Notes on Quantum Field Theory

Jie Ren

February 20, 2022

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Chapter 1

Relativistic Quantum Field Theory

1.1 Lorentz Invariance

1.1.1 The Lorentz Algebra

The metric is chosen to be

$$g_{\mu\nu} = g^{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1). \tag{1.1}$$

The Lorentz transformation $\Lambda^{\mu}{}_{\nu}$ satisfies

$$\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}g_{\mu\nu} = g_{\alpha\beta}. \tag{1.2}$$

From this we have

$$g^{\gamma\alpha}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}g_{\mu\nu} = g^{\gamma\alpha}g_{\alpha\beta} \quad \Longrightarrow \quad \Lambda_{\nu}{}^{\gamma}\Lambda^{\nu}{}_{\beta} = \delta^{\gamma}{}_{\beta},$$

The inverse Lorentz transformation satisfies:

$$(\Lambda^{-1})^{\mu}{}_{\nu} = \Lambda_{\nu}{}^{\mu}.$$

The infinitesimal transformation is denoted as

$$\begin{array}{r}\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \delta\omega^{\mu}{}_{\nu} \\ (\Lambda^{-1})^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} - \delta\omega^{\mu}{}_{\nu}\end{array} \quad \Longrightarrow \quad g_{\alpha\nu}\delta\omega^{\nu}{}_{\beta} + \delta\omega^{\mu}{}_{\alpha}g_{\mu\beta} = \delta\omega_{\alpha\beta} + \delta\omega_{\beta\alpha} = 0.$$

A representation of Lorentz group $U(\Lambda)$ can be parametrized as:

$$U(\Lambda) = \exp\left(\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right). \tag{1.3}$$

Another useful parametrization is

$$\theta_i \equiv \frac{1}{2}\varepsilon_{ijk}\omega_{jk}, \;\; \beta_i \equiv \omega_{0i}.$$

A new set of generators are:

$$J_i \equiv \frac{1}{2}\varepsilon_{ijk}M^{jk}, \;\; K_i \equiv M^{i0}, \tag{1.4}$$

where J_i's are the generators of the spatial rotations, and K_i's are the generators of Lorentz boosts.

In the fundamental representation, the generators are represented by

$$J_1 = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 0 & -i \\ & & i & 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & & & \\ & 0 & & i \\ & & 0 & \\ & -i & & 0 \end{bmatrix}, \quad J_3 = \begin{bmatrix} 0 & & & \\ & 0 & -i & \\ & i & 0 & \\ & & & 0 \end{bmatrix},$$
$$K_1 = \begin{bmatrix} 0 & -i & & \\ -i & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0 & & -i & \\ & 0 & & \\ -i & & 0 & \\ & & & 0 \end{bmatrix}, \quad K_3 = \begin{bmatrix} 0 & & & -i \\ & 0 & & \\ & & 0 & \\ -i & & & 0 \end{bmatrix}.$$

The Lie algebra of the Lorentz algebra can be explicitly done using the fundamental representation. The result is

$$\begin{aligned} [J_i, J_j] &= i\varepsilon_{ijk} J_k, \\ [J_i, K_j] &= i\varepsilon_{ijk} K_k, \\ [K_i, K_j] &= -i\varepsilon_{ijk} J_k. \end{aligned} \tag{1.5}$$

By defining a new set of generators:

$$N_i^L \equiv \frac{J_i - iK_i}{2}, \; N_i^R \equiv \frac{J_i + iK_i}{2}. \tag{1.6}$$

They satisfies two independent $\mathfrak{su}(2)$ algebra:

$$\begin{aligned} \left[N_i^L, N_j^L\right] &= i\varepsilon_{ijk} N_k^L, \\ \left[N_i^R, N_j^R\right] &= i\varepsilon_{ijk} N_k^R, \\ \left[N_i^L, N_j^R\right] &= 0. \end{aligned}$$

That is, the Lorentz algebra is isomorphic to two $\mathfrak{su}(2)$ algebra,

$$\mathfrak{so}(3,1) \approx \mathfrak{su}_L(2) \oplus \mathfrak{su}_R(2). \tag{1.7}$$

From Eq. (1.7), we know that the representation of the Lorentz algebra can be labelled by j_L and j_R. Note that the fundamental representation correspond to

$$\left(j_L = \frac{1}{2}, j_R = \frac{1}{2}\right).$$

The specific form of the group is

$$\Lambda(\vec{\theta}, \vec{\beta}) = \exp\left[i(\vec{\theta} + i\vec{\beta}) \cdot \vec{N}^L + i(\vec{\theta} - i\vec{\beta}) \cdot \vec{N}^R\right].$$

The spinor representations are those with $j_L = 1/2$ or $j_R = 1/2$. Specifically, we define the left-hand spinor ψ_L and right-hand spinor ψ_R that transform as:

$$\begin{aligned} \Lambda_L(\vec{\theta}, \vec{\beta})\psi_L &= \exp\left(\frac{i}{2}\vec{\theta} \cdot \vec{\sigma} - \frac{1}{2}\vec{\beta} \cdot \vec{\sigma}\right)\psi_L, \\ \Lambda_R(\vec{\theta}, \vec{\beta})\psi_R &= \exp\left(\frac{i}{2}\vec{\theta} \cdot \vec{\sigma} + \frac{1}{2}\vec{\beta} \cdot \vec{\sigma}\right)\psi_R. \end{aligned} \tag{1.8}$$

Using the fact $\sigma^2 \cdot \vec{\sigma}^* \cdot \sigma^2 = -\vec{\sigma}$, the left-hand and the right-hand representations are related by:

$$\begin{aligned} \sigma^2\Lambda_L^*\sigma^2 = \Lambda_R, \quad \sigma^2\Lambda_L^T\sigma^2 = \Lambda_L^{-1}, \\ \sigma^2\Lambda_R^*\sigma^2 = \Lambda_L, \quad \sigma^2\Lambda_R^T\sigma^2 = \Lambda_R^{-1}. \end{aligned} \tag{1.9}$$

For this reason, the left-hand and right-hand spinor can be interchanged by

$$\begin{aligned} \sigma^2\psi_L^* \sim \chi_R, \quad \psi_L^\dagger\sigma^2 \sim \chi_R^\dagger \\ \sigma^2\psi_R^* \sim \chi_L, \quad \psi_R^\dagger\sigma^2 \sim \chi_L^\dagger. \end{aligned} \tag{1.10}$$

1.1.2 The Invariant Symbols

The invariant symbols can be thought as the Clebsch-Gordan coefficients that help to form singlets. The first singlet comes from the decomposition

$$\frac{1}{2} \otimes \frac{1}{2} \approx 0 \oplus 1.$$

Correspondingly, we can check that for each-hand-side spinor, the quadratic forms

$$\psi_L^T\sigma^2\chi_L \quad \text{or} \quad \psi_R^T\sigma^2\chi_R \tag{1.11}$$

are singlets. We can define the first invariant symbol as[1]

$$\varepsilon^{ab} = \varepsilon^{\dot{a}\dot{b}} = i(\sigma^2)_{ab}, \quad \varepsilon_{ab} = \varepsilon_{\dot{a}\dot{b}} = -i(\sigma^2)_{ab}. \tag{1.12}$$

The symbol ε^{ab} or ε_{ab} also serve as the index raising/lowering symbol, i.e.,

$$\varepsilon^{ab}\psi_b = \psi^a, \;\; \varepsilon_{ab}\psi^b = \psi_a. \tag{1.13}$$

The singlet (1.11) is then defined as the inner product of two spinors:

$$\psi \cdot \chi \equiv \varepsilon_{ab}\psi^a\chi^b = \psi^a\chi_a = -\varepsilon_{ba}\psi^a\chi^b = -\psi_b\chi^b. \tag{1.14}$$

In addition, because of (1.10), the expressions

$$\psi_L^\dagger\chi_R \quad \text{and} \quad \psi_R^\dagger\chi_L$$

are also singlets.

Besides, we know there should be another invariant symbol from the decomposition

$$\left(\frac{1}{2}, 0\right) \otimes \left(0, \frac{1}{2}\right) \approx (0,0) \oplus \cdots.$$

For this reason, we are searching for the symbol M that the expression

$$M^\mu_{a\dot{b}}\psi_L^a\chi_R^{\dot{b}}$$

transforms as the Lorentz vector. The matrix M^μ should transform as

$$M^\mu \longrightarrow \Lambda_L^T \cdot M^\mu \cdot \Lambda_R = \Lambda^\mu{}_\nu M^\nu.$$

[1] We use the dotted symbol to denote the right-hand spinor indices.

Use the fact that $\sigma^2 \cdot \Lambda_L^T \cdot \sigma^2 = \Lambda_L^{-1}$, the above equation transforms to

$$\left(\sigma^2 M^\mu\right) \longrightarrow \Lambda_L^{-1} \cdot \left(\sigma^2 M^\mu\right) \cdot \Lambda_R.$$

We then show the matrices $\sigma^\mu = (\sigma^0, \vec{\sigma})$ satisfies the requirement. Firstly, for the spatial rotation,

$$\Lambda_L(\vec{\theta}, \vec{0}) = \Lambda_R(\theta, \vec{0}) = \exp\left(i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right)$$

The Pauli matrix transform as

$$\left(1 - i\delta\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) \sigma^j \left(1 + i\delta\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) = \sigma^j + i\delta\theta_i \left(-i\varepsilon_{ijk}\sigma^k\right)$$

Secondly, for the boosts,

$$\Lambda_L(\vec{0}, \vec{\beta}) = \exp\left(-\vec{\beta} \cdot \frac{\vec{\sigma}}{2}\right), \ \Lambda_R(\vec{0}, \vec{\beta}) = \exp\left(+\vec{\beta} \cdot \frac{\vec{\sigma}}{2}\right)$$

The Pauli matrix transform as

$$\left(1 + \delta\vec{\beta} \cdot \frac{\vec{\sigma}}{2}\right) \sigma^\mu \left(1 + \delta\vec{\beta} \cdot \frac{\vec{\sigma}}{2}\right) = \begin{cases} \sigma^0 + i\delta\beta_i \cdot (-i\sigma^i), & \mu = 0 \\ \sigma^j + i\delta\beta_j(-i\sigma^0), & \mu = j \end{cases}.$$

We thus have shown indeed that

$$\psi_L^T \sigma^2 \sigma^\mu \chi_R \tag{1.15}$$

is a Lorentz vector. Further more, from (1.10), we know that

$$\eta_R^\dagger \sigma^\mu \chi_R \tag{1.16}$$

is also a Lorentz vector. Similarly, consider the Lorentz vector

$$N^\mu_{\dot{a}b} \psi_R^{\dot{a}} \chi_R^b,$$

which together with σ^2 should transforms as

$$\left(\sigma^2 N^\mu\right) \longrightarrow \Lambda_R^{-1} \cdot \left(\sigma^2 N^\mu\right) \cdot \Lambda_L.$$

We can check that $\bar{\sigma}^\mu = (\sigma^0, -\vec{\sigma})$ satisfies the requirement, and thus

$$\eta_L^\dagger \bar{\sigma}^\mu \chi_L \tag{1.17}$$

is also a Lorentz vector.

1.2 Klein-Gordon Field

In relativistic quantum field theory, the Lagrangian should be a singlet under Lorentz transformation. Different free fields correspond to different representation of the Lorentz algebra. The symmetry under Lorentz transformation also restrict the possible terms that can appear in the Lagrangian.

The simplest case is when $j_L = j_R = 0$, corresponding to the scalar field, which we denote as $\phi(x)$. Since the field it self is singlet, any polynomial of the field in principle can appear in the theory. When considering the free theory, we restrict our attention to the quadratic terms. We require the field theory to have a dynamical term, which contains derivative the the field. The derivative operator ∂^μ transforms as the fundamental representation. To be Lorentz invariant, the allowed free theory can only be

$$\mathcal{L}_{\mathrm{K-G}} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{m^2}{2}\phi^2 \simeq -\frac{1}{2}\phi(\partial^2 + m^2)\phi. \tag{1.18}$$

1.2.1 Path-integral Formalism

In this note, the space-time Fourier transformation is defined as

$$\begin{aligned}\tilde{\phi}(k) &= \int d^d x e^{ik\cdot x}\phi(x),\\ \phi(x) &= \int \frac{d^d k}{(2\pi)^d} e^{-ik\cdot x}\tilde{\phi}(k),\end{aligned} \tag{1.19}$$

where the inner product of two 4-momentum and 4-coordinate is

$$k\cdot x = \omega t - \vec{k}\cdot\vec{x}. \tag{1.20}$$

Consider the action for free field with source

$$S_0[\phi, J] = \int d^d x \left[\mathcal{L}_0(\phi) + J(x)\cdot\phi(x)\right]. \tag{1.21}$$

In momentum space, the free Lagrangian (with source) is

$$\tilde{\mathcal{L}}_0[\phi_k, J] = \tilde{\phi}(k)(k^2 - m^2)\tilde{\phi}(-k) + \tilde{J}(k)\cdot\tilde{\phi}(-k) + \tilde{\phi}(k)\cdot\tilde{J}(-k).$$

In the path integral formalism, we consider the partition function

$$Z_0[J] = \int D[\phi]\exp(iS_0[\phi, J]). \tag{1.22}$$

The partition function for free field:

$$\begin{aligned}\frac{Z_0[J]}{Z_0[0]} &= \exp\left(-\frac{i}{2}\int \frac{d^d k}{(2\pi)^d}\frac{\tilde{J}(k)\tilde{J}(-k)}{k^2 - m^2 + i\epsilon}\right)\\ &= \exp\left(-\frac{i}{2}\int d^d x_1 d^d x_2 J(x_1)\Delta_0(x_1 - x_2)J(x_2)\right).\end{aligned}$$

where the propagator is

$$\Delta_0(x_1 - x_2) = \int \frac{d^d k}{(2\pi)^d}\frac{e^{-ik\cdot x}}{k^2 - m^2 + i\epsilon}, \tag{1.23}$$

where the extra $i\epsilon$ term is use to bring the singularities infinitesimally below the real axis. This infinitesimal value can be absorbed into the mass term, by regarding the mass term m^2 as $m^2 - i\epsilon$.

