Notes on Quantum Field Theory

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Chapter 1

Relativistic Quantum Field Theory

1.1 Lorentz Invariance

1.1.1 The Lorentz Algebra

The metric is chosen to be

$$g_{\mu\nu} = g^{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1). \tag{1.1}$$

The Lorentz transformation $\Lambda^{\mu}{}_{\nu}$ satisfies

$$\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}g^{\alpha\beta} = g^{\mu\nu}. \tag{1.2}$$

From this we have

$$g^{\gamma\alpha}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}g_{\mu\nu} = g^{\gamma\alpha}g_{\alpha\beta} \quad \Longrightarrow \quad \Lambda_{\nu}{}^{\gamma}\Lambda^{\nu}{}_{\beta} = \delta^{\gamma}{}_{\beta}, \tag{1.3}$$

The inverse Lorentz transformation satisfies:

$$(\Lambda^{-1})^{\mu}{}_{\nu} = \Lambda_{\nu}{}^{\mu}. \tag{1.4}$$

The infinitesimal transformation is denoted as

$$\begin{aligned} \Lambda^{\mu}{}_{\nu} &= \delta^{\mu}{}_{\nu} + \delta\omega^{\mu}{}_{\nu} \\ (\Lambda^{-1})^{\mu}{}_{\nu} &= \delta^{\mu}{}_{\nu} - \delta\omega^{\mu}{}_{\nu} \end{aligned} \quad \Longrightarrow \quad g_{\alpha\nu}\delta\omega^{\nu}{}_{\beta} + \delta\omega^{\mu}{}_{\alpha}g_{\mu\beta} = \delta\omega_{\alpha\beta} + \delta\omega_{\beta\alpha} = 0. \tag{1.5}$$

A representation of Lorentz group $U(\Lambda)$ can be parametrized as:

$$U(\Lambda) = \exp\left(\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right). \tag{1.6}$$

Another useful parametrization is

$$\theta_i \equiv \frac{1}{2}\varepsilon_{ijk}\omega_{jk},\ \ \beta_i \equiv \omega_{i0}. \tag{1.7}$$

A new set of generators are:

$$J_i \equiv \frac{1}{2}\varepsilon_{ijk}M^{jk},\ \ K_i \equiv M^{i0}, \tag{1.8}$$

where J_i's are the generators of the spatial rotations, and K_i's are the generators of Lorentz boosts.

In the fundamental representation, the generators are represented by

$$\begin{aligned} J_1 &= \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 0 & -i \\ & & i & 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & & & \\ & 0 & & i \\ & & 0 & \\ & -i & & 0 \end{bmatrix}, \quad J_3 = \begin{bmatrix} 0 & & & \\ & 0 & -i & \\ & i & 0 & \\ & & & 0 \end{bmatrix}, \\ K_1 &= \begin{bmatrix} 0 & -i & & \\ -i & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0 & & -i & \\ & 0 & & \\ -i & & 0 & \\ & & & 0 \end{bmatrix}, \quad K_3 = \begin{bmatrix} 0 & & & -i \\ & 0 & & \\ & & 0 & \\ -i & & & 0 \end{bmatrix}. \end{aligned} \tag{1.9}$$

The Lie algebra of the Lorentz algebra can be explicitly done using the fundamental representation. The result is

$$\begin{aligned} [J_i, J_j] &= i\varepsilon_{ijk} J_k, \\ [J_i, K_j] &= i\varepsilon_{ijk} K_k, \\ [K_i, K_j] &= -i\varepsilon_{ijk} J_k. \end{aligned} \tag{1.10}$$

By defining a new set of generators:

$$N_i^L \equiv \frac{J_i - iK_i}{2}, \; N_i^R \equiv \frac{J_i + iK_i}{2}. \tag{1.11}$$

They satisfies two independent $\mathfrak{su}(2)$ algebra:

$$\begin{aligned} \left[N_i^L, N_j^L\right] &= i\varepsilon_{ijk} N_k^L, \\ \left[N_i^R, N_j^R\right] &= i\varepsilon_{ijk} N_k^R, \\ \left[N_i^L, N_j^R\right] &= 0. \end{aligned} \tag{1.12}$$

That is, the Lorentz algebra is isomorphic to two $\mathfrak{su}(2)$ algebra,

$$\mathfrak{so}(3,1) \approx \mathfrak{su}_L(2) \oplus \mathfrak{su}_R(2). \tag{1.13}$$

From Eq. (1.13), we know that the representation of the Lorentz algebra can be labelled by j_L and j_R. Note that the fundamental representation correspond to

$$\left(j_L = \frac{1}{2}, j_R = \frac{1}{2}\right).$$

The specific form of the group is

$$\Lambda(\vec{\theta}, \vec{\beta}) = \exp\left[i(\vec{\theta} + i\vec{\beta}) \cdot \vec{N}^L + i(\vec{\theta} - i\vec{\beta}) \cdot \vec{N}^R\right]. \tag{1.14}$$

The spinor representations are those with $j_L = 1/2$ or $j_R = 1/2$. Specifically, we define the left-hand spinor ψ_L and right-hand spinor ψ_R that transform as:

$$\begin{aligned} \Lambda_L(\vec{\theta}, \vec{\beta})\psi_L &= \exp\left(\frac{i}{2}\vec{\theta} \cdot \vec{\sigma} - \frac{1}{2}\vec{\beta} \cdot \vec{\sigma}\right)\psi_L, \\ \Lambda_R(\vec{\theta}, \vec{\beta})\psi_R &= \exp\left(\frac{i}{2}\vec{\theta} \cdot \vec{\sigma} + \frac{1}{2}\vec{\beta} \cdot \vec{\sigma}\right)\psi_R. \end{aligned} \tag{1.15}$$

Using the fact $\sigma^2 \cdot \vec{\sigma}^* \cdot \sigma^2 = -\vec{\sigma}$, the left-hand and the right-hand representations are related by:

$$\begin{aligned} \sigma^2\Lambda_L^*\sigma^2 &= \Lambda_R, \quad \sigma^2\Lambda_L^T\sigma^2 = \Lambda_L^{-1}, \\ \sigma^2\Lambda_R^*\sigma^2 &= \Lambda_L, \quad \sigma^2\Lambda_R^T\sigma^2 = \Lambda_R^{-1}. \end{aligned} \tag{1.16}$$

For this reason, the left-hand and right-hand spinor can be interchanged by

$$\begin{aligned} \sigma^2\psi_L^* &\sim \chi_R, \quad \psi_L^\dagger\sigma^2 \sim \chi_R^\dagger, \\ \sigma^2\psi_R^* &\sim \chi_L, \quad \psi_R^\dagger\sigma^2 \sim \chi_L^\dagger. \end{aligned} \tag{1.17}$$

1.1.2 The Invariant Symbols

The invariant symbols can be thought as the Clebsch-Gordan coefficients that help to form singlets. The first singlet comes from the decomposition

$$\frac{1}{2} \otimes \frac{1}{2} \approx 0 \oplus 1.$$

Correspondingly, we can check that for each-hand-side spinor, the quadratic forms

$$\psi_L^T\sigma^2\chi_L \quad \text{or} \quad \psi_R^T\sigma^2\chi_R \tag{1.18}$$

are singlets. We can define the first invariant symbol as[1]

$$\varepsilon^{ab} = \varepsilon^{\dot{a}\dot{b}} = i(\sigma^2)_{ab}, \quad \varepsilon_{ab} = \varepsilon_{\dot{a}\dot{b}} = -i(\sigma^2)_{ab}. \tag{1.19}$$

The symbol ε^{ab} or ε_{ab} also serve as the index raising/lowering symbol, i.e.,

$$\varepsilon^{ab}\psi_b = \psi^a, \;\; \varepsilon_{ab}\psi^b = \psi_a. \tag{1.20}$$

The singlet (1.18) is then defined as the inner product of two spinors:

$$\psi \cdot \chi \equiv \varepsilon_{ab}\psi^a\chi^b = \psi^a\chi_a = -\varepsilon_{ba}\psi^a\chi^b = -\psi_b\chi^b. \tag{1.21}$$

In addition, because of (1.17), the expressions

$$\psi_L^\dagger\chi_R \quad \text{and} \quad \psi_R^\dagger\chi_L$$

are also singlets.

Besides, we know there should be another invariant symbol from the decomposition

$$\left(\frac{1}{2}, 0\right) \otimes \left(0, \frac{1}{2}\right) \approx (0,0) \oplus \cdots .$$

For this reason, we are searching for the symbol M that the expression

$$M^\mu_{a\dot{b}}\psi_L^a\chi_R^{\dot{b}}$$

transforms as the Lorentz vector. The matrix M^μ should transform as

$$M^\mu \longrightarrow \Lambda_L^T \cdot M^\mu \cdot \Lambda_R = \Lambda^\mu{}_\nu M^\nu.$$

[1] We use the dotted symbol to denote the right-hand spinor indices.

Use the fact that $\sigma^2 \cdot \Lambda_L^T \cdot \sigma^2 = \Lambda_L^{-1}$, the above equation transforms to

$$\left(\sigma^2 M^\mu\right) \longrightarrow \Lambda_L^{-1} \cdot \left(\sigma^2 M^\mu\right) \cdot \Lambda_R.$$

We then show the matrices $\sigma^\mu = (\sigma^0, \vec{\sigma})$ satisfies the requirement. Firstly, for the spatial rotation,

$$\Lambda_L(\vec{\theta}, \vec{0}) = \Lambda_R(\theta, \vec{0}) = \exp\left(i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) \tag{1.22}$$

The Pauli matrix transform as

$$\left(1 - i\delta\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) \sigma^j \left(1 + i\delta\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) = \sigma^j + i\delta\theta_i \left(-i\varepsilon_{ijk}\sigma^k\right)$$

Secondly, for the boosts,

$$\Lambda_{L/R}(\vec{0}, \vec{\beta}) = \exp\left(\mp\vec{\beta} \cdot \frac{\vec{\sigma}}{2}\right). \tag{1.23}$$

The Pauli matrix transform as

$$\left(1 + \delta\vec{\beta} \cdot \frac{\vec{\sigma}}{2}\right) \sigma^\mu \left(1 + \delta\vec{\beta} \cdot \frac{\vec{\sigma}}{2}\right) = \begin{cases} \sigma^0 + i\delta\beta_i \cdot (-i\sigma^i), & \mu = 0 \\ \sigma^j + i\delta\beta_j(-i\sigma^0), & \mu = j \end{cases}.$$

We thus have shown indeed that

$$\psi_L^T \sigma^2 \sigma^\mu \chi_R \tag{1.24}$$

is a Lorentz vector. Further more, from (1.17), we know that

$$\eta_R^\dagger \sigma^\mu \chi_R \tag{1.25}$$

is also a Lorentz vector. Similarly, consider the Lorentz vector

$$N^\mu_{\dot{a}b} \psi_R^{\dot{a}} \chi_R^b,$$

which together with σ^2 should transforms as

$$\left(\sigma^2 N^\mu\right) \longrightarrow \Lambda_R^{-1} \cdot \left(\sigma^2 N^\mu\right) \cdot \Lambda_L.$$

We can check that $\bar{\sigma}^\mu = (\sigma^0, -\vec{\sigma})$ satisfies the requirement, and thus

$$\eta_L^\dagger \bar{\sigma}^\mu \chi_L \tag{1.26}$$

is also a Lorentz vector.

1.2 Klein-Gordon Field

In relativistic quantum field theory, the Lagrangian should be a singlet under Lorentz transformation. Different free fields correspond to different representation of the Lorentz algebra. The symmetry under Lorentz transformation also restrict the possible terms that can appear in the Lagrangian.

The simplest case is when $j_L = j_R = 0$, corresponding to the scalar field, which we denote as $\phi(x)$. Since the field it self is singlet, any polynomial of the field in principle can appear in the theory. When considering the free theory, we restrict our attention to the quadratic terms. We require the field theory to have a dynamical term, which contains derivative the the field. The derivative operator ∂^μ transforms as the fundamental representation. To be Lorentz invariant, the allowed free theory can only be

$$\mathcal{L}_{\mathrm{K-G}} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{m^2}{2}\phi^2 \simeq -\frac{1}{2}\phi(\partial^2 + m^2)\phi. \tag{1.27}$$

For general discussion, we consider the field theory on d-dimensional space-time. Note that the space-time Fourier transformation is defined as

$$\begin{aligned}\tilde{\phi}(k) &= \int d^d x e^{ik\cdot x}\phi(x),\\ \phi(x) &= \int \frac{d^d k}{(2\pi)^d} e^{-ik\cdot x}\tilde{\phi}(k),\end{aligned} \tag{1.28}$$

where the inner product of two 4-momentum and 4-coordinate is

$$k\cdot x \equiv \omega t - \vec{k}\cdot\vec{x}. \tag{1.29}$$

1.2.1 Canonical Quantization

The classical equation of motion for Klein-Gordon field is:

$$\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right] - \frac{\partial\mathcal{L}}{\partial\phi} = 0 \quad\Longrightarrow\quad (\partial_t^2 - \nabla^2 + m^2)\phi(\vec{x}, t) = 0. \tag{1.30}$$

The solution to Eq. (1.30) is proportional to the plane wave:

$$\phi_{\boldsymbol{k}}(\boldsymbol{x}, t) \propto e^{-i\omega_{\boldsymbol{k}} t + i\boldsymbol{k}\cdot\boldsymbol{x}} + e^{i\omega_{\boldsymbol{k}} t - i\boldsymbol{k}\cdot\boldsymbol{x}},$$

where the energy is $\omega_{\boldsymbol{k}} = \boldsymbol{k}^2 + m^2$ and $\boldsymbol{k}$ is the momentum as the conserved quantity. The general solution to the EOM is

$$\phi(\boldsymbol{x}, t) \propto \int \frac{d^3 k}{(2\pi)^3}\left(a_k e^{-i\omega_{\boldsymbol{k}} t + i\boldsymbol{k}\cdot\boldsymbol{x}} + a_k^* e^{i\omega_{\boldsymbol{k}} t - i\boldsymbol{k}\cdot\boldsymbol{x}}\right). \tag{1.31}$$

The canonical quantization promote the coefficient a_k/a_k^* to the particle annihilation/creation operator $a_k/a_k^\dagger$, with the commutation relation

$$[a_k, a_p^\dagger] = (2\pi)^3\delta^3(\boldsymbol{k} - \boldsymbol{p}). \tag{1.32}$$

Single Particle State

The single-particle state with momentum $\boldsymbol{k}$ is created by $a_k^\dagger$ operators acting on the vacuum:

$$|\boldsymbol{k}\rangle \equiv \sqrt{2\omega_{\boldsymbol{k}}} a_k^\dagger|0\rangle, \tag{1.33}$$

where $|\boldsymbol{k}\rangle$ is a state with a single particle of momentum $\boldsymbol{k}$. The factor of $\sqrt{2\omega_{\boldsymbol{k}}}$ in (1.33) is a convention to ensure Lorenz invariant. To compute the normalization of one-particle states, we start by requiring the vacuum state to be of unit norm:

$$\langle 0|0\rangle = 1, \tag{1.34}$$

which, together with the canonical commutation relation of particle annihilation and creation operators leads to

$$\langle \boldsymbol{p}|\boldsymbol{k}\rangle = 2\sqrt{\omega_{\boldsymbol{p}}\omega_{\boldsymbol{k}}}\left\langle 0\left|a_p a_k^\dagger\right|0\right\rangle = 2\omega_{\boldsymbol{p}}(2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{k}). \tag{1.35}$$

The identity operator for one-particle states under such norm is

$$1 = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\boldsymbol{p}}}|\boldsymbol{p}\rangle\langle\boldsymbol{p}|, \tag{1.36}$$

which we can check with

$$|\boldsymbol{k}\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\boldsymbol{p}}}|\boldsymbol{p}\rangle\langle\boldsymbol{p}|\boldsymbol{k}\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\boldsymbol{p}}}2\omega_{\boldsymbol{p}}(2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{k})|\boldsymbol{p}\rangle = |\boldsymbol{k}\rangle.$$

We see that the identity operator (1.36) under such convention is Lorentz invariant, since it can be expressed as

$$1 = 2\pi\int \frac{d^3p\,d\omega}{(2\pi)^4}\delta(\omega^2-\boldsymbol{p}^2-m^2)|\boldsymbol{p}\rangle\langle\boldsymbol{p}|. \tag{1.37}$$

Field Expansion

We fix the normalization by requiring

$$\langle\boldsymbol{k}|\phi(\boldsymbol{x},0)|0\rangle = e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{1.38}$$

and the quantized field operator is

$$\phi(\boldsymbol{x},t) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{k}}}}\left(a_k e^{-ik\cdot x} + a_k^\dagger e^{ik\cdot x}\right). \tag{1.39}$$

Consider the two-point correlation (propagator):

$$\begin{aligned} i\Delta(x_1-x_2) &= \langle 0|T\phi(x_1)\phi(x_2)|0\rangle \\ &= \theta(t_1-t_2)\langle 0|\phi(x_1)\phi(x_2)|0\rangle + \theta(t_2-t_1)\langle 0|\phi(x_2)\phi(x_1)|0\rangle. \end{aligned}$$

Note that

$$\langle 0|\phi(x_1)\phi(x_2)|0\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}e^{i\boldsymbol{k}\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)-i\omega_{\boldsymbol{k}}\tau}, \tag{1.40}$$

where $\tau = t_1 - t_2$. The propagator can be written as

$$\begin{aligned} i\Delta(x_1-x_2) &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}e^{i\boldsymbol{k}\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)}\left[e^{-i\omega_{\boldsymbol{k}}\tau}\theta(\tau)+e^{i\omega_{\boldsymbol{k}}\tau}\theta(-\tau)\right] \\ &= \int \frac{d^3k}{(2\pi)^3}e^{i\boldsymbol{k}\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)}\int\frac{d\omega}{2\pi i}\frac{-e^{i\omega\tau}}{\omega^2-\omega_k^2+i\epsilon} \\ &= \int \frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x_1-x_2)}\frac{i}{k^2-m^2+i\epsilon}. \end{aligned} \tag{1.41}$$

We have used the identity

$$\frac{1}{2\omega_k}\left[e^{-i\omega_{\boldsymbol{k}}\tau}\theta(\tau)+e^{i\omega_{\boldsymbol{k}}\tau}\theta(-\tau)\right]=\int\frac{d\omega}{2\pi i}\frac{-e^{i\omega\tau}}{\omega^2-\omega_k^2+i\epsilon},$$

where an infinitesimal number ϵ is included to move the singularities away from the real axis. Any final result shall take the $(\epsilon\to 0^+)$ limit. Sometimes the infinitesimal ϵ will be absorbed into the mass, i.e., $m^2\to m^2-i\epsilon$.

1.2.2 Path-integral Formalism

Consider the action for free field with source

$$S_0[\phi,J]=\int d^dx\left[\mathcal{L}_0+J(x)\cdot\phi(x)\right]. \tag{1.42}$$

In the path integral formalism, we consider the partition function

$$Z_0[J]=\int D[\phi]\exp(iS_0[\phi,J])\equiv Z[0]\exp(iW_0[J]). \tag{1.43}$$

where we have introduced a new quantity

$$W_0[J]=-\frac{1}{2}\int d^dx_1d^dx_2J(x_1)\Delta_0(x_1-x_2)J(x_2). \tag{1.44}$$

For free field, the free propagator $\Delta_0(x_1-x_2)$ is:

$$i\Delta_0(x_1-x_2)=\langle 0|T\phi(x_1)\phi(x_2)|0\rangle=\frac{\delta}{i\delta J(x_1)}\frac{\delta}{i\delta J(x_2)}iW_0[J]. \tag{1.45}$$

Now we evaluate the propagator in the path-integral formalism. In momentum space, the free action (with source) is

$$\frac{1}{V}\sum_k\left[\frac{1}{2}\tilde{\phi}^*(k)(k^2-m^2)\tilde{\phi}(k)+\tilde{J}^*(k)\cdot\tilde{\phi}(k)+\tilde{\phi}^*(k)\cdot\tilde{J}(k)\right].$$

For real field, $\tilde{\phi}^*(k)=\tilde{\phi}(-k)$. For our convenience, we have expressed the momentum integral as summation. Actually, consider the d-dimensional box of size L^d, the momentum along each axis is multiple of $2\pi/L$, so when $L\to\infty$, the summation approaches in integral,

$$\frac{1}{V}\sum_k\to\int\frac{d^dk}{(2\pi)^d}.$$

Let us omit the $1/V$ factor, the summation can be formally expressed as

$$\frac{1}{4}\mathbf{v}^T\cdot\mathbf{M}\cdot\mathbf{v}+\frac{1}{2}\mathbf{j}^T\cdot\mathbf{v} \tag{1.46}$$

where

$$\mathbf{v}=\bigoplus_{|\mathbf{k}|}\begin{bmatrix}\tilde{\phi}(k)\\ \tilde{\phi}^*(k)\end{bmatrix},\ \mathbf{M}=\bigoplus_{|\mathbf{k}|}\begin{bmatrix}0 & k^2-m^2\\ k^2-m^2 & 0\end{bmatrix},\ \mathbf{j}=\bigoplus_{|\mathbf{k}|}\begin{bmatrix}\tilde{J}^*(k)\\ \tilde{J}(k)\end{bmatrix}.$$

Note that in the above expression, we have made an infinitesimal shift of mass ($m^2 \to m^2 - i\epsilon$) to ensure the convergence of the Gaussian integral. The integrated variables v_i is not real. To use the real Gaussian integral formula, we make use of a unitary transformation:

$$\mathbf{U} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}, \quad \mathbf{U} \cdot \begin{bmatrix} \tilde{\phi}(k) \\ \tilde{\phi}^*(k) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \tilde{\phi}(k) + \tilde{\phi}^*(k) \\ -i\tilde{\phi}(k) + i\tilde{\phi}^*(k) \end{bmatrix} \equiv \begin{bmatrix} \tilde{\phi}_1(k) \\ \tilde{\phi}_2(k) \end{bmatrix}$$

The path integral then becomes a real field integral. Recall the real Gaussian integral formula:

$$\int d\mathbf{x} \exp\left(-\frac{1}{2}\mathbf{x}^T \cdot \mathbf{A} \cdot \mathbf{x} + \mathbf{B}^T \cdot \mathbf{x}\right) = \sqrt{\frac{(2\pi)^N}{\det \mathbf{A}}} \exp\left(\frac{1}{2}\mathbf{B}^T \cdot \mathbf{A}^{-1} \cdot \mathbf{B}\right), \tag{1.47}$$

For the field integral, we absorbed the $(2\pi)^{N/2}$ term into the measure, and express the path integral for the Gaussian field as:

$$W_0[J] = -\frac{i}{4} \int \frac{d^d k}{(2\pi)^d} \mathbf{j}_k^T \cdot \mathbf{M}_k^{-1} \cdot \mathbf{j}_k = -\frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \tilde{J}^*(k) \tilde{\Delta}_0(k) \tilde{J}(k). \tag{1.48}$$

This gives the propagator in the momentum space:

$$\tilde{\Delta}_0(k) = \frac{i}{k^2 - m^2} \quad \Longrightarrow \quad \Delta_0(x_1 - x_2) = i \int \frac{d^4 k}{(2\pi)^4} \frac{e^{-ik\cdot(x_1 - x_2)}}{k^2 - m^2}. \tag{1.49}$$

From Field to Force

Consider two separate particle described by the delta function $J_a(x) = \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}_a)$, together the source is

$$J(x) = J_1(x) + J_2(x). \tag{1.50}$$

Adding the source,

$$W_0[J] = -\frac{1}{2} \int d^4 x_1 d^4 x_2 J(x_1) \Delta_0(x_1 - x_2) J(x_2)$$

Omit the self energy terms $J_1^2(x)$, $J_2^2(x)$, $W_0[J]$ is

$$\begin{aligned} W_0[J] &= -\int d^4 y_1 d^4 y_2 \; e^{-ik^0(y_1^0 - y_2^0)} \int \frac{d^4 k}{(2\pi)^4} J_1(y_1) \frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{y}_1 - \boldsymbol{y}_2)}}{k^2 - m^2} J_2(y_2) \\ &= -\int dt \int d(y_1^0 - y_2^0) \; e^{-ik^0(y_1^0 - y_2^0)} \int \frac{d^4 k}{(2\pi)^4} \frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{y}_1 - \boldsymbol{y}_2)}}{k^2 - m^2} \\ &= \left(\int dt\right) \int \frac{d^3 k}{(2\pi)^3} \frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{y}_1 - \boldsymbol{y}_2)}}{\boldsymbol{k}^2 + m^2} \end{aligned} \tag{1.51}$$

Recall that the partition function is actually infinite:

$$Z_0 \sim \langle 0 | e^{-iH_0 T} | 0 \rangle \quad \Longrightarrow \quad W_0 = -iET, \tag{1.52}$$

where E is the energy. Writing $\boldsymbol{r} \equiv \boldsymbol{y}_1 - \boldsymbol{y}_2$, and $u \equiv \cos\theta$ with θ the angle between $\boldsymbol{k}$ and $\boldsymbol{r}$, the volume form is $dk \cdot kd\theta \cdot 2\pi k \sin\theta = 2\pi k^2 dk du$, and the integral is

$$\begin{aligned} E &= -\int \frac{d^3k}{(2\pi)^3} \frac{e^{ikru}}{k^2+m^2} \\ &= -\frac{1}{(2\pi)^2} \int_0^\infty k^2 dk \int_{-1}^1 du \frac{e^{ikru}}{k^2+m^2} \\ &= -\frac{1}{2\pi^2 r} \int_0^\infty k \frac{\sin kr}{k^2+m^2} dk. \end{aligned} \tag{1.53}$$

Since the integral is even, we can extend the integral to

$$\begin{aligned} E &= -\frac{1}{4\pi^2 r} \int_{-\infty}^\infty k \frac{\sin kr}{k^2+m^2} dk \\ &= \frac{i}{4\pi^2 r} \int_{-\infty}^\infty \frac{ke^{ikr}}{k^2+m^2} dk \end{aligned} \tag{1.54}$$

The residue theorem gives

$$\int_{-\infty}^\infty \frac{ke^{ikr}}{k^2+m^2} dk = \pi i e^{-mr} \tag{1.55}$$

So we get the potential of two particles:

$$E(r) = -\frac{e^{-mr}}{4\pi r}, \tag{1.56}$$

and the attractive force is

$$F(r) = -\frac{dE}{dr} = -(1+mr)\frac{e^{-mr}}{4\pi r^2}. \tag{1.57}$$

We see that in the massless case, the force gives the long-range Coulomb force $F \propto 1/r^2$, while in the massful field theory, the force is short-ranged, with the decay length proportional to the mass.

