Gravity

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I. RIEMANN GEOMETRY

A. Connection

For a general coordinate system, we can choose a coordinate basis $\{e_\mu \equiv \partial_\mu\}$ and define the connection as $\nabla_\mu e_\nu = \Gamma^\lambda_{\mu\nu} e_\lambda$, where ∇_μ is the covariant derivative along the x^μ direction. We immediately know the covariant derivative for the vector field:

$$\nabla_\mu (W^\nu e_\nu) = \frac{\partial W^\nu}{\partial x^\mu} e_\nu + W^\nu e_\lambda \Gamma^\lambda_{\mu\nu} = \left(\frac{\partial W^\lambda}{\partial x^\mu} + \Gamma^\lambda_{\mu\nu} W^\nu \right) e_\lambda,$$

For the dual vector, consider the expression $\nabla_\mu (W^\nu V_\nu)$. Since $W^\nu V_\nu$ is a scalar, the covariant derivative is the same as the ordinary derivative, $\nabla_\mu (W^\nu V_\nu) = \partial_\mu (W^\nu V_\nu)$. On the other hand,

$$\nabla_\mu (W^\nu V_\nu) = (\partial_\mu W^\nu) V_\nu + W^\nu (\partial_\mu V_\nu) = \left(\frac{\partial W^\lambda}{\partial x^\mu} + \Gamma^\lambda_{\mu\nu} W^\nu \right) V_\lambda + W^\nu (\nabla_\mu V)_\nu,$$

which leads to $\nabla_\mu V_\nu = \partial_\mu V_\nu - V_\lambda \Gamma^\lambda{}_{\mu\nu}$. In general, the covariant derivative on a tensor T is

$$\nabla_\rho T^{\mu_1 \cdots \mu_p}_{\nu_1 \cdots \nu_q} = \partial_\rho T^{\mu_1 \cdots \mu_p}_{\nu_1 \cdots \nu_q} + (\Gamma^{\mu_1}_{\rho\sigma} T^{\sigma \mu_2 \cdots \mu_p}_{\nu_1 \ldots \nu_q} + \cdots + \Gamma^{\mu_p}_{\rho\sigma} T^{\mu_1 \cdots \mu_{p-1}\sigma}_{\nu_1 \cdots \nu_q}) - (\Gamma^\sigma_{\rho\nu_1} T^{\mu_1 \cdots \mu_p}_{\sigma \nu_2 \cdots \nu_q} + \cdots + \Gamma^\sigma_{\rho\nu_q} T^{\mu_1 \cdots \mu_p}_{\nu_1 \cdots \nu_{q-1}\sigma}). \tag{1}$$

In the space-time manifold with a metric $g_{\mu\nu}$, there exists a unique, torsion-free connection such that $\nabla_\rho g_{\mu\nu} = 0$. To see this, let us first write the $\nabla_\rho g_{\mu\nu} = 0$ condition in three equivalent ways:

$$\partial_\rho g_{\mu\nu} - \Gamma^\sigma_{\rho\mu} g_{\sigma\nu} - \Gamma^\sigma_{\rho\nu} g_{\mu\sigma} = 0, \tag{2}$$

$$\partial_\mu g_{\nu\rho} - \Gamma^\sigma_{\mu\nu} g_{\sigma\rho} - \Gamma^\sigma_{\mu\rho} g_{\nu\sigma} = 0, \tag{3}$$

$$\partial_\nu g_{\rho\mu} - \Gamma^\sigma_{\nu\rho} g_{\sigma\mu} - \Gamma^\sigma_{\nu\mu} g_{\rho\sigma} = 0. \tag{4}$$

The torsion is defined as $T^\sigma_{\mu\nu} = \Gamma^\sigma_{\mu\nu} - \Gamma^\sigma_{\nu\mu}$. The torsion-free condition helps reduced the above equations. We simply add Eq. (3) and Eq. (4) and subtract Eq. (2), then we have

$$2 g_{\rho\sigma} \Gamma^\sigma_{\mu\nu} = \partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu} \quad \Longrightarrow \quad \Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}). \tag{5}$$

