Entanglement in Field Theory

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In this note, we discuss the entanglement in the context of conformal field theory. We focus on the $(1+1)$D system, where the conformal symmetry can be used to produce exact results.

Contents

I. PATH-INTEGRAL FORMALISM

A. Reduced Density Matrix

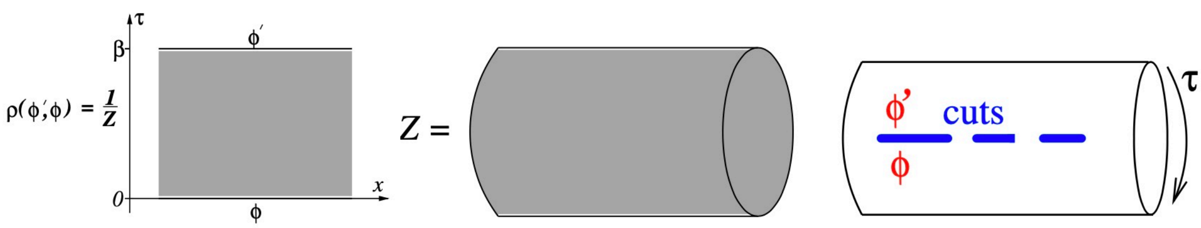
In this section, we are mostly interested in the entanglement entropy (von Neumann) $S \equiv \mathrm{Tr}\, \rho \ln \rho$. We denote the complete set of local field operators as $\{\phi_x\}$, the density operator can be written as

$$\rho\left(\{\phi_x\}|\{\phi'_{x'}\}\right) = \frac{1}{Z(\beta)} \langle\{\phi_x\}|\, e^{-\beta H}\, |\{\phi'_{x'}\}\rangle\,, \tag{1}$$

where $Z(\beta) = \mathrm{Tr}\, e^{-\beta H}$ is the partition function. This can be expressed as the path integral

$$\rho\left(\{\phi_x\} \mid \{\phi'_{x'}\}\right) = Z^{-1} \int [\mathrm{d}\phi(y,\tau)] \prod_{x'} \delta\left(\phi(y,0) - \phi'_{x'}\right) \prod_{x} \delta\left(\phi(y,\beta) - \phi_x\right) \mathrm{e}^{-S_E} \tag{2}$$

defined on the manifold with imaginary time interval $(0,\beta)$. We assume the system is in pure state, the reduced density matrix for subsystem is obtained by taking the partial trace. In the path-integral formalism, the density matrix can be represented by a cylinder with boundaries:



We see that the reduced density matrix ρ_A is obtained by sewing together only those points x which are not in A. If we consider the ground state, just take β to infinity and the manifold becomes the infinite plane.

B. Replica Trick

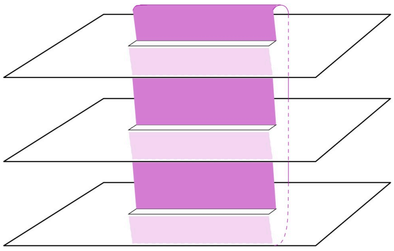
A closely related quantity for entanglement entropy is the Renyi entropy

$$S_A^{(n)} = \frac{1}{1-n} \ln \mathrm{Tr}\, \rho_A^n. \tag{3}$$

If the Renyi entropy is a analytic function of n, then the (von Neumann) entropy can be obtain by

$$S_A = \lim_{n\to 1} S_A^{(n)}. \tag{4}$$

To proceed, we first consider the case where n is integer, and then take the analytic continuation. This correspond to the "replica" system where we make n copies of the fields, with proper boundary condition. For the single-interval case, the boundary condition can be graphically represented as an n-sheeted Riemann surface:



Each layer correspond to different copy of field. The bulk theory of the the system is just n copy of the original Lagrangian, while the boundary condition introduced a brach cut to the theory.

