Lindblad Equation

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1 Lindblad Master Equation

1.1 General Markovian Form

For general open quantum evolution, suppose the system and environment are separable initially: $\rho_T = \rho \otimes \rho_B$, where we assume $\rho_B = \sum_\alpha \lambda_\alpha |\phi_\alpha\rangle\langle\phi_\alpha|$. Then the evolution of system-bath is unitary: $\rho_T(t) = U(t)\rho_T U^\dagger(t)$. Trace out the environment's degrees of freedom, we have the quantum channel expression: $\rho(t) = \sum_{\alpha\beta} W_{\alpha\beta}\rho W^\dagger_{\alpha\beta}$, where the **Kraus operator** W can be expressed formally as $W_{\alpha\beta} = \sqrt{\lambda_\beta}\langle\phi_\alpha|U(t)|\phi_\beta\rangle$.

The Lindblad equation assumes a semi-group relation:

$$\rho_t = \mathcal{L}_t \rho_0 = \lim_{N\to\infty} \mathcal{L}_{t/N} \cdot \mathcal{L}_{t/N} \cdots \mathcal{L}_{t/N}\rho_0. \tag{1}$$

Such time decimation implies that the evolution is Markovian. We will show that Markovian approximation leads directly to the Lindblad equation. First, we choose a complete operator basis $\{F_i\}$ in N-dimensional Hilbert space, satisfying $\mathrm{Tr}[F_i^\dagger F_j] = \delta_{ij}$, where we choose $F_0 = N^{-1/2} \cdot \mathbb{I}$. For a quantum channel, the channel operator K_μ can be expanded as

$$K_\mu = \sum_i \mathrm{Tr}[F_i^\dagger K_\mu] F_i. \tag{2}$$

In general, we have: $\mathcal{L}_t[\rho] = \sum_{ij} c_{ij}(t) F_i \rho F_j^\dagger$, where the Hermitian coefficient $c_{ij}(t)$ is

$$c_{ij}(t) = \sum_\mu \mathrm{Tr}[F_i^\dagger K_\mu] \cdot \mathrm{Tr}[F_j^\dagger K_\mu]^*. \tag{3}$$

Our target is to compute the limit $\partial_t \rho \equiv \lim_{t\to 0} \frac{1}{t}(\mathcal{L}_t[\rho] - \rho)$. For this purpose, we define the (Hermitian) coefficient a_{ij} as:

$$a_{00} = \lim_{t\to 0} \frac{c_{00}(t) - N}{t}, \quad a_{ij} = \lim_{t\to 0} \frac{c_{ij}(t)}{t}.$$

The limit is then

$$\frac{d}{dt}\rho = \frac{a_{00}}{N}\rho + \frac{1}{\sqrt{N}}\sum_{i>0}\left(a_{i0}F_i\rho + a_{i0}^*\rho F_i^\dagger\right) + \sum_{i,j>0} a_{ij}F_i\rho F_j^\dagger.$$

To further simplify the expression, we define

$$F = \frac{1}{\sqrt{N}}\sum_{i=1}^{N^2-1} a_{i0}F_i, \quad G = \frac{a_{00}}{2N}\mathbb{I} + \frac{1}{2}(F^\dagger + F), \quad H = \frac{1}{2i}(F^\dagger - F).$$

The limit can be expressed by G, H in a compact form:

$$\frac{d\rho}{dt} = -i[H,\rho] + \{G,\rho\} + \sum_{i,j=1}^{N^2-1} a_{ij}F_i\rho F_j^\dagger.$$

Note the $[H,\rho]$ part is the traceless part, and the $\{G,\rho\}$ is the trace part. Since the quantum channel preserves the trace (for any ρ):

$$\mathrm{Tr}\left[\frac{d\rho}{dt}\right] = \mathrm{Tr}\left[\left(2G + \sum_{i,j=1}^{N^2-1} a_{ij}F_j^\dagger F_i\right)\rho\right] = 0.$$

Therefore $G = -\frac{1}{2}\sum_{i,j=1}^{N^2-1} a_{ij}F_j^\dagger F_i$. We thus obtain the Lindblad form:

$$\frac{d\rho}{dt} = -i[H,\rho] + \sum_{i,j=1}^{N^2-1} a_{ij}\left(F_i\rho F_j^\dagger - \frac{1}{2}\{F_j^\dagger F_i, \rho\}\right). \tag{4}$$

