The Standard Model

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The standard model (SM) describes all the elementary particles experimentally observed until now. It includes the electromagnetic, weak, and strong interactions mediated by corresponding gauge fields. Also, an additional scalar Higgs field in SM gives masses to fermions and W, Z gauge bosons.

Contents

I. Structure of the Standard Model

A. Elementary Particles

The elementary particles in SM can be divided into the fermions, the gauge bosons, and the Higgs boson. The fermions describe the matters, the gauge boson mediates forces, and the Higgs couples to the matter and gauge field to create masses.

Composite particles like nucleons (neutron/proton) and mesons (pion, kaon, etc.) consist of quarks, which come in six flavors: up, down, charm, strange, top, and bottom, and can also be grouped into three generations. Besides, electrons and neutrinos are known as the lepton, which comes in three generations.

The gauge bosons that mediate electromagnetic force are the photons; for the weak interaction, they are the $W^{\pm}$ and Z bosons, and for the strong force, they are gluons, which come in 8 colors.

B. Gauge Group

The standard model has $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ gauge symmetry. For a general field ϕ_{iI}, i labels the electroweak degrees of freedom, and I labels the color degrees of freedom. The covariant derivative for ϕ_{iI} is then

$$D_\mu = \partial_\mu - i\left(g_1 B_\mu Y + g_2 W_\mu^a T_2^a + g_3 A_\mu^b T_3^b\right),$$

where the operator Y, T_2^a and T_3^b is depend on the representation of U(1), SU(2) and SU(3) groups respectively.

The weak interaction only involves left-handed spinor, so the largest fermion sector is

$$Q_I = \left[\begin{pmatrix} u_L \\ d_L \end{pmatrix}\begin{pmatrix} c_L \\ s_L \end{pmatrix}\begin{pmatrix} t_L \\ b_L \end{pmatrix}\right]_I. \tag{1}$$

Each Q_I is in the $\left(3,2,\frac{1}{6}\right)$ representation, where 3 is the fundamental representation of SU(3), 2 is the fundamental representation of SU(2), and $\frac{1}{6}$ is the hypercharge of the U(1) gauge group. For example, the sector of first-generation left-handed quarks are

$$Q_1 = \begin{bmatrix} u_L^r & u_L^g & u_L^y \\ d_L^r & d_L^g & d_L^y \end{bmatrix}. \tag{2}$$

Note that the electric charge satisfies $q = I_3 + Y$, where I_3 is the z-component of the isospin.

Besides, the left-handed leptons also involve the weak interaction but not the strong interaction

$$L_I = \left[\begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}\begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix}\begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix}\right]_I, \tag{3}$$

and thus each L_I is in the $\left(1,2,\frac{1}{2}\right)$ representation.

The right-handed quarks form $(3,1,\frac{2}{3})$ and $(3,1,-\frac{1}{3})$ representations, the right-handed electrons/muons/tauons form $(1,1,-1)$ representations, and the right-handed neutrinos form trivial $(1,1,0)$ representations.

In addition to the fermion, there is also a scalar Higgs field in the representation $(1,2,\frac{1}{2})$. Together, we say that the standard model contains the following representations (only fermions in the first generation are listed):

Particles		Representation
Quarks	Q_1	$(3,2,1/6)$
	u_R	$(3,1,2/3)$
	d_R	$(3,1,-1/3)$
Leptons	L_1	$(1,2,-1/2)$
	e_R	$(1,1,-1)$
	ν_{eR}	$(1,1,0)$
Higgs	φ	$(1,2,1/2)$

The standard model described the interaction of quarks, leptons, and gauge bosons. The gauge boson intermediate electromagnetic, weak, and strong interaction. In the standard model, the gauge group is

$$G = \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1), \tag{4}$$

where SU(3) concerns the color degrees of freedom, and the $\mathrm{SU}(2) \times \mathrm{U}(1)$ is gauge group of the electroweak interaction.

The gauge field is coupled to the matter fields and consists of 12 elementary fermion particles and a Higgs boson.

The gauge theory is written as a nonabelian or the Yang-Mills theory, which forbids the gauge field to have a mass term. Also, the weak interaction breaks the parity – it only involves the left-handed spinor field, and such single-handed gauge symmetry forbids the mass term for fermions. The Higgs field should be introduced to obtain the mass, which involves spontaneous symmetry breaking.

