}^{-1}\cdot\mathbf{B}\right), \tag{1.114}$$

For the field integral, we absorbed the $(2\pi)^{N/2}$ term into the measure, and express the path integral for the Gaussian field as:

$$W_0[J]=-\frac{i}{4}\int\frac{d^dk}{(2\pi)^d}\mathbf{j}_k^T\cdot\mathbf{M}_k^{-1}\cdot\mathbf{j}_k=-\frac{1}{2}\int\frac{d^dk}{(2\pi)^d}\tilde{J}^*(k)\tilde{\Delta}_0(k)\tilde{J}(k). \tag{1.115}$$

This gives the propagator in the momentum space:

$$\tilde{\Delta}_0(k)=\frac{i}{k^2-m^2}\quad\Longrightarrow\quad\Delta_0(x_1-x_2)=i\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot(x_1-x_2)}}{k^2-m^2}. \tag{1.116}$$

From Field to Force

Consider two separate particle described by the delta function $J_a(x) = \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}_a)$, together the source is

$$J(x) = J_1(x) + J_2(x). \tag{1.117}$$

Adding the source,

$$W_0[J] = -\frac{1}{2}\int d^4x_1 d^4x_2 J(x_1)\Delta_0(x_1 - x_2)J(x_2)$$

Omit the self energy terms $J_1^2(x)$, $J_2^2(x)$, $W_0[J]$ is

$$\begin{aligned}
W_0[J] &= -\int d^4y_1 d^4y_2\ e^{-ik^0(y_1^0 - y_2^0)} \int \frac{d^4k}{(2\pi)^4} J_1(y_1) \frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{y}_1 - \boldsymbol{y}_2)}}{k^2 - m^2} J_2(y_2) \\
&= -\int dt \int d(y_1^0 - y_2^0)\ e^{-ik^0(y_1^0 - y_2^0)} \int \frac{d^4k}{(2\pi)^4} \frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{y}_1 - \boldsymbol{y}_2)}}{k^2 - m^2} \\
&= \left(\int dt\right) \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{y}_1 - \boldsymbol{y}_2)}}{\boldsymbol{k}^2 + m^2}
\end{aligned} \tag{1.118}$$

Recall that the partition function is actually infinite:

$$Z_0 \sim \langle 0|e^{-iH_0T}|0\rangle \quad \Longrightarrow \quad W_0 = -iET, \tag{1.119}$$

where E is the energy. Writing $\boldsymbol{r} \equiv \boldsymbol{y}_1 - \boldsymbol{y}_2$, and $u \equiv \cos\theta$ with θ the angle between $\boldsymbol{k}$ and $\boldsymbol{r}$, the volume form is $dk \cdot kd\theta \cdot 2\pi k\sin\theta = 2\pi k^2 dkdu$, and the integral is

$$\begin{aligned}
E &= -\int \frac{d^3k}{(2\pi)^3} \frac{e^{ikru}}{k^2 + m^2} \\
&= -\frac{1}{(2\pi)^2}\int_0^\infty k^2dk \int_{-1}^{1} du \frac{e^{ikru}}{k^2 + m^2} \\
&= -\frac{1}{2\pi^2 r}\int_0^\infty k\frac{\sin kr}{k^2 + m^2}dk.
\end{aligned} \tag{1.120}$$

Since the integral is even, we can extend the integral to

$$\begin{aligned}
E &= -\frac{1}{4\pi^2 r}\int_{-\infty}^{\infty} k\frac{\sin kr}{k^2 + m^2}dk \\
&= \frac{i}{4\pi^2 r}\int_{-\infty}^{\infty} \frac{ke^{ikr}}{k^2 + m^2}dk
\end{aligned} \tag{1.121}$$

The residue theorem gives

$$\int_{-\infty}^{\infty} \frac{ke^{ikr}}{k^2 + m^2}dk = \pi i e^{-mr} \tag{1.122}$$

So we get the potential of two particles:

$$V(r) = -\frac{e^{-mr}}{4\pi r}, \tag{1.123}$$

and the attractive force is

$$F(r) = -\frac{dV}{dr} = -(1 + mr)\frac{e^{-mr}}{4\pi r^2}. \tag{1.124}$$

We see that in the massless case, the force gives the long-range Coulomb force $F \propto 1/r^2$, while in the massful field theory, the force is short-ranged, with the decay length proportional to the mass.

1.3.2 Vector Field

We define the gauge fixing function

$$G(A) = \partial_\mu A^\mu(x) - \omega(x) = 0$$

The gauge transformation has the form:

$$A_\mu^\alpha(x) = A_\mu(x) + \partial_\mu \alpha(x).$$

We then have

$$1 \propto \int D[\alpha] \det\left(\frac{\delta G(A^\alpha)}{\delta \alpha}\right) \delta(G(A)).$$

Inset the identity operator into the path integral formula

$$Z[J] \propto \det\left(\partial^2\right) \int D[\alpha] D[A] e^{iS[A,J]} \delta(\partial_\mu A^\mu - \omega(x)).$$

The above equation does not depend on $\omega(x)$. We can then integrate over $\omega(x)$ with gaussian weight

$$\begin{aligned} Z[J] &\propto \int D[\omega] e^{-i\int d^dx \frac{\omega^2}{2\xi}} \int D[\alpha] D[A] e^{iS[A,J]} \delta(\partial_\mu A^\mu - \omega) \\ &= \int D[A] e^{iS[A,J]} \exp\left\{ i \left[S[A,J] - \int d^dx \frac{1}{2\xi} (\partial_\mu A^\mu)^2 \right] \right\}. \end{aligned}$$

In momentum space, the modified Langriangian is

$$\tilde{\mathcal{L}}_\xi(k) = \tilde{A}^\mu(k) \left[-k^2 g_{\mu\nu} + \left(1 - \frac{1}{\xi}\right) k_\mu k_\nu \right] \tilde{A}^\nu(-k) + \tilde{J}_\mu(k) \tilde{A}^\mu(-k) + \tilde{A}^\mu(k) \tilde{J}_\mu(-k).$$

In the momentum space, the photon propagator is

$$\begin{aligned} \tilde{G}^{\mu\nu}(k) &= \left[-k^2 g_{\mu\nu} + \left(1 - \frac{1}{\xi}\right) k_\mu k_\nu \right]^{-1} \\ &= \frac{-g^{\mu\nu} + (1-\xi) k^\mu k^\nu}{k^2}. \end{aligned} \tag{1.125}$$

Thus, the partition function is

$$\frac{Z_{\text{maxwell}}[J]}{Z_{\text{maxwell}}[0]} = \exp\left[-\frac{i}{2} \int d^dx_1 d^dx_2 J_\mu(x_1) G^{\mu\nu}(x_1 - x_2) J_\nu(x_2) \right], \tag{1.126}$$

where the real-space propagator is

$$G^{\mu\nu}(x_1 - x_2) = \int \frac{d^dk}{(2\pi)^d} e^{-ik\cdot(x_1 - x_2)} \frac{-g^{\mu\nu} + (1-\xi) k^\mu k^\nu}{k^2}. \tag{1.127}$$

Note that the propagator is related to the two-point correaltion:

$$\begin{aligned} \langle 0 | T A^\mu(x_1) A^\nu(x_2) | 0 \rangle &= \frac{1}{Z_{\text{Maxwell}}[0]} \frac{\delta}{iJ_\mu(x_1)} \frac{\delta}{iJ_\nu(x_2)} Z_{\text{Maxwell}}[J] \Big|_{J=0} \\ &= iG^{\mu\nu}(x_1 - x_2). \end{aligned} \tag{1.128}$$

1.3.3 Spinor Field

Consider the partition function with source

$$Z_{\text{Dirac}}[J] = \int D[\bar{\psi}, \psi] \exp\left[i \int d^d x \left(\mathcal{L}_{\text{Dirac}} + \bar{\eta}\psi + \bar{\psi}\eta\right)\right]. \tag{1.129}$$

In momentum space:

$$S = \int \frac{d^d k}{(2\pi)^d} \left[\tilde{\bar{\psi}}(k)(\not{k} - m)\tilde{\psi}(k) + \tilde{\bar{\eta}}(k)\tilde{\psi}(k) + \tilde{\bar{\psi}}(k)\tilde{\eta}(k)\right]. \tag{1.130}$$

Using the Gaussian integral formula (for Grassman variables), the partition function is:

$$\begin{aligned}\frac{Z_{\text{Dirac}}[J]}{Z_{\text{Dirac}}[0]} &= \exp\left[-i \int \frac{d^d k}{(2\pi)^d} \tilde{\bar{\eta}}(k) \frac{1}{\not{k} - m} \tilde{\eta}(k)\right] \\ &= \exp\left[-i \int d^d x_1 d^d x_2 \bar{\eta}(x_1) \cdot D_F(x_1 - x_2) \cdot \eta(x_2)\right]\end{aligned} \tag{1.131}$$

where

$$D_F(x_1 - x_2) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{-ik\cdot(x_1 - x_2)}}{\not{k} - m} = \int \frac{d^d k}{(2\pi)^d} \frac{\not{k} + m}{k^2 - m^2} e^{-ik\cdot(x_1 - x_2)}. \tag{1.132}$$

Note that the propagator is

$$\begin{aligned}\langle 0|T\psi^\alpha(x_1)\bar{\psi}^\beta(x_2)|0\rangle &= \frac{1}{Z_{\text{Dirac}}[0]} \frac{\delta}{i\delta\bar{\eta}_\alpha(x_1))} \frac{i\delta}{\delta\eta_\beta(x_2)} Z_{\text{Dirac}}[\bar{\eta}, \eta]\bigg|_{\eta=\bar{\eta}=0} \\ &= iD_F^{\alpha\beta}(x_1 - x_2),\end{aligned} \tag{1.133}$$

where the sign in the variational derivative comes from the anti-commutation relation of the fermionic fields.

Chapter 2

Interactions

In this chapter, we are going to investigate the effects of interactions. When the field theory is no longer free, the notion of free particle is not very well defined. Also, the interactions actually gives the physical quantities that can be measured. For example the scattering amplitudes.

We will mainly focus on the scalar field theory, as the complexity of the vector or spinor field mainly comes from the algebraic structures of themselves. After explaining the stories of the scalar filed, we will try to generalize them to the vector and spinor cases.

2.1 Perturbation Theory

2.1.1 Real-space Formalism

For interaction theory, the partition function can be formally expressed as:

$$Z[J] = \exp\left(i \int d^d x \mathcal{L}_{\text{int}} \left[\frac{\delta}{i\delta J(x)}\right]\right) Z_0[J]. \tag{2.1}$$

The expectation values for a generic operator of the form $O(\phi)$ can be evaluated by the true partition function

$$\langle O(\phi)\rangle = \frac{1}{Z[0]}\, O\left[\frac{\delta}{i\delta J(x)}\right] Z[J]\bigg|_{J=0}. \tag{2.2}$$

The expression (2.2) can be expanded order by order using the Feynman diagram. Since the unconnected diagram can be absorbed into $Z[0]$, we only need to calculate the connected diagram.

The procedure of perturbative expansion with only connected diagrams can be formally represented by introducing the quantity

$$Z[J] = Z[0]\exp\left(iW[J]\right). \tag{2.3}$$

The perturbative expansion of $W[J]$ contain only the connected diagrams. Note that for the free theory,

$$\frac{Z_0[J]}{Z_0[0]} = \exp\left[-\frac{i}{2}\int d^d x_1 d^d x_2 J(x_1)\Delta(x_1 - x_2)J(x_2)\right],$$

which means

$$W_0 = -\frac{1}{2}\int d^d x_1 d^d x_2 J(x_1)\Delta(x_1 - x_2)J(x_2).$$

For the interaction theory, the expectation (2.2) can then be replaced by the connected expectation:

$$\langle O(\phi)\rangle_c \equiv i\, O\left[\frac{\delta}{i\delta J(x)}\right] W[J]\bigg|_{J=0}. \tag{2.4}$$

Consider the two-point connected correlation (propagator):

$$\begin{aligned} i\Delta(x_1 - x_2) &= \langle \mathcal{T}\phi(x_1)\phi(x_2)\rangle_c \\ &= i\,\frac{\delta^2 W[J]}{i\delta J(x_1) i\delta J(x_2)}\bigg|_{J=0} \\ &= \frac{\delta^2 \ln Z[J]}{i\delta J(x_1) i\delta J(x_2)}\bigg|_{J=0} \\ &= \frac{1}{Z[0]}\,\frac{\delta^2 Z[J]}{i\delta J(x_1) i\delta J(x_2)}\bigg|_{J=0}, \end{aligned} \tag{2.5}$$

where we have used the fact that

$$\frac{\delta Z^n[J]}{\delta J(x_1)\cdots\delta J(x_n)} = 0,\ \ \forall n = 1 \bmod 2. \tag{2.6}$$

The result is the same as the original definition.

Further, we can consider the four-point connected correlation:

$$iV_4 \equiv \langle \mathcal{T}\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle_c \tag{2.7}$$

Following the same procedure,

$$\begin{aligned} iV_4 = i\,&\frac{\delta^4 W[J]}{i\delta J(x_1) i\delta J(x_2) i\delta J(x_3) i\delta J(x_4)}\bigg|_{J=0} \\ =\ &\frac{1}{Z[0]}\,\frac{\delta^4 Z[J]}{i\delta J(x_1) i\delta J(x_2) i\delta J(x_3) i\delta J(x_4)}\bigg|_{J=0} \\ &- i\Delta(x_1 - x_2) i\Delta(x_3 - x_4) \\ &- i\Delta(x_1 - x_3) i\Delta(x_2 - x_4) \\ &- i\Delta(x_1 - x_4) i\Delta(x_2 - x_3). \end{aligned} \tag{2.8}$$

The connected correlation function automatically omit those disconnected components.

2.1.2 Momentum-space Formalism

In momentum space, the theory is expressed as

$$\begin{aligned}
S_0[\phi(k)] &= \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\ \phi_R^*(k)(k^2 - m_R^2)\phi_R(k), \\
S_{\text{int}}[\phi(k)] &= \frac{g_R}{4!}\left(\prod_{i=1}^{4}\int \frac{d^4k_i}{(2\pi)^4}\right)\left[\prod_{i=1}^{4}\phi_R(k_i)\right]\delta^{(4)}\left(\sum_{i=1}^{4}k_i\right), \\
S_{\text{ct}}[\phi(k)] &= \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\ \tilde{\phi}_R^*(k)(Ak^2 - Bm_R^2)\tilde{\phi}_R(k) - C\cdot S_{\text{int}}.
\end{aligned} \tag{2.9}$$

For the free theory:

$$\begin{aligned}
\frac{Z_0[J]}{Z_0[0]} &= \frac{1}{Z_0[0]}\int D[\phi]\exp\left\{-S_0[\phi(k)] + i\int \frac{d^4k}{(2\pi)^4}J_k^*\phi(k)\right\} \\
&= \exp\left\{-i\int \frac{d^4k}{(2\pi)^4}J_k^*\Delta(k)J_k\right\}.
\end{aligned} \tag{2.10}$$

Similarly, the expectation in momentum space is

$$\langle O(\phi_k)\rangle = \frac{1}{Z[0]}\,O\left[\frac{\delta}{i\delta J_k^*}\right]Z[J]\bigg|_{J=0}. \tag{2.11}$$

The Feynman diagrams in momentum space is the same as that in real space, just replace the propagator from the real space to the momentum space. Also, at each vertex, the momentum conservation is automatically satisfied.

2.2 Renormalized Field Theory

For the interacting scalar field, the Hamiltonian do not conserve particle number any more, and the ground state $|\Omega\rangle$ is no longer the vacuum $|0\rangle$. Consider the Green's function

$$iG(x_1 - x_2) = \langle\Omega|T\phi(x_1)\phi(x_2)|\Omega\rangle \tag{2.12}$$

We can insert a complete basis into the correlation function:[1]

$$1 = |\Omega\rangle\langle\Omega| + \sum_\lambda\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}|\lambda_{\boldsymbol{k}}\rangle\langle\lambda_{\boldsymbol{k}}|, \tag{2.13}$$

and the Green's function takes the form:

$$iG(x_1 - x_2) = \sum_\lambda\int\frac{d^3k}{(2\pi)^3}\left[\theta(t_1 - t_2)\langle\Omega|\phi(x_1)|\lambda_{\vec{k}}\rangle\langle\lambda_{\vec{k}}|\phi(x_2)|\Omega\rangle + (t_1 \leftrightarrow t_2, x_1 \leftrightarrow x_2)\right].$$

Note that $\phi(x) = e^{iP\cdot x}\phi(0)e^{-iP\cdot x}$, so that

$$\langle\lambda_{\boldsymbol{k}}|\phi(x)|\Omega\rangle = e^{ik\cdot x}\ \langle\lambda_0|\phi(0)|\Omega\rangle|_{k^0=\omega_{\boldsymbol{k}}}. \tag{2.14}$$

[1]Here we assume $\langle\Omega|\phi(x)|\Omega\rangle = 0$ unless there is spontaneously symmetry breaking happening.

Following the same procedure as we do for the free field theory,

$$G(x_1 - x_2) = \int_0^\infty \frac{dM^2}{2\pi} \rho(M^2) G_0(x_1 - x_2; M^2), \tag{2.15}$$

where the *spectral function* $\rho(M^2)$ is

$$\rho(M^2) = \sum_\lambda (2\pi)\delta(M^2 - m_\lambda^2) |\langle \Omega | \phi(0) | \lambda_0 \rangle|^2. \tag{2.16}$$

In particle, near the one-particle state the Green's function looks like:

$$i\tilde{G}(k) = \frac{iZ_\phi}{k^2 - m^2 + i\epsilon} + \text{regular terms.} \tag{2.17}$$

Physically, Eq. (2.17) states that in the interacting theory, the field operator $\tilde{\phi}(k)$ acting on the vacuum only only generate a single particle state, but also multi-particle states with total momentum k. However, those multi-particle state have different singularity structure in the Greens function, as they only contribute regular terms. If we only care about the propagator of the single particle states, we simply need to extract the singular part of of the Green's function. That is, the singularity of $\tilde{G}(k)$ gives the (addresses) mass, and the residue

$$\lim_{k^2 \to m^2} (k^2 - m^2) \tilde{G}(k)$$

gives the wave-function normalization factor Z_ϕ. Trying to restore the original form of the free theory, we consider a renormalized field:

$$\phi_R(x) = \frac{1}{\sqrt{Z_\phi}} \phi_0(x). \tag{2.18}$$

The Green's function of ϕ_R has the same form as free theory. For this reason, we generate the asymptotic single-particle state using the renormalized field operator:

$$\phi_R(k)|\Omega\rangle = \frac{1}{2\omega_{\boldsymbol{k}}} |k\rangle + \text{multi-particle states.} \tag{2.19}$$

If we want to create a single-particle state, say at time $t = 0$. We can do this by acting the operator $\tilde{\phi}(k)$ on the vacuum state at time $-T$, then we know when the system evolves for time T, it becomes:

$$e^{-iE_{\boldsymbol{k}}T} |k\rangle + e^{-iHT} \cdot \text{multi-particle states.} \tag{2.20}$$

Here comes the trick. Assuming the theory is gapped (with mass $m^2 > 0$), the multi-particle states have higher energy than the single particle states. We then replace the t by $(1 - i\epsilon)t$, which effectives impose a suppression factor $e^{-\epsilon HT}$ to the state. In the $T \to \infty$ limit, the amplitude of the multi-particle states vanishes.

The story for the spinor field is exactly the same as the scalar field (also assume the particle has nonzero mass). However, the story for the photon field is different, since the photon is massless. A quick escape from the conundrum is to assume the photon has a small mass m_γ, and latter set $m_\gamma \to 0$.

2.3 Cross Section and Decay Rates

One important physical observable is the transition amplitude from initial state $|i; t_i\rangle$ and initial time t_i to the finial state $|f; t_f\rangle$ at t_f. In the scattering experiment, the initial and final states are assumed to be "free". For this reason, we can think of the process as start from $t = -\infty$ to $t = +\infty$, where free states at $t = \pm\infty$ are known as *asymptotic states.* We give the time-evolution operator a special name: the *S-matrix*, defined as:

$$\langle f|S|i\rangle_{\text{Heisenberg}} = \langle f; \infty \mid i; -\infty\rangle. \tag{2.21}$$

The S-matrix is related to quantities experimentally measurable, for example the cross sections or decay rates, as discussed in the following.

Cross Sections

The *cross section* is an analogy from classical scattering experiment. For example, Rutherford was interested in the size r of an atomic nucleus. By colliding α-particles with gold foil and measuring how many α-particles were scattered, he could determine the cross-sectional area $\sigma = \pi r^2$ of the nucleus.

Imagine there is just a single nucleus. Then the *cross-sectional area* is given by

$$\sigma = \frac{\text{number of particles scattered}}{\text{time } \times \text{ number density in beam } \times \text{ velocity of beam}} = \frac{1}{T}\frac{1}{\Phi}N, \tag{2.22}$$

where T is the time for the experiment and Φ is the incoming flux:

$$\Phi = \text{number density} \times \text{velocity of beam},$$

and N is the number of particles scattered.

In quantum mechanical generalization of the notion of cross-sectional area is the cross section, which still has units of area, but has a more abstract meaning as a measure of the interaction strength. While classically an α-particle either scatters off the nucleus or it does not scatter, quantum mechanically it has a probability for scattering. The classical differential probability is

$$P = \frac{N}{N_{\text{inc}}},$$

where N is the number of particles scattering into a given area and N_{inc} is the number of incident particles. So the quantum mechanical cross section is then naturally

$$d\sigma = \frac{1}{T}\frac{1}{\Phi}dP, \tag{2.23}$$

where Φ is the flux, now normalized as if the beam has just one particle, and P is now the quantum mechanical probability of scattering. The differential quantities $d\sigma$ and dP are differential in kinematical variables, such as the angles and energies of the final state particles. The differential number of scattering events measured in a collider experiment is

$$dN = L \times d\sigma, \tag{2.24}$$

where L is the *luminosity*, which is defined by this equation.

Now let us relate the formula for the differential cross section to S-matrix elements. From a practical point of view it is impossible to collide more than two particles at a time, thus we can focus on the special case of S-matrix elements where $|i\rangle$ is a two-particle state. So, we are interested in the differential cross section for the $(2 \to n)$ process:

$$p_1 + p_2 \to \{p_j\} \, . \tag{2.25}$$

In the rest frame of one of the colliding particles, the flux is just the magnitude of the velocity of the incoming particle divided by the total volume: $\Phi = |\vec{v}|/V$. In a different frame, such as the center-of-mass frame, beams of particles come in from both sides, and the flux is then determined by the difference between the particles' velocities. So, $\Phi = |\vec{v}_1 - \vec{v}_2| \, /V$. This should be familiar from classical scattering. Thus,

$$d\sigma = \frac{V}{T} \frac{1}{|\vec{v}_1 - \vec{v}_2|} dP. \tag{2.26}$$

From quantum mechanics we know that probabilities are given by the square of amplitudes. Since quantum field theory is just quantum mechanics with a lot of fields, the normalized differential probability is

$$dP = \frac{|\langle f|S|i\rangle|^2}{\langle f|f\rangle\langle i|i\rangle} d\Pi. \tag{2.27}$$

Here, $d\Pi$ is the region of final state momenta at which we are looking. It is proportional to the product of the differential momentum, d^3p_j, of each final state and must integrate to 1. So

$$d\Pi = \prod_j \frac{V}{(2\pi)^3} d^3 p_j. \tag{2.28}$$

This has $\int d\Pi = 1$, since $\int \frac{dp}{2\pi} = \frac{1}{L}$ (by dimensional analysis and our 2π convention). According to our normalization convention for single-particle state,

$$\langle p|p\rangle = (2\omega_p)(2\pi)^3 \delta^{(3)}(0) = 2\omega_p V. \tag{2.29}$$

Now let us turn to the S-matrix element $\langle f|S|i\rangle$. We usually calculate S-matrix elements perturbatively. In a free theory, where there are no interactions, the S-matrix is simply the identity matrix. We can therefore write

$$S = 1 + i\mathcal{T}, \tag{2.30}$$

where $\mathcal{T}$ is called the transfer matrix and describes deviations from the free theory. Since the S-matrix should vanish unless the initial and final states have the same total 4-momentum, it is helpful to factor an overall momentum-conserving δ-function:

$$\mathcal{T} = (2\pi)^4 \delta^4(\Sigma p) \mathcal{M} \tag{2.31}$$

Here, $\delta^4(\Sigma p)$ is shorthand for $\delta^4\left(\Sigma p_i - \Sigma p_f\right)$, where p_i are the initial particles' momenta and p_f are the final particles' momenta. In this way, we can focus on computing the nontrivial part of the S-matrix, $\mathcal{M}$. In quantum field theory, "matrix elements" usually means $\langle f|\mathcal{M}|i\rangle$. Thus we have

$$\langle f|\mathcal{T}|i\rangle = (2\pi)^4 \delta^4(\Sigma p) \langle f|\mathcal{M}|i\rangle. \tag{2.32}$$

So,

$$
\begin{aligned}
dP &= \frac{\delta^4(\Sigma p)TV(2\pi)^4}{(2E_1V)(2E_2V)} \frac{|\mathcal{M}|^2}{\prod_j(2E_jV)} \prod_j \frac{V}{(2\pi)^3} d^3p_j \\
&= \frac{T}{V}\frac{1}{(2E_1)(2E_2)}|\mathcal{M}|^2 d\Pi_{\text{LIPS}}
\end{aligned} \tag{2.33}
$$

where

$$
d\Pi_{\text{LIPS}} \equiv \prod_{\text{final states } j} \frac{d^3p_j}{(2\pi)^3}\frac{1}{2E_{p_j}}(2\pi)^4\delta^4(\Sigma p) \tag{2.34}
$$

is called the *Lorentz-invariant phase space* (LIPS). Putting everything together, we have

$$
d\sigma = \frac{1}{(2E_1)(2E_2)|\vec{v}_1 - \vec{v}_2|}|\mathcal{M}|^2 d\Pi_{\text{LIPS}} \tag{2.35}
$$

All the factors of V and T have dropped out, so now it is trivial to take $V \to \infty$ and $T \to \infty$. Recall also that velocity is related to momentum by $\vec{v} = \vec{p}/p_0$.

Decay Rates

An unstable particle may decays to other particle(s), the rate of which is called the *decay rate*. A *differential decay rate* is the probability that a one-particle state with momentum p_1 turns into a multi-particle state with momenta $\{p_j\}$ over a time T:

$$
d\Gamma = \frac{1}{T}dP. \tag{2.36}
$$

Of course, it is impossible for the incoming particle to be an asymptotic state at $-\infty$ if it is to decay, and so we should not be able to use the S-matrix to describe decays. The reason this is not a problem is that we calculate the decay rate in perturbation theory assuming the interactions happen only over a finite time T. Thus, a decay is really just like a $(1 \to n)$ scattering process.

Following the same steps as for the differential cross section, the decay rate can be written as

$$
d\Gamma = \frac{1}{2E_1}|\mathcal{M}|^2 d\Pi_{\text{LIPS}} \tag{2.37}
$$

Note that this is the decay rate in the rest frame of the particle. If the particle is moving at relativistic velocities, it will decay much slower due to time dilation. The rate in the boosted frame can be calculated from the rest-frame decay rate using special relativity.

2.4 LSZ Reduction Formula

The LSZ reduction formula is used to simplify the calculation of the S-matrix in the momentum space. It essentially states that for the S-matrix of an $(n \to m)$ process, the matrix element equals to the *amputated Green's function*, which is the Green's function with in and out states propagators amputated:

$$\tilde{G}(k_1, \cdots, k_n) = \left[\prod_{i=1}^{n} \tilde{G}(k_i)\right] \tilde{G}_{\text{amp}}(k_1, \cdots, k_n). \tag{2.38}$$

Or, in the coordinate space (for scalar field),

$$\tilde{G}_{\text{amp}}(k_1, \cdots, k_n) = \left[\prod_{i=1}^{n} \int dx_i e^{-ik_i x_i} \frac{-\partial^2 - m^2}{i\sqrt{Z}}\right] G(x_1, \cdots, x_n). \tag{2.39}$$

Note that since the in and out states are on-shell, the factor $-\partial^2 - m^2$ effectively filter out the singularity $\frac{i}{k^2 - m^2}$, and any regular term without singularity will not affect the result.

2.4.1 Asymptotic Process

To get the basis idea how it happens, consider the correlation function

$$iG(y_m, \cdots, y_1, x_1, \cdots, x_n) = \langle \Omega | \phi(y_m) \cdots \phi(y_1) \phi(x_1) \cdots \phi(x_n) | \Omega \rangle. \tag{2.40}$$

Now we are going to Fourier transform this function for the variable x_1. First we split the time to three domains: $(-\infty, T_-]$, (T_-, T_+), and $[T_+, +\infty)$ such that at time $T_\pm$ the particles are well-separated. Consider first the integral over the first domain:

$$\int_{-\infty}^{T_-} dx_1^0 \int d^3x \; e^{ik\cdot x_1} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2\omega_q} \langle \Omega | \phi(y_m) \cdots \phi(y_1) \phi(x_2) \cdots \phi(x_n) | q \rangle \langle q | \phi(x_1) | \Omega \rangle, \tag{2.41}$$

where we have inserted the complete set of intermediate states.[2] Then use the fact $\langle q | \phi(x_1) | \Omega \rangle = \sqrt{Z_\phi} e^{iq\cdot x_1}$,

$$\int_{-\infty}^{T_-} dx_1^0 \; e^{i(k^0 + \omega_q - i\epsilon)\cdot x_1^0} \frac{\sqrt{Z_\phi}}{2\omega_k} \langle \Omega | \phi(y_m) \cdots \phi(y_1) \phi(x_2) \cdots \phi(x_n) | k \rangle, \tag{2.42}$$

The time integral gives the singularity at $k^0 = -\omega_k$:

$$\frac{1}{2\omega_k} \frac{i}{\omega_k + k^0 + i\epsilon} = \frac{i\sqrt{Z_\phi}}{k^2 - m^2 + i\epsilon} + \text{regular terms}. \tag{2.43}$$

Now consider the integral over the third time domain. The calculation is basically the same, the difference is the insertion gives

$$\langle \Omega | \phi(x_1) | q \rangle = \sqrt{Z_\phi} e^{iq\cdot x_1},$$

which leads to a singularity at $k^0 = \omega_k$:

$$\frac{1}{2\omega_k} \frac{i}{\omega_k - k^0 + i\epsilon} = \frac{i\sqrt{Z_\phi}}{k^2 - m^2 + i\epsilon} + \text{regular terms}. \tag{2.44}$$

[2] Note that the multi-particle state are discarded as discussed. Also, the single particle state $|k\rangle$ shall be think as a concentrated wave packet near the particle at $\boldsymbol{x}_1$, so that it has negligible overlap with other particle states.

Note that although for the above two cases, the final singular expression can be brought to the same form, the location of the singularity is different, which indicate whether it is the in or out state. Specific frequency filter can be chosen to select out the component accordingly.