1.3 Vector Field

If we can choose $j_L = j_R = 1/2$, the field is transformed as Lorentz vector. We denote the field as $A^\mu(x)$. Some possible quadratic forms for the vector field that forms singlets are

$$A^\mu A_\mu,\ (\partial_\mu A^\mu)^2,\ A^\nu \partial^2 A_\nu,\ \varepsilon_{\mu\nu\rho\lambda} \partial^\mu A^\nu \partial^\rho A^\lambda.$$

For the field theory describe the electromagnetic field, we require the theory to further have gauge symmetry, i.e., invariant under

$$A^\mu(x) \to A^\mu(x) + \partial^\mu \alpha(x). \tag{1.58}$$

The gauge invariant forbids the first term, and forces the second and third term to combine as

$$(\partial_\mu A^\mu)^2 - A^\nu \partial^2 A_\nu \sim \frac{1}{2}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A^\nu - \partial_\nu A_\mu) \equiv \frac{1}{2} F^{\mu\nu} F_{\mu\nu}.$$

where we have define a field-strength tensor

$$F^{\mu\nu} \equiv (\partial^\mu A^\nu - \partial^\nu A^\mu) = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{bmatrix}, \tag{1.59}$$

where we notice that from Maxwell equations:

$$E^i = \partial_t \vec{A} = -\vec{\nabla} A^0, \quad B^i = \nabla \times \vec{A}. \tag{1.60}$$

Note that the fourth term is called the *theta term*, which can be written as a boundary term

$$\varepsilon_{\mu\nu\rho\lambda}\partial^\mu A^\nu \partial^\rho A^\lambda = \partial^\mu(\varepsilon_{\mu\nu\rho\lambda} A^\nu \partial^\rho A^\lambda).$$

The Lagrangian describing the electromagnetic field is given by

$$\mathcal{L}_{\text{Maxwell}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{1.61}$$

1.3.1 Path-integral Formalism

We define the gauge fixing function

$$G(A) = \partial_\mu A^\mu(x) - \omega(x) = 0$$

The gauge transformation has the form:

$$A_\mu^\alpha(x) = A_\mu(x) + \partial_\mu \alpha(x).$$

We then have

$$1 \propto \int D[\alpha] \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right) \delta(G(A)).$$

Inset the identity operator into the path integral formula

$$Z[J] \propto \det\left(\partial^2\right) \int D[\alpha] D[A] e^{iS[A,J]} \delta(\partial_\mu A^\mu - \omega(x)).$$

The above equation does not depend on $\omega(x)$. We can then integrate over $\omega(x)$ with gaussian weight

$$\begin{aligned} Z[J] &\propto \int D[\omega] e^{-i\int d^d x \frac{\omega^2}{2\xi}} \int D[\alpha] D[A] e^{iS[A,J]} \delta(\partial_\mu A^\mu - \omega) \\ &= \int D[A] e^{iS[A,J]} \exp\left\{ i \left[S[A,J] - \int d^d x \frac{1}{2\xi} (\partial_\mu A^\mu)^2 \right] \right\}. \end{aligned}$$

In momentum space, the modified Langriangian is

$$\tilde{\mathcal{L}}_\xi(k) = \tilde{A}^\mu(k) \left[-k^2 g_{\mu\nu} + \left(1 - \frac{1}{\xi}\right) k_\mu k_\nu \right] \tilde{A}^\nu(-k) + \tilde{J}_\mu(k) \tilde{A}^\mu(-k) + \tilde{A}^\mu(k) \tilde{J}_\mu(-k).$$

In the momentum space, the photon propagator is

$$\begin{aligned}\tilde{\Pi}^{\mu\nu}(k) &= \left[-k^2 g_{\mu\nu} + \left(1-\frac{1}{\xi}\right) k_\mu k_\nu\right]^{-1} \\ &= \frac{-g^{\mu\nu} + (1-\xi)k^\mu k^\nu}{k^2}.\end{aligned} \tag{1.62}$$

Thus, the partition function is

$$\frac{Z_{\text{maxwell}}[J]}{Z_{\text{maxwell}}[0]} = \exp\left[-\frac{i}{2}\int d^d x_1 d^d x_2 J_\mu(x_1)\Pi^{\mu\nu}(x_1-x_2)J_\nu(x_2)\right], \tag{1.63}$$

where the real-space propagator is

$$\Pi^{\mu\nu}(x_1-x_2) = \int \frac{d^d k}{(2\pi)^d} e^{-ik\cdot(x_1-x_2)} \frac{-g^{\mu\nu} + (1-\xi)k^\mu k^\nu}{k^2}. \tag{1.64}$$

Note that the propagator is related to the two-point correaltion:

$$\begin{aligned}\langle 0|TA^\mu(x_1)A^\nu(x_2)|0\rangle &= \frac{1}{Z_{\text{Maxwell}}[0]}\frac{\delta}{iJ_\mu(x_1)}\frac{\delta}{iJ_\nu(x_2)} Z_{\text{Maxwell}}[J]\bigg|_{J=0} \\ &= i\Pi^{\mu\nu}(x_1-x_2).\end{aligned} \tag{1.65}$$

1.3.2 Canonical Quantization

In momentum space, the Lagrangian transforms to

$$\tilde{\mathcal{L}}(k) = \tilde{A}^\mu(-k)\left(-k^2 g_{\mu\nu} + k_\mu k_\nu\right)\tilde{A}^\nu(k). \tag{1.66}$$

The EOM in momentum space is

$$(-k^2 g_{\mu\nu} + k_\mu k_\nu)\tilde{A}^\nu(k) = 0. \tag{1.67}$$

The gauge fixing condition $\partial_\mu A^\mu = 0$ in momentum space requires

$$k_\mu A^\mu(k) = 0. \tag{1.68}$$

The gauge freedom can be used to further restrict $A^0 = 0$. We can choose a coordinate frame such that $\vec{k}$ is along z axis, i.e., $k = (E, 0, 0, E)$. In this way, there are only two independent polarization for EOM solution

$$A^\mu = e^{-ik\cdot x}\epsilon_j^\mu, \quad j = 1, 2, \tag{1.69}$$

where $\vec{\epsilon}_1 = (0, 1, 0, 0)$ and $\vec{\epsilon}_2 = (0, 0, 1, 0)$ are two polarization basis vectors. The field expansion is then

$$A^\mu = \int \frac{d^3 k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\sum_{j=1}^{2}\left(\epsilon_j^\mu a_{k,j}e^{-ik\cdot x} + \epsilon_j^{\mu *} a_{k,j}^\dagger e^{ik\cdot x}\right). \tag{1.70}$$

A single-particle state with polarization vector $\vec{\epsilon}_j$ is defined as

$$|k, \epsilon_j\rangle = \sqrt{2\omega_k}\vec{\epsilon}_j a_{k,j}^\dagger|0\rangle. \tag{1.71}$$

In addition, we can define two complement basis vectors $\vec{\epsilon}_0 = (1,0,0,0)$ and $\vec{\epsilon}_3 = (0,0,0,1)$. Correspondingly, we we can add two types of virtual particles generated by $a^\dagger_{k,0}$ and $a^\dagger_{k,3}$ respectively, which are usually called the *scalar photons* and *longitudinal photons*. However, the gauge fixing condition (1.68) requires

$$\partial_\mu A^\mu(x)|\psi\rangle = 0 \quad \Longrightarrow \quad (a_{k,0} - a_{k,3})|\psi\rangle = 0 \tag{1.72}$$

for all state $|\psi\rangle$ in the gauge-fixed Hilbert space.

To obtain the propagator, we consider a modified Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\xi}{2}(\partial_\mu A^\mu)^2. \tag{1.73}$$

In the momentum space:

$$\tilde{\mathcal{L}}_k = \tilde{A}^\mu(-k)\left(-k^2 g_{\mu\nu} + (1-\xi)k_\mu k_\nu\right)\tilde{A}^\nu(k) \tag{1.74}$$

To construct the inverse matrix we make a general symmetric ansatz

$$(\Pi^{-1})^{\mu\nu}(k) = A\left(k^2\right)g^{\mu\nu} + B\left(k^2\right)k^\mu k^\nu. \tag{1.75}$$

Requiring that

$$\Pi_{\mu\sigma}(k)(\Pi^{-1})^{\sigma\nu}(k) = \delta^\nu_\mu, \tag{1.76}$$

and comparing the coefficients, we get the conditions

$$\begin{aligned} &-k^2 A\left(k^2\right) = 1, \\ &\xi k^2 B\left(k^2\right) = (\xi - 1)A\left(k^2\right). \end{aligned} \tag{1.77}$$

In the case $\xi = 0$ these equations are not compatible. Without the gauge-fixing term the matrix $\Pi_{\mu\nu}(k)$ cannot be inverted (since the determinant vanishes) and the Feynman propagator cannot be constructed. If $\xi \neq 0$, however, no problems arise and the system of equations (1.77) is solved by

$$A\left(k^2\right) = -\frac{1}{k^2} \quad , \quad B\left(k^2\right) = \frac{\xi - 1}{\xi}\frac{1}{\left(k^2\right)^2}, \tag{1.78}$$

which leads to (1.62) after a suitable manipulation.

1.4 Dirac Field

Based on previous discussion, the Lagrangian for spinor field can have

$$\psi^\dagger_L \bar{\sigma}^\mu \partial_\mu \psi_L,\ \psi^\dagger_R \sigma^\mu \partial_\mu \psi_R,\ \psi^\dagger_L \psi_R,\ \psi^\dagger_R \psi_L,\ \psi_L \cdot \psi_L,\ \psi_R \cdot \psi_R.$$

The Dirac field describe the theory with both left-hand and right-hand spinors. The Lagrangian is

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi, \tag{1.79}$$

where

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix},\ \bar{\psi} = \begin{pmatrix} \psi^\dagger_R & \psi^\dagger_L \end{pmatrix},\ \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}. \tag{1.80}$$

In addition, we could consider using the last two terms as the mass, the result theory is the *Majorana field theory*:

$$\begin{aligned}\mathcal{L}^L_{\text{Majorana}} &= \psi_L^\dagger\left(i\bar{\sigma}^\mu\partial_\mu - m\sigma^2\right)\psi_L,\\ \mathcal{L}^R_{\text{Majorana}} &= \psi_R^\dagger\left(i\sigma^\mu\partial_\mu - m\sigma^2\right)\psi_R.\end{aligned} \tag{1.81}$$

For the spinor basis, the Dirac Algebra is generated by

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]. \tag{1.82}$$

The Lorentz group is represented by

$$\Lambda_{\frac{1}{2}} = \exp\left(\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right). \tag{1.83}$$

Using the familiar parametrization,

$$S^{i0} = \frac{i}{2}\begin{bmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{bmatrix}, \quad S^{ij} = \frac{1}{2}\epsilon^{ijk}\begin{bmatrix} \sigma^k & 0 \\ 0 & -\sigma^k \end{bmatrix}, \tag{1.84}$$

which agree with the transformation property (1.15).

1.4.1 Path-integral Formalism

Consider the partition function with source

$$Z_{\text{Dirac}}[J] = \int D[\bar{\psi}, \psi]\exp\left[i\int d^dx\left(\mathcal{L}_{\text{Dirac}} + \bar{\eta}\psi + \bar{\psi}\eta\right)\right]. \tag{1.85}$$

In momentum space:

$$S = \int \frac{d^dk}{(2\pi)^d}\left[\bar{\tilde{\psi}}(k)(\not{k} - m)\tilde{\psi}(k) + \tilde{\bar{\eta}}(k)\tilde{\psi}(k) + \bar{\tilde{\psi}}(k)\tilde{\eta}(k)\right]. \tag{1.86}$$

Using the Gaussian integral formula (for Grassman variables), the partition function is:

$$\begin{aligned}\frac{Z_{\text{Dirac}}[J]}{Z_{\text{Dirac}}[0]} &= \exp\left[-i\int \frac{d^dk}{(2\pi)^d}\tilde{\bar{\eta}}(k)\frac{1}{\not{k} - m}\tilde{\eta}(k)\right]\\ &= \exp\left[-i\int d^dx_1 d^dx_2 \bar{\eta}(x_1)\cdot D_F(x_1 - x_2)\cdot \eta(x_2)\right]\end{aligned} \tag{1.87}$$

where

$$D_F(x_1 - x_2) = \int \frac{d^dk}{(2\pi)^d}\frac{e^{-ik\cdot(x_1 - x_2)}}{\not{k} - m} = \int \frac{d^dk}{(2\pi)^d}\frac{\not{k} + m}{k^2 - m^2}e^{-ik\cdot(x_1 - x_2)}. \tag{1.88}$$

Note that the propagator is

$$\begin{aligned}\langle 0|T\psi^\alpha(x_1)\bar{\psi}^\beta(x_2)|0\rangle &= \frac{1}{Z_{\text{Dirac}}[0]}\frac{\delta}{i\delta\bar{\eta}_\alpha(x_1))}\frac{i\delta}{\delta\eta_\beta(x_2)}Z_{\text{Dirac}}[\bar{\eta}, \eta]\bigg|_{\eta = \bar{\eta} = 0}\\ &= iD_F^{\alpha\beta}(x_1 - x_2),\end{aligned} \tag{1.89}$$

where the sign in the variational derivative comes from the anti-commutation relation of the fermionic fields.

1.4.2 Canonical Quantization

The equation of motion for Dirac field is

$$\begin{aligned}
\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} - \frac{\partial \mathcal{L}}{\partial \psi} = 0 \quad &\Longrightarrow \quad \bar{\psi}(i\overleftarrow{\not{\partial}} - m) = 0, \\
\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} - \frac{\partial \mathcal{L}}{\partial \bar{\psi}} = 0 \quad &\Longrightarrow \quad (i\overrightarrow{\not{\partial}} - m)\psi = 0.
\end{aligned} \tag{1.90}$$

This EOM is a matrix equation. The general solution of the Dirac equation can be written as a linear combination of plane waves (with positive and negative energy):[2]

$$\psi_p(x) = \begin{cases} u(p)e^{-ip\cdot x} & p^0 > 0 \\ v(p)e^{+ip\cdot x} & p^0 < 0 \end{cases}, \quad p^2 = m^2. \tag{1.91}$$

In momentum space, $u(p)$ and $v(p)$ satisfies:

$$\begin{bmatrix} -m & p\cdot\sigma \\ p\cdot\bar{\sigma} & -m \end{bmatrix} u_s(p) = \begin{bmatrix} -m & -p\cdot\sigma \\ -p\cdot\bar{\sigma} & -m \end{bmatrix} v_s(p) = 0 \tag{1.92}$$

For massive Dirac field, we can choose the rest frame where $p = (m, 0, 0, 0)$, the matrix equation is[3]

$$\begin{aligned}
\begin{bmatrix} -m & m \\ m & -m \end{bmatrix} u_s = 0 \quad &\Longrightarrow \quad u_s = \sqrt{m}\begin{bmatrix} \xi_s \\ \xi_s \end{bmatrix}, \\
\begin{bmatrix} m & m \\ m & m \end{bmatrix} v_s = 0 \quad &\Longrightarrow \quad v_s = \sqrt{m}\begin{bmatrix} \eta_s \\ -\eta_s \end{bmatrix},
\end{aligned} \tag{1.93}$$

where ξ and η has two independent solutions. For example, four linearly independent solutions are

$$u_\uparrow = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad u_\downarrow = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad v_\uparrow = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_\downarrow = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}. \tag{1.94}$$

The Dirac spinor is a complex four-component object, with eight real degrees of freedom. The equations of motion reduce it to four degrees of freedom, which, as we will see, can be interpreted as spin up and spin down for particle and antiparticle.

Solution in General Frame

To derive a more general expression, we can solve the equations again in the boosted frame and match the normalization. If $p = (E, 0, 0, p_z)$ then

$$p\cdot\sigma = \begin{bmatrix} E - p_z & 0 \\ 0 & E + p_z \end{bmatrix}, \quad p\cdot\bar{\sigma} = \begin{bmatrix} E + p_z & 0 \\ 0 & E - p_z \end{bmatrix}. \tag{1.95}$$

[2]Note that we have chosen to put the + sign into the exponential, rather than having $p^0 < 0$.

[3]We first consider the case where there is only one spatial dimension. It correspond to the choice of coordinate such that the momentum point to the z direction.

Let $a = \sqrt{E - p_z}$ and $b = \sqrt{E + p_z}$, then $m^2 = (E - p_z)(E + p_z) = a^2 b^2$ and Dirac equation becomes

$$\begin{bmatrix} -ab & 0 & a^2 & 0 \\ 0 & -ab & 0 & b^2 \\ b^2 & 0 & -ab & 0 \\ 0 & a^2 & 0 & -ab \end{bmatrix} u_s(p) = \begin{bmatrix} ab & 0 & a^2 & 0 \\ 0 & ab & 0 & b^2 \\ b^2 & 0 & ab & 0 \\ 0 & a^2 & 0 & ab \end{bmatrix} v_s(p) = 0. \tag{1.96}$$

The solutions are

$$u_s = \begin{pmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \xi_s \\ \begin{bmatrix} b & 0 \\ 0 & a \end{bmatrix} \xi_s \end{pmatrix}, \quad v_s = \begin{pmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \eta_s \\ -\begin{bmatrix} b & 0 \\ 0 & a \end{bmatrix} \eta_s \end{pmatrix}. \tag{1.97}$$

Using

$$\sqrt{p \cdot \sigma} = \begin{bmatrix} \sqrt{E - p_z} & 0 \\ 0 & \sqrt{E + p_z} \end{bmatrix}, \quad \sqrt{p \cdot \bar{\sigma}} = \begin{bmatrix} \sqrt{E + p_z} & 0 \\ 0 & \sqrt{E - p_z} \end{bmatrix}, \tag{1.98}$$

we can write more generally

$$u_s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma}\xi_s \\ \sqrt{p \cdot \bar{\sigma}}\xi_s \end{pmatrix}, \quad v_s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma}\eta_s \\ -\sqrt{p \cdot \bar{\sigma}}\eta_s \end{pmatrix}, \tag{1.99}$$

where the square root of a matrix can be defined by changing to the diagonal basis, taking the square root of the eigenvalues, then changing back to the original basis. In practice, we will usually pick p along the z axis, so we do not need to know how to make sense of $\sqrt{p \cdot \sigma}$. Then the four solutions are

$$\begin{aligned} u^1(p) &= \begin{pmatrix} \sqrt{E - p_z} \\ 0 \\ \sqrt{E + p_z} \\ 0 \end{pmatrix}, \quad u^2(p) = \begin{pmatrix} 0 \\ \sqrt{E - p_z} \\ 0 \\ \sqrt{E + p_z} \end{pmatrix}, \\ v^1(p) &= \begin{pmatrix} \sqrt{E - p_z} \\ 0 \\ -\sqrt{E + p_z} \\ 0 \end{pmatrix}, \quad v^2(p) = \begin{pmatrix} 0 \\ \sqrt{E - p_z} \\ 0 \\ -\sqrt{E + p_z} \end{pmatrix}. \end{aligned} \tag{1.100}$$

In any frame u^s are the positive frequency electrons, and the v^s are negative frequency electrons, or equivalently, positive frequency positrons.

For massless spinors, $p_z = \pm E$ and the explicit solutions in Eq. (1.100) are 4-vectors with one non-zero component describing spinors with fixed helicity. The spinor solutions for massless electrons are sometimes called polarizations, and are useful for computing polarized electron scattering amplitudes.

For Weyl spinors, there are only four real degrees of freedom off-shell and two real degrees of freedom on-shell. Recalling that the Dirac equation splits up into separate equations for

ψ_L and ψ_R, the Dirac spinors with zeros in the bottom two rows will be ψ_L and those with zeros in the top two rows will be ψ_R. Since ψ_L and ψ_R have two degrees of freedom each, these must be particle and antiparticle for the same helicity. The embedding of Weyl spinors into fields this way induces irreducible unitary representations of the Poincare group for $m = 0$.