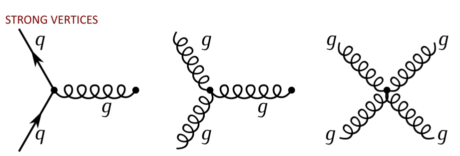
C. Interactions

The Lagrangian of the SM consists of

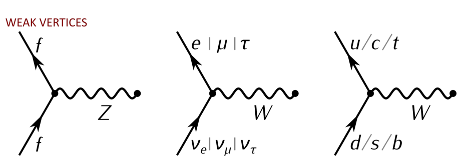
$$\mathcal{L}_{\mathrm{SM}} = \mathcal{L}_{\mathrm{QED}} + \mathcal{L}_{\mathrm{QCD}} + \mathcal{L}_{\mathrm{W}} + \mathcal{L}_{\mathrm{H}}. \tag{5}$$

In each sector, the free theory is described by the free fermions fields, free scalar fields, or free vector fields. The Feynman diagrams can represent the interaction among them.[1]

The strong interaction involves the vertices:



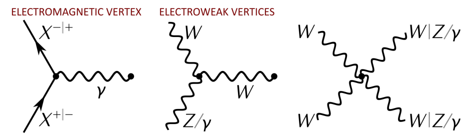
where q is any quark, g is a gluon. The weak interaction involves the vertices:



where f is any fermion.

[1] All Feynman diagrams in the model are built from combinations of these vertices. The conjugate of each listed vertex (reversing the direction of arrows) is also allowed.

The electro-weak interaction involves the vertices:



where X is any charged particle, γ is a photon. One particle label is chosen in diagrams with multiple particle labels separated by "/". In diagrams with particle labels separated by "—", the labels must be chosen in the same order. For example, in the four boson electroweak case, the valid diagrams are WWWW, WWZZ, WW$\gamma\gamma$, and WWZγ.

The Higgs vertices are:

where m is any particle with mass, m_B is any boson with mass.

II. Electroweak Interaction

A. Spontaneous Symmetry Breaking of Higgs Field

Besides the quarks, leptons, and gauge bosons, a doublet scalar field (Higgs) is also introduced to generate mass for the particles:

$$\varphi \equiv \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}, \tag{6}$$

which is also an isospin doublet with hypercharge $Y = \frac{1}{2}I$:

Field	I_3	Y	q
φ_1	1/2	1/2	1
φ_2	−1/2	1/2	0

The Lagrangian of the Higgs field is

$$\mathcal{L} = -\frac{1}{4}(W^a_{\mu\nu})^2 - \frac{1}{4}B^2_{\mu\nu} + (D^\mu\varphi)^\dagger D_\mu\varphi - V(\varphi), \tag{7}$$

where the field strength tensors are

$$\begin{aligned} W^a_{\mu\nu} &= \partial_\mu W^1_\nu - \partial_\nu W^1_\mu + g_2(W^2_\mu W^3_\nu - W^3_\mu W^2_\nu), \\ W^a_{\mu\nu} &= \partial_\mu W^1_\nu - \partial_\nu W^1_\mu + g_2(W^3_\mu W^1_\nu - W^1_\mu W^3_\nu), \\ W^a_{\mu\nu} &= \partial_\mu W^1_\nu - \partial_\nu W^1_\mu + g_2(W^1_\mu W^2_\nu - W^2_\mu W^1_\nu), \\ B^a_{\mu\nu} &= \partial_\mu B^1_\nu - \partial_\nu B^1_\mu, \end{aligned}$$

and the Higgs interaction is

$$V(\varphi) = \frac{\lambda}{4}\left(\varphi^\dagger\varphi - \frac{v^2}{2}\right)^2. \tag{8}$$

This potential gives φ a nonzero vacuum expectation value (VEV). We can make a gauge transformation so that

$$\varphi_0 = \langle 0|\varphi|0\rangle = \frac{v}{\sqrt{2}}\begin{bmatrix}0\\1\end{bmatrix}. \tag{9}$$

The kinetic Lagrangian is described by the SU(2) gauge-invariant Lagrangian

$$\mathcal{L}_{\text{kin}} = (D^\mu\varphi)^\dagger D_\mu\varphi, \tag{10}$$

where the covariant derivative is

$$D_\mu = \partial_\mu - i\left(g_1 B_\mu Y + g_2 W_\mu^a T_2^a\right). \tag{11}$$

We can introduce the Weinberg angle $\theta_W = \arctan\frac{g_1}{g_2}$, and the fields

$$\begin{aligned} W_\mu^\pm &\equiv \frac{1}{\sqrt{2}}(W_\mu^1 \mp iA_\mu^2),\\ Z_\mu &\equiv \cos\theta_W W_\mu^3 - \sin\theta_W B_\mu,\\ A_\mu &\equiv \sin\theta_W W_\mu^3 + \cos\theta_W B_\mu. \end{aligned} \tag{12}$$

