Notes on Quantum Field Theory

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Chapter 1

Relativistic Quantum Field Theory

1.1 Lorentz Invariance

1.1.1 The Lorentz Algebra

The metric is chosen to be

$$g_{\mu\nu} = g^{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1). \tag{1.1}$$

The Lorentz transformation $\Lambda^\mu{}_\nu$ satisfies

$$\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta g^{\alpha\beta} = g^{\mu\nu}. \tag{1.2}$$

From this we have

$$g^{\gamma\alpha} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta g_{\mu\nu} = g^{\gamma\alpha} g_{\alpha\beta} \quad \Longrightarrow \quad \Lambda_\nu{}^\gamma \Lambda^\nu{}_\beta = \delta^\gamma{}_\beta,$$

The inverse Lorentz transformation satisfies:

$$(\Lambda^{-1})^\mu{}_\nu = \Lambda_\nu{}^\mu. \tag{1.3}$$

The infinitesimal transformation is denoted as

$$\begin{aligned} \Lambda^\mu{}_\nu &= \delta^\mu{}_\nu + \delta\omega^\mu{}_\nu \\ (\Lambda^{-1})^\mu{}_\nu &= \delta^\mu{}_\nu - \delta\omega^\mu{}_\nu \end{aligned} \quad \Longrightarrow \quad g_{\alpha\nu}\delta\omega^\nu{}_\beta + \delta\omega^\mu{}_\alpha g_{\mu\beta} = \delta\omega_{\alpha\beta} + \delta\omega_{\beta\alpha} = 0.$$

A representation of Lorentz group $U(\Lambda)$ can be parametrized as:

$$U(\Lambda) = \exp\left(\frac{i}{2}\omega_{\mu\nu} M^{\mu\nu}\right). \tag{1.4}$$

Another useful parametrization is

$$\theta_i \equiv \frac{1}{2}\varepsilon_{ijk}\omega_{jk}, \;\; \beta_i \equiv \omega_{0i}. \tag{1.5}$$

A new set of generators are:

$$J_i \equiv \frac{1}{2}\varepsilon_{ijk} M^{jk}, \;\; K_i \equiv M^{i0}, \tag{1.6}$$

where J_i's are the generators of the spatial rotations, and K_i's are the generators of Lorentz boosts.

In the fundamental representation, the generators are represented by

$$J_1 = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 0 & -i \\ & & i & 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & & & \\ & 0 & & i \\ & & 0 & \\ & -i & & 0 \end{bmatrix}, \quad J_3 = \begin{bmatrix} 0 & & & \\ & 0 & -i & \\ & i & 0 & \\ & & & 0 \end{bmatrix},$$
$$K_1 = \begin{bmatrix} 0 & -i & & \\ -i & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0 & & -i & \\ & 0 & & \\ -i & & 0 & \\ & & & 0 \end{bmatrix}, \quad K_3 = \begin{bmatrix} 0 & & & -i \\ & 0 & & \\ & & 0 & \\ -i & & & 0 \end{bmatrix}.$$

The Lie algebra of the Lorentz algebra can be explicitly done using the fundamental representation. The result is

$$\begin{aligned} [J_i, J_j] &= i\varepsilon_{ijk} J_k, \\ [J_i, K_j] &= i\varepsilon_{ijk} K_k, \\ [K_i, K_j] &= -i\varepsilon_{ijk} J_k. \end{aligned} \tag{1.7}$$

By defining a new set of generators:

$$N_i^L \equiv \frac{J_i - iK_i}{2}, \ N_i^R \equiv \frac{J_i + iK_i}{2}. \tag{1.8}$$

They satisfies two independent $\mathfrak{su}(2)$ algebra:

$$\begin{aligned} \left[N_i^L, N_j^L\right] &= i\varepsilon_{ijk} N_k^L, \\ \left[N_i^R, N_j^R\right] &= i\varepsilon_{ijk} N_k^R, \\ \left[N_i^L, N_j^R\right] &= 0. \end{aligned} \tag{1.9}$$

That is, the Lorentz algebra is isomorphic to two $\mathfrak{su}(2)$ algebra,

$$\mathfrak{so}(3,1) \approx \mathfrak{su}_L(2) \oplus \mathfrak{su}_R(2). \tag{1.10}$$

From Eq. (1.10), we know that the representation of the Lorentz algebra can be labelled by j_L and j_R. Note that the fundamental representation correspond to

$$\left(j_L = \frac{1}{2}, j_R = \frac{1}{2}\right).$$

The specific form of the group is

$$\Lambda(\vec{\theta}, \vec{\beta}) = \exp\left[i(\vec{\theta} + i\vec{\beta}) \cdot \vec{N}^L + i(\vec{\theta} - i\vec{\beta}) \cdot \vec{N}^R\right]. \tag{1.11}$$

The spinor representations are those with $j_L = 1/2$ or $j_R = 1/2$. Specifically, we define the left-hand spinor ψ_L and right-hand spinor ψ_R that transform as:

$$\begin{aligned} \Lambda_L(\vec{\theta}, \vec{\beta})\psi_L &= \exp\left(\frac{i}{2}\vec{\theta} \cdot \vec{\sigma} - \frac{1}{2}\vec{\beta} \cdot \vec{\sigma}\right)\psi_L, \\ \Lambda_R(\vec{\theta}, \vec{\beta})\psi_R &= \exp\left(\frac{i}{2}\vec{\theta} \cdot \vec{\sigma} + \frac{1}{2}\vec{\beta} \cdot \vec{\sigma}\right)\psi_R. \end{aligned} \tag{1.12}$$

Using the fact $\sigma^2 \cdot \vec{\sigma}^* \cdot \sigma^2 = -\vec{\sigma}$, the left-hand and the right-hand representations are related by:

$$\begin{aligned} \sigma^2 \Lambda_L^* \sigma^2 = \Lambda_R, \quad \sigma^2 \Lambda_L^T \sigma^2 = \Lambda_L^{-1}, \\ \sigma^2 \Lambda_R^* \sigma^2 = \Lambda_L, \quad \sigma^2 \Lambda_R^T \sigma^2 = \Lambda_R^{-1}. \end{aligned} \tag{1.13}$$

For this reason, the left-hand and right-hand spinor can be interchanged by

$$\begin{aligned} \sigma^2 \psi_L^* \sim \chi_R, \quad \psi_L^\dagger \sigma^2 \sim \chi_R^\dagger \\ \sigma^2 \psi_R^* \sim \chi_L, \quad \psi_R^\dagger \sigma^2 \sim \chi_L^\dagger. \end{aligned} \tag{1.14}$$

1.1.2 The Invariant Symbols

The invariant symbols can be thought as the Clebsch-Gordan coefficients that help to form singlets. The first singlet comes from the decomposition

$$\frac{1}{2} \otimes \frac{1}{2} \approx 0 \oplus 1.$$

Correspondingly, we can check that for each-hand-side spinor, the quadratic forms

$$\psi_L^T \sigma^2 \chi_L \quad \text{or} \quad \psi_R^T \sigma^2 \chi_R \tag{1.15}$$

are singlets. We can define the first invariant symbol as[1]

$$\varepsilon^{ab} = \varepsilon^{\dot{a}\dot{b}} = i(\sigma^2)_{ab}, \quad \varepsilon_{ab} = \varepsilon_{\dot{a}\dot{b}} = -i(\sigma^2)_{ab}. \tag{1.16}$$

The symbol ε^{ab} or ε_{ab} also serve as the index raising/lowering symbol, i.e.,

$$\varepsilon^{ab}\psi_b = \psi^a, \;\; \varepsilon_{ab}\psi^b = \psi_a. \tag{1.17}$$

The singlet (1.15) is then defined as the inner product of two spinors:

$$\psi \cdot \chi \equiv \varepsilon_{ab}\psi^a\chi^b = \psi^a\chi_a = -\varepsilon_{ba}\psi^a\chi^b = -\psi_b\chi^b. \tag{1.18}$$

In addition, because of (1.14), the expressions

$$\psi_L^\dagger \chi_R \quad \text{and} \quad \psi_R^\dagger \chi_L$$

are also singlets.

Besides, we know there should be another invariant symbol from the decomposition

$$\left(\frac{1}{2}, 0\right) \otimes \left(0, \frac{1}{2}\right) \approx (0,0) \oplus \cdots .$$

For this reason, we are searching for the symbol M that the expression

$$M^\mu_{a\dot{b}} \psi_L^a \chi_R^{\dot{b}}$$

transforms as the Lorentz vector. The matrix M^μ should transform as

$$M^\mu \longrightarrow \Lambda_L^T \cdot M^\mu \cdot \Lambda_R = \Lambda^\mu{}_\nu M^\nu.$$

[1] We use the dotted symbol to denote the right-hand spinor indices.

Use the fact that $\sigma^2 \cdot \Lambda_L^T \cdot \sigma^2 = \Lambda_L^{-1}$, the above equation transforms to

$$\left(\sigma^2 M^\mu\right) \longrightarrow \Lambda_L^{-1} \cdot \left(\sigma^2 M^\mu\right) \cdot \Lambda_R.$$

We then show the matrices $\sigma^\mu = (\sigma^0, \vec{\sigma})$ satisfies the requirement. Firstly, for the spatial rotation,

$$\Lambda_L(\vec{\theta}, \vec{0}) = \Lambda_R(\theta, \vec{0}) = \exp\left(i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right)$$

The Pauli matrix transform as

$$\left(1 - i\delta\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) \sigma^j \left(1 + i\delta\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) = \sigma^j + i\delta\theta_i \left(-i\varepsilon_{ijk}\sigma^k\right)$$

Secondly, for the boosts,

$$\Lambda_L(\vec{0}, \vec{\beta}) = \exp\left(-\vec{\beta} \cdot \frac{\vec{\sigma}}{2}\right), \; \Lambda_R(\vec{0}, \vec{\beta}) = \exp\left(+\vec{\beta} \cdot \frac{\vec{\sigma}}{2}\right)$$

The Pauli matrix transform as

$$\left(1 + \delta\vec{\beta} \cdot \frac{\vec{\sigma}}{2}\right) \sigma^\mu \left(1 + \delta\vec{\beta} \cdot \frac{\vec{\sigma}}{2}\right) = \begin{cases} \sigma^0 + i\delta\beta_i \cdot (-i\sigma^i), & \mu = 0 \\ \sigma^j + i\delta\beta_j(-i\sigma^0), & \mu = j \end{cases}.$$

We thus have shown indeed that

$$\psi_L^T \sigma^2 \sigma^\mu \chi_R \tag{1.19}$$

is a Lorentz vector. Further more, from (1.14), we know that

$$\eta_R^\dagger \sigma^\mu \chi_R \tag{1.20}$$

is also a Lorentz vector. Similarly, consider the Lorentz vector

$$N^\mu_{\dot{a}b} \psi_R^{\dot{a}} \chi_R^b,$$

which together with σ^2 should transforms as

$$\left(\sigma^2 N^\mu\right) \longrightarrow \Lambda_R^{-1} \cdot \left(\sigma^2 N^\mu\right) \cdot \Lambda_L.$$

We can check that $\bar{\sigma}^\mu = (\sigma^0, -\vec{\sigma})$ satisfies the requirement, and thus

$$\eta_L^\dagger \bar{\sigma}^\mu \chi_L \tag{1.21}$$

is also a Lorentz vector.

1.2 Klein-Gordon Field

In relativistic quantum field theory, the Lagrangian should be a singlet under Lorentz transformation. Different free fields correspond to different representation of the Lorentz algebra. The symmetry under Lorentz transformation also restrict the possible terms that can appear in the Lagrangian.

The simplest case is when $j_L = j_R = 0$, corresponding to the scalar field, which we denote as $\phi(x)$. Since the field it self is singlet, any polynomial of the field in principle can appear in the theory. When considering the free theory, we restrict our attention to the quadratic terms. We require the field theory to have a dynamical term, which contains derivative the the field. The derivative operator ∂^μ transforms as the fundamental representation. To be Lorentz invariant, the allowed free theory can only be

$$\mathcal{L}_{\text{K-G}} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{m^2}{2}\phi^2 \simeq -\frac{1}{2}\phi(\partial^2 + m^2)\phi. \tag{1.22}$$

For general discussion, we consider the field theory on d-dimensional space-time. Note that the space-time Fourier transformation is defined as

$$\begin{aligned} \tilde{\phi}(k) &= \int d^d x e^{ik\cdot x}\phi(x), \\ \phi(x) &= \int \frac{d^d k}{(2\pi)^d} e^{-ik\cdot x}\tilde{\phi}(k), \end{aligned} \tag{1.23}$$

where the inner product of two 4-momentum and 4-coordinate is

$$k \cdot x = \omega t - \vec{k}\cdot\vec{x}. \tag{1.24}$$

1.2.1 Canonical Quantization

The classical equation of motion

$$\partial_\mu \left[\frac{\partial}{\partial(\partial_\mu\phi)}\right] - \frac{\partial\mathcal{L}}{\partial\phi} = 0$$

for Klein-Gordon field is

$$(\partial_t^2 - \nabla^2 + m^2)\phi(\vec{x}, t) = 0. \tag{1.25}$$

The solution to Eq. (1.25) is proportional to the plane wave:

$$\phi(\vec{x}, t) \propto e^{-i\omega_{\boldsymbol{k}} t + i\vec{p}\cdot\vec{x}} + e^{i\omega_{\vec{k}} t - i\boldsymbol{p}\cdot\vec{x}},$$

where the energy is $\omega_{\boldsymbol{k}} = \boldsymbol{k}^2 + m^2$ and $\vec{k}$ is the momentum as the conserved quantity. The general solution to the EOM is

$$\phi(\vec{x}, t) \propto \int \frac{d^3k}{(2\pi)^3}\left(a_k e^{-i\omega_{\boldsymbol{k}} t + i\vec{k}\cdot\vec{x}} + a_k^* e^{i\omega_{\boldsymbol{k}} t - i\vec{k}\cdot\vec{x}}\right). \tag{1.26}$$

Single Particle State

The canonical quantization promote the coefficient a_k/a_k^* to the particle annihilation/creation operator $a_k/a_k^\dagger$, with the commutation relation

$$[a_k, a_p^\dagger] = (2\pi)^3\delta^3(\vec{k} - \vec{p}). \tag{1.27}$$

The single-particle state with momentum $\vec{k}$ is created by $a_k^\dagger$ operators acting on the vacuum:

$$|\vec{k}\rangle \equiv \sqrt{2\omega_{\boldsymbol{k}}}\, a_k^\dagger|0\rangle, \tag{1.28}$$

where $|\vec{k}\rangle$ is a state with a single particle of momentum $\vec{k}$. The factor of $\sqrt{2\omega_{\boldsymbol{k}}}$ in Eq. (1.28) is just a convention, but it will make some calculations easier. To compute the normalization of one-particle states, we start with

$$\langle 0|0\rangle = 1, \tag{1.29}$$

which leads to

$$\langle \vec{p}|\vec{k}\rangle = 2\sqrt{\omega_{\boldsymbol{p}}\omega_{\boldsymbol{k}}}\left\langle 0\left|a_p a_k^\dagger\right|0\right\rangle = 2\omega_{\boldsymbol{p}}(2\pi)^3\delta^3(\vec{p}-\vec{k}). \tag{1.30}$$

The identity operator for one-particle states is

$$1 = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\boldsymbol{p}}}|\vec{p}\rangle\langle\vec{p}|, \tag{1.31}$$

which we can check with

$$|\vec{k}\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\boldsymbol{p}}}|\vec{p}\rangle\langle\vec{p}|\vec{k}\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\boldsymbol{p}}}2\omega_{\boldsymbol{p}}(2\pi)^3\delta^3(\vec{p}-\vec{k})|\vec{p}\rangle = |\vec{k}\rangle.$$

The identity operator Eq. (1.31) is Lorentz invariant since it can be expressed as

$$1 = \int \frac{d^3p\,d\omega}{(2\pi)^4}2\pi\delta(\omega^2-\boldsymbol{p}^2-m^2)|\vec{p}\rangle\langle\vec{p}|. \tag{1.32}$$

Field Expansion

We fix the normalization by requiring

$$\langle\vec{k}|\phi(\vec{x},0)|0\rangle = e^{-i\vec{k}\cdot\vec{x}}, \tag{1.33}$$

and the quantized field operator is

$$\phi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{k}}}}\left(a_k e^{-ik\cdot x} + a_k^\dagger e^{ik\cdot x}\right). \tag{1.34}$$

Consider the two-point correlation (propagator):

$$\begin{aligned} i\Delta(x_1-x_2) &= \langle 0|T\phi(x_1)\phi(x_2)|0\rangle \\ &= \theta(t_1-t_2)\langle 0|\phi(x_1)\phi(x_2)|0\rangle + \theta(t_2-t_1)\langle 0|\phi(x_2)\phi(x_1)|0\rangle. \end{aligned}$$

Note that

$$\langle 0|\phi(x_1)\phi(x_2)|0\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)-i\omega_{\vec{k}}\tau}, \tag{1.35}$$

where $\tau = t_1 - t_2$. The propagator can be written as

$$\begin{aligned} i\Delta(x_1-x_2) &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}\left[e^{-i\omega_{\vec{k}}\tau}\theta(\tau) + e^{i\omega_{\vec{k}}\tau}\theta(-\tau)\right] \\ &= \int \frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}\int \frac{d\omega}{2\pi i}\frac{-e^{i\omega\tau}}{\omega^2-\omega_k^2+i\epsilon} \\ &= \int \frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x_1-x_2)}\frac{i}{k^2-m^2+i\epsilon}. \end{aligned} \tag{1.36}$$

1.2.2 Path-integral Formalism

Consider the action for free field with source

$$S_0[\phi, J] = \int d^d x \left[\mathcal{L}_0 + J(x)\cdot\phi(x)\right]. \tag{1.37}$$

In the path integral formalism, we consider the partition function

$$Z[J] = \int D[\phi] \exp(iS[\phi, J]) \equiv Z[0] \exp(iW[J]). \tag{1.38}$$

where we have introduced a new quantity

$$W[J] = -\frac{1}{2}\int d^d x_1 d^d x_2 J(x_1)\Delta(x_1 - x_2)J(x_2). \tag{1.39}$$

For free field, the free propagator $\Delta_0(x_1 - x_2)$ is:

$$i\Delta_0(x_1 - x_2) = \int \frac{d^d k}{(2\pi)^d}\frac{e^{-ik\cdot x}}{k^2 - m^2 + i\epsilon}, \tag{1.40}$$

where the extra $i\epsilon$ term is use to bring the singularities infinitesimally below the real axis. This infinitesimal value can be absorbed into the mass term, by regarding the mass term m^2 as $m^2 - i\epsilon$.