C. Twisted Fields

Cardy et al. proposed that the field theory can be well described by inserting the *twist fields* $\mathcal{T}_n$, $\tilde{\mathcal{T}}_n$ to each of the n disconnected sheets. The partition function and other expectation values (on a single sheet) can be expressed as:

$$Z_n = \langle \mathcal{T}_n(u)\tilde{\mathcal{T}}_n(v)\rangle, \quad \langle O(x,\tau)\rangle = \frac{1}{Z_n}\langle \mathcal{T}_n(u)\tilde{\mathcal{T}}_n(v) O(x,\tau)\rangle. \tag{5}$$

One of the most important operator in the conformal field theory is the energy momentum tensor $T(z)$. The expectation value for energy momentum tensor can be obtain by consider the conformal mapping

$$z \to w(z) = \left(\frac{z-u}{z-v}\right)^{\frac{1}{n}}. \tag{6}$$

One can check that w maps the Riemann surface to a single complex plane. The energy-momentum tensors on different manifold are related by

$$\langle T(z)\rangle = \left(\frac{dw}{dz}\right)^2 \langle T(w)\rangle + \frac{c}{12}\{w,z\}, \tag{7}$$

where the Schwarzian derivative $\{z,w\}$ is

$$\begin{aligned} \{w,z\} &= \left(\frac{dw}{dz}\right)^{-2}\left[\frac{d^3w}{dz^3}\frac{dw}{dz} - \frac{3}{2}\left(\frac{d^2w}{dz^2}\right)^2\right] \\ &= \frac{1}{2}\left(1-\frac{1}{n^2}\right)\frac{(u-v)^2}{(z-u)^2(z-v)^2}. \end{aligned} \tag{8}$$

The energy-momentum tensor on the complex plane is zero: $\langle T(w)\rangle = 0$, so that on the Riemann surface is

$$\frac{\langle \mathcal{T}_n(u)\tilde{\mathcal{T}}_n(v)T(z)\rangle}{\langle \mathcal{T}_n(u)\tilde{\mathcal{T}}_n(v)\rangle} = \frac{c}{24}\left(1-\frac{1}{n^2}\right)\frac{(u-v)^2}{(z-u)^2(z-v)^2}. \tag{9}$$

If we assume the twist field is primary, with dimension d_n, the two point function is[1]

$$\langle \mathcal{T}_n(u)\tilde{\mathcal{T}}_n(v)\rangle = (u-v)^{-2d_n}. \tag{10}$$

And we also have the conformal Ward identity

$$\begin{aligned}\langle \mathcal{T}_n(u)\tilde{\mathcal{T}}_n(v)T(z)\rangle &= \left[\frac{\partial_u}{z-u} + \frac{d_n}{(z-u)^2} + \frac{\partial_v}{z-v} + \frac{d_n}{(z-v)^2}\right]\left\langle \mathcal{T}_n(u)\tilde{\mathcal{T}}_n(v)\right\rangle \\ &= d|u-v|^{2-2d}\end{aligned} \tag{11}$$

Together we know

$$d_n = \frac{c}{24}\left(1 - \frac{1}{n^2}\right). \tag{12}$$

II. ENTANGLEMENT ENTROPY

A. Single interval on an infinite chain

Now note that the partition function for the replica system is related to the Renyi entropy by:

$$\frac{Z_n}{Z^n} \propto \mathrm{Tr}\,\rho_A^n = c_n \left|\frac{u-v}{a}\right|^{-4nd_n}, \tag{13}$$

where a is the UV cutoff introduced by Z^n. The n-th order Renyi entropy is then

$$S_A^{(n)} = \frac{c}{6}\left(1 + \frac{1}{n}\right)\ln\left|\frac{u-v}{a}\right| + \frac{\ln c_n}{1-n}. \tag{14}$$