The reverse transformation for W_μ^3 and B_μ is

$$\begin{aligned} W_\mu^3 &= \cos\theta_W Z_\mu + \sin\theta_W A_\mu,\\ B_\mu &= -\sin\theta_W Z_\mu + \cos\theta_W A_\mu. \end{aligned} \tag{13}$$

The off-diagonal part of the Lie algebra becomes

$$\frac{1}{\sqrt{2}}\begin{bmatrix}0 & W_\mu^+\\ W_\mu^- & 0\end{bmatrix}, \tag{14}$$

and the diagonal part becomes

$$\begin{aligned} g_2 W_\mu^3 T_2^z + g_1 B_\mu Y &= g_2(\cos\theta_W Z_\mu + \sin\theta_W A_\mu)T_2^3 + g_1 Y(-\sin\theta_W Z_\mu + \cos\theta_W A_\mu)\\ &= g_2\cos\theta_W A_\mu\left(T_2^3 + Y\right) + eZ_\mu\left(\cot\theta_W T_2^3 - \tan\theta_W Y\right)\\ &= eA_\mu q + \frac{e}{\sin\theta_W\cos\theta_W}Z_\mu\left(\cos^2\theta_W T_2^3 - \sin^2\theta_W Y\right)\\ &= eA_\mu q + \frac{e}{\sin\theta_W\cos\theta_W}Z_\mu\left(T_2^3 - \sin^2\theta_W q\right). \end{aligned} \tag{15}$$

Note that the gauge field A_μ describes QED, so we identify the coupling constant with the electric charge:

$$e = g_2\sin\theta_W = g_1\cos\theta_W. \tag{16}$$

For Higgs field,

$$T_2^3 = \frac{1}{2}\sigma^z, \quad Y = \frac{1}{2}I, \quad q = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}. \tag{17}$$

$T_2^3 = \frac{1}{2}\sigma^z$, $Y = \frac{1}{2}I$, so With the new definition of the field, the Lie algebra becomes

$$\begin{aligned} &g_2 W_\mu^a T_2^a + g_1 B_\mu Y\\ &= \frac{e}{2\sin\theta_W}\begin{bmatrix}2\sin\theta_W A_\mu + \frac{\cos 2\theta_W}{\cos\theta_W}Z_\mu & \sqrt{2}W_\mu^+\\ \sqrt{2}W_\mu^- & \frac{1}{\cos\theta_W}Z_\mu\end{bmatrix}. \end{aligned}$$

We can now expand the kinetic term to get the effective mass of the gauge field in a unitary gauge. The mass term is the quadratic part of φ:

$$\begin{aligned} \mathcal{L}_{\text{mass}} &= \varphi_0^\dagger\left(g_2 W_\mu^a T_2^a + g_1 B_\mu Y\right)^2\varphi_0\\ &= \frac{e^2v^2}{8\sin^2\theta_W}\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}\cdots & \sqrt{2}W_\mu^+\\ \sqrt{2}W_\mu^- & \frac{1}{\cos\theta_W}Z_\mu\end{bmatrix}^2\begin{bmatrix}0\\1\end{bmatrix}\\ &= m_W W^{+\mu}W_\mu^- + \frac{1}{2}m_Z^2 Z^\mu Z_\mu, \end{aligned}$$

where the mass for W and Z bosons are

$$m_W = \frac{ev}{2\sin\theta_W}, \quad m_Z = \frac{ev}{2\sin\theta_W\cos\theta_W} = \frac{m_W}{\cos\theta_W}.$$

Note that the photon field A_μ does not obtain mass from the Higgs field.

B. Lagrangian of Higgs-Gauge Sector

We will now express the Lagrangian (7) in terms of the fields we have just introduced. In the unitary gauge, the Higgs field can be written as

$$\varphi = \frac{1}{\sqrt{2}}\begin{bmatrix}v + H(x)\\ 0\end{bmatrix}, \tag{18}$$

where $H(x)$ is a real scalar field. The interaction terms give

$$\begin{aligned} V(H) &= \frac{\lambda v^2}{4}H^2 + \frac{\lambda v}{4}H^3 + \frac{\lambda}{16}H^4\\ &= \frac{1}{2}m_H^2 H^2 + \frac{m_H^2}{2v}H^3 + \frac{m_H^2}{8v^2}H^4, \end{aligned} \tag{19}$$

where we have defined the Higgs mass

$$m_H^2 = \frac{\lambda v^2}{2}. \tag{20}$$

Also, the fluctuation around the φ_0 creates coupling between the Higgs field and the gauge fields. To capture the coupling, we can replace v^2 with $(v+H)^2$ in the gauge field mass term. Thus, the Lagrangian is

$$\mathcal{L}_{\mathrm{H}} = \partial^\mu H^\dagger \partial_\mu H - V(H) + \left(m_W W^{+\mu} W^-_\mu + \frac{m_Z^2}{2} Z^\mu Z_\mu \right) \left(H^2 + 2vH \right).$$