Finally, consider the integral over time interval (T_-, T_+), where the particle are interacting and single particles are not well defined. On this interval the correlation will not have any singularity.[3] We then know that if we choose $\phi(x_1)$ to create the in state, and we only care about the singular structure, then the Fourier transformation produce the factor

$$\frac{i\sqrt{Z_\phi}}{k_1^2 - m_1^2}. \tag{2.45}$$

The same procedure applies to every field operator, and the final result is

$$\begin{aligned} S &= \langle p_1, \cdots, p_1; T_+ | k_1, \cdots, k_n; T_- \rangle \\ &= i\tilde{G}_{\text{amp}}(p_m, \cdots, p_1; -k_1, \cdots, -k_n)\delta^{(4)}\left(\sum p - \sum k\right). \end{aligned} \tag{2.46}$$

Or, the matrix element satisfies

$$\mathcal{M}_{fi} = \tilde{G}_{\text{amp}}(p_m, \cdots, p_1; -k_1, \cdots, -k_n). \tag{2.47}$$

2.4.2 Operator Proof for Scalar Field

Here we choose another way to prove the LSZ formula. We think a single-particle state to be created by the particle creation operator $a^\dagger$. For free theory, we have

$$\begin{aligned} \sqrt{2\omega_k} a_k &= i\int d^3x\ e^{ik\cdot x}(-i\omega_k + \partial_t)\phi(x), \\ \sqrt{2\omega_k} a_k^\dagger &= -i\int d^3x\ e^{-ik\cdot x}(i\omega_k + \partial_t)\phi(x). \end{aligned} \tag{2.48}$$

When interaction is turned on, the field operator $\phi(x)$ is renormalized as

$$\phi_R(x) \sim \sqrt{Z_\phi}\phi_{\text{in}}(x) \sim \sqrt{Z_\phi}\phi_{\text{out}}(x),$$

so we define the particle creation operator as

$$a_R^\dagger \equiv -i\int d^3x\ e^{-ik\cdot x}(i\omega_k + \partial_t)\phi_R(x). \tag{2.49}$$

When acting on the vacuum:

$$\sqrt{2\omega_k} a_R^\dagger(k)|\Omega\rangle = |k\rangle + \text{multi-particle states}. \tag{2.50}$$

On may wonder why $a_{\text{in}}(k)$ do not contribute to the single-particle state. To see that, one can think of the original particle-creation operator $a^\dagger(k)$ in the frequency domain to have a delta function peak at ω_k. While for the $a(k)$ in the interacting theory, although it can have weight at the frequency ω_k, there will be no delta-function-like peak.

[3]some branch cuts are possible, but they will also be annihilated by $k^2 - m^2$ term.

The in and out state are though to be created by the operator $a_R^\dagger(k)$. Note that as discussed, the multi-particle contribution is discarded. In the Heisenberg picture, the particle-creation operator satisfies:

$$
\begin{aligned}
a_R^\dagger(-\infty) - a_R^\dagger(+\infty) &= \frac{i}{\sqrt{2\omega_k}} \int dt\ \partial_t \left[\int d^3x\ e^{-ikx}(i\omega_k + \partial_t)\phi_R(x) \right] \\
&= \frac{i}{\sqrt{2\omega_k}} \int d^4x e^{-ik\cdot x}(\omega_k^2 + \partial_t^2)\phi_R(x) \\
&= \frac{i}{\sqrt{2\omega_k}} \int d^4x e^{-ik\cdot x}\partial_t^2\phi_0(x) + \phi_R(x)(-\nabla^2 + m^2)e^{-ik\cdot x} \\
&= \frac{i}{\sqrt{2\omega_k}} \int d^4x e^{-ik\cdot x}(\partial^2 + m^2)\phi_R(x)
\end{aligned}
\tag{2.51}
$$

The initial and final states are:

$$
\begin{aligned}
|k_1, \cdots, k_m; \text{in}\rangle &= \left[\prod_{j=1}^{m} \sqrt{2\omega_{k_j}} a_R^\dagger(k_j; -\infty) \right] |\Omega\rangle, \\
|p_1, \cdots, p_n, \text{out}\rangle &= \left[\prod_{j=1}^{n} \sqrt{2\omega_{p_j}} a_R^\dagger(p_j; +\infty) \right] |\Omega\rangle.
\end{aligned}
\tag{2.52}
$$

The S-matrix is

$$
\begin{aligned}
S_{fi} &= \langle p_1, \cdots, p_n; \text{out}|S|k_1, \cdots, k_m; \text{in}\rangle \\
&= \frac{\langle 0|T \left(\prod \sqrt{2\omega_{p_j}} a_{p_j;\text{out}}\right) \int d^4x \exp(i\mathcal{L}_{\text{int}}) \left(\prod \sqrt{2\omega_{k_j}} a_{k_j;\text{in}}^\dagger\right) |0\rangle}{\langle 0|T \int d^4x \exp(i\mathcal{L}_{\text{int}})|0\rangle}
\end{aligned}
$$

Since the scattering process correspond to the connected diagram, meaning that the initial and final state has distinct momentum particles. We are free to make the substitution

$$
a_{\text{in}}^\dagger \to (a_{\text{in}}^\dagger - a_{\text{out}}^\dagger),\ \ a_{\text{out}} \to -(a_{\text{in}}^\dagger - a_{\text{out}}^\dagger)^\dagger.
$$

In this way, the S-matrix is

$$
\begin{aligned}
&\langle p_1, \cdots, p_n|S|k_1, \cdots, k_m\rangle \\
=&\prod_{i=1}^{m} \left[\int d^dx_i\ e^{ip_i\cdot x_i} i(\partial^2 + m_i^2) \right] \prod_{j=m+1}^{m+n} \left[\int d^dx_j\ e^{-ik_j\cdot x_j} i(\partial^2 + m_j^2) \right] iG(\{x\}).
\end{aligned}
\tag{2.53}
$$

In momentum space

$$
\mathcal{M} = \prod_{i=1}^{m} \left[\frac{p_i^2 - m_i^2}{i\sqrt{Z_\phi}} \right] \prod_{j=m+1}^{m+n} \left[\frac{k_j^2 - m_j^2}{i\sqrt{Z_\phi}} \right] \tilde{G}(\{p_i\}; \{-k_j\}).
\tag{2.54}
$$

We thus proved the LSZ reduction formula again.

Note that in the second equality, we move the operator ∂^2 out of the time-ordering operator, which will actually create *contact terms*. We will show the contact term can be safely neglected. To see this, first consider the time-ordered two-point function:

$$
\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = \theta(t_1 - t_2)\langle 0|\phi(x_1)\phi(x_2)|0\rangle - \theta(t_2 - t_1)\langle 0|\phi(x_2)\phi(x_1)|0\rangle.
\tag{2.55}
$$

Take time derivative on both side:

$$\begin{aligned}\partial_{t_1}\langle 0|T\phi(x_1)\phi(x_2)|0\rangle &= \langle 0|T\partial_{t_1}\phi(x_1)\phi(x_2)|0\rangle + \delta(t_1-t_2)\langle 0|[\phi(x_1),\phi(x_2)]|0\rangle \\ &= \langle 0|T\partial_{t_1}\phi(x_1)\phi(x_2)|0\rangle.\end{aligned}$$

The second equality follows from the fact that x_1, x_2 is equal-time. Take the the time derivative once more:

$$\partial_{t_1}^2\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = \langle 0|T\partial_{t_1}^2\phi(x_1)\phi(x_2)|0\rangle + \delta(t_1-t_2)\langle 0|[\partial_{t_1}\phi(x_1),\phi(x_2)]|0\rangle.$$

The second term on the right hand side is the contact term. For free theory, $\partial_{t_1}\phi(x_1)$ is the canonical momentum, meaning that

$$[\phi(\vec{x}_1,t),\partial_t\phi(\vec{x}_1,t)] = i\delta^3(\vec{x}_1-\vec{x}_2). \tag{2.56}$$

In general, for n-point correlation,

$$\partial_{t_1}^2\langle T\phi_{x_1}\cdots\phi_{x_n}\rangle = \langle T\partial_{t_1}^2\phi_{x_1}\cdots\phi_{x_n}\rangle - i\sum_j \delta^4(x_1-x_j)\langle T\phi_{x_2}\cdots\cancel{\phi_{x_j}}\cdots\phi_{x_n}\rangle. \tag{2.57}$$

In the LSZ formula, the contact term do not have any singularity. When the external legs approach to momentum shell, these regular terms vanishes, so the contact will not contribute to the S-matrix.

2.4.3 LSZ for Dirac Field

Use the field expansion

$$\begin{aligned}\psi(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\sum_s\left(a^s_{\mathbf{p}}u^s(p)e^{-ip\cdot x} + b^{s\dagger}_{\mathbf{p}}v^s(p)e^{ip\cdot x}\right), \\ \bar{\psi}(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\sum_s\left(b^s_{\mathbf{p}}\bar{v}^s(p)e^{-ip\cdot x} + a^{s\dagger}_{\mathbf{p}}\bar{u}^s(p)e^{ip\cdot x}\right),\end{aligned} \tag{2.58}$$

and the orthogonality relation

$$\begin{aligned}u^{r\dagger}(p)u^s(p) = 2\omega_{\boldsymbol{p}}\delta^{rs}, &\quad u^{r\dagger}(\boldsymbol{p},\omega_{\boldsymbol{p}})v^s(-\boldsymbol{p},\omega_{\boldsymbol{p}}) = 0, \\ v^{r\dagger}(p)v^s(p) = 2\omega_{\boldsymbol{p}}\delta^{rs}, &\quad v^{r\dagger}(\boldsymbol{p},\omega_{\boldsymbol{p}})u^s(-\boldsymbol{p},\omega_{\boldsymbol{p}}) = 0.\end{aligned} \tag{2.59}$$

The spatial Fourier transformation gives:

$$\int d^3x e^{ip\cdot x}\psi(x) = \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\sum_s a^s_{\boldsymbol{p}}u^s(p) + \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\sum_s b^{s\dagger}_{\boldsymbol{p}}v^s(-\boldsymbol{p},\omega)e^{2i\omega t} \tag{2.60}$$

Left-multiply on both hand side by $\bar{u}^s(p)\gamma^0$, we then get

$$\begin{aligned}\sqrt{2\omega_{\boldsymbol{p}}}a^s_{\boldsymbol{p}} &= \int d^3x e^{ip\cdot x}\bar{u}^s(p)\gamma^0\psi(x), \\ \sqrt{2\omega_{\boldsymbol{p}}}a^{s\dagger}_{\boldsymbol{p}} &= \int d^3x e^{-ip\cdot x}\bar{\psi}(x)\gamma^0 u^s(p).\end{aligned} \tag{2.61}$$

Similarly, we consider

$$\int d^3x e^{ip\cdot x}\bar{\psi}(x) = \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\sum_s b^s_{\boldsymbol{p}}\bar{v}^s(p) + \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\sum_s a^{s\dagger}_{\boldsymbol{p}}\bar{u}^s(-\boldsymbol{p},\omega)e^{2i\omega t} \tag{2.62}$$

Right-multiply on both hand side by $\gamma^0 v^s(p)$, we then get

$$\begin{aligned}\sqrt{2\omega_{\boldsymbol{p}}}b^s_{\boldsymbol{p}} &= \int d^3x e^{ip\cdot x}\bar{\psi}(x)\gamma^0 v^s(p),\\ \sqrt{2\omega_{\boldsymbol{p}}}b^{s\dagger}_{\boldsymbol{p}} &= \int d^3x e^{-ip\cdot x}\bar{v}^s(p)\gamma^0\psi(x).\end{aligned} \tag{2.63}$$

Following the same strategy as we did for the scalar field, we consider

$$\begin{aligned}\sqrt{2\omega_{\boldsymbol{p}}}a^s_{\boldsymbol{p};\text{out}} - \sqrt{2\omega_{\boldsymbol{p}}}a^s_{\boldsymbol{p};\text{in}} &= \int dt\ \partial_t\sqrt{2\omega_{\boldsymbol{p}}}a^s_{\boldsymbol{p}}\\ &= \int dt\ \int d^3x e^{ip\cdot x}\bar{u}(p)(\gamma^0\partial_t + i\gamma^0 p^0)\psi(x)\\ &= \int d^4x e^{ip\cdot x}\bar{u}(p)(\gamma^0\partial_t + i\gamma^i p^i + im)\psi(x)\\ &= i\int d^4x e^{ip\cdot x}\bar{u}(p)(-i\not{\partial} + m)\psi(x)\end{aligned} \tag{2.64}$$

where we have used the fact $\bar{u}(p)(\not{p} - m) = 0$. Take hermitian conjugate,

$$\begin{aligned}\sqrt{2\omega_{\boldsymbol{p}}}a^{s\dagger}_{\boldsymbol{p};\text{in}} - \sqrt{2\omega_{\boldsymbol{p}}}a^{s\dagger}_{\boldsymbol{p};\text{out}} &= i\int d^4x e^{-ip\cdot x}\bar{\psi}(x)\gamma^0(-i\not{\partial} + m)^\dagger\gamma^0 u(p)\\ &= i\int d^4x e^{-ip\cdot x}\bar{\psi}(x)(i\overleftarrow{\not{\partial}} + m)u(p)\end{aligned} \tag{2.65}$$

Similarly, using the fact $(\not{p} + m)v(p) = 0$,

$$\begin{aligned}\sqrt{2\omega_{\boldsymbol{p}}}b^s_{\boldsymbol{p};\text{out}} - \sqrt{2\omega_{\boldsymbol{p}}}b^s_{\boldsymbol{p};\text{in}} &= \int d^4x e^{ip\cdot x}\bar{\psi}(x)(\gamma^0\overleftarrow{\partial_t} + i\gamma^0 p^0)v(p)\\ &= \int d^4x e^{ip\cdot x}\bar{\psi}(x)(\gamma^0\overleftarrow{\partial_t} + i\gamma^i p^i - im)v(p)\\ &= -i\int d^4x e^{ip\cdot x}\bar{\psi}(x)(i\overleftarrow{\not{\partial}} + m)v(p).\end{aligned} \tag{2.66}$$

Again, take the hermitian conjugate,

$$\begin{aligned}\sqrt{2\omega_{\boldsymbol{p}}}b^{s\dagger}_{\boldsymbol{p};\text{in}} - \sqrt{2\omega_{\boldsymbol{p}}}b^{s\dagger}_{\boldsymbol{p};\text{out}} &= -i\int d^4x e^{ip\cdot x}\bar{v}(p)\gamma^0(i\overleftarrow{\not{\partial}} + m)^\dagger\gamma^0\psi(x)\\ &= -i\int d^4x e^{-ip\cdot x}\bar{v}(p)(-i\not{\partial} + m)\psi(x)\end{aligned} \tag{2.67}$$

The same strategy gives the LSZ reduction formula for Dirac field. Consider the S-matrix for particles:

$$\begin{aligned}&\langle p_1,\cdots,p_n|S|k_1,\cdots,k_m\rangle\\ =&\prod_{i=1}^{m}\left[\int d^dx_i\ e^{ip_i\cdot x_i}u^{s_1}(p_i)\frac{i\not{\partial} - m_i}{i\sqrt{Z_\phi}}\right] iG(\{x\})\prod_{j=m+1}^{m+n}\left[\int d^dx_j\ e^{-ik_j\cdot x_j}\frac{-i\overleftarrow{\not{\partial}} - m_j}{i\sqrt{Z_\phi}}u^{s_j}(k_j)\right].\end{aligned} \tag{2.68}$$

In the momentum space:

$$\mathcal{M} = \prod_{i=1}^{m} \left[\frac{\not{p} - m_i}{i\sqrt{Z_\phi}} u^{s_i}(p_i) \right] \tilde{G}(\{p_i\}; \{-k_j\}) \prod_{j=m+1}^{m+n} \left[u^{s_j}(k_j) \frac{\not{k} - m_j}{i\sqrt{Z_\phi}} \right]. \tag{2.69}$$

Chapter 3

Scalar Field Theory

In this chapter, we study the interacting scalar field theory

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_{\text{int}},\\ \mathcal{L}_0 &= \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{m^2}{2}\phi^2,\\ \mathcal{L}_{\text{int}} &= -\frac{g}{4!}\phi^4.\end{aligned} \tag{3.1}$$

As we have discussed, because of the interaction, the field and coefficients will be renormalized:

$$\begin{aligned}\phi &= \sqrt{Z_\phi}\phi_R,\\ m &= \sqrt{Z_m}m_R,\\ g &= Z_g g_R.\end{aligned} \tag{3.2}$$

The renormalized Lagrangian becomes:

$$\mathcal{L} = Z_\phi\frac{1}{2}(\Box\phi_R)^2 - Z_m Z_\phi\frac{m_R^2}{2}\phi_R^2 - Z_g Z_\phi^2\frac{g_R}{4!}\phi_R^4. \tag{3.3}$$

We can formally divided the renormalized theory into three parts: free theory part, interaction part, and the counter terms:

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_{\text{int}} + \mathcal{L}_{\text{ct}},\\ \mathcal{L}_0 &= \frac{1}{2}(\Box\phi_R)^2 - \frac{m_R^2}{2}\phi_R^2,\\ \mathcal{L}_{\text{int}} &= -\frac{g_R}{4}\phi_R^4,\\ \mathcal{L}_{\text{ct}} &= A\frac{1}{2}(\Box\phi_R)^2 - B\frac{m_R^2}{2}\phi_R^2 - C\frac{g_R}{4!}\phi_R^4,\end{aligned} \tag{3.4}$$

where the coefficients in the counter terms are:

$$A = Z_\phi - 1, \quad B = Z_m Z_\phi - 1, \quad C = Z_g Z_\phi^2 - 1. \tag{3.5}$$

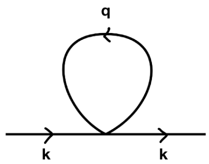
The counter terms come from the renormalization factors, which can be formally infinity but are perturbatively of order $O(g)$, and thus are regarded as additional perturbations to the free theory.

3.1 Perturbative Renormalization

3.1.1 First-order Correction to the Propagator

Consider the first order perturbation to the propagator:

$$i\Delta^{(1)}(k) =$$



$$+$$

$$= i\Delta_0(k)\left[-\frac{g}{2}\int\frac{d^4q}{(2\pi)^4}\frac{1}{q^2-m_R^2}+i(Ak^2-Bm_R^2)\right]i\Delta_0(k). \tag{3.6}$$

Note that in the loop diagram, there is in total $(4\times 3 = 12)$ identical diagrams, and so gives the coefficients. A simpler counter rule is that if the diagram has no symmetry, there is in total 4! identical diagrams and the denominator cancels out exactly, while for any remaining symmetries, the symmetry factor will remain in the denominator. For the loop diagram we considered here, the symmetry factor is 2.

Note that the second order correction to the Green's function has one "ingoing" leg and one "outgoing" leg, the core (or the amputated Green's function) in this case is called the *self-energy* $i\Sigma(k^2)$:

$$i\Sigma(k^2) = -\frac{g}{2}\int\frac{d^4q}{(2\pi)^4}\frac{1}{q^4-m_R^2}+i(Ak^2-Bm_R^2). \tag{3.7}$$

The integral is divergent, that when the counter terms come to rescue. The divergent part of the integral can be absorbed into the coefficients, or be canceled by the counter terms. In the following, we will see that the divergent integral

$$I \equiv -\frac{g}{2}\int\frac{d^4q}{(2\pi)^4}\frac{1}{q^4-m_R^2} \tag{3.8}$$

can be regularized. The regularization is typically controlled by a parameter which will recover the infinity when taking a specific limit. The physical observable shall not depend explicitly on the regularization parameter, so the final result is free from divergent even if we take the limit.

Regularization of the Divergent Integral

Consider the integral

$$I = -\frac{g}{2}\int\frac{d\omega}{2\pi}\int\frac{d^3q}{(2\pi)^3}\frac{1}{\omega^2-\boldsymbol{q}^2-m_R^2-i\epsilon} \tag{3.9}$$

Since the singularity locates at $\pm(\sqrt{\boldsymbol{q}^2+m_R^2}-i\epsilon)$, we can analytically change the integral of ω from real axis to the imaginary axis anti-clock-wisely, the result is equivalent to the substitution $\omega \to i\omega$. The new integral is defined on the 4D Euclidean space:

$$\begin{aligned} I &= i\frac{g}{2}\int \frac{d^4q}{(2\pi)^4}\frac{1}{q^2+m_R^2} \\ &= i\frac{g}{2}\frac{\Omega_4}{(2\pi)^4}\int_0^\infty dq\ \frac{q^3}{q^2+m_R^2}, \end{aligned} \tag{3.10}$$

where

$$\Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \tag{3.11}$$

is the d-dimensional spherical area.

Till now the integral is essentially the same and thus still divergent. Now we are going to regularize the expression. One most frequently used regularization scheme is the *dimensional regularization*. It take note of the fact that the divergence of the integral only happens at integer dimension. When we put the field theory to $(d = 4 - \varepsilon)$-dimensional space, the integral becomes:

$$I_\varepsilon = i\frac{g\tilde{\mu}^\varepsilon}{2}\frac{\Omega_{4-\varepsilon}}{(2\pi)^{4-\varepsilon}}\int_0^\infty dq\ \frac{q^{3-\varepsilon}}{q^2+m_R^2}. \tag{3.12}$$

Note that we have introduced a mass scale $\tilde{\mu}$ to get the correct dimensionality. The integral is now convergent. The specific form the the integral is not important, but we are concerned about the expansion near $\varepsilon = 0$. All the computation here can be done automatically (in `Mathematica`), and the result is

$$\begin{aligned} I_\varepsilon &= i\frac{gm_R^2}{32\pi^2}\left[\frac{2}{\varepsilon}+1+\log\left(\frac{4\pi\tilde{\mu}^2e^{-\gamma_E}}{m_R^2}\right)\right]+O(\varepsilon) \\ &\equiv i\frac{gm_R^2}{32\pi^2}\left[\frac{2}{\varepsilon}+1+\log\left(\frac{\mu^2}{m_R^2}\right)\right]+O(\varepsilon), \end{aligned} \tag{3.13}$$

where γ_E is the Euler constant. We have seen that the integral is controlled by the parameter ε. In the $\varepsilon \to 0$ limit, the integral is divergent.

Renormalization Using the Counter Terms

We are now going to renormalize the theory. Note that in the self-energy definition

$$\Sigma(k^2) = \frac{1}{i}I + Ak^2 - B^2m_R^2, \tag{3.14}$$

we have the freedom to choose the counter terms that cancel the infinity. To the first order, the coefficients can be

$$A = O(g^2), \quad B = \frac{g}{16\pi^2\varepsilon} + O(g^2). \tag{3.15}$$

The result is

$$\Sigma(k^2) = \frac{gm_R^2}{16\pi^2}\log\left(\frac{\mu}{m_R}\right) + \frac{gm_R^2}{32\pi^2} + O(\varepsilon). \tag{3.16}$$

The one-loop correction also leads to a infinite *Dyson series*:

$$\begin{aligned} i\Delta(k) &= i\Delta_0(k) + i\Delta_0(k)\sum_{n=1}^{\infty}\left[i\Sigma(k^2)i\Delta_0(k)\right] \\ &= \frac{i}{k^2-m_R^2+\Sigma(k^2)}. \end{aligned} \tag{3.17}$$

Physical Observables

The physical observable here is the rest mass m_0, which is experimentally measurable. It equals to the pole of the propagator, i.e.,

$$m_R^2 - \Sigma(m_0^2) = m_0^2. \tag{3.18}$$

Depending on the mass scale μ we choose, the renormalized mass m_R may or may not equal to the rest mass. Specifically, we can choose mass scale so that $m_R = m_0$, that is

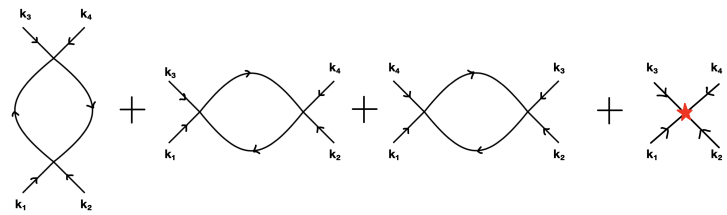
$$\Sigma(m_0^2) = 0 \quad \Longrightarrow \quad \mu = e^{-\frac{1}{2}} m_0. \tag{3.19}$$

To the first order, the self-energy has no momentum dependence, so there is no physical prediction.

3.1.2 Second-order Correction to the Vertex

Now consider the second order correction to the vertex function, which is the connected, amputated 4-point Green's function. The perturbation correspond to 3 different Feynman diagrams together with a vertex counter term. The explicit form is:

$$i\Gamma_4(k_1, k_2, k_3, k_4) =$$



$$= \frac{g_R^2}{2}\left[iF(s) + iF(t) + iF(u)\right] - iCg_R, \tag{3.20}$$

where we have introduced three momentum parameters s, t, and u:

$$s = (k_1 + k_2)^2, \quad t = (k_1 + k_3)^2, \quad u = (k_1 + k_4)^2. \tag{3.21}$$

The loop integrals for three channels are the same, denoted by

$$iF(k^2) = \int \frac{d^4q}{(2\pi)^4} \Delta_0(q)\Delta_0(q + k). \tag{3.22}$$

The integral is again divergent.

Regularization

The denominator in the integral

$$iF(k^2) = \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m_R^2} \frac{i}{(q + k)^2 - m_R^2} \tag{3.23}$$

involves multiplication of two polynomial. The expression can be simplified by the *Feynman parametrization*, which is essentially the identity:

$$\frac{1}{A_1 \ldots A_n} = \int dF_n \left(x_1 A_1 + \ldots + x_n A_n\right)^{-n}, \tag{3.24}$$

where the integration measure over the Feynman parameters x_i is

$$\int dF_n = (n-1)! \int_0^1 dx_1 \ldots dx_n \delta\left(x_1 + \ldots + x_n - 1\right). \tag{3.25}$$

This measure is normalized so that $\int dF_n = 1$. The simplest case is

$$\frac{1}{AB} = \int_0^1 \frac{dx}{[A + (B-A)x]^2} = \int_0^1 \frac{\delta(x+y-1)}{[xA + yB]^2} dxdy. \tag{3.26}$$

Other useful identities are

$$\begin{aligned} \frac{1}{AB^n} &= \int_0^1 dxdy \frac{\delta(x+y-1)ny^{n-1}}{[xA+yB]^{n+1}}, \\ \frac{1}{ABC} &= \int_0^1 dxdydz \frac{2\delta(x+y+z-1)}{[xA+yB+zC]^3}. \end{aligned} \tag{3.27}$$

Using the Feynman parameters, the integral is

$$iF(k^2) = \frac{i\Omega_4}{(2\pi)^4} \int_0^1 dx \int dq \, \frac{q^3}{\left[q^2 + m_R^2 + x(1-x)k^2\right]^2}. \tag{3.28}$$

Using the dimensional regularization, the integral is

$$iF_\varepsilon(k^2) = \frac{i\tilde{\mu}^\varepsilon \Omega_{4-\varepsilon}}{(2\pi)^{4-\varepsilon}} \int_0^1 dx \int dq \, \frac{q^{3-\varepsilon}}{\left[q^2 + m_R^2 + x(1-x)k^2\right]^2}. \tag{3.29}$$

Then we carry out the calculation, the result is:

$$\begin{aligned} F_\varepsilon(s) &= \frac{1}{8\pi^2\varepsilon} + \frac{1}{16\pi^2} \int_0^1 dx \ln\left(\frac{4\pi\tilde{\mu}^2 e^{-\gamma_E}}{D_{s,x}}\right) \\ &= \frac{1}{8\pi^2\varepsilon} + \frac{1}{8\pi^2} \ln\left(\frac{\mu}{m_R}\right) - \frac{1}{16\pi^2} \int_0^1 dx \ln\left(\frac{D_{s,x}}{m_R^2}\right), \end{aligned} \tag{3.30}$$

where we have denote

$$D_{k^2,x} \equiv m_R^2 + x(1-x)k^2. \tag{3.31}$$

Now sum up the contribution from all channel, the result looks like:

$$\Gamma_4^{(2)} = \frac{3g_R^2}{16\pi^2}\left[\frac{1}{\varepsilon} + \ln\left(\frac{\mu}{m_R}\right)\right] - \frac{g_R^2}{32\pi^2} \int_0^1 dx \ln\left(\frac{D_{s,x}D_{t,x}D_{u,x}}{m_R^6}\right) - Cg_R. \tag{3.32}$$

Renormalization and Physical Observables

To absorb the divergence, we can choose the counter term coefficient as

$$C = \frac{3g_R}{16\pi^2}. \tag{3.33}$$

So, to the second order, the vertex function is:

$$\Gamma_4(k_1, k_2, k_3, k_4) = -g_R + \frac{g_R^2}{32\pi^2} \int_0^1 dx \ln\left(\frac{\mu^6}{D_{s,x}D_{t,x}D_{u,x}}\right). \tag{3.34}$$

The vertex function is directly related to the physical observables. It can be measured in the scattering experiment or just just by the repulsive force it generated. We can choose the scale μ_0 so that

$$\Gamma(m_R, m_R, m_R, m_R) = -g_R. \tag{3.35}$$

Then the corrected vertex function gives the physical predictions, for example on the scattering amplitude for different k_i's.