We can simplify the form by diagonalizing the matrix a_{ij}. It is a convention to take the norm of a_{ij} out to indicate the strength of the system-bath coupling, and the diagonalized Lindblad equation is

$$\frac{d\rho}{dt} = -i[H,\rho] + \gamma\sum_m\left(L_m\rho L_m^\dagger - \frac{1}{2}\{L_m^\dagger L_m, \rho\}\right). \tag{5}$$

1.2 First Principal Deduction

In this section, we consider a general system-bath coupling:[1]

$$H_T = H + H_B + V, \quad V = \sum_k A_k \otimes B_k. \tag{6}$$

Under certain conditions, we will show that the dynamics of the system are well approximated by the Lindblad equation. We first assume that initially, the total system is a product state

$$\rho_T(0) = \rho(0)\otimes\rho_B.$$

In the following, we will adopt the interacting picture, where the density operator evolves as

$$\partial_t\rho_T(t) = -i[V(t),\rho_T(t)] \equiv -i\mathcal{V}(t)|\rho_T(t)\rangle.$$

[1]Without loss of generality, we can also assume $\|A_k\| = 1$, $\mathrm{Tr}[\rho_B B_k] = 0$.

In the last equality, ρ_T is expressed as a ket in the Hilbert space of linear operator, and the commutator with V is expressed as a superoperator $\mathcal{V}$. This notation can simplify the expression. For example, the inner product in the operator space is the trace so that the partial trace operation can be denoted as $|\rho\rangle = \langle \mathbb{I}_B|\rho_T\rangle$. The evolution of the system is then

$$\begin{aligned}\partial_t|\rho(t)\rangle = -i\langle\mathbb{I}_B|\mathcal{V}(t)|\rho_T(t)\rangle = -i\langle\mathbb{I}_B|\mathcal{V}(t)|\rho_T(0)\rangle - \int_0^t \langle\mathbb{I}_B|\mathcal{V}(t)\mathcal{V}(\tau)|\rho_T(\tau)\rangle d\tau \\ = -\int_0^t \langle\mathbb{I}_B|\mathcal{V}(t)\mathcal{V}(\tau)|\rho_T(\tau)\rangle d\tau.\end{aligned} \tag{7}$$

Now we are taking the **Born approximation**, which states when the coupling is weak enough compared with the energy scale of the system and the bath, the total density matrix is approximated by the product state $|\rho_T(t)\rangle \approx |\rho(t)\rangle \otimes |\rho_B\rangle$. The evolution is now

$$\begin{aligned}\frac{d}{dt}\rho(t) &\approx \int_0^t \mathrm{Tr}_B[V(t)\rho_T(\tau)V(\tau) - \rho_T(\tau)V(\tau)V(t)]d\tau + h.c. \\ &= \sum_{kl}\int_0^t d\tau\ [C_{lk}(\tau-t)A_k(t)\rho(\tau)A_l(\tau) - C_{lk}(\tau-t)\rho(\tau)A_l(\tau)A_k(t) + h.c.],\end{aligned} \tag{8}$$

where $C_{kl}(t) \equiv \mathrm{Tr}_B[\rho_B B_k(t)B_l]$ is the correlation function of B_k's. We then take the **Markovian approximation**, which assumes that the correlations of the bath decay fast in time. We can thus make the substitution $\rho(\tau) \to \rho(t)$, the result equation of motion is Markovian:

$$\begin{aligned}\frac{d}{dt}\rho(t) &\approx \sum_{kl}\int_0^t dt'\big[C_{lk}(-t')A_k(t)\rho(t)A_l(t-t') - C_{lk}(-t')\rho(t)A_l(t-t')A_k(t) + h.c.\big] \\ &= \sum_k \int_0^t dt\,[A_k\rho B_k - \rho B_k A_k + h.c.],\end{aligned} \tag{9}$$

where we have defined $B_k(t) = \sum_l \int_0^\infty dt' A_l(t-t')C_{lk}(-t')$. Now, we switch to the frequency domain,

$$A_k(t) = \sum_\omega A_k(\omega)e^{-i\omega t}, \quad B_k(t) = \sum_{l,\omega} e^{-i\omega t}A_l(\omega)\Gamma_{lk}(\omega), \quad \Gamma_{kl}(\omega) = \int_0^\infty dt\ e^{i\omega t}C_{kl}(t).$$