The torsion-free connection is called the *Christoffel symbol*. Note that the Christoffel symbol satisfies

$$\Gamma^\mu_{\mu\nu} = \frac{1}{\sqrt{|g|}} \partial_\nu \sqrt{|g|}. \tag{6}$$

The proof is straightforward, since

$$\Gamma^{\mu}_{\mu\nu} = \frac{1}{2} g^{\rho\mu}\partial_\nu g_{\mu\rho} = \frac{1}{2}\,\mathrm{tr}[g^{-1}\partial_\nu g] = \frac{1}{2}\partial_\nu\,\mathrm{tr}[\log g] = \partial_\nu \log\sqrt{|g|} = \frac{1}{\sqrt{|g|}}\partial_\nu\sqrt{|g|},$$

where we have used the fact $\mathrm{tr}\log A = \log\det A$, and we can replace $\det g$ with $|\det g| = |g|$ since the additional phase, upon the action of logarithm and derivative, vanished.

The vielbeins There is a neat way to represent the connection. First, we introduce a set of local frame called *vielbeins* or *tetrads*:

$$\hat{e}_a = e^{\mu}_a\partial_\mu, \quad g_{\mu\nu}e^{\mu}_a e^{\nu}_b = \eta_{ab}.$$

The vielbeins convert a general metric to the Minkowski metric (locally). We can also raise and lower the indices by $e^a_\mu = \eta^{ab}e^{\mu}_b g_{\mu\nu}$. Now consider the one form $\theta^a \equiv e^a_\mu dx^\mu$, satisfying $\eta_{ab}\theta^a\theta^b = g_{\mu\nu}dx^\mu dx^\nu$. We define the matrix-valued connection one-form as

$$\omega^a{}_b = \Gamma^a_{bc}\theta^c, \tag{7}$$

where Γ^c_{ab} is defined by $\nabla_{\hat{e}_a}\hat{e}_b = \Gamma^c_{ab}\hat{e}_c$. There is a rather simple way to compute the connection one-forms, at least for a torsion-free connection. This follows from the first of two Cartan structure relations.

Claim: for torsion-free connection,

$$d\theta^a + \omega^a{}_b \wedge \theta^b = 0. \tag{8}$$

Proof: We first look at the second term $\omega^a_b \wedge \hat{\theta}^b = \Gamma^a_{bc}\left(e^c_\mu dx^\mu\right)\wedge\left(e^b_\nu dx^\nu\right)$. According to its definition, the components of Γ^a_{cb} are related to the coordinate basis components by

$$\Gamma^c_{ab} = e^c_\rho e^\mu_a \nabla_\mu e^\rho_b = e^c_\rho e^\mu_a(\partial_\mu e^\rho_b + \Gamma^\rho_{\mu\nu}e^\nu_b).$$

So $\omega^a{}_b \wedge \theta^b = e^a_\rho e^\lambda_c e^c_\mu e^b_\nu\left(\partial_\lambda e^\rho_b + e^\sigma_b\Gamma^\rho_{\lambda\sigma}\right)dx^\mu \wedge dx^\nu$. We can further simplify the expression using the fact $e^\lambda_c e^c_\mu = \delta^\lambda_\mu$ and the fact that the connection is torsion-free. Therefore, the connection term vanished:

$$\omega^a{}_b \wedge \theta^b = e^a_\rho e^b_\nu \partial_\mu e^\rho_b dx^\mu \wedge dx^\nu$$

Now we use the fact that $e^b_\nu e^\rho_b = \delta^\rho_\nu$, so $e^b_\nu\partial_\mu e_b{}^\rho = -e^\rho_b\partial_\mu e^b{}_\nu$. We have

$$\omega^a{}_b \wedge \theta^b = -e^a_\rho e^\rho_b \partial_\mu e^b_\nu dx^\mu \wedge dx^\nu = -\partial_\mu e^a_\nu dx^\mu \wedge dx^\nu = -d\theta^a,$$

which completes the proof.

