Gaussian Stochastic Schrödinger Equation

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Contents

1 Gaussian System

1.1 BdG Hamiltonian

The BdG Hamiltonian has the form

$$\hat{H}=\sum_{i,j=1}^{n}A_{ij}c_i^\dagger c_j+\frac{1}{2}\sum_{i,j=1}^{n}(B_{ij}c_i^\dagger c_j^\dagger-B_{ij}^*c_ic_j)=\frac{1}{2}\sum_{i,j=1}^{2n}\Psi_i^\dagger H_{ij}\Psi_j, \tag{1}$$

where A is Hermitian and B is antisymmetric, the Hamiltonian matrix is

$$H=\begin{bmatrix} A & B \\ -B^* & -A^* \end{bmatrix}, \tag{2}$$

and the Nambu spinor Ψ is

$$\Psi_i=\begin{cases} c_i & 1\le i\le n \\ c_i^\dagger & n+1\le i\le 2n \end{cases}. \tag{3}$$

A fermionic Gaussian state can be regarded as the ground state of a BdG Hamiltonian.

The action of particle-hole symmetry $\mathcal{C}$ on the Nambu spinor is $\mathcal{C}\cdot\Psi=\sigma^x\cdot\Psi^*$, on Hamiltonian is $\mathcal{C}H\mathcal{C}^{-1}=\sigma^xH^*\sigma^x=-H$. The unitary transformation conserving particle-hole symmetry is

$$T^\dagger HT=\mathrm{diag}(\varepsilon_1,\cdots,\varepsilon_N,-\varepsilon_1,\cdots,-\varepsilon_N),\quad T=\begin{bmatrix} U & V^* \\ V & U^* \end{bmatrix}. \tag{4}$$

Define a set of new fermionic modes

$$d_k^\dagger=\sum_{j=1}^{2n}\Psi_j^\dagger T_{jk}=\sum_{j=1}^{n}(U_{jk}c_j^\dagger+V_{jk}c_j), \tag{5}$$

where $k = 1, 2, \ldots, n$. The Hamiltonian is

$$\hat{H} = \frac{1}{2}\sum_n \varepsilon_n (d_n^\dagger d_n - d_n d_n^\dagger) = \sum_n \varepsilon_n \left(d_n^\dagger d_n - \frac{1}{2} \right). \tag{6}$$

The following code does the diagonalization in the Nambu basis:

```
function bdg_eigen(A, B)
    n = size(A, 1)
    H = [A B; -conj(B) -conj(A)]
    vals, vecs = eigen(Hermitian(H))
    e = real(vals[n+1:2n])
    U, V = vecs[1:n, n+1:2n], vecs[n+1:2n, n+1:2n]
    T = [U conj(V); V conj(U)]
    e, T
end
```

1.2 Majorana Basis

Under the Majorana basis $\underline{\Omega} = (\omega_1, \ldots, \omega_{2n})^T$, where

$$\omega_i = \begin{cases} c_i + c_i^\dagger & 1 \le i \le n \\ i(c_i - c_i^\dagger) & n+1 \le i \le 2n \end{cases}. \tag{7}$$

the Hamiltonian is

$$\hat{H} = \frac{1}{8}\underline{\Omega} \begin{bmatrix} \mathbb{I} & \mathbb{I} \\ i\mathbb{I} & -i\mathbb{I} \end{bmatrix} \cdot \begin{bmatrix} A & B \\ -B^* & -A^* \end{bmatrix} \cdot \begin{bmatrix} \mathbb{I} & -i\mathbb{I} \\ \mathbb{I} & i\mathbb{I} \end{bmatrix} \underline{\Omega} = -\frac{i}{4}\sum_{i,j=1}^{2n} \Omega_i H_{ij} \Omega_j. \tag{8}$$