Note that $\Delta_0(x_1 - x_2)$ is related to the correlation function:

$$\Delta_0(x_1 - x_2) = \frac{1}{i}\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = \frac{\delta}{i\delta J(x_1)}\frac{\delta}{i\delta J(x_2)}W_0[J]. \tag{1.41}$$

Gaussian Integral

Now we evaluate the propagator in the path-integral formalism. In momentum space, the free action (with source) is

$$\frac{1}{V}\sum_k \left[\tilde{\phi}^*(k)(k^2 - m^2)\tilde{\phi}(k) + \tilde{J}^*(k)\cdot\tilde{\phi}(k) + \tilde{\phi}^*(k)\cdot\tilde{J}(k)\right].$$

For real field, $\tilde{\phi}^*(k) = \tilde{\phi}(-k)$. For our convenience, we have expressed the momentum integral as summation. Actually, consider the d-dimensional box of size L^d, the momentum along each axis is multiple of $2\pi/L$, so when $L \to \infty$, the summation approaches in integral,

$$\frac{1}{V}\sum_k \to \int \frac{d^d k}{(2\pi)^d}.$$

Let us omit the $1/V$ factor, the summation can be formally expressed as

$$-\frac{1}{2}\mathbf{v}^T\cdot\mathbf{A}\cdot\mathbf{v} + \mathbf{b}^T\cdot\mathbf{v} \tag{1.42}$$

where

$$\mathbf{v} = \bigoplus_{|\mathbf{k}|}\begin{bmatrix}\tilde{\phi}(k)\\ \tilde{\phi}^*(k)\end{bmatrix},\ \mathbf{A} = \bigoplus_{|\mathbf{k}|}\begin{bmatrix}0 & k^2 - m^2\\ k^2 - m^2 & 0\end{bmatrix},\ \mathbf{b} = \bigoplus_{|\mathbf{k}|}\begin{bmatrix}\tilde{J}^*(k)\\ \tilde{J}(k)\end{bmatrix}.$$

We can use a unitary transformation to tranform

$$\mathbf{U} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}, \quad \mathbf{U}\cdot\begin{bmatrix} \tilde{\phi}(k) \\ \tilde{\phi}^*(k) \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} \tilde{\phi}(k) + \tilde{\phi}^*(k) \\ -i\tilde{\phi}(k) + i\tilde{\phi}^*(k) \end{bmatrix} \equiv \begin{bmatrix} \tilde{\phi}_1(k) \\ \tilde{\phi}_2(k) \end{bmatrix}$$

The path integral then becomes a real field integral. Recall the real Gaussian integral formula:

$$\int d\mathbf{v} \exp\left(-\frac{1}{2}\mathbf{v}^T\cdot\mathbf{A}\cdot\mathbf{v} + \mathbf{b}^T\cdot\boldsymbol{v}\right) = \sqrt{\frac{(2\pi)^N}{\det\mathbf{A}}}\exp\left(\frac{1}{2}\mathbf{b}^T\cdot\mathbf{A}^{-1}\cdot\mathbf{b}\right), \tag{1.43}$$

For the field integral, we absorbed the $(2\pi)^{N/2}$ term into the measure, and express the path integral for the Gaussian field as:

$$W_0[J] = -\frac{i}{2}\int\frac{d^dk}{(2\pi)^d}\mathbf{b}_k^T\cdot\mathbf{A}_k^{-1}\cdot\mathbf{b}_k. \tag{1.44}$$

This gives the propagator in the momentum space:

$$\tilde{\Delta}_0(k) = \frac{1}{k^2 - m^2}.$$

From Field to Force

Consider two separate particle described by the delta function $J_a(x) = \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}_a)$, together the source is

$$J(x) = J_1(x) + J_2(x). \tag{1.45}$$

Adding the source,

$$W[J] = -\frac{1}{2}\int d^4x_1 d^4x_2 J(x_1)\Delta(x_1 - x_2)J(x_2)$$

Omit the self energy terms $J_1^2(x)$, $J_2^2(x)$, $W[J]$ is

$$\begin{aligned} W[J] &= -\int d^4y_1 d^4y_2\, e^{-ik^0(y_1^0 - y_2^0)}\int\frac{d^4k}{(2\pi)^4}J_1(y_1)\frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{y}_1 - \boldsymbol{y}_2)}}{k^2 - m^2}J_2(y_2) \\ &= -\int dt\int d(y_1^0 - y_2^0)\, e^{-ik^0(y_1^0 - y_2^0)}\int\frac{d^4k}{(2\pi)^4}\frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{y}_1 - \boldsymbol{y}_2)}}{k^2 - m^2} \\ &= \left(\int dt\right)\int\frac{d^3k}{(2\pi)^3}\frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{y}_1 - \boldsymbol{y}_2)}}{\boldsymbol{k}^2 + m^2} \end{aligned} \tag{1.46}$$

Recall that the partition function is actually infinite:

$$Z \sim \langle 0|e^{-iHT}|0\rangle \quad \Longrightarrow \quad W = -iET, \tag{1.47}$$

where E is the energy. Writing $\boldsymbol{r} \equiv \boldsymbol{y}_1 - \boldsymbol{y}_2$, and $u \equiv \cos\theta$ with θ the angle between $\boldsymbol{k}$ and $\boldsymbol{r}$, the volume form is $dk \cdot kd\theta \cdot 2\pi k\sin\theta = 2\pi k^2 dkdu$, and the integral is

$$\begin{aligned} E &= -\int\frac{d^3k}{(2\pi)^3}\frac{e^{ikru}}{k^2 + m^2} \\ &= -\frac{1}{(2\pi)^2}\int_0^\infty k^2dk\int_{-1}^1 du\frac{e^{ikru}}{k^2 + m^2} \\ &= -\frac{1}{2\pi^2 r}\int_0^\infty k\frac{\sin kr}{k^2 + m^2}dk. \end{aligned} \tag{1.48}$$

Since the integral is even, we can extend the integral to

$$\begin{aligned} E &= -\frac{1}{4\pi^2 r}\int_{-\infty}^{\infty} k\frac{\sin kr}{k^2+m^2}dk \\ &= \frac{i}{4\pi^2 r}\int_{-\infty}^{\infty} \frac{ke^{ikr}}{k^2+m^2}dk \end{aligned} \tag{1.49}$$

The residue theorem gives

$$\int_{-\infty}^{\infty} \frac{ke^{ikr}}{k^2+m^2}dk = \pi i e^{-mr}$$

So we get the potential of two particles:

$$E(r) = -\frac{e^{-mr}}{4\pi r}, \tag{1.50}$$

and the attractive force is

$$F(r) = -\frac{dE}{dr} = -(1+mr)\frac{e^{-mr}}{4\pi r^2}. \tag{1.51}$$

1.3 Vector Field

If we can choose $j_L = j_R = 1/2$, the field is transformed as Lorentz vector. We denote the field as $A^\mu(x)$. Some possible quadratic forms for the vector field that forms singlets are

$$A^\mu A_\mu,\ (\partial_\mu A^\mu)^2,\ A^\nu\partial^2 A_\nu,\ \varepsilon_{\mu\nu\rho\lambda}\partial^\mu A^\nu \partial^\rho A^\lambda.$$

For the field theory describe the electromagnetic field, we require the theory to further have gauge symmetry, i.e., invariant under

$$A^\mu(x) \to A^\mu(x) + \partial^\mu \alpha(x). \tag{1.52}$$

The gauge invariant forbids the first term, and forces the second and third term to combine as

$$(\partial_\mu A^\mu)^2 - A^\nu\partial^2 A_\nu \sim \frac{1}{2}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A^\nu - \partial_\nu A_\mu) \equiv \frac{1}{2}F^{\mu\nu}F_{\mu\nu}.$$

where we have define a field-strength tensor

$$F^{\mu\nu} \equiv (\partial^\mu A^\nu - \partial^\nu A^\mu) = \begin{bmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & E_3 & -E_2 \\ -B_2 & -E_3 & 0 & E_1 \\ -B_3 & E_2 & -E_1 & 0 \end{bmatrix}. \tag{1.53}$$

Note that the fourth term is called the *theta term*, which can be written as a boundary term

$$\varepsilon_{\mu\nu\rho\lambda}\partial^\mu A^\nu \partial^\rho A^\lambda = \partial^\mu(\varepsilon_{\mu\nu\rho\lambda} A^\nu \partial^\rho A^\lambda).$$

The Lagrangian describing the electromagnetic field is given by

$$\mathcal{L}_{\text{Maxwell}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{1.54}$$

1.3.1 Path-integral Formalism

We define the gauge fixing function

$$G(A) = \partial_\mu A^\mu(x) - \omega(x) = 0$$

The gauge transformation has the form:

$$A^\alpha_\mu(x) = A_\mu(x) + \partial_\mu \alpha(x).$$

We then have

$$1 \propto \int D[\alpha] \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)\delta(G(A)).$$

Inset the identity operator into the path integral formula

$$Z[J] \propto \det\left(\partial^2\right) \int D[\alpha]D[A]e^{iS[A,J]}\delta(\partial_\mu A^\mu - \omega(x)).$$

The above equation does not depend on $\omega(x)$. We can then integrate over $\omega(x)$ with gaussian weight

$$\begin{aligned} Z[J] &\propto \int D[\omega]e^{-i\int d^dx \frac{\omega^2}{2\xi}} \int D[\alpha]D[A]e^{iS[A,J]}\delta(\partial_\mu A^\mu - \omega) \\ &= \int D[A]e^{iS[A,J]} \exp\left\{i\left[S[A,J] - \int d^dx \frac{1}{2\xi}(\partial_\mu A^\mu)^2\right]\right\}. \end{aligned}$$

In momentum space, the modified Langriangian is

$$\tilde{\mathcal{L}}_\xi(k) = \tilde{A}^\mu(k)\left[-k^2 g_{\mu\nu} + \left(1 - \frac{1}{\xi}\right)k_\mu k_\nu\right]\tilde{A}^\nu(-k) + \tilde{J}_\mu(k)\tilde{A}^\mu(-k) + \tilde{A}^\mu(k)\tilde{J}_\mu(-k).$$

We can check that

$$\left[-k^2 g_{\mu\nu} + \left(1 - \frac{1}{\xi}\right)k_\mu k_\nu\right]^{-1} = \frac{-g^{\mu\nu} + (1-\xi)k^\mu k^\nu}{k^2}. \tag{1.55}$$

Thus, the partition function is

$$\frac{Z_{\text{maxwell}}[J]}{Z_{\text{maxwell}}[0]} = \exp\left[-\frac{i}{2}\int d^dx_1 d^dx_2 J_\mu(x_1)\Pi^{\mu\nu}(x_1 - x_2)J_\nu(x_2)\right], \tag{1.56}$$

where

$$\Pi^{\mu\nu}(x_1 - x_2) = \int \frac{d^dk}{(2\pi)^d} e^{-ik\cdot(x_1 - x_2)} \frac{-g^{\mu\nu} + (1-\xi)k^\mu k^\nu}{k^2}. \tag{1.57}$$

The propagator is

$$\begin{aligned} \langle 0|TA^\mu(x_1)A^\nu(x_2)|0\rangle &= \frac{1}{Z_{\text{Maxwell}}[0]} \frac{\delta}{iJ_\mu(x_1)} \frac{\delta}{iJ_\nu(x_2)} Z_{\text{Maxwell}}[J]\bigg|_{J=0} \\ &= i\Pi^{\mu\nu}(x_1 - x_2). \end{aligned} \tag{1.58}$$

1.3.2 Canonical Quantization

In momentum space, the Lagrangian transforms to

$$\tilde{A}^{\mu}(k)\left(-k^2 g_{\mu\nu} + k_\mu k_\nu\right)\tilde{A}^{\nu}(-k). \tag{1.59}$$

The EOM in momentum space is

$$(-k^2 g_{\mu\nu} + k_\mu k_\nu)\tilde{A}^{\nu}(k) = 0.$$

Since the linear operator $(-k^2 g_{\mu\nu} + k_\mu k_\nu)$ is singular, i.e.,

$$(-k^2 g_{\mu\nu} + k_\mu k_\nu)k^{\nu} = 0.$$

The gauge freedom can be used to further restrict

$$A^0 = 0.$$

In this way, there are only two independent polarization for EOM solution

$$A^{\mu} = e^{-ik\cdot x}\epsilon_j^{\mu},\ j = 1, 2, \tag{1.60}$$

where

$$\epsilon_1 = (0, 1, 0, 0),\ \epsilon_2 = (0, 0, 1, 0).$$

The field expansion is then

$$A^{\mu} = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\sum_{j=1}^{2}\left(\epsilon_j^{\mu} a_{k,j}e^{-ik\cdot x} + \epsilon_j^{\mu *}a_{k,j}^{\dagger}e^{ik\cdot x}\right). \tag{1.61}$$

A single-particle state with polarization vector ϵ_j is defined as

$$|k, \epsilon_j\rangle = \sqrt{2\omega_k}\vec{\epsilon}_j a_{k,j}^{\dagger}|0\rangle. \tag{1.62}$$

Note that then the field is off shell (internal photon line), the photon can be space-like or time-like, and then there are an additional polarization. In general,

$$\sum_{j=1}^{3}\epsilon_j^{\mu *}\epsilon_j^{\nu} = -(1 - P_k) = -(g^{\mu\nu} - k^{\mu}k^{\nu}),$$

where P_k is the projection to 4-momentum k. The propagator is then

$$i\Pi(x_1 - x_2) = \int \frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x_1 - x_2)}\frac{-i(g^{\mu\nu} - k^{\mu}k^{\nu})}{k^2 + i\epsilon}. \tag{1.63}$$

1.4 Dirac Field

Based on previous discussion, the Lagrangian for spinor field can have

$$\psi_L^{\dagger}\bar{\sigma}^{\mu}\partial_\mu\psi_L,\ \psi_R^{\dagger}\sigma^{\mu}\partial_\mu\psi_R,\ \psi_L^{\dagger}\psi_R,\ \psi_R^{\dagger}\psi_L,\ \psi_L\cdot\psi_L,\ \psi_R\cdot\psi_R.$$

The Dirac field describe the theory with both left-hand and right-hand spinors. The Lagrangian is

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi, \tag{1.64}$$

where

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \ \bar{\psi} = \begin{pmatrix} \psi_R^\dagger & \psi_L^\dagger \end{pmatrix}, \ \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}. \tag{1.65}$$

In addition, we could consider using the last two terms as the mass, the result theory is the *Majorana field theory*:

$$\begin{aligned} \mathcal{L}^L_{\text{Majorana}} &= \psi_L^\dagger\left(i\bar{\sigma}^\mu \partial_\mu - m\sigma^2\right)\psi_L, \\ \mathcal{L}^R_{\text{Majorana}} &= \psi_R^\dagger\left(i\sigma^\mu \partial_\mu - m\sigma^2\right)\psi_R. \end{aligned} \tag{1.66}$$

1.4.1 Path-integral Formalism

Consider the partition function with source

$$Z_{\text{Dirac}}[J] = \int D[\bar{\psi}, \psi] \exp\left[i \int d^d x \left(\mathcal{L}_{\text{Dirac}} + \bar{\eta}\psi + \bar{\psi}\eta\right)\right]. \tag{1.67}$$

In momentum space:

$$S = \int \frac{d^d k}{(2\pi)^d}\left[\tilde{\bar{\psi}}(k)(\not{k} - m)\tilde{\psi}(k) + \tilde{\bar{\eta}}(k)\tilde{\psi}(k) + \tilde{\bar{\psi}}(k)\tilde{\eta}(k)\right]. \tag{1.68}$$

Using the Gaussian integral formula (for Grassman variables), the partition function is:

$$\begin{aligned} \frac{Z_{\text{Dirac}}[J]}{Z_{\text{Dirac}}[0]} &= \exp\left[-i\int \frac{d^d k}{(2\pi)^d}\tilde{\bar{\eta}}(k)\frac{1}{\not{k} - m}\tilde{\eta}(k)\right] \\ &= \exp\left[-i\int d^d x_1 d^d x_2 \bar{\eta}(x_1)\cdot D_F(x_1 - x_2)\cdot \eta(x_2)\right] \end{aligned} \tag{1.69}$$

where

$$D_F(x_1 - x_2) = \int \frac{d^d k}{(2\pi)^d}\frac{e^{-ik\cdot(x_1 - x_2)}}{\not{k} - m} = \int \frac{d^d k}{(2\pi)^d}\frac{\not{k} + m}{k^2 - m^2}e^{-ik\cdot(x_1 - x_2)}. \tag{1.70}$$

Note that the propagator is

$$\begin{aligned} \langle 0|T\psi^\alpha(x_1)\bar{\psi}^\beta(x_2)|0\rangle &= \frac{1}{Z_{\text{Dirac}}[0]}\frac{\delta}{i\delta\bar{\eta}_\alpha(x_1))}\frac{i\delta}{\delta\eta_\beta(x_2)}Z_{\text{Dirac}}[\bar{\eta}, \eta]\bigg|_{\eta=\bar{\eta}=0} \\ &= iD_F^{\alpha\beta}(x_1 - x_2), \end{aligned} \tag{1.71}$$

where the sign in the variational derivative comes from the anti-commutation relation of the fermionic fields.

1.4.2 Canonical Quantization

In momentum space, the Lagrangian ie=s:

$$\bar{\tilde{\psi}}(p)(\not{p} - m)\tilde{\psi}(p),$$

The EOM is

$$(\not{p} - m)\tilde{\psi}(p) = 0 \tag{1.72}$$

The general solution of the Dirac equation can be written as a linear combination of plane waves. The positive frequency waves are of the form

$$\psi(x) = u(p)e^{-ip\cdot x}, \quad p^2 = m^2, \quad p^0 > 0$$

There are two linearly independent solutions for $u(p)$,

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\xi^s \end{pmatrix}, \quad s = 1, 2$$

which we normalize according to

$$\bar{u}^r(p)u^s(p) = 2m\delta^{rs} \quad \text{or} \quad u^{r\dagger}(p)u^s(p) = 2\omega_{\boldsymbol{p}}\delta^{rs}$$

In exactly the same way, we can find the negative-frequency solutions:

$$\psi(x) = v(p)e^{+ip\cdot x}, \quad p^2 = m^2, \quad p^0 > 0. \ (3.61)$$

Note that we have chosen to put the + sign into the exponential, rather than having $p^0 < 0$. There are two linearly independent solutions for $v(p)$,

$$v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\eta^s \end{pmatrix}, \quad s = 1, 2$$

where η^s is another basis of two-component spinors. These solutions are normalized according to

$$\bar{v}^r(p)v^s(p) = -2m\delta^{rs} \quad \text{or} \quad v^{r\dagger}(p)v^s(p) = +2\omega_{\boldsymbol{p}}\delta^{rs}$$

The u's and v's are also orthogonal to each other:

$$\begin{aligned} \bar{u}^r(p)v^s(p) = 0, \quad u^{r\dagger}(\boldsymbol{p}, \omega_{\boldsymbol{p}})v^s(-\boldsymbol{p}, \omega_{\boldsymbol{p}}) = 0, \\ \bar{v}^r(p)u^s(p) = 0, \quad v^{r\dagger}(\boldsymbol{p}, \omega_{\boldsymbol{p}})u^s(-\boldsymbol{p}, \omega_{\boldsymbol{p}}) = 0. \end{aligned}$$

A useful identity is

$$\begin{aligned} \sum_s u^s(p)\bar{u}^s(p) &= \not{p} + m, \\ \sum_s v^s(p)\bar{v}^s(p) &= \not{p} - m. \end{aligned}$$

The Dirac field expansion is

$$\begin{aligned} \psi(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\sum_s \left(a^s_{\mathbf{p}}u^s(p)e^{-ip\cdot x} + b^{s\dagger}_{\mathbf{p}}v^s(p)e^{ip\cdot x}\right), \\ \bar{\psi}(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\sum_s \left(b^s_{\mathbf{p}}\bar{v}^s(p)e^{-ip\cdot x} + a^{s\dagger}_{\mathbf{p}}\bar{u}^s(p)e^{ip\cdot x}\right). \end{aligned} \tag{1.73}$$

Now let us investigate the propagator

$$\begin{aligned} iD_{F,\alpha\beta}(x_1 - x_2) &= \langle 0|T\psi_\alpha(x_1)\bar{\psi}_\beta(x_2)|0\rangle \\ &= \theta(\tau)\langle 0|\psi_\alpha(x_1)\bar{\psi}_\beta(x_2)|0\rangle - \theta(-\tau)\langle 0|\bar{\psi}_\beta(x_2)\psi_\alpha(x_1)|0\rangle. \end{aligned} \tag{1.74}$$

On the RHS, the first term is

$$\begin{aligned} \langle 0|\psi_\alpha(x_1)\bar{\psi}_\beta(x_2)|0\rangle &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left[\sum_s u^s_\alpha(p)\bar{u}^s_\beta(p)\right] e^{-ip\cdot(x_1-x_2)} \\ &= (i\partial\!\!\!/ + m)_{\alpha\beta}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}} e^{-ip\cdot(x_1-x_2)}. \end{aligned}$$

For the second term:

$$\begin{aligned} \langle 0|\bar{\psi}_\beta(x_2)\psi_\alpha(x_1)|0\rangle &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left[\sum_s \bar{v}^s_\beta(p)v^s_\alpha(p)\right] e^{ip\cdot(x_1-x_2)} \\ &= -(i\partial\!\!\!/ + m)_{\alpha\beta}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}} e^{ip\cdot(x_1-x_2)}. \end{aligned}$$

Together, the Dirac propagator is:

$$\begin{aligned} iD_F(x_1 - x_2) &= (i\partial\!\!\!/ + m)i\Delta(x_1 - x_2) \\ &= \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x_1-x_2)}\frac{i(p\!\!\!/ + m)}{p^2 - m^2 + i\epsilon}. \end{aligned}$$

1.5 Symmetries

1.5.1 Global Symmetries and Conserved Quantities

If a field theory has a global symmetry, it means that under the infinitesimal transformation:

$$\begin{aligned} x^\mu &\to x^\mu + \delta x^\mu, \\ \phi_r(x) &\to \phi_r(x) + \delta\phi_r(x), \end{aligned}$$

the action is invariant, i.e.,

$$\int_{\Omega'} d^4x'\mathcal{L}'(x') = \int_\Omega d^4x\mathcal{L}(x). \tag{1.75}$$

Here, instead of writing the infinite integral, we require that for any space-time region Ω (which is transformed to Ω' under symmetry transformation), the Lagrangian integral is invariant under the symmetry action.