We then take the **rotating wave approximation**, where we only keep the contributions from canceling frequency of operator A and B,

$$\begin{aligned}\frac{d}{dt}\rho(t) &= \sum_\omega [\Gamma_{lk}(\omega)A_k(\omega)\rho A_l(\omega) - \Gamma_{lk}(\omega)\rho A_l(\omega)A_k(\omega) + h.c.] \\ &= \sum_\omega \gamma_{kl}(\omega)(A_{l,\omega}\rho A^\dagger_{k,\omega} - \frac{1}{2}\{\rho, A^\dagger_{k,\omega}A_{l,\omega}\}) - i\left[\sum_\omega S_{kl}(\omega)A^\dagger_{k,\omega}A_{l,\omega}, \rho\right],\end{aligned} \tag{10}$$

where we defined

$$\gamma_{kl}(\omega) = \Gamma_{kl}(\omega) + \Gamma^*_{lk}(\omega), \quad S_{kl}(\omega) = \frac{1}{2i}[\Gamma_{kl}(\omega) - \Gamma^*_{lk}(\omega)]. \tag{11}$$

The matrices $\gamma(\omega)$ are positive; we can then take the square root of them. The jump operator is then

$$L_{i,\omega} = \sum_j \sqrt{\gamma_{ij}(\omega)}A_{j,\omega}.$$

The evolution is then in the Lindblad form.

2 Stochastic Schrödinger Equation

The Lindblad form Eq. (5) is equivalent to the stochastic Schrödinger equation (SSE):

$$d|\psi\rangle = -iH|\psi\rangle + A[\psi]dt + \sum_m B[\psi]dW_m, \tag{12}$$

where dW_m is a stochastic infinitesimal element. The expectation value is then the average over all possible evolution path (trajectory):

$$\langle O(t)\rangle = \overline{\langle\psi(t)|O|\psi(t)\rangle}.$$

For simplicity, in this section, we consider the jump operator L_x labeled by coordinate x. The SSE can be Trotterized as

$$\rho' = \left(\prod_x \mathcal{M}_x\right)\left[e^{-iH\Delta t}\rho e^{iH\Delta t}\right]. \tag{13}$$

2.1 Poisson SSE

Consider a small time interval Δt; the Lindblad equation is equivalent to the quantum channel

$$\mathcal{L}_{\Delta t}[\rho] = M_0\rho M_0^\dagger + M_x\rho M_x^\dagger, \tag{14}$$

where the jump operators are:

$$M_x = \sqrt{\gamma\Delta t}L_x, \quad M_0 = \sqrt{1-\gamma\Delta t L_x^\dagger L_x}. \tag{15}$$

A quantum channel can be simulated by a stochastic evolution of pure states:

$$|\psi(t+\Delta t)\rangle \propto \left(\prod_x \mathcal{M}_x\right)e^{-iH\Delta t}|\psi(t)\rangle \tag{16}$$

where each weak measurement is

$$\mathcal{M}_x|\psi\rangle \propto \begin{cases} M_x|\psi\rangle & p = \langle L_x^\dagger L_x\rangle\gamma\Delta t \\ M_0|\psi\rangle & p = 1-\langle L_x^\dagger L_x\rangle\gamma\Delta t \end{cases}, \tag{17}$$

We can introduce a Poisson variable dW_x satisfying

$$dW_x dW_y = \delta_{xy}dW_x, \quad \overline{dW_x} = \langle L_x^\dagger L_x\rangle\gamma dt, \tag{18}$$

and the evolution can be cast into the stochastic differential equation

$$d|\psi\rangle = -iHdt|\psi\rangle + \sum_x \frac{L_x - \sqrt{\langle L_x^\dagger L_x\rangle}}{\sqrt{\langle L_x^\dagger L_x\rangle}}dW_x|\psi\rangle - \frac{\gamma}{2}\sum_x\left(L_x^\dagger L_x - \langle L_x^\dagger L_x\rangle\right)dt|\psi\rangle. \tag{19}$$

The $-\langle L_x^\dagger L_x\rangle dt|\psi\rangle$ comes from the renormalization. For numerical simulation, we can ignore it.