Normalization and Spin Sum

The normalization chosen this way gives the orthogonal relation:

$$\begin{aligned}\bar{u}^r(p)u^s(p) &= +2m\delta^{rs},\\ \bar{v}^r(p)v^s(p) &= -2m\delta^{rs}.\end{aligned} \tag{1.101}$$

This is the (conventional) normalization for the spinor inner product for massive Dirac spinors. It is also easy to check that

$$\bar{u}_s(p)v_{s'}(p) = \bar{v}_s(p)u_{s'}(p) = 0. \tag{1.102}$$

We can further check that an additional orthogonal relation hold

$$\begin{aligned}u^{r\dagger}(p)u^s(p) &= -2\omega_{\boldsymbol{p}}\delta^{rs},\\ v^{r\dagger}(p)v^s(p) &= +2\omega_{\boldsymbol{p}}\delta^{rs}.\end{aligned} \tag{1.103}$$

And if we define $\bar{p} \equiv (E, -\vec{p})$, there is another set of orthogonal relation:

$$u^{r\dagger}(p)v^s(\bar{p}) = v^{r\dagger}(p)u^s(\bar{p}) = 0. \tag{1.104}$$

A useful identity is the spin sum identity:

$$\begin{aligned}\sum_s u^s(p)\bar{u}^s(p) &= \not{p} + m,\\ \sum_s v^s(p)\bar{v}^s(p) &= \not{p} - m.\end{aligned} \tag{1.105}$$

Field Expansion

The Dirac field expansion is

$$\begin{aligned}\psi(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\sum_s \left(a^s_{\mathbf{p}}u^s(p)e^{-ip\cdot x} + b^{s\dagger}_{\mathbf{p}}v^s(p)e^{ip\cdot x}\right),\\ \bar{\psi}(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\sum_s \left(b^s_{\mathbf{p}}\bar{v}^s(p)e^{-ip\cdot x} + a^{s\dagger}_{\mathbf{p}}\bar{u}^s(p)e^{ip\cdot x}\right).\end{aligned} \tag{1.106}$$

Now let us investigate the propagator

$$\begin{aligned}iD_{F,\alpha\beta}(x_1 - x_2) &= \langle 0|T\psi_\alpha(x_1)\bar{\psi}_\beta(x_2)|0\rangle\\ &= \theta(\tau)\langle 0|\psi_\alpha(x_1)\bar{\psi}_\beta(x_2)|0\rangle - \theta(-\tau)\langle 0|\bar{\psi}_\beta(x_2)\psi_\alpha(x_1)|0\rangle.\end{aligned} \tag{1.107}$$

On the RHS, the first term is

$$
\begin{aligned}
\langle 0|\psi_\alpha(x_1)\bar{\psi}_\beta(x_2)|0\rangle &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left[\sum_s u^s_\alpha(p)\bar{u}^s_\beta(p)\right] e^{-ip\cdot(x_1-x_2)} \\
&= (i\partial\!\!\!/ + m)_{\alpha\beta}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}} e^{-ip\cdot(x_1-x_2)}.
\end{aligned}
$$

For the second term:

$$
\begin{aligned}
\langle 0|\bar{\psi}_\beta(x_2)\psi_\alpha(x_1)|0\rangle &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left[\sum_s \bar{v}^s_\beta(p)v^s_\alpha(p)\right] e^{ip\cdot(x_1-x_2)} \\
&= -(i\partial\!\!\!/ + m)_{\alpha\beta}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}} e^{ip\cdot(x_1-x_2)}.
\end{aligned}
$$

Together, the Dirac propagator is:

$$
\begin{aligned}
iD_F(x_1-x_2) &= (i\partial\!\!\!/ + m)i\Delta(x_1-x_2) \\
&= \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x_1-x_2)}\frac{i(p\!\!\!/ + m)}{p^2-m^2+i\epsilon}.
\end{aligned} \tag{1.108}
$$

1.5 Scattering Theory

Quantum mechanics consists of an elaborate collection of rules for manipulating states in a Hilbert space. The experimentally measurable quantities that are predicted in quantum mechanics are differential probabilities. These probabilities are given by the modulus squared of inner products of states. We can write such inner products as

$$
\langle f; t_f \mid i; t_i\rangle\,,
$$

where $|i;t_i\rangle$ is the initial state we start with at time t_i and $|f;t_f\rangle$ is the final state we are interested in at some later time t_f. Since quantum field theory is just quantum mechanics with lots of fields, the experimental quantities we will be able to predict are also of the form

$$
|\langle f; t_f \mid i; t_i\rangle|^2\,.
$$

The notation $\langle f;t_f|i;t_i\rangle$ refers to the Schrödinger picture representation, where the states evolve in time. In the Heisenberg picture, which will be the default picture for quantum field theory, we leave the states alone and put all the time evolution into an operator. In the special case where we evolve momentum eigenstates from $t=-\infty$ to $t=+\infty$, relevant for collider physics applications, we give the time-evolution operator a special name: the scattering or *S-matrix*. The S-matrix is defined as

$$
\langle f|S|i\rangle_{\text{Heisenberg}} = \langle f;\infty \mid i;-\infty\rangle \tag{1.109}
$$

The S-matrix has all the information about how the initial and final states evolve in time. Quantum field theory will tell us how to calculate S-matrix elements. As we will explain in this chapter and the next, the S-matrix is defined assuming that all of the things that change the state (the interactions) happen in a finite time interval, so that at asymptotic times, $t = \pm\infty$, the states are free of interactions. Free states at $t = \pm\infty$ are known as *asymptotic states.*

S-matrix elements are the primary objects of interest for high-energy physics. In this chapter, we will relate S-matrix elements to scattering cross sections, which are directly measured in collider experiments. We will also derive an expression for decay rates, which are also straightforward to measure experimentally. Quantum field theory is capable of calculating other quantities besides S-matrix elements, such as thermodynamic properties of condensed matter systems. However, since the tools we develop for S-matrix calculations, such as Feynman rules, are also relevant for these applications, it is logical to focus on S-matrix elements for concreteness.

1.5.1 Cross Sections and Decay Rates

A cross section is a natural quantity to measure experimentally. For example, Rutherford was interested in the size r of an atomic nucleus. By colliding α-particles with gold foil and measuring how many α-particles were scattered, he could determine the cross-sectional area

$$\sigma = \pi r^2$$

of the nucleus.

Imagine there is just a single nucleus. Then the cross-sectional area is given by

$$\sigma = \frac{\text{number of particles scattered}}{\text{time } \times \text{ number density in beam } \times \text{ velocity of beam}} = \frac{1}{T}\frac{1}{\Phi}N, \tag{1.110}$$

where T is the time for the experiment and Φ is the incoming flux:

$$\Phi = \text{number density} \times \text{velocity of beam},$$

and N is the number of particles scattered.

In quantum mechanics, we generalize the notion of cross-sectional area to a *cross section*, which still has units of area, but has a more abstract meaning as a measure of the interaction strength. While classically an α-particle either scatters off the nucleus or it does not scatter, quantum mechanically it has a probability for scattering. The classical differential probability is

$$P = \frac{N}{N_{\text{inc}}},$$

where N is the number of particles scattering into a given area and N_{inc} is the number of incident particles. So the quantum mechanical cross section is then naturally

$$d\sigma = \frac{1}{T}\frac{1}{\Phi}dP, \tag{1.111}$$

where Φ is the flux, now normalized as if the beam has just one particle, and P is now the quantum mechanical probability of scattering. The differential quantities $d\sigma$ and dP are differential in kinematical variables, such as the angles and energies of the final state particles. The differential number of scattering events measured in a collider experiment is

$$dN = L \times d\sigma, \tag{1.112}$$

where L is the *luminosity*, which is defined by this equation.

Now let us relate the formula for the differential cross section to S-matrix elements. From a practical point of view it is impossible to collide more than two particles at a time, thus we can focus on the special case of S-matrix elements where $|i\rangle$ is a two-particle state. So, we are interested in the differential cross section for the $(2 \to n)$ process:

$$p_1 + p_2 \to \{p_j\}\,. \tag{1.113}$$

In the rest frame of one of the colliding particles, the flux is just the magnitude of the velocity of the incoming particle divided by the total volume: $\Phi = |\vec{v}|/V$. In a different frame, such as the center-of-mass frame, beams of particles come in from both sides, and the flux is then determined by the difference between the particles' velocities. So, $\Phi = |\vec{v}_1 - \vec{v}_2|\,/V$. This should be familiar from classical scattering. Thus,

$$d\sigma = \frac{V}{T}\frac{1}{|\vec{v}_1 - \vec{v}_2|}dP. \tag{1.114}$$

From quantum mechanics we know that probabilities are given by the square of amplitudes. Since quantum field theory is just quantum mechanics with a lot of fields, the normalized differential probability is

$$dP = \frac{|\langle f|S|i\rangle|^2}{\langle f|f\rangle\langle i|i\rangle}d\Pi. \tag{1.115}$$

Here, $d\Pi$ is the region of final state momenta at which we are looking. It is proportional to the product of the differential momentum, d^3p_j, of each final state and must integrate to 1. So

$$d\Pi = \prod_j \frac{V}{(2\pi)^3}d^3p_j. \tag{1.116}$$

This has $\int d\Pi = 1$, since $\int \frac{dp}{2\pi} = \frac{1}{L}$ (by dimensional analysis and our 2π convention). According to our normalization convention for single-particle state,

$$\langle p|p\rangle = (2\omega_p)(2\pi)^3\delta^{(3)}(0) = 2\omega_p V. \tag{1.117}$$

Now let us turn to the S-matrix element $\langle f|S|i\rangle$. We usually calculate S-matrix elements perturbatively. In a free theory, where there are no interactions, the S-matrix is simply the identity matrix. We can therefore write

$$S = 1 + i\mathcal{T}, \tag{1.118}$$

where $\mathcal{T}$ is called the transfer matrix and describes deviations from the free theory. Since the S-matrix should vanish unless the initial and final states have the same total 4-momentum, it is helpful to factor an overall momentum-conserving δ-function:

$$\mathcal{T} = (2\pi)^4\delta^4(\Sigma p)\mathcal{M} \tag{1.119}$$

Here, $\delta^4(\Sigma p)$ is shorthand for $\delta^4\left(\Sigma p_i - \Sigma p_f\right)$, where p_i are the initial particles' momenta and p_f are the final particles' momenta. In this way, we can focus on computing the nontrivial part of the S-matrix, $\mathcal{M}$. In quantum field theory, "matrix elements" usually means $\langle f|\mathcal{M}|i\rangle$. Thus we have

$$\langle f|\mathcal{T}|i\rangle = (2\pi)^4\delta^4(\Sigma p)\langle f|\mathcal{M}|i\rangle. \tag{1.120}$$

So,

$$\begin{aligned} dP &= \frac{\delta^4(\Sigma p)TV(2\pi)^4}{(2E_1V)\,(2E_2V)}\frac{|\mathcal{M}|^2}{\prod_j(2E_jV)}\prod_j\frac{V}{(2\pi)^3}d^3p_j \\ &= \frac{T}{V}\frac{1}{(2E_1)\,(2E_2)}|\mathcal{M}|^2 d\Pi_{\text{LIPS}} \end{aligned} \tag{1.121}$$

where

$$d\Pi_{\text{LIPS}} \equiv \prod_{\text{final states } j}\frac{d^3p_j}{(2\pi)^3}\frac{1}{2E_{p_j}}(2\pi)^4\delta^4(\Sigma p) \tag{1.122}$$

is called the *Lorentz-invariant phase space* (LIPS). Putting everything together, we have

$$d\sigma = \frac{1}{(2E_1)\,(2E_2)\,|\vec{v}_1-\vec{v}_2|}|\mathcal{M}|^2 d\Pi_{\text{LIPS}} \tag{1.123}$$

All the factors of V and T have dropped out, so now it is trivial to take $V\to\infty$ and $T\to\infty$. Recall also that velocity is related to momentum by $\vec{v}=\vec{p}/p_0$.

A differential decay rate is the probability that a one-particle state with momentum p_1 turns into a multi-particle state with momenta $\{p_j\}$ over a time T:

$$d\Gamma = \frac{1}{T}dP. \tag{1.124}$$

Of course, it is impossible for the incoming particle to be an asymptotic state at $-\infty$ if it is to decay, and so we should not be able to use the S-matrix to describe decays. The reason this is not a problem is that we calculate the decay rate in perturbation theory assuming the interactions happen only over a finite time T. Thus, a decay is really just like a $(1\to n)$ scattering process.

Following the same steps as for the differential cross section, the decay rate can be written as

$$d\Gamma = \frac{1}{2E_1}|\mathcal{M}|^2 d\Pi_{\text{LIPS}} \tag{1.125}$$

Note that this is the decay rate in the rest frame of the particle. If the particle is moving at relativistic velocities, it will decay much slower due to time dilation. The rate in the boosted frame can be calculated from the rest-frame decay rate using special relativity.

1.5.2 LSZ for Klein-Gordon Field

For free theory, the particle annihilation operator is

$$\begin{aligned} \sqrt{2\omega_k}a_k &= i\int d^3x\; e^{ik\cdot x}(-i\omega_k+\partial_t)\phi(x), \\ \sqrt{2\omega_k}a_k^\dagger &= -i\int d^3x\; e^{-ik\cdot x}(i\omega_k+\partial_t)\phi(x). \end{aligned} \tag{1.126}$$

When interaction is turned on, the field operator $\phi(x)$ is renormalized as

$$\phi(x) \sim \sqrt{Z_\phi}\phi_{\text{in}}(x) \sim \sqrt{Z_\phi}\phi_{\text{out}}(x).$$

In this way, we have

$$\begin{aligned}
\sqrt{2\omega_k}(a^\dagger_{\text{in}} - a^\dagger_{\text{out}}) &= iZ_\phi^{-1/2} \int dt\partial_t \left(\int d^3x \; e^{-ikx}(i\omega_k + \partial_t)\phi_0(x) \right) \\
&= iZ_\phi^{-1/2} \int d^4x e^{-ik\cdot x}(\omega_k^2 + \partial_t^2)\phi_0(x) \\
&= iZ_\phi^{-1/2} \int d^4x e^{-ik\cdot x}\partial_t^2\phi_0(x) + \phi_0(x)(-\nabla^2 + m^2)e^{-ik\cdot x} \\
&= iZ_\phi^{-1/2} \int d^4x e^{-ik\cdot x}(\partial^2 + m^2)\phi_0(x)
\end{aligned}$$

The initial and final states are:

$$\begin{aligned}
|k_1, \cdots, k_m; \text{in}\rangle &= \left[\prod_{j=1}^m \sqrt{2\omega_{k_j}} a^\dagger_{\text{in}}(k_j) \right] |0\rangle, \\
|p_1, \cdots, p_n, \text{out}\rangle &= \left[\prod_{j=1}^n \sqrt{2\omega_{p_j}} a^\dagger_{\text{out}}(p_j) \right] |0\rangle.
\end{aligned} \tag{1.127}$$

The S-matrix is

$$\begin{aligned}
S_{fi} &= \langle p_1, \cdots, p_n; \text{out}|S|k_1, \cdots, k_m; \text{in}\rangle \\
&= \frac{\langle 0|T\left(\prod \sqrt{2\omega_{p_j}} a_{p_j;\text{out}}\right) \int d^4x \exp(i\mathcal{L}_{\text{int}}) \left(\prod \sqrt{2\omega_{k_j}} a^\dagger_{k_j;\text{in}}\right)|0\rangle}{\langle 0|T \int d^4x \exp(i\mathcal{L}_{\text{int}})|0\rangle}
\end{aligned}$$

Since the scattering process correspond to the connected diagram, meaning that the initial and final state has distinct momentum particles. We are free to make the substitution

$$a^\dagger_{\text{in}} \to (a^\dagger_{\text{in}} - a^\dagger_{\text{out}}), \;\; a_{\text{out}} \to -(a^\dagger_{\text{in}} - a^\dagger_{\text{out}})^\dagger.$$

In this way, the S-matrix is

$$\begin{aligned}
&\langle p_1, \cdots, p_n|S|k_1, \cdots, k_m\rangle \\
=&\prod_{i=1}^m \left[\int d^dx_i \; e^{ip_i\cdot x_i} \frac{-\partial^2 - m_i^2}{i\sqrt{Z_\phi}} \right] \prod_{j=m+1}^{m+n} \left[\int d^dx_j \; e^{-ik_j\cdot x_j} \frac{-\partial^2 - m_j^2}{i\sqrt{Z_\phi}} \right] \\
&\times \frac{\langle 0|T\phi_0(x_1)\cdots\phi_0(x_{m+n}) \int d^dx \exp(i\mathcal{L}_{\text{int}})|0\rangle}{\langle 0|T \int d^dx \exp(i\mathcal{L}_{\text{int}})|0\rangle} \\
=&\prod_{i=1}^m \left[\frac{p_i^2 - m_i^2}{i\sqrt{Z_\phi}} \right] \prod_{j=m+1}^{m+n} \left[\frac{k_j^2 - m_j^2}{i\sqrt{Z_\phi}} \right] \tilde{G}(-p_1, \cdots, -p_n, k_1, \cdots, k_m).
\end{aligned}$$

Note that in the second equality, we move the operator ∂^2 out of the time-ordering operator, which will actually create contact terms. The contact terms can be shown to have no contribution to the S-matrix. Also, the Green function is defined as

$$\begin{aligned}
G(x_1, \cdots, x_{m+n}) &\equiv \langle \Omega|T\phi(x_1)\cdots\phi(x_{m+n})|\Omega\rangle \\
&= \frac{\langle 0|T\phi_0(x_1)\cdots\phi_0(x_{m+n}) \int d^dx \exp(i\mathcal{L}_{\text{int}})|0\rangle}{\langle 0|T \int d^dx \exp(i\mathcal{L}_{\text{int}})|0\rangle}.
\end{aligned}$$

The extra factor before the momentum-space Green's function effectively cancel out the external propagator. Thus the LSZ formula (1.5.2) means that the S-matrix is the amputated Green's function.

Remark 1. Contact Terms

We first consider the time-ordered two-point function:

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = \theta(t_1-t_2)\langle 0|\phi(x_1)\phi(x_2)|0\rangle - \theta(t_2-t_1)\langle 0|\phi(x_2)\phi(x_1)|0\rangle.$$

Take time derivative on both side:

$$\begin{aligned}\partial_{t_1}\langle 0|T\phi(x_1)\phi(x_2)|0\rangle &= \langle 0|T\partial_{t_1}\phi(x_1)\phi(x_2)|0\rangle + \delta(t_1-t_2)\langle 0|[\phi(x_1),\phi(x_2)]|0\rangle \\ &= \langle 0|T\partial_{t_1}\phi(x_1)\phi(x_2)|0\rangle.\end{aligned}$$

The second equality follows from the fact that x_1, x_2 is equal-time. Take the the time derivative once more:

$$\partial_{t_1}^2\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = \langle 0|T\partial_{t_1}^2\phi(x_1)\phi(x_2)|0\rangle + \delta(t_1-t_2)\langle 0|[\partial_{t_1}\phi(x_1),\phi(x_2)]|0\rangle.$$

The second term on the right hand side is the contact term. For free theory, $\partial_{t_1}\phi(x_1)$ is the canonical momentum, meaning that

$$[\phi(\vec{x}_1,t),\partial_t\phi(\vec{x}_1,t)] = i\hbar\delta^3(\vec{x}_1-\vec{x}_2).$$

In general, for n-point correlation,

$$\begin{aligned}\partial_{t_1}^2\langle 0|T\phi_{x_1}\cdots\phi_{x_n}|0\rangle =& \langle 0|T\partial_{t_1}^2\phi_{x_1}\cdots\phi_{x_n}|0\rangle \\ &- i\hbar\sum_j\delta^4(x_1-x_j)\langle 0|T\phi_{x_2}\cdots\cancel{\phi_{x_j}}\cdots\phi_{x_n}|0\rangle.\end{aligned}$$

In the LSZ formula, the contact term do not have any singularity. When the external legs approach to momentum shell, these regular terms vanishes, so the contact will not contribute to the S-matrix.

1.5.3 LSZ for Dirac Field

Use the field expansion

$$\begin{aligned}\psi(x) &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\sum_s\left(a_{\mathbf{p}}^s u^s(p)e^{-ip\cdot x} + b_{\mathbf{p}}^{s\dagger}v^s(p)e^{ip\cdot x}\right), \\ \bar{\psi}(x) &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\sum_s\left(b_{\mathbf{p}}^s\bar{v}^s(p)e^{-ip\cdot x} + a_{\mathbf{p}}^{s\dagger}\bar{u}^s(p)e^{ip\cdot x}\right),\end{aligned} \tag{1.128}$$

and the orthogonality relation

$$\begin{aligned}u^{r\dagger}(p)u^s(p) = 2\omega_{\boldsymbol{p}}\delta^{rs}, \quad u^{r\dagger}(\boldsymbol{p},\omega_{\boldsymbol{p}})v^s(-\boldsymbol{p},\omega_{\boldsymbol{p}}) = 0, \\ v^{r\dagger}(p)v^s(p) = 2\omega_{\boldsymbol{p}}\delta^{rs}, \quad v^{r\dagger}(\boldsymbol{p},\omega_{\boldsymbol{p}})u^s(-\boldsymbol{p},\omega_{\boldsymbol{p}}) = 0.\end{aligned} \tag{1.129}$$

The spatial Fourier transformation gives:

$$\int d^3x e^{ip\cdot x}\psi(x) = \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\sum_s a^s_{\boldsymbol{p}} u^s(p) + \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\sum_s b^{s\dagger}_{\boldsymbol{p}} v^s(-\boldsymbol{p},\omega)e^{2i\omega t} \tag{1.130}$$

Left-multiply on both hand side by $\bar{u}^s(p)\gamma^0$, we then get

$$\begin{aligned}
\sqrt{2\omega_{\boldsymbol{p}}}a^s_{\boldsymbol{p}} &= \int d^3x e^{ip\cdot x}\bar{u}^s(p)\gamma^0\psi(x), \\
\sqrt{2\omega_{\boldsymbol{p}}}a^{s\dagger}_{\boldsymbol{p}} &= \int d^3x e^{-ip\cdot x}\bar{\psi}(x)\gamma^0 u^s(p).
\end{aligned} \tag{1.131}$$

Similarly, we consider

$$\int d^3x e^{ip\cdot x}\bar{\psi}(x) = \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\sum_s b^s_{\boldsymbol{p}} \bar{v}^s(p) + \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\sum_s a^{s\dagger}_{\boldsymbol{p}} \bar{u}^s(-\boldsymbol{p},\omega)e^{2i\omega t} \tag{1.132}$$

Right-multiply on both hand side by $\gamma^0 v^s(p)$, we then get

$$\begin{aligned}
\sqrt{2\omega_{\boldsymbol{p}}}b^s_{\boldsymbol{p}} &= \int d^3x e^{ip\cdot x}\bar{\psi}(x)\gamma^0 v^s(p), \\
\sqrt{2\omega_{\boldsymbol{p}}}b^{s\dagger}_{\boldsymbol{p}} &= \int d^3x e^{-ip\cdot x}\bar{v}^s(p)\gamma^0\psi(x).
\end{aligned} \tag{1.133}$$

Following the same strategy as we did for the scalar field, we consider

$$\begin{aligned}
\sqrt{2\omega_{\boldsymbol{p}}}a^s_{\boldsymbol{p};\mathrm{out}} - \sqrt{2\omega_{\boldsymbol{p}}}a^s_{\boldsymbol{p};\mathrm{in}} &= \int dt\ \partial_t\sqrt{2\omega_{\boldsymbol{p}}}a^s_{\boldsymbol{p}} \\
&= \int dt\ \int d^3x e^{ip\cdot x}\bar{u}(p)(\gamma^0\partial_t + i\gamma^0 p^0)\psi(x) \\
&= \int d^4x e^{ip\cdot x}\bar{u}(p)(\gamma^0\partial_t + i\gamma^i p^i + im)\psi(x) \\
&= i\int d^4x e^{ip\cdot x}\bar{u}(p)(-i\not{\partial} + m)\psi(x)
\end{aligned} \tag{1.134}$$

where we have used the fact $\bar{u}(p)(\not{p} - m) = 0$. Take hermitian conjugate,

$$\begin{aligned}
\sqrt{2\omega_{\boldsymbol{p}}}a^{s\dagger}_{\boldsymbol{p};\mathrm{in}} - \sqrt{2\omega_{\boldsymbol{p}}}a^{s\dagger}_{\boldsymbol{p};\mathrm{out}} &= i\int d^4x e^{-ip\cdot x}\bar{\psi}(x)\gamma^0(-i\not{\partial} + m)^\dagger\gamma^0 u(p) \\
&= i\int d^4x e^{-ip\cdot x}\bar{\psi}(x)(i\overleftarrow{\not{\partial}} + m)u(p)
\end{aligned} \tag{1.135}$$