Claim: For the Levi-Civita connection, the connection one-form is anti-symmetric:

$$\omega_{ab} = -\omega_{ba}. \tag{9}$$

Proof: This follows from the expression for the components Γ^a_{bc}. Lowering an index, we have

$$\Gamma_{abc} = \eta_{ad}e^d_\rho e^\mu_b \nabla_\mu e^\rho_c = -\eta_{ad}e^\rho_c e^\mu_b \nabla_\mu e^d_\rho = -\eta_{cf}e^f_\sigma e^\mu_b \nabla_\mu\left(\eta_{ad}g^{\rho\sigma}e^d_\rho\right)$$

where, in the final equality, we've used the fact that the connection is compatible with the metric to raise the indices of e^d_ρ inside the covariant derivative. Finishing off the derivation, we then have

$$\Gamma_{abc} = -\eta_{cf}e^f_\rho e^\mu_b \nabla_\mu e^\rho_a = -\Gamma_{cba}.$$

The result then follows from the definition $\omega_{ab} = \Gamma_{acb}\hat{\theta}^c$.

As a concrete example, consider the metric of the general form

$$ds^2 = -f(r)^2dt^2 + f(r)^{-2}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{10}$$

The basis of coordinate one-forms is[1]

$$\hat{\theta} = \left(f(r)dt,\ \frac{1}{f(r)}dr,\ rd\theta,\ r\sin\theta d\phi\right).$$

[1] Note that we have put a hat on the one-form to avoid confusion with the θ angle.

The exterior derivatives are

$$d\hat{\theta} = \left(\frac{d}{dr} f(r)\ dr \wedge dt,\ 0,\ dr \wedge d\theta,\ \sin\theta\ dr \wedge d\phi + r\cos\theta\ d\theta \wedge d\phi \right).$$

Then we can simply read out the non-vanishing component of the connection one form:

$$\omega^0{}_1 = \omega^1{}_0 = f'(r)\hat{\theta}^0, \quad \omega^2{}_1 = -\omega^1{}_2 = \frac{f}{r}\hat{\theta}^2, \quad \omega^3{}_1 = -\omega^1{}_3 = \frac{f}{r}\hat{\theta}^3, \quad \omega^3{}_2 = -\omega^2{}_3 = \frac{\cot\theta}{r}\hat{\theta}^3.$$

B. Curvature

The curvature R can be viewed as a map from $\mathfrak{X}(M) \times \mathfrak{X}(M)$ to a differential operator acting on $\mathfrak{X}(M)$,

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \tag{11}$$

We can evaluate these tensors in a coordinate basis $\{e_\mu\} = \{\partial_\mu\}$, with the dual basis $\{f^\mu\} = \{dx^\mu\}$. The components of R are

$$\begin{aligned} R^\sigma_{\rho\mu\nu} &= f^\sigma \left(\nabla_\mu \nabla_\nu e_\rho - \nabla_\nu \nabla_\mu e_\rho - \nabla_{[e_\mu, e_\nu]} e_\rho \right) = f^\sigma \left(\nabla_\mu \nabla_\nu e_\rho - \nabla_\nu \nabla_\mu e_\rho \right) \\ &= f^\sigma \left[\nabla_\mu \left(\Gamma^\lambda_{\nu\rho} e_\lambda \right) - \nabla_\nu \left(\Gamma^\lambda_{\mu\rho} e_\lambda \right) \right] = f^\sigma \left[\left(\partial_\mu \Gamma^\lambda_{\nu\rho} \right) e_\lambda + \Gamma^\lambda_{\nu\rho} \Gamma^\tau_{\mu\lambda} e_\tau - \left(\partial_\nu \Gamma^\lambda_{\mu\rho} \right) e_\lambda - \Gamma^\lambda_{\mu\rho} \Gamma^\tau_{\nu\lambda} e_\tau \right] \\ &= \partial_\mu \Gamma^\sigma_{\nu\rho} - \partial_\nu \Gamma^\sigma_{\mu\rho} + \Gamma^\lambda_{\nu\rho} \Gamma^\sigma_{\mu\lambda} - \Gamma^\lambda_{\mu\rho} \Gamma^\sigma_{\nu\lambda}, \end{aligned} \tag{12}$$

where we've used the fact that, in a coordinate basis, $[e_\mu, e_\nu] = [\partial_\mu, \partial_\nu] = 0$.