Note that in the numerical simulation, after each quantum jump, the state should be renormalized so that the jump probability for other L_x can be computed correctly; this requires several renormalization procedures in a single time step.

2.2 Gaussian SSE

Gaussian SSE is another way to unravel the Lindblad evolution. It is often numerically more efficient since it only requires one renormalization in a single time step. We first introduce the Wiener processes dW_x satisfying

$$\overline{dW_x} = 0, \quad \overline{dW_x dW_y} = \delta_{xy}\gamma dt. \tag{20}$$

Here we assume L_x is Hermitian. The Gaussian SSE is then

$$d|\psi\rangle = -iHdt|\psi\rangle + \sum_x (L_x - \langle L_x\rangle)\, dW_x\, |\psi\rangle - \frac{\gamma}{2}\sum_x (L_x - \langle L_x\rangle)^2 dt\, |\psi\rangle\,. \tag{21}$$

To retain the Lindblad, note that

$$d\rho = \overline{|d\psi\rangle\langle\psi|} + \overline{|\psi\rangle\langle d\psi|} + \overline{|d\psi\rangle\langle d\psi|}.$$

The first two terms give

$$-i[H,\rho]dt - \frac{\gamma}{2}\sum_x \left\{L_x^2 - 2\langle L_x\rangle L_x + \langle L_x\rangle^2, \rho\right\} dt.$$

The second term gives

$$\gamma \sum_x \left[L_x \rho L_x - \left\{ \langle L_x\rangle L_x - \frac{\langle L_x\rangle^2}{2}, \rho \right\} \right] dt + O(dt^2).$$

We, therefore, recover the Lindblad equation after averaging the SSE.

In numerical simulation, we exponentiate the expression,

$$|\psi'\rangle \sim \exp\left(Adt + \sum_x B_x dW_x \right) e^{-iH\Delta t}|\psi\rangle. \tag{22}$$

The Taylor expansion of the exponent to the lowest order gives

$$e^{Adt + B\sum_x dW_x} = 1 + \left(A + \frac{1}{2}\sum_x B_x^2 \right) dt + \sum_x B_x dW_x$$

Compared with the SSE, we get

$$A = -\gamma \sum_x (L_x - \langle L_x\rangle)^2, \quad B_x = L_x - \langle L_x\rangle.$$

The simplest example is when $L_x - n_x$ is a (quasi-)particle number operator, $n_x^2 = n_x$, then we can simulate the SSE by the following form

$$|\psi'\rangle \propto \exp\left\{ \sum_x [dW_x + \gamma(2\langle n_x\rangle - 1)] n_x dt \right\} e^{-iH\Delta t}|\psi\rangle, \tag{23}$$

where we have ignored the normalization term. After each time step, there should be a normalization procedure.

3 Quadratic Lindbladian

Consider the Lindblad in the Heisenberg picture:

$$\frac{d}{dt}\hat{O} = i[\hat{H}, \hat{O}] + \sum_{\mu} \hat{L}_{\mu}^{\dagger}\hat{O}\hat{L}_{\mu} - \frac{1}{2}\sum_{\mu}\{\hat{L}_{\mu}^{\dagger}\hat{L}_{\mu}, \hat{O}\}, \tag{24}$$

where we choose $\hat{O}_{ij} = \omega_i\omega_j$ satisfying the relation $\hat{O}^T = 2\mathbb{I} - \hat{O}$. The covariance matrix is then $\Gamma_{ij} = i\langle\hat{O}\rangle - i\delta_{ij}$.

We assume that the jump operator has up to quadratic Majorana terms. In particular, we denote the linear terms and the Hermitian quadratic terms as

$$\hat{L}_r = \sum_{j=1}^{2N} L_j^r \omega_j, \quad \hat{L}_s = -\frac{i}{4}\sum_{j,k=1}^{2N} M_{jk}^s \omega_j\omega_k. \tag{25}$$

When the **jump operator** $\hat{L}_\mu$ contains only the linear Majorana operator, the Lindblad equation preserves Gaussianity. The evolution will break the Gaussian form for jump operators containing up to quadratic Majorana terms. However, the $2n$-point correlation is still solvable for free fermion systems.