As discussed, the entanglement entropy is obtained by taking the limit

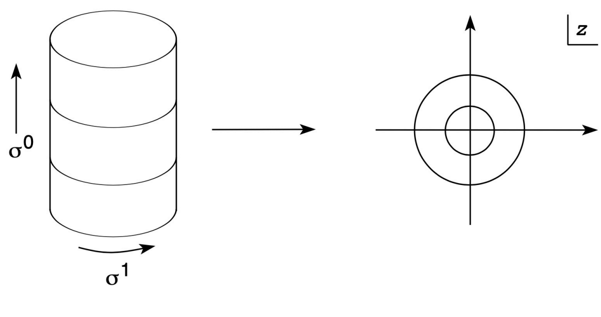
$$\lim_{n\to 1} S_A^{(n)} = \frac{c}{3}\ln\left|\frac{u-v}{a}\right| - c_1'. \tag{15}$$

We thus obtained the entanglement entropy behavior of the single interval with length l:

$$S_A = \frac{c}{3}\ln\frac{l}{a} + O(1). \tag{16}$$

B. Finite size and finite temperature

We can also obtain the exact form of entanglement entropies for finite size or finite temperature, using a special conformal mapping that map the complex plane to a infinite cylinder with circumference β:



[1] We focus on the holomorphic part, and assume the anti-holomorphic part has the same dimensionality. Also note that we have chosen a proper normalization for the twist fields.

Specifically, w and z are related by:

$$w = \frac{\beta}{2\pi} \ln z, \quad z = \exp\left(\frac{2\pi w}{\beta}\right). \tag{17}$$

There are different ways to place u and v. If we place them along the radius, their image then lie parallel to the axis of the cylinder (the images are denoted as w_1 and w_2 respectively). This correspond to the infinite chain with finite temperature. The two point function is

$$\begin{aligned}
\langle \mathcal{T}_n(w_1)\tilde{\mathcal{T}}_n(w_2)\rangle &= [w'(z_1)w'(z_2)]^{-d_n} \langle \mathcal{T}_n(z_1)\tilde{\mathcal{T}}_n(z_2)\rangle \\
&= \exp\left[\frac{2\pi}{\beta} d_n(w_1+w_2)\right]\left[\exp\left(\frac{2\pi}{\beta}w_1\right) - \exp\left(\frac{2\pi}{\beta}w_2\right)\right]^{-2d_n} \\
&= \left\{2\sinh\left[\frac{\pi(w_1-w_2)}{\beta}\right]\right\}^{-2d_n}.
\end{aligned} \tag{18}$$

Similarly we have

$$\operatorname{Tr}\rho_A^n = c_n \left[\frac{2}{b(\beta)}\sinh\left(\frac{\pi l}{\beta}\right)\right]^{-4nd_n}. \tag{19}$$

The $b(\beta)$ is a β-dependent cutoff that should have the asymptotic behavior

$$\lim_{\beta\to\infty} \frac{2}{b(\beta)}\sinh\left(\frac{\pi l}{\beta}\right) = \frac{l}{a} \quad \Longrightarrow \quad b(\beta) = \frac{2\pi a}{\beta}. \tag{20}$$

Thus the Renyi entropy for the replica system is

$$S_A^{(n)} = \frac{c}{6}\left(1+\frac{1}{n}\right)\ln\left[\frac{\beta}{\pi a}\sinh\left(\frac{\pi l}{\beta}\right)\right] + \frac{\ln c_n}{1-n}. \tag{21}$$

The entanglement entropy from the replica limit is

$$S_A = \frac{c}{3}\ln\left[\frac{\beta}{\pi a}\sinh\left(\frac{\pi l}{\beta}\right)\right] + O(1). \tag{22}$$

Apart from this, we can also place u and v on the same circle so that their image is perpendicular to the axis of the cylinder. The calculation carries out without a change, but now β is regarded as the system size (periodic boundary condition) while the temperature is zero. We usually denote the size of the finite chain as L, so the finite-size result is:

$$S_A = \frac{c}{3}\ln\left[\frac{L}{\pi a}\sinh\left(\frac{\pi l}{L}\right)\right] + O(1). \tag{23}$$

III. QUENCH DYNAMICS