Note that $\Delta_0(x_1 - x_2)$ is related to the correlation function:

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = \frac{\delta}{i\delta J(x_1)}\frac{\delta}{i\delta J(x_2)}Z_0[J] = i\Delta(x_1 - x_2). \tag{1.24}$$

Remark 1. Gaussian Integral for Real Scalar Field

The real Gaussian integral formula is

$$\int d\boldsymbol{v} \exp\left(-\frac{1}{2}\boldsymbol{v}^T \cdot A \cdot \boldsymbol{v} + \boldsymbol{b}^T \cdot \boldsymbol{v}\right) = \sqrt{\frac{(2\pi)^N}{\det A}} \exp\left(\frac{1}{2}\boldsymbol{b}^T \cdot A^{-1} \cdot \boldsymbol{b}\right), \tag{1.25}$$

where $\boldsymbol{v}, \boldsymbol{b}$ are two N-dimensional vector, and A is an $N \times N$ matrix. For the field integral, we absorbed the $(2\pi)^{N/2}$ term into the measure, and express the path integral for the Gaussian field as:

$$\begin{aligned} Z[J] &= \int D[\phi] \exp\left(\frac{i}{2}\int d^d x \phi \hat{A} \phi + i \int d^d x J\phi\right) \\ &= Z[0] \exp\left[-\frac{i}{2}\int d^d x_1 d^d x_2 J(x_1) A^{-1}(x_1 - x_2) J(x_2)\right]. \end{aligned}$$

We make use of (1.25) by making the identification

$$A = \bigoplus_{|k|} \begin{pmatrix} 0 & k^2 - m^2 \\ k^2 - m^2 & 0 \end{pmatrix}, \quad b = \bigoplus_{|k|} \begin{pmatrix} \tilde{J}(k) \\ \tilde{J}(-k) \end{pmatrix}.$$

This gives the propagator in the momentum space:

$$\tilde{\Delta}_0(k) = \frac{1}{k^2 - m^2}.$$

1.2.2 Canonical Quantization

The classical equation of motion for Klein-Gordon field is

$$(-\partial_t^2 + \nabla^2 - m^2)\phi(\vec{x}, t) = 0. \tag{1.26}$$

The solution to Eq. (1.26) is proportional to the plane wave:

$$\phi(\vec{x}, t) \propto e^{-i\omega_{\boldsymbol{k}} t + i\vec{p}\cdot\vec{x}} + e^{i\omega_{\vec{k}} t - i\boldsymbol{p}\cdot\vec{x}},$$

where the energy is $\omega_{\boldsymbol{k}} = \boldsymbol{k}^2 + m^2$ and $\vec{k}$ is the momentum as the conserved quantity. The general solution to the EOM is

$$\phi(\vec{x}, t) \propto \int \frac{d^3k}{(2\pi)^3} \left(a_k e^{-i\omega_{\boldsymbol{k}} t + i\vec{k}\cdot\vec{x}} + a_k^* e^{i\omega_{\boldsymbol{k}} t - i\vec{k}\cdot\vec{x}}\right). \tag{1.27}$$

The canonical quantization promote the coefficient a_k/a_k^* to the particle annihilation/creation operator $a_k/a_k^\dagger$, with the commutation relation

$$[a_k, a_p^\dagger] = (2\pi)^3 \delta^3(\vec{k} - \vec{p}). \tag{1.28}$$

The single-particle state with momentum $\vec{k}$ is created by $a_k^\dagger$ operators acting on the vacuum:

$$|\vec{k}\rangle \equiv \sqrt{2\omega_{\boldsymbol{k}}} a_k^\dagger |0\rangle, \tag{1.29}$$

where $|\vec{k}\rangle$ is a state with a single particle of momentum $\vec{k}$.

Remark 2. Lorentz Invariance of Single-particle State

The factor of $\sqrt{2\omega_{\boldsymbol{k}}}$ in Eq. (1.29) is just a convention, but it will make some calculations easier. To compute the normalization of one-particle states, we start with

$$\langle 0|0\rangle = 1, \tag{1.30}$$

which leads to

$$\langle \vec{p}|\vec{k}\rangle = 2\sqrt{\omega_{\boldsymbol{p}}\omega_{\boldsymbol{k}}}\left\langle 0 \left| a_p a_k^\dagger \right| 0 \right\rangle = 2\omega_{\boldsymbol{p}}(2\pi)^3\delta^3(\vec{p}-\vec{k}). \tag{1.31}$$

The identity operator for one-particle states is

$$1 = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\boldsymbol{p}}}|\vec{p}\rangle\langle\vec{p}|, \tag{1.32}$$

which we can check with

$$|\vec{k}\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\boldsymbol{p}}}|\vec{p}\rangle\langle\vec{p}|\vec{k}\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\boldsymbol{p}}}2\omega_{\boldsymbol{p}}(2\pi)^3\delta^3(\vec{p}-\vec{k})|\vec{p}\rangle = |\vec{k}\rangle.$$

The identity operator Eq. (1.32) is Lorentz invariant since it can be expressed as

$$1 = \int \frac{d^3p\,d\omega}{(2\pi)^4}2\pi\delta(\omega^2 - \boldsymbol{p}^2 - m^2)|\vec{p}\rangle\langle\vec{p}|. \tag{1.33}$$

We fix the normalization by requiring

$$\langle\vec{k}|\phi(\vec{x},0)|0\rangle = e^{-i\vec{k}\cdot\vec{x}},$$

and the quantized field operator is

$$\phi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{k}}}}\left(a_k e^{-ik\cdot x} + a_k^\dagger e^{ik\cdot x}\right). \tag{1.34}$$

Consider the two-point correlation:

$$\begin{aligned} i\Delta(x_1 - x_2) &= \langle 0|T\phi(x_1)\phi(x_2)|0\rangle \\ &= \theta(t_1 - t_2)\langle 0|\phi(x_1)\phi(x_2)|0\rangle + \theta(t_2 - t_1)\langle 0|\phi(x_2)\phi(x_1)|0\rangle. \end{aligned}$$

Note that

$$\langle 0|\phi(x_1)\phi(x_2)|0\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2) - i\omega_{\vec{k}}\tau},$$

where $\tau = t_1 - t_2$. The propagator can be written as

$$\begin{aligned} i\Delta(x_1 - x_2) &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}\left[e^{-i\omega_{\vec{k}}\tau}\theta(\tau) + e^{i\omega_{\vec{k}}\tau}\theta(-\tau)\right] \\ &= \int \frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}\int \frac{d\omega}{2\pi i}\frac{-e^{i\omega\tau}}{\omega^2 - \omega_k^2 + i\epsilon} \\ &= \int \frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x_1-x_2)}\frac{i}{k^2 - m^2 + i\epsilon}. \end{aligned}$$

1.3 Vector Field

If we can choose $j_L = j_R = 1/2$, the field is transformed as Lorentz vector. We denote the field as $A^\mu(x)$. Some possible quadratic forms for the vector field that forms singlets are

$$A^\mu A_\mu,\ (\partial_\mu A^\mu)^2,\ A^\nu \partial^2 A_\nu,\ \varepsilon_{\mu\nu\rho\lambda}\partial^\mu A^\nu \partial^\rho A^\lambda.$$

For the field theory describe the electromagnetic field, we require the theory to further have gauge symmetry, i.e., invariant under

$$A^\mu(x) \to A^\mu(x) + \partial^\mu \alpha(x). \tag{1.35}$$

The gauge invariant forbids the first term, and forces the second and third term to combine as

$$(\partial_\mu A^\mu)^2 - A^\nu \partial^2 A_\nu \sim \frac{1}{2}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A^\nu - \partial_\nu A_\mu) \equiv \frac{1}{2} F^{\mu\nu} F_{\mu\nu}.$$

where we have define a field-strength tensor

$$F^{\mu\nu} \equiv (\partial^\mu A^\nu - \partial^\nu A^\mu) = \begin{bmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & E_3 & -E_2 \\ -B_2 & -E_3 & 0 & E_1 \\ -B_3 & E_2 & -E_1 & 0 \end{bmatrix}. \tag{1.36}$$

Note that the fourth term is called the *theta term*, which can be written as a boundary term

$$\varepsilon_{\mu\nu\rho\lambda}\partial^\mu A^\nu \partial^\rho A^\lambda = \partial^\mu(\varepsilon_{\mu\nu\rho\lambda} A^\nu \partial^\rho A^\lambda).$$

The Lagrangian describing the electromagnetic field is given by

$$\mathcal{L}_{\text{Maxwell}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{1.37}$$

1.3.1 Path-integral Formalism

We define the gauge fixing function

$$G(A) = \partial_\mu A^\mu(x) - \omega(x) = 0$$

The gauge transformation has the form:

$$A^\alpha_\mu(x) = A_\mu(x) + \partial_\mu \alpha(x).$$

We then have

$$1 \propto \int D[\alpha] \det\left(\frac{\delta G(A^\alpha)}{\delta \alpha}\right) \delta(G(A)).$$

Inset the identity operator into the path integral formula

$$Z[J] \propto \det\left(\partial^2\right) \int D[\alpha] D[A] e^{iS[A,J]} \delta(\partial_\mu A^\mu - \omega(x)).$$

The above equation does not depend on $\omega(x)$. We can then integrate over $\omega(x)$ with gaussian weight

$$\begin{aligned} Z[J] &\propto \int D[\omega] e^{-i\int d^d x \frac{\omega^2}{2\xi}} \int D[\alpha] D[A] e^{iS[A,J]} \delta(\partial_\mu A^\mu - \omega) \\ &= \int D[A] e^{iS[A,J]} \exp\left\{ i \left[S[A,J] - \int d^d x \frac{1}{2\xi} (\partial_\mu A^\mu)^2 \right] \right\}. \end{aligned}$$

In momentum space, the modified Langriangian is

$$\tilde{\mathcal{L}}_\xi(k) = \tilde{A}^\mu(k) \left[-k^2 g_{\mu\nu} + \left(1 - \frac{1}{\xi}\right) k_\mu k_\nu \right] \tilde{A}^\nu(-k) + \tilde{J}_\mu(k) \tilde{A}^\mu(-k) + \tilde{A}^\mu(k) \tilde{J}_\mu(-k).$$

We can check that

$$\left[-k^2 g_{\mu\nu} + \left(1 - \frac{1}{\xi}\right) k_\mu k_\nu \right]^{-1} = \frac{-g^{\mu\nu} + (1-\xi) k^\mu k^\nu}{k^2}. \tag{1.38}$$

Thus, the partition function is

$$\frac{Z_{\text{maxwell}}[J]}{Z_{\text{maxwell}}[0]} = \exp\left[-\frac{i}{2} \int d^d x_1 d^d x_2 J_\mu(x_1) \Pi^{\mu\nu}(x_1 - x_2) J_\nu(x_2) \right], \tag{1.39}$$

where

$$\Pi^{\mu\nu}(x_1 - x_2) = \int \frac{d^d k}{(2\pi)^d} e^{-ik\cdot(x_1 - x_2)} \frac{-g^{\mu\nu} + (1-\xi) k^\mu k^\nu}{k^2}. \tag{1.40}$$

The propagator is

$$\begin{aligned} \langle 0 | T A^\mu(x_1) A^\nu(x_2) | 0 \rangle &= \frac{1}{Z_{\text{Maxwell}}[0]} \frac{\delta}{iJ_\mu(x_1)} \frac{\delta}{iJ_\nu(x_2)} Z_{\text{Maxwell}}[J] \bigg|_{J=0} \\ &= i\Pi^{\mu\nu}(x_1 - x_2). \end{aligned} \tag{1.41}$$

1.3.2 Canonical Quantization

In momentum space, the Lagrangian transforms to

$$\tilde{A}^\mu(k) \left(-k^2 g_{\mu\nu} + k_\mu k_\nu \right) \tilde{A}^\nu(-k). \tag{1.42}$$

The EOM in momentum space is

$$(-k^2 g_{\mu\nu} + k_\mu k_\nu) \tilde{A}^\nu(k) = 0.$$

Since the linear operator $(-k^2 g_{\mu\nu} + k_\mu k_\nu)$ is singular, i.e.,

$$(-k^2 g_{\mu\nu} + k_\mu k_\nu) k^\nu = 0.$$

The gauge freedom can be used to further restrict

$$A^0 = 0.$$

In this way, there are only two independent polarization for EOM solution

$$A^\mu = e^{-ik\cdot x} \epsilon_j^\mu, \ j = 1, 2, \tag{1.43}$$

where

$$\epsilon_1 = (0,1,0,0),\ \epsilon_2 = (0,0,1,0).$$

The field expansion is then

$$A^\mu = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \sum_{j=1}^{2} \left(\epsilon_j^\mu a_{k,j} e^{-ik\cdot x} + \epsilon_j^{\mu *} a_{k,j}^\dagger e^{ik\cdot x} \right). \tag{1.44}$$

A single-particle state with polarization vector ϵ_j is defined as

$$|k, \epsilon_j\rangle = \sqrt{2\omega_k}\vec{\epsilon}_j a_{k,j}^\dagger |0\rangle. \tag{1.45}$$

Note that then the field is off shell (internal photon line), the photon can be space-like or time-like, and then there are an additional polarization. In general,

$$\sum_{j=1}^{3} \epsilon_j^{\mu *} \epsilon_j^\nu = -(1 - P_k) = -(g^{\mu\nu} - k^\mu k^\nu),$$

where P_k is the projection to 4-momentum k. The propagator is then

$$i\Pi(x_1 - x_2) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x_1 - x_2)} \frac{-i(g^{\mu\nu} - k^\mu k^\nu)}{k^2 + i\epsilon}. \tag{1.46}$$

1.4 Dirac Field

Based on previous discussion, the Lagrangian for spinor field can have

$$\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L,\ \psi_R^\dagger \sigma^\mu \partial_\mu \psi_R,\ \psi_L^\dagger \psi_R,\ \psi_R^\dagger \psi_L,\ \psi_L \cdot \psi_L,\ \psi_R \cdot \psi_R.$$

The Dirac field describe the theory with both left-hand and right-hand spinors. The Lagrangian is

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi} \left(i\gamma^\mu \partial_\mu - m \right) \psi, \tag{1.47}$$

where

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix},\ \bar{\psi} = \begin{pmatrix} \psi_R^\dagger & \psi_L^\dagger \end{pmatrix},\ \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}. \tag{1.48}$$

In addition, we could consider using the last two terms as the mass, the result theory is the *Majorana field theory*:

$$\begin{aligned} \mathcal{L}^L_{\text{Majorana}} &= \psi_L^\dagger \left(i\bar{\sigma}^\mu \partial_\mu - m\sigma^2 \right) \psi_L, \\ \mathcal{L}^R_{\text{Majorana}} &= \psi_R^\dagger \left(i\sigma^\mu \partial_\mu - m\sigma^2 \right) \psi_R. \end{aligned} \tag{1.49}$$

1.4.1 Path-integral Formalism

Consider the partition function with source

$$Z_{\text{Dirac}}[J] = \int D[\bar{\psi}, \psi] \exp\left[i \int d^dx \left(\mathcal{L}_{\text{Dirac}} + \bar{\eta}\psi + \bar{\psi}\eta \right) \right]. \tag{1.50}$$

In momentum space:

$$S = \int \frac{d^d k}{(2\pi)^d} \left[\tilde{\bar{\psi}}(k)(\not{k} - m)\tilde{\psi}(k) + \tilde{\bar{\eta}}(k)\tilde{\psi}(k) + \tilde{\bar{\psi}}(k)\tilde{\eta}(k) \right]. \tag{1.51}$$

Using the Gaussian integral formula (for Grassman variables), the partition function is:

$$\begin{aligned} \frac{Z_{\text{Dirac}}[J]}{Z_{\text{Dirac}}[0]} &= \exp\left[-i \int \frac{d^d k}{(2\pi)^d} \tilde{\bar{\eta}}(k) \frac{1}{\not{k} - m} \tilde{\eta}(k) \right] \\ &= \exp\left[-i \int d^d x_1 d^d x_2 \bar{\eta}(x_1) \cdot D_F(x_1 - x_2) \cdot \eta(x_2) \right] \end{aligned} \tag{1.52}$$

where

$$D_F(x_1 - x_2) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{-ik\cdot(x_1 - x_2)}}{\not{k} - m} = \int \frac{d^d k}{(2\pi)^d} \frac{\not{k} + m}{k^2 - m^2} e^{-ik\cdot(x_1 - x_2)}. \tag{1.53}$$

Note that the propagator is

$$\begin{aligned} \langle 0 | T \psi^\alpha(x_1) \bar{\psi}^\beta(x_2) | 0 \rangle &= \frac{1}{Z_{\text{Dirac}}[0]} \frac{\delta}{i\delta\bar{\eta}_\alpha(x_1))} \frac{i\delta}{\delta\eta_\beta(x_2)} Z_{\text{Dirac}}[\bar{\eta}, \eta] \Big|_{\eta = \bar{\eta} = 0} \\ &= i D_F^{\alpha\beta}(x_1 - x_2), \end{aligned} \tag{1.54}$$

where the sign in the variational derivative comes from the anti-commutation relation of the fermionic fields.