Here, the Hamiltonian matrix under the Majorana basis is

$$H = \begin{bmatrix} -\mathrm{Im}A - \mathrm{Im}B & \mathrm{Re}A - \mathrm{Re}B \\ -\mathrm{Re}A - \mathrm{Re}B & -\mathrm{Im}A + \mathrm{Im}B \end{bmatrix}, \tag{9}$$

where $\mathrm{Re}A = \mathrm{Re}[A]$, $\mathrm{Im}A = \mathrm{Im}[A]$, $\mathrm{Re}B = \mathrm{Re}[B]$, $\mathrm{Im}B = \mathrm{Im}[B]$. The canonical form of antisymmetric matrix H is:

$$H = R \begin{bmatrix} 0 & \mathrm{diag}(\varepsilon_1, \cdots, \varepsilon_n) \\ -\mathrm{diag}(\varepsilon_1, \cdots, \varepsilon_n) & 0 \end{bmatrix} R^T, \tag{10}$$

where $R \in \mathrm{SO}(2n)$. By defining the new Majorana operator

$$\gamma_k = \sum_{j=1}^{2n} \omega_j R_{jk}, \quad k = 1, \cdots, 2n. \tag{11}$$

The Hamiltonian becomes diagonal

$$H = -\frac{i}{2}\sum_{k=1}^{n} \varepsilon_k \gamma_k \gamma_{k+n} = \sum_{k=1}^{n} \varepsilon_k \left(d_k^\dagger d_k - \frac{1}{2} \right), \tag{12}$$

The diagonalization is done by

```
function majorana_eigen(A, B)
    n = size(A, 1)
    S, V, vals = schur(majoranaform(A, B))
    e = abs.(imag(vals))
    R = Matrix{Float64}(undef, 2n, 2n)
    for i in 1:n
        if S[2i-1, 2i] > 0
            R[:, i] = V[:, 2i-1]
            R[:, i+n] = V[:, 2i]
        else
            R[:, i] = V[:, 2i]
            R[:, i+n] = V[:, 2i-1]
        end
    end
    e, R
end
```

Note that the ground state $|\psi\rangle$ of H is annihilated by all d's,

$$\sum_{j=1}^{n} U_{jk}^{*} c_j |\psi\rangle = -\sum_{j=1}^{n} V_{jk}^{*} c_j^{\dagger} |\psi\rangle, \quad \forall k = 1, 2, \ldots, n. \tag{13}$$

For this linear form, we immediately see that $|\psi\rangle$ has the Gaussian form:

$$|\psi\rangle = \frac{1}{\mathcal{N}} \exp\left(\sum_{ij} M_{ij} c_i^{\dagger} c_j^{\dagger}\right) |0\rangle. \tag{14}$$

The equation leads to

$$c_j |\psi\rangle = e^{\sum_{kl} M_{kl} c_k^{\dagger} c_l^{\dagger}} e^{-\sum_{kl} M_{kl} c_k^{\dagger} c_l^{\dagger}} c_j e^{\sum_{kl} M_{kl} c_k^{\dagger} c_l^{\dagger}} |0\rangle = -2 M_{kj} c_k^{\dagger} |\psi\rangle.$$

Therefore $M = \frac{1}{2} V^{*} \cdot (U^{*})^{-1}$.

2 Dirac SSE

For particle number conserving systems, the Gaussian state can be represented as a matrix:

$$|B\rangle \equiv \prod_{j=1}^{N} \sum_{i} B_{ij} c_i^{\dagger} |0\rangle \equiv \bigotimes_{j=1}^{N} |B_j\rangle. \tag{15}$$

Note that the matrix B representing the Gaussian state has the unitary degree of freedom

$$|B\rangle = |B'\rangle, \quad B'_{ij} = \sum_{k} B_{ik} U_{kj},$$

where U_{kj} is an $N \times N$ unitary matrix. It means that the Gaussian state is determined by the linear subspace that columns of B span. The columns of B need not be orthogonal, while the canonical form can be obtained by the QR decomposition: $B_{L\times N} = Q_{L\times N} \cdot R_{N\times N}$, where the Q matrix is orthonormal and we can set $B' = Q$.