Chapter 2

Scalar Field Theory

2.1 Interaction and Scattering

2.1.1 Lehmann Representation

The interacting Hamiltonian do not conserve particle number, and the ground state $|\Omega\rangle$ is no longer the vacuum $|0\rangle$. Consider the Green's function

$$iG(x_1 - x_2) = \langle\Omega|T\phi(x_1)\phi(x_2)|\Omega\rangle$$

We can insert a complete basis

$$1 = |\Omega\rangle\langle\Omega| + \sum_\lambda \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}|\lambda_{\vec{k}}\rangle\langle\lambda_{\vec{k}}|$$

into the correlation function,[1] the Green's function takes the form (take K-G field as the example):

$$iG(x_1 - x_2) = \sum_\lambda \int \frac{d^3k}{(2\pi)^3}\left[\theta(t_1 - t_2)\langle\Omega|\phi(x_1)|\lambda_{\vec{k}}\rangle\langle\lambda_{\vec{k}}|\phi(x_2)|\Omega\rangle + (t_1 \leftrightarrow t_2, x_1 \leftrightarrow x_2)\right].$$

Note that

$$\langle\lambda_{\vec{k}}|\phi(x)|\Omega\rangle = \langle\lambda_{\vec{k}}|e^{iP\cdot x}\phi(0)e^{-iP\cdot x}|\Omega\rangle = e^{ik\cdot x}\left.\langle\lambda_0|\phi(0)|\Omega\rangle\right|_{k^0=\omega_{\vec{k}}}.$$

Following the same procedure as we do for the free field theory,

$$G(x_1 - x_2) = \int_0^\infty \frac{dM^2}{2\pi}\rho(M^2)G_0(x_1 - x_2; M^2), \tag{2.1}$$

where the spectral function $\rho(M^2)$ is

$$\rho(M^2) = \sum_\lambda (2\pi)\delta(M^2 - m_\lambda^2)|\langle\Omega|\phi(0)|\lambda_0\rangle|^2.$$

[1] Here we assume $\langle\Omega|\phi(x)|\Omega\rangle = 0$ unless there is spontaneously symmetry breaking happening.

In particle, near the one-particle state the Green's function looks like:

$$i\tilde{G}(k) = \frac{iZ_\phi}{k^2 - m^2 + i\epsilon} + \text{regular terms.}$$

If we renormalize the field strength

$$\phi_R(x) = \frac{1}{\sqrt{Z_\phi}}\phi_0(x),$$

the Green's function then has exactly the same form as free theory. This normalization factor Z_ϕ is exactly what we obtained in the loop correction to the propagator.

2.1.2 Scattering Amplitude

Consider the scattering process in the interaction picture,

$$\begin{aligned}\langle f|e^{-iHt}|i\rangle &= \langle f|T\exp\left(-i\int dt V_{\text{int}}(t)\right)|i\rangle \\ &= \langle f|T\exp\left(i\int d^dx\mathcal{L}_{\text{int}}(t)\right)|i\rangle.\end{aligned} \tag{2.2}$$

The S-matrix is defined as

$$\langle f|S|i\rangle = \langle f|\mathcal{T}\exp\left(i\int d^dx\mathcal{L}_{\text{int}}(t)\right)|i\rangle = 1 + i\langle f|\mathcal{T}|i\rangle. \tag{2.3}$$

Because of the additional momentum conservation,

$$\langle f|\mathcal{T}|i\rangle = (2\pi)^d\delta^d\left(\sum p\right)\mathcal{M}_{fi}. \tag{2.4}$$

2.1.3 LSZ for Klein-Gordon Field

For free theory, the particle annihilation operator is

$$\begin{aligned}\sqrt{2\omega_k}a_k &= i\int d^3x\ e^{ik\cdot x}(-i\omega_k + \partial_t)\phi(x), \\ \sqrt{2\omega_k}a_k^\dagger &= -i\int d^3x\ e^{-ik\cdot x}(i\omega_k + \partial_t)\phi(x).\end{aligned} \tag{2.5}$$

When interaction is turned on, the field operator $\phi(x)$ is renormalized as

$$\phi(x) \sim \sqrt{Z_\phi}\phi_{\text{in}}(x) \sim \sqrt{Z_\phi}\phi_{\text{out}}(x).$$

In this way, we have

$$\begin{aligned}\sqrt{2\omega_k}(a_{\text{in}}^\dagger - a_{\text{out}}^\dagger) &= iZ_\phi^{-1/2}\int dt\partial_t\left(\int d^3x\ e^{-ikx}(i\omega_k + \partial_t)\phi_0(x)\right) \\ &= iZ_\phi^{-1/2}\int d^4xe^{-ik\cdot x}(\omega_k^2 + \partial_t^2)\phi_0(x) \\ &= iZ_\phi^{-1/2}\int d^4xe^{-ik\cdot x}\partial_t^2\phi_0(x) + \phi_0(x)(-\nabla^2 + m^2)e^{-ik\cdot x} \\ &= iZ_\phi^{-1/2}\int d^4xe^{-ik\cdot x}(\partial^2 + m^2)\phi_0(x)\end{aligned}$$

The initial and final states are:

$$\begin{aligned} |k_1, \cdots, k_m; \text{in}\rangle &= \left[\prod_{j=1}^{m} \sqrt{2\omega_{k_j}} a_{\text{in}}^\dagger(k_j)\right] |0\rangle, \\ |p_1, \cdots, p_n, \text{out}\rangle &= \left[\prod_{j=1}^{n} \sqrt{2\omega_{p_j}} a_{\text{out}}^\dagger(p_j)\right] |0\rangle. \end{aligned} \tag{2.6}$$

The S-matrix is

$$\begin{aligned} S_{fi} &= \langle p_1, \cdots, p_n; \text{out}|S|k_1, \cdots, k_m; \text{in}\rangle \\ &= \frac{\langle 0|T\left(\prod \sqrt{2\omega_{p_j}} a_{p_j;\text{out}}\right) \int d^4x \exp(i\mathcal{L}_{\text{int}}) \left(\prod \sqrt{2\omega_{k_j}} a_{k_j;\text{in}}^\dagger\right)|0\rangle}{\langle 0|T \int d^4x \exp(i\mathcal{L}_{\text{int}})|0\rangle} \end{aligned}$$

Since the scattering process correspond to the connected diagram, meaning that the initial and final state has distinct momentum particles. We are free to make the substitution

$$a_{\text{in}}^\dagger \to (a_{\text{in}}^\dagger - a_{\text{out}}^\dagger),\ a_{\text{out}} \to -(a_{\text{in}}^\dagger - a_{\text{out}}^\dagger)^\dagger.$$

In this way, the S-matrix is

$$\begin{aligned} &\langle p_1, \cdots, p_n|S|k_1, \cdots, k_m\rangle \\ =&\prod_{i=1}^{m}\left[\int d^dx_i\ e^{ip_i\cdot x_i}\frac{-\partial^2 - m_i^2}{i\sqrt{Z_\phi}}\right]\prod_{j=m+1}^{m+n}\left[\int d^dx_j\ e^{-ik_j\cdot x_j}\frac{-\partial^2 - m_j^2}{i\sqrt{Z_\phi}}\right] \\ &\times \frac{\langle 0|T\phi_0(x_1)\cdots\phi_0(x_{m+n})\int d^dx \exp(i\mathcal{L}_{\text{int}})|0\rangle}{\langle 0|T\int d^dx \exp(i\mathcal{L}_{\text{int}})|0\rangle} \\ =&\prod_{i=1}^{m}\left[\frac{p_i^2 - m_i^2}{i\sqrt{Z_\phi}}\right]\prod_{j=m+1}^{m+n}\left[\frac{k_j^2 - m_j^2}{i\sqrt{Z_\phi}}\right]\tilde{G}(-p_1, \cdots, -p_n, k_1, \cdots, k_m). \end{aligned}$$

Note that in the second equality, we move the operator ∂^2 out of the time-ordering operator, which will actually create contact terms. The contact terms can be shown to have no contribution to the S-matrix. Also, the Green function is defined as

$$\begin{aligned} G(x_1, \cdots, x_{m+n}) &\equiv \langle\Omega|T\phi(x_1)\cdots\phi(x_{m+n})|\Omega\rangle \\ &= \frac{\langle 0|T\phi_0(x_1)\cdots\phi_0(x_{m+n})\int d^dx \exp(i\mathcal{L}_{\text{int}})|0\rangle}{\langle 0|T\int d^dx \exp(i\mathcal{L}_{\text{int}})|0\rangle}. \end{aligned}$$

The extra factor before the momentum-space Green's function effectively cancel out the external propagator. Thus the LSZ formula (2.1.3) means that the S-matrix is the amputated Green's function.

Remark 1. Contact Terms

We first consider the time-ordered two-point function:

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = \theta(t_1-t_2)\langle 0|\phi(x_1)\phi(x_2)|0\rangle - \theta(t_2-t_1)\langle 0|\phi(x_2)\phi(x_1)|0\rangle.$$

Take time derivative on both side:

$$\begin{aligned}\partial_{t_1}\langle 0|T\phi(x_1)\phi(x_2)|0\rangle &= \langle 0|T\partial_{t_1}\phi(x_1)\phi(x_2)|0\rangle + \delta(t_1-t_2)\langle 0|[\phi(x_1),\phi(x_2)]|0\rangle \\ &= \langle 0|T\partial_{t_1}\phi(x_1)\phi(x_2)|0\rangle.\end{aligned}$$

The second equality follows from the fact that x_1, x_2 is equal-time. Take the the time derivative once more:

$$\partial_{t_1}^2\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = \langle 0|T\partial_{t_1}^2\phi(x_1)\phi(x_2)|0\rangle + \delta(t_1-t_2)\langle 0|[\partial_{t_1}\phi(x_1),\phi(x_2)]|0\rangle.$$

The second term on the right hand side is the contact term. For free theory, $\partial_{t_1}\phi(x_1)$ is the canonical momentum, meaning that

$$[\phi(\vec{x}_1,t),\partial_t\phi(\vec{x}_1,t)] = i\hbar\delta^3(\vec{x}_1-\vec{x}_2).$$

In general, for n-point correlation,

$$\begin{aligned}\partial_{t_1}^2\langle 0|T\phi_{x_1}\cdots\phi_{x_n}|0\rangle =&\langle 0|T\partial_{t_1}^2\phi_{x_1}\cdots\phi_{x_n}|0\rangle \\ &- i\hbar\sum_j \delta^4(x_1-x_j)\langle 0|T\phi_{x_2}\cdots\cancel{\phi_{x_j}}\cdots\phi_{x_n}|0\rangle.\end{aligned}$$

In the LSZ formula, the contact term do not have any singularity. When the external legs approach to momentum shell, these regular terms vanishes, so the contact will not contribute to the S-matrix.

2.1.4 Perturbation Theory

For interaction theory, the partition function can be formally expressed as:

$$Z[J] = \exp\left(i\int d^dx\mathcal{L}_{\text{int}}\left[\frac{\delta}{i\delta J(x)}\right]\right)Z_0[J]. \tag{2.7}$$

The expectation values for a generic operator of the form $O(\phi)$ can be evaluated by the true partition function

$$\langle O(\phi)\rangle = \frac{1}{Z[0]}\,O\left[\frac{\delta}{i\delta J(x)}\right]Z[J]\bigg|_{J=0}. \tag{2.8}$$

The expression (2.8) can be expanded order by order using the Feynman diagram. Since the unconnected diagram can be absorbed into $Z[0]$, we only need to calculate the connected diagram.

The procedure of perturbative expansion with only connected diagrams can be formally represented by introducing the quantity

$$Z[J] = Z[0]\exp\left(iW[J]\right). \tag{2.9}$$

The perturbative expansion of $W[J]$ contain only the connected diagrams. Note that for the free theory,

$$\frac{Z_0[J]}{Z_0[0]} = \exp\left[-\frac{i}{2}\int d^dx_1 d^dx_2 J(x_1)\Delta(x_1-x_2)J(x_2)\right],$$

which means

$$W_0 = -\frac{1}{2}\int d^dx_1 d^dx_2 J(x_1)\Delta(x_1-x_2)J(x_2).$$

For the interaction theory, the expectation (2.8) can then be replaced by the connected expectation:

$$\langle O(\phi)\rangle_c \equiv i\, O\left[\frac{\delta}{i\delta J(x)}\right] W[J]\bigg|_{J=0}. \tag{2.10}$$

Consider the two-point connected correlation (propagator):

$$\begin{aligned} i\Delta(x_1-x_2) &= \langle \mathcal{T}\phi(x_1)\phi(x_2)\rangle_c \\ &= i\,\frac{\delta^2 W[J]}{i\delta J(x_1) i\delta J(x_2)}\bigg|_{J=0} \\ &= \frac{\delta^2 \ln Z[J]}{i\delta J(x_1) i\delta J(x_2)}\bigg|_{J=0} \\ &= \frac{1}{Z[0]}\,\frac{\delta^2 Z[J]}{i\delta J(x_1) i\delta J(x_2)}\bigg|_{J=0}, \end{aligned} \tag{2.11}$$

where we have used the fact that

$$\frac{\delta Z^n[J]}{\delta J(x_1)\cdots\delta J(x_n)} = 0,\ \ \forall n = 1 \bmod 2.$$

The result is the same as the original definition.

Further, we can consider the four-point connected correlation:

$$iV_4 \equiv \langle \mathcal{T}\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle_c$$

Following the same procedure,

$$\begin{aligned} iV_4 &= i\,\frac{\delta^4 W[J]}{i\delta J(x_1) i\delta J(x_2) i\delta J(x_3) i\delta J(x_4)}\bigg|_{J=0} \\ &= \frac{1}{Z[0]}\,\frac{\delta^4 Z[J]}{i\delta J(x_1) i\delta J(x_2) i\delta J(x_3) i\delta J(x_4)}\bigg|_{J=0} \\ &\quad - i\Delta(x_1-x_2)i\Delta(x_3-x_4) \\ &\quad - i\Delta(x_1-x_3)i\Delta(x_2-x_4) \\ &\quad - i\Delta(x_1-x_4)i\Delta(x_2-x_3). \end{aligned} \tag{2.12}$$

The connected correlation function automatically omit those disconnected components.

2.2 Real ϕ^3 Theory

Now consider the interaction theory with additional Lagrangian

$$\mathcal{L}_{\text{int}}[\phi] = \frac{g}{3!}\phi^3. \tag{2.13}$$

Note that the field ϕ has the mass dimension $[\frac{d-2}{2}]$. The critical dimension is $d = 6$ where the coupling constant g is dimensionless. In this section, we consider the real Klein-Gordon field with ϕ^3 interaction on 6-dimensional space-time.

For interaction theory, the renormalized Lagrangian has the form:

$$\begin{aligned} \mathcal{L} &= Z_\phi \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - Z_m\frac{m^2}{2}\phi^2 + Z_g\frac{g}{3!}\phi^3 \\ &= \mathcal{L}_0 + \mathcal{L}_{\text{int}} + \mathcal{L}_{\text{ct}}, \end{aligned} \tag{2.14}$$

where the counter terms are:

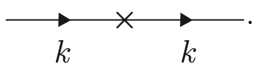
$$\begin{aligned} \mathcal{L}_{\text{ct}}[\phi] &= \frac{A}{2}\partial^\mu\phi\partial_\mu\phi - \frac{B}{2}m^2\phi^2 + \frac{C}{3!}g\phi^3 \\ &\simeq -\frac{A}{2}\phi\partial^2\phi - \frac{B}{2}m^2\phi^2 + \frac{C}{3!}g\phi^3, \end{aligned} \tag{2.15}$$

where

$$A = Z_\phi - 1, B = Z_m - 1, C = Z_g - 1.$$

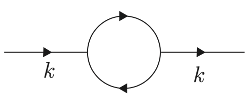
The counter term for the the free field gives additional corrrection

$$i\tilde{\Delta}^{(\text{ct})}(k) = i\tilde{\Delta}_0(k)(Ak^2 - Bm^2)i\tilde{\Delta}_0(k)$$

$=$  . (2.16)

2.2.1 Self Energy Correction

To second order, we consider the one-loop correction to the propagator:

$i\tilde{\Delta}^{(2)}(k) =$ 

$$= i\tilde{\Delta}_0(k)\left[i\Sigma^{(2)}(k^2)\right]i\tilde{\Delta}_0(k), \tag{2.17}$$

where the self energy term to the second order $i\Pi^{(2)}(k)$ is defined as:

$$i\Sigma^{(2)}(k^2) \equiv \frac{g^2}{2}\int\frac{d^dq}{(2\pi)^d}\tilde{\Delta}_0(q)\tilde{\Delta}_0(k-q) + (Ak^2 - Bm^2). \tag{2.18}$$

Remark 2. Symmetry Factor

The coefficient $g^2/2$ comes from the symmetry factor in the diagram. We can also check the coefficient explicitly, by considering the expansion to the second order (we denote $\delta/\delta J(x_i)$ as δ_i):

$$\delta_1\delta_2\frac{1}{2!4!}\left[\frac{ig}{3!}\int d^dy\left(\frac{\delta}{\delta J(y)}\right)^3\right]^2\left[-\frac{i}{2}\int d^dy_1d^dy_2J(y_1)\Delta(y_1-y_2)J(y_2)\right]^4.$$

The expansion gives the coefficient

$$\left(\frac{ig}{6}\right)^2\times\frac{1}{2!\times4!\times2^4}.$$

Now consider the combinatorial factor, which comes from the exchange of $\phi(x_i)$ in the propagator, the exchange of $\phi(x_i)$ in the vertex, the exchange of propagator in the diagram, and the change of vertices in the diagram:

$$(2!)^4\times(3!)^2\times(4\times3)\times2.$$

Those two factors produce a $-g^2/2$ coefficient. Note that in the self energy expression (2.17), we put a i factor in front of each propagator, which absorbs the minus sign.

Once we obtain the self energy, the one-loop corrected propagator has the form:

$$\begin{aligned}i\tilde{\Delta}(k)&=i\tilde{\Delta}_0(k)+i\tilde{\Delta}_0(k)\left[\sum_{n=1}^{\infty}i\Sigma(k^2)\right]i\tilde{\Delta}_0(k)\\&=\frac{i}{\tilde{\Delta}_0^{-1}(k)-\Sigma(k^2)}\\&=\frac{i}{k^2-m^2-\Sigma(k^2)}.\end{aligned}\tag{2.19}$$

Now we are going to evaluate the divergent integral in the self energy expression, using the Feynman parameters:

$$\begin{aligned}&\int\frac{d^dq}{(2\pi)^d}\frac{1}{q^2-m^2}\frac{1}{(k-q)^2-m^2}\\&=\int\frac{d^dq}{(2\pi)^d}\int_0^1dx\frac{1}{\left[q^2-m^2+x\left((q-k)^2-q^2\right)\right]^2}\\&=\int_0^1dx\int\frac{d^dq}{(2\pi)^d}\frac{1}{\left[(q-kx)^2-D\right]^2},\end{aligned}$$

where $D=m^2-k^2x(1-x)$. Then we can shift $q\to q+kx$ leaving an integral that only depends on q^2. In this way,

$$\Sigma(k^2)=\int_0^1I(x)dx.$$

To evaluate the self-energy, it suffices to obtain the integral

$$I(x)=\frac{g^2}{2i}\int\frac{d^dq}{(2\pi)^d}\frac{1}{\left[q^2-D\right]^2}.$$

Remark 3. Feyman Parameters

We use Feynman's formula to combine denominators,

$$\frac{1}{A_1 \dots A_n} = \int dF_n \left(x_1 A_1 + \dots + x_n A_n\right)^{-n}, \tag{2.20}$$

where the integration measure over the Feynman parameters x_i is

$$\int dF_n = (n-1)! \int_0^1 dx_1 \dots dx_n \delta\left(x_1 + \dots + x_n - 1\right). \tag{2.21}$$

This measure is normalized so that $\int dF_n = 1$. The simplest case is

$$\frac{1}{AB} = \int_0^1 \frac{dx}{[A + (B-A)x]^2} = \int_0^1 \frac{\delta(x+y-1)}{[xA+yB]^2} dxdy. \tag{2.22}$$

Other useful identities are

$$\begin{aligned} \frac{1}{AB^n} &= \int_0^1 dxdy \frac{\delta(x+y-1) n y^{n-1}}{[xA+yB]^{n+1}}, \\ \frac{1}{ABC} &= \int_0^1 dxdydz \frac{2\delta(x+y+z-1)}{[xA+yB+zC]^3}. \end{aligned} \tag{2.23}$$

By making the Wick rotation $q^0 \to i q^0_E$, the integral becomes:[2]

$$I(x) = \frac{g}{2} \int \frac{d^d q_E}{(2\pi)^d} \frac{1}{\left(q_E^2 + D\right)^2} = \frac{g\Omega_d}{2(2\pi)^d} \int dq \frac{q^{d-1}}{\left(q^2 + D\right)^2}.$$

Dimensional Regularization

We set the dimension to $d = 6 - \epsilon$, and rewrite the Lagrangian as

$$\mathcal{L} = Z_\phi \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - Z_m \frac{m^2}{2} \phi^2 + Z_g \frac{g \tilde{\mu}^{\epsilon/2}}{3!} \phi^3. \tag{2.26}$$

Note that the coupling constant should be changed to $g \to g\tilde{\mu}^{\epsilon/2}$ where μ is of mass dimension [1] in order to get the correct dimensionality. We then expand the expression to zeroth order of ϵ. A useful identity is:

$$\int dk \frac{k^a}{\left(k^2 + D\right)^b} = D^{\frac{a+1}{2} - b} \frac{\Gamma\left(\frac{a+1}{2}\right) \Gamma\left(b - \frac{a+1}{2}\right)}{2\Gamma(b)}. \tag{2.27}$$

[2] The d-dimensional solid angle is

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}, \tag{2.24}$$

where $\Gamma(x)$ is the gamma function, satisfing

$$\Gamma(1+x) = x\Gamma(x), \;\; \Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + O(\epsilon). \tag{2.25}$$

In particular, $\Gamma(n+1) = n!$.