1.4.2 Canonical Quantization

In momentum space, the Lagrangian ie=s:

$$\tilde{\bar{\psi}}(p)(\not{p} - m)\tilde{\psi}(p),$$

The EOM is

$$(\not{p} - m)\tilde{\psi}(p) = 0 \tag{1.55}$$

The general solution of the Dirac equation can be written as a linear combination of plane waves. The positive frequency waves are of the form

$$\psi(x) = u(p)e^{-ip\cdot x}, \quad p^2 = m^2, \quad p^0 > 0$$

There are two linearly independent solutions for $u(p)$,

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\xi^s \end{pmatrix}, \quad s = 1, 2$$

which we normalize according to

$$\bar{u}^r(p)u^s(p) = 2m\delta^{rs} \quad \text{or} \quad u^{r\dagger}(p)u^s(p) = 2\omega_{\boldsymbol{p}}\delta^{rs}$$

In exactly the same way, we can find the negative-frequency solutions:

$$\psi(x) = v(p)e^{+ip\cdot x}, \quad p^2 = m^2, \quad p^0 > 0. \ (3.61)$$

Note that we have chosen to put the + sign into the exponential, rather than having $p^0 < 0$. There are two linearly independent solutions for $v(p)$,

$$v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\eta^s \end{pmatrix}, \quad s = 1,2$$

where η^s is another basis of two-component spinors. These solutions are normalized according to

$$\bar{v}^r(p)v^s(p) = -2m\delta^{rs} \quad \text{or} \quad v^{r\dagger}(p)v^s(p) = +2\omega_{\boldsymbol{p}}\delta^{rs}$$

The u's and v's are also orthogonal to each other:

$$\begin{aligned} \bar{u}^r(p)v^s(p) &= 0, \quad u^{r\dagger}(\boldsymbol{p},\omega_{\boldsymbol{p}})v^s(-\boldsymbol{p},\omega_{\boldsymbol{p}}) = 0, \\ \bar{v}^r(p)u^s(p) &= 0, \quad v^{r\dagger}(\boldsymbol{p},\omega_{\boldsymbol{p}})u^s(-\boldsymbol{p},\omega_{\boldsymbol{p}}) = 0. \end{aligned}$$

A useful identity is

$$\begin{aligned} \sum_s u^s(p)\bar{u}^s(p) &= \not{p} + m, \\ \sum_s v^s(p)\bar{v}^s(p) &= \not{p} - m. \end{aligned}$$

The Dirac field expansion is

$$\begin{aligned} \psi(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\sum_s \left(a^s_{\mathbf{p}}u^s(p)e^{-ip\cdot x} + b^{s\dagger}_{\mathbf{p}}v^s(p)e^{ip\cdot x}\right), \\ \bar{\psi}(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\sum_s \left(b^s_{\mathbf{p}}\bar{v}^s(p)e^{-ip\cdot x} + a^{s\dagger}_{\mathbf{p}}\bar{u}^s(p)e^{ip\cdot x}\right). \end{aligned} \tag{1.56}$$

Now let us investigate the propagator

$$\begin{aligned} iD_{F,\alpha\beta}(x_1 - x_2) &= \langle 0|T\psi_\alpha(x_1)\bar{\psi}_\beta(x_2)|0\rangle \\ &= \theta(\tau)\langle 0|\psi_\alpha(x_1)\bar{\psi}_\beta(x_2)|0\rangle - \theta(-\tau)\langle 0|\bar{\psi}_\beta(x_2)\psi_\alpha(x_1)|0\rangle. \end{aligned} \tag{1.57}$$

On the RHS, the first term is

$$\begin{aligned} \langle 0|\psi_\alpha(x_1)\bar{\psi}_\beta(x_2)|0\rangle &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left[\sum_s u^s_\alpha(p)\bar{u}^s_\beta(p)\right]e^{-ip\cdot(x_1-x_2)} \\ &= (i\not{\partial} + m)_{\alpha\beta}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}e^{-ip\cdot(x_1-x_2)}. \end{aligned}$$

For the second term:

$$\begin{aligned} \langle 0|\bar{\psi}_\beta(x_2)\psi_\alpha(x_1)|0\rangle &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left[\sum_s \bar{v}^s_\beta(p)v^s_\alpha(p)\right]e^{ip\cdot(x_1-x_2)} \\ &= -(i\not{\partial} + m)_{\alpha\beta}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}e^{ip\cdot(x_1-x_2)}. \end{aligned}$$

Together, the Dirac propagator is:

$$\begin{aligned} iD_F(x_1 - x_2) &= (i\not{\partial} + m)i\Delta(x_1 - x_2) \\ &= \int \frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x_1-x_2)}\frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}. \end{aligned}$$

Chapter 2

Scalar Field Theory

2.1 Interaction and Scattering

2.1.1 Lehmann Representation

The interacting Hamiltonian do not conserve particle number, and the ground state $|\Omega\rangle$ is no longer the vacuum $|0\rangle$.

Consider the Green's function

$$iG(x_1 - x_2) = \begin{cases} \langle\Omega|T\phi(x_1)\phi(x_2)|\Omega\rangle & \text{Klein} - \text{Gordon} \\ \langle\Omega|TA(x_1)A(x_2)|\Omega\rangle & \text{Maxwell} \\ \langle\Omega|T\psi(x_1)\bar{\psi}(x_2)|\Omega\rangle & \text{Dirac} \end{cases}.$$

We can insert a complete basis

$$1 = |\Omega\rangle\langle\Omega| + \sum_\lambda \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} |\lambda_{\vec{k}}\rangle\langle\lambda_{\vec{k}}|$$

into the correlation function,[1] the Green's function takes the form (take K-G field as the example):

$$iG(x_1 - x_2) = \sum_\lambda \int \frac{d^3k}{(2\pi)^3} \left[\theta(t_1 - t_2)\langle\Omega|\phi(x_1)|\lambda_{\vec{k}}\rangle\langle\lambda_{\vec{k}}|\phi(x_2)|\Omega\rangle + (t_1 \leftrightarrow t_2, x_1 \leftrightarrow x_2)\right].$$

Note that

$$\langle\lambda_{\vec{k}}|\phi(x)|\Omega\rangle = \langle\lambda_{\vec{k}}|e^{iP\cdot x}\phi(0)e^{-iP\cdot x}|\Omega\rangle = e^{ik\cdot x} \left.\langle\lambda_0|\phi(0)|\Omega\rangle\right|_{k^0=\omega_{\vec{k}}}.$$

Following the same procedure as we do for the free field theory,

$$G(x_1 - x_2) = \int_0^\infty \frac{dM^2}{2\pi} \rho(M^2) G_0(x_1 - x_2; M^2), \tag{2.1}$$

[1] Here we assume $\langle\Omega|\phi(x)|\Omega\rangle = 0$ unless there is spontaneously symmetry breaking happening.

where the spectral function $\rho(M^2)$ is

$$\rho(M^2) = \sum_\lambda (2\pi)\delta(M^2 - m_\lambda^2)|\langle\Omega|\phi(0)|\lambda_0\rangle|^2.$$

In particle, near the one-particle state the Green's function looks like:

$$i\tilde{G}(k) = \frac{iZ_\phi}{k^2 - m^2 + i\epsilon} + \text{regular terms}.$$

If we renormalize the field strength

$$\phi_R(x) = \frac{1}{\sqrt{Z_\phi}}\phi_0(x),$$

the Green's function then has exactly the same form as free theory. This normalization factor Z_ϕ is exactly what we obtained in the loop correction to the propagator.

2.1.2 Scattering Amplitude

Consider the scattering process in the interaction picture,

$$\begin{aligned}\langle f|e^{-iHt}|i\rangle &= \langle f|T\exp\left(-i\int dtV_{\text{int}}(t)\right)|i\rangle \\ &= \langle f|T\exp\left(i\int d^dx\mathcal{L}_{\text{int}}(t)\right)|i\rangle.\end{aligned} \tag{2.2}$$

The S-matrix is defined as

$$\langle f|S|i\rangle = \langle f|\mathcal{T}\exp\left(i\int d^dx\mathcal{L}_{\text{int}}(t)\right)|i\rangle = 1 + i\langle f|\mathcal{T}|i\rangle. \tag{2.3}$$

Because of the additional momentum conservation,

$$\langle f|\mathcal{T}|i\rangle = (2\pi)^d\delta^d\left(\sum p\right)\mathcal{M}_{fi}. \tag{2.4}$$

2.1.3 LSZ for Klein-Gordon Field

For free theory, the particle annihilation operator is

$$\begin{aligned}\sqrt{2\omega_k}a_k &= i\int d^3x\ e^{ik\cdot x}(-i\omega_k + \partial_t)\phi(x), \\ \sqrt{2\omega_k}a_k^\dagger &= -i\int d^3x\ e^{-ik\cdot x}(i\omega_k + \partial_t)\phi(x).\end{aligned} \tag{2.5}$$

When interaction is turned on, the field operator $\phi(x)$ is renormalized as

$$\phi(x) \sim \sqrt{Z_\phi}\phi_{\text{in}}(x) \sim \sqrt{Z_\phi}\phi_{\text{out}}(x).$$

In this way, we have

$$
\begin{aligned}
\sqrt{2\omega_k}(a^\dagger_{\text{in}} - a^\dagger_{\text{out}}) &= iZ_\phi^{-1/2}\int dt\partial_t\left(\int d^3x\ e^{-ikx}(i\omega_k+\partial_t)\phi_0(x)\right)\\
&= iZ_\phi^{-1/2}\int d^4xe^{-ik\cdot x}(\omega_k^2+\partial_t^2)\phi_0(x)\\
&= iZ_\phi^{-1/2}\int d^4xe^{-ik\cdot x}\partial_t^2\phi_0(x) + \phi_0(x)(-\nabla^2+m^2)e^{-ik\cdot x}\\
&= iZ_\phi^{-1/2}\int d^4xe^{-ik\cdot x}(\partial^2+m^2)\phi_0(x)
\end{aligned}
$$

The initial and final states are:

$$
\begin{aligned}
|k_1,\cdots,k_m;\text{in}\rangle &= \left[\prod_{j=1}^m \sqrt{2\omega_{k_j}}a^\dagger_{\text{in}}(k_j)\right]|0\rangle,\\
|p_1,\cdots,p_n,\text{out}\rangle &= \left[\prod_{j=1}^n \sqrt{2\omega_{p_j}}a^\dagger_{\text{out}}(p_j)\right]|0\rangle.
\end{aligned}
\tag{2.6}
$$

The S-matrix is

$$
\begin{aligned}
S_{fi} &= \langle p_1,\cdots,p_n;\text{out}|S|k_1,\cdots,k_m;\text{in}\rangle\\
&= \frac{\langle 0|T\left(\prod\sqrt{2\omega_{p_j}}a_{p_j;\text{out}}\right)\int d^4x\exp(i\mathcal{L}_{\text{int}})\left(\prod\sqrt{2\omega_{k_j}}a^\dagger_{k_j;\text{in}}\right)|0\rangle}{\langle 0|T\int d^4x\exp(i\mathcal{L}_{\text{int}})|0\rangle}
\end{aligned}
$$

Since the scattering process correspond to the connected diagram, meaning that the initial and final state has distinct momentum particles. We are free to make the substitution

$$
a^\dagger_{\text{in}} \to (a^\dagger_{\text{in}} - a^\dagger_{\text{out}}),\ \ a_{\text{out}} \to -(a^\dagger_{\text{in}} - a^\dagger_{\text{out}})^\dagger.
$$

In this way, the S-matrix is

$$
\begin{aligned}
&\langle p_1,\cdots,p_n|S|k_1,\cdots,k_m\rangle\\
=&\prod_{i=1}^m\left[\int d^dx_i\ e^{ip_i\cdot x_i}\frac{-\partial^2-m_i^2}{i\sqrt{Z_\phi}}\right]\prod_{j=m+1}^{m+n}\left[\int d^dx_j\ e^{-ik_j\cdot x_j}\frac{-\partial^2-m_j^2}{i\sqrt{Z_\phi}}\right]\\
&\times\frac{\langle 0|T\phi_0(x_1)\cdots\phi_0(x_{m+n})\int d^dx\exp(i\mathcal{L}_{\text{int}})|0\rangle}{\langle 0|T\int d^dx\exp(i\mathcal{L}_{\text{int}})|0\rangle}\\
=&\prod_{i=1}^m\left[\frac{p_i^2-m_i^2}{i\sqrt{Z_\phi}}\right]\prod_{j=m+1}^{m+n}\left[\frac{k_j^2-m_j^2}{i\sqrt{Z_\phi}}\right]\tilde{G}(-p_1,\cdots,-p_n,k_1,\cdots,k_m).
\end{aligned}
$$

Note that in the second equality, we move the operator ∂^2 out of the time-ordering operator, which will actually create contact terms. The contact terms can be shown to have no contribution to the S-matrix. Also, the Green function is defined as

$$
\begin{aligned}
G(x_1,\cdots,x_{m+n}) &\equiv \langle\Omega|T\phi(x_1)\cdots\phi(x_{m+n})|\Omega\rangle\\
&= \frac{\langle 0|T\phi_0(x_1)\cdots\phi_0(x_{m+n})\int d^dx\exp(i\mathcal{L}_{\text{int}})|0\rangle}{\langle 0|T\int d^dx\exp(i\mathcal{L}_{\text{int}})|0\rangle}.
\end{aligned}
$$

The extra factor before the momentum-space Green's function effectively cancel out the external propagator. Thus the LSZ formula (2.1.3) means that the S-matrix is the amputated Green's function.