Similarly, using the fact $(\not{p} + m)v(p) = 0$,

$$\begin{aligned}
\sqrt{2\omega_{\boldsymbol{p}}}b^s_{\boldsymbol{p};\mathrm{out}} - \sqrt{2\omega_{\boldsymbol{p}}}b^s_{\boldsymbol{p};\mathrm{in}} &= \int d^4x e^{ip\cdot x}\bar{\psi}(x)(\gamma^0\overleftarrow{\partial_t} + i\gamma^0 p^0)v(p) \\
&= \int d^4x e^{ip\cdot x}\bar{\psi}(x)(\gamma^0\overleftarrow{\partial_t} + i\gamma^i p^i - im)v(p) \\
&= -i\int d^4x e^{ip\cdot x}\bar{\psi}(x)(i\overleftarrow{\not{\partial}} + m)v(p).
\end{aligned} \tag{1.136}$$

Again, take the hermitian conjugate,

$$\begin{aligned}\sqrt{2\omega_{\boldsymbol{p}}}b^{s\dagger}_{\boldsymbol{p};\text{in}}-\sqrt{2\omega_{\boldsymbol{p}}}b^{s\dagger}_{\boldsymbol{p};\text{out}}&=-i\int d^4xe^{ip\cdot x}\bar{v}(p)\gamma^0(i\overleftarrow{\not{\partial}}+m)^\dagger\gamma^0\psi(x)\\&=-i\int d^4xe^{-ip\cdot x}\bar{v}(p)(-i\not{\partial}+m)\psi(x)\end{aligned}\tag{1.137}$$

The same strategy gives the LSZ reduction formula for Dirac field. Consider the S-matrix for particles:

$$\begin{aligned}&\langle p_1,\cdots,p_n|S|k_1,\cdots,k_m\rangle\\&=\prod_{i=1}^{m}\left[\int d^dx_i\ e^{ip_i\cdot x_i}u^{s_1}(p_i)\frac{i\not{\partial}-m_i}{i\sqrt{Z_\phi}}\right]iG(x)\prod_{j=m+1}^{m+n}\left[\int d^dx_j\ e^{-ik_j\cdot x_j}\frac{-i\overleftarrow{\not{\partial}}-m_j}{i\sqrt{Z_\phi}}u^{s_j}(k_j)\right]\\&=\prod_{i=1}^{m}\left[\frac{\not{p}-m_i}{i\sqrt{Z_\phi}}u^{s_i}(p_i)\right]i\tilde{G}(p_1,\cdots,p_n,-k_1,\cdots,-k_m)\prod_{j=m+1}^{m+n}\left[u^{s_j}(k_j)\frac{\not{k}-m_j}{i\sqrt{Z_\phi}}\right].\end{aligned}\tag{1.138}$$

Chapter 2

Scalar Field Theory

In this chapter, we study the interacting scalar field theory

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}, \quad \mathcal{L}_0 = \frac{1}{2}\partial^\mu \phi \partial_\mu \phi - \frac{m^2}{2}\phi^2. \tag{2.1}$$

We will study the case where the interaction is $\frac{g}{3!}\phi^3$ or $\frac{g}{4!}\phi^4$. To make the coupling constant marginal, as it is the physically interesting case, we assume the space-time dimension is 6 and 4 respectively.

2.1 Turning on Interaction

2.1.1 Lehmann Representation

The interacting Hamiltonian do not conserve particle number, and the ground state $|\Omega\rangle$ is no longer the vacuum $|0\rangle$. Consider the Green's function

$$iG(x_1 - x_2) = \langle \Omega | T\phi(x_1)\phi(x_2)|\Omega\rangle \tag{2.2}$$

We can insert a complete basis

$$1 = |\Omega\rangle\langle\Omega| + \sum_\lambda \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}|\lambda_{\boldsymbol{k}}\rangle\langle\lambda_{\boldsymbol{k}}| \tag{2.3}$$

into the correlation function,[1] the Green's function takes the form (take K-G field as the example):

$$iG(x_1 - x_2) = \sum_\lambda \int \frac{d^3k}{(2\pi)^3}\left[\theta(t_1 - t_2)\langle\Omega|\phi(x_1)|\lambda_{\vec{k}}\rangle\langle\lambda_{\vec{k}}|\phi(x_2)|\Omega\rangle + (t_1 \leftrightarrow t_2, x_1 \leftrightarrow x_2)\right].$$

Note that

$$\langle\lambda_{\boldsymbol{k}}|\phi(x)|\Omega\rangle = \langle\lambda_{\boldsymbol{k}}|e^{iP\cdot x}\phi(0)e^{-iP\cdot x}|\Omega\rangle = e^{ik\cdot x}\left.\langle\lambda_0|\phi(0)|\Omega\rangle\right|_{k^0=\omega_{\boldsymbol{k}}}. \tag{2.4}$$

[1] Here we assume $\langle\Omega|\phi(x)|\Omega\rangle = 0$ unless there is spontaneously symmetry breaking happening.

Following the same procedure as we do for the free field theory,

$$G(x_1 - x_2) = \int_0^\infty \frac{dM^2}{2\pi} \rho(M^2) G_0(x_1 - x_2; M^2), \tag{2.5}$$

where the spectral function $\rho(M^2)$ is

$$\rho(M^2) = \sum_\lambda (2\pi)\delta(M^2 - m_\lambda^2)|\langle\Omega|\phi(0)|\lambda_0\rangle|^2. \tag{2.6}$$

In particle, near the one-particle state the Green's function looks like:

$$i\tilde{G}(k) = \frac{iZ_\phi}{k^2 - m^2 + i\epsilon} + \text{regular terms}. \tag{2.7}$$

If we renormalize the field strength

$$\phi_R(x) = \frac{1}{\sqrt{Z_\phi}} \phi_0(x), \tag{2.8}$$

the Green's function then has exactly the same form as free theory. This normalization factor Z_ϕ is exactly what we obtained in the loop correction to the propagator.

2.1.2 Perturbation Theory

For interaction theory, the partition function can be formally expressed as:

$$Z[J] = \exp\left(i\int d^dx \mathcal{L}_{\text{int}}\left[\frac{\delta}{i\delta J(x)}\right]\right) Z_0[J]. \tag{2.9}$$

The expectation values for a generic operator of the form $O(\phi)$ can be evaluated by the true partition function

$$\langle O(\phi)\rangle = \frac{1}{Z[0]}\, O\left[\frac{\delta}{i\delta J(x)}\right] Z[J]\bigg|_{J=0}. \tag{2.10}$$

The expression (2.10) can be expanded order by order using the Feynman diagram. Since the unconnected diagram can be absorbed into $Z[0]$, we only need to calculate the connected diagram.

The procedure of perturbative expansion with only connected diagrams can be formally represented by introducing the quantity

$$Z[J] = Z[0] \exp\left(iW[J]\right). \tag{2.11}$$

The perturbative expansion of $W[J]$ contain only the connected diagrams. Note that for the free theory,

$$\frac{Z_0[J]}{Z_0[0]} = \exp\left[-\frac{i}{2}\int d^dx_1 d^dx_2 J(x_1)\Delta(x_1 - x_2)J(x_2)\right],$$

which means

$$W_0 = -\frac{1}{2}\int d^dx_1 d^dx_2 J(x_1)\Delta(x_1 - x_2)J(x_2).$$

For the interaction theory, the expectation (2.10) can then be replaced by the connected expectation:

$$\langle O(\phi)\rangle_c \equiv i\, O\left[\frac{\delta}{i\delta J(x)}\right] W[J]\bigg|_{J=0}. \tag{2.12}$$

Consider the two-point connected correlation (propagator):

$$\begin{aligned} i\Delta(x_1 - x_2) &= \langle \mathcal{T}\phi(x_1)\phi(x_2)\rangle_c \\ &= i\,\frac{\delta^2 W[J]}{i\delta J(x_1) i\delta J(x_2)}\bigg|_{J=0} \\ &= \frac{\delta^2 \ln Z[J]}{i\delta J(x_1) i\delta J(x_2)}\bigg|_{J=0} \\ &= \frac{1}{Z[0]}\,\frac{\delta^2 Z[J]}{i\delta J(x_1) i\delta J(x_2)}\bigg|_{J=0}, \end{aligned} \tag{2.13}$$

where we have used the fact that

$$\frac{\delta Z^n[J]}{\delta J(x_1)\cdots\delta J(x_n)} = 0,\ \forall n = 1 \bmod 2. \tag{2.14}$$

The result is the same as the original definition.

Further, we can consider the four-point connected correlation:

$$iV_4 \equiv \langle \mathcal{T}\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle_c \tag{2.15}$$

Following the same procedure,

$$\begin{aligned} iV_4 = i\,&\frac{\delta^4 W[J]}{i\delta J(x_1) i\delta J(x_2) i\delta J(x_3) i\delta J(x_4)}\bigg|_{J=0} \\ = &\frac{1}{Z[0]}\,\frac{\delta^4 Z[J]}{i\delta J(x_1) i\delta J(x_2) i\delta J(x_3) i\delta J(x_4)}\bigg|_{J=0} \\ &- i\Delta(x_1 - x_2) i\Delta(x_3 - x_4) \\ &- i\Delta(x_1 - x_3) i\Delta(x_2 - x_4) \\ &- i\Delta(x_1 - x_4) i\Delta(x_2 - x_3). \end{aligned} \tag{2.16}$$

The connected correlation function automatically omit those disconnected components.

2.2 Real ϕ^3 Theory

Now consider the interaction theory with additional Lagrangian

$$\mathcal{L}_{\text{int}}[\phi] = \frac{g}{3!}\phi^3. \tag{2.17}$$

Note that the field ϕ has the mass dimension $[\frac{d-2}{2}]$. The critical dimension is $d = 6$ where the coupling constant g is dimensionless. In this section, we consider the real Klein-Gordon field with ϕ^3 interaction on 6-dimensional space-time.

For interaction theory, the renormalized Lagrangian has the form:

$$\begin{aligned}\mathcal{L} &= Z_\phi \frac{1}{2}\partial^\mu \phi \partial_\mu \phi - Z_m \frac{m^2}{2}\phi^2 + Z_g \frac{g}{3!}\phi^3 \\ &= \mathcal{L}_0 + \mathcal{L}_{\text{int}} + \mathcal{L}_{\text{ct}},\end{aligned} \tag{2.18}$$

where the counter terms are:

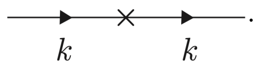
$$\begin{aligned}\mathcal{L}_{\text{ct}}[\phi] &= \frac{A}{2}\partial^\mu \phi \partial_\mu \phi - \frac{B}{2} m^2 \phi^2 + \frac{C}{3!} g \phi^3 \\ &\simeq -\frac{A}{2}\phi \partial^2 \phi - \frac{B}{2} m^2 \phi^2 + \frac{C}{3!} g \phi^3,\end{aligned} \tag{2.19}$$

where

$$A = Z_\phi - 1, B = Z_m - 1, C = Z_g - 1.$$

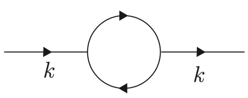
The counter term for the the free field gives additional correction

$$i\tilde{\Delta}^{(\text{ct})}(k) = i\tilde{\Delta}_0(k)(Ak^2 - Bm^2) i\tilde{\Delta}_0(k) \tag{2.20}$$

$=$ .

2.2.1 Self Energy Correction

To second order, we consider the one-loop correction to the propagator:

$i\tilde{\Delta}^{(2)}(k) =$ 

$$= i\tilde{\Delta}_0(k)\left[i\Sigma^{(2)}(k^2)\right] i\tilde{\Delta}_0(k), \tag{2.21}$$

where the self energy term to the second order $i\Pi^{(2)}(k)$ is defined as:

$$i\Sigma^{(2)}(k^2) \equiv \frac{g^2}{2}\int \frac{d^d q}{(2\pi)^d}\tilde{\Delta}_0(q)\tilde{\Delta}_0(k-q) + (Ak^2 - Bm^2). \tag{2.22}$$

The coefficient $g^2/2$ comes from the symmetry factor in the diagram. We can also check the coefficient explicitly, by considering the expansion to the second order (we denote $\delta/\delta J(x_i)$ as δ_i):

$$\delta_1 \delta_2 \frac{1}{2!4!}\left[\frac{ig}{3!}\int d^d y \left(\frac{\delta}{\delta J(y)}\right)^3\right]^2 \left[-\frac{i}{2}\int d^d y_1 d^d y_2 J(y_1)\Delta(y_1 - y_2) J(y_2)\right]^4.$$

The expansion gives the coefficient

$$\left(\frac{ig}{6}\right)^2 \times \frac{1}{2! \times 4! \times 2^4}.$$

Now consider the combinatorial factor, which comes from the exchange of $\phi(x_i)$ in the propagator, the exchange of $\phi(x_i)$ in the vertex, the exchange of propagator in the diagram, and the change of vertices in the diagram:

$$(2!)^4 \times (3!)^2 \times (4 \times 3) \times 2.$$

Those two factors produce a $-g^2/2$ coefficient. Note that in the self energy expressio, we put a i factor in front of each propagator, which absorbs the minus sign.

Once we obtain the self energy, the one-loop corrected propagator has the form:

$$\begin{aligned}
i\tilde{\Delta}(k) &= i\tilde{\Delta}_0(k) + i\tilde{\Delta}_0(k)\left[\sum_{n=1}^{\infty} i\Sigma(k^2)\right] i\tilde{\Delta}_0(k) \\
&= \frac{i}{\tilde{\Delta}_0^{-1}(k) - \Sigma(k^2)} \\
&= \frac{i}{k^2 - m^2 - \Sigma(k^2)}.
\end{aligned} \tag{2.23}$$

Now we are going to evaluate the divergent integral in the self energy expression, using the Feynman parameters:

$$\begin{aligned}
&\int \frac{d^dq}{(2\pi)^d}\frac{1}{q^2-m^2}\frac{1}{(k-q)^2-m^2} \\
&= \int \frac{d^dq}{(2\pi)^d}\int_0^1 dx \frac{1}{\left[q^2 - m^2 + x\left((q-k)^2 - q^2\right)\right]^2} \\
&= \int_0^1 dx \int \frac{d^dq}{(2\pi)^d}\frac{1}{\left[(q-kx)^2 - D\right]^2},
\end{aligned}$$

where $D = m^2 - k^2x(1-x)$. Then we can shift $q \to q + kx$ leaving an integral that only depends on q^2. In this way,

$$\Sigma(k^2) = \int_0^1 I(x)dx.$$

To evaluate the self-energy, it suffices to obtain the integral

$$I(x) = \frac{g^2}{2i}\int \frac{d^dq}{(2\pi)^d}\frac{1}{\left[q^2 - D\right]^2}.$$

Remark 2. Feyman Parameters

We use Feynman's formula to combine denominators,

$$\frac{1}{A_1 \ldots A_n} = \int dF_n \left(x_1 A_1 + \ldots + x_n A_n\right)^{-n}, \tag{2.24}$$

where the integration measure over the Feynman parameters x_i is

$$\int dF_n = (n-1)! \int_0^1 dx_1 \ldots dx_n \delta\left(x_1 + \ldots + x_n - 1\right). \tag{2.25}$$

This measure is normalized so that $\int dF_n = 1$. The simplest case is

$$\frac{1}{AB} = \int_0^1 \frac{dx}{[A + (B-A)x]^2} = \int_0^1 \frac{\delta(x+y-1)}{[xA+yB]^2} dxdy. \tag{2.26}$$

Other useful identities are

$$\begin{aligned} \frac{1}{AB^n} &= \int_0^1 dxdy \frac{\delta(x+y-1) n y^{n-1}}{[xA+yB]^{n+1}}, \\ \frac{1}{ABC} &= \int_0^1 dxdydz \frac{2\delta(x+y+z-1)}{[xA+yB+zC]^3}. \end{aligned} \tag{2.27}$$

By making the Wick rotation $q^0 \to i q_E^0$, the integral becomes:[2]

$$I(x) = \frac{g}{2} \int \frac{d^d q_E}{(2\pi)^d} \frac{1}{\left(q_E^2 + D\right)^2} = \frac{g \Omega_d}{2(2\pi)^d} \int dq \frac{q^{d-1}}{\left(q^2 + D\right)^2}.$$

Dimensional Regularization

We set the dimension to $d = 6 - \epsilon$, and rewrite the Lagrangian as

$$\mathcal{L} = Z_\phi \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - Z_m \frac{m^2}{2} \phi^2 + Z_g \frac{g \tilde{\mu}^{\epsilon/2}}{3!} \phi^3. \tag{2.30}$$

Note that the coupling constant should be changed to $g \to g\tilde{\mu}^{\epsilon/2}$ where μ is of mass dimension [1] in order to get the correct dimensionality. We then expand the expression to zeroth order of ϵ. A useful identity is:

$$\int dk \frac{k^a}{\left(k^2 + D\right)^b} = D^{\frac{a+1}{2} - b} \frac{\Gamma\left(\frac{a+1}{2}\right) \Gamma\left(b - \frac{a+1}{2}\right)}{2\Gamma(b)}. \tag{2.31}$$

---

[2] The d-dimensional solid angle is

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}, \tag{2.28}$$

where $\Gamma(x)$ is the gamma function, satisfing

$$\Gamma(1+x) = x\Gamma(x), \ \Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + O(\epsilon). \tag{2.29}$$

In particular, $\Gamma(n+1) = n!$.

Actually, we can compute the integral and series expansion in `Mathematica` all together:

```
omg=(2*Pi^(d/2))/(Gamma[d/2]);
cof=g^2*\[Mu]^(6-d)/2*omg/(2*Pi)^d;
int=cof*Integrate[q^(d-1)/(q^2+D)^2,{q,0,Infinity}][[1]];
map={g^2->\[Alpha]*(4*Pi)^3,EulerGamma->Subscript[\[Gamma],E]};
ans=Series[int/.{d->6-\[Epsilon]},{\[Epsilon],0,0}];
ans/.map//Simplify
```

The result is (where $\alpha \equiv g^2/(4\pi)^3$)

$$I(x) = \frac{\alpha D}{2}\left[\ln\left(\frac{De^{\gamma_E}}{4\pi\tilde{\mu}^2}\right) - \left(\frac{2}{\epsilon}+1\right)\right] + O(\epsilon).$$

Now insert $D = m^2 - k^2 x(1-x)$. Note that

$$\int_0^1 dx D = m^2 - \frac{k^2}{6}.$$

This simplifies the result to

$$\Sigma^{(2)}(k^2) = \frac{\alpha}{2}\left(\frac{2}{\epsilon}+1\right)\left(\frac{k^2}{2} - m^2\right) + \frac{\alpha}{2}\int_0^1 dx D(x)\ln\left(\frac{D(x)}{\mu^2}\right), \tag{2.32}$$

where we have replace $\tilde{\mu}$ with

$$\mu \equiv \sqrt{\frac{4\pi}{e^{\gamma_E}}}\tilde{\mu}. \tag{2.33}$$

Renormalization

The counter terms also contribute to the perturbative correction,

$$\begin{aligned}\Sigma^{(2)}\left(k^2\right) =& \frac{\alpha}{2}\int_0^1 dx D\ln\left(\frac{D}{m^2}\right) + \left\{\frac{\alpha}{6}\left[\frac{1}{\varepsilon} + \ln\left(\frac{\mu}{m}\right) + \frac{1}{2}\right] + A\right\}k^2 \\ &- \left\{\alpha\left[\frac{1}{\varepsilon} + \ln\left(\frac{\mu}{m}\right) + \frac{1}{2}\right] + B\right\}m^2 + O\left(\alpha^2\right).\end{aligned}$$

Consider the on-shell condition for the subtraction:

$$\Sigma(m^2) = \Sigma'(m^2) = 0. \tag{2.34}$$

Set $D_0 \equiv D(x)|_{k^2=m^2} = m^2(1-x+x^2)$, the self energy has the form:

$$\Sigma^{(2)}(k^2) = \frac{\alpha}{2}\int_0^1 dx D(x)\ln\left(\frac{D(x)}{D_0(x)}\right) + C_k k^2 + C_m m^2. \tag{2.35}$$

The condition $\Pi(m^2) = 0$ requires

$$\Sigma^{(2)}(k^2) = \frac{\alpha}{2}\int_0^1 dx D(x)\ln\left(\frac{D(x)}{D_0(x)}\right) + C_k(k^2 - m^2).$$

The condition $\Pi'(m^2) = 0$ requires

$$\begin{aligned}
\left.\frac{d\Sigma^{(2)}(k^2)}{dk^2}\right|_{k^2=m^2} &= \frac{\alpha}{2}\int_0^1 dx \left.\left[\frac{D(x)}{dk^2}\ln\left(\frac{D(x)}{D_0(x)}\right) + D_0(x)\right]\right|_{q^2=m^2} + C_k \\
&= \frac{\alpha}{2}\int_0^1 dx(x^2 - x) + C_k \\
&= C_k - \frac{\alpha}{12} = 0.
\end{aligned}$$

In this way, we obtained the renormalized self-energy:

$$\Sigma^{(2)}(k^2) = \frac{\alpha}{2}\int_0^1 dx D(x)\ln\left(\frac{D(x)}{D_0(x)}\right) + \frac{\alpha}{12}(k^2 - m^2). \tag{2.36}$$

On the other hand, we chan choose the $\overline{\text{MS}}$ subtraction scheme, i.e.,

$$A = -\frac{\alpha}{6\epsilon},\ B = -\frac{\alpha}{\epsilon}. \tag{2.37}$$

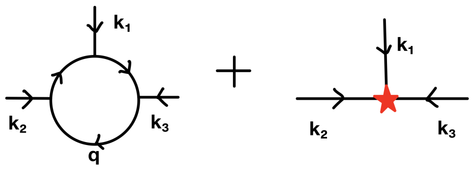
The self energy under $\overline{\text{MS}}$ scheme will depend on the the mass scale μ we choose:

$$\Sigma^{(2)}\left(k^2\right) = \frac{\alpha}{2}\int_0^1 dxD\ln\left(\frac{D}{m^2}\right) + \alpha\left[\ln\left(\frac{\mu}{m}\right) + \frac{1}{2}\right]\left(\frac{k^2}{6} - m^2\right). \tag{2.38}$$

2.2.2 Vertex Correction

Now consider the simplest one-loop correction to the vertex function (together with the counter term):

$iV_3^{(3)}(k_1, k_2, k_3) =$



(2.39)

$$= (ig)^3 i^3 \int \frac{d^a q}{(2\pi)^d}\tilde{\Delta}(q - k_1)\tilde{\Delta}(q + k_2)\tilde{\Delta}(q) + iCg,$$

Using the Feynman parameter, the integrant is

$$\tilde{\Delta}(q - k_1)\tilde{\Delta}(q + k_2)\tilde{\Delta}(q) = \int dF_3 \frac{1}{(q^2 - D)^3}$$

where we have shift the value of q, and D can be evaluate by the following code:

```
A1=(l-k1)^2-m^2;
A2=(l+k2)^2-m^2;
A3=(l)^2-m^2;
{c,b,a}=CoefficientList[x1*A1+x2*A2+(1-x1-x2)*A3,{l}];
-c+b^2/(4*a)//Expand
```