3.1.3 Renormalization Group

Now consider the RG equation for the one-loop correction. The bare parameters are:

$$g_0 = Z_g g \tilde{\mu}^\epsilon, \; m_0 = Z_m^{1/2} m, \tag{3.36}$$

The RG conditions are:

$$\frac{dg_0}{d\ln\mu} = \left(\frac{3}{16\pi^2\epsilon} + \frac{1}{g}\right)\frac{dg}{d\ln\mu} + \epsilon = 0, \tag{3.37}$$

$$\frac{dm_0}{d\ln\mu} = \frac{1}{32\pi^2\epsilon}\frac{dg}{d\ln\mu} + \frac{1}{m}\frac{dm}{d\ln\mu} = 0. \tag{3.38}$$

Consider the series expansion of beta function:

$$\beta(g) = \frac{dg}{d\ln\mu} = \beta_1 g + \beta_2 g^2 + O(g^3). \tag{3.39}$$

The beta function is

$$\beta(g) = -\epsilon g + \frac{3g^2}{16\pi^2} + O(g^3). \tag{3.40}$$

The anomalous dimension of mass is

$$\gamma_m = \frac{1}{m}\frac{dm}{d\ln\mu} = \frac{g}{32\pi^2} + O(g^2) \tag{3.41}$$

3.2 Wilsonian Renormalization Group

Now we consider the two-loop correction to the self energy

$$\begin{aligned} i\Sigma^{(2)}(k) &= -\frac{g_R^2}{3!}\int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{i}{p_1^2 - m_R^2}\frac{i}{p_2^2 - m_R^2}\frac{i}{(k-p_1-p_2)^2 - m_R^2} \\ &= i\frac{g_R^2}{3!}\int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\int_0^1 dxdydz\, \frac{2\delta(1-x-y-z)}{D_{x,y,z}^3}, \end{aligned} \tag{3.42}$$

where the denominator is a bilinear

$$D = \begin{pmatrix} p_1 & p_2 \end{pmatrix} \begin{bmatrix} x+z & z \\ z & y+z \end{bmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} - 2kz(p_1+p_2) + k^2 z - m_R^2. \tag{3.43}$$

For a generic bilinear, we can simplify it by shifting the variable and diagonalized the quadratic coefficient matrix (which is always symmetric):

$$\begin{aligned} D &= \sum_{ij} A_{ij} p_i p_j + \sum_i B_i p_i - C \\ &= \sum_{ij} A_{ij} p'_i p'_j - C - \frac{1}{4} \sum_{ij} B_i A^{-1}_{ij} B_j \\ &= \sum_n a_n p''_n p''_n - C', \end{aligned} \tag{3.44}$$

We do not really need to carry out the diagonalization explicitly, but the above general form tells tells us that the integral can be transformed to

$$\begin{aligned} \frac{1}{D^3_{x,y,z}} &\to \frac{1}{(\det A)^2} \frac{1}{[p_1^2 + p_2^2 - C']^3} \\ &= \frac{1}{[xy + yz + zx]^2} \frac{1}{\left[p_1^2 + p_2^2 - \left(-\frac{xyz}{xy+yz+zx} k^2 + m_R^2\right)\right]^3} \end{aligned} \tag{3.45}$$

The self energy is then

$$\Sigma^{(2)}(k) = -\frac{g_R^2}{3!} \int_0^1 \frac{dF_3}{[xy + (x+y)z]^2} I(p_1^2, p_2^2), \tag{3.46}$$

where

$$I_\varepsilon(p_1^2, p_2^2) = \frac{\tilde{\mu}^{2\varepsilon} \Omega_d^2}{(2\pi)^{2d}} \int dp_1 dp_2 \frac{p_1^{d-1} p_2^{d-2}}{(p_1^2 + p_2^2 + C')^3} \tag{3.47}$$

3.2.1 Regularization and Renormalization

The ε expansion gives:

$$I_\varepsilon(p_1^2, p_2^2) = -\frac{C'}{512\pi^4 \varepsilon} + \text{finite part.} \tag{3.48}$$

Back to the

$$\frac{g_R^2}{3 \cdot 2^{10} \pi^4 \varepsilon} \int_0^1 dF_3 \, \frac{xyz}{(xy + yz + zx)^3} k^2 \tag{3.49}$$

3.3 Spontaneously Symmetry Breaking

Chapter 4

Quantum Electrodynamics

The Lagrangian for quantum electrodynamics is

$$\begin{aligned}\mathcal{L}_{\text{QED}} &= \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - eA_\mu\bar{\psi}\gamma^\mu\psi \\ &= \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{Maxwell}} + \mathcal{L}_{\text{int}},\end{aligned} \tag{4.1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\nu = (dA)_{\mu\nu}. \tag{4.2}$$

The Lagrangian is invariant under the gauge transformation:

$$\begin{aligned}\psi(x) &\to e^{-ie\alpha(x)}\psi(x), \\ A^\mu(x) &\to A^\mu(x) + \partial^\mu\alpha(x).\end{aligned} \tag{4.3}$$

It is convenient to rewrite Lagrangian as

$$\mathcal{L}_{\text{QED}} = \bar{\psi}\left(i\not{D} - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{4.4}$$

where we have define the covariant derivative as:

$$\not{D} = \gamma^\mu D_\mu = \gamma^\mu[\partial_\mu + ieA_\mu(x)] = \not{\partial} + ie\not{A}. \tag{4.5}$$

As with the scalar field, the partition function with source is defined as

$$Z[\bar{\eta},\eta,J] = \exp\left\{i\int d^dx\mathcal{L}_{\text{int}}\left[\frac{\delta}{i\delta J},\frac{\delta}{i\delta\eta},\frac{i\delta}{\delta\bar{\eta}}\right]\right\}Z_0[\bar{\eta},\eta,J]. \tag{4.6}$$

In the following, we will use the dimensional regularization by default. Note that the mass dimensionality for the fermion and gauge field is

$$[\psi] = \left[\frac{d-1}{2}\right], \quad [A] = \left[\frac{d}{2} - 1\right],$$

which lead to the coupling constant e to have the dimension

$$[e] = \left[2 - \frac{d}{2}\right].$$

When $d = 4 - \epsilon$, we make the replacement

$$e \to e\tilde{\mu}^{\epsilon/2},$$

so that to make the coupling constant e dimensionless.

4.1 Perturbative Renormalization

The interaction will renormalize the field and coefficients to

$$\begin{aligned} \psi &= \sqrt{Z_\psi}\psi_R, \quad m = Z_m m_R, \\ A &= \sqrt{Z_A}A_R, \quad\ e = Z_e e_R. \end{aligned} \tag{4.7}$$

The renormalized Lagrangian is

$$\begin{aligned} \mathcal{L} &= Z_\psi \bar{\psi}_R(i\gamma^\mu\partial_\mu)\psi_R - Z_m Z_\psi m_R\bar{\psi}_R\psi_R + Z_A\frac{1}{4}F_{R,\mu\nu}F_R^{\mu\nu} - Z_e Z_\psi\sqrt{Z_A}e_R A_R^\mu\bar{\psi}_R\gamma^\mu\psi_R \\ &= \mathcal{L}_0 + \mathcal{L}_{\text{int}} + \mathcal{L}_{\text{ct}}. \end{aligned} \tag{4.8}$$

The we define the coefficients

$$\delta_\psi = Z_\phi - 1, \quad \delta_m = Z_m - 1, \quad \delta_A = Z_A - 1, \quad \delta_1 = Z_e Z_\psi\sqrt{Z_A} - 1. \tag{4.9}$$

The counter term also contribute to the perturbative expansion like the interactions. The counter terms for the fermion propagator come from the diagram expression:

$$iG_F^{(\text{ct})}(p) =$$

$$= G_F^{(0)}(p)\left[\delta_\psi \not{p} - (\delta_m + \delta_\psi)m_R\right]G_F^{(0)}(p). \tag{4.10}$$

The contribution to the electron self energy is

$$i\Sigma^{(\text{ct})}(p) \equiv \delta_\psi\not{p} - (\delta_m + \delta_\psi)m_R. \tag{4.11}$$

Similarly, the counter term contribution to the photon self energy is

$$i\Pi_{\mu\nu}^{(\text{ct})}(k) = \ \mu$$

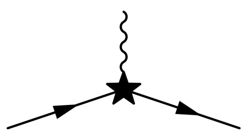
$$\nu$$

$$= \delta_A[-p^2 g_{\mu\nu} + (1-\xi)p_\mu p_\nu]. \tag{4.12}$$

In Landau gauge, $\xi = 1$ and we can ignore the $p_\mu p_\nu$ terms, and the photon self-energy is always proportional to $g_{\mu\nu}$. Whenever other gauge choices are concerned, we can simply replace $g_{\mu\nu}$ with $g_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}$.

The counter term contribution to the QED vertex is

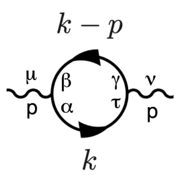
$$\Gamma_{\alpha\beta}^{(\text{ct})\mu} =$$



$$= -\delta_1\gamma^\mu_{\alpha\beta}. \tag{4.13}$$

4.1.1 Vacuum Polarization

Consider the one-loop correction to the photon propagator:



$$\simeq (-ie_R)^2 A_\mu\bar{\psi}_\alpha\gamma^\mu_{\alpha\beta}\psi_\beta A_\nu\bar{\psi}_\gamma\gamma^\nu_{\gamma\tau}\psi_\tau \tag{4.14}$$

$$\equiv iA_\mu\Pi^{\mu\nu}(p)A_\nu.$$

The self energy is:

$$
\begin{aligned}
i\Pi^{\mu\nu}(p) &= -e_R^2 \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left[\gamma^\mu G_F^{(0)}(k-p)\gamma^\nu G_F^{(0)}(k)\right] \\
&= -e_R^2 \int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{Tr}\left[\gamma^\mu(\not{k}-\not{p}+m_R)\gamma^\nu(\not{k}+m)\right]}{(k^2-m_R^2)[(p-k)^2-m_R^2]}.
\end{aligned}
\tag{4.15}
$$

The trace of the Dirac matrices can be evaluated in `Mathematic` using the `FeynCalc` package:

```
(*Dirac trace using FeynCalc`*)
res=DiracTrace[GA[\[Mu]].(GS[k-p]+m).GA[\[Nu]].(GS[k]+m)];
DiracSimplify[res]
```

The Dirac trace is:

$$
\begin{aligned}
&\mathrm{Tr}\left[\gamma^\mu(\not{k}-\not{p}+m_R)\gamma^\nu(\not{k}+m_R)\right] \\
&= 4\left[g^{\mu\nu}\left(k\cdot p-k^2+m_R^2\right)+2k^\mu k^\nu-k^\mu p^\nu-p^\mu k^\nu\right].
\end{aligned}
\tag{4.16}
$$

Using the Feynman parameters, the denominator is:

$$
\begin{aligned}
\frac{1}{(k^2-m_R^2)[(p-k)^2-m_R^2]} &= \frac{1}{\left\{[k-p(1-x)]^2-[m_R^2+p^2x(x-1)]\right\}^2} \\
&\equiv \frac{1}{\left\{[k-p(1-x)]^2-D_x\right\}^2}.
\end{aligned}
\tag{4.17}
$$

Since the Ward identity requires that the p^μ term in the propagator do not contribute to any scattering process, we then shift $k \to k+p(1-x)$ and drop all p^μ linear term. The final result is simplified to:

$$
\begin{aligned}
i\Pi^{\mu\nu}(p) &= -4e_R^2 \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{2k^\mu k^\nu - g^{\mu\nu}\left[k^2-x(1-x)p^2-m_R^2\right]}{\left[k^2-D_x\right]^2} \\
&\simeq 4e_R^2 g^{\mu\nu} \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{\frac{1}{2}k^2-x(1-x)p^2-m_R^2}{\left[k^2-D_x\right]^2} \\
&\simeq -ie_R^2 g^{\mu\nu} \int_0^1 dx\, \frac{\Omega_d \tilde{\mu}^\varepsilon}{(2\pi)^d} \int_0^\infty dk\; k^{d-1} \frac{\left(4-\frac{2}{d}\right)k^2+4x(1-x)p^2+4m_R^2}{\left[k^2+D_x\right]^2}.
\end{aligned}
\tag{4.18}
$$

where we have made the Wick rotation, shifted the dimensionality to ($d = 4-\varepsilon$), and made the substitution (since the self-energy $i\Pi^{\mu\nu} \propto g^{\mu\nu}$):

$$
k^\mu k^\nu \to \frac{1}{d}k^2 g^{\mu\nu}. \tag{4.19}
$$

The remaining problem is to regularize and renormalize the divergent integral

$$
I_\varepsilon(x) \equiv \frac{\Omega_{4-\varepsilon}\tilde{\mu}^\varepsilon}{(2\pi)^{4-\varepsilon}} \int_0^\infty dk\; k^{3-\varepsilon} \frac{\left(4-\frac{8}{4-\varepsilon}\right)k^2+4x(1-x)p^2+4m_R^2}{\left[k^2+D_x\right]^2}. \tag{4.20}
$$

Regularization and Renormalization

In $(4-\varepsilon)$-dimensional Euclidean space, the integral is convergent. The ε-expansion is carried out in Mathematica using the following code:

```
omg = (2*Pi^(d/2))/(Gamma[d/2]);
cof = \[Mu]^(4-d)*omg/(2*Pi)^d;
nom = k^(d-1)*((4-8/(4-\[Epsilon]))k^2+4x*(1-x)p^2+4m^2);
int = cof*Integrate[nom/(k^2+D)^2,{k,0,Infinity}][[1]];
map = D->m^2-p^2*x*(1-x);
ans = Series[int/.{d->4-\[Epsilon]},{\[Epsilon],0,0}];
ans /. map // Simplify
```

The result is

$$I_\varepsilon(x) = \frac{p^2x(1-x)}{2\pi^2}\left[\frac{2}{\varepsilon} + \ln\left(\frac{4\pi e^{-\gamma_E}\tilde{\mu}^2}{m_R^2 - p^2x(1-x)}\right)\right] \tag{4.21}$$

So the photon self-energy is (also denote $\mu^2 = 4\pi e^{-\gamma_E}\tilde{\mu}^2$):

$$\Pi^{\mu\nu}(p) = -\frac{e_R^2p^2g^{\mu\nu}}{6\pi^2\varepsilon} - \frac{e_R^2p^2g^{\mu\nu}}{2\pi^2}\int_0^1 dx\ x(1-x)\ln\left[\frac{\mu^2}{m_R^2 - p^2x(1-x)}\right] \tag{4.22}$$

The counter term coefficient can be chosen as

$$\delta_A = -\frac{e_R^2}{6\pi^2\epsilon}. \tag{4.23}$$

The renormalized photon self-energy is then

$$\begin{aligned}\Pi^{\mu\nu}(p) &= -\frac{e_R^2p^2g^{\mu\nu}}{2\pi^2}\int_0^1 dx\ x(1-x)\ln\left[\frac{\mu^2}{m_R^2 - p^2x(1-x)}\right] \\ &= \frac{e_R^2p^2g^{\mu\nu}}{2\pi^2}\left\{\frac{1}{3}\ln\left(\frac{m_R}{\mu}\right) + \int_0^1 dx\ x(1-x)\ln\left[1 - \frac{p^2x(1-x)}{m_R^2}\right]\right\}.\end{aligned} \tag{4.24}$$

Physical Observable

The photon self-energy has the form

$$\Pi^{\mu\nu}(p) = -e_R^2\left[g^{\mu\nu} - (1-\xi)p^\mu p^\nu\right]g^{\mu\nu}\Pi_2(p), \tag{4.25}$$

where

$$\Pi_2(p) = \frac{1}{2\pi^2}\int_0^1 dx\ x(1-x)\ln\left[\frac{\mu^2}{m_R^2 - p^2x(1-x)}\right] \tag{4.26}$$

The one-loop correction to photon propagator is

$$\begin{aligned}iG_\gamma^{\mu\nu}(p) &= -i\frac{g^{\mu\nu}}{p^2}\left(1 + \sum_{n=1}^{\infty}(-e_R^2)^n\Pi_2^n(p)\right) \\ &= -i\frac{g^{\mu\nu}}{p^2\left[1 + e_R^2\Pi_2(p)\right]}.\end{aligned} \tag{4.27}$$

We can choose the on-shell condition that the photon has no rest mass:

$$\Pi_2(0) = 0 \quad \Longrightarrow \quad \mu = m_R. \tag{4.28}$$

Note that the propagator is related to the Coulomb potential.[1] To the second order,

$$\begin{aligned} V(p) &= e_R^2 \frac{1 - e_R^2 \Pi_2(p)}{p^2} + O(e_R^6) \\ &= \frac{e_R^2}{p^2} \left\{ 1 + \frac{e_R^2}{2\pi^2} \int_0^1 dx\ x(1-x) \ln \left[1 - \frac{p^2 x(1-x)}{m_R^2} \right] + O(e_R^4) \right\}. \end{aligned} \tag{4.29}$$

Consider the small momentum limit, where the integral in approximated by

$$\int_0^1 dx\ x(1-x) \ln \left[1 - \frac{p^2 x(1-x)}{m_R^2} \right] \approx -\frac{p^2}{m_R^2} \int_0^1 dx\ x^2 (1-x)^2 = -\frac{p^2}{30 m_R^2}. \tag{4.30}$$

This implies

$$V(p) = \frac{e_R^2}{p^2} - \frac{e_R^4}{60\pi^2 m_R^2}. \tag{4.31}$$

The Fourier transformation gives

$$V(r) = -\frac{e_R^2}{4\pi r} - \frac{e_R^4}{60\pi^2 m_R^2} \delta^{(3)}(r). \tag{4.32}$$

For atomic orbit, since only the $(L = 0)$-orbit have support at $r = 0$, this extra potential will shift the spectrum. This effect is called the *Lamb shift*.

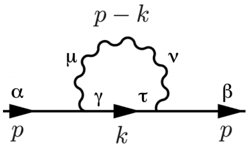
On the other hand, the the large momentum limit,

$$V(p) \approx \frac{e_R^2}{p^2} \left[1 + \frac{e_R^2}{12\pi^2} \ln \frac{-p^2}{m_R^2} \right], \tag{4.33}$$

which predicts a *Landau pole* beyond which perturbation theory breaks down.

4.1.2 One-loop Correction to Electron Propagator

Consider the one-loop correction to the particle propagator:



$$\simeq (-ie_R)^2 A_\mu \bar{\psi}_\alpha \gamma^\mu_{\alpha\gamma} \psi_\gamma A_\nu \bar{\psi}_\tau \gamma^\nu_{\tau\beta} \psi_\beta \equiv i\bar{\psi}_\alpha \Sigma^{\alpha\beta}(p) \psi_\beta. \tag{4.34}$$

The self energy is

$$\begin{aligned} i\Sigma_{\alpha\beta}(p) &= e_R^2 \int \frac{d^4 k}{(2\pi)^4} G_\gamma^{\mu\nu}(p-k) \left[\gamma_\mu G_F(k) \gamma_\nu \right]_{\alpha\beta} \\ &= -e_R^2 \int \frac{d^d k}{(2\pi)^d} \frac{\gamma^\mu (\not{k} + m_R) \gamma_\mu}{(p-k)^2 (k^2 - m_R^2)}. \end{aligned} \tag{4.35}$$

[1] The Coulomb potential arises just like we derive the force (1.123), but the sources have additional charge e_R, and the photon is mass less, so $V(p) = \frac{e_R^2}{p^2}$ for free field.

The seconded equality comes from the contraction:

$$(-ie)^2 A_\mu \bar{\psi}_\alpha \gamma^\mu_{\alpha\gamma} \psi_\gamma A_\nu \bar{\psi}_\tau \gamma^\nu_{\tau\beta} \psi_\beta \tag{4.36}$$

The nominator can be simplified using the Dirac matrix identities:

$$\gamma^\mu \gamma_\mu = d, \quad \gamma^\mu \gamma^\nu \gamma_\mu = (2-d)\gamma^\nu \quad \Longrightarrow \quad \gamma^\mu (\not{k} + m_R)\gamma_\mu = d m_R + (2-d)\not{k}. \tag{4.37}$$

The denominator can be simplify using the Feynman parameter:

$$\begin{aligned} \frac{1}{(p-k)^2(k^2-m_R^2)} &= \int_0^1 \frac{dx}{\left[(k-px)^2 - (1-x)(m_R^2 - p^2 x)\right]^2} \\ &\to \int_0^1 \frac{dx}{\left(k^2 - D_x\right)^2} \end{aligned} \tag{4.38}$$

where we have shifted $k \to k + px$ (note this shift also change the numerator).

The self energy becomes (including a $\tilde{\mu}$ mass scale):

$$\begin{aligned} i\Sigma(p) &= e_R^2 \tilde{\mu}^\varepsilon \int_0^1 [(2-\varepsilon)x\not{p} - (4-\varepsilon)m_R]dx \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - D_x)^2} \\ &= i e_R^2 \int_0^1 dx\, [(2-\varepsilon)x\not{p} - (4-\varepsilon)m_R]\, \frac{\tilde{\mu}^\varepsilon \Omega_d}{(2\pi)^d} \int \frac{k^{d-1} dk}{(k^2 + D_x)^2}. \end{aligned} \tag{4.39}$$

Regularization and Renormalization

The regularization procedure is carried out by the following code:

```
omg=(2*Pi^(d/2))/(Gamma[d/2]);
cof=\[Mu]^(4-d)*omg/(2*Pi)^d;
int=cof*Integrate[q^(d-1)/(q^2+D)^2,{q,0,Infinity}][[1]];
map={e->Sqrt[4*Pi*\[Alpha]],EulerGamma->Subscript[\[Gamma],E]};
ans=Series[int/.{d->4-\[Epsilon]},{\[Epsilon],0,0}];
ans/.map//Simplify
```

The result is ($\mu^2 = 4\pi e^{-\gamma_E} \tilde{\mu}^2$)

$$\begin{aligned} \Sigma(p) &= \frac{e_R^2}{16\pi^2} \int_0^1 dx\, [(2-\varepsilon)x\not{p} - (4-\varepsilon)m_R] \left[\frac{2}{\varepsilon} + \ln \frac{\mu^2}{(1-x)(m_R^2 - p^2 x)}\right] \\ &= \frac{e_R^2}{16\pi^2} \left\{ \int_0^1 dx\, [2x\not{p} - 4m_R] \left[\frac{2}{\varepsilon} + \ln \frac{\mu^2}{(1-x)(m_R^2 - p^2 x)}\right] - \not{p} + 2m_R \right\}. \end{aligned} \tag{4.40}$$

The infinity comes from

$$\frac{e_R^2}{4\pi^2 \varepsilon} \int_0^1 dx (x\not{p} - 2m_R) = \frac{e_R^2}{8\pi^2 \varepsilon}\not{p} - \frac{e_R^2}{2\pi^2 \varepsilon} m_R. \tag{4.41}$$

Using the $\overline{\text{MS}}$ subtraction scheme, we choose

$$\delta_\psi = -\frac{e_R^2}{8\pi^2 \varepsilon}, \quad \delta_m = -\frac{3e_R^2}{8\pi^2 \varepsilon}, \tag{4.42}$$

and the self energy is

$$\Sigma(p) = \frac{e_R^2}{16\pi^2} \left\{ \int_0^1 dx (2x\not{p} - 4m_R) \ln \left[\frac{\mu^2}{(1-x)(m_R^2 - p^2 x)}\right] - \not{p} + 2m_R \right\}. \tag{4.43}$$

Physical Observables

The Dyson series gives:

$$iG_F(p) = \frac{i}{\not{p} - m_R + \Sigma(p)} \tag{4.44}$$

Experimentally, for a given The on-shell subtraction requires that the m_R equals to the physical mass:

$$\Sigma(\not{p})\big|_{\not{p}=m_R} = 0, \quad \frac{d}{d\not{p}}\Sigma(\not{p})\bigg|_{\not{p}=m_R} = 0. \tag{4.45}$$

To implement the on-shell condition, we have to modify the subtraction scheme to

$$\begin{aligned} \delta_\psi &= -\frac{e_R^2}{8\pi^2}\left(\frac{1}{\varepsilon} + \ln\frac{\mu}{m_R} + A\right), \\ \delta_m &= -\frac{e_R^2}{8\pi^2}\left(\frac{3}{\varepsilon} + 3\ln\frac{\mu}{m_R} + B\right), \end{aligned} \tag{4.46}$$

and the self energy is

$$\begin{aligned} \Sigma(\not{p}) = & -\frac{e_R^2}{8\pi^2}\int_0^1 dx(x\not{p} - 2m_R)\ln\left[(1-x)\left(1 - \frac{p^2}{m_R^2}x\right)\right] \\ & -\frac{e_R^2}{8\pi^2}\left[\left(A + \frac{1}{2}\right)\not{p} - (A+B+1)m_R\right]. \end{aligned} \tag{4.47}$$

The first condition

$$\begin{aligned} \Sigma(\not{p})\big|_{\not{p}=m_R} &= -\frac{e_R^2}{8\pi^2}m_R\left[\int_0^1 dx\ (x-2)\ln(1-x)^2 - B - \frac{1}{2}\right] \\ &= -\frac{e_R^2}{8\pi^2}m_R\,(2-B) = 0 \end{aligned} \tag{4.48}$$

gives the mass renormalization coefficient

$$\delta_m = -\frac{e_R^2}{8\pi^2}\left(\frac{3}{\varepsilon} + 3\ln\frac{\mu}{m_R} + 2\right). \tag{4.49}$$

While in the derivative of the self-energy:

$$\frac{d}{d\not{p}}\Sigma(m_R) = -\frac{e_R^2}{8\pi^2}\left\{\int_0^1 dx\ \left[x\ln(1-x)^2 - \frac{2x(x-2)}{1-x}\ln(1-x)\right] + A + \frac{1}{2}\right\}, \tag{4.50}$$

there is a divergent integral:

$$\int_0^1 dx\,\frac{2x(x-2)}{1-x}\ln(1-x), \tag{4.51}$$

indicating an IR divergence. We can never the less get rid of it by introducing a small mass m_γ for photon (which will be set to zero). This mass term change the denominator in the loop integral:

$$\frac{1}{\left[(p-k)^2 - m_\gamma^2\right](k^2 - m_R^2)} = \int_0^1 \frac{dx}{\left[(k-px)^2 - D_x - xm_\gamma^2\right]^2}. \tag{4.52}$$

Most derivation remains the same, we just need to make a substitution in the finial result:

$$D_x \to D_x + x m_\gamma^2. \tag{4.53}$$

Especially, the introducing of the photon mass will not change the result of the mass renormalization factor we have computed.

The modified self-energy is then

$$\begin{aligned}\Sigma(\not{p}) = & -\frac{e_R^2}{8\pi^2}\int_0^1 dx(x\not{p} - 2m_R)\ln\left[(1-x)\left(1-\frac{p^2}{m_R^2}x\right) + x\frac{m_\gamma^2}{m_R^2}\right] \\ & -\frac{e_R^2}{8\pi^2}\left[\left(A+\frac{1}{2}\right)\not{p} - (A+B+1)m_R\right].\end{aligned} \tag{4.54}$$

The derivative is now

$$\frac{d}{d\not{p}}\Sigma(m_R) = -\frac{e_R^2}{8\pi^2}\left\{\int_0^1 dx\ \left[x\ln(1-x)^2 + \frac{2x(2-x)(1-x)}{(1-x)^2 + x\frac{m_\gamma^2}{m_R^2}}\right] + A + \frac{1}{2}\right\}, \tag{4.55}$$

Note that in the ($m_\gamma \to 0$) limit, the asymptotic behavior of the originally divergent integral is

$$\lim_{m_\gamma\to 0}\int_0^1 dx\ \frac{2x(2-x)(1-x)}{(1-x)^2 + x\frac{m_\gamma^2}{m_R^2}} = -1 - 2\ln\frac{m_\gamma}{m_R}. \tag{4.56}$$

So the second subtraction condition is:

$$\frac{d}{d\not{p}}\Sigma(m_R) = -\frac{e_R^2}{8\pi^2}\left(A - 2 - 2\frac{m_\gamma}{m_R}\right) = 0. \tag{4.57}$$

The field strength renormalization is

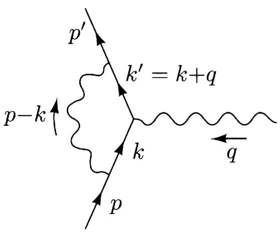
$$\delta_\psi = -\frac{e_R^2}{8\pi^2}\left(\frac{1}{\varepsilon} + \ln\frac{\mu}{m_R} + 2 + 2\ln\frac{m_\gamma}{m_R}\right). \tag{4.58}$$

The final self energy is (shall take the $m_\gamma \to 0$ limit):

$$\begin{aligned}\Sigma(\not{p}) = & -\frac{e_R^2}{16\pi^2}\int_0^1 dx(2x\not{p} - 4m_R)\ln\left[(1-x)\left(1-\frac{p^2}{m_R^2}x\right) + 2x\ln\frac{m_\gamma}{m_R}\right] \\ & -\frac{e_R^2}{16\pi^2}\left[\left(5+4\ln\frac{m_\gamma}{m_R}\right)\not{p} - \left(10 + 4\ln\frac{m_\gamma}{m_R}\right)m_R\right].\end{aligned} \tag{4.59}$$

4.1.3 One-loop Correction to Vertex

Consider the one-loop correction to interaction:



$$\simeq (-ie_R)^3 A_\nu\bar{\psi}_\alpha\gamma^\nu_{\alpha\beta}\psi_\beta A_\mu\bar{\psi}_\lambda\gamma^\mu_{\lambda\tau}\psi_\tau A_\xi\bar{\psi}_\rho\gamma^\nu_{\rho\sigma}\psi_\sigma \tag{4.60}$$

$$\equiv -ieA_\mu\Gamma^\mu_{\alpha\beta}(q,p,p')\bar{\psi}_\alpha\psi_\beta.$$

The vertex function is:

$$
\begin{aligned}
i\Gamma^{\mu}_{\alpha\beta}(q,p,p') &= -e_R^2 \int \frac{d^4k}{(2\pi)^4} G_\gamma^{\nu\lambda}(p-k)[\gamma_\nu G_F(k')\gamma^\mu G_F(k)\gamma_\lambda]_{\alpha\beta} \\
&= e_R^2 \int \frac{d^4k}{(2\pi)^4} \frac{[\gamma^\nu(\not{k}'+m_R)\gamma^\mu(\not{k}+m_R)\gamma_\nu]_{\alpha\beta}}{(k^2-m_R^2)(k'^2-m_R^2)(p-k)^2}
\end{aligned}
\tag{4.61}
$$

Using the following code

```
(*numerator*)
den=Contract[GA[\[Nu]].(GS[kp]+m).GA[\[Mu]].(GS[k]+m).GA[\[Nu]]];
DiracSimplify[den]

(*Feynman parameter*)
A1=k^2-m^2;
A2=(k+q)^2-m^2;
A3=(p-k)^2;
{c,b,a}=CoefficientList[x*A1+y*A2+z*A3,{k}];
-b/2//Simplify
-c+b^2/4//Simplify
```

The numerator is

$$
-2\not{k}\gamma^\mu\not{k}' - 2m_R^2\gamma^\mu + 4m_R(k+k')^\mu. \tag{4.62}
$$

The denominator is

$$
\int \frac{dF_3}{[(k+yq-zp)^2 - D_{xyz}]^3}, \tag{4.63}
$$

where

$$
\begin{aligned}
D_{xyz} &= (x+y)m^2 - z(1-z)p^2 - y(1-y)q^2 - 2yzpq \\
&= (x+y)m_R^2 - xyq^2 - yzp'^2 - xzp^2.
\end{aligned}
$$

Shift $k^\mu \to k^\mu + zq_1^\mu - yp^\mu$, throw away all terms with linear k^μ, and replace $k^\mu k^\nu$ with $\frac{1}{d}k^2 g^{\mu\nu}$, the result is

$$
\frac{4}{d}k^2\gamma^\mu - 2(-y\not{q}+z\not{p})\gamma^\mu[(1-y)\not{q}+z\not{p}] + 4m_R^2\gamma^\mu - 2m_R\left[(1-2y)q^\mu + 2zp^\mu\right].
$$

Note that only the quadratic term is divergent.

$$
\Gamma^\mu(p,q_1,q_2) = -i\frac{4e^2\tilde{\mu}^\epsilon\gamma^\mu}{d}\int dF_3 \int \frac{d^dk}{(2\pi)^d}\frac{k^2}{(k^2-D)^3} + \delta\Gamma^\mu(p,q_1,q_2).
$$

where $\delta\Gamma^\mu$ stores all the finite part

$$
\begin{aligned}
&\delta\Gamma^\mu(p,q_1,q_2) \\
&= \int \frac{e^2k^3dkdF_3}{(2\pi)^2(k^2+D)^3}\left\{(-y\not{q}+z\not{p})\gamma^\mu[(1-y)\not{q}+z\not{p}] - 2m_R^2\gamma^\mu + m_R\left[(1-2y)q^\mu + 2zp^\mu\right]\right\}.
\end{aligned}
$$

The divergent part is

$$
\frac{4e^2\tilde{\mu}^\epsilon\Omega_d\gamma^\mu}{d(2\pi)^d}\int dF_3\int\frac{k^{d+1}dk}{(k^2+D)^3} = \frac{e_R^2}{16\pi^2}\gamma^\mu\int dF_3\left(\frac{2}{\epsilon} + \ln\frac{\mu^2}{D_{xyz}}\right). \tag{4.64}
$$

Using the $\overline{\text{MS}}$ scheme, the coefficient of the counter term is

$$
\delta_e = -\frac{e_R^2}{8\pi^2\varepsilon}. \tag{4.65}
$$

4.2 Systematic Renormalization

4.2.1 Renormalization Group

In summery, the renormalization factors are

$$\begin{aligned} Z_\psi &= 1 - \frac{e_R^2}{8\pi^2\epsilon} + O(e_R^3), \\ Z_A &= 1 - \frac{e_R^2}{6\pi^2\epsilon} + O(e_R^3), \\ Z_m &= 1 - \frac{e_R^2}{2\pi^2\epsilon} + O(e_R^3), \\ Z_e &= 1 - \frac{e_R^2}{8\pi^2\epsilon} + O(e_R^3), \end{aligned} \tag{4.66}$$

which means

$$\begin{aligned} \frac{d\ln Z_\phi}{de_R} &= -\frac{e_R}{4\pi^2\epsilon} + O(e_R^2), \\ \frac{d\ln Z_A}{de_R} &= -\frac{e_R}{3\pi^2\epsilon} + O(e_R^2), \\ \frac{d\ln Z_m}{de_R} &= -\frac{e_R}{\pi^2\epsilon} + O(e_R^2), \\ \frac{d\ln Z_e}{de_R} &= -\frac{e_R}{4\pi^2\epsilon} + O(e_R^2). \end{aligned} \tag{4.67}$$

The bare parameters are

$$\begin{aligned} \psi_0 &= Z_\psi^{1/2}\psi_R, \\ A_0 &= Z_A^{1/2}A_R, \\ m_0 &= Z_m Z_\psi^{-1} m_R, \\ e_0 &= Z_e Z_\psi^{-1} Z_A^{-1/2} e_R \tilde{\mu}^{\epsilon/2}. \end{aligned} \tag{4.68}$$

The RG equation for e_0 is

$$\frac{d\ln e_0}{d\ln\mu} = \left(\frac{\ln Z_e}{de_R} - \frac{\ln Z_\psi}{de_R} - \frac{1}{2}\frac{\ln Z_A}{de_R} + \frac{1}{e_R}\right)\frac{de_R}{d\ln\mu} + \frac{\epsilon}{2} = 0. \tag{4.69}$$

The beta function is

$$\beta(e_R) = \frac{de_R}{d\ln\mu} = -\frac{\epsilon}{2}e_R + \frac{e_R^3}{12\pi^2} + O(e_R^4). \tag{4.70}$$

The RG equation for m_0 is

$$\frac{d\ln m_0}{d\ln\mu} = \left(\frac{\ln Z_m}{de_R} - \frac{\ln Z_\psi}{de_R}\right)\frac{de_R}{d\ln\mu} + +\frac{1}{m_R}\frac{dm_R}{d\ln\mu} = 0. \tag{4.71}$$

The anomalous dimension of mass is

$$\gamma_m = \frac{d\ln m_R}{d\ln\mu} = -\frac{3e_R^2}{8\pi^2} + O(e_R^3). \tag{4.72}$$

Chapter 5

The Standard Model

The standard model described the interaction of quarks, leptons and gauge bosons. The gauge boson intermediate electromagnetic, weak, and strong interaction. In the standard model, the gauge group is $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$, where $\mathrm{SU}(3)$ concerns the color degrees of freedom, the $\mathrm{SU}(2) \times \mathrm{U}(1)$ is gauge group of the electroweak interaction.