Remark 3. Contact Terms

We first consider the time-ordered two-point function:

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = \theta(t_1 - t_2)\langle 0|\phi(x_1)\phi(x_2)|0\rangle - \theta(t_2 - t_1)\langle 0|\phi(x_2)\phi(x_1)|0\rangle.$$

Take time derivative on both side:

$$\begin{aligned}\partial_{t_1}\langle 0|T\phi(x_1)\phi(x_2)|0\rangle &= \langle 0|T\partial_{t_1}\phi(x_1)\phi(x_2)|0\rangle + \delta(t_1 - t_2)\langle 0|[\phi(x_1), \phi(x_2)]|0\rangle \\ &= \langle 0|T\partial_{t_1}\phi(x_1)\phi(x_2)|0\rangle.\end{aligned}$$

The second equality follows from the fact that x_1, x_2 is equal-time. Take the the time derivative once more:

$$\partial_{t_1}^2\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = \langle 0|T\partial_{t_1}^2\phi(x_1)\phi(x_2)|0\rangle + \delta(t_1 - t_2)\langle 0|[\partial_{t_1}\phi(x_1), \phi(x_2)]|0\rangle.$$

The second term on the right hand side is the contact term. For free theory, $\partial_{t_1}\phi(x_1)$ is the canonical momentum, meaning that

$$[\phi(\vec{x}_1, t), \partial_t\phi(\vec{x}_1, t)] = i\hbar\delta^3(\vec{x}_1 - \vec{x}_2).$$

In general, for n-point correlation,

$$\begin{aligned}\partial_{t_1}^2\langle 0|T\phi_{x_1}\cdots\phi_{x_n}|0\rangle =&\langle 0|T\partial_{t_1}^2\phi_{x_1}\cdots\phi_{x_n}|0\rangle \\ &- i\hbar\sum_j \delta^4(x_1 - x_j)\langle 0|T\phi_{x_2}\cdots\cancel{\phi_{x_j}}\cdots\phi_{x_n}|0\rangle.\end{aligned}$$

In the LSZ formula, the contact term do not have any singularity. When the external legs approach to momentum shell, these regular terms vanishes, so the contact will not contribute to the S-matrix.

2.1.4 Perturbation Theory

For interaction theory, the partition function can be formally expressed as:

$$Z[J] = \exp\left(i\int d^dx\mathcal{L}_{\text{int}}\left[\frac{\delta}{i\delta J(x)}\right]\right)Z_0[J]. \tag{2.7}$$

The expectation values for a generic operator of the form $O(\phi)$ can be evaluated by the true partition function

$$\langle O(\phi)\rangle = \frac{1}{Z[0]}\,O\left[\frac{\delta}{i\delta J(x)}\right]Z[J]\bigg|_{J=0}. \tag{2.8}$$

The expression (2.8) can be expanded order by order using the Feynman diagram. Since the unconnected diagram can be absorbed into $Z[0]$, we only need to calculate the connected diagram.

The procedure of perturbative expansion with only connected diagrams can be formally represented by introducing the quantity

$$Z[J] = \exp\left(iW[J]\right). \tag{2.9}$$

The perturbative expansion of $W[J]$ contain only the connected diagrams. Eq. (2.8) can then be rephrased as

$$\langle O(\phi)\rangle = i\, O\left[\frac{\delta}{i\delta J(x)}\right] W[J]\bigg|_{J=0}. \tag{2.10}$$

Example 1. Two-point Correlation

Consider the two-point connected correlation (propagator):

$$\begin{aligned} i\Delta(x_1 - x_2) &= \langle \mathcal{T}\phi(x_1)\phi(x_2)\rangle_c \\ &= i\, \frac{\delta^2 W[J]}{i\delta J(x_1) i\delta J(x_2)}\bigg|_{J=0} \\ &= \frac{\delta^2 \ln Z[J]}{i\delta J(x_1) i\delta J(x_2)}\bigg|_{J=0} \\ &= \frac{1}{Z[0]}\, \frac{\delta^2 Z[J]}{i\delta J(x_1) i\delta J(x_2)}\bigg|_{J=0}, \end{aligned}$$

where we have used the fact that

$$\frac{\delta Z^n[J]}{\delta J(x_1)\cdots\delta J(x_n)} = 0,\ \ \forall n = 1 \bmod 2.$$

The result is the same as the original definition.

Example 2. Four-point Correlation

Consider the four-point connected correlation:

$$iV_4 \equiv \langle \mathcal{T}\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle_c$$

Following the same procedure,

$$\begin{aligned} iV_4 = i\, &\frac{\delta^4 W[J]}{i\delta J(x_1) i\delta J(x_2) i\delta J(x_3) i\delta J(x_4)}\bigg|_{J=0} \\ = &\frac{1}{Z[0]}\, \frac{\delta^4 Z[J]}{i\delta J(x_1) i\delta J(x_2) i\delta J(x_3) i\delta J(x_4)}\bigg|_{J=0} \\ &- i\Delta(x_1 - x_2) i\Delta(x_3 - x_4) \\ &- i\Delta(x_1 - x_3) i\Delta(x_2 - x_4) \\ &- i\Delta(x_1 - x_4) i\Delta(x_2 - x_3). \end{aligned}$$

The connected correlation function automatically omit those disconnected components.

2.2 Real ϕ^3 Theory

Now consider the interaction theory with additional Lagrangian

$$\mathcal{L}_{\text{int}}[\phi] = \frac{g}{3!}\phi^3. \tag{2.11}$$

Note that the field ϕ has the mass dimension $[\frac{d-2}{2}]$. The critical dimension is $d = 6$ where the coupling constant g is dimensionless. In this section, we consider the real Klein-Gordon field with ϕ^3 interaction on 6-dimensional space-time.

For interaction theory, the renormalized Lagrangian has the form:

$$\begin{aligned} \mathcal{L} &= Z_\phi \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - Z_m\frac{m^2}{2}\phi^2 + Z_g\frac{g}{3!}\phi^3 \\ &= \mathcal{L}_0 + \mathcal{L}_{\text{int}} + \mathcal{L}_{\text{ct}}, \end{aligned} \tag{2.12}$$

where the counter terms are:

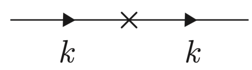
$$\begin{aligned} \mathcal{L}_{\text{ct}}[\phi] &= \frac{A}{2}\partial^\mu\phi\partial_\mu\phi - \frac{B}{2}m^2\phi^2 + \frac{C}{3!}g\phi^3 \\ &\simeq -\frac{A}{2}\phi\partial^2\phi - \frac{B}{2}m^2\phi^2 + \frac{C}{3!}g\phi^3, \end{aligned} \tag{2.13}$$

where

$$A = Z_\phi - 1, B = Z_m - 1, C = Z_g - 1.$$

The counter term for the the free field gives additional corrrection

$$i\tilde{\Delta}^{(\text{ct})}(k) = i\tilde{\Delta}_0(k)(Ak^2 - Bm^2)i\tilde{\Delta}_0(k) \tag{2.14}$$

$=$ 

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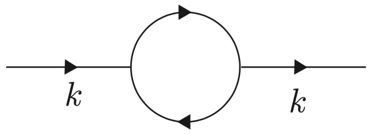
2.2.1 Self Energy Correction

To second order, we consider the one-loop correction to the propagator with the diagram:



This correspond to

$$i\tilde{\Delta}^{(2)}(k) = i\tilde{\Delta}_0(k)\left[i\Sigma^{(2)}(k^2)\right]i\tilde{\Delta}_0(k), \tag{2.15}$$

where the self energy term to the second order $i\Pi^{(2)}(k)$ is defined as:

$$i\Sigma^{(2)}(k^2) \equiv \frac{g^2}{2}\int\frac{d^dq}{(2\pi)^d}\tilde{\Delta}_0(q)\tilde{\Delta}_0(k-q) + (Ak^2 - Bm^2). \tag{2.16}$$

Remark 4. Symmetry Factor

The coefficient $g^2/2$ comes from the symmetry factor in the diagram. We can also check the coefficient explicitly, by considering the expansion to the second order (we denote $\delta/\delta J(x_i)$ as δ_i):

$$\delta_1\delta_2 \frac{1}{2!4!}\left[\frac{ig}{3!}\int d^d y \left(\frac{\delta}{\delta J(y)}\right)^3\right]^2 \left[-\frac{i}{2}\int d^d y_1 d^d y_2 J(y_1)\Delta(y_1-y_2)J(y_2)\right]^4.$$

The expansion gives the coefficient

$$\left(\frac{ig}{6}\right)^2 \times \frac{1}{2!\times 4!\times 2^4}.$$

Now consider the combinatorial factor, which comes from the exchange of $\phi(x_i)$ in the propagator, the exchange of $\phi(x_i)$ in the vertex, the exchange of propagator in the diagram, and the change of vertices in the diagram:

$$(2!)^4 \times (3!)^2 \times (4\times 3)\times 2.$$

Those two factors produce a $-g^2/2$ coefficient. Note that in the self energy expression (2.15), we put a i factor in front of each propagator, which absorbs the minus sign.

Once we obtain the self energy, the one-loop corrected propagator has the form:

$$\begin{aligned} i\tilde{\Delta}(k) &= i\tilde{\Delta}_0(k) + i\tilde{\Delta}_0(k)\left[\sum_{n=1}^{\infty} i\Sigma(k^2)\right] i\tilde{\Delta}_0(k) \\ &= \frac{i}{\tilde{\Delta}_0^{-1}(k) - \Sigma(k^2)} \\ &= \frac{i}{k^2 - m^2 - \Sigma(k^2)}. \end{aligned} \tag{2.17}$$

Now we are going to evaluate the divergent integral in the self energy expression, using the Feynman parameters:

$$\begin{aligned} &\int \frac{d^d q}{(2\pi)^d}\frac{1}{q^2-m^2}\frac{1}{(k-q)^2-m^2} \\ &= \int \frac{d^d q}{(2\pi)^d}\int_0^1 dx \frac{1}{\left[q^2-m^2+x\left((q-k)^2-q^2\right)\right]^2} \\ &= \int_0^1 dx \int \frac{d^d q}{(2\pi)^d}\frac{1}{\left[(q-kx)^2-D\right]^2}, \end{aligned}$$

where $D = m^2 - k^2 x(1-x)$. Then we can shift $q \to q + kx$ leaving an integral that only depends on q^2. In this way,

$$\Sigma(k^2) = \int_0^1 I(x)dx.$$

To evaluate the self-energy, it suffices to obtain the integral

$$I(x) = \frac{g^2}{2i}\int \frac{d^d q}{(2\pi)^d}\frac{1}{\left[q^2-D\right]^2}.$$

Remark 5. Feyman Parameters

We use Feynman's formula to combine denominators,

$$\frac{1}{A_1 \ldots A_n} = \int dF_n \left(x_1 A_1 + \ldots + x_n A_n\right)^{-n}, \tag{2.18}$$

where the integration measure over the Feynman parameters x_i is

$$\int dF_n = (n-1)! \int_0^1 dx_1 \ldots dx_n \delta\left(x_1 + \ldots + x_n - 1\right). \tag{2.19}$$

This measure is normalized so that $\int dF_n = 1$. The simplest case is

$$\frac{1}{AB} = \int_0^1 \frac{dx}{[A+(B-A)x]^2} = \int_0^1 \frac{\delta(x+y-1)}{[xA+yB]^2} dxdy. \tag{2.20}$$

Other useful identities are

$$\begin{aligned} \frac{1}{AB^n} &= \int_0^1 dxdy \frac{\delta(x+y-1)ny^{n-1}}{[xA+yB]^{n+1}}, \\ \frac{1}{ABC} &= \int_0^1 dxdydz \frac{2\delta(x+y+z-1)}{[xA+yB+zC]^3}. \end{aligned} \tag{2.21}$$

By making the Wick rotation $q^0 \to iq_E^0$, the integral becomes:[2]

$$I(x) = \frac{g}{2} \int \frac{d^d q_E}{(2\pi)^d} \frac{1}{\left(q_E^2 + D\right)^2} = \frac{g\Omega_d}{2(2\pi)^d} \int dq \frac{q^{d-1}}{\left(q^2+D\right)^2}.$$

Dimensional Regularization

We set the dimension to $d = 6 - \epsilon$, and rewrite the Lagrangian as

$$\mathcal{L} = Z_\phi \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - Z_m \frac{m^2}{2} \phi^2 + Z_g \frac{g\tilde{\mu}^{\epsilon/2}}{3!} \phi^3. \tag{2.24}$$

Note that the coupling constant should be changed to $g \to g\tilde{\mu}^{\epsilon/2}$ where μ is of mass dimension [1] in order to get the correct dimensionality. We then expand the expression to zeroth order of ϵ. A useful identity is:

$$\int dk \frac{k^a}{\left(k^2+D\right)^b} = D^{\frac{a+1}{2}-b} \frac{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(b - \frac{a+1}{2}\right)}{2\Gamma(b)}. \tag{2.25}$$

[2] The d-dimensional solid angle is

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}, \tag{2.22}$$

where $\Gamma(x)$ is the gamma function, satisfing

$$\Gamma(1+x) = x\Gamma(x), \;\; \Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + O(\epsilon). \tag{2.23}$$

In particular, $\Gamma(n+1) = n!$.

Actually, we can compute the integral and series expansion in `Mathematica` all together:

```
omg=(2*Pi^(d/2))/(Gamma[d/2]);
cof=g^2*\[Mu]^(6-d)/2*omg/(2*Pi)^d;
int=cof*Integrate[q^(d-1)/(q^2+D)^2,{q,0,Infinity}][[1]];
map={g^2->\[Alpha]*(4*Pi)^3,EulerGamma->Subscript[\[Gamma],E]};
ans=Series[int/.{d->6-\[Epsilon]},{\[Epsilon],0,0}];
ans/.map//Simplify
```

The result is (where $\alpha \equiv g^2/(4\pi)^3$)

$$I(x) = \frac{\alpha D}{2}\left[\ln\left(\frac{De^{\gamma_E}}{4\pi\tilde{\mu}^2}\right) - \left(\frac{2}{\epsilon}+1\right)\right] + O(\epsilon).$$

Now insert $D = m^2 - k^2 x(1-x)$. Note that

$$\int_0^1 dx D = m^2 - \frac{k^2}{6}.$$

This simplifies the result to

$$\Sigma^{(2)}(k^2) = \frac{\alpha}{2}\left(\frac{2}{\epsilon}+1\right)\left(\frac{k^2}{2} - m^2\right) + \frac{\alpha}{2}\int_0^1 dx D(x)\ln\left(\frac{D(x)}{\mu^2}\right), \tag{2.26}$$

where we have replace $\tilde{\mu}$ with

$$\mu \equiv \sqrt{\frac{4\pi}{e^{\gamma_E}}}\tilde{\mu}. \tag{2.27}$$

Renormalization

The counter terms also contribute to the perturbative correction,

$$\begin{aligned}\Sigma^{(2)}\left(k^2\right) =& \frac{\alpha}{2}\int_0^1 dx D\ln\left(\frac{D}{m^2}\right) + \left\{\frac{\alpha}{6}\left[\frac{1}{\varepsilon} + \ln\left(\frac{\mu}{m}\right) + \frac{1}{2}\right] + A\right\}k^2 \\ &- \left\{\alpha\left[\frac{1}{\varepsilon} + \ln\left(\frac{\mu}{m}\right) + \frac{1}{2}\right] + B\right\}m^2 + O\left(\alpha^2\right).\end{aligned}$$

Consider the on-shell condition for the subtraction:

$$\Sigma(m^2) = \Sigma'(m^2) = 0. \tag{2.28}$$

Set $D_0 \equiv D(x)|_{k^2=m^2} = m^2(1-x+x^2)$, the self energy has the form:

$$\Sigma^{(2)}(k^2) = \frac{\alpha}{2}\int_0^1 dx D(x)\ln\left(\frac{D(x)}{D_0(x)}\right) + C_k k^2 + C_m m^2. \tag{2.29}$$