Now consider the Lagrangian for the gauge field

$$\mathcal{L}_{\mathrm{W}} = -\frac{1}{4}(W^a_{\mu\nu})^2 - \frac{1}{4}B^2_{\mu\nu} + m_W W^{+\mu}W^-_\mu + \frac{1}{2}m_Z^2 Z^\mu Z_\mu.$$

We can regroup the field strength as

$$\begin{aligned}
W^+_{\mu\nu} &= D_\mu W^+_\nu - D_\nu W^+_\mu, \\
W^-_{\mu\nu} &= D^\dagger_\mu W^-_\nu - D^\dagger_\nu W^-_\mu, \\
W^3_{\mu\nu} &= \sin\theta_W F_{\mu\nu} + \cos\theta_W Z_{\mu\nu} - ig_2(W^+_\mu W^-_\nu - W^-_\mu W^+_\nu), \\
B^a_{\mu\nu} &= \cos\theta_W F_{\mu\nu} - \sin\theta_W Z_{\mu\nu},
\end{aligned}$$

where we have defined

$$\begin{aligned}
W^\pm_{\mu\nu} &= \frac{1}{\sqrt{2}}(W^1_{\mu\nu} \mp iW^2_{\mu\nu}), \\
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu, \\
Z_{\mu\nu} &= \partial_\mu Z_\nu - \partial_\nu Z_\mu.
\end{aligned}$$

The covariant derivative for the $W^\pm$ field is

$$\begin{aligned}
D_\mu &= \partial_\mu - ig_2 W^3_\mu \\
&= \partial_\mu - ig_2\left(\sin\theta_W A_\mu + \cos\theta_W Z_\mu\right) \\
&= \partial_\mu - ie\left(A_\mu + \cot\theta_W Z_\mu\right)
\end{aligned}$$

So, the Lagrangian for the gauge field is

$$\begin{aligned}
\mathcal{L}_{\mathrm{W}} = &-\frac{1}{4}(2W^+_{\mu\nu}W^{-\mu\nu} + W^3_{\mu\nu}W^{3\mu\nu}) - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} + m_W W^{+\mu}W^-_\mu + \frac{1}{2}m_Z^2 Z^\mu Z_\mu \\
= &-\frac{1}{4}(F_{\mu\nu}F^{\mu\nu} + Z_{\mu\nu}Z^{\mu\nu}) - D^\mu W^{+\nu} D^\dagger_\mu W^-_\nu + D^\mu W^{+\nu} D^\dagger_\nu W^-_\mu \\
&+ ie(F^{\mu\nu} + \cot\theta_W Z^{\mu\nu})W^+_\mu W^-_\nu + m_W W^{+\mu}W^-_\mu + \frac{1}{2}m_Z^2 Z^\mu Z_\mu \\
&- \frac{e^2}{2\sin^2\theta_W}\left(W^{+\mu}W^-_\mu W^{+\nu}W^-_\nu - W^{+\mu}W^+_\mu W^{-\nu}W^-_\nu\right).
\end{aligned} \tag{21}$$

We remark that in the unitary gauge, there is no redundancy of the gauge field. No ghost field is needed in the quantization procedure.

III. Yukawa Couplings

The fermion fields in the electroweak interaction

Field	I_3	Y	q
u_L	$1/2$	$1/6$	$2/3$
d_L	$-1/2$	$1/6$	$-1/3$
u_R	0	$2/3$	$2/3$
d_R	0	$-1/3$	$-1/3$
e_L	$-1/2$	$-1/2$	-1
ν_{eL}	$1/2$	$-1/2$	0
e_R	0	-1	-1
ν_{eR}	0	0	0

(22)