The matters in the standard model consists of quarks and leptons, each of them have three generations:

Generation	Quarks		Leptons	
I	u	d	e	ν_e
II	c	s	μ	ν_μ
III	t	b	τ	ν_τ

(5.1)

The gauge theory is written as a nonabelian gauge theory, or the Yang-Mills theory, which forbid the gauge field to have mass term. Also, the weak interaction breaks the parity – it only involve the left-handed spinor field, and such single-handed gauge symmetry forbid the mass term for fermions. In order to obtain the mass, the Higgs field should be introduced, which involves the spontaneous symmetry breaking.

5.1 Nonabelian Gauge Theory

The U(1)-gauge-invariant QED Lagrangian consists of the gauge part and the matter part:

$$\begin{aligned} \mathcal{L}_{\mathrm{QED}} &= \mathcal{L}_\gamma + \mathcal{L}_\psi \\ &= -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi. \end{aligned} \tag{5.2}$$

where the gauge-covariant derivative

$$D_\mu = \partial_\mu - iqA_\mu \tag{5.3}$$

and the field strength tensor

$$F_{\mu\nu} = \frac{i}{q}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{5.4}$$

are all manifestly invariant under U(1) gauge transformation.

The nonabelian gauge theory generalize the gauge group to be a general (nonabelian) Lie group parametrized as

$$U(x) = \exp\left[-ig\pi^a(x)T^a\right], \tag{5.5}$$

where $\{T^a\}$ are the generators of the gauge group. For the nonabelian Lie group, the commutation relation, the Lie algebra, specify the structure of the infinitesimal gauge transformation.

5.1.1 Yang-Mills Theory

The gauge-invariant Lagrangian can be:

$$\begin{aligned}\mathcal{L}_{\text{n.a.}} &= \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{matter}} \\ &= -\frac{1}{4}F^{a,\mu\nu}F^a_{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi.\end{aligned} \tag{5.6}$$

The form of Yang-Mills Lagrangian similar to the Maxwell field:

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4}F^{a,\mu\nu}F^a_{\mu\nu}. \tag{5.7}$$

To get gauge-invariant Lagrangian for the matter fields, we simply replace the ordinary derivative with the gauge-covariant derivative:

$$\mathcal{L}_{\text{matter}} = \begin{cases}\psi(i\gamma^\mu D_\mu - m)\psi & \text{Fermion} \\ (D^\mu\phi)^\dagger D_\mu\phi - m^2\phi^\dagger\phi & \text{Scalar}\end{cases}. \tag{5.8}$$

The covariant derivative D_μ is now defined as

$$D_\mu = \partial_\mu - igA^a_\mu T^a, \tag{5.9}$$

and the field-strength tensor is then

$$\begin{aligned}F^a_{\mu\nu} &= \frac{i}{g}[D_\mu, D_\nu]^a \\ &= \partial_\mu A^a_\nu - \partial_\nu A^a_\nu - igf^{abc}A^b_\mu A^c_\nu,\end{aligned} \tag{5.10}$$

The field strength tensor $F_{\mu\nu}$ transform as

$$F_{\mu\nu}(x) \to U(x)F_{\mu\nu}(x)U^\dagger(x). \tag{5.11}$$

The Lagrangian

$$\mathcal{L}_{\text{GI}} = -\frac{1}{2}\text{Tr}\left[F_{\mu\nu}F^{\mu\nu}\right]$$

is manifestly gauge-invariant. Also, we can also choose the renormalization of the generators so that $\text{Tr}[T^aT^b] = \delta^{ab}$. In this way, when express the field strength as $F_{\mu\nu} = F^a_{\mu\nu}T^a$, the above gauge-invariant Lagrangian is exactly the Yang-Mills Lagrangian $\mathcal{L}_{\text{YM}}$.

The covariant derivative transforms as

$$D_\mu \to U(x)D_\mu U^\dagger(x) = \partial_\mu - ig\left[U(x)A_\mu(x)U^\dagger(x) + \frac{i}{g}U(x)\partial_\mu U^\dagger(x)\right]. \tag{5.12}$$

The matter part $\mathcal{L}_{\text{matter}}$ is invariant if we define the transformation of the gauge field as:

$$A_\mu(x) \to U(x)A_\mu(x)U^\dagger(x) + \frac{i}{g}U(x)\partial_\mu U^\dagger(x). \tag{5.13}$$

Similar to QED case, the gauge redundancy causes trouble in quantization. We follow the Faddeev-Popov procedure to quantize the Yang-Mills Lagrangian.

The partition function of the Yang-Mills Lagriangian is

$$Z = \int D[A] \exp\left(i\int d^4x\mathcal{L}[A]\right) = \int D[A_f]\ \mathcal{V}_G[A] \exp\left(i\int d^4x\mathcal{L}_{\text{YM}}[A_f]\right) \tag{5.14}$$

where A_f is the gauge-fixed field, and $\mathcal{V}_G[A]$ denotes the phase volume of the gauge redundancy. The task is to determine $\mathcal{V}_G[A]$. This can be done by introducing a gauge-fixing condition:

$$Z = \int D[\pi]\mathcal{V}_G[A] \int D[A]\delta(G[A^\pi]) \exp\left(i\int d^4x\mathcal{L}_{\text{YM}}[A^\pi]\right), \tag{5.15}$$

where the gauge-fixing function is

$$G[A] = \partial^\mu A_\mu - X. \tag{5.16}$$

Here $D[\pi]$ is the Haar measure over the Lie group. A^π denotes the gauge transformed field. Since the Lagrangian is gauge-invariant, we can simply drop a superscript, and the integral over π field is just a number depending on A:

$$\int D[\pi]\delta(G[A]) = \det\left(\frac{\delta G[A^\pi]}{\delta\pi}\right)^{-1} \tag{5.17}$$

The infinitesimal version of the gauge transformation (5.13) is

$$A^a_\mu \to A^a_\mu + \frac{1}{g}\partial_\mu\pi^a + f^{abc}A^b_\mu\pi^b \equiv A_\mu + \frac{1}{g}D_\mu\pi^a, \tag{5.18}$$

where the covariant derivative on the operator involves the adjoint representation:

$$D_\mu\pi^a = \partial_\mu\pi^a + gf^{abc}A^b_\mu\pi^c = \partial_\mu - igA^a_\mu\left(T^a_{\text{adj}}\right)_{bc}\pi^c. \tag{5.19}$$

The integral is then formally the operator determinant, and we thus know

$$\mathcal{V}_G[A] = \det(\partial^\mu D_\mu). \tag{5.20}$$

This functional determinant can be regarded as the functional integral on a "ghost" field:

$$\det(\partial^\mu D_\mu) = \int D[\bar{c}, c] \exp\left[i\int d^4x\ \bar{c}(-\partial^\mu D_\mu)c\right]. \tag{5.21}$$

The gauge-fixing condition (5.16) impose a hard constraint on the field configuration. For our convenience, we

$$\begin{aligned} Z &= \int DXe^{-\frac{X}{2\xi}} \int D[A_X] \exp\left\{i\int d^4x\left[\mathcal{L}_{\text{YM}} - \bar{c}\partial^\mu D_\mu c\right]\right\} \\ &= \int D[A]e^{-\frac{i}{2\xi}(\partial^\mu A_\mu)^2} \exp\left\{i\int d^4x\left[\mathcal{L}_{\text{YM}} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2 - \bar{c}\partial^\mu D_\mu c\right]\right\}. \end{aligned} \tag{5.22}$$

The gauge-fixing results in a modified Lagrangian with gauge-fixing term and ghost term:

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{gh}},\\ \mathcal{L}_{\text{gf}} &= -\frac{1}{2\xi}(\partial^\mu A_\mu)^2,\\ \mathcal{L}_{\text{gh}} &= -\bar{c}\partial^\mu D_\mu c.\end{aligned} \tag{5.23}$$

In the standard model, the weak and strong interaction is described by the SU(2) and SU(3) Yang-Mills theory respectively, so we mainly focus on the those two cases.

5.1.2 SU(3) Lie Algebra

For the fundamental representation, the generators for the SU(3) is one-half of the *Gell-Mann matrices* $T^a = \frac{1}{2}\lambda^a$, $a = 1, \ldots, 8$, where the 8 matrices are:

$$\begin{aligned}
\lambda^1 &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \lambda^2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\\
\lambda^3 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \lambda^4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},\\
\lambda^5 &= \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \quad \lambda^6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},\\
\lambda^7 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.
\end{aligned} \tag{5.24}$$

The commutation relation among the generator defines the *structure constant*:

$$\left[T^a, T^b\right] = i f^{abc} T^c. \tag{5.25}$$

The product of two SU(3) generators is

$$T^a T^b = \frac{1}{6}\delta^{ab} + \frac{1}{2}d^{abc}T^c + \frac{i}{2}f^{abc}T^c, \tag{5.26}$$

where

$$\frac{i}{2}f^{abc} = \operatorname{Tr}\left([T^a, T^b]T^c\right), \quad \frac{1}{2}d^{abc} = \operatorname{Tr}\left(\{T^a, T^b\}T^c\right). \tag{5.27}$$

The cycling properties of the trace expression leads to

$$f^{abc} = f^{bca} = f^{cab}, \quad d^{abc} = d^{bca} = d^{cab}. \tag{5.28}$$

It also manifest that f^{abc} is totally anti-symmetric and d^{abc} totally symmetric. This also leads to the trace identities:

$$\begin{aligned}\operatorname{Tr}(T^aT^b) &= \frac{1}{2}\delta^{ab},\\ \operatorname{Tr}(T^aT^bT^c) &= \frac{1}{4}(d^{abc} + i f^{abc}),\\ \operatorname{Tr}(T^aT^bT^cT^d) &= \frac{1}{12}\delta^{ab}\delta^{cd} + \frac{1}{8}(d^{abe} + i f^{abe})(d^{cde} + i f^{cde}).\end{aligned} \tag{5.29}$$

The generator in representation labeled by R is denoted as T_R^a. One important representation apart from the fundamental representation is the adjoint representation, which is a 8-dimensional representation defined as

$$(T_{\text{adj}}^a)^{bc} = -if^{abc}. \tag{5.30}$$

Also, different representations can be characterized by the quadratic Casimir $C_2(R)$ defined as

$$T_R^a T_R^a = C_2(R)\mathbb{1}\,. \tag{5.31}$$

We know

$$C_2(\text{fund}) = \frac{4}{3}, \quad C_2(\text{adj}) = 3. \tag{5.32}$$

This also means

$$f^{acd}f^{bcd} = 3\delta^{ab}. \tag{5.33}$$

Roots and Weights

For a simple Lie algebra, a standard set of generators (called the *Cartan-Weyl basis*) can be chosen so that it contains a maximal number of mutually commuting subset. The maximum number of such generators is defined as the *rank* of the algebra. The SU(3) Lie algebra is rank-2, with the Cartan-Weyl basis

$$H_1 = \frac{1}{2}\lambda_3, \quad H_2 = \frac{1}{2}\lambda_8 \tag{5.34}$$

The common eigenstates of Cartan sub-algebra is denoted as $\{|\boldsymbol{m}\rangle\}$, where each $\boldsymbol{m}$ is a vector, called the *weight*, defined by

$$H_i|\boldsymbol{m}\rangle = m_i|\boldsymbol{m}\rangle. \tag{5.35}$$

Apart from the Cartan sub-algebra, other generators can be linearly combined to form a standard non-Hermitian basis $\{E_{\boldsymbol{\alpha}}^{\pm}\}$. Each element is labeled by an r-dimensional vectors $\pm\boldsymbol{\alpha}$, called the *root*, which is defined by the commutation relation:

$$[H_i, E_{\boldsymbol{\alpha}}^{\pm}] = \pm\alpha_i E_{\boldsymbol{\alpha}}^{\pm}. \tag{5.36}$$

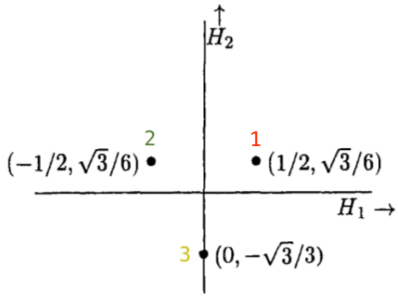
We can also denote $E_{\pm\boldsymbol{\alpha}} \equiv E_{\boldsymbol{\alpha}}^{\pm}$ for notational simplicity. Note that the action of E_α will change the weight,

$$\begin{aligned} H_i E_{\boldsymbol{\alpha}}|\boldsymbol{m}\rangle &= [H_i, E_{\boldsymbol{\alpha}}]|\boldsymbol{m}\rangle + E_{\boldsymbol{\alpha}} H_i|\boldsymbol{m}\rangle \\ &= (m_i + \alpha_i)E_{\boldsymbol{\alpha}}|\boldsymbol{m}\rangle, \end{aligned} \tag{5.37}$$

For SU(3), since $\{H_i\}$ are all diagonal, the common eigenstates are

$$\begin{bmatrix}1\\0\\0\end{bmatrix} = \left|\left(\frac{1}{2}, \frac{\sqrt{3}}{6}\right)\right\rangle, \quad \begin{bmatrix}0\\1\\0\end{bmatrix} = \left|\left(-\frac{1}{2}, \frac{\sqrt{3}}{6}\right)\right\rangle, \quad \begin{bmatrix}0\\0\\1\end{bmatrix} = \left|\left(0, -\frac{\sqrt{3}}{3}\right)\right\rangle. \tag{5.38}$$

These vectors, plotted in a plane, form a equilateral triangle:



(5.39)

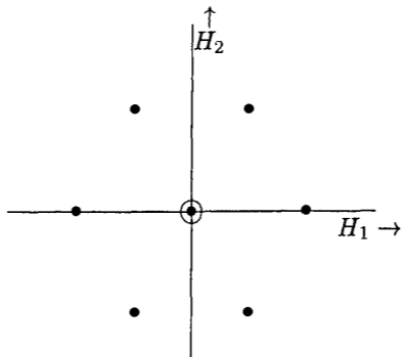
Other generators are labeled by

$$E_{\boldsymbol{\alpha}_1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_{\boldsymbol{\alpha}_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_{\boldsymbol{\alpha}_3} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{5.40}$$

where the root vectors are

$$\boldsymbol{\alpha}_1 = (1, 0), \quad \boldsymbol{\alpha}_2 = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \quad \boldsymbol{\alpha}_3 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right). \tag{5.41}$$

The root vectors plotted in a plane form a regular hexagon:



(5.42)

The roots of $\mathfrak{su}(3)$ are not linearly independent. We can find a set of linear independent roots called the *simple roots*. All other roots can be expressed as a linear combination of simples roots with all positive/negative coefficients. The SU(3) algebra, for example, has $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ as its simple roots.

Irreducible Representations

An irreducible representation (Irrep) of a Lie algebra is entirely specified by the *highest weight* state $|\boldsymbol{M}\rangle$ in the Irrep, which satisfies

$$E_{\boldsymbol{\alpha}}|\boldsymbol{M}\rangle = 0 \tag{5.43}$$

for all simple root. The Irrep then can be constructing by applying the lowering operators to the highest weight state:

$$\text{Irrep} = \text{span}\left\{E^-_{\boldsymbol{\alpha}} E^-_{\boldsymbol{\beta}} \cdots E^-_{\boldsymbol{\gamma}}|\boldsymbol{M}\rangle, \text{ all possible sequence}\right\}. \tag{5.44}$$

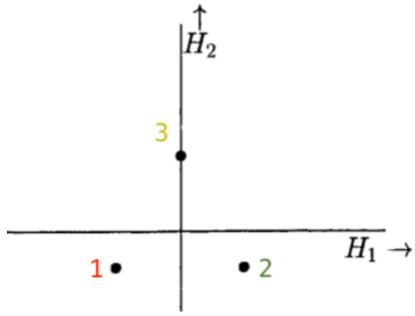
The highest weight state can be systematically constructed using the tensor method. For SU(3), we first need to introduce the complex of the fundamental representation $\bar{3}$, defined by

$$\exp(-i\pi^a T^a_{\bar{3}}) = \exp(-i\pi^a T^a_3)^*, \tag{5.45}$$

which gives

$$T^a_{\bar{3}} = -(T^a_3)^T. \tag{5.46}$$

The definition flip the sign of $\{H_i\}$, the weight plotted in the plane becomes the upside-down triangle:



(5.47)

The highest wight state in $\bar{3}$ is $|3\rangle$. For a general Irrep (which can be labeled by (m,n)-representation), the highest weight states is

$$|\boldsymbol{M}\rangle = \bigotimes_{i=1}^{m} |1\rangle_i \bigotimes_{j=m+1}^{m+n} |3\rangle_j. \tag{5.48}$$

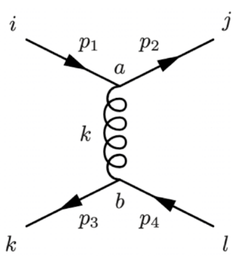
The ladder operators are now

$$E_{\boldsymbol{\alpha}} = \sum_{i=1}^{m} E^{(i)}_{\boldsymbol{\alpha}} - \sum_{j=m+1}^{m+n} E^{(j)}_{-\boldsymbol{\alpha}}. \tag{5.49}$$

The adjoint representation of the SU(3) is the $(1,1)$-representation.

5.1.3 Quantum Chromodynamics

The SU(3) Yang-Mills theory describe the strong interaction, also known as the quantum chromodynamics (QCD). The Feynman rule in QCD is very complicated. Here we just consider the simplest tree-level process $u\bar{d} \to u\bar{d}$, corresponding to the diagram



$$= T^a_{ji} T^a_{kl} \times (\text{QED-like term}), \tag{5.50}$$

where the index i labels the color:

Index	Color	Simbol
1	red	r
2	green	g
3	yellow	y

(5.51)

Since the total color is conserved in the scattering process, we can divide the color degrees of freedom to the singlet and octet, corresponding to

$$3 \otimes \bar{3} = 1 \otimes 8. \tag{5.52}$$

The singlet states is

$$|s\rangle = \frac{1}{\sqrt{3}} \left(|r\bar{r}\rangle + |g\bar{g}\rangle + |y\bar{y}\rangle \right) \tag{5.53}$$

The factor then becomes

$$\left(\frac{1}{\sqrt{3}} \right)^2 (T^a T^a)_{jl} = \frac{4}{9} \delta_{jl}. \tag{5.54}$$

For the octet state $|r\bar{y}\rangle$

$$T^a_{i1} T^a_{3l} = -\frac{1}{6} \delta_{i1} \delta_{l3}. \tag{5.55}$$

In QED, we know that two particle with opposite charge are attractive to each other. From the factors above, we know that color singlet channel is attractive while the color octet channel is repulsive. Indeed, in QCD there is phenomenon named *color confinement*, which means quarks tend to form composite colorless bound state.

5.1.4 Gauge Group of the Standard Model

The standard model has SU(3) × SU(2) × U(1) gauge symmetry. For a general field ϕ_{iI}, where i labels the electroweak degrees of freedom and I labels the color degrees of freedom. The covariant derivative for ϕ_{iI} is then

$$D_\mu = \partial_\mu - i \left(g_1 B_\mu Y + g_2 W^a_\mu T^a_2 + g_3 A^b_\mu T^b_3 \right), \tag{5.56}$$

where the operator Y, T^a_2 and T^b_3 is depend on the representation of U(1), SU(2) and SU(3) groups respectively.

The weak interaction only involve left-handed spinor, so the largest fermion sector is

$$Q_I = \left[\begin{pmatrix} u_L \\ d_L \end{pmatrix} \begin{pmatrix} c_L \\ s_L \end{pmatrix} \begin{pmatrix} t_L \\ b_L \end{pmatrix} \right]_I. \tag{5.57}$$

Each Q_I is in the $\left(3, 2, \frac{1}{6}\right)$ representation, where 3 is the fundamental representation of SU(3), 2 is the fundamental representation of SU(2), and $\frac{1}{6}$ is the hypercharge of the U(1) gauge group. For example the sector of first generation left-handed quarks are

$$Q_1 = \begin{bmatrix} u^r_L & u^g_L & u^y_L \\ d^r_L & d^g_L & d^y_L \end{bmatrix}. \tag{5.58}$$

Note that the electric charge satisfies

$$q = I_3 + Y, \tag{5.59}$$

where I_3 is the z-component of the isospin.

Besides, the left-handed leptons also involves the weak interaction but not the strong interaction

$$L_I = \left[\begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix} \begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix} \begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix} \right]_I, \tag{5.60}$$

and thus each L_I is in the $\left(1, 2, \frac{1}{2}\right)$ representation.

The right-handed quarks form $\left(3, 1, \frac{2}{3}\right)$ and $\left(3, 1, -\frac{1}{3}\right)$ representations, the right-handed electrons/muons/tauons form $(1, 1, -1)$ representations, and the right-handed neutrinos form trivial $(1, 1, 0)$ representations.

In addition to the fermion, there is also a scalar Higgs field in the representation $\left(1, 2, \frac{1}{2}\right)$. Together, we say that the standard model contains the representations (only fermions in the first generation are listed):

Particles		Representation
Quarks	Q_1	$(3, 2, 1/6)$
	u_R	$(3, 1, 2/3)$
	d_R	$(3, 1, -1/3)$
Leptons	L_1	$(1, 2, -1/2)$
	e_R	$(1, 1, -1)$
	ν_{eR}	$(1, 1, 0)$
Higgs	φ	$(1, 2, 1/2)$

(5.61)

5.2 Lagrangian of the Standard Model

5.2.1 Electroweak Symmetry Breaking

Beside the quarks, leptons and gauge bosons, a doublet scalar field (Higgs) is also introduced in order to generate mass for the particles:

$$\varphi \equiv \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}, \tag{5.62}$$

which is also an isospin doublet with hyperchage $Y = \frac{1}{2}I$:

Field	I_3	Y	q
φ_1	$1/2$	$1/2$	1
φ_2	$-1/2$	$1/2$	0

(5.63)

The Lagrangian of the Higgs field is

$$\mathcal{L} = -\frac{1}{4}(W^a_{\mu\nu})^2 - \frac{1}{4}B^2_{\mu\nu} + (D^\mu\varphi)^\dagger D_\mu\varphi - V(\varphi), \tag{5.64}$$

where the field strength tensors are

$$\begin{aligned} W^a_{\mu\nu} &= \partial_\mu W^1_\nu - \partial_\nu W^1_\mu + g_2(W^2_\mu W^3_\nu - W^3_\mu W^2_\nu), \\ W^a_{\mu\nu} &= \partial_\mu W^1_\nu - \partial_\nu W^1_\mu + g_2(W^3_\mu W^1_\nu - W^1_\mu W^3_\nu), \\ W^a_{\mu\nu} &= \partial_\mu W^1_\nu - \partial_\nu W^1_\mu + g_2(W^1_\mu W^2_\nu - W^2_\mu W^1_\nu), \\ B^a_{\mu\nu} &= \partial_\mu B^1_\nu - \partial_\nu B^1_\mu, \end{aligned} \tag{5.65}$$

and the Higgs interaction is

$$V(\varphi) = \frac{\lambda}{4}\left(\varphi^\dagger\varphi - \frac{v^2}{2}\right)^2. \tag{5.66}$$

This potential gives φ a nonzero vacuum expectation value (VEV). We can make a gauge transformation so that

$$\varphi_0 = \langle 0|\varphi|0\rangle = \frac{v}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{5.67}$$

The kinetic Lagrangian is described by the SU(2) gauge-invariant Lagrangian

$$\mathcal{L}_{\text{kin}} = (D^\mu\varphi)^\dagger D_\mu\varphi, \tag{5.68}$$

where the covariant derivative is

$$D_\mu = \partial_\mu - i\left(g_1 B_\mu Y + g_2 W_\mu^a T_2^a\right). \tag{5.69}$$

We can introducing the Weinberg angle

$$\theta_W \equiv \arctan\frac{g_1}{g_2}, \tag{5.70}$$

and the fields

$$\begin{aligned} W_\mu^\pm &\equiv \frac{1}{\sqrt{2}}(W_\mu^1 \mp iA_\mu^2), \\ Z_\mu &\equiv \cos\theta_W W_\mu^3 - \sin\theta_W B_\mu, \\ A_\mu &\equiv \sin\theta_W W_\mu^3 + \cos\theta_W B_\mu. \end{aligned} \tag{5.71}$$

The reverse transformation for W_μ^3 and B_μ is

$$\begin{aligned} W_\mu^3 &= \cos\theta_W Z_\mu + \sin\theta_W A_\mu, \\ B_\mu &= -\sin\theta_W Z_\mu + \cos\theta_W A_\mu. \end{aligned} \tag{5.72}$$

The off-diagonal part of the Lie algebra becomes

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & W_\mu^+ \\ W_\mu^- & 0 \end{bmatrix}, \tag{5.73}$$

and the diagonal part becomes

$$\begin{aligned} g_2 W_\mu^3 T_2^z + g_1 B_\mu Y &= g_2(\cos\theta_W Z_\mu + \sin\theta_W A_\mu)T_2^3 + g_1 Y(-\sin\theta_W Z_\mu + \cos\theta_W A_\mu) \\ &= g_2\cos\theta_W A_\mu\left(T_2^3 + Y\right) + eZ_\mu\left(\cot\theta_W T_2^3 - \tan\theta_W Y\right) \\ &= eA_\mu q + \frac{e}{\sin\theta_W\cos\theta_W}Z_\mu\left(\cos^2\theta_W T_2^3 - \sin^2\theta_W Y\right) \\ &= eA_\mu q + \frac{e}{\sin\theta_W\cos\theta_W}Z_\mu\left(T_2^3 - \sin^2\theta_W q\right). \end{aligned} \tag{5.74}$$

Note that the gauge field A_μ describe QED, so we identify the coupling constant with the electric charge:

$$e = g_2\sin\theta_W = g_1\cos\theta_W. \tag{5.75}$$

For Higgs field,

$$T_2^3 = \frac{1}{2}\sigma^z, \quad Y = \frac{1}{2}I, \quad q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{5.76}$$

$T_2^3 = \frac{1}{2}\sigma^z$, $Y = \frac{1}{2}I$, so With the new definition of the field, the Lie algebra of becomes

$$g_2 W_\mu^a T_2^a + g_1 B_\mu Y = \frac{e}{2\sin\theta_W} \begin{bmatrix} 2\sin\theta_W A_\mu + \frac{\cos 2\theta_W}{\cos\theta_W} Z_\mu & \sqrt{2}W_\mu^+ \\ \sqrt{2}W_\mu^- & \frac{1}{\cos\theta_W} Z_\mu \end{bmatrix}. \tag{5.77}$$

We can now expand the kinetic term to get the effective mass of the the gauge field in unitary gauge. The mass term is the quadratic part of φ:

$$\begin{aligned} \mathcal{L}_{\text{mass}} &= \varphi_0^\dagger \left(g_2 W_\mu^a T_2^a + g_1 B_\mu Y\right)^2 \varphi_0 \\ &= \frac{e^2 v^2}{8\sin^2\theta_W} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \cdots & \sqrt{2}W_\mu^+ \\ \sqrt{2}W_\mu^- & \frac{1}{\cos\theta_W} Z_\mu \end{bmatrix}^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= m_W W^{+\mu} W_\mu^- + \frac{1}{2} m_Z^2 Z^\mu Z_\mu, \end{aligned} \tag{5.78}$$

where the mass for W and Z bosons are

$$m_W = \frac{ev}{2\sin\theta_W}, \quad m_Z = \frac{ev}{2\sin\theta_W\cos\theta_W} = \frac{m_W}{\cos\theta_W}. \tag{5.79}$$

Note that the photon field A_μ does not obtain mass from the Higgs field.