2.1 Evolution

The Stochastic Schrödinger equation can be Trotterized as

$$|\psi'\rangle = \left(\prod_x \mathcal{M}_x\right) e^{-iH\Delta t}|\psi\rangle. \tag{16}$$

That is, the monitored dynamics can be regarded as a two-step process, where the state $|\psi\rangle$ first undergoes a coherent Hamiltonian evolution and is then subject to a generalized measurement controlled by parameter Δt. Using the Baker-Campbell-Hausdorff formula,

$$e^X Y e^{-X} = \exp(\mathrm{ad}\, X)Y, \quad e^{-iH\Delta t} c_j^\dagger e^{iH\Delta t} = c_k^\dagger [e^{-iH\Delta t}]_{kj}.$$

Therefore,[1]

$$e^{-iH\Delta t}|B\rangle = \prod_{j=1}^{N}\sum_i \left[e^{-iH\Delta t}\right]_{ki} B_{ij} c_k^\dagger |0\rangle = \left|e^{-iH\Delta t}\cdot B\right\rangle. \tag{17}$$

The Krause operator of $\mathcal{M}$ is

$$M_x = L_x\sqrt{\gamma\Delta t}, \quad M_0 = \sqrt{1 - L_x^\dagger L_x \gamma \Delta t}. \tag{18}$$

We consider the case where the jump operator L_x the form:

$$L_x = U_x d_x^\dagger d_x, \quad L_x^\dagger L_x = d_x^\dagger d_x, \tag{19}$$

where U_x is a Gaussian unitary operator. In this case, the Kraus operator is

$$M_0 = \sqrt{(1-\gamma\Delta t)d_x^\dagger d_x + d_x d_x^\dagger} = \sqrt{1-\gamma\Delta t} d_x^\dagger d_x + d_x d_x^\dagger = 1 - \left(1 - \sqrt{1-\gamma\Delta t}\right) d_x^\dagger d_x.$$

This operator preserve Gaussianity since $\exp(-\alpha d_x^\dagger d_x) = 1 - (1 - e^{-\alpha}) d_x^\dagger d_x$.

2.2 Quantum Jump

When applied to a quasi-particle creation/annihilation operator, a free fermion state maintains its structure. Consider a general quasi-particle $b^\dagger = \sum_i b_i c_i^\dagger$, creating a quasiparticle is simply adding a column to B, since

$$b^\dagger|B\rangle = \sum_k b_k c_k^\dagger \prod_{j=1}^{N}\sum_i c_i^\dagger B_{ij}|0\rangle = \prod_{j=1}^{N+1}\sum_i c_i^\dagger \left[b|B\right]_{ij}|0\rangle. \tag{20}$$

The new column b is not orthogonal to linear space B. Therefore, an orthogonalization procedure is needed to obtain canonical form.

For the quasiparticle annihilation operator b,

$$b|B\rangle = \sum_k b_k^* c_k \prod_j \sum_i c_i^\dagger B_{ij}|0\rangle = \sum_j \langle b|B_j\rangle \bigotimes_{l\neq j} |B_l\rangle.$$

We can use the gauge freedom to restrict $\langle b|B_j'\rangle = 0$ for $j > 1$. Such matrix B' always exists since we can always find a column j that $\langle b|B_j\rangle \neq 0$ (otherwise $p_m = 0$ and the jump is impossible). We then move the column to the first and define the column as

$$|B_j'\rangle = |B_j\rangle - \frac{\langle a|B_j\rangle}{\langle a|B_1\rangle}|B_1\rangle, \quad j > 1. \tag{21}$$

Such column transformations do not alter the linear space B spans, while the orthogonality and the normalization might be affected.

[1] Note that the Hamiltonian is not necessarily Hermitian, so the vectors will no longer be orthogonal.