The kinetic term for the fermion Lagrangian is

$$\begin{aligned}
\mathcal{L}_{\mathrm{kin}} &= iQ^\dagger_J \sigma^\mu D_\mu Q_J + iu^\dagger_{RJ}\bar{\sigma}^\mu D_\mu u_{RJ} + id^\dagger_{RJ}\bar{\sigma}^\mu D_\mu d_{RJ} \\
&\quad + iL^\dagger_J \sigma^\mu D_\mu L_J + ie^\dagger_{RJ}\bar{\sigma}^\mu D_\mu e_{RJ} + i\nu^\dagger_{RJ}\bar{\sigma}^\mu D_\mu \nu_{RJ},
\end{aligned}$$

where we have used the index J to label the generation of the fermion:

$$\begin{aligned}
u_{RJ} &= \{u_R, c_R, t_R\}, \quad d_{RJ} = \{d_R, s_R, b_R\}, \\
e_{RJ} &= \{e_R, \mu_R, \tau_R\}, \quad \nu_{RJ} = \{\nu_{eR}, \nu_{\mu R}, \nu_{\tau R}\}.
\end{aligned} \tag{23}$$

The left-handed SU(2) gauge symmetry forbids any fermion mass term incompatible with reality. However, fermion can obtain fermion mass by introducing Yukawa coupling between the Higgs and fermion fields.

A. Quark Sector

The Yukawa coupling between quark and Higgs is

$$\mathcal{L}_{q,\mathrm{Yuk}} = -\varepsilon^{ij} Y^u_{IJ} Q^\dagger_{iI}\varphi^\dagger_j u_{RJ} - Y^d_{IJ} Q^\dagger_I \varphi d_{RJ} + h.c.. \tag{24}$$

The first term corresponds to

$$\left(\bar{3},\bar{2},-\frac{1}{6}\right)\times\left(1,\bar{2},-\frac{1}{2}\right)\times\left(3,1,\frac{2}{3}\right)=(1,1,0)\oplus\cdots,$$

with ε^{ij} being the Clebsch-Gordan coefficient, and the second term corresponds to

$$\left(\bar{3},\bar{2},-\frac{1}{6}\right)\times\left(1,2,\frac{1}{2}\right)\times\left(3,1,-\frac{1}{3}\right)=(1,1,0)\,.$$

Now substitute φ with

$$\varphi=\frac{1}{\sqrt{2}}\begin{bmatrix}0\\ v+H\end{bmatrix}. \tag{25}$$

The Yukawa potential then gives quark mass term:

$$\mathcal{L}_{Q,\mathrm{Yuk}}=-\frac{v+H}{\sqrt{2}}\left(Y^u_{IJ}u^\dagger_{LI}u_{RJ}+Y^d_{IJ}d^\dagger_{LI}d_{RJ}\right)+h.c..$$

We can then use a basis transformation (singular value decomposition) to diagonalize the coupling matrix:

$$Y^u_{IJ}=U_uM_uK_u^\dagger,\quad Y^d_{IJ}=U_dM_dK_d^\dagger, \tag{26}$$

where M_u and M_d are diagonal matrices. We can change the basis

$$\begin{aligned}u_L\to U_uu_L,\quad u_R\to K_uu_R,\\ d_L\to U_dd_L,\quad d_R\to K_dd_R,\end{aligned}$$

and define the diagonal masses:

$$m_{u_I}=\frac{v}{\sqrt{2}}(M_u)_{II},\quad m_{d_I}=\frac{v}{\sqrt{2}}(M_d)_{II},$$

then the Lagrangian in the quark sector can be expressed as:

$$\begin{aligned}\mathcal{L}_Q=&\sum_I\bar{u}_I\left(i\not{D}^c-m_{u_I}\right)u_I+\sum_I\bar{d}_I\left(i\not{D}^c-m_{d_I}\right)d_I\\ &-\frac{H}{v}\sum_I\left(m_{u_I}\bar{u}_Iu_I+m_{d_I}\bar{d}_Id_I\right)+\mathcal{L}_{Q-G},\end{aligned}$$

where $\not{D}^c$ is the covariant derivative in QCD (neglecting the electroweak gauge), and $\mathcal{L}_{Q-G}$ denotes the coupling between quarks and the gauge fields. Using the expression of the Lie algebra

$$\begin{aligned}g_2\sum_{a=1}^2W^a_\mu T^a_2&=\frac{e}{\sqrt{2}\sin\theta_W}\begin{bmatrix}0&W^+_\mu\\ W^-_\mu&0\end{bmatrix},\\ g_2W^3_\mu T^z_2+g_1B_\mu Y&=eA_\mu q+\frac{e}{\sin\theta_W\cos\theta_W}Z_\mu\left(T^3_2-\sin^2\theta_Wq\right),\end{aligned} \tag{27}$$