The condition $\Pi(m^2) = 0$ requires

$$\Sigma^{(2)}(k^2) = \frac{\alpha}{2}\int_0^1 dx D(x)\ln\left(\frac{D(x)}{D_0(x)}\right) + C_k(k^2 - m^2).$$

The condition $\Pi'(m^2) = 0$ requires

$$\begin{aligned}
\left.\frac{d\Sigma^{(2)}(k^2)}{dk^2}\right|_{k^2=m^2} &= \frac{\alpha}{2}\int_0^1 dx \left.\left[\frac{D(x)}{dk^2}\ln\left(\frac{D(x)}{D_0(x)}\right) + D_0(x)\right]\right|_{q^2=m^2} + C_k \\
&= \frac{\alpha}{2}\int_0^1 dx(x^2 - x) + C_k \\
&= C_k - \frac{\alpha}{12} = 0.
\end{aligned}$$

In this way, we obtained the renormalized self-energy:

$$\Sigma^{(2)}(k^2) = \frac{\alpha}{2}\int_0^1 dx D(x)\ln\left(\frac{D(x)}{D_0(x)}\right) + \frac{\alpha}{12}(k^2 - m^2). \tag{2.30}$$

On the other hand, we chan choose the $\overline{\text{MS}}$ subtraction scheme, i.e.,

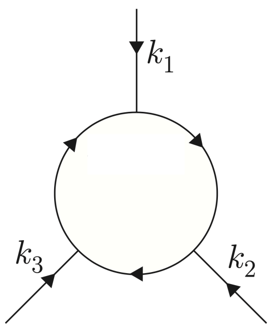
$$A = -\frac{\alpha}{6\epsilon},\ B = -\frac{\alpha}{\epsilon}. \tag{2.31}$$

The self energy under $\overline{\text{MS}}$ scheme will depend on the the mass scale μ we choose:

$$\Sigma^{(2)}\left(k^2\right) = \frac{\alpha}{2}\int_0^1 dx D \ln\left(\frac{D}{m^2}\right) + \alpha\left[\ln\left(\frac{\mu}{m}\right) + \frac{1}{2}\right]\left(\frac{k^2}{6} - m^2\right). \tag{2.32}$$

2.2.2 Vertex Correction

Now consider the simplest one-loop correction to the vertex function from the diagram:



The vertex function corresponding to such correction, together with the counter term, can be expressed as:

$$iV_3^{(3)}(k_1, k_2, k_3) = (ig)^3 i^3 \int \frac{d^a q}{(2\pi)^d}\tilde{\Delta}(q - k_1)\tilde{\Delta}(q + k_2)\tilde{\Delta}(q) + iCg, \tag{2.33}$$

Using the Feynman parameter, the integrant is

$$\tilde{\Delta}(q - k_1)\tilde{\Delta}(q + k_2)\tilde{\Delta}(q) = \int dF_3 \frac{1}{(q^2 - D)^3}$$

where we have shift the value of q, and D can be evaluate by the following code:

```
A1=(l-k1)^2-m^2;
A2=(l+k2)^2-m^2;
A3=(l)^2-m^2;
{c,b,a}=CoefficientList[x1*A1+x2*A2+(1-x1-x2)*A3,{l}];
-c+b^2/(4*a)//Expand
```

The result is

$$D = m^2 - k_1^2 x_1(1-x_1) - k_2^2 x_2(1-x_2) - 2k_1 k_2 x_1 x_2.$$

The same procedure gives:

$$V_3^{(3)}/g = \int dF_3 I(x_1, x_2, x_3) + C, \tag{2.34}$$

where

$$I(x_1, x_2, x_3) = \frac{g^2 \Omega_d}{(2\pi)^d} \int dq \frac{q^{d-1}}{(q^2+D)^3}.$$

The same regularization procedure in `Mathematica`:

```
omg=(2*Pi^(d/2))/(Gamma[d/2]);
cof=g^2*\[Mu]^(6-d)*omg/(2*Pi)^d;
int=cof*Integrate[q^(d-1)/(q^2+D)^3,{q,0,Infinity}][[1]];
map={g^2->\[Alpha]*(4*Pi)^3,EulerGamma->Subscript[\[Gamma],E]};
ans=Series[int/.{d->6-\[Epsilon]},{\[Epsilon], 0, 0}];
ans/.map//Simplify
```

The result is

$$\begin{aligned} V_3^{(3)}/g &= \frac{\alpha}{\epsilon} + \frac{\alpha}{2} \int dF_3 \ln\left(\frac{4\pi\tilde{\mu}^2 e^{-\gamma_E}}{D}\right) + C + O(\epsilon) \\ &= \frac{\alpha}{\epsilon} + \alpha \ln\left(\frac{\mu}{m}\right) - \frac{\alpha}{2} \int dF_3 \ln\left(\frac{D}{m}\right) + C. \end{aligned} \tag{2.35}$$

The on-shell subtraction requires

$$V_3(0,0,0) = g, \tag{2.36}$$

which gives

$$C = -\frac{\alpha}{\epsilon} - \alpha \ln\left(\frac{\mu}{m}\right). \tag{2.37}$$

So the vertex function to the third order is

$$V_3(k_1, k_2, k_3) = g\left\{1 - \frac{\alpha}{2}\int dF_3 \ln\left[\frac{D(x_1, x_2, x_3)}{m}\right]\right\}. \tag{2.38}$$

The $\overline{\text{MS}}$ scheme, on the other hand, sets

$$C = -\frac{\alpha}{\epsilon}. \tag{2.39}$$

2.2.3 Renormalization Group

We fist summarize the normalization factor obtained on the one-loop level (with $\overline{\text{MS}}$ subtraction scheme):

$$\begin{aligned} Z_\phi &= 1 - \frac{\alpha}{6\epsilon} + O(\alpha^2), \\ Z_m &= 1 - \frac{\alpha}{\epsilon} + O(\alpha^2), \\ Z_g &= 1 - \frac{\alpha}{\epsilon} + O(\alpha^2). \end{aligned} \tag{2.40}$$

For the renormalized Lagrangian in $(6-\epsilon)$-dimension

$$\mathcal{L} = Z_\phi \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - Z_m \frac{m^2}{2} \phi^2 + Z_g \frac{g\tilde{\mu}^{\epsilon/2}}{3!} \phi^3, \tag{2.41}$$

the factors relate the original field and bare coefficients

$$\phi_0 = Z_\phi^{1/2} \phi, \ m_0 = Z_m^{1/2} Z_\phi^{-1/2} m, \ g_0 = Z_g Z_\phi^{-3/2} \tilde{\mu}^{\epsilon/2} g. \tag{2.42}$$

The renormalization group requires that the bare parameter is independent of the mass scale μ we choose, that is:

$$\frac{d\phi_0}{d\ln\mu} = \frac{dm_0}{d\ln\mu} = \frac{dg_0}{d\ln\mu} = 0. \tag{2.43}$$

Beta Function

Star with g_0, it is more convenient to use

$$\alpha_0 \equiv \frac{g_0^2}{4\pi} = Z_g^2 Z_\phi^{-3} \tilde{\mu}^\epsilon \alpha. \tag{2.44}$$

Take logarithm on both side:

$$\ln\alpha_0 = \ln(Z_g^2 Z_\phi^{-3}) + \ln\alpha + \epsilon \ln\tilde{\mu}. \tag{2.45}$$

The RG equation is

$$\frac{d\ln\alpha_0}{d\ln\mu} = \frac{d\ln(Z_g^2 Z_\phi^{-3})}{d\alpha} \frac{d\alpha}{d\ln\mu} + \frac{1}{\alpha} \frac{d\alpha}{d\ln\mu} + \epsilon = 0. \tag{2.46}$$

To the first order of α:

$$\frac{d\ln(Z_g^2 Z_\phi^{-3})}{d\alpha} = \frac{d}{d\alpha} \left(-\frac{2\alpha}{\epsilon} + \frac{\alpha}{2\epsilon} \right) = -\frac{3}{2\epsilon}, \tag{2.47}$$

which leads to

$$\frac{d\alpha}{d\ln\mu} \left(1 - \frac{3\alpha}{2\epsilon} + O(\alpha^2) \right) + \epsilon\alpha = 0. \tag{2.48}$$

The beta function is defined as

$$\beta(\alpha) = \frac{d\alpha}{d\ln\mu} = \beta_1 \alpha + \beta_2 \alpha^2 + O(\alpha^3). \tag{2.49}$$

Insert such definition into the original expression, and keep track of the order of α, we get

$$(\beta_1 + \epsilon)\alpha + \left(\beta_2 - \frac{3\beta_1}{2\epsilon}\right)\alpha^2 + O(\alpha^3) = 0. \tag{2.50}$$

The beta function is

$$\beta(\alpha) = -\epsilon\alpha - \frac{3}{2}\alpha^2 + O(\alpha^3). \tag{2.51}$$

Anomalous Dimension

Consider the RG equation with bare mass:

$$\begin{aligned}\frac{d\ln m_0}{d\ln\mu} &= \frac{1}{2}\frac{d(\ln Z_m - \ln Z_\phi)}{d\alpha}\frac{d\alpha}{d\ln\mu} + \frac{1}{m}\frac{dm}{d\ln\mu} \\ &= \frac{5\alpha}{12} + \frac{1}{m}\frac{dm}{d\ln\mu} + O(\alpha^2) = 0.\end{aligned} \tag{2.52}$$

We get the anomalous dimension of the mass:

$$\gamma_m(\alpha) \equiv \frac{1}{m}\frac{dm}{d\ln\mu} = -\frac{5\alpha}{12} + O(\alpha^2). \tag{2.53}$$

Also, for the bare field

$$\frac{d\ln\phi_0}{d\ln\mu} = \frac{1}{2}\frac{d\ln Z_\phi}{d\ln\mu} + \frac{d\ln\phi}{d\ln\mu} = 0. \tag{2.54}$$

We can define the anomalous dimension of the field as

$$\gamma_\phi \equiv \frac{1}{2}\frac{d\ln Z_\phi}{d\ln\mu} = \frac{1}{2}\frac{d\ln Z_\phi}{d\alpha}\frac{d\alpha}{d\ln\mu} = \frac{\alpha}{12} + O(\alpha^2). \tag{2.55}$$

Callan-Symanzik Equation

Consider the bare propagator:

$$\tilde{\Delta}_0(k) = Z_\phi\tilde{\Delta}(k) \tag{2.56}$$

The RG condition for the bare propagator gives:

$$\frac{d\ln\tilde{\Delta}_0(k)}{d\ln\mu} = \frac{d\ln Z_\phi}{d\ln\mu} + \frac{1}{\tilde{\Delta}(k)}\left(\frac{\partial}{\partial\ln\mu} + \frac{d\alpha}{d\ln\mu}\frac{\partial}{\partial\alpha} + \frac{dm}{d\ln\mu}\frac{\partial}{\partial m}\right)\tilde{\Delta}(k) = 0.$$

The Callan-Symanzik equation is

$$\left(2\gamma_\phi + \frac{\partial}{\partial\ln\mu} + \beta(\alpha)\frac{\partial}{\partial\alpha} + \gamma_m(\alpha)m\frac{\partial}{\partial m}\right)\tilde{\Delta}(k) = 0. \tag{2.57}$$

2.3 Real ϕ^4 Theory

In this section, we consider the real Klein-Gordon field with ϕ^4 interaction. The field ϕ has mass dimension $[\frac{d-2}{2}] = [1]$, so the critical dimension is $d = 4$, where the coupling constant g is dimensionless. For dimensional regulation purpose, we write the renormalized Lagrangian on $(4-\epsilon)$-dimensional space-time as

$$\mathcal{L} = Z_\phi \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - Z_m\frac{m^2}{2}\phi^2 - Z_g\frac{g\tilde{\mu}^\epsilon}{4!}\phi^4, \tag{2.58}$$

where we have introduced a mass scale $\tilde{\mu}$. As the ϕ^3 theory, we can rewrite the Lagrangian as:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}} + \mathcal{L}_{\text{ct}}. \tag{2.59}$$

In the following we investigate the loop correction to the mass and the coupling constant.

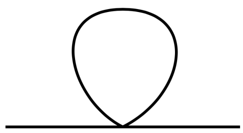
2.3.1 One-loop Correction

Self-energy

Following the same procedure, the one-loop self-energy correction is

$$i\Sigma(k^2) = -\frac{g\tilde{\mu}^\epsilon}{2}\int\frac{d^dq}{(2\pi)^d}\frac{1}{q^2-m^2} + i(Ak^2 - Bm^2). \tag{2.60}$$

The first term comes from the diagram



and the second term comes from the counter terms. After the Wick rotation,

$$\Sigma(k^2) = -\frac{g\tilde{\mu}^\epsilon}{2}\frac{\Omega_d}{(2\pi)^d}\int\frac{q^{d-1}dq}{q^2+m^2} + (Ak^2 - Bm^2). \tag{2.61}$$

The dimensional regulation is carried out using the following code:

```
omg=(2*Pi^(d/2))/(Gamma[d/2]);
cof=g*\[Mu]^(4-d)/2*omg/(2*Pi)^d;
int=cof*Integrate[q^(d-1)/(q^2+m^2),{q,0,Infinity}][[1]];
map={EulerGamma->Subscript[\[Gamma],E]};
ans=Series[int/.{d->4-\[Epsilon]},{\[Epsilon],0,0}];
ans/.map//Simplify
```

The result is

$$\Sigma(k^2) = \frac{gm^2}{32\pi^2}\left[\frac{2}{\epsilon} + 1 + \log\left(\frac{4\pi\tilde{\mu}^2 e^{-\gamma_E}}{m^2}\right)\right] + (Ak^2 - Bm^2) + O(\epsilon). \tag{2.62}$$

Using the $\overline{\text{MS}}$ renormalization scheme, we set

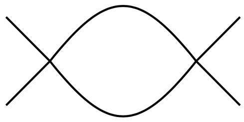
$$A = 0,\ \ B = \frac{g}{16\pi^2\epsilon}. \tag{2.63}$$

The result is

$$\Sigma(k^2) = \frac{gm^2}{16\pi^2}\log\left(\frac{\mu}{m}\right) + \frac{gm^2}{32\pi^2} + O(\epsilon). \tag{2.64}$$

Vertex Correction

Now consider the vertex correction. To the lowest order the diagram is



Together with the counter term, the vertex function is

$$iV_4^{(2)}(k_1, k_2, k_3, k_4) = \frac{g^2}{2}\left[iF(s) + iF(t) + iF(u)\right] - iCg, \tag{2.65}$$

where

$$s = (k_1 + k_2)^2,\ \ t = (k_1 + k_3)^2,\ \ u = (k_1 + k_4)^2, \tag{2.66}$$

and

$$iF(k^2) = \tilde{\mu}^\epsilon \int \frac{d^dq}{(2\pi)^d}\tilde{\Delta}_0(q)\tilde{\Delta}_0(q + k) \tag{2.67}$$

$$= \frac{i\tilde{\mu}^\epsilon\Omega_d}{(2\pi)^d}\int_0^1 dx \int \frac{q^{d-1}dq}{\left[q^2 + m^2 + x(1-x)k^2\right]^2}. \tag{2.68}$$