Actually, we can compute the integral and series expansion in `Mathematica` all together:

```
omg=(2*Pi^(d/2))/(Gamma[d/2]);
cof=g^2*\[Mu]^(6-d)/2*omg/(2*Pi)^d;
int=cof*Integrate[q^(d-1)/(q^2+D)^2,{q,0,Infinity}][[1]];
map={g^2->\[Alpha]*(4*Pi)^3,EulerGamma->Subscript[\[Gamma],E]};
ans=Series[int/.{d->6-\[Epsilon]},{\[Epsilon],0,0}];
ans/.map//Simplify
```

The result is (where $\alpha \equiv g^2/(4\pi)^3$)

$$I(x) = \frac{\alpha D}{2}\left[\ln\left(\frac{De^{\gamma_E}}{4\pi\tilde{\mu}^2}\right) - \left(\frac{2}{\epsilon}+1\right)\right] + O(\epsilon).$$

Now insert $D = m^2 - k^2x(1-x)$. Note that

$$\int_0^1 dxD = m^2 - \frac{k^2}{6}.$$

This simplifies the result to

$$\Sigma^{(2)}(k^2) = \frac{\alpha}{2}\left(\frac{2}{\epsilon}+1\right)\left(\frac{k^2}{2}-m^2\right) + \frac{\alpha}{2}\int_0^1 dxD(x)\ln\left(\frac{D(x)}{\mu^2}\right), \tag{2.28}$$

where we have replace $\tilde{\mu}$ with

$$\mu \equiv \sqrt{\frac{4\pi}{e^{\gamma_E}}}\tilde{\mu}. \tag{2.29}$$

Renormalization

The counter terms also contribute to the perturbative correction,

$$\begin{aligned}\Sigma^{(2)}\left(k^2\right) =& \frac{\alpha}{2}\int_0^1 dxD\ln\left(\frac{D}{m^2}\right) + \left\{\frac{\alpha}{6}\left[\frac{1}{\varepsilon}+\ln\left(\frac{\mu}{m}\right)+\frac{1}{2}\right]+A\right\}k^2 \\ &- \left\{\alpha\left[\frac{1}{\varepsilon}+\ln\left(\frac{\mu}{m}\right)+\frac{1}{2}\right]+B\right\}m^2 + O\left(\alpha^2\right).\end{aligned}$$

Consider the on-shell condition for the subtraction:

$$\Sigma(m^2) = \Sigma'(m^2) = 0. \tag{2.30}$$

Set $D_0 \equiv D(x)|_{k^2=m^2} = m^2(1-x+x^2)$, the self energy has the form:

$$\Sigma^{(2)}(k^2) = \frac{\alpha}{2}\int_0^1 dxD(x)\ln\left(\frac{D(x)}{D_0(x)}\right) + C_k k^2 + C_m m^2. \tag{2.31}$$

The condition $\Pi(m^2) = 0$ requires

$$\Sigma^{(2)}(k^2) = \frac{\alpha}{2}\int_0^1 dxD(x)\ln\left(\frac{D(x)}{D_0(x)}\right) + C_k(k^2 - m^2).$$

The condition $\Pi'(m^2) = 0$ requires

$$\begin{aligned}
\left.\frac{d\Sigma^{(2)}(k^2)}{dk^2}\right|_{k^2=m^2} &= \frac{\alpha}{2}\int_0^1 dx \left.\left[\frac{D(x)}{dk^2}\ln\left(\frac{D(x)}{D_0(x)}\right) + D_0(x)\right]\right|_{q^2=m^2} + C_k \\
&= \frac{\alpha}{2}\int_0^1 dx(x^2 - x) + C_k \\
&= C_k - \frac{\alpha}{12} = 0.
\end{aligned}$$

In this way, we obtained the renormalized self-energy:

$$\Sigma^{(2)}(k^2) = \frac{\alpha}{2}\int_0^1 dx D(x)\ln\left(\frac{D(x)}{D_0(x)}\right) + \frac{\alpha}{12}(k^2 - m^2). \tag{2.32}$$

On the other hand, we chan choose the $\overline{\text{MS}}$ subtraction scheme, i.e.,

$$A = -\frac{\alpha}{6\epsilon},\ B = -\frac{\alpha}{\epsilon}. \tag{2.33}$$

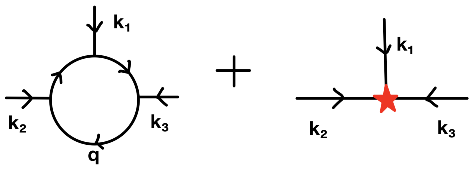
The self energy under $\overline{\text{MS}}$ scheme will depend on the the mass scale μ we choose:

$$\Sigma^{(2)}\left(k^2\right) = \frac{\alpha}{2}\int_0^1 dx D\ln\left(\frac{D}{m^2}\right) + \alpha\left[\ln\left(\frac{\mu}{m}\right) + \frac{1}{2}\right]\left(\frac{k^2}{6} - m^2\right). \tag{2.34}$$

2.2.2 Vertex Correction

Now consider the simplest one-loop correction to the vertex function (together with the counter term):

$$iV_3^{(3)}(k_1, k_2, k_3) =$$



$$= (ig)^3 i^3 \int \frac{d^a q}{(2\pi)^d}\tilde{\Delta}(q - k_1)\tilde{\Delta}(q + k_2)\tilde{\Delta}(q) + iCg, \tag{2.35}$$

Using the Feynman parameter, the integrant is

$$\tilde{\Delta}(q - k_1)\tilde{\Delta}(q + k_2)\tilde{\Delta}(q) = \int dF_3 \frac{1}{(q^2 - D)^3}$$

where we have shift the value of q, and D can be evaluate by the following code:

```
A1=(l−k1)^2−m^2;
A2=(l+k2)^2−m^2;
A3=(l)^2−m^2;
{c,b,a}=CoefficientList[x1*A1+x2*A2+(1−x1−x2)*A3,{l}];
−c+b^2/(4*a)//Expand
```

The result is

$$D = m^2 - k_1^2 x_1(1-x_1) - k_2^2 x_2(1-x_2) - 2k_1k_2x_1x_2.$$

The same procedure gives:

$$V_3^{(3)}/g = \int dF_3 I(x_1,x_2,x_3) + C, \tag{2.36}$$

where

$$I(x_1,x_2,x_3) = \frac{g^2\Omega_d}{(2\pi)^d}\int dq \frac{q^{d-1}}{(q^2+D)^3}.$$

The same regularization procedure in `Mathematica`:

```
omg=(2*Pi^(d/2))/(Gamma[d/2]);
cof=g^2*\[Mu]^(6-d)*omg/(2*Pi)^d;
int=cof*Integrate[q^(d-1)/(q^2+D)^3,{q,0,Infinity}][[1]];
map={g^2->\[Alpha]*(4*Pi)^3,EulerGamma->Subscript[\[Gamma],E]};
ans=Series[int/.{d->6-\[Epsilon]},{\[Epsilon], 0, 0}];
ans/.map//Simplify
```

The result is

$$\begin{aligned} V_3^{(3)}/g &= \frac{\alpha}{\epsilon} + \frac{\alpha}{2}\int dF_3 \ln\left(\frac{4\pi\tilde{\mu}^2 e^{-\gamma_E}}{D}\right) + C + O(\epsilon) \\ &= \frac{\alpha}{\epsilon} + \alpha\ln\left(\frac{\mu}{m}\right) - \frac{\alpha}{2}\int dF_3 \ln\left(\frac{D}{m}\right) + C. \end{aligned} \tag{2.37}$$

The on-shell subtraction requires

$$V_3(0,0,0) = g, \tag{2.38}$$

which gives

$$C = -\frac{\alpha}{\epsilon} - \alpha\ln\left(\frac{\mu}{m}\right). \tag{2.39}$$

So the vertex function to the third order is

$$V_3(k_1,k_2,k_3) = g\left\{1 - \frac{\alpha}{2}\int dF_3 \ln\left[\frac{D(x_1,x_2,x_3)}{m}\right]\right\}. \tag{2.40}$$

The $\overline{\text{MS}}$ scheme, on the other hand, sets

$$C = -\frac{\alpha}{\epsilon}. \tag{2.41}$$

2.2.3 Renormalization Group

We fist summarize the normalization factor obtained on the one-loop level (with $\overline{\text{MS}}$ subtraction scheme):

$$\begin{aligned} Z_\phi &= 1 - \frac{\alpha}{6\epsilon} + O(\alpha^2), \\ Z_m &= 1 - \frac{\alpha}{\epsilon} + O(\alpha^2), \\ Z_g &= 1 - \frac{\alpha}{\epsilon} + O(\alpha^2). \end{aligned} \tag{2.42}$$

For the renormalized Lagrangian in $(6-\epsilon)$-dimension

$$\mathcal{L} = Z_\phi \frac{1}{2}\partial^\mu \phi \partial_\mu \phi - Z_m \frac{m^2}{2}\phi^2 + Z_g \frac{g\tilde{\mu}^{\epsilon/2}}{3!}\phi^3, \tag{2.43}$$

the factors relate the original field and bare coefficients

$$\phi_0 = Z_\phi^{1/2}\phi,\ m_0 = Z_m^{1/2} Z_\phi^{-1/2} m,\ g_0 = Z_g Z_\phi^{-3/2}\tilde{\mu}^{\epsilon/2} g. \tag{2.44}$$

The renormalization group requires that the bare parameter is independent of the mass scale μ we choose, that is:

$$\frac{d\phi_0}{d\ln\mu} = \frac{dm_0}{d\ln\mu} = \frac{dg_0}{d\ln\mu} = 0. \tag{2.45}$$

Beta Function

Star with g_0, it is more convenient to use

$$\alpha_0 \equiv \frac{g_0^2}{4\pi} = Z_g^2 Z_\phi^{-3}\tilde{\mu}^\epsilon \alpha. \tag{2.46}$$

Take logarithm on both side:

$$\ln\alpha_0 = \ln(Z_g^2 Z_\phi^{-3}) + \ln\alpha + \epsilon\ln\tilde{\mu}. \tag{2.47}$$

The RG equation is

$$\frac{d\ln\alpha_0}{d\ln\mu} = \frac{d\ln(Z_g^2 Z_\phi^{-3})}{d\alpha}\frac{d\alpha}{d\ln\mu} + \frac{1}{\alpha}\frac{d\alpha}{d\ln\mu} + \epsilon = 0. \tag{2.48}$$

To the first order of α:

$$\frac{d\ln(Z_g^2 Z_\phi^{-3})}{d\alpha} = \frac{d}{d\alpha}\left(-\frac{2\alpha}{\epsilon} + \frac{\alpha}{2\epsilon}\right) = -\frac{3}{2\epsilon}, \tag{2.49}$$

which leads to

$$\frac{d\alpha}{d\ln\mu}\left(1 - \frac{3\alpha}{2\epsilon} + O(\alpha^2)\right) + \epsilon\alpha = 0. \tag{2.50}$$

The beta function is defined as

$$\beta(\alpha) = \frac{d\alpha}{d\ln\mu} = \beta_1\alpha + \beta_2\alpha^2 + O(\alpha^3). \tag{2.51}$$

Insert such definition into the original expression, and keep track of the order of α, we get

$$(\beta_1 + \epsilon)\alpha + \left(\beta_2 - \frac{3\beta_1}{2\epsilon}\right)\alpha^2 + O(\alpha^3) = 0. \tag{2.52}$$

The beta function is

$$\beta(\alpha) = -\epsilon\alpha - \frac{3}{2}\alpha^2 + O(\alpha^3). \tag{2.53}$$

Anomalous Dimension

Consider the RG equation with bare mass:

$$\begin{aligned}\frac{d\ln m_0}{d\ln\mu} &= \frac{1}{2}\frac{d(\ln Z_m - \ln Z_\phi)}{d\alpha}\frac{d\alpha}{d\ln\mu} + \frac{1}{m}\frac{dm}{d\ln\mu} \\ &= \frac{5\alpha}{12} + \frac{1}{m}\frac{dm}{d\ln\mu} + O(\alpha^2) = 0.\end{aligned} \tag{2.54}$$

We get the anomalous dimension of the mass:

$$\gamma_m(\alpha) \equiv \frac{1}{m}\frac{dm}{d\ln\mu} = -\frac{5\alpha}{12} + O(\alpha^2). \tag{2.55}$$

Also, for the bare field

$$\frac{d\ln\phi_0}{d\ln\mu} = \frac{1}{2}\frac{d\ln Z_\phi}{d\ln\mu} + \frac{d\ln\phi}{d\ln\mu} = 0. \tag{2.56}$$

We can define the anomalous dimension of the field as

$$\gamma_\phi \equiv \frac{1}{2}\frac{d\ln Z_\phi}{d\ln\mu} = \frac{1}{2}\frac{d\ln Z_\phi}{d\alpha}\frac{d\alpha}{d\ln\mu} = \frac{\alpha}{12} + O(\alpha^2). \tag{2.57}$$

Callan-Symanzik Equation

Consider the bare propagator:

$$\tilde{\Delta}_0(k) = Z_\phi\tilde{\Delta}(k) \tag{2.58}$$

The RG condition for the bare propagator gives:

$$\frac{d\ln\tilde{\Delta}_0(k)}{d\ln\mu} = \frac{d\ln Z_\phi}{d\ln\mu} + \frac{1}{\tilde{\Delta}(k)}\left(\frac{\partial}{\partial\ln\mu} + \frac{d\alpha}{d\ln\mu}\frac{\partial}{\partial\alpha} + \frac{dm}{d\ln\mu}\frac{\partial}{\partial m}\right)\tilde{\Delta}(k) = 0.$$

The Callan-Symanzik equation is

$$\left(2\gamma_\phi + \frac{\partial}{\partial\ln\mu} + \beta(\alpha)\frac{\partial}{\partial\alpha} + \gamma_m(\alpha)m\frac{\partial}{\partial m}\right)\tilde{\Delta}(k) = 0. \tag{2.59}$$

2.3 Real ϕ^4 Theory

In this section, we consider the real Klein-Gordon field with ϕ^4 interaction. The field ϕ has mass dimension $[\frac{d-2}{2}] = [1]$, so the critical dimension is $d = 4$, where the coupling constant g is dimensionless. For dimensional regulation purpose, we write the renormalized Lagrangian on $(4-\epsilon)$-dimensional space-time as

$$\mathcal{L} = Z_\phi\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - Z_m\frac{m^2}{2}\phi^2 - Z_g\frac{g\tilde{\mu}^\epsilon}{4!}\phi^4, \tag{2.60}$$

where we have introduced a mass scale $\tilde{\mu}$. As the ϕ^3 theory, we can rewrite the Lagrangian as:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}} + \mathcal{L}_{\text{ct}}. \tag{2.61}$$

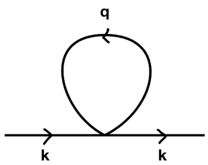
In the following we investigate the loop correction to the mass and the coupling constant.

2.3.1 One-loop Correction

Self-energy

Following the same procedure, the one-loop self-energy correction is (with counter terms):

$$i\Sigma(k^2) =$$



$$+$$

$$= -\frac{g\tilde{\mu}^\epsilon}{2}\int \frac{d^dq}{(2\pi)^d}\frac{1}{q^2-m^2} + i(Ak^2 - Bm^2). \tag{2.62}$$

After the Wick rotation,

$$\Sigma(k^2) = -\frac{g\tilde{\mu}^\epsilon}{2}\frac{\Omega_d}{(2\pi)^d}\int \frac{q^{d-1}dq}{q^2+m^2} + (Ak^2 - Bm^2). \tag{2.63}$$

The dimensional regulation is carried out using the following code:

```
omg=(2*Pi^(d/2))/(Gamma[d/2]);
cof=g*\[Mu]^(4-d)/2*omg/(2*Pi)^d;
int=cof*Integrate[q^(d-1)/(q^2+m^2),{q,0,Infinity}][[1]];
map={EulerGamma->Subscript[\[Gamma],E]};
ans=Series[int/.{d->4-\[Epsilon]},{\[Epsilon],0,0}];
ans/.map//Simplify
```

The result is

$$\Sigma(k^2) = \frac{gm^2}{32\pi^2}\left[\frac{2}{\epsilon} + 1 + \log\left(\frac{4\pi\tilde{\mu}^2 e^{-\gamma_E}}{m^2}\right)\right] + (Ak^2 - Bm^2) + O(\epsilon). \tag{2.64}$$

Using the $\overline{\mathrm{MS}}$ renormalization scheme, we set

$$A = 0,\ \ B = \frac{g}{16\pi^2\epsilon}. \tag{2.65}$$

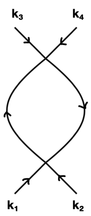
The result is

$$\Sigma(k^2) = \frac{gm^2}{16\pi^2}\log\left(\frac{\mu}{m}\right) + \frac{gm^2}{32\pi^2} + O(\epsilon). \tag{2.66}$$

Vertex Correction

Now consider the vertex correction. The vertex correction to the lowest order (with the counter term) is

$$iV_4^{(2)}(k_1,k_2,k_3,k_4) =$$



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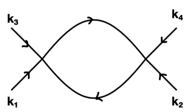

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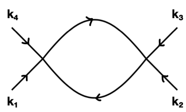

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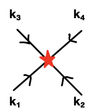


$$= \frac{g^2}{2}\left[iF(s) + iF(t) + iF(u)\right] - iCg, \tag{2.67}$$

where

$$s = (k_1 + k_2)^2, \ t = (k_1 + k_3)^2, \ u = (k_1 + k_4)^2, \tag{2.68}$$

and

$$\begin{aligned} iF(k^2) &= \tilde{\mu}^\epsilon \int \frac{d^d q}{(2\pi)^d} \tilde{\Delta}_0(q) \tilde{\Delta}_0(q+k) && (2.69) \\ &= \frac{i\tilde{\mu}^\epsilon \Omega_d}{(2\pi)^d} \int_0^1 dx \int \frac{q^{d-1} dq}{[q^2 + m^2 + x(1-x)k^2]^2}. && (2.70) \end{aligned}$$

Then we carry out the calculation (set $D(k^2, x) = m^2 + x(1-x)k^2$)

```
omg=(2*Pi^(d/2))/(Gamma[d/2]);
cof=g^2*\[Mu]^(4-d)/2*omg/(2*Pi)^d;
int=cof*Integrate[q^(d-1)/(q^2+D)^2,{q,0,Infinity}][[1]];
map={EulerGamma->Subscript[\[Gamma],E]};
ans=Series[int/.{d->4-\[Epsilon]},{\[Epsilon],0,0}];
ans/.map//Simplify
```

The result is:

$$\begin{aligned} F(s) &= \frac{1}{8\pi^2 \epsilon} + \frac{1}{16\pi^2} \int_0^1 dx \ln \left(\frac{4\pi \tilde{\mu}^2 e^{-\gamma_E}}{D} \right) \\ &= \frac{1}{8\pi^2 \epsilon} + \frac{1}{8\pi^2} \ln \left(\frac{\mu}{m} \right) - \frac{1}{16\pi^2} \int_0^1 dx \ln \left(\frac{D(s,x)}{m^2} \right). \end{aligned} \tag{2.71}$$

The $\overline{\text{MS}}$ scheme absorbs the $\frac{1}{8\pi^2\epsilon}$ term, i.e.,

$$C = \frac{3g}{16\pi^2}. \tag{2.72}$$

The result is:

$$V_4(k_1, k_2, k_3, k_4) = -g + \frac{g^2}{32\pi^2} \int_0^1 dx \ln \left(\frac{\mu^6}{D(s,x)D(t,x)D(u,x)} \right). \tag{2.73}$$

To summarize, the normalization is:

$$\begin{aligned} Z_\phi &= 1, && (2.74) \\ Z_m &= 1 + \frac{g}{16\pi^2 \epsilon}, && (2.75) \\ Z_g &= 1 + \frac{3g}{16\pi^2 \epsilon}. && (2.76) \end{aligned}$$

2.3.2 Renormalization Group

Now consider the RG equation for the one-loop correction. The bare parameters are:

$$g_0 = Z_g g \tilde{\mu}^\epsilon, \ m_0 = Z_m^{1/2} m, \tag{2.77}$$

The RG conditions are:

$$\frac{dg_0}{d\ln\mu} = \left(\frac{3}{16\pi^2\epsilon} + \frac{1}{g}\right)\frac{dg}{d\ln\mu} + \epsilon = 0, \tag{2.78}$$

$$\frac{dm_0}{d\ln\mu} = \frac{1}{32\pi^2\epsilon}\frac{dg}{d\ln\mu} + \frac{1}{m}\frac{dm}{d\ln\mu} = 0. \tag{2.79}$$