The result is

$$D = m^2 - k_1^2 x_1(1-x_1) - k_2^2 x_2(1-x_2) - 2k_1k_2x_1x_2.$$

The same procedure gives:

$$V_3^{(3)}/g = \int dF_3 I(x_1,x_2,x_3) + C, \tag{2.40}$$

where

$$I(x_1,x_2,x_3) = \frac{g^2\Omega_d}{(2\pi)^d}\int dq \frac{q^{d-1}}{(q^2+D)^3}.$$

The same regularization procedure in `Mathematica`:

```
omg=(2*Pi^(d/2))/(Gamma[d/2]);
cof=g^2*\[Mu]^(6-d)*omg/(2*Pi)^d;
int=cof*Integrate[q^(d-1)/(q^2+D)^3,{q,0,Infinity}][[1]];
map={g^2->\[Alpha]*(4*Pi)^3,EulerGamma->Subscript[\[Gamma],E]};
ans=Series[int/.{d->6-\[Epsilon]},{\[Epsilon], 0, 0}];
ans/.map//Simplify
```

The result is

$$\begin{aligned} V_3^{(3)}/g &= \frac{\alpha}{\epsilon} + \frac{\alpha}{2}\int dF_3 \ln\left(\frac{4\pi\tilde{\mu}^2 e^{-\gamma_E}}{D}\right) + C + O(\epsilon) \\ &= \frac{\alpha}{\epsilon} + \alpha \ln\left(\frac{\mu}{m}\right) - \frac{\alpha}{2}\int dF_3 \ln\left(\frac{D}{m}\right) + C. \end{aligned} \tag{2.41}$$

The on-shell subtraction requires

$$V_3(0,0,0) = g, \tag{2.42}$$

which gives

$$C = -\frac{\alpha}{\epsilon} - \alpha \ln\left(\frac{\mu}{m}\right). \tag{2.43}$$

So the vertex function to the third order is

$$V_3(k_1,k_2,k_3) = g\left\{1 - \frac{\alpha}{2}\int dF_3 \ln\left[\frac{D(x_1,x_2,x_3)}{m}\right]\right\}. \tag{2.44}$$

The $\overline{\text{MS}}$ scheme, on the other hand, sets

$$C = -\frac{\alpha}{\epsilon}. \tag{2.45}$$

2.2.3 Renormalization Group

We fist summarize the normalization factor obtained on the one-loop level (with $\overline{\text{MS}}$ subtraction scheme):

$$\begin{aligned} Z_\phi &= 1 - \frac{\alpha}{6\epsilon} + O(\alpha^2), \\ Z_m &= 1 - \frac{\alpha}{\epsilon} + O(\alpha^2), \\ Z_g &= 1 - \frac{\alpha}{\epsilon} + O(\alpha^2). \end{aligned} \tag{2.46}$$

For the renormalized Lagrangian in $(6-\epsilon)$-dimension

$$\mathcal{L} = Z_\phi \frac{1}{2}\partial^\mu \phi \partial_\mu \phi - Z_m \frac{m^2}{2}\phi^2 + Z_g \frac{g\tilde{\mu}^{\epsilon/2}}{3!}\phi^3, \tag{2.47}$$

the factors relate the original field and bare coefficients

$$\phi_0 = Z_\phi^{1/2}\phi,\ m_0 = Z_m^{1/2} Z_\phi^{-1/2} m,\ g_0 = Z_g Z_\phi^{-3/2}\tilde{\mu}^{\epsilon/2} g. \tag{2.48}$$

The renormalization group requires that the bare parameter is independent of the mass scale μ we choose, that is:

$$\frac{d\phi_0}{d\ln\mu} = \frac{dm_0}{d\ln\mu} = \frac{dg_0}{d\ln\mu} = 0. \tag{2.49}$$

Beta Function

Star with g_0, it is more convenient to use

$$\alpha_0 \equiv \frac{g_0^2}{4\pi} = Z_g^2 Z_\phi^{-3}\tilde{\mu}^\epsilon \alpha. \tag{2.50}$$

Take logarithm on both side:

$$\ln\alpha_0 = \ln(Z_g^2 Z_\phi^{-3}) + \ln\alpha + \epsilon\ln\tilde{\mu}. \tag{2.51}$$

The RG equation is

$$\frac{d\ln\alpha_0}{d\ln\mu} = \frac{d\ln(Z_g^2 Z_\phi^{-3})}{d\alpha}\frac{d\alpha}{d\ln\mu} + \frac{1}{\alpha}\frac{d\alpha}{d\ln\mu} + \epsilon = 0. \tag{2.52}$$

To the first order of α:

$$\frac{d\ln(Z_g^2 Z_\phi^{-3})}{d\alpha} = \frac{d}{d\alpha}\left(-\frac{2\alpha}{\epsilon} + \frac{\alpha}{2\epsilon}\right) = -\frac{3}{2\epsilon}, \tag{2.53}$$

which leads to

$$\frac{d\alpha}{d\ln\mu}\left(1 - \frac{3\alpha}{2\epsilon} + O(\alpha^2)\right) + \epsilon\alpha = 0. \tag{2.54}$$

The beta function is defined as

$$\beta(\alpha) = \frac{d\alpha}{d\ln\mu} = \beta_1\alpha + \beta_2\alpha^2 + O(\alpha^3). \tag{2.55}$$

Insert such definition into the original expression, and keep track of the order of α, we get

$$(\beta_1 + \epsilon)\alpha + \left(\beta_2 - \frac{3\beta_1}{2\epsilon}\right)\alpha^2 + O(\alpha^3) = 0. \tag{2.56}$$

The beta function is

$$\beta(\alpha) = -\epsilon\alpha - \frac{3}{2}\alpha^2 + O(\alpha^3). \tag{2.57}$$

Anomalous Dimension

Consider the RG equation with bare mass:

$$\begin{aligned}\frac{d\ln m_0}{d\ln\mu} &= \frac{1}{2}\frac{d(\ln Z_m - \ln Z_\phi)}{d\alpha}\frac{d\alpha}{d\ln\mu} + \frac{1}{m}\frac{dm}{d\ln\mu} \\ &= \frac{5\alpha}{12} + \frac{1}{m}\frac{dm}{d\ln\mu} + O(\alpha^2) = 0.\end{aligned} \tag{2.58}$$

We get the anomalous dimension of the mass:

$$\gamma_m(\alpha) \equiv \frac{1}{m}\frac{dm}{d\ln\mu} = -\frac{5\alpha}{12} + O(\alpha^2). \tag{2.59}$$

Also, for the bare field

$$\frac{d\ln\phi_0}{d\ln\mu} = \frac{1}{2}\frac{d\ln Z_\phi}{d\ln\mu} + \frac{d\ln\phi}{d\ln\mu} = 0. \tag{2.60}$$

We can define the anomalous dimension of the field as

$$\gamma_\phi \equiv \frac{1}{2}\frac{d\ln Z_\phi}{d\ln\mu} = \frac{1}{2}\frac{d\ln Z_\phi}{d\alpha}\frac{d\alpha}{d\ln\mu} = \frac{\alpha}{12} + O(\alpha^2). \tag{2.61}$$

Callan-Symanzik Equation

Consider the bare propagator:

$$\tilde{\Delta}_0(k) = Z_\phi\tilde{\Delta}(k) \tag{2.62}$$

The RG condition for the bare propagator gives:

$$\frac{d\ln\tilde{\Delta}_0(k)}{d\ln\mu} = \frac{d\ln Z_\phi}{d\ln\mu} + \frac{1}{\tilde{\Delta}(k)}\left(\frac{\partial}{\partial\ln\mu} + \frac{d\alpha}{d\ln\mu}\frac{\partial}{\partial\alpha} + \frac{dm}{d\ln\mu}\frac{\partial}{\partial m}\right)\tilde{\Delta}(k) = 0.$$

The Callan-Symanzik equation is

$$\left(2\gamma_\phi + \frac{\partial}{\partial\ln\mu} + \beta(\alpha)\frac{\partial}{\partial\alpha} + \gamma_m(\alpha)m\frac{\partial}{\partial m}\right)\tilde{\Delta}(k) = 0. \tag{2.63}$$

2.3 Real ϕ^4 Theory

In this section, we consider the real Klein-Gordon field with ϕ^4 interaction. The field ϕ has mass dimension $[\frac{d-2}{2}] = [1]$, so the critical dimension is $d = 4$, where the coupling constant g is dimensionless. For dimensional regulation purpose, we write the renormalized Lagrangian on $(4-\epsilon)$-dimensional space-time as

$$\mathcal{L} = Z_\phi\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - Z_m\frac{m^2}{2}\phi^2 - Z_g\frac{g\tilde{\mu}^\epsilon}{4!}\phi^4, \tag{2.64}$$

where we have introduced a mass scale $\tilde{\mu}$. As the ϕ^3 theory, we can rewrite the Lagrangian as:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}} + \mathcal{L}_{\text{ct}}. \tag{2.65}$$

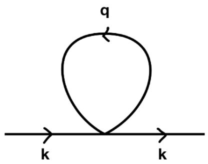
In the following we investigate the loop correction to the mass and the coupling constant.

2.3.1 One-loop Correction

Self-energy

Following the same procedure, the one-loop self-energy correction is (with counter terms):

$$i\Sigma(k^2) =$$



$$+$$

$$= -\frac{g\tilde{\mu}^{\epsilon}}{2}\int \frac{d^dq}{(2\pi)^d}\frac{1}{q^2-m^2} + i(Ak^2 - Bm^2). \tag{2.66}$$

After the Wick rotation,

$$\Sigma(k^2) = -\frac{g\tilde{\mu}^{\epsilon}}{2}\frac{\Omega_d}{(2\pi)^d}\int \frac{q^{d-1}dq}{q^2+m^2} + (Ak^2 - Bm^2). \tag{2.67}$$

The dimensional regulation is carried out using the following code:

```
omg=(2*Pi^(d/2))/(Gamma[d/2]);
cof=g*\[Mu]^(4-d)/2*omg/(2*Pi)^d;
int=cof*Integrate[q^(d-1)/(q^2+m^2),{q,0,Infinity}][[1]];
map={EulerGamma->Subscript[\[Gamma],E]};
ans=Series[int/.{d->4-\[Epsilon]},{\[Epsilon],0,0}];
ans/.map//Simplify
```

The result is

$$\Sigma(k^2) = \frac{gm^2}{32\pi^2}\left[\frac{2}{\epsilon} + 1 + \log\left(\frac{4\pi\tilde{\mu}^2 e^{-\gamma_E}}{m^2}\right)\right] + (Ak^2 - Bm^2) + O(\epsilon). \tag{2.68}$$

Using the $\overline{\mathrm{MS}}$ renormalization scheme, we set

$$A = 0,\ \ B = \frac{g}{16\pi^2\epsilon}. \tag{2.69}$$

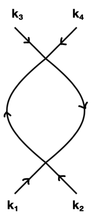
The result is

$$\Sigma(k^2) = \frac{gm^2}{16\pi^2}\log\left(\frac{\mu}{m}\right) + \frac{gm^2}{32\pi^2} + O(\epsilon). \tag{2.70}$$

Vertex Correction

Now consider the vertex correction. The vertex correction to the lowest order (with the counter term) is

$$iV_4^{(2)}(k_1, k_2, k_3, k_4) =$$



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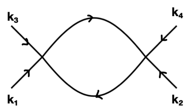

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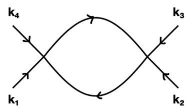

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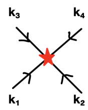


$$= \frac{g^2}{2}\left[iF(s) + iF(t) + iF(u)\right] - iCg, \tag{2.71}$$

where

$$s = (k_1 + k_2)^2,\ t = (k_1 + k_3)^2,\ u = (k_1 + k_4)^2, \tag{2.72}$$

and

$$\begin{aligned} iF(k^2) &= \tilde{\mu}^\epsilon \int \frac{d^d q}{(2\pi)^d} \tilde{\Delta}_0(q)\tilde{\Delta}_0(q+k) && (2.73) \\ &= \frac{i\tilde{\mu}^\epsilon \Omega_d}{(2\pi)^d} \int_0^1 dx \int \frac{q^{d-1} dq}{[q^2 + m^2 + x(1-x)k^2]^2}. && (2.74) \end{aligned}$$

Then we carry out the calculation (set $D(k^2, x) = m^2 + x(1-x)k^2$)

```
omg=(2*Pi^(d/2))/(Gamma[d/2]);
cof=g^2*\[Mu]^(4-d)/2*omg/(2*Pi)^d;
int=cof*Integrate[q^(d-1)/(q^2+D)^2,{q,0,Infinity}][[1]];
map={EulerGamma->Subscript[\[Gamma],E]};
ans=Series[int/.{d->4-\[Epsilon]},{\[Epsilon],0,0}];
ans/.map//Simplify
```

The result is:

$$\begin{aligned} F(s) &= \frac{1}{8\pi^2\epsilon} + \frac{1}{16\pi^2} \int_0^1 dx \ln\left(\frac{4\pi\tilde{\mu}^2 e^{-\gamma_E}}{D}\right) \\ &= \frac{1}{8\pi^2\epsilon} + \frac{1}{8\pi^2} \ln\left(\frac{\mu}{m}\right) - \frac{1}{16\pi^2} \int_0^1 dx \ln\left(\frac{D(s,x)}{m^2}\right). \end{aligned} \tag{2.75}$$

The $\overline{\text{MS}}$ scheme absorbs the $\frac{1}{8\pi^2\epsilon}$ term, i.e.,

$$C = \frac{3g}{16\pi^2}. \tag{2.76}$$

The result is:

$$V_4(k_1, k_2, k_3, k_4) = -g + \frac{g^2}{32\pi^2} \int_0^1 dx \ln\left(\frac{\mu^6}{D(s,x)D(t,x)D(u,x)}\right). \tag{2.77}$$

To summarize, the normalization is:

$$\begin{aligned} Z_\phi &= 1, && (2.78) \\ Z_m &= 1 + \frac{g}{16\pi^2\epsilon}, && (2.79) \\ Z_g &= 1 + \frac{3g}{16\pi^2\epsilon}. && (2.80) \end{aligned}$$

2.3.2 Renormalization Group

Now consider the RG equation for the one-loop correction. The bare parameters are:

$$g_0 = Z_g g \tilde{\mu}^\epsilon,\ m_0 = Z_m^{1/2} m, \tag{2.81}$$

The RG conditions are:

$$\frac{dg_0}{d\ln\mu} = \left(\frac{3}{16\pi^2\epsilon} + \frac{1}{g}\right)\frac{dg}{d\ln\mu} + \epsilon = 0, \tag{2.82}$$

$$\frac{dm_0}{d\ln\mu} = \frac{1}{32\pi^2\epsilon}\frac{dg}{d\ln\mu} + \frac{1}{m}\frac{dm}{d\ln\mu} = 0. \tag{2.83}$$

Consider the series expansion of beta function:

$$\beta(g) = \frac{dg}{d\ln\mu} = \beta_1 g + \beta_2 g^2 + O(g^3). \tag{2.84}$$

The beta function is

$$\beta(g) = -\epsilon g + \frac{3g^2}{16\pi^2} + O(g^3). \tag{2.85}$$

The anomalous dimension of mass is

$$\gamma_m = \frac{1}{m}\frac{dm}{d\ln\mu} = \frac{g}{32\pi^2} + O(g^2) \tag{2.86}$$

Chapter 3

Quantum Electrodynamics

The Lagrangian for quantum electrodynamics is

$$\begin{aligned}\mathcal{L}_{\text{QED}} &= \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - eA_\mu\bar{\psi}\gamma^\mu\psi \\ &= \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{Maxwell}} + \mathcal{L}_{\text{int}},\end{aligned} \tag{3.1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\nu = (dA)_{\mu\nu}. \tag{3.2}$$

The Lagrangian is invariant under the gauge transformation:

$$\begin{aligned}\psi(x) &\to e^{-ie\alpha(x)}\psi(x), \\ A^\mu(x) &\to A^\mu(x) + \partial^\mu\alpha(x).\end{aligned} \tag{3.3}$$

It is convenient to rewrite Lagrangian as

$$\mathcal{L}_{\text{QED}} = \bar{\psi}\left(i\not{D} - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{3.4}$$

where we have define the covariant derivative as:

$$\not{D} = \gamma^\mu D_\mu = \gamma^\mu[\partial_\mu + ieA_\mu(x)] = \not{\partial} + ie\not{A}. \tag{3.5}$$

3.1 Perturbative Renormalization

As with the scalar field,

$$Z[\bar{\eta}, \eta, J] = \exp\left\{i\int d^dx\mathcal{L}_{\text{int}}\left[\frac{\delta}{i\delta J}, \frac{\delta}{i\delta\eta}, \frac{i\delta}{\delta\bar{\eta}}\right]\right\}Z_0[\bar{\eta}, \eta, J]. \tag{3.6}$$

We use the dimensional regularization by default. Note that ψ has the mass dimension $[\frac{d-1}{2}]$, A^μ had the mass dimension $[\frac{d}{2}-1]$, and e has the mass dimension $[2-\frac{d}{2}]$. When $d = 4-\epsilon$, we make the replacement

$$e \to e\tilde{\mu}^{\epsilon/2}, \tag{3.7}$$

so that to make the coupling constant e dimensionless.

The renormalized Lagrangian is

$$
\begin{aligned}
\mathcal{L} &= Z_\psi \bar{\psi}_R (i\gamma^\mu \partial_\mu)\psi_R - Z_m m \bar{\psi}_R \psi_R + \frac{1}{4} Z_A F_{R,\mu\nu} F_R^{\mu\nu} - Z_e e_R A_R^\mu \bar{\psi}_R \gamma^\mu \psi_R \\
&= \mathcal{L}_0 + \mathcal{L}_{\text{int}} + \mathcal{L}_{\text{ct}}.
\end{aligned}
\tag{3.8}
$$

The we define the coefficients

$$
\delta_\psi = Z_\phi - 1,\ \delta_m = Z_m - 1,\ \delta_Z = Z_A - 1,\ \delta_e = Z_e - 1. \tag{3.9}
$$

The counter term also contribute to the perturbative expansion like the interactions. The counter terms for the fermion propagator come from the diagram expression:

$$
iD_F^{(\text{ct})} =
$$

$$
\sim \frac{\delta^2}{\delta\bar{\eta}\delta\eta} i(\delta_\psi \gamma^\mu_{\alpha\beta} k_\mu - \delta_m \mathbb{I}_{\alpha\beta}) \frac{\delta^2}{\delta\eta_\alpha \delta\bar{\eta}_\beta} \frac{1}{2!} \left(-i\bar{\eta}_\alpha D_F^{\alpha\beta} \eta_\beta\right)^2. \tag{3.10}
$$

The contribution to the electron self energy is

$$
i\Sigma(p) = \delta_\psi \not{p} - \delta_m m_R. \tag{3.11}
$$

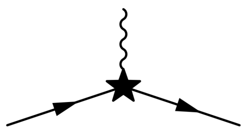
Similarly, the counter term contribution to the photon self energy is

$$
i\Sigma^{\mu\nu}(k) = \mu \quad \nu
$$

$$
= \delta_A[-p^2 g^{\mu\nu} + (1-\xi)p^\mu p^\nu]. \tag{3.12}
$$

And the counter term contribution to the QED vertex is

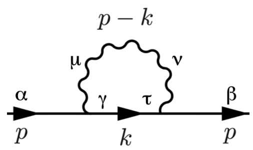
$$
i\Gamma^\mu_{\alpha\beta} =
$$



$$
= \delta_e \gamma^\mu_{\alpha\beta}. \tag{3.13}
$$

3.1.1 One-loop Correction to Electron Propagator

Consider the one-loop correction to the particle propagator:



$$
\simeq (-ie)^2 A_\mu \bar{\psi}_\alpha \gamma^\mu_{\alpha\gamma} \psi_\gamma A_\nu \bar{\psi}_\tau \gamma^\nu_{\tau\beta} \psi_\beta \equiv i\bar{\psi}_\alpha \Sigma^{\alpha\beta}(p)\psi_\beta. \tag{3.14}
$$

The self energy is

$$
\begin{aligned}
i\Sigma_{\alpha\beta}(p) &= e^2 \int \frac{d^4k}{(2\pi)^4} \Pi_{\mu\nu}(p-k)\left[\gamma^\mu D_F(k)\gamma^\nu\right]_{\alpha\beta} \\
&= -e^2 \int \frac{d^dk}{(2\pi)^d} \frac{\gamma^\mu(\not{k}+m)\gamma_\mu}{(p-k)^2(k^2-m^2)}.
\end{aligned}
\tag{3.15}
$$

The seconded equality comes from the contraction:

$$
(-ie)^2 A_\mu \bar{\psi}_\alpha \gamma^\mu_{\alpha\gamma} \psi_\gamma A_\nu \bar{\psi}_\tau \gamma^\nu_{\tau\beta} \psi_\beta \tag{3.16}
$$

The nominator can be simplified using the Dirac matrix identities:

$$\gamma^\mu\gamma_\mu = 4,\ \gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu \quad \Rightarrow \quad \gamma^\mu(\not{k}+m)\gamma_\mu = 4m - 2\not{k}. \tag{3.17}$$

The denominator can be simplify using the Feynman parameter:

$$\frac{1}{(p-k)^2(k^2-m^2)} = \int_0^1 \frac{dx}{[(k-b)^2-D]^2}$$

where b and D can be calculated by

```
A1=(k-p)^2;
A2=k^2-m^2;
{c,b,a}=CoefficientList[x*A1+(1-x)*A2,{k}];
-b/2
-c+b^2/(4*a)//Simplify
```

The result is $b = px$ and $D = (1-x)(m^2 - p^2 x)$. Shift $k \to k + px$, the self energy becomes (including a $\tilde{\mu}$ mass scale):

$$\begin{aligned} i\Sigma(p) &= 2e^2\tilde{\mu}^\epsilon \int_0^1 (x\not{p} - 2m)dx \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2-D)^2} \\ &= i\frac{2e^2\tilde{\mu}^\epsilon\Omega_d}{(2\pi)^d} \int_0^1 (x\not{p} - 2m)dx \int \frac{k^{d-1}dk}{(k^2+D)^2}. \end{aligned} \tag{3.18}$$

The regularization procedure is

```
omg=(2*Pi^(d/2))/(Gamma[d/2]);
cof=2*e^2*\[Mu]^(4-d)*omg/(2*Pi)^d;
int=cof*Integrate[q^(d-1)/(q^2+D)^2,{q,0,Infinity}][[1]];
map={e->Sqrt[4*Pi*\[Alpha]],EulerGamma->Subscript[\[Gamma],E]};
ans=Series[int/.{d->4-\[Epsilon]},{\[Epsilon],0,0}];
ans/.map//Simplify
```

The result is ($\mu^2 = 4\pi\tilde{\mu}^2 e^{-\gamma_E}$)

$$\Sigma(p) = \frac{e_R^2}{8\pi^2}\int_0^1 dx(x\not{p} - 2m_R)\left[\frac{2}{\epsilon} + \ln\left(\frac{\mu^2}{D}\right)\right]. \tag{3.19}$$

The infinity comes from

$$\frac{e_R^2}{4\pi^2\epsilon}\int_0^1 dx(x\not{p} - 2m_R) = \frac{e_R^2}{8\pi^2\epsilon}\not{p} - \frac{e_R^2}{2\pi^2\epsilon}m_R. \tag{3.20}$$