5.2.2 Higgs-Gauge Sector

We are now going to express the Lagrangian (5.64) in terms of the fields we have just introduced. In the unitary gauge, the Higgs field can be written as

$$\varphi = \frac{1}{\sqrt{2}} \begin{bmatrix} v + H(x) \\ 0 \end{bmatrix}, \tag{5.80}$$

where $H(x)$ is a real scalar field. The interaction terms gives

$$\begin{aligned} V(H) &= \frac{\lambda v^2}{4} H^2 + \frac{\lambda v}{4} H^3 + \frac{\lambda}{16} H^4 \\ &= \frac{1}{2} m_H^2 H^2 + \frac{m_H^2}{2v} H^3 + \frac{m_H^2}{8v^2} H^4, \end{aligned} \tag{5.81}$$

where we have defined the Higgs mass

$$m_H^2 = \frac{\lambda v^2}{2}. \tag{5.82}$$

Also, the fluctuation around the φ_0 also create coupling between Higgs field and the gauge fields. We can simply replace v^2 with $(v+H)^2$ in the gauge field mass term to capture the coupling. Thus, the Lagrangian is

$$\mathcal{L}_{\text{H}} = \partial^\mu H^\dagger \partial_\mu H - V(H) + \left(m_W W^{+\mu} W_\mu^- + \frac{1}{2} m_Z^2 Z^\mu Z_\mu\right) \left(H^2 + 2vH\right). \tag{5.83}$$

Now consider the Lagrangian for the gauge field

$$\mathcal{L}_{\mathrm{G}} = -\frac{1}{4}(W^a_{\mu\nu})^2 - \frac{1}{4}B^2_{\mu\nu} + m_W W^{+\mu}W^-_\mu + \frac{1}{2}m_Z^2 Z^\mu Z_\mu. \tag{5.84}$$

We can regroup the field strength as

$$\begin{aligned} W^+_{\mu\nu} &= D_\mu W^+_\nu - D_\nu W^+_\mu, \\ W^-_{\mu\nu} &= D^\dagger_\mu W^-_\nu - D^\dagger_\nu W^-_\mu, \\ W^3_{\mu\nu} &= \sin\theta_W F_{\mu\nu} + \cos\theta_W Z_{\mu\nu} - ig_2(W^+_\mu W^-_\nu - W^-_\mu W^+_\nu), \\ B^a_{\mu\nu} &= \cos\theta_W F_{\mu\nu} - \sin\theta_W Z_{\mu\nu}, \end{aligned} \tag{5.85}$$

where we have defined

$$\begin{aligned} W^\pm_{\mu\nu} &= \frac{1}{\sqrt{2}}(W^1_{\mu\nu} \mp iW^2_{\mu\nu}), \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu, \\ Z_{\mu\nu} &= \partial_\mu Z_\nu - \partial_\nu Z_\mu. \end{aligned} \tag{5.86}$$

The covariant derivative for $W^\pm$ field is

$$\begin{aligned} D_\mu &= \partial_\mu - ig_2 W^3_\mu \\ &= \partial_\mu - ig_2\left(\sin\theta_W A_\mu + \cos\theta_W Z_\mu\right) \\ &= \partial_\mu - ie\left(A_\mu + \cot\theta_W Z_\mu\right) \end{aligned} \tag{5.87}$$

So the Lagrangian for the gauge field is

$$\begin{aligned} \mathcal{L}_{\mathrm{G}} = & -\frac{1}{4}(2W^+_{\mu\nu}W^{-\mu\nu} + W^3_{\mu\nu}W^{3\mu\nu}) - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} + m_W W^{+\mu}W^-_\mu + \frac{1}{2}m_Z^2 Z^\mu Z_\mu \\ = & -\frac{1}{4}(F_{\mu\nu}F^{\mu\nu} + Z_{\mu\nu}Z^{\mu\nu}) - D^\mu W^{+\nu} D^\dagger_\mu W^-_\nu + D^\mu W^{+\nu} D^\dagger_\nu W^-_\mu \\ & + ie(F^{\mu\nu} + \cot\theta_W Z^{\mu\nu})W^+_\mu W^-_\nu + m_W W^{+\mu}W^-_\mu + \frac{1}{2}m_Z^2 Z^\mu Z_\mu \\ & - \frac{e^2}{2\sin^2\theta_W}\left(W^{+\mu}W^-_\mu W^{+\nu}W^-_\nu - W^{+\mu}W^+_\mu W^{-\nu}W^-_\nu\right). \end{aligned} \tag{5.88}$$

We remark that in the unitary gauge, there is no redundancy of the gauge field. No ghost field is needed in the quantization procedure.

5.2.3 Yukawa Coupling and Fermion Mass

The fermion fields in the electroweak interaction

Field	I_3	Y	q
u_L	$1/2$	$1/6$	$2/3$
d_L	$-1/2$	$1/6$	$-1/3$
u_R	0	$2/3$	$2/3$
d_R	0	$-1/3$	$-1/3$
e_L	$-1/2$	$-1/2$	-1
ν_{eL}	$1/2$	$-1/2$	0
e_R	0	-1	-1
ν_{eR}	0	0	0

(5.89)

The kinetic term for the fermion Lagrangian is

$$\begin{aligned}\mathcal{L}_{\text{kin}} &= iQ^\dagger_J\sigma^\mu D_\mu Q_J + iu^\dagger_{RJ}\bar{\sigma}^\mu D_\mu u_{RJ} + id^\dagger_{RJ}\bar{\sigma}^\mu D_\mu d_{RJ} \\ &\quad + iL^\dagger_J\sigma^\mu D_\mu L_J + ie^\dagger_{RJ}\bar{\sigma}^\mu D_\mu e_{RJ} + i\nu^\dagger_{RJ}\bar{\sigma}^\mu D_\mu \nu_{RJ},\end{aligned} \tag{5.90}$$

where we have use the index J to label the generation of the fermion:

$$\begin{aligned} u_{RJ} &= \{u_R, c_R, t_R\}, \quad d_{RJ} = \{d_R, s_R, b_R\}, \\ e_{RJ} &= \{e_R, \mu_R, \tau_R\}, \quad \nu_{RJ} = \{\nu_{eR}, \nu_{\mu R}, \nu_{\tau R}\}. \end{aligned} \tag{5.91}$$

The left-handed SU(2) gauge symmetry forbid any fermion mass term, which is incompatible with the reality. However, fermion can obtain fermion mass by introducing Yukawa coupling between the Higgs and fermion fields.

Quark Sector

The Yukawa coupling between quark and Higgs is

$$\mathcal{L}_{q,\text{Yuk}} = -\varepsilon^{ij} Y^u_{IJ} Q^\dagger_{iI}\varphi^\dagger_j u_{RJ} - Y^d_{IJ} Q^\dagger_I \varphi d_{RJ} + h.c.. \tag{5.92}$$

The first term corresponds to

$$\left(\bar{3}, \bar{2}, -\frac{1}{6}\right) \times \left(1, \bar{2}, -\frac{1}{2}\right) \times \left(3, 1, \frac{2}{3}\right) = (1,1,0) \oplus \cdots\,, \tag{5.93}$$

with ε^{ij} being the Clebsch-Gordan coefficient, and the second term corresponds to

$$\left(\bar{3}, \bar{2}, -\frac{1}{6}\right) \times \left(1, 2, \frac{1}{2}\right) \times \left(3, 1, -\frac{1}{3}\right) = (1,1,0)\,. \tag{5.94}$$

Now substitute φ with

$$\varphi = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ v + H \end{bmatrix}. \tag{5.95}$$

The Yukawa potential then gives quark mass term:

$$\mathcal{L}_{Q,\text{Yuk}} = -\frac{v+H}{\sqrt{2}} \left(Y^u_{IJ} u^\dagger_{LI} u_{RJ} + Y^d_{IJ} d^\dagger_{LI} d_{RJ} \right) + h.c.. \tag{5.96}$$

We can then use a basis transformation (singular value decomposition) to diagonalize the coupling matrix:

$$Y^u_{IJ} = U_u M_u K^\dagger_u, \quad Y^d_{IJ} = U_d M_d K^\dagger_d, \tag{5.97}$$

where M_u and M_d are diagonal matrices. We can change the basis

$$\begin{aligned} u_L &\to U_u u_L, \quad u_R \to K_u u_R, \\ d_L &\to U_d d_L, \quad d_R \to K_d d_R, \end{aligned} \tag{5.98}$$

and define the diagonal masses:

$$m_{u_I} = \frac{v}{\sqrt{2}} (M_u)_{II}, \quad m_{d_I} = \frac{v}{\sqrt{2}} (M_d)_{II}, \tag{5.99}$$

then the Lagrangian in the quark sector can be expressed as:

$$\begin{aligned}\mathcal{L}_Q = & \sum_I \bar{u}_I \left(i\not{D}^c - m_{u_I}\right) u_I + \sum_I \bar{d}_I \left(i\not{D}^c - m_{d_I}\right) d_I \\ & - \frac{H}{v} \sum_I \left(m_{u_I}\bar{u}_I u_I + m_{d_I}\bar{d}_I d_I\right) + \mathcal{L}_{Q-G},\end{aligned} \tag{5.100}$$

where $\not{D}^c$ is the covariant derivative in QCD (neglecting the electroweak gauge), and $\mathcal{L}_{Q-G}$ denotes the coupling between quarks and the gauge fields.

Lepton Sector

The coupling between Higgs and e_R has the general Yukawa form:

$$\mathcal{L}_{e,\text{Yuk}} = -Y^e_{IJ} L^\dagger_{iI} \varphi_j e_{RiJ} + h.c., \tag{5.101}$$

where y_{IJ} is the Hermitian coupling matrix on generation space. The Yukawa coupling term corresponds to

$$\left(\bar{2}, \frac{1}{2}\right) \times \left(2, \frac{1}{2}\right) \times (1, -1) = (1, 0)\,, \tag{5.102}$$

which is indeed a singlet under gauge transformation (it is also a singlet under Lorentz transformation). The Higgs field then produce the term

$$\mathcal{L}_{e,\text{Yuk}} = -\frac{Y^e_{IJ}}{\sqrt{2}}(v+H) e^\dagger_{LI} e_{RJ} + h.c., \tag{5.103}$$

We then diagonalize the coupling matrix:

$$Y^e_{IJ} = \left[U_e M_e K_e^\dagger\right]_{IJ}, \tag{5.104}$$

where M_e is diagonal. We can then change the basis by

$$\begin{aligned} e_R &\to K_e e_R, \\ e_L &\to U_e e_L, \end{aligned} \tag{5.105}$$

Then the coupling gives the mass:

$$\mathcal{L}_{e,\text{Yuk}} = -\sum_I m_{e_I} \bar{e}_I e_I - \frac{1}{v} \sum_I m_{e_I} H \bar{e}_I e_I, \tag{5.106}$$

where

$$m_{e_I} = \frac{v}{\sqrt{2}} (M_e)_{II}. \tag{5.107}$$

Now we consider the neutrino mass, which is much smaller than electrons. The neutrinos can similarly obtain mass by coupling to Higgs field. However, as there is no gauge symmetry constraint on neutrinos, the Yukawa coupling can be

$$\mathcal{L}_{\nu,\text{Yuk}} = -Y^\nu_{IJ} \varepsilon^{ij} L_{iI} \varphi_j \nu_{RJ} - \frac{1}{2} M_{IJ} \nu_{RI} \nu_{RJ} + h.c.. \tag{5.108}$$

The first term corresponds to

$$\left(2,-\frac{1}{2}\right)\times\left(2,\frac{1}{2}\right)\times(1,0)=(1,0)\oplus(3,0), \tag{5.109}$$

and ε^{ij} is the Clebsch-Gordan coefficients. The second term is automatically a gauge singlet. We note here that the left-handed neutrino do not participate in any interaction, so we can neglect it in the standard model. The only role it plays is to generate neutrino mass.

The Yukawa potential gives

$$\mathcal{L}_{\nu,\text{mass}}=-\frac{1}{2}\begin{bmatrix}\nu_L & \nu_R\end{bmatrix}\begin{bmatrix}0 & Y^{\nu}\\ Y^{\nu\dagger} & M\end{bmatrix}\begin{bmatrix}\nu_L\\ \nu_R\end{bmatrix}. \tag{5.110}$$

If $M\gg m$, the eigenvalues of the mass term will be a huge masses plus some tiny masses. In general, the low-energy part of the mass term is

$$\mathcal{L}_{\nu,\text{mass}}=-\frac{1}{2}(M_\nu)_{IJ}\left(\nu_{LI}\nu_{LJ}+\nu_{LI}^{\dagger}\nu_{LJ}^{\dagger}\right)\left(1+\frac{H}{v}\right)^2, \tag{5.111}$$

where

$$(M_\nu)_{IJ}=\frac{v^2}{2}\left(Y^{\nu T}M^{-1}Y\right)_{IJ}. \tag{5.112}$$

We can also diagonalize the matrix M_ν by a unitary matrix U_ν, corresponding to

$$\nu_L\to U_\nu\nu_L. \tag{5.113}$$

With the discussion above, we can write down the Lagrangian in the lepton sector:

$$\begin{aligned}\mathcal{L}_L=&\sum_I\bar{e}_I(i\partial\!\!\!/-m_{e_I})e_I+\sum_I\left[i\nu_{LI}^{\dagger}\bar{\sigma}^{\mu}\partial_\mu\nu_{LI}-\frac{1}{2}m_{\nu_I}\left(\nu_L\nu_L+\nu_L^{\dagger}\nu_L^{\dagger}\right)\right]\\&-\frac{H}{v}\sum_I\left(m_{e_I}\bar{e}_Ie_I+m_{\nu_I}\bar{\nu}_{LI}\nu_{LI}\right)+\mathcal{L}_{L-G},\end{aligned} \tag{5.114}$$

where $\mathcal{L}_{L-G}$ is the lepton-gauge coupling term.

5.2.4 Gauge-Current Coupling

Using the formula

$$\begin{aligned}g_2\sum_{a=1}^{2}W_\mu^aT_2^a&=\frac{e}{\sqrt{2}\sin\theta_W}\begin{bmatrix}0 & W_\mu^+\\ W_\mu^- & 0\end{bmatrix},\\ g_2W_\mu^3T_2^z+g_1B_\mu Y&=eA_\mu q+\frac{e}{\sin\theta_W\cos\theta_W}Z_\mu\left(T_2^3-\sin^2\theta_Wq\right).\end{aligned} \tag{5.115}$$

The coupling terms $\mathcal{L}_{Q-G}$ and $\mathcal{L}_{L-G}$ are then of the form:

$$\begin{aligned}\mathcal{L}_{Q-G}&=\frac{e}{\sqrt{2}\sin\theta_W}\left(W_\mu^+J_Q^{-\mu}+W_\mu^-J_Q^{+\mu}\right)+eA_\mu J_{\text{EM},Q}^{\mu}+\frac{e}{\sin\theta_W\cos\theta_W}Z_\mu J_{z,Q}^{\mu},\\ \mathcal{L}_{L-G}&=\frac{e}{\sqrt{2}\sin\theta_W}\left(W_\mu^+J_L^{-\mu}+W_\mu^-J_L^{+\mu}\right)+eA_\mu J_{\text{EM},L}^{\mu}+\frac{e}{\sin\theta_W\cos\theta_W}Z_\mu J_{z,L}^{\mu}.\end{aligned} \tag{5.116}$$

Note that the off-diagonal current $J^{\pm}$ will change under the basis transformation U_{IJ}, i.e., the transformation mixes different generations:

$$\begin{aligned}
J_Q^{-\mu} &= (V_Q)_{IJ} u_{LI}^\dagger \bar{\sigma}^\mu d_{LJ}, \\
J_Q^{+\mu} &= (V_Q^\dagger)_{IJ} d_{LI}^\dagger \bar{\sigma}^\mu u_{LJ}, \\
J_L^{-\mu} &= (V_L)_{IJ} \nu_{LI}^\dagger \bar{\sigma}^\mu e_{LJ}, \\
J_L^{+\mu} &= (V_L^\dagger)_{IJ} e_{LI}^\dagger \bar{\sigma}^\mu \nu_{LJ},
\end{aligned} \tag{5.117}$$

where the mixing matrices are defined as

$$V_Q \equiv U_u^\dagger U_d, \quad V_L \equiv U_\nu^\dagger U_e. \tag{5.118}$$

The diagonal part is invariant under the basis transformation:

$$\begin{aligned}
J^\mu_{\text{EM},Q} &= \frac{2}{3} u_{LI}^\dagger \bar{\sigma}^\mu u_{LI} - \frac{1}{3} d_{LI}^\dagger \bar{\sigma}^\mu d_{LI} \\
J^\mu_{\text{EM},L} &= -e_{LI}^\dagger \bar{\sigma}^\mu e_{LI} \\
J^\mu_{z,Q} &= \frac{1}{2} u_{LI}^\dagger \bar{\sigma}^\mu u_{LI} - \frac{1}{2} d_{LI}^\dagger \bar{\sigma}^\mu d_{LI} J_3^\mu - \sin^2\theta_W J^\mu_{\text{EM},Q}, \\
J^\mu_{z,L} &= \frac{1}{2} \nu_{LI}^\dagger \bar{\sigma}^\mu \nu_{LI} - \frac{1}{2} e_{LI}^\dagger \bar{\sigma}^\mu e_{LI} J_3^\mu - \sin^2\theta_W J^\mu_{\text{EM},L}.
\end{aligned} \tag{5.119}$$

Finally, we can sum over the current, and write down the total fermion-gauge coupling:

$$\mathcal{L}_{F-G} = \frac{e}{\sqrt{2}\sin\theta_W}\left(W_\mu^+ J^{-\mu} + W_\mu^- J^{+\mu}\right) + eA_\mu J^\mu_{\text{EM}} + \frac{e}{\sin\theta_W \cos\theta_W} Z_\mu J_z^\mu. \tag{5.120}$$

From the current-gauge coupling, we can recover the effective 4-fermion theory by integrating out the gauge field. In the low energy regime ($p^2 \ll m_W^2$), the vertex function for electron decay is:

$$2\left(\frac{e}{\sqrt{2}\sin\theta_W}\right)^2 J_\mu^+ \frac{g^{\mu\nu}}{p^2 - m_W^2} J_\nu^- \simeq \frac{e^2}{m_W^2 \sin^2\theta_W} J^{+\mu} J_\mu^-. \tag{5.121}$$

The diagonal part of the vertex is

$$\left(\frac{e}{\sin\theta_W \cos\theta_W}\right)^2 J_\mu^z \frac{g^{\mu\nu}}{p^2 - m_Z^2} J_\nu^z \simeq -\frac{e^2}{m_Z^2 \sin^2\theta_W \cos^2\theta_W} J^{z\mu} J_\mu^z. \tag{5.122}$$

Note that $m_Z^2 \cos^2\theta_W = m_W^2$, we can then identify the fermion constant

$$\frac{4G_F}{\sqrt{2}} = -\frac{e^2}{m_W^2 \sin^2\theta_W}, \tag{5.123}$$

and the 4-Fermion vertex is

$$\mathcal{L}_{4F} = -\frac{4G_F}{\sqrt{2}}\left(J^{+\mu} J_\mu^- + J^{z\mu} J_\mu^z\right). \tag{5.124}$$

Chapter 6

Lattice Systems

6.1 Lattice Spins

6.1.1 The Ising Model

The Ising model on the Euclidean space is described by the action

$$S[\{s_k\}] = -K \sum_{\langle ij \rangle} s_i s_j, \quad s_k = \pm 1. \tag{6.1}$$

where $\langle ij \rangle$ is the nearest-neighbor sites and the coupling is

$$K = \beta J = \frac{J}{T}. \tag{6.2}$$

The phase of the Ising model can be revealed by considering the correlation function:

$$G_{ij} \equiv \langle \sigma_i^z \sigma_j^z \rangle = \frac{1}{Z} \sum_{\{s_k\}} e^{-S[\{s_k\}]} s_i s_j, \tag{6.3}$$

where the partition function is

$$Z = \sum_{\{s_k\}} e^{-S[\{s_k\}]} \tag{6.4}$$

Series Expansion

The behavior of correlation in the high- and low- temperature limit can be seen using the series expansion. Consider first the high-temperature limit where $K \to 0$, the partition function can be expanded in different order of K:

$$\begin{aligned} Z &= \sum_{\{s_k\}} \prod_{\langle ij \rangle} \cosh K (1 + s_i s_j \tanh K) \\ &\sim \sum_{\{s_k\}} \prod_{\langle ij \rangle} (1 + s_i s_j K). \end{aligned} \tag{6.5}$$

The only terms that survive the averaging form a non-crossing path from site i to site j. In the small K limit, the main contribution comes from the shortest path, i.e.,

$$G_{ij} \propto K^{-r_{ij}}, \tag{6.6}$$

where r_{ij} is the distance (Manhattan metric) from i to j. We see in high temperature the correlation is exponentially decaying, indicating a disorder phase. Note that for $d = 1$ case, there is only one path from i to j, and the exact result is

$$G_{ij} = \frac{(2\cosh K \tanh K)^{|i-j|}}{(2\cosh K)^{|i-j|}} = (\tanh K)^{|i-j|}, \tag{6.7}$$

independent of the temperature, so the 1d Ising model has only one phase.

For $d \geq 2$ case, in the lower temperature limit, the dominant contribution to the partition function comes from the ferromagnetic configuration. The excitations are the spin domains, whose energy is proportional to their perimeters. For higher dimensional system, the creation of the domain is suppressed, leading to an ordered phase.

6.2 Lattice Fermions

In this section, we consider the system whose Hamiltonian composed of quadratic fermionic operators, i.e.,

$$\hat{H}_{\text{free}} = \sum_{i,j=1}^{N} A_{ij} c_i^\dagger c_j + \frac{1}{2}\sum_{i,j=1}^{N} B_{ij} c_i c_j + \frac{1}{2}\sum_{i,j=1}^{N} B_{ij}^* c_j^\dagger c_i^\dagger, \tag{6.8}$$

where A is a Hermitian matrix, and matrix B is anti-symmetric. Without loss of generality, in the following we always assume that the sum of chemical potential is zero, i.e., $\text{Tr}A = 0$. In the Nambu basis

$$\Psi = (c_1, \ldots, c_N, c_1^\dagger, \ldots, c_N^\dagger)^T, \tag{6.9}$$

the Hamiltonian has the BdG form:

$$\hat{H}_{\text{free}} = \frac{1}{2}\sum_{i,j=1}^{2N} \Psi_i^\dagger H_{ij}^\Psi \Psi_j + \frac{1}{2}\text{Tr}A, \tag{6.10}$$

where the single-body matrix H^Ψ is a $2N \times 2N$ Hermitian matrix

$$H^\Psi = \begin{bmatrix} A & B \\ -B^* & -A^* \end{bmatrix}. \tag{6.11}$$

Note that in the Nambu basis, the single-body Hamiltonian matrix has the particle-hole symmetry

$$P = \sigma_x \mathcal{K}, \quad \Longrightarrow \quad P H^\Psi P = -H^\Psi. \tag{6.12}$$

This means the spectrum of the BdG Hamiltonian is symmetric with respect to zero.

6.2.1 Majorana Representation

The Majorana operators are defined as:

$$\begin{bmatrix} \omega_i \\ \omega_{i+N} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} c_i \\ c_i^\dagger \end{bmatrix}, \quad \begin{bmatrix} c_i \\ c_i^\dagger \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \begin{bmatrix} \omega_i \\ \omega_{i+N} \end{bmatrix}. \tag{6.13}$$

The Majorana operator satisfies the Fermion-like commutation relation

$$\{\omega_i, \omega_j\} = 2\delta_{ij}. \tag{6.14}$$

The fermionic bilinear in the Majorana basis has the form

$$\hat{H} = -\frac{i}{4} \sum_{i,j=1}^{2N} H_{ij} \omega_i \omega_j \tag{6.15}$$

where the single-body matrix H is a $2N \times 2N$ real anti-symmetric matrix:

$$H = \begin{bmatrix} -A^I - B^I & A^R - B^R \\ -A^R - B^R & -A^I + B^I \end{bmatrix}. \tag{6.16}$$

where we have define $A^{R/I} = \mathrm{Re}A/\mathrm{Im}A$ and $B^{R/I} = \mathrm{Re}B/\mathrm{Im}B$. Conversely, if we have a Majorana bilinear

$$\frac{i}{2} \sum_{i,j=1}^{2N} M_{ij} \omega_i \omega_j, \quad M = \begin{bmatrix} M^{11} & M^{12} \\ M^{21} & M^{22} \end{bmatrix}, \tag{6.17}$$

it can be transformed back to ordinary fermionic bilinear (6.8) where

$$\begin{aligned} A &= M^{21} - M^{12} + iM^{11} + iM^{22}, \\ B &= M^{21} + M^{12} + iM^{11} - iM^{22}. \end{aligned} \tag{6.18}$$

A real anti-symmetric matrix can be transformed to standard form by an orthogonal transformation O:

$$\begin{aligned} H &= O \cdot \Sigma(\boldsymbol{\lambda}) \cdot O^T, \\ \Sigma(\boldsymbol{\lambda}) &= i\sigma_y \otimes \mathrm{diag}(\lambda_1, \cdots, \lambda_n). \end{aligned} \tag{6.19}$$

Make the basis transformation

$$\gamma_n = \sum_{j=1}^{2N} O_{jn} \omega_j, \tag{6.20}$$

the Hamiltonian becomes the standard form:

$$\begin{aligned} H &= -\frac{i}{4} \sum_{i=1}^{N} \lambda_i (\gamma_i \gamma_{i+N} - \gamma_{i+N} \gamma_i) \\ &= -\frac{i}{2} \sum_{i=1}^{N} \lambda_i \gamma_i \gamma_{i+N}. \end{aligned} \tag{6.21}$$

Each $\gamma_i \gamma_{i+N}$ pair can then transforms to independent fermion mode:

$$\begin{aligned} -\frac{i}{2} \gamma_i \gamma_{i+N} &= -\frac{i}{2} (d_i + d_i^\dagger)(id_i - id_i^\dagger) \\ &= d_i^\dagger d_i - \frac{1}{2}. \end{aligned} \tag{6.22}$$

6.2.2 Gaussian States

The Fermionic Gaussian states are those states with Gaussian form density operator:

$$\hat{\rho} \propto \exp\left(\frac{i}{2}\sum_{i,j=1}^{2N} M_{ij}\omega_i\omega_j\right), \tag{6.23}$$

where the matrix M is real and anti-symmetric.[1] If we expand the Gaussian form, the density operator becomes a Majorana polynomial:[2]

$$\hat{\rho} = \frac{\mathbb{I}}{2^N} + \sum_{n=1}^{N}\frac{i^n}{2^N}\sum_{1\le i_1<\cdots<i_{2n}\le 2N}\Gamma_{i_1\cdots i_{2n}}\omega_{i_1}\cdots\omega_{i_{2n}}, \tag{6.24}$$

where the coefficient $\Gamma_{i_1\cdots i_{2n}}$ is the $2n$-point correlation function:

$$\Gamma_{i_1\cdots i_{2n}} = i^n\langle\omega_{i_1}\cdots\omega_{i_{2n}}\rangle, \quad i_m \neq i_n. \tag{6.25}$$

In particular, the 2-point function

$$\Gamma_{ij} = i\langle\omega_i\omega_j\rangle - i\delta_{ij} = \frac{i}{2}\langle[\omega_i,\omega_j]\rangle \tag{6.26}$$

is also called the *covariance matrix*. For Gaussian state all $2n$-point correlation is determined by the covariance matrix by the Wick theorem.

Remark 1. Two-point Correlation Function

We are usually more familiar with the ordinary fermionic two-point correlation function $\langle c_i^\dagger c_j\rangle$ or $\langle c_i c_j\rangle$, which is related to the Majorana covariance matrix by:

$$\begin{aligned}
\langle c_i^\dagger c_j\rangle &= \frac{1}{4}(\Gamma_{ij}^{21} - \Gamma_{ij}^{12} + i\Gamma_{ij}^{11} + i\Gamma_{ij}^{22}) + \frac{1}{2}\delta_{ij},\\
\langle c_i c_j\rangle &= \frac{1}{4}(\Gamma_{ij}^{21} + \Gamma_{ij}^{12} + i\Gamma_{ij}^{11} - i\Gamma_{ij}^{22}),\\
\langle c_i^\dagger c_j^\dagger\rangle &= \frac{1}{4}(-\Gamma_{ij}^{21} - \Gamma_{ij}^{12} + i\Gamma_{ij}^{11} - i\Gamma_{ij}^{22}).
\end{aligned} \tag{6.27}$$

The relation of the correlation in each order can be neatly captured by the Grassmannian Gaussian form:

$$\begin{aligned}
\omega(\hat{\rho},\theta) &= \frac{1}{2^N}\exp\left(\frac{i}{2}\sum_{i,j=1}^{2N}\Gamma_{ij}\theta_i\theta_j\right)\\
&= \frac{1}{2^N} + \sum_{n=1}^{N}\frac{i^n}{2^N}\sum_{1\le i_1<\cdots<i_{2n}\le 2N}\Gamma_{i_1\cdots i_{2n}}\theta_{i_1}\cdots\theta_{i_{2n}}.
\end{aligned} \tag{6.28}$$

[1] In particular, any thermal state has this form, with $M = \beta H/2$. The ground state of the free fermion system, though being pure state, can be regarded as the Gaussian state in the limit $M = \lim_{\beta\to\infty}\beta H$.

[2] Note that the coefficient Γ in each order is not the direct expansion of the matrix M, since the direct expansion contains identical Majorana operators. That is, the n-th order expansion of the Majorana Gaussian form may contribute to the $(n-2m)$-th order term in the Majorana polynomial.