Consider the series expansion of beta function:

$$\beta(g) = \frac{dg}{d\ln\mu} = \beta_1 g + \beta_2 g^2 + O(g^3). \tag{2.80}$$

The beta function is

$$\beta(g) = -\epsilon g + \frac{3g^2}{16\pi^2} + O(g^3). \tag{2.81}$$

The anomalous dimension of mass is

$$\gamma_m = \frac{1}{m}\frac{dm}{d\ln\mu} = \frac{g}{32\pi^2} + O(g^2) \tag{2.82}$$

Chapter 3

Quantum Electrodynamics

The Lagrangian for quantum electrodynamics is

$$\begin{aligned}\mathcal{L}_{\text{QED}} &= \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - eA_\mu\bar{\psi}\gamma^\mu\psi \\ &= \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{Maxwell}} + \mathcal{L}_{\text{int}},\end{aligned} \tag{3.1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\nu = (dA)_{\mu\nu}. \tag{3.2}$$

The Lagrangian is invariant under the gauge transformation:

$$\begin{aligned}\psi(x) &\to e^{-ie\alpha(x)}\psi(x), \\ A^\mu(x) &\to A^\mu(x) + \partial^\mu\alpha(x).\end{aligned} \tag{3.3}$$

It is convenient to rewrite Lagrangian as

$$\mathcal{L}_{\text{QED}} = \bar{\psi}\left(i\not{D} - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{3.4}$$

where we have define the covariant derivative as:

$$\not{D} = \gamma^\mu D_\mu = \gamma^\mu[\partial_\mu + ieA_\mu(x)] = \not{\partial} + ie\not{A}. \tag{3.5}$$

3.1 Perturbation Theory

3.1.1 LSZ for Dirac Field

Use the field expansion

$$\begin{aligned}\psi(x) &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\sum_s\left(a^s_{\mathbf{p}}u^s(p)e^{-ip\cdot x} + b^{s\dagger}_{\mathbf{p}}v^s(p)e^{ip\cdot x}\right), \\ \bar{\psi}(x) &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\sum_s\left(b^s_{\mathbf{p}}\bar{v}^s(p)e^{-ip\cdot x} + a^{s\dagger}_{\mathbf{p}}\bar{u}^s(p)e^{ip\cdot x}\right),\end{aligned}$$

and the orthogonality relation

$$\begin{aligned} u^{r\dagger}(p)u^s(p) = 2\omega_{\boldsymbol{p}}\delta^{rs}, \quad u^{r\dagger}(\boldsymbol{p},\omega_{\boldsymbol{p}})v^s(-\boldsymbol{p},\omega_{\boldsymbol{p}}) = 0, \\ v^{r\dagger}(p)v^s(p) = 2\omega_{\boldsymbol{p}}\delta^{rs}, \quad v^{r\dagger}(\boldsymbol{p},\omega_{\boldsymbol{p}})u^s(-\boldsymbol{p},\omega_{\boldsymbol{p}}) = 0. \end{aligned}$$

The spatial Fourier transformation gives:

$$\int d^3x e^{ip\cdot x}\psi(x) = \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\sum_s a^s_{\boldsymbol{p}} u^s(p) + \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\sum_s b^{s\dagger}_{\boldsymbol{p}} v^s(-\boldsymbol{p},\omega)e^{2i\omega t}$$

Left-multiply on both hand side by $\bar{u}^s(p)\gamma^0$, we then get

$$\begin{aligned} \sqrt{2\omega_{\boldsymbol{p}}}a^s_{\boldsymbol{p}} &= \int d^3x e^{ip\cdot x}\bar{u}^s(p)\gamma^0\psi(x), \\ \sqrt{2\omega_{\boldsymbol{p}}}a^{s\dagger}_{\boldsymbol{p}} &= \int d^3x e^{-ip\cdot x}\bar{\psi}(x)\gamma^0 u^s(p). \end{aligned}$$

Similarly, we consider

$$\int d^3x e^{ip\cdot x}\bar{\psi}(x) = \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\sum_s b^s_{\boldsymbol{p}} \bar{v}^s(p) + \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\sum_s a^{s\dagger}_{\boldsymbol{p}} \bar{u}^s(-\boldsymbol{p},\omega)e^{2i\omega t}$$

Right-multiply on both hand side by $\gamma^0 v^s(p)$, we then get

$$\begin{aligned} \sqrt{2\omega_{\boldsymbol{p}}}b^s_{\boldsymbol{p}} &= \int d^3x e^{ip\cdot x}\bar{\psi}(x)\gamma^0 v^s(p), \\ \sqrt{2\omega_{\boldsymbol{p}}}b^{s\dagger}_{\boldsymbol{p}} &= \int d^3x e^{-ip\cdot x}\bar{v}^s(p)\gamma^0\psi(x). \end{aligned}$$

Following the same strategy as we did for the scalar field, we consider

$$\begin{aligned} \sqrt{2\omega_{\boldsymbol{p}}}a^s_{\boldsymbol{p};\text{out}} - \sqrt{2\omega_{\boldsymbol{p}}}a^s_{\boldsymbol{p};\text{in}} &= \int dt\ \partial_t\sqrt{2\omega_{\boldsymbol{p}}}a^s_{\boldsymbol{p}} \\ &= \int dt\ \int d^3x e^{ip\cdot x}\bar{u}(p)(\gamma^0\partial_t + i\gamma^0 p^0)\psi(x) \\ &= \int d^4x e^{ip\cdot x}\bar{u}(p)(\gamma^0\partial_t + i\gamma^i p^i + im)\psi(x) \\ &= i\int d^4x e^{ip\cdot x}\bar{u}(p)(-i\not{\partial} + m)\psi(x) \end{aligned}$$

where we have used the fact $\bar{u}(p)(\not{p} - m) = 0$. Take hermitian conjugate,

$$\begin{aligned} \sqrt{2\omega_{\boldsymbol{p}}}a^{s\dagger}_{\boldsymbol{p};\text{in}} - \sqrt{2\omega_{\boldsymbol{p}}}a^{s\dagger}_{\boldsymbol{p};\text{out}} &= i\int d^4x e^{-ip\cdot x}\bar{\psi}(x)\gamma^0(-i\not{\partial} + m)^\dagger\gamma^0 u(p) \\ &= i\int d^4x e^{-ip\cdot x}\bar{\psi}(x)(i\overleftarrow{\not{\partial}} + m)u(p) \end{aligned}$$

Similarly, using the fact $(\not{p} + m)v(p) = 0$,

$$\begin{aligned} \sqrt{2\omega_{\boldsymbol{p}}}b^s_{\boldsymbol{p};\text{out}} - \sqrt{2\omega_{\boldsymbol{p}}}b^s_{\boldsymbol{p};\text{in}} &= \int d^4x e^{ip\cdot x}\bar{\psi}(x)(\gamma^0\overleftarrow{\partial_t} + i\gamma^0 p^0)v(p) \\ &= \int d^4x e^{ip\cdot x}\bar{\psi}(x)(\gamma^0\overleftarrow{\partial_t} + i\gamma^i p^i - im)v(p) \\ &= -i\int d^4x e^{ip\cdot x}\bar{\psi}(x)(i\overleftarrow{\not{\partial}} + m)v(p). \end{aligned}$$

Again, take the hermitian conjugate,

$$
\begin{aligned}
\sqrt{2\omega_{\boldsymbol{p}}} b^{s\dagger}_{\boldsymbol{p};\text{in}} - \sqrt{2\omega_{\boldsymbol{p}}} b^{s\dagger}_{\boldsymbol{p};\text{out}} &= -i \int d^4x e^{ip\cdot x} \bar{v}(p) \gamma^0 (i\overleftarrow{\not\partial} + m)^\dagger \gamma^0 \psi(x) \\
&= -i \int d^4x e^{-ip\cdot x} \bar{v}(p)(-i\not\partial + m)\psi(x)
\end{aligned}
$$

The same strategy gives the LSZ reduction formula for Dirac field. Consider the S-matrix for particles:

$$
\begin{aligned}
&\langle p_1, \cdots, p_n | S | k_1, \cdots, k_m \rangle \\
=&\prod_{i=1}^{m} \left[\int d^dx_i \; e^{ip_i\cdot x_i} u^{s_1}(p_i) \frac{i\not\partial - m_i}{i\sqrt{Z_\phi}} \right] iG(x) \prod_{j=m+1}^{m+n} \left[\int d^dx_j \; e^{-ik_j\cdot x_j} \frac{-i\overleftarrow{\not\partial} - m_j}{i\sqrt{Z_\phi}} u^{s_j}(k_j) \right] \\
=&\prod_{i=1}^{m} \left[\frac{\not p - m_i}{i\sqrt{Z_\phi}} u^{s_i}(p_i) \right] i\tilde{G}(p_1, \cdots, p_n, -k_1, \cdots, -k_m) \prod_{j=m+1}^{m+n} \left[u^{s_j}(k_j) \frac{\not k - m_j}{i\sqrt{Z_\phi}} \right].
\end{aligned}
$$

3.1.2 Perturbative Corrections

As with the scalar field,

$$
Z[\bar{\eta}, \eta, J] = \exp\left\{ i \int d^dx \mathcal{L}_{\text{int}} \left[\frac{\delta}{i\delta J}, \frac{\delta}{i\delta\eta}, \frac{i\delta}{\delta\bar{\eta}} \right] \right\} Z_0[\bar{\eta}, \eta, J]. \tag{3.6}
$$

We use the dimensional regularization by default. Note that ψ has the mass dimension $[\frac{d-1}{2}]$, A^μ had the mass dimension $[\frac{d}{2}-1]$, and e has the mass dimension $[2-\frac{d}{2}]$. When $d = 4-\epsilon$, we replace e with $e\tilde{\mu}^{\epsilon/2}$, so that to make the coupling constant e dimensionless.

The renormalized Lagrangian is

$$
\begin{aligned}
\mathcal{L} &= Z_\psi \bar{\psi}_R (i\gamma^\mu \partial_\mu)\psi_R - Z_m m \bar{\psi}_R \psi_R + \frac{1}{4} Z_A F_{R,\mu\nu} F_R^{\mu\nu} - Z_e e_R A_R^\mu \bar{\psi}_R \gamma^\mu \psi_R \\
&= \mathcal{L}_0 + \mathcal{L}_{\text{int}} + \mathcal{L}_{\text{ct}}.
\end{aligned} \tag{3.7}
$$

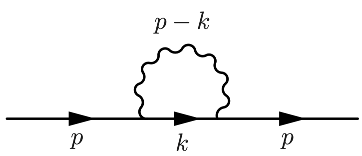
The we define the coefficients

$$
\delta_\psi = Z_\phi - 1, \;\delta_m = Z_m - 1, \;\delta_Z = Z_A - 1, \;\delta_e = Z_e - 1. \tag{3.8}
$$

The counter term also contribute to the perturbative expansion like the interactions.

One-loop Correction to Electron Propagator

Consider the diagram



This contains 3 electron propagator, 1 photon propagator, and 2 vertices. The coefficient is (omit all the integration and summation for simplicity):

$$iD_F^{(2)}(p) \sim \frac{\delta^2}{\delta\bar{\eta}\delta\eta}\frac{1}{2!}\left(\frac{-ie\gamma^\mu_{\alpha\beta}\delta^3}{i\delta J^\mu \delta\eta_\alpha\delta\bar{\eta}_\beta}\right)^2 \frac{1}{3!}\left(-i\bar{\eta}_\alpha D_F^{\alpha\beta}\eta_\beta\right)^3\left(-\frac{i}{2}J^\mu\Pi_{\mu\nu}J^\nu\right).$$

First consider the scalar coefficient. Since there is no additional symmetry, the abstract value is e^2. There is an additional sign factor by the proper order of the fermion operators:

$$\frac{\delta^2}{\delta\bar{\eta}_f\delta\eta_i}\frac{\delta^2}{\delta\eta_1\delta\bar{\eta}_1}\frac{\delta^2}{\delta\eta_2\delta\bar{\eta}_2} = -\frac{\delta}{\delta\eta_i}\frac{\delta^2}{\delta\bar{\eta}_1\delta\eta_1}\frac{\delta^2}{\delta\bar{\eta}_2\delta\eta_2}\frac{\delta}{\delta\bar{\eta}_f}.$$

Then consider the tensor contraction,

$$\Pi_{\mu\nu}D_F^{\alpha\lambda}\gamma^\mu_{\lambda\rho}D_F^{\rho\tau}\gamma^\nu_{\tau\sigma}D_F^{\sigma\beta}.$$

The total amplitude is

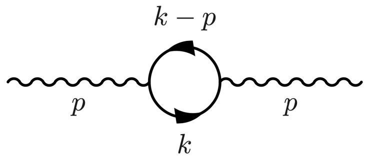
$$\begin{aligned} iD_F^{(2)}(p) &= -e^2\int\frac{d^4k}{(2\pi)^4}\Pi_{\mu\nu}(p-k)\left[D_F(p)\gamma^\mu D_F(k)\gamma^\nu D_F(p)\right]_{\alpha\beta} \\ &= iD_F(p)i\Sigma(p^2)iD_F(p), \end{aligned}$$

where $i\Sigma(p^2)$ is the self energy:

$$i\Sigma(p^2) = e^2\int\frac{d^4k}{(2\pi)^4}\Pi_{\mu\nu}(p-k)\gamma^\mu D_F(k)\gamma^\nu, \tag{3.9}$$

One-loop Correction to Photon Propagator

Consider the diagram



There is 2 electron propagator, 2 photon propagator, and 2 vertices. Consider the perturbative expansion:

$$i\Pi^{(2)}(p) \sim \frac{\delta^2}{i\delta J i\delta J}\frac{1}{2!}\left(\frac{-e\gamma^\mu_{\alpha\beta}\delta^3}{\delta J^\mu\delta\eta_\alpha\delta\bar{\eta}_\beta}\right)^2\frac{1}{2!}\left(-i\bar{\eta}_\alpha D_F^{\alpha\beta}\eta_\beta\right)^2\frac{1}{2!}\left(\frac{i}{2}J^\mu\Pi_{\mu\nu}J^\nu\right)^2.$$

The diagram has no symmetry factor, but with a -1 sign, which is canceled out by the operator reordering:

$$\bar{\eta}_\beta D_F^{\beta\tau}\eta_\tau\bar{\eta}_\sigma D_F^{\sigma\alpha}\eta_\alpha = -\eta_\alpha\bar{\eta}_\beta D_F^{\beta\tau}\eta_\tau\bar{\eta}_\sigma D_F^{\sigma\alpha}. \tag{3.10}$$

The overall value is e^2.

Then consider the tensor contraction,

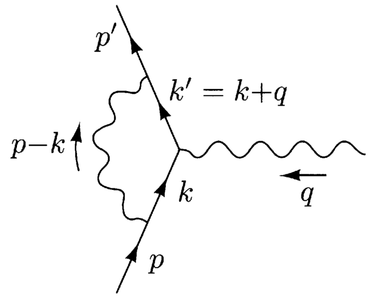
$$-i\Pi^{\mu\nu}_{(2)} \sim e^2\Pi_{\mu\rho}\gamma^\rho_{\alpha\beta}D_F^{\beta\tau}\gamma^\eta_{\tau\sigma}D_F^{\sigma\alpha}\Pi_{\eta\nu} \quad \sim i\Pi_{\mu\rho}i\Sigma^{\rho\sigma}i\Pi_{\sigma\nu}.$$

The photon self-energy is

$$i\Sigma^{\mu\nu}(p^2) = -e^2\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left[\gamma^\mu D_F(k-p)\gamma^\nu D_F(k)\right]. \tag{3.11}$$

One-loop Correction to Vertex

Consider the diagram



There is 4 electron propagator, 2 photon propagator, and 3 vertices. Consider the perturbative expansion:

$$\frac{\delta^3}{i\delta J\delta\bar{\eta}\delta\eta}\frac{1}{2!}\left(\frac{-e\gamma^{\mu}_{\alpha\beta}\delta^3}{\delta J^{\mu}\delta\eta_{\alpha}\delta\bar{\eta}_{\beta}}\right)^3\frac{1}{2!}\left(-i\bar{\eta}_{\alpha}D_F^{\alpha\beta}\eta_{\beta}\right)^4\frac{1}{2!}\left(-\frac{i}{2}J^{\mu}\Pi_{\mu\nu}J^{\nu}\right)^2.$$

There is not symmetry factor, and an additional $-i$ factor. The total coefficient is $-ie^3$.

Then consider the tensor contraction

$$D_F^{\alpha\gamma}\gamma^{\nu}_{\gamma\rho}D_F^{\rho\sigma}\gamma^{\zeta}_{\sigma\tau}D_F^{\tau\eta}\gamma^{\lambda}_{\eta\xi}D_F^{\xi\beta}\Pi_{\nu\lambda}\Pi_{\mu\zeta}.$$

The vertex correction is:

$$iV_3(q,p,p') = [iD_F(p)][iD_F(p')][i\Pi^{\mu\nu}(q)][-ie\Gamma^{\nu}(q,p,p')]$$

where

$$i\Gamma^{\mu}(q,p,p') = -e^2\int\frac{d^4k}{(2\pi)^4}\Pi_{\nu\lambda}(p-k)\gamma^{\nu}D_F(k')\gamma^{\mu}D_F(k)\gamma^{\lambda}. \tag{3.12}$$

Counter Terms

The counter terms for the fermion propagator come from the diagram expression:

$$iD_F^{(\mathrm{ct})} =$$

$$\sim \frac{\delta^2}{\delta\bar{\eta}\delta\eta}i(\delta_{\psi}\gamma^{\mu}_{\alpha\beta}k_{\mu} - \delta_m\mathbb{I}_{\alpha\beta})\frac{\delta^2}{\delta\eta_{\alpha}\delta\bar{\eta}_{\beta}}\frac{1}{2!}\left(-i\bar{\eta}_{\alpha}D_F^{\alpha\beta}\eta_{\beta}\right)^2. \tag{3.13}$$

The contribution to the electron self energy is

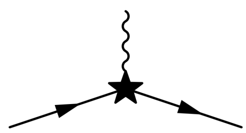
$$\delta_{\psi}\not{k} - \delta_m m_R. \tag{3.14}$$

Similarly, the counter term contribution to the photon self energy is

μ  ν

$$= \delta_A[-p^2g^{\mu\nu} + (1-\xi)p^{\mu}p^{\nu}]. \tag{3.15}$$

And the counter term contribution to the QED vertex is



$$= \delta_e\gamma^{\mu}. \tag{3.16}$$

3.2 One-loop Correction

In this section, we consider the QED in $(d = 4 - \epsilon)$ dimensional space-time.