Using the $\overline{\text{MS}}$ subtraction scheme, we choose

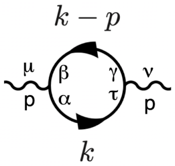
$$\delta_\psi = -\frac{e_R^2}{8\pi^2\epsilon},\ \delta_m = -\frac{e_R^2}{2\pi^2\epsilon}, \tag{3.21}$$

and the self energy is

$$\begin{aligned} \Sigma(p) &= \frac{e_R^2}{8\pi^2}\int_0^1 dx(x\not{p} - 2m_R)\ln\left[\frac{\mu^2}{(1-x)(m_R^2 - p^2x)}\right] \\ &= \frac{e_R^2}{8\pi^2}(\not{p} - 4m_R)\ln\left(\frac{\mu}{m_R}\right) - \int_0^1 dx \ln\left[(1-x)\left(1 - \frac{p^2x}{m_R^2}\right)\right]. \end{aligned} \tag{3.22}$$

3.1.2 One-loop Correction to Photon Propagator

Consider the one-loop correction to the photon propagator:



$$\simeq (-ie)^2 A_\mu \bar{\psi}_\alpha \gamma^\mu_{\alpha\beta} \psi_\beta A_\nu \bar{\psi}_\gamma \gamma^\nu_{\gamma\tau} \psi_\tau \equiv i A_\mu \Pi^{\mu\nu}(p) A_\nu. \tag{3.23}$$

The self energy is:

$$\begin{aligned} i\Sigma^{\mu\nu}(p) &= -e^2 \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left[\gamma^\mu D_F(k-p)\gamma^\nu D_F(k)\right] \\ &= -e^2 \int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{Tr}\left[\gamma^\mu (\not{k} - \not{p} + m)\gamma^\nu (\not{k} + m)\right]}{(k^2-m^2)[(p-k)^2-m^2]}. \end{aligned} \tag{3.24}$$

The Dirac trace and Feynman parameter is calculated by the following code:

```
(*Dirac trace using FeynCalc`*)
DiracTrace[GA[\[Mu]].(GS[k-p]+m).GA[\[Nu]].(GS[k]+m)]//DiracSimplify

(*Feynman paramater*)
A1=k^2-m^2;
A2=(k-p)^2-m^2;
{c,b,a}=CoefficientList[x*A1+(1-x)*A2,{k}];
-b/2
-c+b^2/(4*a)//Simplify
```

The nominator is

$$4\left[g^{\mu\nu}\left(k\cdot p - k^2 + m^2\right) + 2k^\mu k^\nu - k^\mu p^\nu - p^\mu k^\nu\right]. \tag{3.25}$$

The denominator is:

$$\frac{1}{(k^2-m^2)[(p-k)^2-m^2]} = \frac{1}{\{[k-p(1-x)]^2 - [m^2 + p^2 x(x-1)]\}^2}. \tag{3.26}$$

Let $D = m^2 - p^2 x(1-x)$, shift $k \to k + p(1-x)$, and drop all p^μ linear term,[1] the result is

$$i\Sigma^{\mu\nu} = -4e^2\tilde{\mu}^\epsilon \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{2k^\mu k^\nu - g^{\mu\nu}\left[k^2 - x(1-x)p^2 - m^2\right]}{\left[k^2 - D\right]^2}. \tag{3.27}$$

The self-energy $i\Sigma^\mu \propto g^{\mu\nu}$, we can make the substitution

$$k^\mu k^\nu \to \frac{1}{d} k^2 g^{\mu\nu}.$$

We then need to consider the integral

$$\begin{aligned} iI(x) &= 4e^2\tilde{\mu}^\epsilon \int \frac{d^dk}{(2\pi)^d} \frac{(1-\frac{2}{d})k^2 - x(1-x)p^2 - m^2}{\left[k^2 - D\right]^2}, \\ I(x) &= -\frac{4e^2\tilde{\mu}^\epsilon \Omega_d}{(2\pi)^d} \int k^{d-1} dk \frac{(1-\frac{2}{d})k^2 + x(1-x)p^2 + m^2}{\left[k^2 + D\right]^2}. \end{aligned}$$

The regulation is carried out by the following code:

[1] The Ward identity requires that the p^μ term in the propagator do not contribute to any scattering process.

```
omg=(2*Pi^(d/2))/(Gamma[d/2]);
cof=-4*e^2*\[Mu]^(4-d)*omg/(2*Pi)^d;
den=q^(d-1)*((1-2/d)q^2+x*(1-x)p^2+m^2);
int=cof*Integrate[den/(q^2+D)^2,{q,0,Infinity}][[1]];
map={EulerGamma->Subscript[\[Gamma],E],D->m^2-p^2*x*(1-x)};
ans=Series[int/.{d->4-\[Epsilon]},{\[Epsilon], 0, 0}];
ans/.map//Simplify
```

The result is

$$\Sigma^{\mu\nu}(p) = -\frac{e_R^2 p^2 g^{\mu\nu}}{2\pi^2}\int_0^1 dx\ x(1-x)\left[\frac{2}{\epsilon} + \ln\left(\frac{\mu^2}{m_R^2 - p^2 x(1-x)}\right)\right] \tag{3.28}$$

The divergent part is

$$-\frac{e_R^2 p^2 g^{\mu\nu}}{\pi^2\epsilon}\int_0^1 dx\ x(1-x) = -\frac{e_R^2 p^2 g^{\mu\nu}}{6\pi^2\epsilon}.$$

The counter term coefficient is

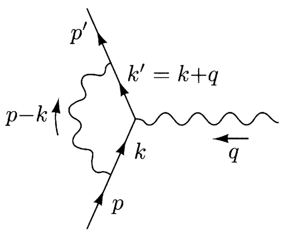
$$\delta_A = -\frac{e_R^2}{6\pi^2\epsilon}. \tag{3.29}$$

The photon self-energy is then

$$\begin{aligned}\Sigma^{\mu\nu}(p) &= -\frac{e_R^2 p^2 g^{\mu\nu}}{2\pi^2}\int_0^1 dx\ x(1-x)\ln\left[\frac{\mu^2}{m_R^2 - p^2 x(1-x)}\right] \\ &= -\frac{e_R^2 p^2 g^{\mu\nu}}{12\pi^2}\ln\left(\frac{\mu}{m}\right) + \frac{e_R^2 p^2 g^{\mu\nu}}{2\pi^2}\int_0^1 dx\ x(1-x)\ln\left[1 - \frac{p^2}{m_R^2}x(1-x)\right].\end{aligned} \tag{3.30}$$

3.1.3 One-loop Correction to Vertex

Consider the one-loop correction to interaction:



$$\begin{aligned}&\simeq (-ie)^3 A_\nu \bar{\psi}_\alpha \gamma^\nu_{\alpha\beta}\psi_\beta A_\mu \bar{\psi}_\lambda \gamma^\mu_{\lambda\tau}\psi_\tau A_\xi \bar{\psi}_\rho \gamma^\nu_{\rho\sigma}\psi_\sigma \\ &\equiv -ieA_\mu \Gamma^\mu_{\alpha\beta}(q,p,p')\bar{\psi}_\alpha\psi_\beta.\end{aligned} \tag{3.31}$$

The vertex function is:

$$\begin{aligned}i\Gamma^\mu_{\alpha\beta}(q,p,p') &= -e^2\int\frac{d^4k}{(2\pi)^4}\Pi_{\nu\lambda}(p-k)[\gamma^\nu D_F(k')\gamma^\mu D_F(k)\gamma^\lambda]_{\alpha\beta} \\ &= e^2\int\frac{d^4k}{(2\pi)^4}\frac{[\gamma^\nu(\not{k}' + m)\gamma^\mu(\not{k}+m)\gamma_\nu]_{\alpha\beta}}{(k^2-m^2)(k'^2-m^2)(p-k)^2}\end{aligned} \tag{3.32}$$

Using the following code

```
(*numerator*)
den=Contract[GA[\[Nu]].(GS[kp]+m).GA[\[Mu]].(GS[k]+m).GA[\[Nu]]];
DiracSimplify[den]

(*Feynman parameter*)
A1=k^2-m^2;
A2=(k+q)^2-m^2;
A3=(p-k)^2;
{c,b,a}=CoefficientList[x*A1+y*A2+z*A3,{k}];
-b/2//Simplify
-c+b^2/4//Simplify
```

The numerator is

$$-2\not{k}\gamma^\mu \not{k}' - 2m^2\gamma^\mu + 4m(k+k')^\mu.$$

The denominator is

$$\int \frac{dF_3}{[(k+yq-zp)^2 - D]^3},$$

where

$$\begin{aligned} D &= (x+y)m^2 - z(1-z)p^2 - y(1-y)q^2 - 2yzpq \\ &= (x+y)m^2 - xyq^2 - yzp'^2 - xzp^2. \end{aligned}$$

Shift $k^\mu \to k^\mu + zq_1^\mu - yp^\mu$, throw away all terms with linear k^μ, and replace $k^\mu k^\nu$ with $\frac{1}{d}k^2 g^{\mu\nu}$, the result is

$$\frac{4}{d}k^2\gamma^\mu - 2(-y\not{q} + z\not{p})\gamma^\mu[(1-y)\not{q} + z\not{p}] + 4m^2\gamma^\mu - 2m\left[(1-2y)q^\mu + 2zp^\mu\right].$$

Note that only the quadratic term is divergent.

$$\Gamma^\mu(p, q_1, q_2) = -i\frac{4e^2\tilde{\mu}^\epsilon\gamma^\mu}{d}\int dF_3 \int \frac{d^dk}{(2\pi)^d}\frac{k^2}{(k^2-D)^3} + \delta\Gamma^\mu(p, q_1, q_2).$$

where $\delta\Gamma^\mu$ stores all the finite part

$$\begin{aligned} &\delta\Gamma^\mu(p, q_1, q_2) \\ &= \int \frac{e^2k^3dkdF_3}{(2\pi)^2(k^2+D)^3}\left\{(-y\not{q} + z\not{p})\gamma^\mu[(1-y)\not{q} + z\not{p}] - 2m^2\gamma^\mu + m\left[(1-2y)q^\mu + 2zp^\mu\right]\right\}. \end{aligned}$$

The divergent part is

$$\frac{4e^2\tilde{\mu}^\epsilon\Omega_d\gamma^\mu}{d(2\pi)^d}\int dF_3 \int \frac{k^{d+1}dk}{(k^2+D)^3} = \frac{e_R^2}{16\pi^2}\gamma^\mu \int dF_3\left[\frac{2}{\epsilon} + \ln\left(\frac{\mu^2}{D}\right)\right].$$

Using the $\overline{\text{MS}}$ scheme, the coefficient of the counter term is

$$\delta_e = -\frac{e_R^2}{8\pi^2\epsilon}. \tag{3.33}$$

3.1.4 Renormalization Group

In summery, the renormalization factors are

$$\begin{aligned} Z_\psi &= 1 - \frac{e_R^2}{8\pi^2\epsilon} + O(e_R^3), \\ Z_A &= 1 - \frac{e_R^2}{6\pi^2\epsilon} + O(e_R^3), \\ Z_m &= 1 - \frac{e_R^2}{2\pi^2\epsilon} + O(e_R^3), \\ Z_e &= 1 - \frac{e_R^2}{8\pi^2\epsilon} + O(e_R^3), \end{aligned} \tag{3.34}$$

which means

$$\begin{aligned} \frac{d\ln Z_\phi}{de_R} &= -\frac{e_R}{4\pi^2\epsilon} + O(e_R^2), \\ \frac{d\ln Z_A}{de_R} &= -\frac{e_R}{3\pi^2\epsilon} + O(e_R^2), \\ \frac{d\ln Z_m}{de_R} &= -\frac{e_R}{\pi^2\epsilon} + O(e_R^2), \\ \frac{d\ln Z_e}{de_R} &= -\frac{e_R}{4\pi^2\epsilon} + O(e_R^2). \end{aligned} \tag{3.35}$$

The bare parameters are

$$\begin{aligned} \psi_0 &= Z_\psi^{1/2}\psi_R, \\ A_0 &= Z_A^{1/2}A_R, \\ m_0 &= Z_m Z_\psi^{-1} m_R, \\ e_0 &= Z_e Z_\psi^{-1} Z_A^{-1/2} e_R \tilde{\mu}^{\epsilon/2}. \end{aligned} \tag{3.36}$$

The RG equation for e_0 is

$$\frac{d\ln e_0}{d\ln\mu} = \left(\frac{\ln Z_e}{de_R} - \frac{\ln Z_\psi}{de_R} - \frac{1}{2}\frac{\ln Z_A}{de_R} + \frac{1}{e_R}\right)\frac{de_R}{d\ln\mu} + \frac{\epsilon}{2} = 0. \tag{3.37}$$

The beta function is

$$\beta(e_R) = \frac{de_R}{d\ln\mu} = -\frac{\epsilon}{2}e_R + \frac{e_R^3}{12\pi^2} + O(e_R^4). \tag{3.38}$$

The RG equation for m_0 is

$$\frac{d\ln m_0}{d\ln\mu} = \left(\frac{\ln Z_m}{de_R} - \frac{\ln Z_\psi}{de_R}\right)\frac{de_R}{d\ln\mu} + + \frac{1}{m_R}\frac{dm_R}{d\ln\mu} = 0. \tag{3.39}$$

The anomalous dimension of mass is

$$\gamma_m = \frac{d\ln m_R}{d\ln\mu} = -\frac{3e_R^2}{8\pi^2} + O(e_R^3). \tag{3.40}$$

Chapter 4

Non-relativistic Quantum Field Theory

A general non-relativistic field theory is described by the action (with repeated indices automatically summed):

$$\begin{aligned} S = S_0 + S_{\text{int}} &= \int dt \int d^d x \mathcal{L}_0 + \int dt\ \mathcal{V}_{\text{int}}, \\ \mathcal{L}_0 &= \bar{\psi}_a(x)(i\delta_{ab}\partial_t - \hat{H}_{ab})\psi_b(x). \end{aligned} \tag{4.1}$$

where the field operator $\psi(x)$ can be bosonic or fermionic, which is denoted by a number $\zeta = \pm 1$, and $\mathcal{V}_{\text{int}}$ is the interaction Lagrangian. A general interaction has the form

$$\mathcal{V}_{\text{int}} = \sum_{abcd} \int \prod_{i=1}^{4} d^d x_i\ \bar{\psi}_c(x_3)\bar{\psi}_d(x_4)V_{abcd}(x_1, x_2, x_3, x_4)\psi_b(x_2)\psi_a(x_1). \tag{4.2}$$

Note that the classical equation of motion for the free field satisfies the Schrödinger equation:

$$\partial_\mu \frac{\partial \mathcal{L}_0}{\partial(\partial_\mu \bar{\psi}_a(x))} - \frac{\partial \mathcal{L}_0}{\partial \bar{\psi}_a(x)} = -i\partial_t \psi_a(x) + \hat{H}_{ab}\psi_b(x) = 0. \tag{4.3}$$

We are mostly work with finite system size L^d with UV cutoff $\Lambda = 2\pi/a$ (where a is the lattice spacing, and $L = Na$). The spatial Fourier transformation is

$$\tilde{\psi}_a(k) = \int_{L^d} d^d x e^{-ik\cdot x}\psi_a(x), \quad \psi_a(x) = \frac{1}{L^d}\sum_k e^{ik\cdot x}\tilde{\psi}_a(k). \tag{4.4}$$

For the finite size, the momentum is discretized: $k = 2\pi n_k/L$, $n_k \in \mathbb{Z}$. The summation in the thermodynamic limit becomes the integral:

$$\frac{1}{L^d}\sum_k \longrightarrow \int_{|k|<\Lambda} \frac{d^d k}{(2\pi)^d}. \tag{4.5}$$

In the momentum space, the free theory can be simplified:

$$S_0 = \int dt \int \frac{d^d k}{(2\pi)^d} \tilde{\bar{\psi}}_a(k)[i\partial_t - \varepsilon_a(k)]\tilde{\psi}_a(k). \tag{4.6}$$

The interaction in the momentum space is described by the vertex function:

$$\tilde{V}_{abcd}(k_1, k_2, k_3, k_4) = \int \prod_{i=1}^{4} d^d x_i e^{i(k_1x_1+k_2x_2-k_3x_3-k_4x_4)} V_{abcd}(x_1, x_2, x_3, x_4). \tag{4.7}$$

Because of the momentum conservation, the vertex will contain a delta function factor.

Consider the Coulomb repulsive potential e^2/r, in the field theory formalism, the coefficient $V_{abcd}(x_1, x_2, x_3, x_4)$ is

$$\frac{V(x_1 - x_2)}{2!2!} \left[\delta_{ac}\delta_{bd}\delta^{(3)}(x_1 - x_3)\delta^{(3)}(x_2 - x_4) - \delta_{ad}\delta_{bc}\delta^{(3)}(x_1 - x_4)\delta^{(3)}(x_2 - x_3)\right], \tag{4.8}$$

where the factor $\frac{1}{2!2!}$ is the symmetry from interchanging the fermion fields. In momentum space:

$$V_{abab}(k_1, k_2, k_3 + q, k_4 - q) = \frac{1}{2!2!} V_{\text{Coul}}(q), \tag{4.9}$$

where the Coulomb potential in the momentum space is

$$\begin{aligned} V_{\text{Coul}}(q) &= \lim_{\alpha\to 0} e^2 \int_0^\infty dr\ 2\pi r^2 \int_{-1}^{+1} d(\cos\theta) \frac{e^{-iqr\cos\theta-\alpha r}}{r} \\ &= \lim_{\alpha\to 0} \frac{2\pi e^2}{iq} \int_0^\infty dr \left(e^{iqr-\alpha r} - e^{-iqr-\alpha r}\right) \\ &= \lim_{\alpha\to 0} \frac{4\pi e^2}{q^2+\alpha^2} = \frac{4\pi e^2}{q^2}. \end{aligned} \tag{4.10}$$

4.1 Finite Temperature Field Theory

The original real-time partition function is defined as[1]

$$Z[J] = \int D[\bar{\psi}, \psi] \exp\left\{i \int dt \int d^d x \left[\mathcal{L} + \bar{J}_a(x)\psi_a(x) + \bar{\psi}_a(x)J_a(x)\right]\right\}. \tag{4.11}$$

For finite-temperature field theory, after making the wick rotation ($t \to -i\tau$), the partition function for a generic non-relativistic lattice theory is:

$$Z[J] = \int D[\bar{\psi}, \psi] e^{-S[\bar{\psi},\psi]+\bar{J}\cdot\psi+\bar{\psi}\cdot J}, \tag{4.12}$$

where the action is

$$S = \int_0^\beta d\tau \left[\int d^d x\ \bar{\psi}_a(x)(\delta_{ab}\partial_\tau + \hat{H}_{ab})\psi_b(x) + \mathcal{V}_{\text{int}}\right]. \tag{4.13}$$

The Fourier transformation on the imaginary time domain is defined as:

$$\tilde{\psi}(\omega_n) = \int_0^\beta d\tau e^{i\omega_n\tau}\psi(\tau), \quad \psi(\tau) = \frac{1}{\beta}\sum_{\omega_n} e^{-i\omega_n\tau}\tilde{\psi}(\omega_n). \tag{4.14}$$

Under such convention, in the thermodynamic limit and zero-temperature limit, the spatial-temporal Fourier transformation agrees with the relativistic case (up to a Wick rotation).

[1] As with the relativistic case, we introduce an auxiliary source J, which is bosonic/fermionic if the field ψ is bosonic/fermionic.

4.1.1 Free Field Theory

We first consider the action of free field

$$S_0 = \int_0^\beta d\tau \int d^d x\ \bar{\psi}_a(x)(\delta_{ab}\partial_\tau + \hat{H}_{ab})\psi_b(x). \tag{4.15}$$

The Fourier transformation

$$S_0 = \frac{1}{\beta}\sum_{\omega_n}\int_\Lambda \frac{d^d k}{(2\pi)^d}\tilde{\bar{\psi}}_a(k,\omega_n)\left[-i\omega_n + \tilde{H}_{ab}(k)\right]\tilde{\psi}_b(k,\omega_n). \tag{4.16}$$

The partition function with source is

$$\frac{Z_0[J]}{Z_0[0]} = \exp\left[-\frac{1}{\beta}\sum_{\omega_n}\int_\Lambda \frac{d^d k}{(2\pi)^d}\tilde{\bar{J}}_a(k,\omega_n)\tilde{G}_{ab}(k,\omega_n)\tilde{J}_b(k,\omega_n)\right], \tag{4.17}$$

where the Green's function is

$$\tilde{G}_{ab}(k,\omega_n) = \left[\frac{1}{i\omega_n - \tilde{H}(k)}\right]_{ab}. \tag{4.18}$$

Remark 3. Obtaining the Partition Function

Unlike the relativistic case, the value of the value of partition function without source $Z_0[0]$ is related to the free energy. We can express it formally as

$$Z_0[0] = \left[\det(-G_{ab})^{-1}\right]^{-\zeta}.$$

To get the correct dimensionality, we set the determinant as

$$Z_0[0] \equiv \prod_{k,\omega_n}\left\{\beta\det\left[-i\omega_n + \tilde{H}(k)\right]\right\}^{-\zeta}.$$

Thus the free energy is

$$F = -\frac{1}{\beta}\ln Z_0 = \zeta\sum_{k,\omega_n}\ln\left\{\beta\det\left[-i\omega_n + \tilde{H}(k)\right]\right\}. \tag{4.19}$$

4.1.2 Matsubara Summation

Now consider the summation on Matsubara frequency:

$$\sum_{\omega_n} f(\omega_n) = \begin{cases}\sum_n f(\frac{2n\pi}{\beta}) & \text{bosonic}\\ \sum_n f(\frac{(2n+1)\pi}{\beta}) & \text{fermionic}\end{cases}. \tag{4.20}$$

The frequency is capture by the singularities of the density function of the states:

$$\rho(z) = \begin{cases}\frac{1}{\exp(\beta z)-1} & \text{bosonic}\\ \frac{1}{\exp(\beta z)+1} & \text{fermionic}\end{cases}. \tag{4.21}$$

The residue on imaginary frequency $i\omega_n$ is alway $\frac{1}{\beta}$. In this way, the summation is:

$$\frac{1}{\beta}\sum_{\omega_n} f(i\omega_n) = \frac{1}{2\pi i}\oint \rho(z)f(z) = -\sum_k \text{Res } \rho(z)f(z)|_{z=z_k}. \tag{4.22}$$

Summation of Green's function

Consider the frequency summation for the correlation function:

$$\frac{1}{\beta}\sum_{\omega_n} \tilde{G}_0(k) = \frac{1}{\beta}\sum_{\omega_n} \frac{1}{i\omega_n - E_p} = -\text{Res } \frac{\rho(z)}{z - E_p}\bigg|_{z=E_p} = \rho(E_p). \tag{4.23}$$

Summation of Green's function

Consider the frequency summation for the correlation function:

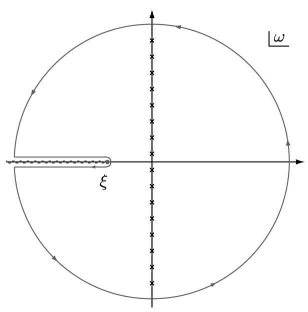
$$\sum_{\omega_n} \langle \bar{\psi}_{\vec{p},\omega_n} \psi_{\vec{p},\omega_n} \rangle = \frac{1}{\beta}\sum_{\omega_n} \frac{1}{-i\omega_n + \epsilon_{\vec{p}}} = \text{Res } \frac{\rho(z)}{z - \epsilon_{\vec{p}}}\bigg|_{z-\epsilon_{\vec{p}}} = \rho(\epsilon_{\vec{p}}). \tag{4.24}$$

Free Energy Summation

Consider the free energy

$$F = -\frac{1}{\beta}\ln Z = -\frac{1}{\beta}\sum_{\omega_n} \ln[\beta(-i\omega_n + E_{\vec{p}})] = \frac{1}{2\pi i}\oint dz \rho(z)\ln[\beta(\xi - z)]. \tag{4.25}$$

To calculate the summation, we consider the line integral along the loop:



The free energy is

$$\begin{aligned} F &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} dx \rho(x)\ln\left(\frac{\xi - x - i\epsilon}{\xi - x + i\epsilon}\right) \\ &= \frac{-\zeta}{2\pi i \beta}\int_{-\infty}^{\infty} dx \ln(1 - \zeta e^{-\beta z})\left(\frac{1}{x + i\epsilon - \xi} - \frac{1}{x - i\epsilon - \xi}\right), \end{aligned} \tag{4.26}$$

where we integrate the expression by part, noticing that

$$\frac{d}{dz}\frac{\zeta}{\beta}\ln(1 - \zeta e^{-\beta z}) = \frac{1}{e^{\beta z} - \zeta} = \rho(z) \tag{4.27}$$

Using the identity

$$\lim_{\epsilon\to 0^+}\frac{1}{x+i\epsilon} = -i\pi\delta(x) + \mathcal{P}\frac{1}{x},$$

the above expression can be simplified to

$$F = \frac{\zeta}{\beta}\ln(1-\zeta e^{-\beta\zeta}). \tag{4.28}$$

4.2 Fermi Liquid Theory

In this section, we are considering the system of weakly interacting Fermi gas. To be specific, we consider the lattice Hamiltonian:

$$H = -\frac{1}{2}\sum_{\langle i,j\rangle}(c_i^\dagger c_j + c_j^\dagger c_i) + \mu\sum_i c_i^\dagger c_i + \sum_{i,j,k,l} u_{ijkl}c_i^\dagger c_j^\dagger c_k c_l. \tag{4.29}$$

In the following, we investigate the effective field theory near the Fermi surface. We discuss the RG flow of the couplings (mainly for two dimensional system). Then we carry out the perturbative calculation for the correlation functions.