Then we carry out the calculation (set $D(k^2, x) = m^2 + x(1-x)k^2$)

```
omg=(2*Pi^(d/2))/(Gamma[d/2]);
cof=g^2*\[Mu]^(4-d)/2*omg/(2*Pi)^d;
int=cof*Integrate[q^(d-1)/(q^2+D)^2,{q,0,Infinity}][[1]];
map={EulerGamma->Subscript[\[Gamma],E]};
ans=Series[int/.{d->4-\[Epsilon]},{\[Epsilon],0,0}];
ans/.map//Simplify
```

The result is:

$$\begin{aligned} F(s) &= \frac{1}{8\pi^2\epsilon} + \frac{1}{16\pi^2}\int_0^1 dx \ln\left(\frac{4\pi\tilde{\mu}^2e^{-\gamma_E}}{D}\right) \\ &= \frac{1}{8\pi^2\epsilon} + \frac{1}{8\pi^2}\ln\left(\frac{\mu}{m}\right) - \frac{1}{16\pi^2}\int_0^1 dx \ln\left(\frac{D(s,x)}{m^2}\right). \end{aligned} \tag{2.69}$$

The $\overline{\text{MS}}$ scheme absorbs the $\frac{1}{8\pi^2\epsilon}$ term, i.e.,

$$C = \frac{3g}{16\pi^2}. \tag{2.70}$$

The result is:

$$V_4(k_1,k_2,k_3,k_4) = -g + \frac{g^2}{32\pi^2}\int_0^1 dx \ln\left(\frac{\mu^6}{D(s,x)D(t,x)D(u,x)}\right). \tag{2.71}$$

To summarize, the normalization is:

$$Z_\phi = 1, \tag{2.72}$$

$$Z_m = 1 + \frac{g}{16\pi^2\epsilon}, \tag{2.73}$$

$$Z_g = 1 + \frac{3g}{16\pi^2\epsilon}. \tag{2.74}$$

2.3.2 Renormalization Group

Now consider the RG equation for the one-loop correction. The bare parameters are:

$$g_0 = Z_g g \tilde{\mu}^\epsilon,\ m_0 = Z_m^{1/2} m, \tag{2.75}$$

The RG conditions are:

$$\frac{dg_0}{d\ln\mu} = \left(\frac{3}{16\pi^2\epsilon} + \frac{1}{g}\right)\frac{dg}{d\ln\mu} + \epsilon = 0, \tag{2.76}$$

$$\frac{dm_0}{d\ln\mu} = \frac{1}{32\pi^2\epsilon}\frac{dg}{d\ln\mu} + \frac{1}{m}\frac{dm}{d\ln\mu} = 0. \tag{2.77}$$

Consider the series expansion of beta function:

$$\beta(g) = \frac{dg}{d\ln\mu} = \beta_1 g + \beta_2 g^2 + O(g^3). \tag{2.78}$$

The beta function is

$$\beta(g) = -\epsilon g + \frac{3g^2}{16\pi^2} + O(g^3). \tag{2.79}$$

The anomalous dimension of mass is

$$\gamma_m = \frac{1}{m}\frac{dm}{d\ln\mu} = \frac{g}{32\pi^2} + O(g^2) \tag{2.80}$$

Chapter 3

Quantum Electrodynamics

The Lagrangian for quantum electrodynamics is

$$\begin{aligned}\mathcal{L}_{\text{QED}} &= \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - eA_\mu\bar{\psi}\gamma^\mu\psi \\ &= \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{Maxwell}} + \mathcal{L}_{\text{int}},\end{aligned} \tag{3.1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\nu = (dA)_{\mu\nu}. \tag{3.2}$$

The Lagrangian is invariant under the gauge transformation:

$$\begin{aligned}\psi(x) &\to e^{-ie\alpha(x)}\psi(x), \\ A^\mu(x) &\to A^\mu(x) + \partial^\mu\alpha(x).\end{aligned} \tag{3.3}$$

It is convenient to rewrite Lagrangian as

$$\mathcal{L}_{\text{QED}} = \bar{\psi}\left(i\not{D} - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{3.4}$$

where we have define the covariant derivative as:

$$\not{D} = \gamma^\mu D_\mu = \gamma^\mu[\partial_\mu + ieA_\mu(x)] = \not{\partial} + ie\not{A}. \tag{3.5}$$

3.1 Perturbation Theory

3.1.1 LSZ for Dirac Field

Use the field expansion

$$\begin{aligned}\psi(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\sum_s \left(a^s_{\mathbf{p}}u^s(p)e^{-ip\cdot x} + b^{s\dagger}_{\mathbf{p}}v^s(p)e^{ip\cdot x}\right), \\ \bar{\psi}(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\sum_s \left(b^s_{\mathbf{p}}\bar{v}^s(p)e^{-ip\cdot x} + a^{s\dagger}_{\mathbf{p}}\bar{u}^s(p)e^{ip\cdot x}\right).\end{aligned}$$

and the orthogonality relation

$$u^{r\dagger}(p)u^s(p) = 2\omega_{\boldsymbol{p}}\delta^{rs}, \quad u^{r\dagger}(\boldsymbol{p},\omega_{\boldsymbol{p}})v^s(-\boldsymbol{p},\omega_{\boldsymbol{p}}) = 0,$$
$$v^{r\dagger}(p)v^s(p) = 2\omega_{\boldsymbol{p}}\delta^{rs}, \quad v^{r\dagger}(\boldsymbol{p},\omega_{\boldsymbol{p}})u^s(-\boldsymbol{p},\omega_{\boldsymbol{p}}) = 0.$$

The spatial Fourier transformation gives:

$$\int d^3x e^{ip\cdot x}\psi(x) = \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\sum_s a^s_{\boldsymbol{p}}u^s(p) + \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\sum_s b^{s\dagger}_{\boldsymbol{p}}v^s(-\boldsymbol{p},\omega)e^{2i\omega t}$$

Left-multiply on both hand side by $\bar{u}^s(p)\gamma^0$, we then get

$$\sqrt{2\omega_{\boldsymbol{p}}}a^s_{\boldsymbol{p}} = \int d^3x e^{ip\cdot x}\bar{u}^s(p)\gamma^0\psi(x),$$
$$\sqrt{2\omega_{\boldsymbol{p}}}a^{s\dagger}_{\boldsymbol{p}} = \int d^3x e^{-ip\cdot x}\bar{\psi}(x)\gamma^0 u^s(p).$$

Similarly, we consider

$$\int d^3x e^{ip\cdot x}\bar{\psi}(x) = \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\sum_s b^s_{\boldsymbol{p}}\bar{v}^s(p) + \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\sum_s a^{s\dagger}_{\boldsymbol{p}}\bar{u}^s(-\boldsymbol{p},\omega)e^{2i\omega t}$$

Right-multiply on both hand side by $\gamma^0 v^s(p)$, we then get

$$\sqrt{2\omega_{\boldsymbol{p}}}b^s_{\boldsymbol{p}} = \int d^3x e^{ip\cdot x}\bar{\psi}(x)\gamma^0 v^s(p),$$
$$\sqrt{2\omega_{\boldsymbol{p}}}b^{s\dagger}_{\boldsymbol{p}} = \int d^3x e^{-ip\cdot x}\bar{v}^s(p)\gamma^0\psi(x).$$

Following the same strategy as we did for the scalar field, we consider

$$\begin{aligned}
\sqrt{2\omega_{\boldsymbol{p}}}a^s_{\boldsymbol{p};\text{out}} - \sqrt{2\omega_{\boldsymbol{p}}}a^s_{\boldsymbol{p};\text{in}} &= \int dt\ \partial_t\sqrt{2\omega_{\boldsymbol{p}}}a^s_{\boldsymbol{p}} \\
&= \int dt \int d^3x e^{ip\cdot x}\bar{u}(p)(\gamma^0\partial_t + i\gamma^0 p^0)\psi(x) \\
&= \int d^4x e^{ip\cdot x}\bar{u}(p)(\gamma^0\partial_t + i\gamma^i p^i + im)\psi(x) \\
&= i\int d^4x e^{ip\cdot x}\bar{u}(p)(-i\partial\!\!\!/ + m)\psi(x)
\end{aligned}$$

where we have used the fact $\bar{u}(p)(p\!\!\!/ - m) = 0$. Take hermitian conjugate,

$$\begin{aligned}
\sqrt{2\omega_{\boldsymbol{p}}}a^{s\dagger}_{\boldsymbol{p};\text{in}} - \sqrt{2\omega_{\boldsymbol{p}}}a^{s\dagger}_{\boldsymbol{p};\text{out}} &= i\int d^4x e^{-ip\cdot x}\bar{\psi}(x)\gamma^0(-i\partial\!\!\!/ + m)^\dagger\gamma^0 u(p) \\
&= i\int d^4x e^{-ip\cdot x}\bar{\psi}(x)(i\overleftarrow{\partial\!\!\!/} + m)u(p)
\end{aligned}$$

Similarly, using the fact $(p\!\!\!/ + m)v(p) = 0$,

$$\begin{aligned}
\sqrt{2\omega_{\boldsymbol{p}}}b^s_{\boldsymbol{p};\text{out}} - \sqrt{2\omega_{\boldsymbol{p}}}b^s_{\boldsymbol{p};\text{in}} &= \int d^4x e^{ip\cdot x}\bar{\psi}(x)(\gamma^0\overleftarrow{\partial_t} + i\gamma^0 p^0)v(p) \\
&= \int d^4x e^{ip\cdot x}\bar{\psi}(x)(\gamma^0\overleftarrow{\partial_t} + i\gamma^i p^i - im)v(p) \\
&= -i\int d^4x e^{ip\cdot x}\bar{\psi}(x)(i\overleftarrow{\partial\!\!\!/} + m)v(p).
\end{aligned}$$

Again, take the hermitian conjugate,

$$\begin{aligned}\sqrt{2\omega_{\boldsymbol{p}}}b^{s\dagger}_{\boldsymbol{p};\text{in}} - \sqrt{2\omega_{\boldsymbol{p}}}b^{s\dagger}_{\boldsymbol{p};\text{out}} &= -i\int d^4x e^{ip\cdot x}\bar{v}(p)\gamma^0(i\overleftarrow{\not\partial} + m)^\dagger\gamma^0\psi(x) \\ &= -i\int d^4x e^{-ip\cdot x}\bar{v}(p)(-i\not\partial + m)\psi(x)\end{aligned}$$

The same strategy gives the LSZ reduction formula for Dirac field. Consider the S-matrix for particles:

$$\begin{aligned}&\langle p_1, \cdots, p_n|S|k_1, \cdots, k_m\rangle \\ =&\prod_{i=1}^{m}\left[\int d^dx_i\ e^{ip_i\cdot x_i}u^{s_1}(p_i)\frac{i\not\partial - m_i}{i\sqrt{Z_\phi}}\right] iG(x)\prod_{j=m+1}^{m+n}\left[\int d^dx_j\ e^{-ik_j\cdot x_j}\frac{-i\overleftarrow{\not\partial} - m_j}{i\sqrt{Z_\phi}}u^{s_j}(k_j)\right] \\ =&\prod_{i=1}^{m}\left[\frac{\not p - m_i}{i\sqrt{Z_\phi}}u^{s_i}(p_i)\right] i\tilde{G}(p_1, \cdots, p_n, -k_1, \cdots, -k_m)\prod_{j=m+1}^{m+n}\left[u^{s_j}(k_j)\frac{\not k - m_j}{i\sqrt{Z_\phi}}\right].\end{aligned}$$

3.1.2 Perturbative Corrections

As with the scalar field,

$$Z[\bar\eta, \eta, J] = \exp\left\{i\int d^dx\mathcal{L}_{\text{int}}\left[\frac{\delta}{i\delta J}, \frac{\delta}{i\delta\eta}, \frac{i\delta}{\delta\bar\eta}\right]\right\} Z_0[\bar\eta, \eta, J]. \tag{3.6}$$

We use the dimensional regularization by default. Note that ψ has the mass dimension $[\frac{d-1}{2}]$, A^μ had the mass dimension $[\frac{d}{2}-1]$, and e has the mass dimension $[2-\frac{d}{2}]$. When $d = 4-\epsilon$, we replace e with $e\tilde{\mu}^{\epsilon/2}$, so that to make the coupling constant e dimensionless.

The renormalized Lagrangian is

$$\begin{aligned}\mathcal{L} &= Z_\psi\bar\psi_R(i\gamma^\mu\partial_\mu)\psi_R - Z_m m\bar\psi_R\psi_R + \frac{1}{4}Z_A F_{R,\mu\nu}F_R^{\mu\nu} - Z_e e_R A_R^\mu\bar\psi_R\gamma^\mu\psi_R \\ &= \mathcal{L}_0 + \mathcal{L}_{\text{int}} + \mathcal{L}_{\text{ct}}.\end{aligned} \tag{3.7}$$

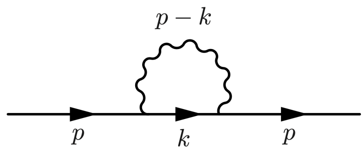
The we define the coefficients

$$\delta_\psi = Z_\phi - 1,\ \delta_m = Z_m - 1,\ \delta_Z = Z_A - 1,\ \delta_e = Z_e - 1. \tag{3.8}$$

The counter term also contribute to the perturbative expansion like the interactions.