2.3 Entanglement Entropy

The density matrix for m-site subregion has the Gaussian form $\rho = e^{-M_{ij}c_i^\dagger c_j}$. If we diagonalize it,

$$\rho = \frac{1}{Z}\exp(-\sum_k \lambda_k a_k^\dagger a_k) = \bigotimes_{k=1}^{n} \frac{1}{1+e^{-\lambda_k}} \begin{bmatrix} 1 & 0 \\ 0 & e^{-\lambda_k} \end{bmatrix} = \bigotimes_{k=1}^{n} \mathrm{diag}(1-\mu_k, \mu_k). \tag{22}$$

Note that $\{a_k\}$ is also the basis that diagonalizes the correlation function $G_{ij} = \mathrm{Tr}[\rho c_i^\dagger c_j]$, with

$$U \cdot G \cdot U^T = \mathrm{diag}(\mu_1, \cdots, \mu_m). \tag{23}$$

The entropy is

$$S(\rho) = -\mathrm{Tr}[\rho \log \rho] = -\sum_k [\mu_k \log \mu_k + (1-\mu_k)\log(1-\mu_k)] = \sum_k H(\mu_k). \tag{24}$$

Note that the eigenvalue μ_k of correlation G is the square of singular values of $B|_{\text{subregion}}$ matrix.

3 Majorana SSE

The Majorana operators are defined as $\omega_j = c_i + c_i^\dagger$, $\omega_{j+n} = i(c_i - c_i^\dagger)$. For the Majorana case, a Gaussian pure state $|\psi$ can be expressed as

$$|\psi\rangle\langle\psi| = \prod_{j=1}^{n} d_j^\dagger d_j. \tag{25}$$

Note that the state is annihilated by $\{d_j^\dagger\}$. We can store the information of $|\psi\rangle$ into a $2n \times n$ complex matrix B. The rest of the procedures parallel those of the Dirac fermion case.

4 Grassmann Representation

In this section, we discuss the general fermionic Gaussian state in the framework of the Grassmann representation. We will closely follow Ref. [1]. A general operator in Fermionic Fock space can be expanded on the Majorana basis:

$$\hat{X} = \alpha \hat{I} + \sum_{p=1}^{2n} \sum_{1\le a_1 < \cdots < a_p \le 2n} \alpha_{a_1\cdots a_p} \hat{\omega}_{a_1} \cdots \hat{\omega}_{a_p}. \tag{26}$$

Define a linear map from Fermionic operator space to Grassmann algebra:

$$\hat{X} \mapsto X(\theta) = \alpha + \sum_{1\le a_1 < \cdots < a_p \le 2n} \alpha_{a_1\cdots a_p} \theta_{a_1} \cdots \theta_{a_p}. \tag{27}$$

This mapping is called the Grassmann representation of $\hat{X}$.

One can formally define calculus on Grassmann algebra:

$$\frac{\partial}{\partial \theta_i}\theta_j = \int d\theta_i \theta_j = \delta_{ij}, \quad \frac{\partial}{\partial \theta_i} 1 = \int d\theta_i 1 = 0. \tag{28}$$

The Gaussian integral of Grassmann algebra is

$$\int D\theta \; e^{\eta^T\theta + \frac{i}{2}\theta^T M \theta} = i^n \,\mathrm{Pf}(M) e^{-\frac{i}{2}\eta^T M^{-1}\eta}. \tag{29}$$

One useful result concerning the expectation value is

Theorem 1. *For two operator $\hat{X}$ and $\hat{Y}$, we have the following identity*

$$\mathrm{Tr}\left(\hat{X}\hat{Y}\right) = (-2)^n \int D[\theta,\mu] e^{\theta^T\cdot\mu} X(\theta)Y(\mu).$$

where $\int D\theta = \int d\theta_{2n}\cdots\int d\theta_1$, $\int D\mu = \int d\mu_{2n}\cdots\int d\mu_1$.