When the covariance matrix is obtained, we can use the same routine to canonicalize the skew-symmetric matrix Γ:

$$\Gamma = O \cdot \Sigma(\boldsymbol{\lambda}) \cdot O^T, \quad \tilde{\theta}_n = \sum_i O_{in}\theta_i,$$

and the density matrix in the Grassmann representation is

$$\omega(\hat{\rho}, \theta) = \prod_{n=1}^{N} \left(\frac{1}{2} e^{i\lambda_n \tilde{\theta}_n \tilde{\theta}_{n+N}} \right) = \prod_{n=1}^{N} \left(\frac{1 + i\lambda_n \tilde{\theta}_n \tilde{\theta}_{n+N}}{2} \right). \tag{6.29}$$

This state correspond to a product state $\rho = \otimes_n \rho_n$ where

$$\rho_n = \frac{1}{2} \begin{bmatrix} 1+\lambda_n & 0 \\ 0 & 1-\lambda_n \end{bmatrix}. \tag{6.30}$$

The entanglement entropy is then

$$S = \sum_n S_n = -\sum_n \left[\left(\frac{1+\lambda_n}{2} \right) \ln \left(\frac{1+\lambda_n}{2} \right) + \left(\frac{1-\lambda_n}{2} \right) \ln \left(\frac{1-\lambda_n}{2} \right) \right]. \tag{6.31}$$

6.2.3 Jordan-Wigner Transformation

Some lattice spin model can be mapped to fermion one by the Jordan-Wigner (J-W) transformation, which defines the isomorphism between the fermion and spin Hilbert space. One a single site, we map the

6.3 Lattice Gauges

Chapter 7

Fermi Liquid

7.1 Non-relativistic Field Theory

A general non-relativistic field theory is described by the action (with repeated indices automatically summed):

$$\begin{aligned} S = S_0 + S_{\text{int}} &= \int dt \int d^d x \mathcal{L}_0 - \int dt\ \mathcal{V}_{\text{int}}, \\ \mathcal{L}_0 &= \bar{\psi}_a(x)(i\delta_{ab}\partial_t - \hat{H}_{ab})\psi_b(x). \end{aligned} \tag{7.1}$$

where the field operator $\psi(x)$ can be bosonic or fermionic, which is denoted by a number $\zeta = \pm 1$, and $\mathcal{V}_{\text{int}}$ is the interaction Lagrangian. A general interaction has the form

$$\mathcal{V}_{\text{int}} = \sum_{abcd} \int \prod_{i=1}^{4} d^d x_i\ \bar{\psi}_c(x_3)\bar{\psi}_d(x_4)V_{abcd}(x_1, x_2, x_3, x_4)\psi_b(x_2)\psi_a(x_1). \tag{7.2}$$

Note that the classical equation of motion for the free field satisfies the Schrödinger equation:

$$\partial_\mu \frac{\partial \mathcal{L}_0}{\partial(\partial_\mu \bar{\psi}_a(x))} - \frac{\partial \mathcal{L}_0}{\partial \bar{\psi}_a(x)} = -i\partial_t \psi_a(x) + \hat{H}_{ab}\psi_b(x) = 0. \tag{7.3}$$

We are mostly work with finite system size L^d with UV cutoff $\Lambda = \pi/a$ (where a is the lattice spacing, and $L = Na$). The spatial Fourier transformation is

$$\tilde{\psi}_a(k) = \int_{L^d} d^d x e^{-ik\cdot x}\psi_a(x), \quad \psi_a(x) = \frac{1}{L^d}\sum_k e^{ik\cdot x}\tilde{\psi}_a(k). \tag{7.4}$$

For the finite size, the momentum is discretized: $k = 2\pi n_k/L$, $n_k \in \mathbb{Z}$. The summation in the thermodynamic limit becomes the integral:

$$\frac{1}{L^d}\sum_k \longrightarrow \int_{|k|<\Lambda} \frac{d^d k}{(2\pi)^d}. \tag{7.5}$$

In the momentum space, the free theory can be simplified:

$$S_0 = \int dt \int \frac{d^d k}{(2\pi)^d} \tilde{\bar{\psi}}_a(k)[i\partial_t - \varepsilon_a(k)]\tilde{\psi}_a(k). \tag{7.6}$$

The interaction in the momentum space is described by the vertex function:

$$\tilde{V}_{abcd}(k_1,k_2,k_3,k_4) = \int \prod_{i=1}^{4} d^d x_i e^{i(k_1x_1+k_2x_2-k_3x_3-k_4x_4)} V_{abcd}(x_1,x_2,x_3,x_4). \tag{7.7}$$

Because of the momentum conservation, the vertex will contain a delta function factor.

Consider the Coulomb repulsive potential e^2/r, in the field theory formalism, the coefficient $V_{abcd}(x_1,x_2,x_3,x_4)$ is

$$\frac{V(x_1-x_2)}{2!2!}\left[\delta_{ac}\delta_{bd}\delta^{(3)}(x_1-x_3)\delta^{(3)}(x_2-x_4)-\delta_{ad}\delta_{bc}\delta^{(3)}(x_1-x_4)\delta^{(3)}(x_2-x_3)\right], \tag{7.8}$$

where the factor $\frac{1}{2!2!}$ is the symmetry from interchanging the fermion fields. In momentum space:

$$V_{abab}(k_1,k_2,k_3+q,k_4-q) = \frac{1}{2!2!}V_{\text{Coul}}(q), \tag{7.9}$$

where the Coulomb potential in the momentum space is

$$\begin{aligned} V_{\text{Coul}}(q) &= \lim_{\alpha\to 0} e^2 \int_0^\infty dr\ 2\pi r^2 \int_{-1}^{+1} d\left(\cos\theta\right) \frac{e^{-iqr\cos\theta-\alpha r}}{r} \\ &= \lim_{\alpha\to 0} \frac{2\pi e^2}{iq}\int_0^\infty dr\left(e^{iqr-\alpha r}-e^{-iqr-\alpha r}\right) \\ &= \lim_{\alpha\to 0}\frac{4\pi e^2}{q^2+\alpha^2} = \frac{4\pi e^2}{q^2}. \end{aligned} \tag{7.10}$$

7.1.1 Action in Euclidean Space

The original real-time partition function is defined as[1]

$$Z[J] = \int D[\bar{\psi},\psi]\exp\left\{i\int dt\int d^dx\left[\mathcal{L}+\bar{J}_a(x)\psi_a(x)+\bar{\psi}_a(x)J_a(x)\right]\right\}. \tag{7.11}$$

If we make a analytic continuation of t to the complex plane:[2]

$$t\to -i\tau, \quad \omega\to i\omega. \tag{7.12}$$

In this way the free action transforms as:

$$iS_0 = i\int dtdx\ \psi(\boldsymbol{x},t)(i\partial_t - E)\psi(\boldsymbol{x},t) \to -\int d\tau dx\ \psi(\boldsymbol{x},\tau)(\partial_\tau + E)\psi(\boldsymbol{x},\tau).$$

The partition function can then be written as:

$$Z[J] = \int D[\bar{\psi},\psi]e^{-S_0[\bar{\psi},\psi]+\bar{J}\cdot\psi+\bar{\psi}\cdot J}, \tag{7.13}$$

[1] As with the relativistic case, we introduce an auxiliary source J, which is bosonic/fermionic if the field ψ is bosonic/fermionic.

[2] Note that in the frequency domain, the singularities for positive frequency lies below the complex plane, as we always include an infinitesimal $-i\epsilon$ to the energy (mass) term of the theory in ensure convergence. So, the rotation of the real axis anti-clock-wisely to the imaginary axis will not cross any singularity, and thus the can be analytically extended.

where the Euclidean free action defined as:

$$S = \int d\tau \left[\int d^d x \; \bar{\psi}_a(\boldsymbol{x}, \tau)(\delta_{ab}\partial_\tau + \hat{H}_{ab})\psi_b(\boldsymbol{x}, \tau) + \mathcal{V}_{\rm int} \right]. \tag{7.14}$$

The Euclidean action is suitable to describe the system both in zero temperature or finite temperature. For finite temperature case, the integral over the imaginary time τ is over $[0, \beta)$. The Fourier transformation of the field on the imaginary time domain is defined as:

$$\tilde{\psi}(\omega_n) = \int_0^\beta d\tau e^{i\omega_n \tau}\psi(\tau), \quad \psi(\tau) = \frac{1}{\beta}\sum_{\omega_n} e^{-i\omega_n \tau}\tilde{\psi}(\omega_n). \tag{7.15}$$

Under such convention, in the thermodynamic limit and zero-temperature limit, the spatial-temporal Fourier transformation agrees with the relativistic case (up to a Wick rotation).

7.1.2 Free Field Theory

The Fourier transformation of the free field action is

$$S_0 = \frac{1}{\beta}\sum_{\omega_n}\int_\Lambda \frac{d^d k}{(2\pi)^d}\tilde{\bar{\psi}}_a(k, \omega_n)\left[-i\omega_n + \tilde{H}_{ab}(k)\right]\tilde{\psi}_b(k, \omega_n). \tag{7.16}$$

The partition function with source is

$$\frac{Z_0[J]}{Z_0[0]} = \exp\left[-\frac{1}{\beta}\sum_{\omega_n}\int_\Lambda \frac{d^d k}{(2\pi)^d}\tilde{\bar{J}}_a(k, \omega_n)\tilde{G}_{ab}(k, \omega_n)\tilde{J}_b(k, \omega_n)\right], \tag{7.17}$$

where the Green's function is

$$\tilde{G}_{ab}(k, \omega_n) = \left[\frac{1}{i\omega_n - \tilde{H}(k)}\right]_{ab}. \tag{7.18}$$

Unlike the relativistic case, the value of the value of partition function without source $Z_0[0]$ is related to the free energy. We can express it formally as

$$Z_0[0] = \left[\det(-G_{ab})^{-1}\right]^{-\zeta}.$$

To get the correct dimensionality, we set the determinant as

$$Z_0[0] \equiv \prod_{k, \omega_n}\left\{\beta \det\left[-i\omega_n + \tilde{H}(k)\right]\right\}^{-\zeta}.$$

Thus the free energy is

$$F = -\frac{1}{\beta}\ln Z_0 = \zeta \sum_{k, \omega_n}\ln\left\{\beta \det\left[-i\omega_n + \tilde{H}(k)\right]\right\}. \tag{7.19}$$

7.1.3 Matsubara Summation

Now consider the summation on Matsubara frequency:

$$\sum_{\omega_n} f(\omega_n) = \begin{cases} \sum_n f(\frac{2n\pi}{\beta}) & \text{bosonic} \\ \sum_n f(\frac{(2n+1)\pi}{\beta}) & \text{fermionic} \end{cases}. \tag{7.20}$$

The frequency is capture by the singularities of the density function of the states:

$$\rho(z) = \begin{cases} \frac{1}{\exp(\beta z)-1} & \text{bosonic} \\ \frac{1}{\exp(\beta z)+1} & \text{fermionic} \end{cases}. \tag{7.21}$$

The residue on imaginary frequency $i\omega_n$ is alway $\frac{1}{\beta}$. In this way, the summation is:

$$\frac{1}{\beta}\sum_{\omega_n} f(i\omega_n) = \frac{1}{2\pi i}\oint \rho(z)f(z) = -\sum_k \text{Res } \rho(z)f(z)|_{z=z_k}. \tag{7.22}$$

Summation of Green's function

Consider the frequency summation for the correlation function:

$$\frac{1}{\beta}\sum_{\omega_n}\tilde{G}_0(k) = \frac{1}{\beta}\sum_{\omega_n}\frac{1}{i\omega_n - E_p} = -\text{Res }\frac{\rho(z)}{z-E_p}\bigg|_{z=E_p} = \rho(E_p). \tag{7.23}$$

Summation of Green's function

Consider the frequency summation for the correlation function:

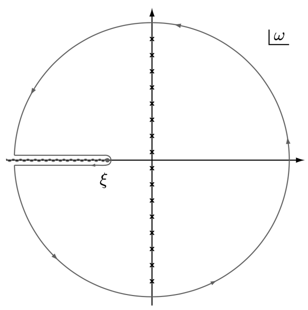
$$\sum_{\omega_n}\langle\bar{\psi}_{\vec{p},\omega_n}\psi_{\vec{p},\omega_n}\rangle = \frac{1}{\beta}\sum_{\omega_n}\frac{1}{-i\omega_n+\epsilon_{\vec{p}}} = \text{Res }\frac{\rho(z)}{z-\epsilon_{\vec{p}}}\bigg|_{z-\epsilon_{\vec{p}}} = \rho(\epsilon_{\vec{p}}). \tag{7.24}$$

Free Energy Summation

Consider the free energy

$$F = -\frac{1}{\beta}\ln Z = -\frac{1}{\beta}\sum_{\omega_n}\ln[\beta(-i\omega_n + E_{\vec{p}})] = \frac{1}{2\pi i}\oint dz\rho(z)\ln[\beta(\xi - z)]. \tag{7.25}$$

To calculate the summation, we consider the line integral along the loop:



The free energy is

$$
\begin{aligned}
F &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} dx\rho(x)\ln\left(\frac{\xi - x - i\epsilon}{\xi - x + i\epsilon}\right) \\
&= \frac{-\zeta}{2\pi i\beta}\int_{-\infty}^{\infty} dx \ln(1-\zeta e^{-\beta z})\left(\frac{1}{x+i\epsilon-\xi} - \frac{1}{x-i\epsilon-\xi}\right),
\end{aligned}
\tag{7.26}
$$

where we integrate the expression by part, noticing that

$$
\frac{d}{dz}\frac{\zeta}{\beta}\ln(1-\zeta e^{-\beta z}) = \frac{1}{e^{\beta z}-\zeta} = \rho(z) \tag{7.27}
$$

Using the identity

$$
\lim_{\epsilon\to 0^+}\frac{1}{x+i\epsilon} = -i\pi\delta(x) + \mathcal{P}\frac{1}{x},
$$

the above expression can be simplified to

$$
F = \frac{\zeta}{\beta}\ln(1-\zeta e^{-\beta\zeta}). \tag{7.28}
$$

7.2 RG of 2D Fermi Liquid

In this section, we are considering the system of weakly interacting Fermi gas. To be specific, we consider the lattice Hamiltonian:

$$
H = -\frac{1}{2}\sum_{\langle i,j\rangle}(c_i^\dagger c_j + c_j^\dagger c_i) + \mu\sum_i c_i^\dagger c_i + \sum_{i,j,k,l} u_{ijkl} c_i^\dagger c_j^\dagger c_k c_l. \tag{7.29}
$$

In the following, we investigate the effective field theory near the Fermi surface. We discuss the RG flow of the couplings (mainly for two dimensional system). Then we carry out the perturbative calculation for the correlation functions.

7.2.1 Effective Field Theory for Interacting Fermi Systems

The low-energy manifold is an annulus of thickness 2Λ symmetrically situated with respect to the Fermi circle $K = K_{\mathrm{F}}$. The dispersion for the free lattice model is

$$
E(\boldsymbol{K}) = -\cos K_x - \cos K_y \simeq -2 + \frac{\boldsymbol{K}^2}{2}. \tag{7.30}
$$

For a given chemical potential μ, the Fermi circle is $K_F = \sqrt{2m\mu}$, we can linearize the dispersion near the Fermi surface:

$$
E(\boldsymbol{K}) = \frac{\boldsymbol{K}^2 - K_F^2}{2m} \simeq \frac{K_F}{m}k \equiv v_F k, \quad k \equiv |K| - K_F \tag{7.31}
$$

The partition function is:

$$
Z_0 = \sum_\theta \sum_{|k|<\Lambda}\int D\left[\bar{\psi}(k,\theta,\omega), \psi(k,\theta,\omega)\right] e^{-S_0}, \tag{7.32}
$$

where the free field action is:[3]

$$S_0 = \int \frac{d\theta}{2\pi} \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \bar{\psi}(k,\theta,\omega)(-i\omega + v_F k)\psi(k,\theta,\omega). \tag{7.33}$$

Consider the quartic interaction

$$\delta S_4 = \frac{1}{4} \int_{\boldsymbol{K},\theta,\omega} \bar{\psi}(4)\bar{\psi}(3)\psi(2)\psi(1)u(4,3,2,1) \tag{7.34}$$

where we eliminate one of the four sets of variables, say, the one numbered 4, by integrating them against the delta functions:

$$\int_{K,\theta,\omega} = \prod_{i=1}^{3} \int_0^{2\pi} \frac{d\theta_i}{2\pi} \int_{-\Lambda}^{\Lambda} \frac{dk_i}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega_i}{2\pi} \theta\left(\Lambda - |k_4|\right), \quad k_4 = |\boldsymbol{K}_4| - K_F. \tag{7.35}$$

The ω integral is easy: since all ω's are allowed, the condition $\omega_4 = \omega_1 + \omega_2 - \omega_3$ is always satisfied for any choice of the first three frequencies. The same would be true for the momenta if all momenta were allowed. But they are not; they are required to lie within the annulus of thickness 2Λ around the Fermi circle. Consequently, if one freely chooses the first three momenta from the annulus, the fourth could have a length as large as $3K_F$. The role of $\delta(\Lambda - |k_4|)$ is to prevent exactly this.

Momentum Constraint

Note that k_4 can be expressed as

$$k_4 = |(K_F + k_1)\boldsymbol{\Omega}_1 + (K_f + k_2)\boldsymbol{\Omega}_2 - (K_F + k_3)\boldsymbol{\Omega}_3| - K_F. \tag{7.36}$$

When doing RG towards the Fermi surface, the integral measure will not preserve the preserve the original form. The situation is clearly is we use a smooth cutoff

$$\theta(\Lambda - |k_4|) \to e^{-|k_4|/\Lambda}, \tag{7.37}$$

and define $\Delta \equiv \boldsymbol{\Omega}_1 + \boldsymbol{\Omega}_2 - \boldsymbol{\Omega}_3$, k_4 in this way behaves as

$$k_4 = (|\Delta| - 1)K_F + O(k). \tag{7.38}$$

The integral then change to:

$$\begin{aligned} &\prod_{i=1}^{3} \int_{-\Lambda}^{\Lambda} \frac{dk_i}{2\pi} \int \frac{d\theta_i}{2\pi} \int \frac{d\omega_i}{2\pi} e^{-||\Delta|-1|\frac{K_F}{\Lambda}} u(k,\theta,\omega)\bar{\psi}\bar{\psi}\psi\psi \\ \xrightarrow{\text{RG}} &\prod_{1}^{3} \int_{-\Lambda}^{\Lambda} \frac{dk_i'}{2\pi} \int \frac{d\theta_i}{2\pi} \int \frac{d\omega_i'}{2\pi} e^{-||\Delta|-1|\frac{sK_F}{\Lambda}} u\left(\frac{k'}{s} \frac{\omega'}{s} \theta\right) \bar{\psi}\bar{\psi}\psi\psi. \end{aligned} \tag{7.39}$$

We can then get the RG transformation of u as

$$u'(k',\theta,\omega') = e^{-||\Delta|-1|\frac{(s-1)K_F}{\Lambda}} u\left(\frac{k'}{s}, \theta, \frac{\omega'}{s}\right). \tag{7.40}$$

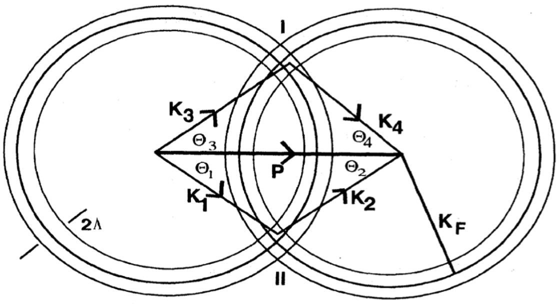


Figure 7.1: The geometric construction for determining the allowed values of momenta. If K_1 and K_2 add up to P, then K_3 and K_4 are constrained as shown, if they are to add up to P and lie within the cutoff. If the incoming momenta K_1 and K_2 are equal and opposite, the two shells coalesce and K_3 and K_4 are free to point in all directions, as long as they are equal and opposite.

By Taylor expansion, we conclude that the only couplings that survive the RG transformation without any decay correspond to the cases in which $|\Delta| = 1$, and without momentum dependence.

This equation has only three solutions (see also Fig. 7.1):

$$\begin{aligned}&\text{Case I:}\quad \boldsymbol{\Omega}_1 = \boldsymbol{\Omega}_3,\\ &\text{Case II:}\quad \boldsymbol{\Omega}_2 = \boldsymbol{\Omega}_3,\\ &\text{Case III:}\quad \boldsymbol{\Omega}_1 = -\boldsymbol{\Omega}_2.\end{aligned} \tag{7.41}$$

Because of the rotational symmetry, the marginal vertex functions are determined solely by two functions:

$$u[\theta_1,\theta_2,\theta_1,\theta_2] \equiv F(\theta_1,\theta_2) = F(\theta_1-\theta_2), \tag{7.42}$$

$$u[\theta_1,\theta_2,\theta_2,\theta_1] = -F(\theta_1-\theta_2), \tag{7.43}$$

$$u[\theta_1,-\theta_1,\theta_3,-\theta_3] \equiv V(\theta_1,\theta_3) = V(\theta_1-\theta_3). \tag{7.44}$$

Note that the manifestation of the Pauli principle on F and V is somewhat subtle: F will not be antisymmetric under $1 \leftrightarrow 2$ since, according to the way it is defined above, we cannot exchange 1 and 2 without exchanging 3 and 4 at the same time. On the other hand, since 3 and 4 can be exchanged without touching 1 and 2 in the definition of V, V must go to $-V$ when $1 \leftrightarrow 3$.

7.2.2 One-loop RG for 2D System

We first consider the loop correction to the chemical potential:

$$\begin{aligned}\mu^{(2)}(k,\theta,\omega) &= \int_{d\Lambda}\frac{dK'}{2\pi}\int\frac{d\omega'}{2\pi}\int\frac{d\theta'}{2\pi}\frac{F(\theta-\theta')}{i\omega - v_F k'}\\ &= \int_{-\Lambda}^{-\Lambda+\Lambda dt}\frac{dK'}{2\pi}\int\frac{d\theta'}{2\pi}F(\theta-\theta')\\ &= \frac{\Lambda}{2\pi}\left[\int\frac{d\phi}{2\pi}F(\phi)\right]dt.\end{aligned} \tag{7.45}$$

[3]A factor of K_{F} has been absorbed in the field.

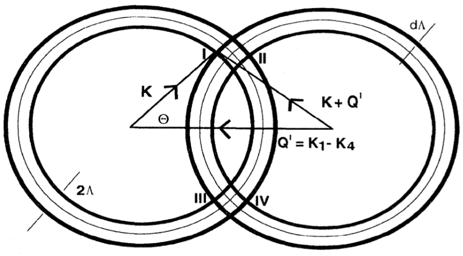
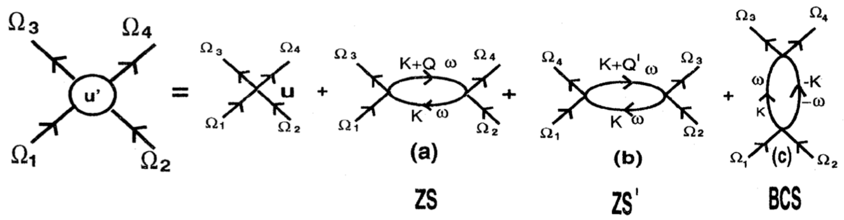


Figure 7.2: Construction for determining the allowed values of loop momenta in ZS'. The requirement that the loop momenta come from the shell and differ by Q' forces them to lie in one of the eight intersection regions of width $d\Lambda^2$.

For the vertex correction, again we should consider three channels corresponding to the diagrams:



(7.46)

First we consider the correction to the $F(\theta)$. The contribution from the ZS channel (the momentum transfer $Q \simeq 0$) is

$$F^{(2)}_{\rm ZS}(\theta_1 - \theta_2) = \int_{d\Lambda} \frac{dk}{2\pi} \int \frac{d\omega}{2\pi} \int \frac{d\theta}{2\pi} \frac{F(\theta_1 - \theta)F(\theta - \theta_2)}{(i\omega - v_F k)^2}. \tag{7.47}$$

Since two poles of the integrant lie at the same half plane, we can alway choose to close the loop integral along the other half, and thus getting zero contribution.

For the ZS' channels, the momentum conservation condition (see Fig. 7.2) restrict the phase space to be of order $d\Lambda^2$, and thus has no relevant contribution to $F(\theta)$. Finally, for the same kinematical reason, the BCS diagram does not renormalize $F(\theta)$ at one loop. Consider Fig. 7.1, with K_3 and K_4 replaced by the two momenta in the BCS loop, K and $P-K$. In each annulus we keep just two shells of thickness $d\Lambda$ at the cutoff corresponding to the modes to be eliminated. The requirement that K and $P-K$ lie in these shells and also add up to P forces them into intersection regions of order $d\Lambda^2$. This means the diagram is just as ineffective as the ZS' diagram in causing a flow. Thus any F is a fixed point to this order.

Now we consider the correction to the $V(\theta)$ function. We choose the external momenta equal and opposite and on the Fermi surface. The ZS and ZS' diagrams do not contribute to any marginal flow for the same reason that BCS and ZS' did not contribute to the flow of $F(\theta)$. But the BCS diagram produces a flow:

$$\begin{aligned} V^{(2)}_{\rm BCS}(\theta_1 - \theta_3) &= -\frac{1}{2} \int_{d\Lambda} \frac{dk}{2\pi} \int \frac{d\omega}{2\pi} \int \frac{d\theta}{2\pi} \frac{V(\theta_1 - \theta)V(\theta - \theta_3)}{(i\omega - v_F k)(-i\omega - v_F k)} \\ &= -\frac{dt}{4\pi v_F} \int \frac{d\theta}{2\pi} V(\theta_1 - \theta)V(\theta - \theta_3). \end{aligned} \tag{7.48}$$

We can simplify the picture by going to angular momentum eigenfunctions,

$$V(\theta) = \sum_l e^{il\theta} V_l, \tag{7.49}$$

which gives the RG flow as

$$\frac{dV_l}{dt} = -\frac{V_l^2}{4\pi v_F}. \tag{7.50}$$

The solution to the RG flow is:

$$V_l(t) = \frac{V_l(0)}{1 + \frac{V_l(0)}{4\pi v_F} t}. \tag{7.51}$$

What these equations tell us is that if the potential in angular momentum channel l is repulsive, it will get renormalized (logarithmically) down to zero, while if it is attractive, it will run off to large negative values signaling the BCS instability. This is the reason the V's are excluded in Landau theory, which assumes we have no phase transitions.[4]

[4] Remember that the sign of any given V_l is not necessarily equal to that of the microscopic interaction. Kohn and Luttinger have shown (PRL, 15, 524 (1965)) that some of them will be always negative. Thus, the BCS instability is inevitable, though possibly at absurdly low temperatures or absurdly high angular momentum l.

Chapter 8

Luttinger Liquid

In this section, we discuss the one-dimensional interacting Fermi system, described by the Hamiltonian

$$H = -\frac{1}{2}\sum_i (c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i) + \mu \sum_i c_i^\dagger c_i + \sum_{i,j,k,l} u_{ijkl} c_i^\dagger c_j^\dagger c_k c_l. \tag{8.1}$$

The dispersion for the free theory is

$$\varepsilon(k) = -\cos k, \quad v_F = \partial_k \varepsilon(k)|_{k=k_F} = \sin k_F. \tag{8.2}$$

Near the Fermi surface with momentum k_F, the spectrum can be approximately linearized (as shown in Fig. 8.1), with the left and right moving fermion modes:

$$\varepsilon_{R/L}(k) = \begin{cases} v_F(k-k_F) & r = R \\ -v_F(k+k_F) & r = L \end{cases}. \tag{8.3}$$

The fermi momentum is (assume $N_L = N_R$)

$$k_F = \frac{\pi N}{2L}. \tag{8.4}$$

The state while all fermion modes filled below the Fermi surface and all modes empty above the Fermi surface is defined as the vacuum state $|0\rangle_0$.

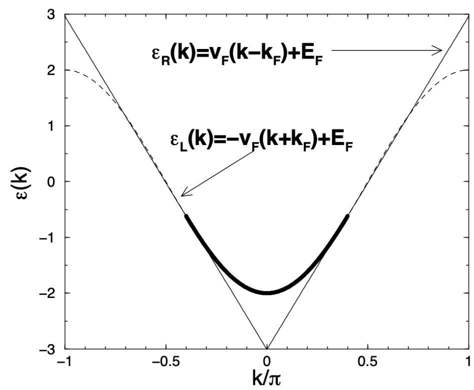


Figure 8.1: Linearized Model.

8.1 Field Theory for Luttinger Liquid

In this section, we gives a field theoretical analysis of the interacting fermion system in 1D. For the notational simplicity, we shift the momentum so that

$$k \to k' = \begin{cases} k - k_F & r = R \\ -k - k_F & r = L \end{cases}. \tag{8.5}$$

The dispersion is then $\varepsilon_r(k) = v_F k$. In this way, two Fermi points are brought to the origin, the left and right moving branches have the same dispersion. The integral over momentum shell for both species of fermion can then be denoted by

$$\int^{\Lambda} \frac{dK}{2\pi} \equiv \int_{-\Lambda}^{\Lambda} \frac{dk_L}{2\pi} + \int_{-\Lambda}^{\Lambda} \frac{dk_R}{2\pi}. \tag{8.6}$$

8.1.1 Effective Field Theory

The effective field theory for the free field is

$$Z_0 = \prod_{r=L/R} \int D\left[\bar{\psi}_r(k,\omega), \psi_r(k,\omega)\right] e^{-S_0}, \tag{8.7}$$

where the free field action is

$$S_0 = \sum_{r=L/R} \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \bar{\psi}_r(k,\omega)[-i\omega + v_F k]\psi_r(k,\omega), \tag{8.8}$$

which gives the free field propagator:

$$G_r(k,\omega) = -\langle \psi_r(k,\omega)\bar{\psi}_r(k,\omega)\rangle = \frac{1}{i\omega - v_F k}. \tag{8.9}$$

We then consider the rescaling of the cut-off $\Lambda \to \Lambda/s$. To make the free action scale invariant, we define the rescaled variables:

$$k' = sk, \quad \omega' = s\omega, \quad \psi_r'(k',\omega') = s^{-3/2}\psi_r(k,\omega). \tag{8.10}$$

Then we consider the perturbation from quadratic and quartic terms:

$$\begin{aligned} \delta S_2 &= \sum_{r=L/R} \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \mu(k,\omega)\bar{\psi}_r(k,\omega)\psi_r(k,\omega), \\ \delta S_4 &= \frac{1}{2!2!} \int_{K,\omega}^{\Lambda} u(4,3,2,1)\bar{\psi}(4)\bar{\psi}(3)\psi(2)\psi(1), \end{aligned} \tag{8.11}$$

where we have suppressed the momentum labels:

$$\psi(i) = \psi_{r_i}(k_i,\omega_i), \quad u(4,3,2,1) = u(K_4,\omega_4; K_3,\omega_3; K_2,\omega_2; K_1,\omega_1), \tag{8.12}$$

and the integral is defined as:[1]

$$\int_{K\omega}^{\Lambda} = \int^{\Lambda} \frac{dK_1 \cdots dK_4}{(2\pi)^4} \int_{-\infty}^{\infty} \frac{d\omega_1 \cdots d\omega_4}{(2\pi)^4} \times 2\pi\delta\left(\omega_1 + \omega_2 - \omega_3 - \omega_4\right) \\ \times 2\pi\bar{\delta}(K_1 + K_2 - K_3 - K_4). \tag{8.13}$$

Since this action separates into slow and fast pieces, the effect of mode elimination is simply to reduce Λ to Λ/s in the integral above. Rescaling moments and fields, we find that

$$\mu'(k', \omega') = s \cdot \mu\left(\frac{k'}{s}, \frac{\omega}{s}\right). \tag{8.14}$$

Expand μ in series:

$$\mu(k, \omega) = \mu_{00} + \mu_{10}k + \mu_{01}i\omega + \cdots + \mu_{nm}k^n(i\omega)^m + \cdots, \tag{8.15}$$

and compare both sides. The constant piece is a relevant perturbation. This relevant flow reflects the readjustment of the Fermi sea to a change in chemical potential. The correct way to deal with this term is to include it in the free-field action by filling the Fermi sea to a point that takes μ_{00} into account. The next two terms are marginal and modify terms that are already present in the action.