One-loop Correction to Electron Propagator

Consider the diagram



This contains 3 electron propagator, 1 photon propagator, and 2 vertices. The coefficient is (omit all the integration and summation for simplicity):

$$iD_F^{(2)}(p) \sim \frac{\delta^2}{\delta\bar{\eta}\delta\eta}\frac{1}{2!}\left(\frac{-ie\gamma^\mu_{\alpha\beta}\delta^3}{i\delta J^\mu\delta\eta_\alpha\delta\bar{\eta}_\beta}\right)^2\frac{1}{3!}\left(-i\bar{\eta}_\alpha D_F^{\alpha\beta}\eta_\beta\right)^3\left(-\frac{i}{2}J^\mu\Pi_{\mu\nu}J^\nu\right).$$

First consider the scalar coefficient. Since there is no additional symmetry, the abstract value is e^2. There is an additional sign factor by the proper order of the fermion operators:

$$\frac{\delta^2}{\delta\bar{\eta}_f\delta\eta_i}\frac{\delta^2}{\delta\eta_1\delta\bar{\eta}_1}\frac{\delta^2}{\delta\eta_2\delta\bar{\eta}_2} = -\frac{\delta}{\delta\eta_i}\frac{\delta^2}{\delta\bar{\eta}_1\delta\eta_1}\frac{\delta^2}{\delta\bar{\eta}_2\delta\eta_2}\frac{\delta}{\delta\bar{\eta}_f}.$$

Then consider the tensor contraction,

$$\Pi_{\mu\nu}D_F^{\alpha\lambda}\gamma^\mu_{\lambda\rho}D_F^{\rho\tau}\gamma^\nu_{\tau\sigma}D_F^{\sigma\beta}.$$

The total amplitude is

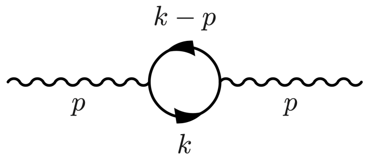
$$\begin{aligned}iD_F^{(2)}(p) &= -e^2\int\frac{d^4k}{(2\pi)^4}\Pi_{\mu\nu}(p-k)\left[D_F(p)\gamma^\mu D_F(k)\gamma^\nu D_F(p)\right]_{\alpha\beta}\\ &= iD_F(p)i\Sigma(p^2)iD_F(p),\end{aligned}$$

where $i\Sigma(p^2)$ is the self energy:

$$i\Sigma(p^2) = e^2\int\frac{d^4k}{(2\pi)^4}\Pi_{\mu\nu}(p-k)\gamma^\mu D_F(k)\gamma^\nu, \tag{3.9}$$

One-loop Correction to Photon Propagator

Consider the diagram



There is 2 electron propagator, 2 photon propagator, and 2 vertices. Consider the perturbative expansion:

$$i\Pi^{(2)}(p) \sim \frac{\delta^2}{i\delta Ji\delta J}\frac{1}{2!}\left(\frac{-e\gamma^\mu_{\alpha\beta}\delta^3}{\delta J^\mu\delta\eta_\alpha\delta\bar{\eta}_\beta}\right)^2\frac{1}{2!}\left(-i\bar{\eta}_\alpha D_F^{\alpha\beta}\eta_\beta\right)^2\frac{1}{2!}\left(\frac{i}{2}J^\mu\Pi_{\mu\nu}J^\nu\right)^2.$$

The diagram has no symmetry factor, but with a -1 sign, which is canceled out by the operator reordering:

$$\bar{\eta}_\beta D_F^{\beta\tau}\eta_\tau\bar{\eta}_\sigma D_F^{\sigma\alpha}\eta_\alpha = -\eta_\alpha\bar{\eta}_\beta D_F^{\beta\tau}\eta_\tau\bar{\eta}_\sigma D_F^{\sigma\alpha}. \tag{3.10}$$

The overall value is e^2.

Then consider the tensor contraction,

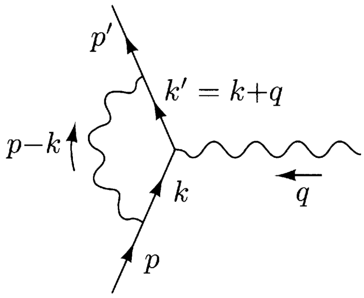
$$-i\Pi^{\mu\nu}_{(2)} \sim e^2\Pi_{\mu\rho}\gamma^\rho_{\alpha\beta}D_F^{\beta\tau}\gamma^\eta_{\tau\sigma}D_F^{\sigma\alpha}\Pi_{\eta\nu} \quad \sim i\Pi_{\mu\rho}i\Sigma^{\rho\sigma}i\Pi_{\sigma\nu}.$$

The photon self-energy is

$$i\Sigma^{\mu\nu}(p^2) = -e^2\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left[\gamma^\mu D_F(k-p)\gamma^\nu D_F(k)\right]. \tag{3.11}$$

One-loop Correction to Vertex

Consider the diagram



There is 4 electron propagator, 2 photon propagator, and 3 vertices. Consider the perturbative expansion:

$$\frac{\delta^3}{i\delta J\delta\bar{\eta}\delta\eta}\frac{1}{2!}\left(\frac{-e\gamma^\mu_{\alpha\beta}\delta^3}{\delta J^\mu\delta\eta_\alpha\delta\bar{\eta}_\beta}\right)^3\frac{1}{2!}\left(-i\bar{\eta}_\alpha D_F^{\alpha\beta}\eta_\beta\right)^4\frac{1}{2!}\left(-\frac{i}{2}J^\mu\Pi_{\mu\nu}J^\nu\right)^2.$$

There is not symmetry factor, and an additional $-i$ factor. The total coefficient is $-ie^3$.

Then consider the tensor contraction

$$D_F^{\alpha\gamma}\gamma^\nu_{\gamma\rho}D_F^{\rho\sigma}\gamma^\zeta_{\sigma\tau}D_F^{\tau\eta}\gamma^\lambda_{\eta\xi}D_F^{\xi\beta}\Pi_{\nu\lambda}\Pi_{\mu\zeta}.$$

The vertex correction is:

$$iV_3(q,p,p') = [iD_F(p)][iD_F(p')][i\Pi^{\mu\nu}(q)][-ie\Gamma^\nu(q,p,p')]$$

where

$$i\Gamma^\mu(q,p,p') = -e^2\int\frac{d^4k}{(2\pi)^4}\Pi_{\nu\lambda}(p-k)\gamma^\nu D_F(k')\gamma^\mu D_F(k)\gamma^\lambda. \tag{3.12}$$

Counter Terms

Consider the counter term in the diagram

The perturbative expansion is

$$iD_F^{(\mathrm{ct})} \sim \frac{\delta^2}{\delta\bar{\eta}\delta\eta}i(\delta_\psi\gamma^\mu_{\alpha\beta}k_\mu - \delta_m\mathbb{I}_{\alpha\beta})\frac{\delta^2}{\delta\eta_\alpha\delta\bar{\eta}_\beta}\frac{1}{2!}\left(-i\bar{\eta}_\alpha D_F^{\alpha\beta}\eta_\beta\right)^2.$$

The contribution to the electron self energy is

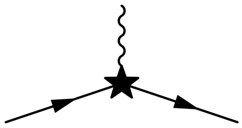
$$\delta_\psi \not{k} - \delta_m m_R.$$

Similarly, the digram

contribute to the photon self energy with term

$$\delta_A[-p^2 g^{\mu\nu} + (1-\xi)p^\mu p^\nu],$$

and the diagram



contribute to the vertex with term

$$\delta_e \gamma^\mu.$$

3.2 One-loop Correction

In this section, we consider the QED in $(d = 4 - \epsilon)$ dimensional space-time.

3.2.1 Electron Propagator

Consider the one-loop correction to the electron propagator, where the self energy (3.9) is

$$\begin{aligned} i\Sigma(p^2) &= e^2\tilde{\mu}^\epsilon \int \frac{d^dk}{(2\pi)^d} \Pi_{\mu\nu}(p-k)\left[\gamma^\mu D_F(k)\gamma^\nu\right]_{\alpha\beta} \\ &= -e^2\tilde{\mu}^\epsilon \int \frac{d^dk}{(2\pi)^d} \frac{\gamma^\mu(\not{k}+m)\gamma_\mu}{(p-k)^2(k^2-m^2)}. \end{aligned} \tag{3.13}$$

The nominator can be simplified using the `FeynCalc` Package:

```
(*load FeynCalc Package*)
<< FeynCalc`

(*simplify the gamma expression*)
Contract[GA[\[Mu]].(GS[k]+m).GA[\[Mu]]]//DiracSimplify
```

The result is

$$4m - 2\not{k}.$$

The denominator can be simplify using the Feynman parameter:

$$\frac{1}{(p-k)^2(k^2-m^2)} = \int_0^1 \frac{dx}{\left[(k-b)^2 - D\right]^2}$$

where b and D can be calculated by

```
A1=(k-p)^2;
A2=k^2-m^2;
{c,b,a}=CoefficientList[x*A1+(1-x)*A2,{k}];
-b/2
-c+b^2/(4*a)//Simplify
```

The result is

$$b = px, \; D = (1-x)(m^2 - p^2 x).$$

Shift $k \to k + px$, the self energy becomes:

$$\begin{aligned} i\Sigma(p^2) &= 2e^2 \tilde{\mu}^\epsilon \int_0^1 (x\not{p} - 2m) dx \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - D)^2} \\ &= i\frac{2e^2 \tilde{\mu}^\epsilon \Omega_d}{(2\pi)^d} \int_0^1 (x\not{p} - 2m) dx \int \frac{k^{d-1} dk}{(k^2 + D)^2}. \end{aligned} \tag{3.14}$$

The regularization procedure

```
omg=(2*Pi^(d/2))/(Gamma[d/2]);
cof=2*e^2*\[Mu]^(4-d)*omg/(2*Pi)^d;
int=cof*Integrate[q^(d-1)/(q^2+D)^2,{q,0,Infinity}][[1]];
map={e->Sqrt[4*Pi*\[Alpha]],EulerGamma->Subscript[\[Gamma],E]};
ans=Series[int/.{d->4-\[Epsilon]},{\[Epsilon],0,0}];
ans/.map//Simplify
```

The result is ($\mu^2 = 4\pi \tilde{\mu}^2 e^{-\gamma_E}$)

$$\Sigma(p^2) = \frac{e_R^2}{8\pi^2} \int_0^1 dx (x\not{p} - 2m_R) \left[\frac{2}{\epsilon} + \ln\left(\frac{\mu^2}{D} \right) \right]. \tag{3.15}$$

The infinity comes from

$$\frac{e_R^2}{4\pi^2 \epsilon} \int_0^1 dx (x\not{p} - 2m_R) = \frac{e_R^2}{8\pi^2 \epsilon} \not{p} - \frac{e_R^2}{2\pi^2 \epsilon} m_R.$$

Using the $\overline{\text{MS}}$ subtraction scheme, we choose

$$\delta_\psi = -\frac{e_R^2}{8\pi^2 \epsilon}, \; \delta_m = -\frac{e_R^2}{2\pi^2 \epsilon}, \tag{3.16}$$

and the self energy is

$$\begin{aligned} \Sigma(p^2) &= \frac{e_R^2}{8\pi^2} \int_0^1 dx (x\not{p} - 2m_R) \ln\left[\frac{\mu^2}{(1-x)(m_R^2 - p^2 x)} \right] \\ &= \frac{e_R^2}{8\pi^2} (\not{p} - 4m_R) \ln\left(\frac{\mu}{m_R} \right) - \int_0^1 dx \ln\left[(1-x)\left(1 - \frac{p^2 x}{m_R^2}\right) \right]. \end{aligned} \tag{3.17}$$

3.2.2 Photon Self-energy

Consider the one-loop correction to the photon propagator, where the self energy (3.11) is

$$i\Sigma^{\mu\nu} = -e^2 \tilde{\mu}^\epsilon \int \frac{d^d k}{(2\pi)^d} \frac{\text{Tr}\left[\gamma^\mu D_F(k-p) \gamma^\nu D_F(k)\right]}{(k^2 - m^2)[(p-k)^2 - m^2]}. \tag{3.18}$$

The Dirac trace and Feynman parameter is calculated by

```
(*Dirac trace*)
DiracTrace[GA[\[Mu]].(GS[k−p]+m).GA[\[Nu]].(GS[k]+m)]//DiracSimplify

(*Feynman paramater*)
A1=k^2−m^2;
A2=(k−p)^2−m^2;
{c,b,a}=CoefficientList[x*A1+(1−x)*A2,{k}];
−b/2
−c+b^2/(4*a)//Simplify
```

The nominator is

$$4\left[g^{\mu\nu}\left(k\cdot p-k^2+m^2\right)+2k^\mu k^\nu-k^\mu p^\nu-p^\mu k^\nu\right]$$

The denominator is:

$$\frac{1}{(k^2-m^2)[(p-k)^2-m^2]}=\frac{1}{\{[k-p(1-x)]^2-[m^2+p^2x(x-1)]\}^2}$$

Let $D=m^2-p^2x(1-x)$, shift $k\to k+p(1-x)$, and drop all p^μ linear term,[1] the result is

$$i\Sigma^{\mu\nu}=-4e^2\tilde{\mu}^\epsilon\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\frac{2k^\mu k^\nu-g^{\mu\nu}\left[k^2-x(1-x)p^2-m^2\right]}{\left[k^2-D\right]^2}\tag{3.19}$$

The self-energy $i\Sigma^\mu\propto g^{\mu\nu}$, we can make the substitution

$$k^\mu k^\nu\to\frac{1}{d}k^2g^{\mu\nu}.$$

We then need to consider the integral

$$\begin{aligned}iI(x)&=4e^2\tilde{\mu}^\epsilon\int\frac{d^dk}{(2\pi)^d}\frac{(1-\frac{2}{d})k^2-x(1-x)p^2-m^2}{\left[k^2-D\right]^2},\\ I(x)&=-\frac{4e^2\tilde{\mu}^\epsilon\Omega_d}{(2\pi)^d}\int k^{d-1}dk\frac{(1-\frac{2}{d})k^2+x(1-x)p^2+m^2}{\left[k^2+D\right]^2}.\end{aligned}$$

The regulation is carried out by the following code:

```
omg=(2*Pi^(d/2))/(Gamma[d/2]);
cof=−4*e^2*\[Mu]^(4−d)*omg/(2*Pi)^d;
den=q^(d−1)*((1−2/d)q^2+x*(1−x)p^2+m^2);
int=cof*Integrate[den/(q^2+D)^2,{q,0,Infinity}][[1]];
map={EulerGamma−>Subscript[\[Gamma],E],D−>m^2−p^2*x*(1−x)};
ans=Series[int/.{d−>4−\[Epsilon]},{\[Epsilon], 0, 0}];
ans/.map//Simplify
```

The result is

$$\Sigma^{\mu\nu}(p^2)=-\frac{e_R^2p^2g^{\mu\nu}}{2\pi^2}\int_0^1 dx\;x(1-x)\left[\frac{2}{\epsilon}+\ln\left(\frac{\mu^2}{m_R^2-p^2x(1-x)}\right)\right]\tag{3.20}$$

[1] The Ward identity requires that the p^μ term in the propagator do not contribute to any scattering process.