The gauge-current coupling is:

$$\mathcal{L}_{Q-G}=\frac{e}{\sqrt{2}\sin\theta_W}\left(W^+_\mu J_Q^{+\mu}+W^-_\mu J_Q^{-\mu}\right)+eA_\mu J^\mu_{\mathrm{EM},Q}+\frac{e}{\sin\theta_W\cos\theta_W}Z_\mu J^\mu_{z,Q}, \tag{28}$$

where the currents are defined as

$$\begin{aligned}J_Q^{+\mu}&=d^\dagger_{LI}\bar{\sigma}^\mu u_{LI},\quad J_Q^{-\mu}=u^\dagger_{LI}\bar{\sigma}^\mu d_{LI},\\ J^\mu_{\mathrm{EM},Q}&=\frac{2}{3}u^\dagger_{LI}\bar{\sigma}^\mu u_{LI}-\frac{1}{3}d^\dagger_{LI}\bar{\sigma}^\mu d_{LI},\\ J^\mu_{z,Q}&=\frac{1}{2}u^\dagger_{LI}\bar{\sigma}^\mu u_{LI}-\frac{1}{2}d^\dagger_{LI}\bar{\sigma}^\mu d_{LI}J^\mu_3-\sin^2\theta_WJ^\mu_{\mathrm{EM},Q}.\end{aligned}$$

We note that in the definition of the currents, the quarks are the eigenstates of the weak interaction, not the mass eigenstates. If we define the quarks as particles with definite mass, the above definition shall be rotation by a unitary matrix.

B. Lepton Sector

The coupling between Higgs and e_R has the general Yukawa form:

$$\mathcal{L}_{e,\mathrm{Yuk}}=-Y^e_{IJ}L^\dagger_{iI}\varphi_je_{RiJ}+h.c., \tag{29}$$

where y_{IJ} is the Hermitian coupling matrix on generation space. The Yukawa coupling term corresponds to

$$\left(\bar{2},\frac{1}{2}\right)\times\left(2,\frac{1}{2}\right)\times(1,-1)=(1,0)\,,$$

which is indeed a singlet under gauge transformation (it is also a singlet under Lorentz transformation). The Higgs field then produces the term

$$\mathcal{L}_{e,\mathrm{Yuk}}=-\frac{Y^e_{IJ}}{\sqrt{2}}(v+H)e^\dagger_{LI}e_{RJ}+h.c., \tag{30}$$

We then diagonalize the coupling matrix:

$$Y^e_{IJ} = \left[U_e M_e K_e^\dagger\right]_{IJ}, \quad (31)$$

where M_e is diagonal. We can then change the basis by

$$e_R \to K_e e_R, \quad e_L \to U_e e_L, \quad (32)$$

Then, the coupling gives the mass:

$$\mathcal{L}_{e,\text{Yuk}} = -\sum_I m_{e_I} \bar{e}_I e_I - \frac{H}{v}\sum_I m_{e_I}\bar{e}_I e_I, \quad (33)$$

where

$$m_{e_I} = \frac{v}{\sqrt{2}}(M_e)_{II}. \quad (34)$$

Now, we consider the neutrino mass much smaller than electrons. The neutrinos can similarly obtain mass by coupling to the Higgs field. However, as the neutrinos are charge neutral, the mass term can come from the Yukawa coupling as well as the Majorana pairing:

$$\begin{aligned}\mathcal{L}_{\nu,\text{Yuk}} &= -Y^\nu_{IJ}\varepsilon^{ij} L_{iI}\varphi_j \nu_{RJ} - \frac{1}{2}M_{IJ}\nu_{RI}\cdot\nu_{RJ} + h.c.\\ &= -\frac{Y^\nu_{IJ}}{\sqrt{2}}(v+H)\nu^\dagger_{LI}\nu_{RJ} - \frac{1}{2}M_{IJ}\nu_{RI}\cdot\nu_{RJ} + h.c.\end{aligned}$$

The first term corresponds to

$$\left(2,-\frac{1}{2}\right)\times\left(2,\frac{1}{2}\right)\times(1,0) = (1,0)\oplus(3,0),$$

and ε^{ij} is the Clebsch-Gordan coefficients. The second term is automatically a gauge singlet. We can make a basis change to make Y diagonal and define a new set of Dirac spinors from single Weyl spinors:

$$\psi_1 = \begin{pmatrix}\nu_L\\ i\sigma^2\nu_L^*\end{pmatrix}, \quad \psi_2 = \begin{pmatrix}-i\sigma^2\nu_R^*\\ \nu_R\end{pmatrix}, \quad (35)$$

and the effective mass term can be written as:

$$\mathcal{L}_{\nu,\text{mass}} = -\sum_I \frac{Y_{II}v}{\sqrt{2}}\bar{\psi}_{1I}\psi_{2I} - \sum_{IJ}\frac{1}{2}M_{IJ}\bar{\psi}_{2I}\psi_{2J}. \quad (36)$$