There is a closely related calculation in which both the torsion and Riemann tensors appears. We look at the commutator of covariant derivatives acting on vector fields. Written in an orgy of anti-symmetrised notation, this calculation gives[2]

$$\begin{aligned} \nabla_{[\mu} \nabla_{\nu]} Z^\sigma =& \partial_{[\mu} \left(\nabla_{\nu]} Z^\sigma \right) + \Gamma^\sigma_{[\mu|\lambda|} \nabla_{\nu]} Z^\lambda - \Gamma^\rho_{[\mu\nu]} \nabla_\rho Z^\sigma \\ =& \partial_{[\mu} \partial_{\nu]} Z^\sigma + \left(\partial_{[\mu} \Gamma^\sigma_{\nu]\rho} \right) Z^\rho + \left(\partial_{[\mu} Z^\rho \right) \Gamma^\sigma_{\nu]\rho} + \Gamma^\sigma_{[\mu|\lambda|} \partial_{\nu]} Z^\lambda + \Gamma^\sigma_{[\mu|\lambda|} \Gamma^\lambda_{\nu]\rho} Z^\rho - \Gamma^\rho_{[\mu\nu]} \nabla_\rho Z^\sigma. \end{aligned}$$

The first term vanishes, while the third and fourth terms cancel against each other. We're left with

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) Z^\sigma = R^\sigma_{\rho\mu\nu} Z^\rho - T^\rho_{\mu\nu} \nabla_\rho Z^\sigma, \tag{13}$$

where the torsion tensor is $T^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu}$ and the Riemann tensor coincides with Eq. (12). The expression Eq. (13) is known as the Ricci identity.

We can compute the components of the Riemann tensor in our non-coordinate basis,

$$R^a_{bcd} = R\left(\hat{\theta}^a; \hat{e}_c, \hat{e}_d, \hat{e}_b \right).$$

The anti-symmetry of the last two indices, $R^a_{bcd} = -R^a_{bdc}$, makes this ripe for turning into a matrix of two-forms,

$$\mathcal{R}^a{}_b = \frac{1}{2} R^a_{bcd} \hat{\theta}^c \wedge \hat{\theta}^d. \tag{14}$$

The second of the two Cartan structure relations states that this can be written in terms of the curvature one-form as

$$\mathcal{R}^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b. \tag{15}$$

Consider the metric in Eq. (10). Now we can use this to compute the curvature two-form. We will focus on $\mathcal{R}^0{}_1 = d\omega^0{}_1 + \omega^0{}_c \wedge \omega^c{}_1$. We have

$$d\omega^0{}_1 = f' d\hat{\theta}^0 + f'' dr \wedge \hat{\theta}^0 = \left[(f')^2 + f'' f \right] dr \wedge dt.$$

The second term in the curvature 2-form is $\omega^0{}_c \wedge \omega^c{}_1 = \omega^0{}_1 \wedge \omega^1{}_1 = 0$. So we're left with

$$\mathcal{R}^0{}_1 = \left[(f')^2 + f'' f \right] dr \wedge dt = \left[(f')^2 + f'' f \right] \hat{\theta}^1 \wedge \hat{\theta}^0.$$

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[2] We use the notation $A_{[\mu\nu]} = \frac{1}{2}(A_{\mu\nu} - A_{\nu\mu})$.

