Nonrelativistic Electrons

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Contents

I. HOMOGENEOUS ELECTRON GAS

In this section, we consider the finite temperature electron system. A general non-relativistic field theory is described by the action (with repeated indices automatically summed):

$$S = S_0 + S_{\text{int}} = \int dt \int d^d x \; \mathcal{L}_0 - \int dt \; \mathcal{V}_{\text{int}}[\bar{\psi}, \psi], \tag{1}$$

where the free theory is described by a quadratic Lagrangian $\mathcal{L}_0 = \bar{\psi}(i\partial_t - \hat{H})\psi$. The classical equation of motion for the free field satisfies the Schrödinger equation:

$$\partial_\mu \frac{\partial \mathcal{L}_0}{\partial(\partial_\mu \bar{\psi}_a)} - \frac{\partial \mathcal{L}_0}{\partial \bar{\psi}_a} = -i\partial_t \psi_a + \hat{H}_{ab}\psi_b = 0. \tag{2}$$

For the homogeneous electron gas, we will mostly work in the momentum and frequency domain. To obtain a similar form as the relativistic field theory, we use the similar definition of the Fourier transformation:

$$\begin{aligned} \psi_a(k, \omega) &= \int dt \int_{L^d} d^d x \; e^{-ik\cdot x + i\omega t} \psi_a(x, t), \\ \psi_a(x, t) &= \frac{1}{L^d} \sum_k e^{ik\cdot x} \int \frac{d\omega}{2\pi} e^{-i\omega t} \psi_a(k, \omega) \sim \int \frac{d\omega}{2\pi} \int_{|k|<\Lambda} \frac{d^d k}{(2\pi)^d} e^{ik\cdot x - i\omega t} \psi_a(k, \omega). \end{aligned} \tag{3}$$

Note that in condensed matter system, we usually use the lattice regularization: we think of the system as a finite d-dimensional cubic with length $L = Na$. The lattice spacing a impose a natural UV cutoff $\Lambda = \pi/a$. In the thermodynamic limit, the summation becomes the integral:

$$\frac{1}{L^d} \sum_k \longrightarrow \int_{|k|<\Lambda} \frac{d^d k}{(2\pi)^d}. \tag{4}$$

In the momentum space, the free theory can be simplified:

$$S_0[\bar{\psi}, \psi] = \int dt \int \frac{d^d k}{(2\pi)^d} \bar{\psi}(k)[i\partial_t - \hat{H}(k)]\psi_b(k) = \int \frac{d\omega}{2\pi} \int \frac{d^d k}{(2\pi)^d} \bar{\psi}(k, \omega)[\omega - \hat{H}(k)]\psi(k, \omega), \tag{5}$$

which gives the real-time Green's function

$$iG_0(k, \omega) = \frac{1}{\omega - \hat{H}(k)} \tag{6}$$

A. Finite Temperature Formalism

The original real-time partition function is defined as[1]

$$Z[J] = \int D[\bar{\psi}, \psi] \exp\left\{ i \int dt \int d^d x \left[\mathcal{L} + \bar{J}_a(x)\psi_a(x) + \bar{\psi}_a(x) J_a(x) \right] \right\}. \tag{7}$$

If we make a analytic continuation of t to the complex plane: $t \to -i\tau,\ \omega \to i\omega$, the free action transforms as:

$$iS_0 = i \int dt dx\ \psi(\boldsymbol{x}, t)(i\partial_t - E)\psi(\boldsymbol{x}, t) \ \longrightarrow\ -\int d\tau dx\ \psi(\boldsymbol{x}, \tau)(\partial_\tau + E)\psi(\boldsymbol{x}, \tau). \tag{8}$$

Note that in the frequency domain, the singularities for positive frequency lies below the complex plane, as we always include an infinitesimal $-i\epsilon$ to the energy (mass) term of the theory in ensure convergence. So, the rotation of the real axis anti-clock-wisely to the imaginary axis will not cross any singularity, and thus the can be analytically extended. The partition function can then be written as $Z[J] = \int D[\bar{\psi}, \psi] e^{-S_0[\bar{\psi}, \psi] + \bar{J}\cdot\psi + \bar{\psi}\cdot J}$, where the Euclidean free action defined as:

$$S = \int d\tau \left[\int d^d x\ \bar{\psi}_a(\boldsymbol{x}, \tau)(\delta_{ab}\partial_\tau + \hat{H}_{ab})\psi_b(\boldsymbol{x}, \tau) + \mathcal{V}_{\text{int}} \right]. \tag{9}$$

The Euclidean action is suitable to describe the system both in zero temperature or finite temperature. For finite temperature case, the integral over the imaginary time τ is over $[0, \beta)$. The Fourier transformation of the field on the imaginary time domain is defined as:[2]

$$\psi(\omega_n) = \int_0^\beta d\tau e^{i\omega_n \tau}\psi(\tau), \quad \psi(\tau) = \frac{1}{\beta}\sum_{\omega_n} e^{-i\omega_n \tau}\psi(\omega_n). \tag{10}$$

Under such convention, in the thermodynamic limit and zero-temperature limit, the spatial-temporal Fourier transformation agrees with the relativistic case (up to a Wick rotation). The Fourier transformation of the free field action is

$$S_0 = \frac{1}{\beta}\sum_{\omega_n} \int_\Lambda \frac{d^d k}{(2\pi)^d} \bar{\psi}_a(k, \omega_n) \left[-i\omega_n + \hat{H}_{ab}(k) \right] \psi_b(k, \omega_n). \tag{11}$$

The partition function with source is

$$\frac{Z_0[J]}{Z_0[0]} = \exp\left[-\frac{1}{\beta}\sum_{\omega_n} \int_\Lambda \frac{d^d k}{(2\pi)^d} \bar{J}_a(k, \omega_n) G_{ab}(k, \omega_n) J_b(k, \omega_n) \right], \tag{12}$$

where the Green's function is

$$G_{ab}(k, \omega_n) = \frac{1}{i\omega_n - \hat{H}(k)}. \tag{13}$$

Unlike the relativistic case, the value of the value of partition function without source $Z_0[0]$ is related to the free energy. We can express it formally as $Z_0[0] = \det(-G_{ab})$. To get the correct dimensionality, we set the determinant as

$$Z_0[0] \equiv \prod_{k, \omega_n} \left\{ \beta \det\left[-i\omega_n + \tilde{H}(k) \right] \right\}. \tag{14}$$

The summation on Matsubara frequencies is capture by the singularities of the density function of the states:

$$\rho(z) = \frac{1}{e^{\beta z} + 1}. \tag{15}$$

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[1] As with the relativistic case, we introduce an auxiliary source J, which is bosonic/fermionic if the field ψ is bosonic/fermionic.

[2] Note that the fermion field satisfies the anti-periodic boundary condition on the interval $[0, \beta)$: it change sign when crossing the boundary. The Matsubara frequencies for fermions are $\omega_n = (2n + 1)\pi/\beta$ for $n \geq 0$.

The residue on imaginary frequency $i\omega_n$ is alway $\frac{1}{\beta}$. In this way, the summation is:

$$\frac{1}{\beta}\sum_{\omega_n} f(i\omega_n) = \frac{1}{2\pi i}\oint \rho(z) f(z). \tag{16}$$

The right-hand side can usually be evaluated using the residue theorem.

B. RPA Correction to Photon Propagator

Under the random phase approximation, the photon

$$\Pi(p,\omega) \tag{17}$$