Proof. We prove the statement by considering only m-th order monomial. On the one hand

$$\mathrm{LHS} = \mathrm{Tr}[\hat{\omega}_1\cdots\hat{\omega}_m\hat{\omega}_1\cdots\hat{\omega}_m] = 2^n(-1)^{m(m-1)/2}.$$

On the other hand,

$$\begin{aligned}\mathrm{RHS} &= (-2)^n \int D[\theta,\mu]\prod_{i=1}^{m}\theta_i\prod_{j=m+1}^{2n}(\theta_j\mu_j)\prod_{k=1}^{m}\mu_k \\ &= (-2)^n(-1)^{(4n-m)m+(m+1+2n)(2n-m)/2} \\ &= 2^n(-1)^{-m(m+3)/2} = 2^n(-1)^{m(m-1)/2}.\end{aligned}$$

We, therefore, proved the statement. □

4.1 Gaussian States

Definition 1. *A quantum state $\hat{\rho}$ is Gaussian if it has Gaussian Grassmann representation:*

$$\rho(\theta) = \frac{1}{2^n}\exp\left(\frac{i}{2}\theta^T M\theta\right), \tag{30}$$

where the antisymmetric matrix $M_{ab} = \frac{i}{2}\mathrm{Tr}(\hat{\rho}[\hat{\omega}_a,\hat{\omega}_b])$ is the ***covariance matrix****.*

All higher correlations of a Gaussian state are determined by the Wick theorem, namely

$$\mathrm{Tr}(i^p\hat{\rho}\hat{\omega}_{a_1}\cdots\hat{\omega}_{a_p}) = \mathrm{Pf}(M|_{a_1,\ldots,a_p}).$$

The canonical form of antisymmetric matrix M is:

$$M = R\begin{bmatrix} 0 & \mathrm{diag}(\lambda_1,\cdots,\lambda_n) \\ -\mathrm{diag}(\lambda_1,\cdots,\lambda_n) & 0\end{bmatrix}R^T, \tag{31}$$

where $R\in\mathrm{SO}(2n)$. Under the new Grassmann variance $\mu = R\theta$, ρ has the form

$$\rho(\mu) = \frac{1}{2^n}\prod_j \exp\left(i\lambda_j\mu_j\mu_{j+n}\right) = \frac{1}{2^n}\prod_j\left(1+i\lambda_j\mu_j\mu_{j+n}\right). \tag{32}$$

We can then obtain the operator form:

$$\hat{\rho} = 2^{-n}\prod_{j=1}^{n}(1+i\lambda_j\hat{\gamma}_j\hat{\gamma}_{j+n}) \tag{33}$$

where $\hat{\gamma}$'s are a new set of Majorana operators. In the fermion basis

$$\hat{d}_j = \frac{\hat{\gamma}_j - i\hat{\gamma}_{j+n}}{2}, \quad \hat{d}_j^\dagger = \frac{\hat{\gamma}_j + i\hat{\gamma}_{j+n}}{2}, \tag{34}$$

the density matrix has the form

$$\hat{\rho} = \prod_j\left(\frac{1+\lambda_j}{2} - \lambda_j d_j^\dagger d_j\right) = \bigotimes_j \begin{bmatrix}\frac{1+\lambda_j}{2} & 0 \\ 0 & \frac{1-\lambda_j}{2}\end{bmatrix}_j. \tag{35}$$

Without loss of generality, we assume $\lambda_i \geq 0$. For pure state, $\lambda_i = 1,\ \forall i$. For mixed state, the entropy of ρ is just

$$S(\hat{\rho}) = \sum_j H\left(\frac{1+\lambda_j}{2}\right), \tag{36}$$

where $H(p) = -p\log p - (1-p)\log(1-p)$.

4.2 Gaussian Operators

Definition 2. *An operator $\hat{X}$ (with nonzero trace) is Gaussian if*

$$X(\theta) = C\exp\left(\frac{i}{2}\theta^T M\theta\right)$$

for some complex number C and some ***complex antisymmetric*** *matrix M. M is called a correlation matrix of $\hat{X}$. If $\hat{X}$ is traceless, it should be thought of as a limit $\hat{X} = \lim_{m\to\infty}\hat{X}_m$ for some converging sequence of Gaussian operators with nonzero trace.*