We now turn on the quartic interaction, the dimensional analysis gives the transformation of u:

$$u'_{i_4,i_3,i_2,i_1}(k'_i, \omega'_i) = u_{i_4,i_3,i_2,i_1}\left(\frac{k'_i}{s}, \frac{\omega'_i}{s}\right). \tag{8.16}$$

If we expand u in a Taylor series in its arguments and compare coefficients, we find that the constant term u0 is marginal and the higher coefficients are irrelevant. Thus, u depends only on its discrete labels and we can limit the problem to just a few coupling constants instead of the coupling function we started with. Furthermore, all reduce to just one coupling constant:

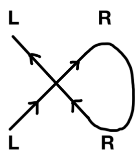
$$u_0 = u_{LRLR} = u_{RLRL} = -u_{RLLR} = -u_{LRRL} \equiv u. \tag{8.17}$$

Other couplings corresponding to the $(LL \to RR)$ process are wiped out by the Pauli principle since they have no momentum dependence and cannot have the desired anti-symmetry.

8.1.2 RG at One-loop Level

Consider the infinitesimal rescale $s = e^{dt}$. The one-loop contribution to the quadratic term is[2]

$$\mu^{(2)}_{LL} =$$



$$= -u \int_{d\Lambda} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{i\omega\eta}}{i\omega - v_F k}, \tag{8.18}$$

[1]The symbol $\bar{\delta}$ enforces momentum conservation mod 2π, as is appropriate to any lattice problem. A process where lattice momentum is violated in multiples of 2π is called an *umklapp process*.

[2]We include an infinitesimal $e^{i\omega\eta}$ to ensure convergence as we do the integral over ω by closing the upper half-plane.

where the integral on the momentum shell is

$$\int_{d\Lambda} \frac{dk}{2\pi} = \int_{-\Lambda}^{-\Lambda(1-dt)} \frac{dk}{2\pi} + \int_{\Lambda(1-dt)}^{\Lambda} \frac{dk}{2\pi}. \tag{8.19}$$

The result gives:

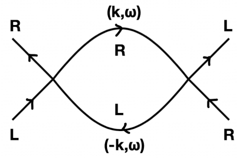
$$\mu_{LL}^{(2)} = -\frac{u\Lambda}{2\pi} dt$$

By the symmetry $L \leftrightarrow R$, we know $\mu_{LL}^{(2)} = \mu_{RR}^{(2)} = \mu^{(2)}$, so the RG flow is

$$\frac{d}{dt}\left[s \cdot \left(\mu + \mu^{(2)}\right)\right] = \mu - \frac{u\Lambda}{2\pi}. \tag{8.20}$$

The one-loop correction to the quartic term ($u_{LRRL} = -u$) have two contributions. One is called ZS' (zero sound) channel:[3]

$$u_{\mathrm{ZS'}}^{(2)} =$$



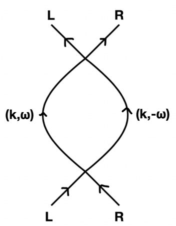
$$\begin{aligned} &= -u^2 \int_{-\infty}^{\infty} \int_{\Lambda/s<|k|<\Lambda} \frac{d\omega dk}{(2\pi)^2} \frac{e^{i\omega\eta}}{(i\omega + v_F k)(i\omega - v_F k)} \\ &= u^2 \int_{\Lambda/s<|k|<\Lambda} \frac{dk}{2\pi} \frac{1}{2|k|} \\ &= \frac{u^2}{2\pi} \frac{d\Lambda}{\Lambda}. \end{aligned} \tag{8.21}$$

The sign is obtained from contracting the Fermion field monomial:

$$\bar{\psi}_L \bar{\psi}_R \psi_L \psi_R \bar{\psi}_L \bar{\psi}_R \psi_L \psi_R = -G_L G_R \bar{\psi}_R \psi_L \bar{\psi}_L \psi_R = -G_L G_R \bar{\psi}_L \bar{\psi}_R \psi_L \psi_R.$$

The other is called the BCS channel:[4]

$$u_{\mathrm{BCS}}^{(2)} =$$



$$= -\frac{u^2}{2} \sum_{r=L/R} \int_{-\infty}^{\infty} \int_{d\Lambda} \frac{d\omega dk_i}{(2\pi)^2} \frac{e^{i\omega\eta}}{(i\omega - v_F k)(-i\omega - v_F k)}. \tag{8.22}$$

The sign is obtained from the contraction:

$$\bar{\psi}_L \bar{\psi}_R \psi_L \psi_R \bar{\psi}_L \bar{\psi}_R \psi_L \psi_R = -G_L G_R \bar{\psi}_L \bar{\psi}_R \psi_L \psi_R.$$

Note that we will obtained a factor of 2 since in this channel, the intermedia propagator can be left mover or right mover. We see that two contributions cancel out:

$$u_{\mathrm{ZS'}}^{(2)} + u_{\mathrm{BCS}}^{(2)} = 0. \tag{8.23}$$

[3]There is actually another zero sound channel ZS, but which has no contribution to the vertex because the diagram contains the vertex of the ($LL \to RR$) process, which has no relevant contribution the the vertex.

[4]The 1/2 factor comes from the symmetry factor of the diagram.

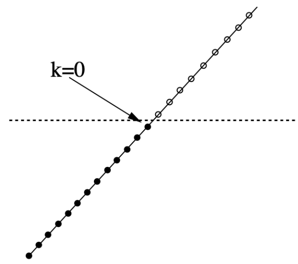


Figure 8.2: The vacuum state $|0\rangle_0$ of a single fermion branch.

Together, the RG flow to the one-loop level is

$$\frac{d\mu}{dt} = \mu - \frac{u\Lambda}{2\pi}, \quad \frac{du}{dt} = 0. \tag{8.24}$$

The fixed point solution to the RG flow is:

$$\mu^* = \frac{u^*\Lambda}{2\pi}, \tag{8.25}$$

where the fixed-point value of u^* is arbitrary. The vanishing beta function predict that at least for small interaction, the system does not immediately develop a CDW order. However, as the vertex function is marginal in the RG analysis, the fix points become a fixed line. When the coupling u is nonzero, the analysis for the Gaussian fix point becomes untrustworthy. Actually when u is getting sufficiently large, the umklapp process becomes relevant and the system flows to CDW phase. A more systematic analysis requires the bosonization techniques.

8.2 Bosonization

8.2.1 Bosonic Hilbert Space

In this section, we map the 1D interacting fermion system to a bosonic one. The low energy excitations are particle-hole modes:

$$\rho_k^\dagger = \sum_q c_{q+k}^\dagger c_q, \quad \rho_k = \sum_q c_q^\dagger c_{q+k} = \rho_{-k}^\dagger. \tag{8.26}$$

To simplify the discussion, here we consider only a single branch of Fermion, as depicted in Fig. 8.2. The generalization to multiple branches is trivial since the dispersion is the same.

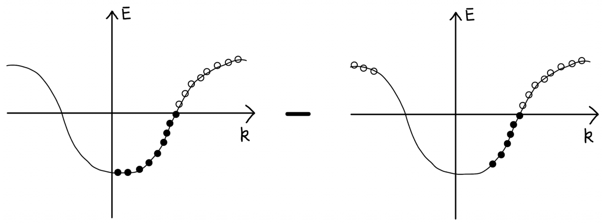


Figure 8.3: Shift of the right-moving modes.

Commutation Relation

The commutation relation between ρ_k and $\rho^\dagger_{k'}$ is:[5]

$$\begin{aligned}
\left[\rho_k, \rho^\dagger_{k'}\right] &= \sum_{q_1,q_2} \left[c^\dagger_{q_1} c_{q_1+k}, c^\dagger_{q_2+k'} c_{q_2}\right] \\
&= \sum_{q_1,q_2} \left\{ c^\dagger_{q_1} \left[c_{q_1+k}, c^\dagger_{q_2+k'} c_{q_2}\right] + \left[c^\dagger_{q_1}, c^\dagger_{q_2+k'} c_{q_2}\right] c_{q_1+k} \right\} \\
&= \sum_{q_1,q_2} \left\{ \delta_{q_1+k,q_2+k'} c^\dagger_{q_1} c_{q_2} - \delta_{q_1,q_2} c^\dagger_{q_2+k'} c_{q_1+k} \right\} \\
&= \sum_{q} \left[c^\dagger_{q+k'-k} c_q - c^\dagger_{q+k'} c_{q+k} \right].
\end{aligned} \tag{8.27}$$

For $k \neq k'$, it is clear that $[\rho_k, \rho^\dagger_{k'}] = 0$. However, when $k = k'$, we should be careful about the subtraction, since it evolve two infinities of which the subtraction is ill-defined.

Here we deal with the infinity with the lattice regularization, i.e., we think of the linearized theory as the low-energy approximation of a lattice mode, where the dispersion form a single energy band. The left/right movers are actually in a single band but with positive/negative momentum. Consider for example the density operator for the right mover, the commutator of the right moving density operator is then

$$\left[\rho_{k,r}, \rho^\dagger_{k,r}\right] = \sum_{0<q<\pi} [n_{q,r} - n_{k+q,r}]. \tag{8.28}$$

Consider for example the case where $r = R, k > 0$, as shown in Fig. 8.3. The subtraction result in a sum of low-lying right-mover number operator minus a sum of high-energy left-mover number operator, which behaves like a constant in the low energy regime. The above analysis gives the commutation relation:

$$\left[\rho_{k,r}, \rho^\dagger_{kr}\right] \simeq \frac{qL}{2\pi} \equiv n_q. \tag{8.29}$$

Another way to deal with the infinity is by the normal-order expression:

$$O = :O: + \langle 0|O|0\rangle. \tag{8.30}$$

[5] We use the identity $[AB, C] = A[B, C] + [A, C]B$ and $[A, BC] = \{A, B\}C - B\{A, C\}$.

For $k \neq k'$, since $\langle 0|c_k^\dagger c_k|0\rangle = 0$, the normal ordering does not affect the result, while for the $k = k'$ case, the normal ordering takes care of the infinity of the particle number operator:

$$\begin{aligned}\sum_q [n_q - n_{q+k}] &= \sum_q [:n_q: - :n_{q+k}: + \langle 0|n_q|0\rangle - \langle 0|n_{q+k}|0\rangle] \\ &= \sum_q [\langle 0|n_q|0\rangle - \langle 0|n_{q+k}|0\rangle].\end{aligned} \tag{8.31}$$

The final result gives the same result as the above discussions.

Particle-hole Excitation

We denote the ground state with N fermions as $|N\rangle_0$, which satisfies:

$$\rho_{p>0}|N\rangle_0 = \rho_{p<0}^\dagger|N\rangle_0 = 0. \tag{8.32}$$

We can thus define a set of canonical bosonic modes:

$$b_p^\dagger = i\frac{\rho_p^\dagger}{\sqrt{n_p}}, \quad b_p = -i\frac{\rho_p}{\sqrt{n_p}}, \quad [b_q, b_{q'}^\dagger] = \delta_{qq'}. \tag{8.33}$$

We will assume $p > 0$, and the $\pm i$ factor is a convention chosen for the future convenience.

Now we discuss the construction of the Hilbert space using the bosonic modes. The N-particle sector is spanned by the states generated by applying $b_q^\dagger$'s to the ground state $|N\rangle_0$. A general N-particle state has the form:

$$|N\rangle = f(\{b_q^\dagger\})|N\rangle_0. \tag{8.34}$$

Note that the bosonic mode b_q does not change the particle number:[6]

$$\left[\hat{N}, b_q\right] = \left[\hat{N}, b_q^\dagger\right] = 0. \tag{8.35}$$

In order to construct the full Fock space, we also need to include a particle-number-changing operator, the *Klein factor* $\hat{F}$, that shifts the total number of fermion by one, and commutes with bosonic operator b_q:

$$\left[\hat{F}, b_q\right] = \left[\hat{F}, b_q^\dagger\right] = 0, \quad \left[\hat{F}, \hat{N}\right] = \hat{F}, \quad \left[\hat{F}^\dagger, \hat{N}\right] = -\hat{F}^\dagger. \tag{8.36}$$

For system with different fermion species (labeled by η), the set of operator

$$\{b_{q,\eta}, b_{q,\eta}^\dagger, \hat{F}_\eta, \hat{N}_\eta\}$$

form a complete operator basis for the Hilbert space within the low-energy regime. However, note that the Klein factor also takes care of the fermionic statistics. That is, for the state denoted as

$$|\{N_i\}\rangle = |N_1, N_2, \cdots, N_m\rangle \tag{8.37}$$

The Klein factor $\hat{F}_\eta$ acting on the state will contribute an additional factor

$$\hat{F}_\eta|\{N_i\}\rangle = \exp\left(i\pi\sum_{j=1}^{\eta-1} N_j\right)|N_1, \cdots, N_\eta - 1, \cdots, N_m\rangle. \tag{8.38}$$

[6]The particle number operator is defined by the normal order expression $\hat{N} = \sum_k :c_k^\dagger c_k:$.

8.2.2 Bosonization of Fermion Field

Now we try to express the fermion operator

$$\psi(x) = \frac{1}{\sqrt{L}} \sum_p e^{ipx} c_p \tag{8.39}$$

by the bosonic operator set. First we consider the commutation relation between the fermion field operator and bosonic mode:

$$\begin{aligned} [b_q, \psi(x)] &= \frac{1}{\sqrt{L}} \frac{i}{\sqrt{n_q}} \sum_k \sum_p e^{ipx} [c_k^\dagger c_{k+q,r'}, c_p] \\ &= -\frac{1}{\sqrt{L}} \frac{i}{\sqrt{n_q}} \sum_p e^{ipx} c_{p+q} = -i \frac{1}{\sqrt{n_q}} e^{-iqx} \psi(x). \end{aligned} \tag{8.40}$$

Similarly,

$$\left[b_q^\dagger, \psi(x)\right] = i \frac{e^{iqx}}{\sqrt{n_q}} \psi(x). \tag{8.41}$$

For future convenience, we define a c-number factor:

$$\alpha_q(x) \equiv \frac{1}{\sqrt{n_q}} e^{iqx}, \quad [b_q^\dagger, \psi(x)] = i\alpha_q(x)\psi(x), \quad [b_q, \psi(x)] = -i\alpha_q^*(x)\psi(x). \tag{8.42}$$

Since $b_q|N\rangle_0 = 0$, the commutation relation (8.40) leads to

$$[b_q, \psi(x)]\,|N\rangle_0 = b_q \psi(x)|N\rangle_0 = -i\alpha_q^*(x)\psi(x)|N\rangle_0. \tag{8.43}$$

Thus, $\psi(x)|N\rangle_0$ is an eigenstate of b_q, i.e., a coherent state:

$$\psi(x)|N\rangle_0 = \Lambda(x)\hat{F} \exp\left[-i \sum_{q>0} \alpha_q^*(x) b_q^\dagger\right] |N\rangle_0 \tag{8.44}$$

where $\Lambda(x)$ is a c-number, which can be determined by

$${}_0\langle N-1|\psi(x)|N\rangle_0 = \Lambda(x)\ {}_0\langle N| \exp\left[-i \sum_{q>0} \alpha_q^*(x) b_q^\dagger\right] |N\rangle_0 = \Lambda(x).$$

The left-hand side can be computed directly, the result is[7]

$$\Lambda(x) = \frac{1}{\sqrt{L}} \sum_k e^{ikx}\ {}_0\langle N-1|c_{k,r}|N\rangle_0 = \frac{1}{\sqrt{L}} e^{i\frac{2\pi N}{L}}, \tag{8.45}$$

In this way, we get

$$\psi(x)|N\rangle_0 = \frac{\hat{F}}{\sqrt{L}} e^{i\frac{2\pi \hat{N} x}{L}} \exp\left[-i \sum_{q>0} \alpha_q^*(x) b_q^\dagger\right] |N\rangle_0 \tag{8.46}$$

[7]Note that the Fermi point locates at $k_F = \frac{2\pi N}{L}$.

The commutation relation (8.41) also leads to:[8]

$$\psi(x)b_q^\dagger = \left[b_q^\dagger - i\alpha_q(x)\right]\psi(x)$$
$$\Rightarrow \psi(x)\left(b_q^\dagger\right)^n = \left[b_q^\dagger - i\alpha_q(x)\right]^n\psi(x)$$
$$\Rightarrow \psi(x)f\left[\left\{b_q^\dagger\right\}\right] = f\left[\left\{b_q^\dagger - i\alpha_q(x)\right\}\right]\psi(x).$$

Then, for a generic N-particle state:

$$\begin{aligned}\psi(x)|N\rangle &= f\left[\left\{b_q^\dagger - i\alpha_q(x)\right\}\right]\psi(x)|N\rangle_0 \\ &= f\left[\left\{b_q^\dagger - i\alpha_q(x)\right\}\right]\frac{\hat{F}}{\sqrt{L}}e^{i\frac{2\pi\hat{N}x}{L}}\exp\left[-i\sum_{q>0}\alpha_q^*(x)b_q^\dagger\right]|N\rangle_0 \\ &= \frac{\hat{F}}{\sqrt{L}}e^{i\frac{2\pi\hat{N}x}{L}}\exp\left[-i\sum_{q>0}\alpha_q^*(x)b_q^\dagger\right]f\left[\left\{b_q^\dagger - \alpha_q(x)\right\}\right]|N\rangle_0.\end{aligned} \tag{8.47}$$

Using the BCH formula

$$e^A B e^{-A} = e^{[A,\cdot]}B = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \cdots, \tag{8.48}$$

we have the identity:

$$\exp\left[-i\sum_{q>0}\alpha_q(x)b_q\right]b_q^\dagger\exp\left[i\sum_{q>0}\alpha_q(x)b_q\right] = b_q^\dagger - i\alpha_q(x)$$
$$\Rightarrow \exp\left[-i\sum_{q>0}\alpha_q(x)b_q\right]f\left[b_q^\dagger\right]\exp\left[i\sum_{q>0}\alpha_q(x)b_q\right] = f\left[\left\{b_q^\dagger - i\alpha_q(x)\right\}\right].$$

Eq. (8.47) can be further simplified to:

$$\begin{aligned}\psi(x)|N\rangle &= \frac{\hat{F}}{\sqrt{L}}e^{i\frac{2\pi\hat{N}x}{L}}e^{-i\sum_{q>0}\alpha_q^*(x)b_q^\dagger}e^{-i\sum_{q>0}\alpha_q(x)b_q}f\left[b_q^\dagger\right]e^{i\sum_{q>0}\alpha_q(x)b_q}|N\rangle_0 \\ &= \frac{\hat{F}}{\sqrt{L}}e^{i\frac{2\pi\hat{N}x}{L}}\exp\left[-i\sum_{q>0}\alpha_q^*(x)b_q^\dagger\right]\exp\left[-i\sum_{q>0}\alpha_q(x)b_q\right]|N\rangle.\end{aligned}$$

We thus express the Fermi field operator in bosonic operator:

$$\psi(x) = \frac{\hat{F}}{\sqrt{L}}e^{i\frac{2\pi\hat{N}x}{L}}e^{-i\sqrt{2\pi}\varphi^\dagger(x)}e^{-i\sqrt{2\pi}\varphi(x)}, \tag{8.49}$$

where we have introduced a bosonic field:[9]

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}\sum_{q>0}e^{-aq/2}a_q(x)b_q. \tag{8.50}$$

[8]We denote $\alpha_q(x) \equiv e^{iqx}/\sqrt{n_q}$ to simplify the notation.

[9]The "converging factor" $e^{-aq/2}$ is important in defining a proper bosonic theory in 1D. These equations should always be viewed as having $e^{-aq/2}$ to ensure convergence at intermediate steps, but final results should be written taking $a \to 0^+$.

Introduction of Bosonic Field

Now we introducing the bosonic field:

$$\phi(x) \equiv \varphi(x) + \varphi^\dagger(x) = \frac{1}{\sqrt{2\pi}} \sum_{q>0} \frac{e^{-aq/2}}{\sqrt{n_q}} \left[e^{iqx} b_q + e^{-iqx} b_q^\dagger \right]. \tag{8.51}$$

We can expressed the fermion field as:[10]

$$\psi(x) = \frac{\hat{F}}{\sqrt{L}} e^{i\frac{2\pi \hat{N} x}{L}} e^{-i\sqrt{2\pi}\phi(x)} e^{\pi[\varphi(x),\varphi^\dagger(x)]}.$$

The commutation relation between φ and $\varphi^\dagger$ is

$$\left[\varphi(x), \varphi^\dagger(y)\right] = \frac{1}{L} \sum_q \frac{e^{iq(x-y+ia)}}{q} = -\frac{1}{2\pi} \ln \left\{ 1 - \exp\left[\frac{2\pi i}{L}(x - y + ia) \right] \right\}.$$

Set $x = y$, take limit $a \to 0^+$,

$$\exp\left\{\pi[\varphi(x), \varphi^\dagger(x)]\right\} = \lim_{a\to 0^+} \left[1 - \exp\left(-\frac{2\pi a}{L} \right) \right]^{-\frac{1}{2}} \to \sqrt{\frac{L}{2\pi a}}.$$

so we have

$$\psi(x) = \frac{\hat{F}}{\sqrt{2\pi a}} e^{i\frac{2\pi \hat{N} x}{L}} e^{-i\sqrt{2\pi}\phi(x)}. \tag{8.52}$$

The divergent factor $1/\sqrt{2\pi a}$ appears because Eq. (8.52) is not normal-ordered. We can check the result by normal-ordering the bilinear term $\psi^\dagger(x+a)\psi(x)$. Insert Eq. (8.52) into the expression, we have:

$$\psi^\dagger(x+a)\psi(x) \simeq \frac{e^{-i\frac{2\pi \hat{N} a}{L}}}{2\pi a} e^{i\sqrt{2\pi}\partial_x\phi(x)a} e^{\pi[\phi(x+a),\phi(x)]}. \tag{8.53}$$

The commutation relation is:

$$\begin{aligned} [\phi(x), \phi(y)] &= -\frac{1}{2\pi} \ln \left\{ \frac{1 - \exp\left[\frac{2\pi i}{L}(x - y + ia)\right]}{1 - \exp\left[-\frac{2\pi i}{L}(x - y - ia)\right]} \right\} \\ &\xrightarrow{a\to 0} \frac{i}{2} \operatorname{sgn}(x - y) - \frac{i}{L}(x - y). \end{aligned} \tag{8.54}$$

To the lowest order of a/L:

$$\psi^\dagger(x+a)\psi(x) \simeq \frac{i}{2\pi a} + \frac{\hat{N}+1}{L} - \frac{1}{\sqrt{2\pi}} \partial_x \phi(x). \tag{8.55}$$

The normal ordering will delaminate all constant, including the divergent one:

$$:\psi^\dagger(x+a)\psi(x): = \frac{\hat{N}}{L} - \frac{1}{\sqrt{2\pi}} \partial_x \phi(x). \tag{8.56}$$

[10] We use the identity $e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]}$ if both A and B commutes with $[A, B]$.

We will show in the following the above equation agrees with the bosonization of fermion bilinear $\psi^\dagger(x)\psi(x)$ term:

$$
\begin{aligned}
:\psi^\dagger(x)\psi(x): &= \frac{1}{L}\sum_q :c_q^\dagger c_q: + \frac{1}{L}\sum_{q>0}[e^{-iqx}\rho_q^\dagger + e^{iqx}\rho_q] \\
&= \frac{\hat{N}_r}{L} - \frac{1}{2\pi}\sum_{q>0}\frac{iq}{\sqrt{n_q}}\left[e^{iqx}b_q - e^{-iqx}b_q^\dagger\right].
\end{aligned}
\tag{8.57}
$$

We then get:

$$
:\psi^\dagger(x)\psi(x): = \frac{\hat{N}}{L} - \frac{1}{\sqrt{2\pi}}\partial_x\phi(x). \tag{8.58}
$$

Effective Dirac Theory

Now we consider the case where both the right-moving and left-moving fermion branches are involved. Note that in the previous convention, the direction of momentum for left-mover is inverted, i.e., $\psi_L(k) = \tilde{\psi}(-k)$ for $k \sim -k_F$. When take the Fourier transformation back to the coordinate space,

$$
\int_{-k_F-\Lambda}^{-k_F+\Lambda}\frac{dk}{2\pi}\tilde{\psi}(k) = \psi_L(-x), \tag{8.59}
$$

which would lead to $\psi(x) = \psi_R(x) + \psi_L(-x)$. To avoid such notational mess, we change the direction of the left-mover by defining $\psi_\pm = \psi_{R/L}(\pm x)$. All the relations we have yet discussed are preserved, sometimes up to a change of coordinate: $x \to -x$, this amounts to

$$
\psi_\pm(x) = \frac{1}{\sqrt{2\pi a}}e^{\pm i\frac{2\pi\hat{N}_\pm x}{L}}e^{-i\sqrt{2\pi}\phi_\pm(x)}. \tag{8.60}
$$

Note that from now on, we will constantly omit the Klein factor since for the calculation of correlation function, the Klein factors will alway cancel out finally.

Furthermore, we can shift the momentum $\pm k_F$ to the origin so that the free theory becomes a Dirac theory, with $N_\pm = 0$. The field expansion is now

$$
\begin{aligned}
\psi(x) &= e^{-ik_Fx}\int_{-\Lambda}^{\Lambda}\frac{dk}{2\pi}e^{ikx}\psi(-k_F+k) + e^{ik_Fx}\int_{-\Lambda}^{\Lambda}\frac{dk}{2\pi}e^{ikx}\psi(k_F+k) \\
&= e^{+ik_Fx}\psi_+(x) + e^{-ik_Fx}\psi_-(x).
\end{aligned}
\tag{8.61}
$$

To better work with two fermion branches, we define a new set of bosonic fields:

$$
\phi(x) \equiv \frac{\phi_-(x)+\phi_+(x)}{\sqrt{2}}, \quad \theta(x) \equiv \frac{\phi_-(x)-\phi_+(x)}{\sqrt{2}}. \tag{8.62}
$$

Note that we can define a new set of canonical variables from these fields. First consider the commutation relation

$$
[\phi(x),\theta(y)] \xrightarrow{a\to 0} -\frac{i}{2}\,\mathrm{sgn}(x-y) + \frac{i}{L}(x-y) \tag{8.63}
$$

Define a new variable $\Pi(x) \equiv \partial_x \theta(x)$, the canonical commutation relation is:

$$[\phi(x), \Pi(y)] = i\delta(x-y) - \frac{i}{L} \overset{L\to\infty}{\longrightarrow} i\delta(x-y). \tag{8.64}$$

Note that the fermion fields can also be written as

$$\psi_\pm(x) = \frac{1}{\sqrt{2\pi a}} \exp\left[-i\sqrt{\pi}\phi(x) \pm i\sqrt{\pi}\int_{-\infty}^{x} dy\ \Pi(y)\right]. \tag{8.65}$$

8.2.3 Bosonization Dictionary

Bosonic Hamiltonian

Now we go back to the Hamiltonian with linear dispersion (for single fermion branch):

$$H_0 = v_F \sum_{k,r} k\ {:}c^\dagger_{k,r} c_{k,r}{:} = v_F \int dx\ {:}\psi^\dagger(x)(-i\partial_x)\psi(x){:}. \tag{8.66}$$

Since $b^\dagger_q$ raise the energy of any eigenstate of H_0 by q unit, we have the commutation relation:

$$\left[H_0, b^\dagger_{q,r}\right] = q b^\dagger_{q,r}. \tag{8.67}$$

The Hamiltonian satisfies such relation can only be the bosonic bilinear:

$$H_0 = v_F \sum_r \sum_{q>0} q b^\dagger_{q,r} b_{q,r} + \frac{\pi v_F}{L}\sum_r \hat{N}_r(\hat{N}_r + 1). \tag{8.68}$$

The constant part comes from the fact that

$$H_0|N\rangle_0 = \frac{2\pi v_F}{L}\frac{\sum_r \hat{N}_r(\hat{N}_r+1)}{2}|N\rangle_0, \quad b^\dagger_{q,r} b_{q,r}|N\rangle_0 = 0. \tag{8.69}$$

Also, the bosonic operator can also be expressed as

$$H_0 = \frac{v_F}{2}\int dx\ {:}\left[\Pi^2 + (\partial_x\phi)^2\right]{:} + \frac{\pi v_F}{L}\sum_r \hat{N}_r(\hat{N}_r+1). \tag{8.70}$$

Now consider the density-density interaction:

$$\mathcal{V}_{\text{int}} = u \int dx\ {:}\psi^\dagger(x)\psi(x){:}^2 \tag{8.71}$$

The fermion density operator is

$$\begin{aligned} \psi^\dagger(x)\psi(x) = {} & \psi^\dagger_-(x)\psi_-(x) + \psi^\dagger_+(x)\psi_+(x) + \\ & e^{-2ik_F x}\psi^\dagger_+(x)\psi_-(x) + e^{2ik_F x}\psi^\dagger_-(x)\psi_+(x). \end{aligned} \tag{8.72}$$

The oscillation term is irrelevant unless the system is half-filled. For the system away from half-filling, the bosonized interaction is also free:

$$\mathcal{V}_{\text{int}} = \frac{u}{2\pi}\sum_r \int dx\ (\partial_x \phi_r)^2. \tag{8.73}$$

The half-filling case is more subtle. In order to be more precise, we consider the lattice Hamiltonian where the interaction is

$$H_I = u\sum_j \left(n_j - \frac{1}{2}\right)\left(n_{j+1} - \frac{1}{2}\right). \tag{8.74}$$

The bosonization procedure gives:

$$\begin{aligned} H_I &= u\int dx\left[:\psi_+^\dagger(x)\psi_+(x) + \psi_-^\dagger(x)\psi_- : +(-1)^j\left(\psi_+^\dagger(x)\psi_-(x) + \psi_-^\dagger(x)\psi_+(x)\right)\right] \\ &\quad\times\left[:\psi_+^\dagger(x)\psi_+(x) + \psi_-^\dagger(x)\psi_-(x) : -(-1)^j\left(\psi_+^\dagger(x)\psi_-(x) + \psi_-^\dagger(x)\psi_+(x)\right)\right] \\ &= u\int dx\left[\left(\frac{1}{\sqrt{\pi}}\partial_x\phi\right)^2 - \left(\psi_+^\dagger\psi_- + \psi_-^\dagger\psi_+\right)^2\right] + \left\{(-1)^j \text{ oscillations}\right\}. \end{aligned} \tag{8.75}$$

The oscillations can be omitted, and the second term is (we omit the Klein factors since they eventually cancel out)

$$\psi_+^\dagger\psi_- + h.c. = \frac{1}{2\pi a}e^{i\sqrt{4\pi}\phi(x)} + h.c. = \frac{1}{\pi a}\cos\left[\sqrt{4\pi}\phi(x)\right]. \tag{8.76}$$