3.1 Third Quantization

Assume only linear terms in jump operators,

$$\partial_t\hat{O} = \left[i\hat{H}, \hat{O}\right] + \mathcal{D}_r[\hat{O}] = \left[i\hat{H} - \frac{1}{2}\sum_r \hat{L}_r^{\dagger}L_r, \hat{O}\right] + \sum_r \left[\hat{L}_r^{\dagger}, \hat{O}\right]\hat{L}_r. \tag{26}$$

For the future convenience, we define

$$\sum_r L_i^r L_j^{r*} = B_{ij} = \mathrm{Re}\,B_{ij} + i\,\mathrm{Im}\,B_{ij}. \tag{27}$$

The first term of EOM is:

$$\left[i\hat{H} - \frac{1}{2}\sum_r \hat{L}_r^{\dagger}L_r, \hat{O}_{ij}\right] = \sum_{kl}\left(\frac{1}{4}H - \frac{1}{2}B\right)_{kl}[\omega_k\omega_l, \omega_i\omega_j].$$

Use the commutation relation $\{\omega_i, \omega_j\} = 2\delta_{ij}$, we have the relation

$$\begin{aligned} [\omega_k, \omega_i\omega_j] &= 2(\delta_{ki}\omega_j - \delta_{kj}\omega_i), \\ [\omega_k\omega_l, \omega_i\omega_j] &= 2(\delta_{ki}\omega_j\omega_l - \delta_{kj}\omega_i\omega_l + \delta_{li}\omega_k\omega_j - \delta_{lj}\omega_k\omega_i). \end{aligned} \tag{28}$$

Therefore

$$\begin{aligned} \left[i\hat{H} - \frac{1}{2}\sum_r \hat{L}_r^{\dagger}L_r, \hat{O}_{ij}\right] &= \left[\left(\frac{H}{2} - B\right)\cdot\hat{O}^T + \left(\frac{H}{2} - B\right)^T\cdot\hat{O} - \hat{O}\cdot\left(\frac{H}{2} - B\right)^T - \hat{O}^T\cdot\left(\frac{H}{2} - B\right)\right]_{ij} \\ &= \left[(-H + 2\,\mathrm{Im}\,B)\cdot\hat{O} + \hat{O}\cdot(H - 2\,\mathrm{Im}\,B)\right]_{ij}. \end{aligned}$$

The second term is

$$\begin{aligned} \sum_r \left[L_r^{\dagger}, \hat{O}_{ij}\right]\hat{L}_r &= \sum_{kl} B_{kl}[\omega_k, \omega_i\omega_j]\omega_l = 2\sum_{kl} B_{kl}\left(\delta_{ki}\omega_j\omega_l - \delta_{kj}\omega_i\omega_l\right) \\ &= \left[2B\cdot\hat{O}^T - 2\hat{O}\cdot B^T\right]_{ij} = \left[-2B\cdot\hat{O} - 2\hat{O}\cdot B^* + 4B\right]_{ij}. \end{aligned}$$

Therefore

$$\partial_t \hat{O}_{ij} = \left[(-H - 2\,\mathrm{Re}\,B)\cdot \hat{O} + \hat{O}\cdot (H - 2\,\mathrm{Re}\,B) + 4B\right]_{ij}. \tag{29}$$

The EOM of the covariance matrix is then

$$\partial_t \Gamma = X^T \cdot \Gamma + \Gamma \cdot X + Y, \quad X = H - 2\,\mathrm{Re}\,B, \quad Y = 4\,\mathrm{Im}\,B. \tag{30}$$

Note that the constant part is replaced by its anti-symmetric part.

The steady state of the system is solved by the Lyapunov equation

$$X^T \cdot \Gamma + \Gamma \cdot X = -Y. \tag{31}$$

Now include the Hermitian quadratic quantum jumps:

$$\begin{aligned} \partial_t \hat{O} &= i[\hat{H}, \hat{O}] + \mathcal{D}_r[\hat{O}] + \mathcal{D}_s[\hat{O}], \\ \mathcal{D}_s[\hat{O}] &= \sum_s \hat{L}_s \hat{O} \hat{L}_s - \frac{1}{2}\sum_r \{\hat{L}_s^2, \hat{O}\} = -\frac{1}{2}\sum_s [\hat{L}_s, [\hat{L}_s, \hat{O}]]. \end{aligned} \tag{32}$$