3.2.1 Electron Propagator

Consider the one-loop correction to the electron propagator, where the self energy (3.9) is

$$\begin{aligned} i\Sigma(p^2) &= e^2\tilde{\mu}^\epsilon \int \frac{d^dk}{(2\pi)^d}\Pi_{\mu\nu}(p-k)\left[\gamma^\mu D_F(k)\gamma^\nu\right]_{\alpha\beta} \\ &= -e^2\tilde{\mu}^\epsilon \int \frac{d^dk}{(2\pi)^d}\frac{\gamma^\mu(\not{k}+m)\gamma_\mu}{(p-k)^2(k^2-m^2)}. \end{aligned} \tag{3.17}$$

The nominator can be simplified using the `FeynCalc` Package:

```
(*load FeynCalc Package*)
<< FeynCalc`

(*simplify the gamma expression*)
Contract[GA[\[Mu]].(GS[k]+m).GA[\[Mu]]]//DiracSimplify
```

The result is

$$4m - 2\not{k}.$$

The denominator can be simplify using the Feynman parameter:

$$\frac{1}{(p-k)^2(k^2-m^2)} = \int_0^1 \frac{dx}{\left[(k-b)^2 - D\right]^2}$$

where b and D can be calculated by

```
A1=(k-p)^2;
A2=k^2-m^2;
{c,b,a}=CoefficientList[x*A1+(1-x)*A2,{k}];
-b/2
-c+b^2/(4*a)//Simplify
```

The result is

$$b = px, \;\; D = (1-x)(m^2 - p^2x).$$

Shift $k \to k + px$, the self energy becomes:

$$\begin{aligned} i\Sigma(p^2) &= 2e^2\tilde{\mu}^\epsilon \int_0^1 (x\not{p} - 2m)dx \int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2-D)^2} \\ &= i\frac{2e^2\tilde{\mu}^\epsilon\Omega_d}{(2\pi)^d}\int_0^1 (x\not{p}-2m)dx\int\frac{k^{d-1}dk}{(k^2+D)^2}. \end{aligned} \tag{3.18}$$

The regularization procedure

```
omg=(2*Pi^(d/2))/(Gamma[d/2]);
cof=2*e^2*\[Mu]^(4-d)*omg/(2*Pi)^d;
int=cof*Integrate[q^(d-1)/(q^2+D)^2,{q,0,Infinity}][[1]];
map={e->Sqrt[4*Pi*\[Alpha]],EulerGamma->Subscript[\[Gamma],E]};
ans=Series[int/.{d->4-\[Epsilon]},{\[Epsilon],0,0}];
ans/.map//Simplify
```

The result is ($\mu^2 = 4\pi\tilde{\mu}^2 e^{-\gamma_E}$)

$$\Sigma(p^2) = \frac{e_R^2}{8\pi^2}\int_0^1 dx(x\not{p} - 2m_R)\left[\frac{2}{\epsilon} + \ln\left(\frac{\mu^2}{D}\right)\right]. \tag{3.19}$$

The infinity comes from

$$\frac{e_R^2}{4\pi^2\epsilon}\int_0^1 dx(x\not{p} - 2m_R) = \frac{e_R^2}{8\pi^2\epsilon}\not{p} - \frac{e_R^2}{2\pi^2\epsilon}m_R.$$

Using the $\overline{\text{MS}}$ subtraction scheme, we choose

$$\delta_\psi = -\frac{e_R^2}{8\pi^2\epsilon},\ \delta_m = -\frac{e_R^2}{2\pi^2\epsilon}, \tag{3.20}$$

and the self energy is

$$\begin{aligned}\Sigma(p^2) &= \frac{e_R^2}{8\pi^2}\int_0^1 dx(x\not{p} - 2m_R)\ln\left[\frac{\mu^2}{(1-x)(m_R^2 - p^2x)}\right] \\ &= \frac{e_R^2}{8\pi^2}(\not{p} - 4m_R)\ln\left(\frac{\mu}{m_R}\right) - \int_0^1 dx\ln\left[(1-x)\left(1 - \frac{p^2x}{m_R^2}\right)\right].\end{aligned} \tag{3.21}$$

3.2.2 Photon Self-energy

Consider the one-loop correction to the photon propagator, where the self energy (3.11) is

$$i\Sigma^{\mu\nu} = -e^2\tilde{\mu}^\epsilon\int\frac{d^dk}{(2\pi)^d}\frac{\mathrm{Tr}\left[\gamma^\mu D_F(k-p)\gamma^\nu D_F(k)\right]}{(k^2-m^2)[(p-k)^2-m^2]}. \tag{3.22}$$

The Dirac trace and Feynman parameter is calculated by

```
(*Dirac trace*)
DiracTrace[GA[\[Mu]].(GS[k-p]+m).GA[\[Nu]].(GS[k]+m)]//DiracSimplify

(*Feynman paramater*)
A1=k^2-m^2;
A2=(k-p)^2-m^2;
{c,b,a}=CoefficientList[x*A1+(1-x)*A2,{k}];
-b/2
-c+b^2/(4*a)//Simplify
```

The nominator is

$$4\left[g^{\mu\nu}\left(k\cdot p - k^2 + m^2\right) + 2k^\mu k^\nu - k^\mu p^\nu - p^\mu k^\nu\right]$$

The denominator is:

$$\frac{1}{(k^2-m^2)[(p-k)^2-m^2]}=\frac{1}{\{[k-p(1-x)]^2-[m^2+p^2x(x-1)]\}^2}$$

Let $D=m^2-p^2x(1-x)$, shift $k\to k+p(1-x)$, and drop all p^μ linear term,[1] the result is

$$i\Sigma^{\mu\nu}=-4e^2\tilde{\mu}^\epsilon\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\frac{2k^\mu k^\nu-g^{\mu\nu}\left[k^2-x(1-x)p^2-m^2\right]}{\left[k^2-D\right]^2} \tag{3.23}$$

The self-energy $i\Sigma^\mu\propto g^{\mu\nu}$, we can make the substitution

$$k^\mu k^\nu\to\frac{1}{d}k^2g^{\mu\nu}.$$

We then need to consider the integral

$$\begin{aligned}iI(x)&=4e^2\tilde{\mu}^\epsilon\int\frac{d^dk}{(2\pi)^d}\frac{(1-\frac{2}{d})k^2-x(1-x)p^2-m^2}{\left[k^2-D\right]^2},\\ I(x)&=-\frac{4e^2\tilde{\mu}^\epsilon\Omega_d}{(2\pi)^d}\int k^{d-1}dk\frac{(1-\frac{2}{d})k^2+x(1-x)p^2+m^2}{\left[k^2+D\right]^2}.\end{aligned}$$

The regulation is carried out by the following code:

```
omg=(2*Pi^(d/2))/(Gamma[d/2]);
cof=-4*e^2*\[Mu]^(4-d)*omg/(2*Pi)^d;
den=q^(d-1)*((1-2/d)q^2+x*(1-x)p^2+m^2);
int=cof*Integrate[den/(q^2+D)^2,{q,0,Infinity}][[1]];
map={EulerGamma->Subscript[\[Gamma],E],D->m^2-p^2*x*(1-x)};
ans=Series[int/.{d->4-\[Epsilon]},{\[Epsilon], 0, 0}];
ans/.map//Simplify
```

The result is

$$\Sigma^{\mu\nu}(p^2)=-\frac{e_R^2p^2g^{\mu\nu}}{2\pi^2}\int_0^1 dx\ x(1-x)\left[\frac{2}{\epsilon}+\ln\left(\frac{\mu^2}{m_R^2-p^2x(1-x)}\right)\right] \tag{3.24}$$

The divergent part is

$$-\frac{e_R^2p^2g^{\mu\nu}}{\pi^2\epsilon}\int_0^1 dx\ x(1-x)=-\frac{e_R^2p^2g^{\mu\nu}}{6\pi^2\epsilon}.$$

The counter term coefficient is

$$\delta_A=-\frac{e_R^2}{6\pi^2\epsilon}. \tag{3.25}$$

The photon self-energy is then

$$\begin{aligned}\Sigma^{\mu\nu}(p^2)&=-\frac{e_R^2p^2g^{\mu\nu}}{2\pi^2}\int_0^1 dx\ x(1-x)\ln\left[\frac{\mu^2}{m_R^2-p^2x(1-x)}\right]\\ &=-\frac{e_R^2p^2g^{\mu\nu}}{12\pi^2}\ln\left(\frac{\mu}{m}\right)+\frac{e_R^2p^2g^{\mu\nu}}{2\pi^2}\int_0^1 dx\ x(1-x)\ln\left[1-\frac{p^2}{m_R^2}x(1-x)\right].\end{aligned} \tag{3.26}$$

[1] The Ward identity requires that the p^μ term in the propagator do not contribute to any scattering process.

3.2.3 Vertex Correction

Consider the loop correction (3.12):

$$i\Gamma^\mu(p, q_1, q_2) = e^2\tilde{\mu}^\epsilon \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\nu(\not{k}' + m)\gamma^\mu(\not{k} + m)\gamma_\nu}{(k^2 - m^2)(k'^2 - m^2)(p - k)^2}. \tag{3.27}$$

Using the following code

```
(*numerator*)
den=Contract[GA[\[Nu]].(GS[kp]+m).GA[\[Mu]].(GS[k]+m).GA[\[Nu]]];
DiracSimplify[den]

(*Feynman parameter*)
A1=k^2-m^2;
A2=(k+q)^2-m^2;
A3=(p-k)^2;
{c,b,a}=CoefficientList[x*A1+y*A2+z*A3,{k}];
-b/2//Simplify
-c+b^2/4//Simplify
```

The numerator is

$$-2\not{k}\gamma^\mu\not{k}' - 2m^2\gamma^\mu + 4m(k + k')^\mu.$$

The denominator is

$$\int \frac{dF_3}{[(k + yq - zp)^2 - D]^3},$$

where

$$\begin{aligned} D &= (x + y)m^2 - z(1 - z)p^2 - y(1 - y)q^2 - 2yzpq \\ &= (x + y)m^2 - xyq^2 - yzp'^2 - xzp^2. \end{aligned}$$

Shift $k^\mu \to k^\mu + zq_1^\mu - yp^\mu$, throw away all terms with linear k^μ, and replace $k^\mu k^\nu$ with $\frac{1}{d}k^2 g^{\mu\nu}$, the result is

$$\frac{4}{d}k^2\gamma^\mu - 2(-y\not{q} + z\not{p})\gamma^\mu[(1 - y)\not{q} + z\not{p}] + 4m^2\gamma^\mu - 2m\left[(1 - 2y)q^\mu + 2zp^\mu\right].$$

Note that only the quadratic term is divergent.

$$\Gamma^\mu(p, q_1, q_2) = -i\frac{4e^2\tilde{\mu}^\epsilon\gamma^\mu}{d} \int dF_3 \int \frac{d^dk}{(2\pi)^d} \frac{k^2}{(k^2 - D)^3} + \delta\Gamma^\mu(p, q_1, q_2).$$

where $\delta\Gamma^\mu$ stores all the finite part

$$\begin{aligned} &\delta\Gamma^\mu(p, q_1, q_2) \\ &= \int \frac{e^2k^3dkdF_3}{(2\pi)^2(k^2 + D)^3} \left\{(-y\not{q} + z\not{p})\gamma^\mu[(1 - y)\not{q} + z\not{p}] - 2m^2\gamma^\mu + m\left[(1 - 2y)q^\mu + 2zp^\mu\right]\right\}. \end{aligned}$$

The divergent part is

$$\frac{4e^2\tilde{\mu}^\epsilon\Omega_d\gamma^\mu}{d(2\pi)^d} \int dF_3 \int \frac{k^{d+1}dk}{(k^2 + D)^3} = \frac{e_R^2}{16\pi^2}\gamma^\mu \int dF_3 \left[\frac{2}{\epsilon} + \ln\left(\frac{\mu^2}{D}\right)\right].$$

Using the $\overline{\text{MS}}$ scheme, the coefficient of the counter term is

$$\delta_e = -\frac{e_R^2}{8\pi^2\epsilon}. \tag{3.28}$$

3.2.4 Renormalization Group

In summery, the renormalization factors are

$$\begin{aligned}
Z_\psi &= 1 - \frac{e_R^2}{8\pi^2\epsilon} + O(e_R^3), \\
Z_A &= 1 - \frac{e_R^2}{6\pi^2\epsilon} + O(e_R^3), \\
Z_m &= 1 - \frac{e_R^2}{2\pi^2\epsilon} + O(e_R^3), \\
Z_e &= 1 - \frac{e_R^2}{8\pi^2\epsilon} + O(e_R^3),
\end{aligned} \tag{3.29}$$

which means

$$\begin{aligned}
\frac{d\ln Z_\phi}{de_R} &= -\frac{e_R}{4\pi^2\epsilon} + O(e_R^2), \\
\frac{d\ln Z_A}{de_R} &= -\frac{e_R}{3\pi^2\epsilon} + O(e_R^2), \\
\frac{d\ln Z_m}{de_R} &= -\frac{e_R}{\pi^2\epsilon} + O(e_R^2), \\
\frac{d\ln Z_e}{de_R} &= -\frac{e_R}{4\pi^2\epsilon} + O(e_R^2).
\end{aligned} \tag{3.30}$$

The bare parameters are

$$\begin{aligned}
\psi_0 &= Z_\psi^{1/2}\psi_R, \\
A_0 &= Z_A^{1/2}A_R, \\
m_0 &= Z_m Z_\psi^{-1} m_R, \\
e_0 &= Z_e Z_\psi^{-1} Z_A^{-1/2} e_R \tilde{\mu}^{\epsilon/2}.
\end{aligned} \tag{3.31}$$

The RG equation for e_0 is

$$\frac{d\ln e_0}{d\ln\mu} = \left(\frac{\ln Z_e}{de_R} - \frac{\ln Z_\psi}{de_R} - \frac{1}{2}\frac{\ln Z_A}{de_R} + \frac{1}{e_R}\right)\frac{de_R}{d\ln\mu} + \frac{\epsilon}{2} = 0. \tag{3.32}$$

The beta function is

$$\beta(e_R) = \frac{de_R}{d\ln\mu} = -\frac{\epsilon}{2}e_R + \frac{e_R^3}{12\pi^2} + O(e_R^4). \tag{3.33}$$

The RG equation for m_0 is

$$\frac{d\ln m_0}{d\ln\mu} = \left(\frac{\ln Z_m}{de_R} - \frac{\ln Z_\psi}{de_R}\right)\frac{de_R}{d\ln\mu} + + \frac{1}{m_R}\frac{dm_R}{d\ln\mu} = 0. \tag{3.34}$$

The anomalous dimension of mass is

$$\gamma_m = \frac{d\ln m_R}{d\ln\mu} = -\frac{3e_R^2}{8\pi^2} + O(e_R^3). \tag{3.35}$$

Chapter 4

Non-relativistic Quantum Field Theory

A general non-relativistic field has the Lagrangian[1]

$$\mathcal{L} = \bar{\psi}_a(x)(i\delta_{ab}\partial_t - \hat{H}_{ab})\psi_b(x) + \mathcal{V}_{\text{int}} \tag{4.1}$$

where the field operator ψ can be bosonic or fermionic, which is denoted by a number $\zeta = \pm 1$, and $\mathcal{V}_{\text{int}}$ is the interaction Lagrangian. A general interaction has the form

$$\mathcal{V}_{\text{int}} = \bar{\psi}_a(x_1)\bar{\psi}_b(x_2)V_{ab}(x_1, x_2)\psi_b(x_2)\psi_a(x_1). \tag{4.2}$$

Note that the classical equation of motion for the free field is

$$\begin{aligned} 0 &= \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi}_a(x))} - \frac{\partial \mathcal{L}}{\partial \bar{\psi}_a(x)} \\ &= -i\partial_t \phi_a(x) + \hat{H}_{ab}\phi_b(x), \end{aligned} \tag{4.3}$$

which satisfies the Schrödinger equation.

We are mostly work with finite system size L^d with UV cutoff $\Lambda = \frac{2\pi}{a}$,[2] in which case the spatial Fourier transformation is

$$\tilde{\psi}_a(k) = \int_{L^d} d^d x e^{-ik\cdot x}\psi(k), \tag{4.4}$$

$$\psi_a(x) = \frac{1}{L^d}\sum_k e^{ik\cdot x}\tilde{\psi}(k). \tag{4.5}$$

Note that for finite size, the momentum is discretized:

$$k_i = \frac{2\pi}{L}n_i, \ n_i = -N, \cdots, N. \tag{4.6}$$

By default, we take the thermodynamic limit. The summation is approximated by the integration:

$$\frac{1}{L^d}\sum_k \longrightarrow \int_{|k|<\Lambda} \frac{d^d k}{(2\pi)^d}. \tag{4.7}$$

[1] The repeated indices are automatically summed.

[2] We can regard a as the lattice spacing, and assume $L = Na$.

4.1 Finite Temperature Field Theory

The original real-time partition function is defined as[3]

$$Z[J] = \int D[\bar{\psi}, \psi] \exp\left\{ i \int dt \int d^d x \left[\mathcal{L} + \bar{J}_a(x)\psi_a(x) + \bar{\psi}_a(x) J_a(x) \right] \right\}. \tag{4.8}$$

For finite-temperature field theory, after making the wick rotation $t \to -i\tau$, the partition function for a generic non-relativistic lattice theory is:

$$Z[J] = \int D[\bar{\psi}, \psi] e^{-S[\bar{\psi},\psi] + \bar{J}\cdot\psi + \bar{\psi}\cdot J}, \tag{4.9}$$

where the action is

$$S = \int_0^\beta d\tau \int d^d x \left[\bar{\psi}_a(x)(\delta_{ab}\partial_\tau + \hat{H}_{ab})\psi_b(x) + \mathcal{V}_{\text{int}} \right]. \tag{4.10}$$

Remark 4. Temporal Fourier Transformation

The Fourier transformation on the imaginary time domain is defined as:

$$\begin{aligned} \tilde{\psi}(\omega_n) &= \int_0^\beta d\tau e^{i\omega_n \tau} \psi(\tau), & (4.11) \\ \psi(\tau) &= \frac{1}{\beta} \sum_{\omega_n} e^{-i\omega_n \tau} \tilde{\psi}(\omega_n). & (4.12) \end{aligned}$$

Under such convention, in the thermodynamic limit and zero-temperature limit, the spatial-temporal Fourier transformation agrees with the relativistic case (up to a Wick rotation).

4.1.1 Free Field Theory

We first consider the action of free field

$$S_0 = \int_0^\beta d\tau \int d^d x \; \bar{\psi}_a(x)(\delta_{ab}\partial_\tau + \hat{H}_{ab})\psi_b(x). \tag{4.13}$$

The Fourier transformation

$$S_0 = \frac{1}{\beta} \sum_{\omega_n} \int_\Lambda \frac{d^d k}{(2\pi)^d} \tilde{\bar{\psi}}_a(k, \omega_n) \left[-i\omega_n + \tilde{H}_{ab}(k) \right] \tilde{\psi}_b(k, \omega_n). \tag{4.14}$$

The partition function with source is

$$\frac{Z_0[J]}{Z_0[0]} = \exp\left[-\frac{1}{\beta} \sum_{\omega_n} \int_\Lambda \frac{d^d k}{(2\pi)^d} \tilde{\bar{J}}_a(k, \omega_n) \tilde{G}_{ab}(k, \omega_n) \tilde{J}_b(k, \omega_n) \right], \tag{4.15}$$

[3] As with the relativistic case, we introduce an auxiliary source J, which is bosonic/fermionic if the field ψ is bosonic/fermionic.

where the Green's function is

$$\tilde{G}_{ab}(k,\omega_n) = \left[\frac{1}{i\omega_n - \tilde{H}(k)}\right]_{ab}. \tag{4.16}$$

Remark 5. Obtaining the Partition Function

Unlike the relativistic case, the value of the value of partition function without source $Z_0[0]$ is related to the free energy. We can express it formally as

$$Z_0[0] = \left[\det(-G_{ab})^{-1}\right]^{-\zeta}.$$

To get the correct dimensionality, we set the determinant as

$$Z_0[0] \equiv \prod_{k,\omega_n} \left\{\beta \det\left[-i\omega_n + \tilde{H}(k)\right]\right\}^{-\zeta}.$$

Thus the free energy is

$$F = -\frac{1}{\beta}\ln Z_0 = \zeta \sum_{k,\omega_n} \ln\left\{\beta \det\left[-i\omega_n + \tilde{H}(k)\right]\right\}. \tag{4.17}$$

4.1.2 Matsubara Summation

Now consider the summation on Matsubara frequency:

$$\sum_{\omega_n} f(\omega_n) = \begin{cases} \sum_n f(\frac{2n\pi}{\beta}) & \text{bosonic} \\ \sum_n f(\frac{(2n+1)\pi}{\beta}) & \text{fermionic} \end{cases}. \tag{4.18}$$

The frequency is capture by the singularities of the density function of the states:

$$\rho(z) = \begin{cases} \frac{1}{\exp(\beta z)-1} & \text{bosonic} \\ \frac{1}{\exp(\beta z)+1} & \text{fermionic} \end{cases}. \tag{4.19}$$

The residue on imaginary frequency $i\omega_n$ is alway $\frac{1}{\beta}$. In this way, the summation is:

$$\frac{1}{\beta}\sum_{\omega_n} f(i\omega_n) = \frac{1}{2\pi i}\oint \rho(z) f(z) = -\sum_k \text{Res}\ \rho(z)f(z)|_{z=z_k}. \tag{4.20}$$

Summation of Green's function

Consider the frequency summation for the correlation function:

$$\frac{1}{\beta}\sum_{\omega_n}\tilde{G}_0(k) = \frac{1}{\beta}\sum_{\omega_n}\frac{1}{i\omega_n - E_p} = -\text{Res}\left.\frac{\rho(z)}{z-E_p}\right|_{z=E_p} = \rho(E_p). \tag{4.21}$$

Summation of Green's function

Consider the frequency summation for the correlation function:

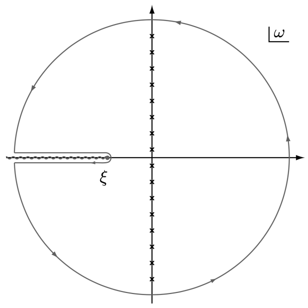
$$\sum_{\omega_n}\langle\bar{\psi}_{\vec{p},\omega_n}\psi_{\vec{p},\omega_n}\rangle = \frac{1}{\beta}\sum_{\omega_n}\frac{1}{-i\omega_n+\epsilon_{\vec{p}}} = \text{Res}\left.\frac{\rho(z)}{z-\epsilon_{\vec{p}}}\right|_{z-\epsilon_{\vec{p}}} = \rho(\epsilon_{\vec{p}}). \tag{4.22}$$

Free Energy Summation

Consider the free energy

$$F = -\frac{1}{\beta}\ln Z = -\frac{1}{\beta}\sum_{\omega_n}\ln[\beta(-i\omega_n+E_{\vec{p}})] = \frac{1}{2\pi i}\oint dz\rho(z)\ln[\beta(\xi-z)]. \tag{4.23}$$

To calculate the summation, we consider the line integral along the loop:



The free energy is

$$\begin{aligned} F &= \frac{1}{2\pi i}\int_{-\infty}^{\infty}dx\rho(x)\ln\left(\frac{\xi-x-i\epsilon}{\xi-x+i\epsilon}\right) \\ &= \frac{-\zeta}{2\pi i\beta}\int_{-\infty}^{\infty}dx\ln(1-\zeta e^{-\beta z})\left(\frac{1}{x+i\epsilon-\xi}-\frac{1}{x-i\epsilon-\xi}\right), \end{aligned} \tag{4.24}$$

where we integrate the expression by part, noticing that

$$\frac{d}{dz}\frac{\zeta}{\beta}\ln(1-\zeta e^{-\beta z}) = \frac{1}{e^{\beta z}-\zeta} = \rho(z) \tag{4.25}$$

Using the identity

$$\lim_{\epsilon\to 0^+}\frac{1}{x+i\epsilon} = -i\pi\delta(x)+\mathcal{P}\frac{1}{x},$$

the above expression can be simplified to

$$F = \frac{\zeta}{\beta}\ln(1-\zeta e^{-\beta\zeta}). \tag{4.26}$$

4.2 Interacting Fermi Gas

In this section, we are considering the system of weakly interacting Fermi gas. To be specific, we consider the lattice Hamiltonian:

$$H = -\frac{1}{2}\sum_{\langle i,j\rangle}(c_i^\dagger c_j + c_j^\dagger c_i) + \mu\sum_i c_i^\dagger c_i + \sum_{i,j,k,l}u_{ijkl}c_i^\dagger c_j^\dagger c_k c_l. \tag{4.27}$$

In the following, we investigate the effective field theory (near the Fermi surface) in one and two dimensions.