C. Dynamics

The covariant derivative defines the *parallel transport*: let X be a vector field defined along curve $c(t)$. X is said to be parallel transported if $\nabla_V X = 0$, which leads to the parallel transportation equation:

$$\frac{dx^\mu}{dt}\left(\frac{\partial X^\lambda}{\partial x^\mu} + \Gamma^\lambda{}_{\mu\nu}X^\nu\right) = \frac{d}{dt}X^\lambda + \Gamma^\lambda{}_{\mu\nu}V^\mu X^\nu = 0, \quad \text{where} \quad V^\mu = \frac{d}{dt}x^\mu|_{c(t)}.$$

Further, a curve $c(t)$ is a geodesic if $\nabla_V V = 0$, which leads to the geodesic equation:

$$\frac{d^2x^\mu}{dt^2} + \Gamma^\mu{}_{\nu\lambda}\frac{dx^\nu}{dt}\frac{dx^\lambda}{dt} = 0.$$

II. THE EINSTEIN EQUATIONS

A. The Einstein-Hilbert Action

Given a Ricci scalar R, the action for the gravitational field is

$$S = \frac{M_{\rm pl}^2}{2}\int d^4x\sqrt{|g|}R. \tag{16}$$

Note that S is non-renormalizable. In the following, we will choose the unit so that $M_{\rm pl}^2/2 = 1$.

We would like to determine the Euler-Lagrange equations arising from the action. We do this in the usual way, by starting with some fixed metric $g_{\mu\nu}(x)$ and seeing how the action changes when we shift $g_{\mu\nu}(x) \to g_{\mu\nu}(x) + \delta g_{\mu\nu}(x)$. Writing the Ricci scalar as $R = g^{\mu\nu}R_{\mu\nu}$, the Einstein-Hilbert action clearly changes as

$$\delta S = \int d^4x\left[(\delta\sqrt{|g|})g^{\mu\nu}R_{\mu\nu} + \sqrt{|g|}\left(\delta g^{\mu\nu}\right)R_{\mu\nu} + \sqrt{|g|}g^{\mu\nu}\delta R_{\mu\nu}\right]$$

It turns out that it's slightly easier to think of the variation in terms of the inverse metric $\delta g^{\mu\nu}$. This is equivalent to the variation of the metric $\delta g_{\mu\nu}$; the two are related by

$$g_{\rho\mu}g^{\mu\nu} = \delta^\nu_\rho \Rightarrow \left(\delta g_{\rho\mu}\right)g^{\mu\nu} + g_{\rho\mu}\delta g^{\mu\nu} = 0 \quad \Rightarrow \quad \delta g^{\mu\nu} = -g^{\mu\rho}g^{\nu\sigma}\delta g_{\rho\sigma}.$$

To proceed, we will need to calculate $\delta\sqrt{|g|}$. Using the identity $\log\det A = \operatorname{tr}\log A$, we have

$$\frac{1}{\det A}\delta(\det A) = \operatorname{tr}\left(A^{-1}\delta A\right).$$

Applying this to the metric, we have

$$\delta\sqrt{|g|} = \frac{1}{2}\frac{1}{\sqrt{|g|}}|g|g^{\mu\nu}\delta g_{\mu\nu} = \frac{1}{2}\sqrt{|g|}g^{\mu\nu}\delta g_{\mu\nu} = -\frac{1}{2}\sqrt{|g|}g_{\mu\nu}\delta g^{\mu\nu}.$$

Now we turn to $\delta R_{\mu\nu}$. We claim that $\delta R_{\mu\nu} = \nabla_\rho\delta\Gamma^\rho_{\mu\nu} - \nabla_\nu\delta\Gamma^\rho_{\mu\rho}$, where

$$\delta\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\left(\nabla_\mu\delta g_{\sigma\nu} + \nabla_\nu\delta g_{\sigma\mu} - \nabla_\sigma\delta g_{\mu\nu}\right).$$

is a tensor. The last expression now becomes a total derivative

$$g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\mu X^\mu \quad \text{with} \quad X^\mu = g^{\rho\nu}\delta\Gamma^\mu_{\rho\nu} - g^{\mu\nu}\delta\Gamma^\rho_{\nu\rho}.$$

The variation of the action can then be written as

$$\delta S = \int d^4x\sqrt{-g}\left[\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)\delta g^{\mu\nu} + \nabla_\mu X^\mu\right].$$

This final term is a total derivative and, by the divergence, we ignore it. Requiring $\delta S = 0$, we have the equations of motion

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0.$$

B. Schwarzschild Spacetime

C. de Sitter Space