Note that for traceless $\hat{X}$, the explicit form of $X(\theta)$ is

$$X(\theta) = C\left(\prod_{a=1}^{2k}\mu_a\right)\exp\left(\frac{i}{2}\sum_{a,b=2k+1}^{2n} M_{ab}\mu_a\mu_b\right), \tag{37}$$

where $\mu_a = \sum_b T_{ab}\theta_b$ for some invertible complex matrix T. The factor is a limiting point of the sequence:

$$\prod_{a=1}^{2k}\mu_a = \lim_{t\to\infty}\prod_{a=1}^{k}\left(\mu_{2a-1}\mu_{2a} + \frac{1}{t}\right) = \lim_{t\to\infty}\frac{1}{t^k}\exp\left(t\sum_{a=1}^{k}\mu_{2a-1}\mu_{2a}\right). \tag{38}$$

Introducing the operator $\hat{\Lambda} \equiv \sum_{a=1}^{2n}\hat{\omega}_a\otimes\hat{\omega}_a$, we have the following theorem:

Theorem 2. *An operator $\hat{X}$ is Gaussian iff $\hat{X}$ is even and satisfies*

$$[\hat{\Lambda}, \hat{X}\otimes\hat{X}] = 0.$$

Proof. The adjoint action of $\hat{\Lambda}$ in the Grassmann representation has the form:

$$\Lambda_{\text{ad}} = 2\sum_a\left(\theta_a\otimes\frac{\partial}{\partial\theta_a} + \frac{\partial}{\partial\theta_a}\otimes\theta_a\right) \equiv \sum_a\Delta_a. \tag{39}$$

That is, $[\hat{\omega}_a\otimes\hat{\omega}_a, Y\otimes Z](\theta) = \Delta_a\cdot Y(\theta)\otimes Z(\theta)$ for any operators Y, Z having the same parity. Without loss of generality, both Y and Z are monomials in $\hat{\omega}$'s. Each commutes or anticommutes with $\hat{\omega}_a$. Consider two cases:

1. Both Y and Z contain $\hat{\omega}_a$, or both Y and Z do not contain $\hat{\omega}_a$. Then the commutator $[\hat{\omega}_a\otimes\hat{\omega}_a, Y\otimes Z]$ is zero since both factors yield the same sign. The right-hand side is also zero since either θ_a or $\partial/\partial\theta_a$ annihilates both Y and Z.

2. Y contains $\hat{\omega}_a$ while Z does not contain $\hat{\omega}_a$ (or vice verse). In this case $\hat{\omega}_a\otimes\hat{\omega}_a$ anticommutes with $Y\otimes Z$. Let us write $Y = \hat{\omega}_a\tilde{Y}$, where $\tilde{Y}$ is a monomial which does not contain $\hat{\omega}_a$. We have:

   $$[\hat{\omega}_a\otimes\hat{\omega}_a, Y\otimes Z] = 2(\hat{\omega}_a\otimes\hat{\omega}_a)(Y\otimes Z) = 2\tilde{Y}\otimes(\hat{\omega}_a Z).$$

   On the other hand,

   $$\theta_a\otimes\frac{\partial}{\partial\theta_a}\cdot Y\otimes Z = 0,\ \frac{\partial}{\partial\theta_a}\otimes\theta_a\cdot Y\otimes Z = \tilde{Y}\otimes\theta_a Z.$$

   We again get equality.

Necessity: Note that Λ_{ad} is invariant under change of variables since $\mu_a = \sum_b T_{ab}\theta_b$,

$$\frac{\partial}{\partial \mu_a} = \sum_b (T^{-1})_{ab}\frac{\partial}{\partial \theta_b} \implies \sum_a \theta_a \otimes \frac{\partial}{\partial \theta_a} = \sum_a \mu_a \otimes \frac{\partial}{\partial \mu_a}.$$

Direct application of the operator to the general Gaussian form will prove the necessity.
Sufficiency: Denote $C = 2^{-n}\operatorname{tr}(X) \equiv X(0)$ and represent $X(\theta)$ as

$$X(\theta) = C \cdot 1 + \frac{iC}{2}\sum_{a,b=1}^{2n} M_{ab}\,\theta_a\theta_b + \text{higher order terms}.$$