4.2.1 One-dimensional Fermi System

We first consider the one-dimensional case at half filling. The free field part of (4.27) gives the dispersion

$$E(K) = -\cos K. \tag{4.28}$$

The property of the system is determined mainly by the degrees of freedom near the Fermi surface (at $K_F = \pm\pi$), where the linearized dispersion is:

$$E(K_i) = k, \tag{4.29}$$

where we have defined the momentum for the left-mover hand the right-mover:

$$K_i = \epsilon_i(K_F + k), \quad \epsilon_i = \begin{cases} +1 & i = R \\ -1 & i = L \end{cases}. \tag{4.30}$$

The effective field theory for the free field is

$$Z_0 = \prod_{i=L/R} \int D\left[\bar{\psi}_i(k,\omega), \psi_i(k,\omega)\right] e^{-S_0}, \tag{4.31}$$

where the free field action is

$$S_0 = \sum_{i=L/R} \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \bar{\psi}(k,\omega)(-i\omega + k)\psi(k,\omega), \tag{4.32}$$

which gives the free field propagator:

$$G(k,\omega) = -\langle \psi(k,\omega)\bar{\psi}(k,\omega)\rangle \; = \frac{1}{i\omega - k}. \tag{4.33}$$

We then consider the rescaling of the cut-off $\Lambda \to \Lambda/s$. To make the free action scale invariant, we define the rescaled variables:

$$k' = sk, \quad \omega' = s\omega, \quad \psi'(k',\omega') = s^{-3/2}\psi(k,\omega). \tag{4.34}$$

Then we consider the perturbation from quadratic and quartic terms:[4]

$$\begin{aligned} \delta S_2 &= \sum_{i=L/R} \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \mu(k,\omega)\bar{\psi}(k,\omega)\psi(k,\omega), \\ \delta S_4 &= \frac{1}{4} \sum_{\{i_j\}=L}^{R} \int_{K,\omega}^{\Lambda} \bar{\psi}_{i_4}(4)\bar{\psi}_{i_3}(3)u(4,3,2,1)\psi_{i_2}(2)\psi_{i_1}(1), \end{aligned} \tag{4.35}$$

[4] We use the notation that $\psi(i) = \psi(K_i, \omega_i)$, and $u(4,3,2,1) = u(\{K_i, \omega_i, i = 1,2,3,4\})$.

where we have defined:[5]

$$\int_{K\omega}^{\Lambda} = \int_{-\Lambda}^{\Lambda} \frac{dk_1 \cdots dk_4}{(2\pi)^4} \int_{-\infty}^{\infty} \frac{d\omega_1 \cdots d\omega_4}{(2\pi)^4} \times 2\pi\delta\left(\omega_1 + \omega_2 - \omega_3 - \omega_4\right)$$
$$\times\, 2\pi\bar{\delta}\left[\sum_{j=1}^{2} \epsilon_{i_j}\left(K_{\mathrm{F}} + k_j\right) - \sum_{j=3}^{4} \epsilon_{i_j}\left(K_{\mathrm{F}} + k_j\right)\right]. \tag{4.36}$$

Since this action separates into slow and fast pieces, the effect of mode elimination is simply to reduce Λ to Λ/s in the integral above. Rescaling moments and fields, we find that

$$\mu'(k', \omega') = s \cdot \mu\left(\frac{k'}{s}, \frac{\omega}{s}\right). \tag{4.37}$$

Expand μ in series:

$$\mu(k, \omega) = \mu_{00} + \mu_{10}k + \mu_{01}i\omega + \cdots + \mu_{nm}k^n(i\omega)^m + \cdots, \tag{4.38}$$

and compare both sides. The constant piece is a relevant perturbation. This relevant flow reflects the readjustment of the Fermi sea to a change in chemical potential. The correct way to deal with this term is to include it in the free-field action by filling the Fermi sea to a point that takes μ_{00} into account. The next two terms are marginal and modify terms that are already present in the action.

We now turn on the quartic interaction, the dimensional analysis gives the transformation of u:

$$u'_{i_4,i_3,i_2,i_1}(k'_i, \omega'_i) = u_{i_4,i_3,i_2,i_1}\left(\frac{k'_i}{s}, \frac{\omega'_i}{s}\right). \tag{4.39}$$

If we expand u in a Taylor series in its arguments and compare coefficients, we find that the constant term u0 is marginal and the higher coefficients are irrelevant. Thus, u depends only on its discrete labels and we can limit the problem to just a few coupling constants instead of the coupling function we started with. Furthermore, all reduce to just one coupling constant:

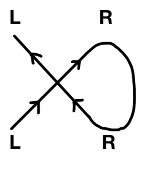
$$u_0 = u_{LRLR} = u_{RLRL} = -u_{RLLR} = -u_{LRRL} \equiv u. \tag{4.40}$$

Other couplings corresponding to the $(LL \to RR)$ process are wiped out by the Pauli principle since they have no momentum dependence and cannot have the desired anti-symmetry.

RG at One-loop Level

Consider the infinitesimal rescale $s = e^{dt}$. The one-loop contribution to the quadratic term is[6]

$$\mu_{LL}^{(2)} =$$



$$= -u \int_{d\Lambda} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{i\omega\eta}}{i\omega - k}, \tag{4.41}$$

[5] The symbol $\bar{\delta}$ enforces momentum conservation mod 2π, as is appropriate to any lattice problem. A process where lattice momentum is violated in multiples of 2π is called an *umklapp process*.

[6] We include an infinitesimal $e^{i\omega\eta}$ to ensure convergence as we do the integral over ω by closing the upper half-plane.

where the integral on the momentum shell is

$$\int_{d\Lambda} \frac{dk}{2\pi} = \int_{-\Lambda}^{-\Lambda(1-dt)} \frac{dk}{2\pi} + \int_{\Lambda(1-dt)}^{\Lambda} \frac{dk}{2\pi}. \tag{4.42}$$

The result gives:

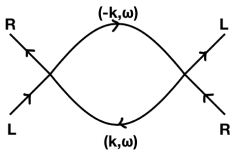
$$\mu_{LL}^{(2)} = -\frac{u\Lambda}{2\pi} dt$$

Assuming that we choose to measure μ in units of Λ, (since we rescale momentum to keep Λ fixed, we can choose $\Lambda = 1$.) By the symmetry $L \leftrightarrow R$, we know $\mu_{LL}^{(2)} = \mu_{RR}^{(2)} = \mu^{(2)}$, so the RG flow is

$$\frac{d}{dt}\left[s \cdot \left(\mu + \mu^{(2)}\right)\right] = \mu - \frac{u}{2\pi}. \tag{4.43}$$

The one-loop correction to the quartic terms have two contributions. One is called ZS' (zero sound) channel:[7]

$$u_{\mathrm{ZS'}}^{(2)} =$$



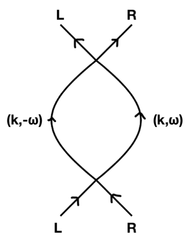
$$\begin{aligned} &= -u^2 \int_{-\infty}^{\infty} \int_{\Lambda/s<|k|<\Lambda} \frac{d\omega dk}{(2\pi)^2} \frac{e^{i\omega\eta}}{(i\omega + k)(i\omega - k)} \\ &= u^2 \int_{\Lambda/s<|k|<\Lambda} \frac{dk}{2\pi} \frac{1}{2|k|} \\ &= \frac{u^2}{2\pi} \frac{d\Lambda}{\Lambda}. \end{aligned} \tag{4.44}$$

The sign is obtained from contracting the Fermion field monomial:

$$\bar{\psi}_L \bar{\psi}_R \psi_L \psi_R \bar{\psi}_L \bar{\psi}_R \psi_L \psi_R = -G_L G_R \bar{\psi}_R \psi_L \bar{\psi}_L \psi_R = -G_L G_R \bar{\psi}_L \bar{\psi}_R \psi_L \psi_R.$$

The other is called the BCS channel:[8]

$$u_{\mathrm{BCS}}^{(2)} =$$



$$= -\frac{u^2}{2} \sum_{i=L/R} \int_{-\infty}^{\infty} \int_{d\Lambda} \frac{d\omega dk_i}{(2\pi)^2} \frac{e^{i\omega\eta}}{(i\omega - k_i)(-i\omega - k_i)}. \tag{4.45}$$

The sign is obtained from the contraction:

$$\bar{\psi}_L \bar{\psi}_R \psi_L \psi_R \bar{\psi}_L \bar{\psi}_R \psi_L \psi_R = -G_L G_R \bar{\psi}_L \bar{\psi}_R \psi_L \psi_R.$$

Note that we will obtained a factor of 2 since in this channel, the intermedia propagator can be left mover or right mover. We see that two contributions cancel out:

$$u_{\mathrm{ZS'}}^{(2)} + u_{\mathrm{BCS}}^{(2)} = 0. \tag{4.46}$$

[7] There is actually another zero sound channel ZS, but which has no contribution to the vertex because the diagram contains the vertex of the $(LL \to RR)$ process, which has no relevant contribution the the vertex.

[8] The 1/2 factor comes from the symmetry factor of the diagram.

Together, the RG flow to the one-loop level is

$$\frac{d\mu}{dt} = \mu - \frac{u}{2\pi}, \quad \frac{du}{dt} = 0. \tag{4.47}$$

The fixed point solution to the RG flow is:

$$\mu = \frac{u^*}{2\pi}, \tag{4.48}$$

where the fixed-point value of u^* is arbitrary. The vanishing beta function predict that the ground state of one-dimensional weakly interacting Fermi gas remains gapless (rather than develops CDW order and becomes gapped).

4.2.2 Two-dimensional Fermi System

The low-energy manifold is an annulus of thickness 2Λ symmetrically situated with respect to the Fermi circle $K = K_\mathrm{F}$. The dispersion for the free lattice model is

$$E(\boldsymbol{K}) = -\cos K_x - \cos K_y \simeq -2 + \frac{\boldsymbol{K}^2}{2}. \tag{4.49}$$

For a given chemical potential μ, the Fermi circle is $K_F = \sqrt{2m\mu}$, we can linearize the dispersion near the Fermi furface:

$$E(\boldsymbol{K}) = \frac{\boldsymbol{K}^2 - K_F^2}{2m} \approx \frac{K_F}{m} k \equiv v_F k, \quad k \equiv |K| - K_F \tag{4.50}$$

The partition function is:

$$Z_0 = \sum_\theta \sum_{|k|<\Lambda} \int D\left[\bar{\psi}(k,\theta,\omega), \psi(k,\theta,\omega)\right] e^{-S_0}, \tag{4.51}$$

where the free field action is:[9]

$$S_0 = \int \frac{d\theta}{2\pi} \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \bar{\psi}(k,\theta,\omega)(-i\omega + v_F k)\psi(k,\theta,\omega). \tag{4.52}$$

Consider the quartic interaction

$$\delta S_4 = \frac{1}{4} \int_{\boldsymbol{K},\theta,\omega} \bar{\psi}(4)\bar{\psi}(3)\psi(2)\psi(1)u(4,3,2,1) \tag{4.53}$$

where we eliminate one of the four sets of variables, say, the one numbered 4, by integrating them against the delta functions:

$$\int_{K,\theta,\omega} = \prod_{i=1}^{3} \int_0^{2\pi} \frac{d\theta_i}{2\pi} \int_{-\Lambda}^{\Lambda} \frac{dk_i}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega_i}{2\pi} \theta\left(\Lambda - |k_4|\right), \quad k_4 = |\boldsymbol{K}_4| - K_F. \tag{4.54}$$

The ω integral is easy: since all ω's are allowed, the condition $\omega_4 = \omega_1 + \omega_2 - \omega_3$ is always satisfied for any choice of the first three frequencies. The same would be true for the momenta if all momenta were allowed. But they are not; they are required to lie within the annulus of thickness 2Λ around the Fermi circle. Consequently, if one freely chooses the first three momenta from the annulus, the fourth could have a length as large as $3K_F$. The role of $\delta(\Lambda - |k_4|)$ is to prevent exactly this.

[9] A factor of K_F has been absorbed in the field.

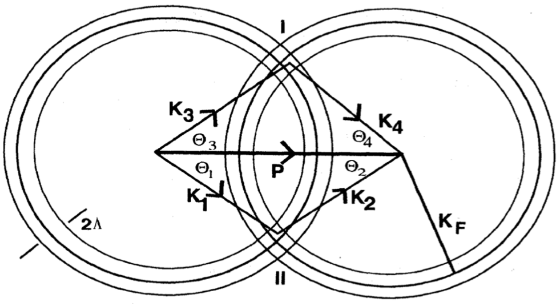


Figure 4.1: The geometric construction for determining the allowed values of momenta. If K_1 and K_2 add up to P, then K_3 and K_4 are constrained as shown, if they are to add up to P and lie within the cutoff. If the incoming momenta K_1 and K_2 are equal and opposite, the two shells coalesce and K_3 and K_4 are free to point in all directions, as long as they are equal and opposite.

Momentum Constraint

Note that k_4 can be expressed as

$$k_4 = |(K_F + k_1)\mathbf{\Omega}_1 + (K_f + k_2)\mathbf{\Omega}_2 - (K_F + k_3)\mathbf{\Omega}_3| - K_F. \tag{4.55}$$

When doing RG towards the Fermi surface, the integral measure will not preserve the preserve the original form. The situation is clearly is we use a smooth cutoff

$$\theta(\Lambda - |k_4|) \to e^{-|k_4|/\Lambda}, \tag{4.56}$$

and define $\Delta \equiv \mathbf{\Omega}_1 + \mathbf{\Omega}_2 - \mathbf{\Omega}_3$, k_4 in this way behaves as

$$k_4 = (|\Delta| - 1)K_F + O(k). \tag{4.57}$$

The integral then change to:

$$\begin{aligned}
&\prod_{i=1}^{3} \int_{-\Lambda}^{\Lambda} \frac{dk_i}{2\pi} \int \frac{d\theta_i}{2\pi} \int \frac{d\omega_i}{2\pi} e^{-||\Delta|-1|\frac{K_F}{\Lambda}} u(k,\theta,\omega)\bar{\psi}\bar{\psi}\psi\psi \\
\xrightarrow{\text{RG}} &\prod_{1}^{3} \int_{-\Lambda}^{\Lambda} \frac{dk_i'}{2\pi} \int \frac{d\theta_i}{2\pi} \int \frac{d\omega_i'}{2\pi} e^{-||\Delta|-1|\frac{sK_F}{\Lambda}} u\left(\frac{k'}{s}\frac{\omega'}{s}\theta\right) \bar{\psi}\bar{\psi}\psi\psi.
\end{aligned} \tag{4.58}$$

We can then get the RG transformation of u as

$$u'(k',\theta,\omega') = e^{-||\Delta|-1|\frac{(s-1)K_F}{\Lambda}} u\left(\frac{k'}{s},\theta,\frac{\omega'}{s}\right). \tag{4.59}$$

By Taylor expansion, we conclude that the only couplings that survive the RG transformation without any decay correspond to the cases in which $|\Delta| = 1$, and without momentum dependence.

This equation has only three solutions (see also Fig. 4.1):

$$\begin{aligned}
&\text{Case I:} \quad \mathbf{\Omega}_1 = \mathbf{\Omega}_3, \\
&\text{Case II:} \quad \mathbf{\Omega}_2 = \mathbf{\Omega}_3, \\
&\text{Case III:} \quad \mathbf{\Omega}_1 = -\mathbf{\Omega}_2.
\end{aligned} \tag{4.60}$$

Because of the rotational symmetry, the marginal vertex functions are determined solely by two functions:

$$
\begin{aligned}
u[\theta_1, \theta_2, \theta_1, \theta_2] &\equiv F(\theta_1, \theta_2) = F(\theta_1 - \theta_2), && (4.61)\\
u[\theta_1, \theta_2, \theta_2, \theta_1] &= -F(\theta_1 - \theta_2), && (4.62)\\
u[\theta_1, -\theta_1, \theta_3, -\theta_3] &\equiv V(\theta_1, \theta_3) = V(\theta_1 - \theta_3). && (4.63)
\end{aligned}
$$

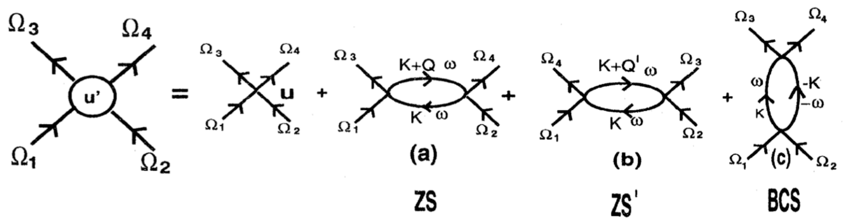
Note that the manifestation of the Pauli principle on F and V is somewhat subtle: F will not be antisymmetric under $1 \leftrightarrow 2$ since, according to the way it is defined above, we cannot exchange 1 and 2 without exchanging 3 and 4 at the same time. On the other hand, since 3 and 4 can be exchanged without touching 1 and 2 in the definition of V, V must go to $-V$ when $1 \leftrightarrow 3$.

RG at One-loop Level

We first consider the loop correction to the chemical potential:

$$
\begin{aligned}
\mu^{(2)}(k, \theta, \omega) &= \int_{d\Lambda} \frac{dK'}{2\pi} \int \frac{d\omega'}{2\pi} \int \frac{d\theta'}{2\pi} \frac{F(\theta - \theta')}{i\omega - v_F k'}\\
&= \int_{-\Lambda}^{-\Lambda + \Lambda dt} \frac{dK'}{2\pi} \int \frac{d\theta'}{2\pi} F(\theta - \theta') \qquad (4.64)\\
&= \frac{\Lambda}{2\pi} \left[\int \frac{d\phi}{2\pi} F(\phi) \right] dt.
\end{aligned}
$$

For the vertex correction, again we should consider three channels corresponding to the diagrams:



(4.65)

First we consider the correction to the $F(\theta)$. The contribution from the ZS channel (the momentum transfer $Q \simeq 0$) is

$$
F_{\mathrm{ZS}}^{(2)}(\theta_1 - \theta_2) = \int_{d\Lambda} \frac{dk}{2\pi} \int \frac{d\omega}{2\pi} \int \frac{d\theta}{2\pi} \frac{F(\theta_1 - \theta) F(\theta - \theta_2)}{(i\omega - v_F k)^2}. \qquad (4.66)
$$

Since two poles of the integrant lie at the same half plane, we can alway choose to close the loop integral along the other half, and thus getting zero contribution.

For the ZS' channels, the momentum conservation condition (see Fig. 4.2) restrict the phase space to be of order $d\Lambda^2$, and thus has no relevant contribution to $F(\theta)$. Finally, for the same kinematical reason, the BCS diagram does not renormalize $F(\theta)$ at one loop. Consider Fig. 4.1, with K_3 and K_4 replaced by the two momenta in the BCS loop, K and $P - K$. In each annulus we keep just two shells of thickness $d\Lambda$ at the cutoff corresponding to the modes to be eliminated. The requirement that K and $P - K$ lie in these shells and also add up to P forces them into intersection regions of order $d\Lambda^2$. This means the

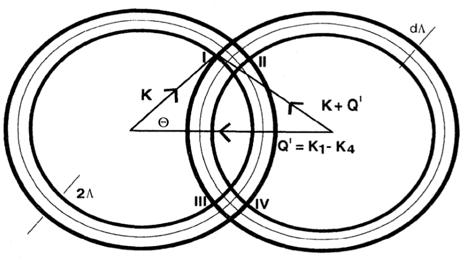


Figure 4.2: Construction for determining the allowed values of loop momenta in ZS'. The requirement that the loop momenta come from the shell and differ by Q' forces them to lie in one of the eight intersection regions of width $d\Lambda^2$.

diagram is just as ineffective as the ZS' diagram in causing a flow. Thus any F is a fixed point to this order.