Now, the Dirac spinor ψ_1 and ψ_2 can mix, and a suitable mixing renders the mass term diagonal. If $M \gg m$, the eigenvalues of the mass term will be huge masses plus some tiny masses. In general, the low-energy part of the mass term is

$$\mathcal{L}_{\nu,\text{mass}} = -\sum_I m_{\nu_I}\bar{\nu}_I\nu_I\left(1+\frac{H}{v}\right)^2, \quad (37)$$

With the discussion above, we can write down the Lagrangian in the lepton sector:

$$\begin{aligned}\mathcal{L}_L = &\sum_I \bar{e}_I(i\partial\!\!\!/ - m_{e_I})e_I + \sum_I \nu^\dagger_{LI}(\bar{\sigma}^\mu\partial_\mu - m_{\nu_I})\nu_{LI}\\ &- \frac{H}{v}\sum_I(m_{e_I}\bar{e}_I e_I + m_{\nu_I}\bar{\nu}_{LI}\nu_{LI}) + \mathcal{L}_{L-G},\end{aligned}$$

where $\mathcal{L}_{L-G}$ is the lepton-gauge coupling term:

$$\mathcal{L}_{L-G} = \frac{e}{\sqrt{2}\sin\theta_W}\left(W^+_\mu J_L^{+\mu} + W^-_\mu J_L^{-\mu}\right) + eA_\mu J^\mu_{\text{EM},L} + \frac{e}{\sin\theta_W\cos\theta_W}Z_\mu J^\mu_{z,L}, \quad (38)$$

where the current is defined as

$$\begin{aligned}&J_L^{+\mu} = e^\dagger_{LI}\bar{\sigma}^\mu\nu_{LJ}, \quad J_L^{-\mu} = \nu^\dagger_{LI}\bar{\sigma}^\mu e_{LJ}, \quad J^\mu_{\text{EM},L} = -e^\dagger_{LI}\bar{\sigma}^\mu e_{LI},\\ &J^\mu_{z,L} = \frac{1}{2}\nu^\dagger_{LI}\bar{\sigma}^\mu\nu_{LI} - \frac{1}{2}e^\dagger_{LI}\bar{\sigma}^\mu e_{LI}J^\mu_3 - \sin^2\theta_W J^\mu_{\text{EM},L}.\end{aligned}$$

Although right-handed neutrinos are introduced, we can only directly observe the left-handed neutrinos. For the same reason, the weak interaction's eigenstate works more naturally than the mass eigenstate for neutrinos. We define the neutrinos as what we directly measured – the left-handed "particle" with no definite mass.

C. Particle Mixing

1. Quark Mixing

In the above discussion, we mean two different things when talking about "particles". The original definition of the particle is an excitation with definite energy. But in the weak interaction process, say

$$u \to d + W^+, \quad (39)$$

the strange and charm quark is the eigenstate of the weak interaction. The weak interaction can change a state called u to a state called d, which does not necessarily have fixed energies. If we stick to the original definition of the particles, it turns out that the isospin doublet Q_I we have just defined is different. It really should be written as

$$Q_I = \left[\begin{pmatrix} u'_L \\ d'_L \end{pmatrix}, \begin{pmatrix} c'_L \\ s'_L \end{pmatrix}, \begin{pmatrix} t'_L \\ b'_L \end{pmatrix} \right], \tag{40}$$

where the primed quarks are mixtures of particle quarks from three generations. Without the loss of generality, we can identify $u' = u$, $c' = c$, $t' = t$, the mixing of quarks happens for the remaining three flavors:

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{bmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{bmatrix} \begin{pmatrix} d \\ s \\ b. \end{pmatrix} \tag{41}$$

The matrix V, being unitary, is called the *Cabbibo-Kobayashi-Maskawa* (CKM) matrix. From the calculation above, we know

$$V \equiv U_u^\dagger U_d. \tag{42}$$

The quark mixing changes the original definition of $J_Q^\pm$ to:

$$\begin{aligned} J_Q^{+\mu} &= {d'}_{LI}^\dagger \bar{\sigma}^\mu u_{LI} = V_{IJ}^\dagger d_{LI}^\dagger \bar{\sigma}^\mu u_{LJ}, \\ J_Q^{-\mu} &= u_{LI}^\dagger \bar{\sigma}^\mu d'_{LI} = V_{IJ} u_{LI}^\dagger \bar{\sigma}^\mu d_{LJ}. \end{aligned} \tag{43}$$

Note that the diagonal current J_{EM} or J_z will not change under the basis transformation.