Applying a differential operator $1 \otimes \frac{\partial}{\partial \theta_b}$ to both sides:

$$\sum_{a=1}^{2n}\left(\theta_a X \otimes \frac{\partial^2 X}{\partial \theta_b \partial \theta_a} - \frac{\partial X}{\partial \theta_a} \otimes \theta_a \frac{\partial X}{\partial \theta_b}\right) + \frac{\partial (X \otimes X)}{\partial \theta_b} = 0.$$

Now let us put $\theta \equiv 0$ in the second factor:

$$\frac{\partial}{\partial \theta_b} X = i \sum_{a=1}^{2n} M_{ba}\theta_a X.$$

This differential equation can be easily solved by $X(\theta) = C\exp\left(\frac{i}{2}\,\theta^T M\theta\right)$.

For general cases, we denote $\mathcal{K} \subseteq \mathcal{M}_1$ a subspace spanned by linear functions which annihilate $\hat{X}$, i.e.

$$\mathcal{K} = \{f \in \mathcal{M}_1 \;:\; f(\theta)X(\theta) = 0\}.$$

Let us perform a linear change of variables $\mu_a = \sum_b T_{ab}\theta_b$, with T being an invertible complex matrix chosen such that the first k variables μ span the subspace $\mathcal{K}$, i.e. $\mathcal{K} = \operatorname{span}[\mu_1, \ldots, \mu_{2k}]$. From equalities $\mu_j X = 0$, $j \in [1, 2k]$ it follows that

$$X(\theta(\mu)) = \left(\prod_a \mu_a\right)\tilde{X}(\mu),$$

where $\tilde{X}(\mu)$ depends only upon $\mu_{2k+1}, \ldots, \mu_{2n}$. The function $\tilde{X}(\mu)$ satisfies the equation

$$\sum_{a=2k+1}^{2n}\left(\mu_a \otimes \frac{\partial}{\partial \mu_a} + \frac{\partial}{\partial \mu_a} \otimes \mu_a\right)\tilde{X} \otimes \tilde{X} = 0.$$

Therefore, we get the general Gaussian form. □

4.3 Gaussian Linear Maps

We define linear maps that preserve Gaussian states as the following:

Definition 3. *A linear map $\mathcal{E}$ is Gaussian iff it admits an integral representation*

$$\mathcal{E}(X)(\theta) = C\int D[\eta, \mu]\exp\left[S(\theta, \eta) + i\eta^T \mu\right]X(\mu), \tag{40}$$

where

$$S(\theta, \eta) = \frac{i}{2}(\theta^T, \eta^T)\begin{pmatrix} A & B \\ -B^T & D \end{pmatrix}\begin{pmatrix} \theta \\ \eta \end{pmatrix} \tag{41}$$

for some complex $2n \times 2n$ matrices A, B, D, and some complex number C.

Consider a Gaussian operator $\hat{X}$, described by a correlation matrix M and a Gaussian map $\mathcal{E}$. Applying the Gaussian integration, one can show that $\mathcal{E}(X)$ has a correlation matrix

$$\mathcal{E}(M) = A + B\left(M^{-1} + D\right)^{-1} B^T = A + B\,(I + MD)^{-1}\,MB^T,$$

while a pre-exponential factor of the operator $\mathcal{E}(X)$ can be found from an identity

$$\mathrm{tr}\,(\mathcal{E}(X)) = C(-1)^n\,\mathrm{Pf}(M)\,\mathrm{Pf}(M^{-1} + D)\,\mathrm{tr}\,(X).$$

The value of $\mathrm{tr}\,(\mathcal{E}(X))$ can be found up to a factor ± 1 using a regularized version:

$$\mathrm{tr}\,(\mathcal{E}(X))^2 = C^2 \det(I + MD)\,\mathrm{tr}\,(X)^2.$$

References

[1] S. Bravyi, *Lagrangian representation for fermionic linear optics* (2004), quant-ph/0404180.