The divergent part is

$$-\frac{e_R^2 p^2 g^{\mu\nu}}{\pi^2\epsilon}\int_0^1 dx\, x(1-x) = -\frac{e_R^2 p^2 g^{\mu\nu}}{6\pi^2\epsilon}.$$

The counter term coefficient is

$$\delta_A = -\frac{e_R^2}{6\pi^2\epsilon}. \tag{3.21}$$

The photon self-energy is then

$$\begin{aligned}\Sigma^{\mu\nu}(p^2) &= -\frac{e_R^2 p^2 g^{\mu\nu}}{2\pi^2}\int_0^1 dx\, x(1-x)\ln\left[\frac{\mu^2}{m_R^2 - p^2 x(1-x)}\right]\\ &= -\frac{e_R^2 p^2 g^{\mu\nu}}{12\pi^2}\ln\left(\frac{\mu}{m}\right) + \frac{e_R^2 p^2 g^{\mu\nu}}{2\pi^2}\int_0^1 dx\, x(1-x)\ln\left[1 - \frac{p^2}{m_R^2}x(1-x)\right].\end{aligned} \tag{3.22}$$

3.2.3 Vertex Correction

Consider the loop correction (3.12):

$$i\Gamma^\mu(p, q_1, q_2) = e^2\tilde{\mu}^\epsilon \int \frac{d^4k}{(2\pi)^4}\frac{\gamma^\nu(\not{k}' + m)\gamma^\mu(\not{k} + m)\gamma_\nu}{(k^2 - m^2)(k'^2 - m^2)(p-k)^2}. \tag{3.23}$$

Using the following code

```
(*numerator*)
den=Contract[GA[\[Nu]].(GS[kp]+m).GA[\[Mu]].(GS[k]+m).GA[\[Nu]]];
DiracSimplify[den]

(*Feynman parameter*)
A1=k^2-m^2;
A2=(k+q)^2-m^2;
A3=(p-k)^2;
{c,b,a}=CoefficientList[x*A1+y*A2+z*A3,{k}];
-b/2//Simplify
-c+b^2/4//Simplify
```

The numerator is

$$-2\not{k}\gamma^\mu\not{k}' - 2m^2\gamma^\mu + 4m(k+k')^\mu.$$

The denominator is

$$\int \frac{dF_3}{[(k + yq - zp)^2 - D]^3},$$

where

$$\begin{aligned}D &= (x+y)m^2 - z(1-z)p^2 - y(1-y)q^2 - 2yzpq\\ &= (x+y)m^2 - xyq^2 - yzp'^2 - xzp^2.\end{aligned}$$

Shift $k^\mu \to k^\mu + zq_1^\mu - yp^\mu$, throw away all terms with linear k^μ, and replace $k^\mu k^\nu$ with $\frac{1}{d}k^2 g^{\mu\nu}$, the result is

$$\frac{4}{d}k^2\gamma^\mu - 2(-y\not{q} + z\not{p})\gamma^\mu[(1-y)\not{q} + z\not{p}] + 4m^2\gamma^\mu - 2m\left[(1-2y)q^\mu + 2zp^\mu\right].$$

Note that only the quadratic term is divergent.

$$\Gamma^\mu(p,q_1,q_2) = -i\frac{4e^2\tilde{\mu}^\epsilon\gamma^\mu}{d}\int dF_3\int\frac{d^dk}{(2\pi)^d}\frac{k^2}{(k^2-D)^3} + \delta\Gamma^\mu(p,q_1,q_2).$$

where $\delta\Gamma^\mu$ stores all the finite part

$$\begin{aligned}&\delta\Gamma^\mu(p,q_1,q_2)\\ &= \int\frac{e^2k^3dkdF_3}{(2\pi)^2(k^2+D)^3}\left\{(-y\not{q}+z\not{p})\gamma^\mu[(1-y)\not{q}+z\not{p}] - 2m^2\gamma^\mu + m\left[(1-2y)q^\mu+2zp^\mu\right]\right\}.\end{aligned}$$

The divergent part is

$$\frac{4e^2\tilde{\mu}^\epsilon\Omega_d\gamma^\mu}{d(2\pi)^d}\int dF_3\int\frac{k^{d+1}dk}{(k^2+D)^3} = \frac{e_R^2}{16\pi^2}\gamma^\mu\int dF_3\left[\frac{2}{\epsilon}+\ln\left(\frac{\mu^2}{D}\right)\right].$$

Using the $\overline{\text{MS}}$ scheme, the coefficient of the counter term is

$$\delta_e = -\frac{e_R^2}{8\pi^2\epsilon}. \tag{3.24}$$

3.2.4 Renormalization Group

In summery, the renormalization factors are

$$\begin{aligned}Z_\psi &= 1-\frac{e_R^2}{8\pi^2\epsilon}+O(e_R^3),\\ Z_A &= 1-\frac{e_R^2}{6\pi^2\epsilon}+O(e_R^3),\\ Z_m &= 1-\frac{e_R^2}{2\pi^2\epsilon}+O(e_R^3),\\ Z_e &= 1-\frac{e_R^2}{8\pi^2\epsilon}+O(e_R^3),\end{aligned} \tag{3.25}$$

which means

$$\begin{aligned}\frac{d\ln Z_\phi}{de_R} &= -\frac{e_R}{4\pi^2\epsilon}+O(e_R^2),\\ \frac{d\ln Z_A}{de_R} &= -\frac{e_R}{3\pi^2\epsilon}+O(e_R^2),\\ \frac{d\ln Z_m}{de_R} &= -\frac{e_R}{\pi^2\epsilon}+O(e_R^2),\\ \frac{d\ln Z_e}{de_R} &= -\frac{e_R}{4\pi^2\epsilon}+O(e_R^2).\end{aligned} \tag{3.26}$$

The bare parameters are

$$\begin{aligned}\psi_0 &= Z_\psi^{1/2}\psi_R,\\ A_0 &= Z_A^{1/2}A_R,\\ m_0 &= Z_mZ_\psi^{-1}m_R,\\ e_0 &= Z_eZ_\psi^{-1}Z_A^{-1/2}e_R\tilde{\mu}^{\epsilon/2}.\end{aligned} \tag{3.27}$$

The RG equation for e_0 is

$$\frac{d\ln e_0}{d\ln\mu} = \left(\frac{\ln Z_e}{de_R} - \frac{\ln Z_\psi}{de_R} - \frac{1}{2}\frac{\ln Z_A}{de_R} + \frac{1}{e_R}\right)\frac{de_R}{d\ln\mu} + \frac{\epsilon}{2} = 0. \tag{3.28}$$

The beta function is

$$\beta(e_R) = \frac{de_R}{d\ln\mu} = -\frac{\epsilon}{2}e_R + \frac{e_R^3}{12\pi^2} + O(e_R^4). \tag{3.29}$$

The RG equation for m_0 is

$$\frac{d\ln m_0}{d\ln\mu} = \left(\frac{\ln Z_m}{de_R} - \frac{\ln Z_\psi}{de_R}\right)\frac{de_R}{d\ln\mu} + + \frac{1}{m_R}\frac{dm_R}{d\ln\mu} = 0. \tag{3.30}$$

The anomalous dimension of mass is

$$\gamma_m = \frac{d\ln m_R}{d\ln\mu} = -\frac{3e_R^2}{8\pi^2} + O(e_R^3). \tag{3.31}$$

Chapter 4

Non-relativistic Quantum Field Theory

A general non-relativistic field has the Lagrangian[1]

$$\mathcal{L} = \bar{\psi}_a(x)(i\delta_{ab}\partial_t - \hat{H}_{ab})\psi_b(x) + \mathcal{V}_{\text{int}} \tag{4.1}$$

where the field operator ψ can be bosonic or fermionic, which is denoted by a number $\zeta = \pm 1$, and $\mathcal{V}_{\text{int}}$ is the interaction Lagrangian. A general interaction has the form

$$\mathcal{V}_{\text{int}} = \bar{\psi}_a(x_1)\bar{\psi}_b(x_2)V_{ab}(x_1, x_2)\psi_b(x_2)\psi_a(x_1). \tag{4.2}$$

Note that the classical equation of motion for the free field is

$$\begin{aligned} 0 &= \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi}_a(x))} - \frac{\partial \mathcal{L}}{\partial \bar{\psi}_a(x)} \\ &= -i\partial_t \phi_a(x) + \hat{H}_{ab}\phi_b(x), \end{aligned} \tag{4.3}$$

which satisfies the Schrödinger equation.

We are mostly work with finite system size L^d with UV cutoff $\Lambda = \frac{2\pi}{a}$,[2] in which case the spatial Fourier transformation is

$$\tilde{\psi}_a(k) = \int_{L^d} d^d x e^{-ik\cdot x}\psi(k), \tag{4.4}$$

$$\psi_a(x) = \frac{1}{L^d}\sum_k e^{ik\cdot x}\tilde{\psi}(k). \tag{4.5}$$

Note that for finite size, the momentum is discretized:

$$k_i = \frac{2\pi}{L} n_i, \; n_i = -N, \cdots, N. \tag{4.6}$$

By default, we take the thermodynamic limit. The summation is approximated by the integration:

$$\frac{1}{L^d}\sum_k \longrightarrow \int_{|k|<\Lambda} \frac{d^d k}{(2\pi)^d}. \tag{4.7}$$

[1] The repeated indices are automatically summed.
[2] We can regard a as the lattice spacing, and assume $L = Na$.

4.1 Finite Temperature Field Theory

The original real-time partition function is defined as[3]

$$Z[J]=\int D[\bar{\psi},\psi]\exp\left\{i\int dt\int d^dx\left[\mathcal{L}+\bar{J}_a(x)\psi_a(x)+\bar{\psi}_a(x)J_a(x)\right]\right\}. \tag{4.8}$$

For finite-temperature field theory, after making the wick rotation $t\to -i\tau$, the partition function for a generic non-relativistic lattice theory is:

$$Z[J]=\int D[\bar{\psi},\psi]e^{-S[\bar{\psi},\psi]+\bar{J}\cdot\psi+\bar{\psi}\cdot J}, \tag{4.9}$$

where the action is

$$S=\int_0^\beta d\tau\int d^dx\left[\bar{\psi}_a(x)(\delta_{ab}\partial_\tau+\hat{H}_{ab})\psi_b(x)+\mathcal{V}_{\text{int}}\right]. \tag{4.10}$$

Remark 6. Temporal Fourier Transformation

The Fourier transformation on the imaginary time domain is defined as:

$$\begin{aligned}\tilde{\psi}(\omega_n) &= \int_0^\beta d\tau e^{i\omega_n\tau}\psi(\tau), &(4.11)\\ \psi(\tau) &= \frac{1}{\beta}\sum_{\omega_n}e^{-i\omega_n\tau}\tilde{\psi}(\omega_n). &(4.12)\end{aligned}$$

Under such convention, in the thermodynamic limit and zero-temperature limit, the spatial-temporal Fourier transformation agrees with the relativistic case (up to a Wick rotation).

4.1.1 Free Field Theory

We first consider the action of free field

$$S_0=\int_0^\beta d\tau\int d^dx\;\bar{\psi}_a(x)(\delta_{ab}\partial_\tau+\hat{H}_{ab})\psi_b(x). \tag{4.13}$$

The Fourier transformation

$$S_0=\frac{1}{\beta}\sum_{\omega_n}\int_\Lambda\frac{d^dk}{(2\pi)^d}\tilde{\bar{\psi}}_a(k,\omega_n)\left[-i\omega_n+\tilde{H}_{ab}(k)\right]\tilde{\psi}_b(k,\omega_n). \tag{4.14}$$

The partition function with source is

$$\frac{Z_0[J]}{Z_0[0]}=\exp\left[-\frac{1}{\beta}\sum_{\omega_n}\int_\Lambda\frac{d^dk}{(2\pi)^d}\tilde{\bar{J}}_a(k,\omega_n)\tilde{G}_{ab}(k,\omega_n)\tilde{J}_b(k,\omega_n)\right], \tag{4.15}$$

[3] As with the relativistic case, we introduce an auxiliary source J, which is bosonic/fermionic if the field ψ is bosonic/fermionic.

where the Green's function is

$$\tilde{G}_{ab}(k,\omega_n) = \left[\frac{1}{i\omega_n - \tilde{H}(k)}\right]_{ab}. \tag{4.16}$$

Remark 7. Obtaining the Partition Function

Unlike the relativistic case, the value of the value of partition function without source $Z_0[0]$ is related to the free energy. We can express it formally as

$$Z_0[0] = \left[\det(-G_{ab})^{-1}\right]^{-\zeta}.$$

To get the correct dimensionality, we set the determinant as

$$Z_0[0] \equiv \prod_{k,\omega_n} \left\{\beta \det\left[-i\omega_n + \tilde{H}(k)\right]\right\}^{-\zeta}.$$

Thus the free energy is

$$F = -\frac{1}{\beta}\ln Z_0 = \zeta \sum_{k,\omega_n} \ln\left\{\beta \det\left[-i\omega_n + \tilde{H}(k)\right]\right\}. \tag{4.17}$$

Remark 8. Matsubara Summation

Now consider the summation on Matsubara frequency:

$$\sum_{\omega_n} f(\omega_n) = \begin{cases} \sum_n f(\frac{2n\pi}{\beta}) & \text{bosonic} \\ \sum_n f(\frac{(2n+1)\pi}{\beta}) & \text{fermionic} \end{cases}. \tag{4.18}$$

The frequency is capture by the singularities of the density function of the states:

$$\rho(z) = \begin{cases} \frac{1}{\exp(\beta z)-1} & \text{bosonic} \\ \frac{1}{\exp(\beta z)+1} & \text{fermionic} \end{cases}. \tag{4.19}$$

The residue on imaginary frequency $i\omega_n$ is alway $\frac{1}{\beta}$. In this way, the summation is:

$$\frac{1}{\beta}\sum_{\omega_n} f(i\omega_n) = \frac{1}{2\pi i}\oint \rho(z) f(z) = -\sum_k \operatorname{Res} \rho(z) f(z)|_{z=z_k}. \tag{4.20}$$

Example 3. Summation of Green's function

Consider the frequency summation for the correlation function:

$$\frac{1}{\beta}\sum_{\omega_n} \tilde{G}_0(k) = \frac{1}{\beta}\sum_{\omega_n} \frac{1}{i\omega_n - E_p} = -\operatorname{Res} \left.\frac{\rho(z)}{z-E_p}\right|_{z=E_p} = \rho(E_p).$$

Example 4. Summation of Green's function

Consider the frequency summation for the correlation function:

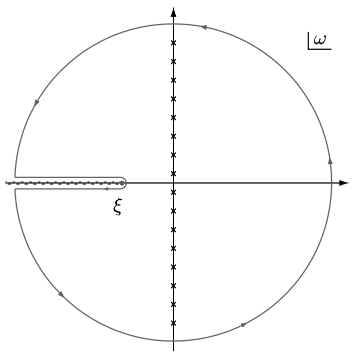
$$\sum_{\omega_n} \langle \bar{\psi}_{\vec{p},\omega_n} \psi_{\vec{p},\omega_n} \rangle = \frac{1}{\beta} \sum_{\omega_n} \frac{1}{-i\omega_n + \epsilon_{\vec{p}}} = \mathrm{Res} \left. \frac{\rho(z)}{z - \epsilon_{\vec{p}}} \right|_{z - \epsilon_{\vec{p}}} = \rho(\epsilon_{\vec{p}}).$$

Example 5. Free Energy Summation

Consider the free energy

$$F = -\frac{1}{\beta} \ln Z = -\frac{1}{\beta} \sum_{\omega_n} \ln[\beta(-i\omega_n + E_{\vec{p}})] = \frac{1}{2\pi i} \oint dz \rho(z) \ln[\beta(\xi - z)].$$

To calculate the summation, we consider the line integral along the loop:



The free energy is

$$\begin{aligned} F &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} dx \rho(x) \ln \left(\frac{\xi - x - i\epsilon}{\xi - x + i\epsilon} \right) \\ &= \frac{-\zeta}{2\pi i \beta} \int_{-\infty}^{\infty} dx \ln(1 - \zeta e^{-\beta z}) \left(\frac{1}{x + i\epsilon - \xi} - \frac{1}{x - i\epsilon - \xi} \right), \end{aligned}$$

where we integrate the expression by part, noticing that

$$\frac{d}{dz} \frac{\zeta}{\beta} \ln(1 - \zeta e^{-\beta z}) = \frac{1}{e^{\beta z} - \zeta} = \rho(z) \tag{4.21}$$

Using the identity

$$\lim_{\epsilon \to 0^+} \frac{1}{x + i\epsilon} = -i\pi\delta(x) + \mathcal{P}\frac{1}{x},$$

the above expression can be simplified to

$$F = \frac{\zeta}{\beta} \ln(1 - \zeta e^{-\beta \zeta}). \tag{4.22}$$