Now we consider the correction to the $V(\theta)$ function. We choose the external momenta equal and opposite and on the Fermi surface. The ZS and ZS' diagrams do not contribute to any marginal flow for the same reason that BCS and ZS' did not contribute to the flow of $F(\theta)$. But the BCS diagram produces a flow:

$$\begin{aligned} V^{(2)}_{\mathrm{BCS}}(\theta_1-\theta_3) &= -\frac{1}{2}\int_{d\Lambda}\frac{dk}{2\pi}\int\frac{d\omega}{2\pi}\int\frac{d\theta}{2\pi}\frac{V(\theta_1-\theta)V(\theta-\theta_3)}{(i\omega-v_Fk)(-i\omega-v_Fk)} \\ &= -\frac{dt}{4\pi v_F}\int\frac{d\theta}{2\pi}V(\theta_1-\theta)V(\theta-\theta_3). \end{aligned} \tag{4.67}$$

We can simplify the picture by going to angular momentum eigenfunctions,

$$V(\theta) = \sum_l e^{il\theta}V_l, \tag{4.68}$$

which gives the RG flow as

$$\frac{dV_l}{dt} = -\frac{V_l^2}{4\pi v_F}. \tag{4.69}$$

The solution to the RG flow is:

$$V_l(t) = \frac{V_l(0)}{1+\frac{V_l(0)}{4\pi v_F}t}. \tag{4.70}$$

What these equations tell us is that if the potential in angular momentum channel l is repulsive, it will get renormalized (logarithmically) down to zero, while if it is attractive, it will run off to large negative values signaling the BCS instability. This is the reason the V's are excluded in Landau theory, which assumes we have no phase transitions.[10]

[10]Remember that the sign of any given V_l is not necessarily equal to that of the microscopic interaction. Kohn and Luttinger have shown (PRL, 15, 524 (1965)) that some of them will be always negative. Thus, the BCS instability is inevitable, though possibly at absurdly low temperatures or absurdly high angular momentum l.

Chapter 5

Lattice Systems

5.1 Free Fermion Systems

In this section, we consider the system whose Hamiltonian composed of quadratic fermionic operators, i.e.,

$$\hat{H}_{\text{free}} = \sum_{i,j=1}^{N} A_{ij} c_i^\dagger c_j + \frac{1}{2}\sum_{i,j=1}^{N} B_{ij} c_i c_j + \frac{1}{2}\sum_{i,j=1}^{N} B_{ij}^* c_j^\dagger c_i^\dagger, \tag{5.1}$$

where t_{ij} is a Hermitian matrix, and Δ_{ij} is anti-symmetric. In the Nambu basis

$$\Psi = (c_1, \ldots, c_N, c_1^\dagger, \ldots, c_N^\dagger)^T, \tag{5.2}$$

the Hamiltonian has the form[1]

$$\hat{H}_{\text{free}} = \frac{1}{2}\sum_{i,j=1}^{2N} \Psi_i^\dagger H_{ij}^\Psi \Psi_j + \frac{1}{2}\text{Tr}A, \tag{5.3}$$

where the single-body matrix H^Ψ is a $2N \times 2N$ Hermitian matrix

$$H^\Psi = \begin{bmatrix} A & B \\ -B^* & -A^* \end{bmatrix}. \tag{5.4}$$

5.1.1 Majorana Representation

The Majorana operators are defined as:

$$\begin{bmatrix} \omega_i \\ \omega_{i+N} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} c_i \\ c_i^\dagger \end{bmatrix}, \quad \begin{bmatrix} c_i \\ c_i^\dagger \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \begin{bmatrix} \omega_i \\ \omega_{i+N} \end{bmatrix}. \tag{5.5}$$

The fermionic bilinear in the Majorana basis has the form

$$\hat{H} = -\frac{i}{4}\sum_{i,j=1}^{2N} H_{ij}\omega_i\omega_j \tag{5.6}$$

[1]Without loss of generality, in the following we always assume that the sum of chemical potential is zero, i.e., $\text{Tr}A = 0$.

where the single-body matrix H is a $2N \times 2N$ real anti-symmetric matrix:

$$H = \begin{bmatrix} -A^I - B^I & A^R - B^R \\ -A^R - B^R & -A^I + B^I \end{bmatrix}. \tag{5.7}$$

where we have define $A^{R/I} = \mathrm{Re}A/\mathrm{Im}A$ and $B^{R/I} = \mathrm{Re}B/\mathrm{Im}B$. Conversely, if we have a Majorana bilinear

$$\frac{i}{2}\sum_{i,j=1}^{2N} M_{ij}\omega_i\omega_j, \quad M = \begin{bmatrix} M^{11} & M^{12} \\ M^{21} & M^{22} \end{bmatrix}, \tag{5.8}$$

it can be transformed back to ordinary fermionic bilinear (5.1) where

$$\begin{aligned} A &= M^{21} - M^{12} + iM^{11} + iM^{22}, \\ B &= M^{21} + M^{12} + iM^{11} - iM^{22}. \end{aligned} \tag{5.9}$$

A real anti-symmetric matrix can be transformed to standard form by an orthogonal transformation O:

$$\begin{aligned} H &= O \cdot \Sigma(\boldsymbol{\lambda}) \cdot O^T, \\ \Sigma(\boldsymbol{\lambda}) &= i\sigma_y \otimes \mathrm{diag}(\lambda_1, \cdots, \lambda_n). \end{aligned} \tag{5.10}$$

Make the basis transformation

$$\gamma_n = \sum_{j=1}^{2N} O_{jn}\omega_j, \tag{5.11}$$

the Hamiltonian becomes the standard form:

$$\begin{aligned} H &= -\frac{i}{4}\sum_{i=1}^{N} \lambda_i(\gamma_i\gamma_{i+N} - \gamma_{i+N}\gamma_i) \\ &= -\frac{i}{2}\sum_{i=1}^{N} \lambda_i\gamma_i\gamma_{i+N}. \end{aligned} \tag{5.12}$$

Each $\gamma_i\gamma_{i+N}$ pair can then transforms to independent fermion mode:

$$\begin{aligned} -\frac{i}{2}\gamma_i\gamma_{i+N} &= -\frac{i}{2}(d_i + d_i^\dagger)(id_i - id_i^\dagger) \\ &= d_i^\dagger d_i - \frac{1}{2}. \end{aligned} \tag{5.13}$$

5.1.2 Gaussian States

The Fermionic Gaussian states are those states with Gaussian form density operator:

$$\hat{\rho} \propto \exp\left(\frac{i}{2}\sum_{i,j=1}^{2N} M_{ij}\omega_i\omega_j\right), \tag{5.14}$$

where the matrix M is real and anti-symmetric.[2] If we expand the Gaussian form, the density operator becomes a Majorana polynomial:[3]

$$\hat{\rho} = \frac{\mathbb{I}}{2^N} + \sum_{n=1}^{N} \frac{i^n}{2^N} \sum_{1 \le i_1 < \cdots < i_{2n} \le 2N} \Gamma_{i_1 \cdots i_{2n}} \omega_{i_1} \cdots \omega_{i_{2n}}, \tag{5.15}$$

where the coefficient $\Gamma_{i_1 \cdots i_{2n}}$ is the $2n$-point correlation function:

$$\Gamma_{i_1 \cdots i_{2n}} = i^n \langle \omega_{i_1} \cdots \omega_{i_{2n}} \rangle, \quad i_m \neq i_n. \tag{5.16}$$

In particular, the 2-point function

$$\Gamma_{ij} = i\langle \omega_i \omega_j \rangle - i\delta_{ij} = \frac{i}{2} \langle [\omega_i, \omega_j] \rangle \tag{5.17}$$

is also called the *covariance matrix*. For Gaussian state all $2n$-point correlation is determined by the covariance matrix by the Wick theorem.

Remark 6. Two-point Correlation Function

We are usually more familiar with the ordinary fermionic two-point correlation function $\langle c_i^\dagger c_j \rangle$ or $\langle c_i c_j \rangle$, which is related to the Majorana covariance matrix by:

$$\begin{aligned} \langle c_i^\dagger c_j \rangle &= \frac{1}{4}(\Gamma_{ij}^{21} - \Gamma_{ij}^{12} + i\Gamma_{ij}^{11} + i\Gamma_{ij}^{22}) + \frac{1}{2}\delta_{ij}, \\ \langle c_i c_j \rangle &= \frac{1}{4}(\Gamma_{ij}^{21} + \Gamma_{ij}^{12} + i\Gamma_{ij}^{11} - i\Gamma_{ij}^{22}), \\ \langle c_i^\dagger c_j^\dagger \rangle &= \frac{1}{4}(-\Gamma_{ij}^{21} - \Gamma_{ij}^{12} + i\Gamma_{ij}^{11} - i\Gamma_{ij}^{22}). \end{aligned} \tag{5.18}$$

The relation of the correlation in each order can be neatly captured by the Grassmannian Gaussian form:

$$\begin{aligned} \omega(\hat{\rho}, \theta) &= \frac{1}{2^N} \exp\left(\frac{i}{2} \sum_{i,j=1}^{2N} \Gamma_{ij} \theta_i \theta_j \right) \\ &= \frac{1}{2^N} + \sum_{n=1}^{N} \frac{i^n}{2^N} \sum_{1 \le i_1 < \cdots < i_{2n} \le 2N} \Gamma_{i_1 \cdots i_{2n}} \theta_{i_1} \cdots \theta_{i_{2n}}. \end{aligned} \tag{5.19}$$

When the covariance matrix is obtained, we can use the same routine to canonicalize the skew-symmetric matrix Γ:

$$\Gamma = O \cdot \Sigma(\boldsymbol{\lambda}) \cdot O^T, \quad \tilde{\theta}_n = \sum_i O_{in} \theta_i,$$

and the density matrix in the Grassmann representation is

$$\omega(\hat{\rho}, \theta) = \prod_{n=1}^{N} \left(\frac{1}{2} e^{i\lambda_n \tilde{\theta}_n \tilde{\theta}_{n+N}} \right) = \prod_{n=1}^{N} \left(\frac{1 + i\lambda_n \tilde{\theta}_n \tilde{\theta}_{n+N}}{2} \right). \tag{5.20}$$

[2] In particular, any thermal state has this form, with $M = \beta H/2$. The ground state of the free fermion system, though being pure state, can be regarded as the Gaussian state in the limit $M = \lim_{\beta \to \infty} \beta H$.

[3] Note that the coefficient Γ in each order is not the direct expansion of the matrix M, since the direct expansion contains identical Majorana operators. That is, the n-th order expansion of the Majorana Gaussian form may contribute to the $(n-2m)$-th order term in the Majorana polynomial.

This state correspond to a product state $\rho = \otimes_n \rho_n$ where

$$\rho_n = \frac{1}{2}\begin{bmatrix} 1+\lambda_n & 0 \\ 0 & 1-\lambda_n \end{bmatrix}. \tag{5.21}$$

The entanglement entropy is then

$$S = \sum_n S_n = -\sum_n \left[\left(\frac{1+\lambda_n}{2}\right)\ln\left(\frac{1+\lambda_n}{2}\right) + \left(\frac{1-\lambda_n}{2}\right)\ln\left(\frac{1-\lambda_n}{2}\right)\right]. \tag{5.22}$$

5.1.3 Lindblad Master Equation

For Lindblad equation

$$\frac{d}{dt}\hat{\rho} = -i[\hat{H},\hat{\rho}] + \sum_{\mu=1}^{m}\hat{L}_\mu\hat{\rho}\hat{L}_\mu^\dagger - \frac{1}{2}\sum_{\mu=1}^{m}\{\hat{L}_\mu^\dagger\hat{L}_\mu,\hat{\rho}\} \tag{5.23}$$

When the *jump operator* $\hat{L}_\mu$ contains only the linear Majorana operator, the Lindblad equation preserve Gaussianity. For *jump operator* contains up to quadratic Majorana terms, the evolution will break the Gaussian form, however, the $2n$-point correlation is still solvable for free fermion system.

Dynamics of Covariance Matrix

We assume that the jump operator has up to quadratic Majorana terms. In particular, we denote the linear terms and the Hermitian quadratic terms as

$$\hat{L}_r = \sum_{j=1}^{2N} L_j^r \omega_j, \quad \hat{L}_s = \sum_{j,k=1}^{2N} M_{jk}^s \omega_j \omega_k. \tag{5.24}$$

Now consider the dynamics of the expectation value $\langle\hat{O}\rangle$:

$$\begin{aligned}\frac{d}{dt}\langle\hat{O}\rangle &= -i\mathrm{Tr}[\hat{O}(\hat{H}\hat{\rho}-\hat{\rho}\hat{H})] + \sum_\mu \mathrm{Tr}[\hat{O}\hat{L}_\mu\hat{\rho}\hat{L}_\mu^\dagger] - \frac{1}{2}\sum_\mu \mathrm{Tr}[\hat{O}\hat{L}_\mu^\dagger\hat{L}_\mu\hat{\rho} + \hat{O}\hat{\rho}\hat{L}_\mu^\dagger\hat{L}_\mu] \\ &= \left\langle i[\hat{H},\hat{O}] + \sum_\mu \hat{L}_\mu^\dagger\hat{O}\hat{L}_\mu - \frac{1}{2}\sum_\mu\{\hat{L}_\mu^\dagger\hat{L}_\mu,\hat{O}\}\right\rangle.\end{aligned} \tag{5.25}$$

We can express the dynamics of operator as in the Heisenberg picture:

$$\frac{d\hat{O}}{dt} = i[\hat{H},\hat{O}] + \mathcal{D}_r[\hat{O}] + \mathcal{D}_s[\hat{O}], \tag{5.26}$$

where

$$\begin{aligned}\mathcal{D}_r[\hat{O}] &= \sum_r \hat{L}_r^\dagger\hat{O}\hat{L}_r - \frac{1}{2}\sum_r\{\hat{L}_r^\dagger\hat{L}_r,\hat{O}\} = \frac{1}{2}\sum_r[\hat{L}_r^\dagger L_r,\hat{O}] - \sum_r \hat{L}_r^\dagger[\hat{L}_r,\hat{O}], \\ \mathcal{D}_s[\hat{O}] &= \sum_s \hat{L}_s\hat{O}\hat{L}_s - \frac{1}{2}\sum_r\{\hat{L}_s^2,\hat{O}\} = -\frac{1}{2}\sum_s[\hat{L}_s,[\hat{L}_s,\hat{O}]].\end{aligned} \tag{5.27}$$

The equation of motion can be further simplified to:

$$\frac{d\hat{O}}{dt} = i[\hat{H}_{\text{eff}}, \hat{O}] - \sum_r \hat{L}_r^\dagger [\hat{L}_r, \hat{O}] - \frac{1}{2}\sum_s [\hat{L}_s, [\hat{L}_s, \hat{O}]], \tag{5.28}$$

where the effective Hamiltonian is

$$\hat{H}_{\text{eff}} = \sum_{ij} \left(-\frac{i}{4} H_{ij} - \frac{1}{2} B_{ij}^I \right) \omega_i \omega_j, \tag{5.29}$$

where we have defined $B_{ij} = \sum_r L_i^r L_j^{r*}$. Using the commutation relation $\{\omega_i, \omega_j\} = 2\delta_{ij}$, we have the following relation

$$\begin{aligned} [\omega_k, \omega_i\omega_j] &= 2(\delta_{ki}\omega_j - \delta_{kj}\omega_i), \\ [\omega_k\omega_l, \omega_i\omega_j] &= 2(\delta_{ki}\omega_j\omega_l - \delta_{kj}\omega_i\omega_l + \delta_{li}\omega_k\omega_j - \delta_{lj}\omega_k\omega_i), \end{aligned} \tag{5.30}$$

and let $\hat{O}_{ij} = \omega_i\omega_j - \delta_{ij}\mathbb{I}$. The first term of EOM is:

$$\begin{aligned} i\langle[\hat{H}_{\text{eff}}, \hat{O}_{ij}]\rangle_t &= \sum_{kl} \left(\frac{1}{4} H - \frac{i}{2} B^I \right)_{kl} \langle [\omega_k\omega_l, \omega_i\omega_j] \rangle_t \\ &= \sum_{kl} \left(\frac{1}{2} H - iB^I \right)_{kl} \langle \delta_{ki}\omega_j\omega_l - \delta_{kj}\omega_i\omega_l + \delta_{li}\omega_k\omega_j - \delta_{lj}\omega_k\omega_i \rangle_t \\ &= \left[(H - 2iB^I)^T \cdot \langle\hat{O}\rangle_t + \langle\hat{O}\rangle_t \cdot (H - 2iB^I) \right]_{ij}. \end{aligned}$$

The second term is

$$\begin{aligned} -\sum_r \langle L_r^\dagger [L_r, \hat{O}_{ij}] \rangle_t &= -\sum_{kl} B_{kl}^* \langle \omega_k [\omega_l, \omega_i\omega_j] \rangle_t \\ &= -2\sum_{kl} B_{kl}^* \langle \delta_{li}\omega_k\omega_j - \delta_{lj}\omega_k\omega_i \rangle_t \\ &= -\left[2B \cdot \langle\hat{O}\rangle_t + 2\langle\hat{O}\rangle_t \cdot B^* + 4iB^I \right]_{ij}. \end{aligned}$$

And the third term is

$$\begin{aligned} -\frac{1}{2}\sum_s \langle [\hat{L}_s, [\hat{L}_s, \hat{O}_{ij}]] \rangle_t &= -\frac{1}{2}\sum_s \sum_{kl} M_{kl}^s \langle [\hat{L}_s, [\omega_k\omega_l, \omega_i\omega_j]] \rangle_t \\ &= 2\sum_s \sum_k \left\langle M_{ik}^s [\hat{L}_s, \omega_k\omega_j] - [\hat{L}_s, \omega_i\omega_k] M_{kj}^s \right\rangle_t \\ &= 8\sum_{s,kl} \left\langle M_{ik}^s [-M_{kl}^s \omega_l\omega_j + \omega_k\omega_l M_{lj}^s] + [M_{il}^s \omega_l\omega_k - \omega_i\omega_l M_{lk}^s] M_{kj}^s \right\rangle_t \\ &= 8\sum_s \left[2M^s \cdot \langle\hat{O}\rangle_t \cdot M^s - (M^s)^2 \cdot \langle\hat{O}\rangle_t - \langle\hat{O}\rangle_t \cdot (M^s)^2 \right]_{ij}. \end{aligned}$$

In together, we get the EOM of variance matrix $\Gamma_{ij}(t) = i\langle\hat{O}_{ij}\rangle_t$, the result is:

$$\partial_t\Gamma = X^T \cdot \Gamma + \Gamma \cdot X + \sum_s (Z^s)^T \cdot \Gamma \cdot Z^s + Y, \tag{5.31}$$

where

$$X = H - 2B^R + 8\sum_s (\text{Im}M^s)^2, \quad Y = 4B^I, \quad Z = 4\text{Im}M^s. \tag{5.32}$$

Chapter 6

Topological Field Theory

6.1 Chern-Simons Theory

Assume the action of the microscopical theory has the form $S[\psi_i]$, where $\{\psi_i\}$ denotes all degrees of microscopical freedom. If the system has the U(1) symmetry, we can always rewrite the field theory as a gauge theory:

$$S[\psi_i; A] = S[\psi_i] + \int d^d x \; j^\mu(x) A_\mu(x), \tag{6.1}$$

where the current j^μ is the Noether current. The gauge field $A^\mu(x)$ is regarded as the back ground field which has no dynamics. If we are interested in the low-energy physics, especially for gapped system, the ground state physics, we can formally integrate out other degrees of freedom, the resulting effective theory has only the gauge degree of freedom:

$$Z_{\text{eff}}[A] = \int D[\psi_i] e^{iS[\psi_i;A]}. \tag{6.2}$$

In this section, we consider the effective gauge field on $(2+1)$-dimensional space-time. The effective action should also be gauge-invariant. The allowed terms include

$$A \wedge dA, \; dA \wedge dA, \; \text{higher order terms.}$$

From dimensional analysis, the first term is most relevant in the low-energy. Such effective theory is the *Chern-Simons theory*:

$$S_{\text{CS}} = \frac{k}{4\pi} \int d^3 x \; \varepsilon_{\mu\nu\rho} A^\mu \partial^\nu A^\rho. \tag{6.3}$$