4.2.1 Effective Field Theory for Interacting Fermi Systems

The low-energy manifold is an annulus of thickness 2Λ symmetrically situated with respect to the Fermi circle $K = K_{\mathrm{F}}$. The dispersion for the free lattice model is

$$E(\boldsymbol{K}) = -\cos K_x - \cos K_y \simeq -2 + \frac{\boldsymbol{K}^2}{2}. \tag{4.30}$$

For a given chemical potential μ, the Fermi circle is $K_F = \sqrt{2m\mu}$, we can linearize the dispersion near the Fermi surface:

$$E(\boldsymbol{K}) = \frac{\boldsymbol{K}^2 - K_F^2}{2m} \simeq \frac{K_F}{m}k \equiv v_F k, \quad k \equiv |K| - K_F \tag{4.31}$$

The partition function is:

$$Z_0 = \sum_\theta \sum_{|k|<\Lambda}\int D\left[\bar\psi(k,\theta,\omega),\psi(k,\theta,\omega)\right]e^{-S_0}, \tag{4.32}$$

where the free field action is:[2]

$$S_0 = \int\frac{d\theta}{2\pi}\int_{-\Lambda}^{\Lambda}\frac{dk}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\bar\psi(k,\theta,\omega)(-i\omega + v_F k)\psi(k,\theta,\omega). \tag{4.33}$$

Consider the quartic interaction

$$\delta S_4 = \frac{1}{4}\int_{\boldsymbol{K},\theta,\omega}\bar\psi(4)\bar\psi(3)\psi(2)\psi(1)u(4,3,2,1) \tag{4.34}$$

[2] A factor of K_{F} has been absorbed in the field.

where we eliminate one of the four sets of variables, say, the one numbered 4, by integrating them against the delta functions:

$$\int_{K,\theta,\omega} = \prod_{i=1}^{3} \int_0^{2\pi} \frac{d\theta_i}{2\pi} \int_{-\Lambda}^{\Lambda} \frac{dk_i}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega_i}{2\pi} \theta\left(\Lambda - |k_4|\right), \quad k_4 = |\boldsymbol{K}_4| - K_F. \tag{4.35}$$

The ω integral is easy: since all ω's are allowed, the condition $\omega_4 = \omega_1 + \omega_2 - \omega_3$ is always satisfied for any choice of the first three frequencies. The same would be true for the momenta if all momenta were allowed. But they are not; they are required to lie within the annulus of thickness 2Λ around the Fermi circle. Consequently, if one freely chooses the first three momenta from the annulus, the fourth could have a length as large as $3K_F$. The role of $\delta(\Lambda - |k_4|)$ is to prevent exactly this.

Momentum Constraint

Note that k_4 can be expressed as

$$k_4 = |(K_F + k_1)\boldsymbol{\Omega}_1 + (K_f + k_2)\boldsymbol{\Omega}_2 - (K_F + k_3)\boldsymbol{\Omega}_3| - K_F. \tag{4.36}$$

When doing RG towards the Fermi surface, the integral measure will not preserve the preserve the original form. The situation is clearly is we use a smooth cutoff

$$\theta(\Lambda - |k_4|) \to e^{-|k_4|/\Lambda}, \tag{4.37}$$

and define $\Delta \equiv \boldsymbol{\Omega}_1 + \boldsymbol{\Omega}_2 - \boldsymbol{\Omega}_3$, k_4 in this way behaves as

$$k_4 = (|\Delta| - 1)K_F + O(k). \tag{4.38}$$

The integral then change to:

$$\begin{aligned} &\prod_{i=1}^{3} \int_{-\Lambda}^{\Lambda} \frac{dk_i}{2\pi} \int \frac{d\theta_i}{2\pi} \int \frac{d\omega_i}{2\pi} e^{-||\Delta|-1|\frac{K_F}{\Lambda}} u(k,\theta,\omega)\bar{\psi}\bar{\psi}\psi\psi \\ \xrightarrow{\text{RG}} &\prod_{1}^{3} \int_{-\Lambda}^{\Lambda} \frac{dk_i'}{2\pi} \int \frac{d\theta_i}{2\pi} \int \frac{d\omega_i'}{2\pi} e^{-||\Delta|-1|\frac{sK_F}{\Lambda}} u\left(\frac{k'}{s}\frac{\omega'}{s}\theta\right)\bar{\psi}\bar{\psi}\psi\psi. \end{aligned} \tag{4.39}$$

We can then get the RG transformation of u as

$$u'(k',\theta,\omega') = e^{-||\Delta|-1|\frac{(s-1)K_F}{\Lambda}} u\left(\frac{k'}{s},\theta,\frac{\omega'}{s}\right). \tag{4.40}$$

By Taylor expansion, we conclude that the only couplings that survive the RG transformation without any decay correspond to the cases in which $|\Delta| = 1$, and without momentum dependence.

This equation has only three solutions (see also Fig. 4.1):

$$\begin{aligned} &\text{Case I:} \quad \boldsymbol{\Omega}_1 = \boldsymbol{\Omega}_3, \\ &\text{Case II:} \quad \boldsymbol{\Omega}_2 = \boldsymbol{\Omega}_3, \\ &\text{Case III:} \quad \boldsymbol{\Omega}_1 = -\boldsymbol{\Omega}_2. \end{aligned} \tag{4.41}$$

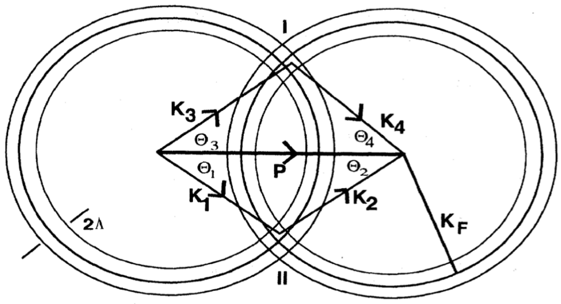


Figure 4.1: The geometric construction for determining the allowed values of momenta. If K_1 and K_2 add up to P, then K_3 and K_4 are constrained as shown, if they are to add up to P and lie within the cutoff. If the incoming momenta K_1 and K_2 are equal and opposite, the two shells coalesce and K_3 and K_4 are free to point in all directions, as long as they are equal and opposite.

Because of the rotational symmetry, the marginal vertex functions are determined solely by two functions:

$$
\begin{aligned}
u[\theta_1, \theta_2, \theta_1, \theta_2] &\equiv F(\theta_1, \theta_2) = F(\theta_1 - \theta_2), && (4.42)\\
u[\theta_1, \theta_2, \theta_2, \theta_1] &= -F(\theta_1 - \theta_2), && (4.43)\\
u[\theta_1, -\theta_1, \theta_3, -\theta_3] &\equiv V(\theta_1, \theta_3) = V(\theta_1 - \theta_3). && (4.44)
\end{aligned}
$$

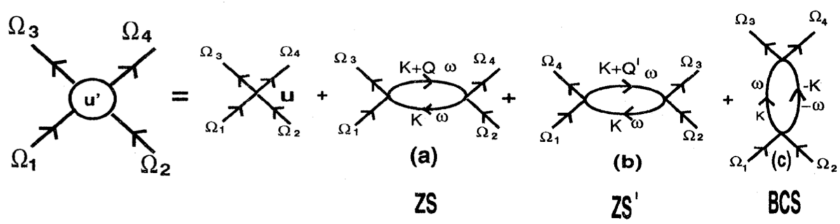
Note that the manifestation of the Pauli principle on F and V is somewhat subtle: F will not be antisymmetric under $1 \leftrightarrow 2$ since, according to the way it is defined above, we cannot exchange 1 and 2 without exchanging 3 and 4 at the same time. On the other hand, since 3 and 4 can be exchanged without touching 1 and 2 in the definition of V, V must go to $-V$ when $1 \leftrightarrow 3$.

4.2.2 One-loop RG for 2D System

We first consider the loop correction to the chemical potential:

$$
\begin{aligned}
\mu^{(2)}(k, \theta, \omega) &= \int_{d\Lambda} \frac{dK'}{2\pi} \int \frac{d\omega'}{2\pi} \int \frac{d\theta'}{2\pi} \frac{F(\theta - \theta')}{i\omega - v_F k'} \\
&= \int_{-\Lambda}^{-\Lambda + \Lambda dt} \frac{dK'}{2\pi} \int \frac{d\theta'}{2\pi} F(\theta - \theta') \\
&= \frac{\Lambda}{2\pi} \left[\int \frac{d\phi}{2\pi} F(\phi) \right] dt.
\end{aligned} \tag{4.45}
$$

For the vertex correction, again we should consider three channels corresponding to the diagrams:



(4.46)

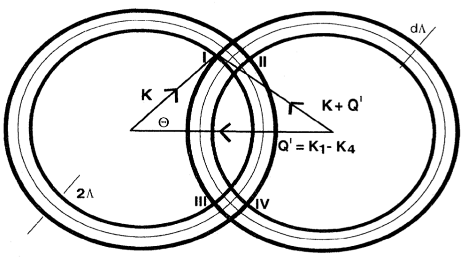


Figure 4.2: Construction for determining the allowed values of loop momenta in ZS'. The requirement that the loop momenta come from the shell and differ by Q' forces them to lie in one of the eight intersection regions of width $d\Lambda^2$.

First we consider the correction to the $F(\theta)$. The contribution from the ZS channel (the momentum transfer $Q \simeq 0$) is

$$F_{\mathrm{ZS}}^{(2)}(\theta_1-\theta_2) = \int_{d\Lambda}\frac{dk}{2\pi}\int\frac{d\omega}{2\pi}\int\frac{d\theta}{2\pi}\frac{F(\theta_1-\theta)F(\theta-\theta_2)}{(i\omega-v_F k)^2}. \tag{4.47}$$

Since two poles of the integrant lie at the same half plane, we can alway choose to close the loop integral along the other half, and thus getting zero contribution.

For the ZS' channels, the momentum conservation condition (see Fig. 4.2) restrict the phase space to be of order $d\Lambda^2$, and thus has no relevant contribution to $F(\theta)$. Finally, for the same kinematical reason, the BCS diagram does not renormalize $F(\theta)$ at one loop. Consider Fig. 4.1, with K_3 and K_4 replaced by the two momenta in the BCS loop, K and $P-K$. In each annulus we keep just two shells of thickness $d\Lambda$ at the cutoff corresponding to the modes to be eliminated. The requirement that K and $P-K$ lie in these shells and also add up to P forces them into intersection regions of order $d\Lambda^2$. This means the diagram is just as ineffective as the ZS' diagram in causing a flow. Thus any F is a fixed point to this order.

Now we consider the correction to the $V(\theta)$ function. We choose the external momenta equal and opposite and on the Fermi surface. The ZS and ZS' diagrams do not contribute to any marginal flow for the same reason that BCS and ZS' did not contribute to the flow of $F(\theta)$. But the BCS diagram produces a flow:

$$\begin{aligned} V_{\mathrm{BCS}}^{(2)}(\theta_1-\theta_3) &= -\frac{1}{2}\int_{d\Lambda}\frac{dk}{2\pi}\int\frac{d\omega}{2\pi}\int\frac{d\theta}{2\pi}\frac{V(\theta_1-\theta)V(\theta-\theta_3)}{(i\omega-v_F k)(-i\omega-v_F k)} \\ &= -\frac{dt}{4\pi v_F}\int\frac{d\theta}{2\pi}V(\theta_1-\theta)V(\theta-\theta_3). \end{aligned} \tag{4.48}$$

We can simplify the picture by going to angular momentum eigenfunctions,

$$V(\theta) = \sum_l e^{il\theta}V_l, \tag{4.49}$$

which gives the RG flow as

$$\frac{dV_l}{dt} = -\frac{V_l^2}{4\pi v_F}. \tag{4.50}$$

The solution to the RG flow is:

$$V_l(t) = \frac{V_l(0)}{1+\frac{V_l(0)}{4\pi v_F}t}. \tag{4.51}$$

What these equations tell us is that if the potential in angular momentum channel l is repulsive, it will get renormalized (logarithmically) down to zero, while if it is attractive, it will run off to large negative values signaling the BCS instability. This is the reason the V's are excluded in Landau theory, which assumes we have no phase transitions.[3]

[3] Remember that the sign of any given V_l is not necessarily equal to that of the microscopic interaction. Kohn and Luttinger have shown (PRL, 15, 524 (1965)) that some of them will be always negative. Thus, the BCS instability is inevitable, though possibly at absurdly low temperatures or absurdly high angular momentum l.

Chapter 5

One Dimensional Fermi System

In this section, we discuss the one-dimensional interacting Fermi system, described by the Hamiltonian

$$H = -\frac{1}{2}\sum_i (c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i) + \mu \sum_i c_i^\dagger c_i + \sum_{i,j,k,l} u_{ijkl} c_i^\dagger c_j^\dagger c_k c_l. \tag{5.1}$$

The dispersion for the free theory is $\varepsilon(k) = -\cos k$, near the Fermi surface with momentum k_F, the spectrum can be approximately linearized (as shown in Fig. 5.1), with the left and right moving fermion modes:

$$\varepsilon_{R/L}(k) = \begin{cases} v_F(k - k_F) & r = R \\ -v_F(k + k_F) & r = L \end{cases}, \quad v_F = \sin k_F. \tag{5.2}$$

The fermi momentum is (assume $N_L = N_R$)

$$k_F = \frac{\pi N}{2L}. \tag{5.3}$$

The state while all fermion modes filled below the Fermi surface and all modes empty above the Fermi surface is defined as the vacuum state $|0\rangle_0$.

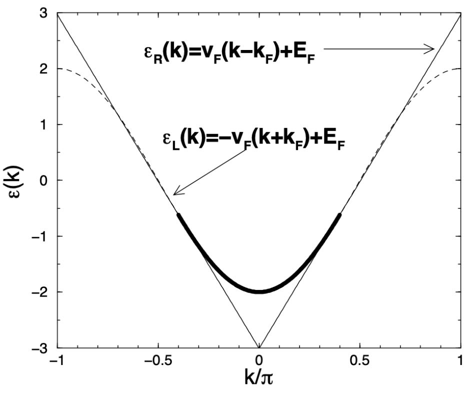


Figure 5.1: Linearized Model.

5.1 Field Theory for Luttinger Liquid

In this section, we gives a field theoretical analysis of the interacting fermion system in 1D. For the notational simplicity, we shift the momentum so that

$$k \to k' = \begin{cases} k - k_F & r = R \\ -k - k_F & r = L \end{cases}. \tag{5.4}$$

The dispersion is then $\varepsilon_r(k) = v_F k$. In this way, two Fermi points are brought to the origin, the left and right moving branches have the same dispersion. The integral over momentum shell for both species of fermion can then be denoted by

$$\int^{\Lambda} \frac{dK}{2\pi} \equiv \int_{-\Lambda}^{\Lambda} \frac{dk_L}{2\pi} + \int_{-\Lambda}^{\Lambda} \frac{dk_R}{2\pi}. \tag{5.5}$$

5.1.1 Effective Field Theory

The effective field theory for the free field is

$$Z_0 = \prod_{r=L/R} \int D\left[\bar{\psi}_r(k,\omega), \psi_r(k,\omega)\right] e^{-S_0}, \tag{5.6}$$

where the free field action is

$$S_0 = \sum_{r=L/R} \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \bar{\psi}_r(k,\omega)[-i\omega + v_F k]\psi_r(k,\omega), \tag{5.7}$$

which gives the free field propagator:

$$G_r(k,\omega) = -\langle \psi_r(k,\omega)\bar{\psi}_r(k,\omega)\rangle = \frac{1}{i\omega - v_F k}. \tag{5.8}$$

We then consider the rescaling of the cut-off $\Lambda \to \Lambda/s$. To make the free action scale invariant, we define the rescaled variables:

$$k' = sk, \quad \omega' = s\omega, \quad \psi'_r(k',\omega') = s^{-3/2}\psi_r(k,\omega). \tag{5.9}$$

Then we consider the perturbation from quadratic and quartic terms:

$$\begin{aligned} \delta S_2 &= \sum_{r=L/R} \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \mu(k,\omega)\bar{\psi}_r(k,\omega)\psi_r(k,\omega), \\ \delta S_4 &= \frac{1}{2!2!} \int_{K,\omega}^{\Lambda} u(4,3,2,1)\bar{\psi}(4)\bar{\psi}(3)\psi(2)\psi(1), \end{aligned} \tag{5.10}$$

where we have suppressed the momentum labels:

$$\psi(i) = \psi_{r_i}(k_i,\omega_i), \quad u(4,3,2,1) = u(K_4,\omega_4;K_3,\omega_3;K_2,\omega_2;K_1,\omega_1), \tag{5.11}$$

and the integral is defined as:[1]

$$\int_{K\omega}^{\Lambda} = \int^{\Lambda} \frac{dK_1 \cdots dK_4}{(2\pi)^4} \int_{-\infty}^{\infty} \frac{d\omega_1 \cdots d\omega_4}{(2\pi)^4} \times 2\pi\delta\left(\omega_1 + \omega_2 - \omega_3 - \omega_4\right) \\ \times 2\pi\bar{\delta}(K_1 + K_2 - K_3 - K_4). \tag{5.12}$$

Since this action separates into slow and fast pieces, the effect of mode elimination is simply to reduce Λ to Λ/s in the integral above. Rescaling moments and fields, we find that

$$\mu'(k', \omega') = s \cdot \mu\left(\frac{k'}{s}, \frac{\omega}{s}\right). \tag{5.13}$$

Expand μ in series:

$$\mu(k, \omega) = \mu_{00} + \mu_{10}k + \mu_{01}i\omega + \cdots + \mu_{nm}k^n(i\omega)^m + \cdots, \tag{5.14}$$

and compare both sides. The constant piece is a relevant perturbation. This relevant flow reflects the readjustment of the Fermi sea to a change in chemical potential. The correct way to deal with this term is to include it in the free-field action by filling the Fermi sea to a point that takes μ_{00} into account. The next two terms are marginal and modify terms that are already present in the action.

We now turn on the quartic interaction, the dimensional analysis gives the transformation of u:

$$u'_{i_4,i_3,i_2,i_1}(k'_i, \omega'_i) = u_{i_4,i_3,i_2,i_1}\left(\frac{k'_i}{s}, \frac{\omega'_i}{s}\right). \tag{5.15}$$

If we expand u in a Taylor series in its arguments and compare coefficients, we find that the constant term u0 is marginal and the higher coefficients are irrelevant. Thus, u depends only on its discrete labels and we can limit the problem to just a few coupling constants instead of the coupling function we started with. Furthermore, all reduce to just one coupling constant:

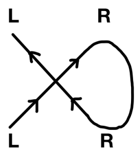
$$u_0 = u_{LRLR} = u_{RLRL} = -u_{RLLR} = -u_{LRRL} \equiv u. \tag{5.16}$$

Other couplings corresponding to the $(LL \to RR)$ process are wiped out by the Pauli principle since they have no momentum dependence and cannot have the desired anti-symmetry.

5.1.2 RG at One-loop Level

Consider the infinitesimal rescale $s = e^{dt}$. The one-loop contribution to the quadratic term is[2]

$$\mu_{LL}^{(2)} =$$



$$= -u \int_{d\Lambda} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{i\omega\eta}}{i\omega - v_F k}, \tag{5.17}$$

[1]The symbol $\bar{\delta}$ enforces momentum conservation mod 2π, as is appropriate to any lattice problem. A process where lattice momentum is violated in multiples of 2π is called an *umklapp process*.

[2]We include an infinitesimal $e^{i\omega\eta}$ to ensure convergence as we do the integral over ω by closing the upper half-plane.

where the integral on the momentum shell is

$$\int_{d\Lambda} \frac{dk}{2\pi} = \int_{-\Lambda}^{-\Lambda(1-dt)} \frac{dk}{2\pi} + \int_{\Lambda(1-dt)}^{\Lambda} \frac{dk}{2\pi}. \tag{5.18}$$

The result gives:

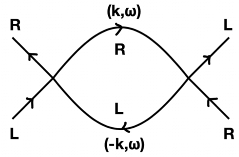
$$\mu_{LL}^{(2)} = -\frac{u\Lambda}{2\pi} dt$$

By the symmetry $L \leftrightarrow R$, we know $\mu_{LL}^{(2)} = \mu_{RR}^{(2)} = \mu^{(2)}$, so the RG flow is

$$\frac{d}{dt}\left[s \cdot \left(\mu + \mu^{(2)}\right)\right] = \mu - \frac{u\Lambda}{2\pi}. \tag{5.19}$$

The one-loop correction to the quartic term ($u_{LRRL} = -u$) have two contributions. One is called ZS' (zero sound) channel:[3]

$$u_{\mathrm{ZS'}}^{(2)} =$$



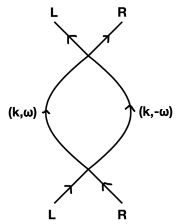
$$\begin{aligned} &= -u^2 \int_{-\infty}^{\infty} \int_{\Lambda/s<|k|<\Lambda} \frac{d\omega dk}{(2\pi)^2} \frac{e^{i\omega\eta}}{(i\omega + v_F k)(i\omega - v_F k)} \\ &= u^2 \int_{\Lambda/s<|k|<\Lambda} \frac{dk}{2\pi} \frac{1}{2|k|} \\ &= \frac{u^2}{2\pi} \frac{d\Lambda}{\Lambda}. \end{aligned} \tag{5.20}$$

The sign is obtained from contracting the Fermion field monomial:

$$\bar\psi_L \bar\psi_R \psi_L \psi_R \bar\psi_L \bar\psi_R \psi_L \psi_R = -G_L G_R \bar\psi_R \psi_L \bar\psi_L \psi_R = -G_L G_R \bar\psi_L \bar\psi_R \psi_L \psi_R.$$

The other is called the BCS channel:[4]

$$u_{\mathrm{BCS}}^{(2)} =$$



$$= -\frac{u^2}{2} \sum_{r=L/R} \int_{-\infty}^{\infty} \int_{d\Lambda} \frac{d\omega dk_i}{(2\pi)^2} \frac{e^{i\omega\eta}}{(i\omega - v_F k)(-i\omega - v_F k)}. \tag{5.21}$$

The sign is obtained from the contraction:

$$\bar\psi_L \bar\psi_R \psi_L \psi_R \bar\psi_L \bar\psi_R \psi_L \psi_R = -G_L G_R \bar\psi_L \bar\psi_R \psi_L \psi_R.$$

Note that we will obtained a factor of 2 since in this channel, the intermedia propagator can be left mover or right mover. We see that two contributions cancel out:

$$u_{\mathrm{ZS'}}^{(2)} + u_{\mathrm{BCS}}^{(2)} = 0. \tag{5.22}$$

[3]There is actually another zero sound channel ZS, but which has no contribution to the vertex because the diagram contains the vertex of the ($LL \to RR$) process, which has no relevant contribution the the vertex.