3.2 Quadratic Lindbladian

A direct calculation gives

$$\begin{aligned} D_s[\hat{O}] &= \frac{i}{8}\sum_s \sum_{kl} M^s_{kl} [\hat{L}_s, [\omega_k \omega_l, \omega_i \omega_j]] \\ &= -\frac{i}{2}\sum_s \sum_k \left\{ M^s_{ik} [\hat{L}_s, \omega_k \omega_j] - [\hat{L}_s, \omega_i \omega_k] M^s_{kj} \right\} \\ &= -\sum_{s,kl} \left[M^s_{ik}(-M^s_{kl}\omega_l \omega_j + \omega_k \omega_l M^s_{lj}) + (M^s_{il}\omega_l \omega_k - \omega_i \omega_l M^s_{lk}) M^s_{kj} \right] \\ &= \frac{1}{2}\sum_s \left[(M^s)^2 \cdot \hat{O} + \hat{O}\cdot (M^s)^2 - 2M^s \cdot \hat{O}\cdot M^s \right]_{ij}. \end{aligned} \tag{33}$$

Since M^s is imaginary anti-symmetric matrix, $(M^s)^2 = -(\mathrm{Im}\,M^s)^2$, and $M^s \cdot O \cdot M^s = -\mathrm{Im}\,M^s \cdot O \cdot \mathrm{Im}\,M^s$. Together, we get the EOM of the variance matrix Γ_{ij}:

$$\partial_t \Gamma = X^T \cdot \Gamma + \Gamma \cdot X - \sum_s M^s \cdot \Gamma \cdot M^s + Y, \tag{34}$$

where

$$X = H - 2\,\mathrm{Re}\,B + \frac{1}{2}\sum_s (M^s)^2, \quad Y = 4\,\mathrm{Im}\,B. \tag{35}$$

3.3 Dirac Fermion Case

This section considers the free fermion system preserving the U(1) charge. The jump operators are assumed to be quadratic: $\hat{L}_s = \sum_{jk} M^s_{jk} c^\dagger_j c_k$ where $\{M^s\}$ are Hermitian matrices.

For the fermion case, we choose $\hat{O}_{ij} = c^\dagger_i c_j$, and consider the Lindbladian

$$\partial_t \hat{O} = i[\hat{H}, \hat{O}] + \mathcal{D}_s[\hat{O}] = i[\hat{H}, \hat{O}] - \frac{1}{2}\sum_s [\hat{L}_s, [\hat{L}_s, \hat{O}]], \tag{36}$$

where each $\hat{L}_s = M^s_{ij} c^\dagger_i c_j$ is a Hermitian fermion bilinear.

The Hamiltonian part is:[2]

$$i\sum_{kl}H_{kl}[c_k^\dagger c_l, c_i^\dagger c_j] = i\sum_{kl}H_{kl}(\delta_{il}c_k^\dagger c_j - \delta_{jk}c_i^\dagger c_l) = i[H^T\cdot\hat{O} - \hat{O}\cdot H^T]_{ij}. \tag{37}$$

Similarly, the double commutation in the second term is:

$$\mathcal{D}_s[\hat{O}] = -\frac{1}{2}\sum_s[(M^{s*})^2\cdot\hat{O} + \hat{O}\cdot(M^{s*})^2 - 2M^{s*}\cdot\hat{O}\cdot M^{s*}]. \tag{38}$$

Together, the EOM of correlation $G_{ij} = \langle c_i^\dagger c_j\rangle$ is

$$\partial_t G = X^\dagger\cdot G + G\cdot X + \sum_s M^{s*}\cdot G\cdot M^{s*}, \tag{39}$$

where

$$X = -iH^* - \frac{1}{2}\sum_s(M^{s*})^2. \tag{40}$$

[2]Using the fact $[c_k^\dagger c_l, c_i^\dagger c_j] = c_k^\dagger[c_l, c_i^\dagger c_j] + [c_k^\dagger, c_i^\dagger c_j]c_l = \delta_{il}c_k^\dagger c_j - \delta_{jk}c_i^\dagger c_l$, we know that for a quadratic form $\hat{A} = \sum_{ij}A_{ij}c_i^\dagger c_j$, $[\hat{A}, \hat{O}_{ij}] = [A^T, \hat{O}]_{ij}$.