The experimental value of the CKM matrix is approximately

$$\begin{bmatrix} |V_{ud}| & |V_{us}| & |V_{ub}| \\ |V_{cd}| & |V_{cs}| & |V_{cb}| \\ |V_{td}| & |V_{ts}| & |V_{tb}| \end{bmatrix} = \begin{bmatrix} 0.97 & 0.22 & 0.004 \\ 0.22 & 0.97 & 0.04 \\ 0.009 & 0.04 & 0.999 \end{bmatrix}. \tag{44}$$

We see the CKM matrix is closed to an identity matrix, and the most mixing happens between d and s quarks, i.e.,

$$\begin{aligned} d' &= \cos\theta_c \; d + \sin\theta_c \; s, \\ s' &= -\sin\theta_c \; d + \cos\theta_c \; s. \end{aligned} \tag{45}$$

Here, the angle $\theta_c \approx 13^\circ$ is called the Cabbibo angle. It corrects the quark decay process to:

$$\begin{aligned} \left[u \to d' + W^+\right] = \; & \cos\theta_c \left[u \to d + W^+\right] \\ & + \sin\theta_c \left[u \to s + W^+\right]. \end{aligned} \tag{46}$$

2. Neutrino Oscillation

Quite similarly, particles in the lepton sector also mix. We identify $e'_L = e_L$ (similarly for muon and tauon). Neutrinos that have specific mass are called ν_1, ν_2, and ν_3 instead, which satisfies

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{bmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{bmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}, \tag{47}$$

the 3×3 matrix is known as the *Pontecorvo-Maki-Nakagawa-Sakata* (PMNS) matrix, or simply the neutrino mixing matrix. Its value is roughly

$$\begin{bmatrix} |U_{e1}| & |U_{e2}| & |U_{e3}| \\ |U_{\mu 1}| & |U_{\mu 2}| & |U_{\mu 3}| \\ |U_{\tau 1}| & |U_{\tau 2}| & |U_{\tau 3}| \end{bmatrix} \approx \begin{bmatrix} 0.8 & 0.5 & 0.1 \\ 0.3 & 0.5 & 0.7 \\ 0.4 & 0.6 & 0.6 \end{bmatrix}. \tag{48}$$

The neutrino mixing means that in the weak process, say $e \to \nu_e + W^-$, the produced neutrino ν_e is not an energy eigenstate of the SM Hamiltonian – it will oscillate. As for the definition of the lepton current $J_L^\pm$. If we stick to the definition of neutrino as the eigenstate of the weak interaction – since that is what we get from the particle decay process – we do not need to change the definition. Still, we should remember that such "neutrinos" do not have fixed mass.

D. Restore 4-Fermion Theory

Finally, we can sum over the current and write down the total fermion-gauge coupling:

$$\begin{aligned} \mathcal{L}_{F-G} = \; & \frac{e}{\sqrt{2}\sin\theta_W}\left(W_\mu^+ J^{+\mu} + W_\mu^- J^{-\mu}\right) \\ & + eA_\mu J_{\text{EM}}^\mu + \frac{e}{\sin\theta_W \cos\theta_W} Z_\mu J_z^\mu. \end{aligned} \tag{49}$$

From the current-gauge coupling, we can recover the effective 4-fermion theory by integrating out the gauge field. In the low energy regime ($p^2 \ll m_W^2$), the vertex function for electron decay is:

$$2\left(\frac{e}{\sqrt{2}\sin\theta_W}\right)^2 J_\mu^+ \frac{g^{\mu\nu}}{p^2 - m_W^2} J_\nu^- \simeq \frac{e^2}{m_W^2 \sin^2\theta_W} J^{+\mu} J_\mu^-.$$

The diagonal part of the vertex is

$$\begin{aligned} & \left(\frac{e}{\sin\theta_W \cos\theta_W}\right)^2 J_\mu^z \frac{g^{\mu\nu}}{p^2 - m_Z^2} J_\nu^z \\ \simeq & \; - \frac{e^2}{m_Z^2 \sin^2\theta_W \cos^2\theta_W} J^{z\mu} J_\mu^z. \end{aligned}$$

Note that $m_Z^2 \cos^2\theta_W = m_W^2$, we can then identify the fermion constant

$$\frac{4G_F}{\sqrt{2}} = \frac{e^2}{m_W^2 \sin^2\theta_W}, \tag{50}$$

and the 4-Fermion vertex is

$$\mathcal{L}_{4F} = -\frac{4G_F}{\sqrt{2}}\left(J^{+\mu} J_\mu^- + J^{z\mu} J_\mu^z\right). \tag{51}$$