[4]The 1/2 factor comes from the symmetry factor of the diagram.

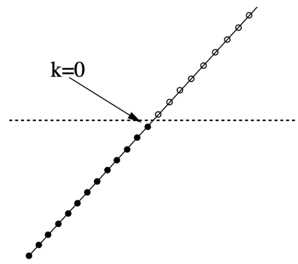


Figure 5.2: The vacuum state $|0\rangle_0$ of a single fermion branch.

Together, the RG flow to the one-loop level is

$$\frac{d\mu}{dt} = \mu - \frac{u\Lambda}{2\pi}, \quad \frac{du}{dt} = 0. \tag{5.23}$$

The fixed point solution to the RG flow is:

$$\mu^* = \frac{u^*\Lambda}{2\pi}, \tag{5.24}$$

where the fixed-point value of u^* is arbitrary. The vanishing beta function predict that the ground state of one-dimensional weakly interacting Fermi gas remains gapless (rather than develops CDW order and becomes gapped).

5.2 Bosonization

In this section, we map the 1D interacting fermion system to a bosonic one. From the RG analysis, we know that the low energy excitations are particle-hole modes:

$$\rho_k^\dagger = \sum_q c_{q+k}^\dagger c_q, \quad \rho_k = \sum_q c_q^\dagger c_{q+k} = \rho_{-k}^\dagger. \tag{5.25}$$

In the following, we restore the notation for the momentum (i.e., use the original momentum instead of the shifted one).

5.2.1 Bosonic Hilbert Space

To simplify the discussion, here we consider only a single branch of Fermion, as depicted in Fig. 5.2. The generalization to multiple branches is trivial since the dispersion is the same.

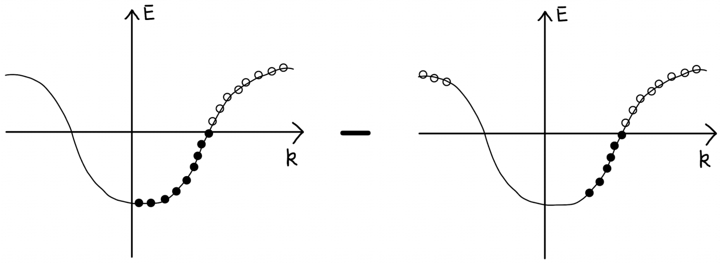


Figure 5.3: Shift of the right-moving modes.

The commutation relation between ρ_k and $\rho^\dagger_{k'}$ is:[5]

$$\begin{aligned}
\left[\rho_k, \rho^\dagger_{k'}\right] &= \sum_{q_1,q_2} \left[c^\dagger_{q_1} c_{q_1+k}, c^\dagger_{q_2+k'} c_{q_2}\right] \\
&= \sum_{q_1,q_2} \left\{c^\dagger_{q_1} \left[c_{q_1+k}, c^\dagger_{q_2+k'} c_{q_2}\right] + \left[c^\dagger_{q_1}, c^\dagger_{q_2+k'} c_{q_2}\right] c_{q_1+k}\right\} \\
&= \sum_{q_1,q_2} \left\{\delta_{q_1+k,q_2+k'} c^\dagger_{q_1} c_{q_2} - \delta_{q_1,q_2} c^\dagger_{q_2+k'} c_{q_1+k}\right\} \\
&= \sum_{q} \left[c^\dagger_{q+k'-k} c_q - c^\dagger_{q+k'} c_{q+k}\right].
\end{aligned} \tag{5.26}$$

For $k \neq k'$, it is clear that $[\rho_k, \rho^\dagger_{k'}] = 0$. However, when $k = k'$, we should be careful about the subtraction, since it evolve two infinities of which the subtraction is ill-defined.

Here we deal with the infinity with the lattice regularization, i.e., we think of the linearized theory as the low-energy approximation of a lattice mode, where the dispersion form a single energy band. The left/right movers are actually in a single band but with positive/negative momentum. Consider for example the density operator for the right mover, the commutator of the right moving density operator is then

$$\left[\rho_{k,r}, \rho^\dagger_{k,r}\right] = \sum_{0<q<\pi} [n_{q,r} - n_{k+q,r}]. \tag{5.27}$$

Consider for example the case where $r = R, k > 0$, as shown in Fig. 5.3. The subtraction result in a sum of low-lying right-mover number operator minus a sum of high-energy left-mover number operator, which behaves like a constant in the low energy regime. The above analysis gives the commutation relation:

$$\left[\rho_{k,r}, \rho^\dagger_{kr}\right] \simeq \frac{qL}{2\pi} \equiv n_q. \tag{5.28}$$

[5] We use the identity $[AB, C] = A[B, C] + [A, C]B$ and $[A, BC] = \{A, B\}C - B\{A, C\}$.

Remark 4. Normal Ordering

Field theoretically, we can also evaluate the infinite subtraction by the normal-order expression:

$$O = {:}O{:} + \langle 0|O|0\rangle. \tag{5.29}$$

For $k \neq k'$, since $\langle 0|c_k^\dagger c_k|0\rangle = 0$, the normal ordering does not affect the result, while for the $k = k'$ case, the normal ordering takes care of the infinity of the particle number operator:

$$\begin{aligned}\sum_q [n_q - n_{q+k}] &= \sum_q [{:}n_q{:} - {:}n_{q+k}{:} + \langle 0|n_q|0\rangle - \langle 0|n_{q+k}|0\rangle] \\ &= \sum_q [\langle 0|n_q|0\rangle - \langle 0|n_{q+k}|0\rangle].\end{aligned} \tag{5.30}$$

The final result gives the same result as the above discussions.

We denote the ground state with N fermions as $|N\rangle_0$, which satisfies:

$$\rho_{p>0}|N\rangle_0 = \rho^\dagger_{p<0}|N\rangle_0 = 0. \tag{5.31}$$

We can thus define a set of canonical bosonic modes:

$$b_p^\dagger = i\frac{\rho_p^\dagger}{\sqrt{n_p}}, \quad b_p = -i\frac{\rho_p}{\sqrt{n_p}}, \quad [b_q, b_{q'}^\dagger] = \delta_{qq'}. \tag{5.32}$$

We will assume $p > 0$, and the $\pm i$ factor is a convention chosen for the future convenience.

Now we discuss the construction of the Hilbert space using the bosonic modes. The N-particle sector is spanned by the states generated by applying $b_q^\dagger$'s to the ground state $|N\rangle_0$. A general N-particle state has the form:

$$|N\rangle = f(\{b_q^\dagger\})|N\rangle_0. \tag{5.33}$$

Note that the bosonic mode b_q does not change the particle number:[6]

$$\left[\hat{N}, b_q\right] = \left[\hat{N}, b_q^\dagger\right] = 0. \tag{5.34}$$

In order to construct the full Fock space, we also need to include a particle-number-changing operator, the *Klein factor* $\hat{F}$, that shifts the total number of fermion by one, and commutes with bosonic operator b_q:

$$\left[\hat{F}, b_q\right] = \left[\hat{F}, b_q^\dagger\right] = 0, \quad \left[\hat{F}, \hat{N}\right] = \hat{F}, \quad \left[\hat{F}^\dagger, \hat{N}\right] = -\hat{F}^\dagger. \tag{5.35}$$

For system with different fermion species (labeled by η), the set of operator

$$\{b_{q,\eta}, b_{q,\eta}^\dagger, \hat{F}_\eta, \hat{N}_\eta\}$$

[6] The particle number operator is defined by the normal order expression $\hat{N} = \sum_k {:}c_k^\dagger c_k{:}$.

form a complete operator basis for the Hilbert space within the low-energy regime. However, note that the Klein factor also takes care of the fermionic statistics. That is, for the state denoted as

$$|\{N_i\}\rangle = |N_1, N_2, \cdots, N_m\rangle \tag{5.36}$$

The Klein factor $\hat{F}_\eta$ acting on the state will contribute an additional factor

$$\hat{F}_\eta|\{N_i\}\rangle = \exp\left(i\pi\sum_{j=1}^{\eta-1}N_j\right)|N_1, \cdots, N_\eta - 1, \cdots, N_m\rangle. \tag{5.37}$$

5.2.2 Bosonization of Fermion Field

Now we try to express the fermion operator

$$\psi(x) = \frac{1}{\sqrt{L}}\sum_p e^{ipx}c_p \tag{5.38}$$

by the bosonic operator set. First we consider the commutation relation between the fermion field operator and bosonic mode:

$$\begin{aligned}[b_q, \psi(x)] &= \frac{1}{\sqrt{L}}\frac{i}{\sqrt{n_q}}\sum_k\sum_p e^{ipx}[c_k^\dagger c_{k+q,r'}, c_p] \\ &= -\frac{1}{\sqrt{L}}\frac{i}{\sqrt{n_q}}\sum_p e^{ipx}c_{p+q} = -i\frac{1}{\sqrt{n_q}}e^{-iqx}\psi(x).\end{aligned} \tag{5.39}$$

Similarly,

$$\left[b_q^\dagger, \psi(x)\right] = i\frac{e^{iqx}}{\sqrt{n_q}}\psi(x). \tag{5.40}$$

For future convenience, we define a c-number factor:

$$\alpha_q(x) \equiv \frac{1}{\sqrt{n_q}}e^{iqx}, \quad [b_q^\dagger, \psi(x)] = i\alpha_q(x)\psi(x), \quad [b_q, \psi(x)] = -i\alpha_q^*(x)\psi(x). \tag{5.41}$$

Since $b_q|N\rangle_0 = 0$, the commutation relation (5.39) leads to

$$[b_q, \psi(x)]\,|N\rangle_0 = b_q\psi(x)|N\rangle_0 = -i\alpha_q^*(x)\psi(x)|N\rangle_0. \tag{5.42}$$

Thus, $\psi(x)|N\rangle_0$ is an eigenstate of b_q, i.e., a coherent state:

$$\psi(x)|N\rangle_0 = \Lambda(x)\hat{F}\exp\left[-i\sum_{q>0}\alpha_q^*(x)b_q^\dagger\right]|N\rangle_0 \tag{5.43}$$

where $\Lambda(x)$ is a c-number, which can be determined by

$${}_0\langle N-1|\psi(x)|N\rangle_0 = \Lambda(x)\;{}_0\langle N|\exp\left[-i\sum_{q>0}\alpha_q^*(x)b_q^\dagger\right]|N\rangle_0 = \Lambda(x).$$

The left-hand side can be computed directly, the result is[7]

$$\Lambda(x) = \frac{1}{\sqrt{L}} \sum_k e^{ikx} \, {}_0\langle N-1|c_{k,r}|N\rangle_0 = \frac{1}{\sqrt{L}} e^{i\frac{2\pi N}{L}}, \tag{5.44}$$

In this way, we get

$$\psi(x)|N\rangle_0 = \frac{\hat{F}}{\sqrt{L}} e^{i\frac{2\pi\hat{N}x}{L}} \exp\left[-i\sum_{q>0} \alpha_q^*(x) b_q^\dagger\right] |N\rangle_0 \tag{5.45}$$

The commutation relation (5.40) also leads to:[8]

$$\begin{aligned} \psi(x) b_q^\dagger &= \left[b_q^\dagger - i\alpha_q(x)\right] \psi(x) \\ \Rightarrow \psi(x) \left(b_q^\dagger\right)^n &= \left[b_q^\dagger - i\alpha_q(x)\right]^n \psi(x) \\ \Rightarrow \psi(x) f\left[\left\{b_q^\dagger\right\}\right] &= f\left[\left\{b_q^\dagger - i\alpha_q(x)\right\}\right] \psi(x). \end{aligned}$$

Then, for a generic N-particle state:

$$\begin{aligned} \psi(x)|N\rangle &= f\left[\left\{b_q^\dagger - i\alpha_q(x)\right\}\right] \psi(x)|N\rangle_0 \\ &= f\left[\left\{b_q^\dagger - i\alpha_q(x)\right\}\right] \frac{\hat{F}}{\sqrt{L}} e^{i\frac{2\pi\hat{N}x}{L}} \exp\left[-i\sum_{q>0} \alpha_q^*(x) b_q^\dagger\right] |N\rangle_0 \\ &= \frac{\hat{F}}{\sqrt{L}} e^{i\frac{2\pi\hat{N}x}{L}} \exp\left[-i\sum_{q>0} \alpha_q^*(x) b_q^\dagger\right] f\left[\left\{b_q^\dagger - \alpha_q(x)\right\}\right] |N\rangle_0. \end{aligned} \tag{5.46}$$

Using the BCH formula

$$e^A B e^{-A} = e^{[A,\cdot]} B = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \cdots, \tag{5.47}$$

we have the identity:

$$\begin{aligned} \exp\left[-i\sum_{q>0} \alpha_q(x) b_q\right] b_q^\dagger \exp\left[i\sum_{q>0} \alpha_q(x) b_q\right] &= b_q^\dagger - i\alpha_q(x) \\ \Rightarrow \exp\left[-i\sum_{q>0} \alpha_q(x) b_q\right] f\left[b_q^\dagger\right] \exp\left[i\sum_{q>0} \alpha_q(x) b_q\right] &= f\left[\left\{b_q^\dagger - i\alpha_q(x)\right\}\right]. \end{aligned}$$

Eq. (5.46) can be further simplified to:

$$\begin{aligned} \psi(x)|N\rangle &= \frac{\hat{F}}{\sqrt{L}} e^{i\frac{2\pi\hat{N}x}{L}} e^{-i\sum_{q>0}\alpha_q^*(x) b_q^\dagger} e^{-i\sum_{q>0}\alpha_q(x) b_q} f\left[b_q^\dagger\right] e^{i\sum_{q>0}\alpha_q(x) b_q} |N\rangle_0 \\ &= \frac{\hat{F}}{\sqrt{L}} e^{i\frac{2\pi\hat{N}x}{L}} \exp\left[-i\sum_{q>0} \alpha_q^*(x) b_q^\dagger\right] \exp\left[-i\sum_{q>0} \alpha_q(x) b_q\right] |N\rangle. \end{aligned}$$

We thus express the Fermi field operator in bosonic operator:

$$\psi(x) = \frac{\hat{F}}{\sqrt{L}} e^{i\frac{2\pi\hat{N}x}{L}} e^{-i\sqrt{2\pi}\varphi^\dagger(x)} e^{-i\sqrt{2\pi}\varphi(x)}, \tag{5.48}$$

[7] Note that the Fermi point locates at $k_F = \frac{2\pi N}{L}$.
[8] We denote $\alpha_q(x) \equiv e^{iqx}/\sqrt{n_q}$ to simplify the notation.

where we have introduced a bosonic field:[9]

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \sum_{q>0} e^{-aq/2} a_q(x) b_q. \tag{5.49}$$

Now we introducing the bosonic field:

$$\phi(x) \equiv \varphi(x) + \varphi^\dagger(x) = \frac{1}{\sqrt{2\pi}} \sum_{q>0} \frac{e^{-aq/2}}{\sqrt{n_q}} \left[e^{iqx} b_q + e^{-iqx} b_q^\dagger \right]. \tag{5.50}$$

We can expressed the fermion field as:[10]

$$\psi(x) = \frac{\hat{F}}{\sqrt{L}} e^{i\frac{2\pi \hat{N} x}{L}} e^{-i\sqrt{2\pi}\phi(x)} e^{\pi[\varphi(x),\varphi^\dagger(x)]}.$$

The commutation relation between φ and $\varphi^\dagger$ is

$$\left[\varphi(x), \varphi^\dagger(y)\right] = \frac{1}{L} \sum_q \frac{e^{iq(x-y+ia)}}{q} = -\frac{1}{2\pi} \ln \left\{ 1 - \exp\left[\frac{2\pi i}{L}(x-y+ia) \right] \right\}.$$

Set $x = y$, take limit $a \to 0^+$,

$$\exp\left\{ \pi[\varphi(x), \varphi^\dagger(x)] \right\} = \lim_{a\to 0^+} \left[1 - \exp\left(-\frac{2\pi a}{L} \right) \right]^{-\frac{1}{2}} \to \sqrt{\frac{L}{2\pi a}}.$$

so we have

$$\psi(x) = \frac{\hat{F}}{\sqrt{2\pi a}} e^{i\frac{2\pi \hat{N} x}{L}} e^{-i\sqrt{2\pi}\phi(x)}. \tag{5.51}$$

The divergent factor $1/\sqrt{2\pi a}$ appears because Eq. (5.51) is not normal-ordered. We can check the result by normal-ordering the bilinear term $\psi^\dagger(x+a)\psi(x)$. Insert Eq. (5.51) into the expression, we have:

$$\psi^\dagger(x+a)\psi(x) \simeq \frac{e^{-i\frac{2\pi \hat{N} a}{L}}}{2\pi a} e^{i\sqrt{2\pi}\partial_x \phi(x) a} e^{\pi[\phi(x+a),\phi(x)]}. \tag{5.52}$$

The commutation relation is:

$$\begin{aligned} [\phi(x), \phi(y)] &= -\frac{1}{2\pi} \ln \left\{ \frac{1 - \exp\left[\frac{2\pi i}{L}(x-y+ia)\right]}{1 - \exp\left[-\frac{2\pi i}{L}(x-y-ia)\right]} \right\} \\ &\xrightarrow{a\to 0} \frac{i}{2}\,\mathrm{sgn}(x-y) - \frac{i}{L}(x-y). \end{aligned} \tag{5.53}$$

To the lowest order of a/L:

$$\psi^\dagger(x+a)\psi(x) \simeq \frac{i}{2\pi a} + \frac{\hat{N}+1}{L} - \frac{1}{\sqrt{2\pi}} \partial_x \phi(x). \tag{5.54}$$

The normal ordering will delaminate all constant, including the divergent one:

$$:\psi^\dagger(x+a)\psi(x): = \frac{\hat{N}}{L} - \frac{1}{\sqrt{2\pi}} \partial_x \phi(x). \tag{5.55}$$

We will show in the following the above equation agrees with the bosonization of fermion bilinear $\psi^\dagger(x)\psi(x)$ term.

[9] The "converging factor" $e^{-aq/2}$ is important in defining a proper bosonic theory in 1D. These equations should always be viewed as having $e^{-aq/2}$ to ensure convergence at intermediate steps, but final results should be written taking $a \to 0^+$.

[10] We use the identity $e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]}$ if both A and B commutes with $[A,B]$.

5.2.3 Bosonization Dictionary

Now we are going to establish a bosonization dictionary. To tell the complete story, we are now working for both the right-moving and left-moving fermion branches. In this case, we express the fermion as a spinor:

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \quad \bar{\psi} = \psi^\dagger \sigma^x = \begin{pmatrix} \psi_R^\dagger & \psi_L^\dagger \end{pmatrix}. \tag{5.56}$$

We also define a new set of bosonic fields

$$\phi(x) \equiv \phi_R(x) + \phi_L(x), \quad \theta(x) \equiv \phi_R(x) - \phi_L(x). \tag{5.57}$$

First we consider the fermion bilinear (for $r = L/R$):

$$\begin{aligned} :\psi_r^\dagger(x)\psi_r(x): &= \frac{1}{L}\sum_q :c_{q,r}^\dagger c_{q,r}: + \frac{1}{L}\sum_{q>0}[e^{-iqx}\rho_{q,r}^\dagger + e^{iqx}\rho_{q,r}] \\ &= \frac{\hat{N}_r}{L} - \frac{1}{2\pi}\sum_{q>0}\frac{iq}{\sqrt{n_q}}\left[e^{iqx}b_{q,r} - e^{-iqx}b_{q,r}^\dagger\right]. \end{aligned} \tag{5.58}$$

We then get:

$$:\psi^\dagger(x)\psi(x): = \frac{\hat{N}_L + \hat{N}_R}{L} - \frac{1}{\sqrt{2\pi}}\partial_x\left[\phi_L(x) + \phi_R(x)\right]. \tag{5.59}$$

Now we go back to the Hamiltonian with linear dispersion (for single fermion branch):

$$H_0 = v_F\sum_{k,r} k\ :c_{k,r}^\dagger c_{k,r}: = v_F\int dx\ :\psi^\dagger(x)(-i\partial_x)\psi(x):. \tag{5.60}$$

Since $b_q^\dagger$ raise the energy of any eigenstate of H_0 by q unit, we have the commutation relation:

$$\left[H_0, b_{q,r}^\dagger\right] = qb_{q,r}^\dagger. \tag{5.61}$$

The Hamiltonian satisfies such relation can only be the bosonic bilinear:

$$H_0 = v_F\sum_r\sum_{q>0} qb_{q,r}^\dagger b_{q,r} + \frac{\pi v_F}{L}\sum_r \hat{N}_r(\hat{N}_r+1). \tag{5.62}$$

The constant part comes from the fact that

$$H_0|N\rangle_0 = \frac{2\pi v_F}{L}\frac{\sum_r \hat{N}_r(\hat{N}_r+1)}{2}|N\rangle_0, \quad b_{q,r}^\dagger b_{q,r}|N\rangle_0 = 0.$$

Also, the bosonic operator can also be expressed as

$$H_0 = \frac{v_F}{2}\int dx\ :[\partial_x\phi(x)]^2: + \frac{\pi v_F}{L}\sum_r \hat{N}_r(\hat{N}_r+1). \tag{5.63}$$

Chapter 6

Topological Field Theory

6.1 Chern-Simons Theory

Assume the action of the microscopical theory has the form $S[\psi_i]$, where $\{\psi_i\}$ denotes all degrees of microscopical freedom. If the system has the U(1) symmetry, we can always rewrite the field theory as a gauge theory:

$$S[\psi_i; A] = S[\psi_i] + \int d^d x\ j^\mu(x) A_\mu(x), \tag{6.1}$$

where the current j^μ is the Noether current. The gauge field $A^\mu(x)$ is regarded as the back ground field which has no dynamics. If we are interested in the low-energy physics, especially for gapped system, the ground state physics, we can formally integrate out other degrees of freedom, the resulting effective theory has only the gauge degree of freedom:

$$Z_{\text{eff}}[A] = \int D[\psi_i] e^{iS[\psi_i;A]}. \tag{6.2}$$

In this section, we consider the effective gauge field on $(2+1)$-dimensional space-time. The effective action should also be gauge-invariant. The allowed terms include

$$A \wedge dA,\ \ dA \wedge dA,\ \ \text{higher order terms.}$$

From dimensional analysis, the first term is most relevant in the low-energy. Such effective theory is the *Chern-Simons theory*:

$$S_{\text{CS}} = \frac{k}{4\pi} \int d^3 x\ \varepsilon_{\mu\nu\rho} A^\mu \partial^\nu A^\rho. \tag{6.3}$$