When square it, we should take note of the fact the microscopically, the square is actually

$$\frac{1}{\pi^2 a^2}\cos\left[\sqrt{4\pi}\phi(x)\right]\cos\left[\sqrt{4\pi}\phi(x+a)\right].$$

Using the identity

$$\cos\alpha\cos\beta = \frac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2}, \tag{8.77}$$

The square is (also note that any constant can be neglected since the final expression is normal-ordered)

$$\frac{1}{2\pi^2 a^2}\left[\cos\left(\sqrt{16\pi}\phi\right) + \cos\left(\sqrt{4\pi}\partial_x\phi\right)\right] \simeq \frac{\cos\left[\sqrt{16\pi}\phi(x)\right]}{2\pi^2 a^2} - \frac{[\partial_x\phi(x)]^2}{\pi}. \tag{8.78}$$

The interaction is then mapped to

$$H_I = u\int dx\left\{\frac{2}{\pi}(\partial_x\phi)^2 - \frac{\cos\left[\sqrt{16\pi}\phi(x)\right]}{2\pi^2 a^2}\right\}. \tag{8.79}$$

Green's Function

Now we calculate the free fermion field Green's function (Matsubara):

$$\begin{aligned} -G_\pm(x,\tau) &= \langle T\psi_r(x,\tau)\psi_r(0)\rangle \\ &= \frac{1}{2\pi a}\left\langle Te^{-i\sqrt{2\pi}\phi_r(x,\tau)}e^{i\sqrt{2\pi}\phi_r(0)}\right\rangle \\ &= \frac{1}{2\pi a}\left\langle Te^{-i\sqrt{2\pi}[\phi_r(x,\tau)-\phi_r(0)]}\right\rangle e^{-i2\pi[\phi_r(x,\tau),\phi_r(0)]} \\ &= \frac{1}{2\pi a}e^{2\pi\langle T\phi_r(x,\tau)\phi_r(0)-\phi_r(0)\phi_r(0)\rangle}, \end{aligned} \tag{8.80}$$

where we have used the fact that for bosonic linear terms B,

$$\begin{aligned}\langle e^{i\lambda B}\rangle &= \sum_{n=0}^{\infty}\frac{(i\lambda B)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-\lambda^2)^n}{(2n)!}\langle B^{2n}\rangle \\ &= \sum_{n=0}^{\infty}\frac{(-\lambda^2)^n}{(2n)!}\frac{(2n)!}{2^n n!}\langle B^2\rangle^n = e^{-\frac{1}{2}\lambda^2\langle B^2\rangle}.\end{aligned} \tag{8.81}$$

The bosonic correlation is then computed as (assume $\tau > 0$)

$$\begin{aligned}\langle \phi_\pm(x,\tau)\phi_\pm(0)\rangle &= \frac{1}{L}\sum_{q>0}\frac{e^{-aq}}{q}e^{\pm iqx - q\tau)} \\ &= -\frac{1}{2\pi}\ln\left[1 - e^{-\frac{2\pi}{L}(a\mp ix+\tau)}\right] \\ &= \frac{1}{2\pi}\ln\frac{L/2\pi}{a\mp ix+\tau}.\end{aligned} \tag{8.82}$$

Furthermore,

$$\langle T\phi_\pm(x,\tau)\phi_\pm(0) - \phi_\pm(0)\phi_\pm(0)\rangle = \frac{1}{2\pi}\ln\frac{a}{a\mp ix+\tau}. \tag{8.83}$$

And thus the fermion correlation is

$$-G_\pm(x,\tau) = \frac{1}{2\pi}\frac{1}{a\mp ix+\tau}. \tag{8.84}$$

Summery of the Results

Here we summerize the bosonization identity we have got so far:

$$\phi_\pm(x) = \frac{1}{\sqrt{2\pi}}\sum_{q>0}\frac{e^{-aq/2}}{\sqrt{n_q}}\left[e^{\pm iqx}b_q + e^{\mp iqx}b_q^\dagger\right] \tag{8.85}$$

$$\psi_\pm(x) \sim \frac{1}{\sqrt{2\pi a}}e^{\pm i\frac{2\pi\hat{N}_\pm x}{L}}\exp\left[-i\sqrt{2\pi}\phi_\pm(x)\right] \tag{8.86}$$

$$\psi_\pm(x) \sim \frac{1}{\sqrt{2\pi a}}\exp\left[-i\sqrt{\pi}\phi(x) \pm i\sqrt{\pi}\int_{-\infty}^{x}dy\ \Pi(y)\right] \tag{8.87}$$

$$[\phi_\pm(x),\phi_\pm(y)] = \pm\frac{i}{2}\operatorname{sgn}(x-y) \mp \frac{i}{L}(x-y) \tag{8.88}$$

$$[\phi(x),\theta(y)] = -\frac{i}{2}\operatorname{sgn}(x-y) + \frac{i}{L}(x-y) \tag{8.89}$$

$$[\phi(x),\Pi(y)] = i\delta(x-y) - \frac{i}{L} \tag{8.90}$$

$$-i\psi^\dagger(x)\partial_x\psi(x) = \frac{v_F}{2}\left[\Pi^2 + (\partial_x\phi)^2\right] \tag{8.91}$$

$$\psi_\pm^\dagger(x)\psi_\pm(x) = \frac{\hat{N}_\pm}{L} - \frac{\partial_x\phi_\pm}{\sqrt{2\pi}} \tag{8.92}$$

$$\psi_\pm^\dagger(x)\psi_\mp(x) = \frac{1}{2\pi a}e^{\pm i\sqrt{4\pi}\phi(x)} \tag{8.93}$$

8.3 Sine-Gordon Model

8.3.1 Effective Theory for Half-filling

In this section we consider the 1D half-filling interacting system. Base on the bosonization dictionary we constructed, the bosonic Hamiltonian is

$$H = \int \frac{dx}{K} \left\{ \frac{1}{2} \left[K\Pi^2 + \frac{1}{K}(\partial_x \phi)^2 \right] + \frac{y}{2\pi^2 a^2} \cos\left[\sqrt{16\pi}\phi\right] \right\}, \tag{8.94}$$

where

$$K = \frac{1}{\sqrt{1+4u/\pi}}, \quad y = \frac{u}{\sqrt{1+4u/\pi}}. \tag{8.95}$$

We can shift the variables as:

$$\Pi \to \frac{\Pi}{\sqrt{K}}, \quad \phi \to \sqrt{K}\phi, \tag{8.96}$$

(note that the canonical commutation relation is preserved in this way) and the Hamiltonian becomes

$$H = \int \frac{dx}{K} \left\{ \frac{1}{2} \left[\Pi^2 + (\partial_x \phi)^2 \right] + \frac{y}{2\pi^2 a^2} \cos\left[\sqrt{16\pi K}\phi\right] \right\}. \tag{8.97}$$

Omit the constant factor, and the Euclidean Lagrangian is

$$\mathcal{L} = \frac{1}{2}(\nabla\phi)^2 + \frac{y}{2\pi^2 a^2} \cos\left[\sqrt{16\pi K}\phi\right] \equiv \frac{1}{2}(\nabla\phi)^2 + \frac{y}{2\pi^2 a^2} \cos\left[\beta\phi\right]. \tag{8.98}$$

8.3.2 RG Analysis

When write ϕ as the sum of slow and fast modes, the partition function is

$$Z = \int D\phi_s e^{-\frac{1}{2}\int d^2x (\nabla\phi_s)^2} \left\langle \exp\left\{ -\frac{y}{2\pi^2 a^2} \int d^2x \cos\left[\beta(\phi_s + \phi_f)\right] \right\} \right\rangle_f \tag{8.99}$$

Under the rescaling $x \to x' = e^{dt}x$, $\phi(x) \to \phi'(x') = \phi(x')$, the free field action is invariant. The effective action can be perturbatively evaluated as

$$S_{\text{eff}} = S_0 + \langle S_1 \rangle_f - \frac{1}{2}(\langle S_1^2 \rangle_f - \langle S_1 \rangle_f^2) + \text{higher orders}. \tag{8.100}$$

To the first order,

$$\begin{aligned} \langle S_1 \rangle &= \frac{y}{2\pi^2 a^2} \int d^2x \, \langle \cos\left[\beta(\phi_s + \phi_f)\right] \rangle_f \\ &= -\frac{y}{2\pi^2 a^2} \int d^2x \cos(\beta\phi_s) \langle \cos(\beta\phi_f) \rangle_f, \end{aligned} \tag{8.101}$$

where we have use the identity

$$\cos\left[\beta(\phi_s)\right] = \cos(\beta\phi_s)\cos(\beta\phi_f) - \sin(\beta\phi_s)\sin(\beta\phi_f) \tag{8.102}$$

and note the fact that $\sin(\beta\phi_f)$ term will not contribute when integrated over the fast field. Also, note of the fact that

$$\langle e^{i\beta\phi} \rangle = e^{-\frac{1}{2}\beta^2\langle\phi^2\rangle}. \tag{8.103}$$

We then compute the expectation

$$\begin{aligned}
\langle\cos(\beta\phi)\rangle_f &= \left\langle \frac{e^{i\beta\phi}+e^{-i\beta\phi}}{2} \right\rangle_f = e^{-\frac{1}{2}\beta^2\langle\phi^2\rangle} \\
&= \exp\left[-\frac{\beta^2}{2}\int_{\Lambda(1-dt)}^{\Lambda} \frac{kdk}{2\pi}\frac{1}{k^2}\right] \\
&= 1-\frac{\beta^2}{4\pi\Lambda}dt.
\end{aligned} \tag{8.104}$$

In this way, the rescaling is

$$y \to e^{2dt}\left(1-\frac{\beta^2}{4\pi\Lambda}dt\right)y. \tag{8.105}$$

In the cut-off unit where Λ is set to 1, the beta function is

$$\frac{dy}{dt} = (2-4K)y = 4\left[\frac{1}{2}-\left(1-\frac{4u}{\pi}\right)^{-\frac{1}{2}}\right]y. \tag{8.106}$$

Setting $x \equiv 2-4K$, the first order RG equation is

$$\frac{dy}{dt} = xy. \tag{8.107}$$

Now consider the second-order expansion

$$\begin{aligned}
\delta S^{(2)} = \frac{y^2}{4\pi^4a^4}\int d^2x_1d^2x_2 \Big\{ &\langle\cos\left[\beta\phi_s(x_1)+\beta\phi_f(x_1)\right]\cos\left[\beta\phi_s(x_2)+\beta\phi_f(x_2)\right]\rangle_f \\
&-\langle\cos\left[\beta\phi_s(x_1)+\beta\phi_f(x_1)\right]\rangle_f\langle\cos\left[\beta\phi_s(x_2)+\beta\phi_f(x_2)\right]\rangle_f \Big\}
\end{aligned} \tag{8.108}$$

The first term is

$$\begin{aligned}
&\cos\left[\beta\phi_s(x_1)+\beta\phi_f(x_1)\right]\cos\left[\beta\phi_s(x_2)+\beta\phi_f(x_2)\right] \\
=&\frac{1}{2}\cos\left[\beta\phi(x_1)+\beta\phi(x_2)\right]+\frac{1}{2}\cos\left[\beta\phi(x_1)-\beta\phi(x_2)\right] \\
=&\frac{1}{2}\cos\left[\beta\phi_s(x_1)+\beta\phi_s(x_2)\right]\cos\left[\beta\phi_f(x_1)+\beta\phi_f(x_2)\right]+ \\
&\frac{1}{2}\cos\left[\beta\phi_s(x_1)-\beta\phi_s(x_2)\right]\cos\left[\beta\phi_f(x_1)-\beta\phi_f(x_2)\right]+\text{sin terms}.
\end{aligned} \tag{8.109}$$

Similarly, we neglect all $\sin(\beta\phi_f)$ terms. The averaging gives:

$$\begin{aligned}
&\langle\cos\left[\beta\phi_s(x_1)+\beta\phi_f(x_1)\right]\cos\left[\beta\phi_s(x_2)+\beta\phi_f(x_2)\right]\rangle_f \\
=&\frac{1}{2}e^{-\frac{\beta^2}{2}\langle[\phi_f(x_1)+\phi_f(x_2)]^2\rangle}\cos[\beta\phi_s(x_1)+\beta\phi_s(x_2)]+ \\
&\frac{1}{2}e^{-\frac{\beta^2}{2}\langle[\phi_f(x_1)-\phi_f(x_2)]^2\rangle}\cos[\beta\phi_s(x_1)-\beta\phi_s(x_2)].
\end{aligned} \tag{8.110}$$

For the second term,

$$\begin{aligned}
&\left\langle \cos\left[\beta\phi_s(x_1)+\beta\phi_f(x_1)\right]\right\rangle_f \left\langle \cos\left[\beta\phi_s(x_2)+\beta\phi_f(x_2)\right]\right\rangle_f \\
=\ & \cos[\beta\phi_s(x_1)]\cos[\beta\phi_s(x_2)]e^{-\frac{\beta^2}{2}\langle\phi_f^2(x_1)+\phi_f^2(x_2)\rangle} \\
=\ & \frac{1}{2}e^{-\beta_f^2\langle\phi^2\rangle}\left\{\cos[\beta\phi_s(x_1)+\beta\phi_s(x_2)]+\cos[\beta\phi_s(x_1)-\beta\phi_s(x_2)]\right\}
\end{aligned} \tag{8.111}$$

The subtraction is

$$\begin{aligned}
&e^{-\beta^2\langle\phi_f^2\rangle}\left\{\left(e^{-\beta^2\langle\phi_f(x_1)\phi_f(x_2)\rangle}-1\right)\cos\left[\beta\phi_s(x_1)+\beta\phi_f(x_2)\right]\right. \\
&\left.+\left(e^{\beta^2\langle\phi_f(x_1)\phi_f(x_2)\rangle}-1\right)\cos\left[\beta\phi_s(x_1)-\beta\phi_f(x_2)\right]\right\}
\end{aligned} \tag{8.112}$$

Consider the bosonic correlation

$$\mathcal{G}(\boldsymbol{x}) \equiv \langle\phi_f(x)\phi_f(0)\rangle_f = F(\boldsymbol{x})dt + O(dt^2). \tag{8.113}$$

A straightforward calculation with the hard cut-off yields $F(\boldsymbol{x})$ a Bessel function with long oscillating tail. However, it can be shown that implementing a smooth cut-off will make $F(\boldsymbol{x})$ short ranged. For this reason, we can switch to the center-of-mass coordinate:

$$\boldsymbol{R} = \frac{\boldsymbol{x}_1+\boldsymbol{x}_2}{2}, \quad \boldsymbol{r} = \boldsymbol{x}_1-\boldsymbol{x}_2. \tag{8.114}$$

In this way, the term $\cos\left[\beta\phi_s(x_1)+\beta\phi_f(x_2)\right]$ is approximated by a $\cos\left[2\beta\phi_s(x)\right]$ term, which is oscillating at double frequency, and thus is regarded as irrelevant.

The remaining term can be simplified by the approximation:

$$\cos\left[\beta\phi_s(x_1)-\beta\phi_f(x_2)\right] \sim 1-\frac{\beta^2}{2}[\boldsymbol{r}\cdot\nabla\phi_s(\boldsymbol{R})]^2 \tag{8.115}$$

The constant term only contributes to the the infinity of free energy. The non-trivial contribution is:

$$\begin{aligned}
&\int d^2R\int d^2r\left(e^{\beta^2\langle\phi_f(x_1)\phi_f(x_2)\rangle}-1\right)\cos\left[\beta\phi_s(x_1)-\beta\phi_f(x_2)\right] \\
\simeq\ & -\frac{\beta^4}{2}\int d^2R\int d^2rF(\boldsymbol{r})[\boldsymbol{r}\cdot\nabla\phi_s(\boldsymbol{R})]^2
\end{aligned} \tag{8.116}$$

Note that

$$\int d^2r\ r_ir_j = \delta_{ij}\int rdrd\theta\ r^2\cos^2(\theta) = \pi\delta_{ij}\int dr\ r^3, \tag{8.117}$$

and the above expression can be formulated as:

$$\begin{aligned}
&\int d^2R\int d^2r\left(e^{\beta^2\langle\phi_f(x_1)\phi_f(x_2)\rangle}-1\right)\cos\left[\beta\phi_s(x_1)-\beta\phi_f(x_2)\right] \\
\simeq\ & -\frac{\pi\beta^4}{2}\left[\int_0^\infty dr\ r^3F(r)\right]\int d^2R[\nabla\phi_s(\boldsymbol{R})]^2 \\
\equiv\ & -\pi\beta^4A\times\frac{1}{2}\int d^2R[\nabla\phi(\boldsymbol{R})]^2
\end{aligned} \tag{8.118}$$

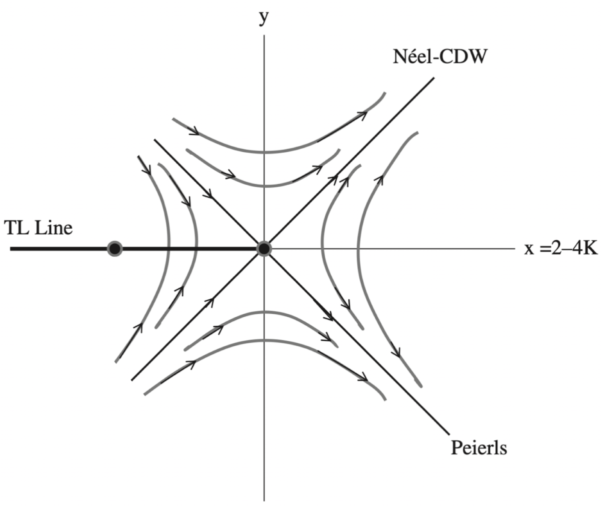


Figure 8.4: KT RG-flow.

We see the second order perturbation renormalize the free field, as it change normalization of the field:

$$\phi \to \phi' = Z^{\frac{1}{2}}\phi, \quad Z^{\frac{1}{2}} = 1 + \frac{1}{2}\pi\beta^4 A\left(\frac{y}{2\pi^2 a^2}\right)^2. \tag{8.119}$$

To preserve the form of the $\cos\beta\phi$ term, we have to shift

$$d\beta = Z^{-\frac{1}{2}}\beta - \beta = -\frac{\beta^5 A}{8\pi^3 a^4}y^2 dt \tag{8.120}$$

The RG equation to the second order is then

$$\frac{dx}{dt} = \frac{d}{dt}\left(2 - \frac{\beta^2}{4\pi}\right) = \frac{\beta^6 A}{16\pi^4 a^4}y^2. \tag{8.121}$$

The $\beta = \sqrt{4\pi(2-x)}$ is dependent on x. While near the fixed point $x = y = 0$, $\beta \simeq \sqrt{8\pi}$, the RG equation is then

$$\begin{aligned} \frac{dy}{dt} &= xy, \\ \frac{dx}{dt} &= \frac{32A}{\pi a^4}y^2. \end{aligned} \tag{8.122}$$

Setting $c^2 = 32A/\pi a^4$, we note that the differential relation implies

$$d(x^2 - c^2 y^2) = 0, \tag{8.123}$$

i.e., the flow is along the hyperbolas, with marginal line described by $x = \pm cy$. The RG-flow is shown in Fig. 8.4.

We see that there are three regimes of the RG flow: in the weak interacting regime, the field theory flows to the Tomonaga-Luttinger (TL) line; for strong interaction, depending on the sign of y, the systems develop two charge-density-wave (CDW) order, namely the Neel-CDW and Peierls-CDW.

8.3.3 Phase Diagram

Tomonaga-Luttinger Liquid

When $K < \frac{1}{2}$, the interacting cosine term become irrelevant. In this case, the system is in a liquid state, called the *Tomonaga-Luttinger liquid.* The system on the TL line is described by the non-interacting Hamiltonian:

$$H = \int \frac{dx}{K}\frac{1}{2}\left[K\Pi^2 + \frac{1}{K}(\partial_x\phi)^2\right], \tag{8.124}$$

As discussed, we can define a new set of variables to map the Hamiltonian to that of the original bosonic model:

$$\begin{aligned}\phi' &= \frac{\phi'_- + \phi'_+}{\sqrt{2}} = \frac{\phi}{\sqrt{K}} = \frac{\phi_- + \phi_+}{\sqrt{2K}} \\ \theta' &= \frac{\phi'_- - \phi'_+}{\sqrt{2}} = \sqrt{K}\theta = \sqrt{\frac{K}{2}}(\phi_- - \phi_+).\end{aligned} \tag{8.125}$$

The fermion mode is

$$\begin{aligned}\psi_\pm(x) &= \frac{1}{\sqrt{2\pi a}}\exp\left[-i\sqrt{2\pi}\phi'_\pm(x)\right] \\ &= \frac{1}{\sqrt{2\pi a}}\exp\left\{-i\sqrt{2\pi}\left[K_\pm\phi_+(x) + K_\mp\phi_-(x)\right]\right\},\end{aligned} \tag{8.126}$$

where

$$K_\pm \equiv \frac{K^{\frac{1}{2}} \pm K^{-\frac{1}{2}}}{2}. \tag{8.127}$$

The Greens function is (assume $\tau > 0$):

$$-G_\pm(x,\tau) = \frac{1}{2\pi a}e^{2\pi\langle\phi'_\pm(x,\tau)\phi'_\pm(0) - \phi'_\pm(0)\phi'_\pm(0)\rangle} \tag{8.128}$$

The bosonic correlation is

$$\begin{aligned}\langle\phi'_\pm(x,\tau)\phi'_\pm(0)\rangle &= K_\pm^2\langle\phi_+(x,\tau)\phi_+(0)\rangle + K_\mp^2\langle\phi_-(x,\tau)\phi_-(0)\rangle \\ &= \frac{K_\pm^2}{2\pi}\ln\frac{L/2\pi}{a - ix + \tau} + \frac{K_\mp^2}{2\pi}\ln\frac{L/2\pi}{a + ix + \tau}\end{aligned} \tag{8.129}$$

So that

$$\begin{aligned}-G_\pm(x,\tau) &= \frac{1}{2\pi a}\left[\frac{a}{a + \tau - ix}\right]^{K_\pm^2}\left[\frac{a}{a + \tau + ix}\right]^{K_\mp^2} \\ &= \frac{1}{2\pi}\frac{1}{a + \tau \mp ix}\left[\frac{a^2}{a^2 + x^2 + \tau^2}\right]^{K_-^2}\end{aligned} \tag{8.130}$$

Note that when $K = 0$, $K_- = 0$, and the theory agrees with the free fermion prediction:

$$-G(x, 0^+) = \frac{1}{2\pi}\frac{1}{a - ix} = \int \frac{dk}{2\pi}e^{ikx - ak}\theta(k) \tag{8.131}$$

However, when $K_- \neq 0$, the additional factor will smear the pole to a brach cut. To see this, note that from the dimensional analysis, the Green's function is of anomalous dimension $-K_-^2$. In the momentum space, this means that the Green's function scales as

$$G(k,\omega) \sim (k,\omega)^{K_-^2}. \tag{8.132}$$

This implies that the density

$$n(k) = n_0 + ck^{K_-^2}. \tag{8.133}$$

Charge Density Wave Order

We first consider the case where $y > 0$ is sufficiently large. Consider

$$\cos\left[\sqrt{16\pi}\phi(x)\right] = \frac{1}{2} - \sin^2\left[\sqrt{4\pi}\phi(x)\right]. \tag{8.134}$$

The energy is minimized if

$$\sin\left[\sqrt{4\pi}\phi(x)\right] = \pm 1. \tag{8.135}$$

Note that in our bosonization dictionary,

$$\psi_\pm^\dagger(x)\psi_\mp(x) = \frac{1}{2\pi a} e^{\pm i\sqrt{4\pi}\phi(x)}, \tag{8.136}$$

so that

$$\frac{i}{\pi a}\sin\left[\sqrt{4\pi}\phi(x)\right] = \psi_+^\dagger(x)\psi_-(x) - \psi_-^\dagger(x)\psi_+(x). \tag{8.137}$$

We know for the Neel-CDW phase, the order parameter is

$$i\left\langle \psi_+^\dagger(x)\psi_-(x) - \psi_-^\dagger(x)\psi_+(x)\right\rangle. \tag{8.138}$$

On the other hand, for $y < 0$, consider

$$\cos\left[\sqrt{16\pi}\phi(x)\right] = -\frac{1}{2} + \cos^2\left[\sqrt{4\pi}\phi(x)\right]. \tag{8.139}$$

The energy is minimized if

$$\cos\left[\sqrt{4\pi}\phi(x)\right] = \pm 1. \tag{8.140}$$

8.3.4 One-dimensional Hubbard Model

Now we consider the fermion with spin. The Hubbard model has a non-interacting part,

$$H_0 = -\frac{1}{2}\sum_{s,n}\left[\psi_s^\dagger(n)\psi_s(n+1) + \text{ h.c. }\right] + \mu\sum_{s,n}\psi_s^\dagger(n)\psi_s(n), \tag{8.141}$$

where $s = \uparrow, \downarrow$ are two possible spin orientations. We do not assume $k_F = \frac{\pi}{2}$ at this point, and use a general chemical potential μ. Following the usual route, we get two copies of the spinless model:

$$H_0 = \sum_s \int \frac{dk}{2\pi}(\mu - \cos k)\psi_s^\dagger(k)\psi_s(k), \tag{8.142}$$

and the continuum version:

$$
\begin{aligned}
H_0 &= v_F \sum_s \int dx \left[\psi^\dagger_{s+}(x)\left(-i\partial_x\right)\psi_{s+}(x) + \psi^\dagger_{s-}(x)\left(i\partial_x\right)\psi_{s-}(x)\right] \\
&= \frac{v_F}{2}\sum_s \int dx \left[\Pi_s^2 + \left(\partial_x\phi_s\right)^2\right].
\end{aligned} \tag{8.143}
$$

Let us now turn on the Hubbard interaction,

$$
H_{\text{int}} = U\sum_n \psi^\dagger_\uparrow(n)\psi_\uparrow(n)\psi^\dagger_\downarrow(n)\psi_\downarrow(n), \tag{8.144}
$$

where $\psi_\uparrow, \psi_\downarrow$ stand for the original non-relativistic fermion. The Hubbard interaction is just the extreme short-range version of the screened Coulomb potential between fermions. Due to the Pauli principle, only opposite-spin electrons can occupy the same site.

Let us now express this interaction in terms of the Dirac fields:

$$
\psi^\dagger_\uparrow\psi_\uparrow\psi^\dagger_\downarrow\psi_\downarrow = \left[\psi^\dagger_{\uparrow+}\psi_{\uparrow+} + \psi^\dagger_{\uparrow-}\psi_{\uparrow-} + \left(\psi^\dagger_{\uparrow+}\psi_{\uparrow-}e^{-2iK_{\text{F}}n} + \text{h.c.}\right)\right] \times (\uparrow \to \downarrow). \tag{8.145}
$$

If we expand out the products and keep only the parts with no rapidly oscillating factors, we will, for generic k_F, get the following terms:

$$
H_{\text{int}} = U\left(j_{0\uparrow}j_{0\downarrow}\right) + U\sum_n \left(\psi^\dagger_{\uparrow+}(n)\psi_{\uparrow-}(n)\psi^\dagger_{\downarrow-}(n)\psi_{\downarrow+}(n) + \text{h.c.}\right), \tag{8.146}
$$

where

$$
U\left(j_{0\uparrow}j_{0\downarrow}\right) = \sum_n \left(\psi^\dagger_{\uparrow+}\psi_{\uparrow+} + \psi^\dagger_{\uparrow-}\psi_{\uparrow-}\right)\left(\psi^\dagger_{\downarrow+}\psi_{\downarrow+} + \psi^\dagger_{\downarrow-}\psi_{\downarrow-}\right) \tag{8.147}
$$

If we now bosonize these terms as per the dictionary, we get, in the continuum,

$$
H_{\text{int}} = U\left[\frac{\partial_x\phi_\uparrow\partial_x\phi_\downarrow}{\pi} + \frac{1}{2\pi^2\alpha^2}\cos\sqrt{4\pi}\left(\phi_\uparrow - \phi_\downarrow\right)\right] \tag{8.148}
$$

We can now separate the theory into two parts by introducing charge and spin fields ϕ_c and ϕ_s:

$$
\phi_{c/s} = \frac{\phi_\uparrow \pm \phi_\downarrow}{\sqrt{2}}. \tag{8.149}
$$

The parametrization bring the first term in the interaction to a free theory:

$$
\begin{aligned}
\partial_x\phi_\uparrow\partial_x\phi_\downarrow &= \frac{1}{2}\partial_x(\phi_c + \phi_s)\partial_x(\phi_c - \phi_s) \\
&= \frac{1}{2}\left[(\partial_x\phi_c)^2 + (\partial_x\phi_s)^2\right].
\end{aligned} \tag{8.150}
$$

The Hamiltonian then become decoupled:

$$
H = H_c + H_s, \tag{8.151}
$$

where the charge and spin part of the Hamiltonian are:

$$
\begin{aligned}
H_c &= \int \frac{dx}{2K_c}\left[K_c\Pi_c^2 + \frac{1}{K_c}\left(\partial\phi_c\right)^2\right], \\
H_s &= \int \frac{dx}{2K_s}\left[K_s\Pi_s^2 + \frac{1}{K_s}\left(\partial\phi_s\right)^2 + \frac{U}{2\pi^2\alpha^2}\cos\sqrt{8\pi}\phi_s\right],
\end{aligned} \tag{8.152}
$$

where

$$K_{c/s} = \frac{1}{\sqrt{1 \pm \frac{U}{\pi}}}. \tag{8.153}$$

The fact that $K_s \neq K_c$ means that charge and spin move at different velocities. This *spin-charge separation* cannot be understood in terms of interacting electrons whose charge and spin would be irrevocably bound. This is more evidence of the demise of the quasiparticle, adiabatically connected to the primordial fermion.

Chapter 9

Topological Field Theory

9.1 Chern-Simons Theory

Assume the action of the microscopical theory has the form $S[\psi_i]$, where $\{\psi_i\}$ denotes all degrees of microscopical freedom. If the system has the U(1) symmetry, we can always rewrite the field theory as a gauge theory:

$$S[\psi_i; A] = S[\psi_i] + \int d^d x \; j^\mu(x) A_\mu(x), \tag{9.1}$$

where the current j^μ is the Noether current. The gauge field $A^\mu(x)$ is regarded as the back ground field which has no dynamics. If we are interested in the low-energy physics, especially for gapped system, the ground state physics, we can formally integrate out other degrees of freedom, the resulting effective theory has only the gauge degree of freedom:

$$Z_{\text{eff}}[A] = \int D[\psi_i] e^{iS[\psi_i;A]}. \tag{9.2}$$

In this section, we consider the effective gauge field on $(2+1)$-dimensional space-time. The effective action should also be gauge-invariant. The allowed terms include

$$A \wedge dA, \;\; dA \wedge dA, \;\; \text{higher order terms.}$$

From dimensional analysis, the first term is most relevant in the low-energy. Such effective theory is the *Chern-Simons theory*:

$$S_{\text{CS}} = \frac{k}{4\pi} \int d^3 x \; \varepsilon_{\mu\nu\rho} A^\mu \partial^\nu A^\rho. \tag{9.3}$$