场论中的对称性

任杰

连续对称性与守恒律

无穷小变换

对称性可以被定义为坐标和场的一组变换，这组变换使得系统在任何区间内的作用量不变，因此在对称变换下，系统的运动规律不变。我们这里只考虑连续对称性，考虑坐标和场的无穷小变换：

$$
\begin{aligned}
x^\mu &\quad\rightarrow\quad x'^\mu = x^\mu + \delta x^\mu, &(1)\\
\phi_r(x) &\quad\rightarrow\quad \phi'_r(x') = \phi_r(x) + \delta\phi_r(x), &(2)
\end{aligned}
$$

注意场的变换定义成由坐标变换前后对应的同一点上，因此与坐标变换无关。也就是说一个连续对称性包含坐标和场这两个独立的无穷小变换，其中如果一个对称变换完全由坐标变换造成，我们称之为时空对称性；反之，如果一个对称变换完全由场变化造成，我们称之为内秉对称性。一个连续对称性要求，在一组无穷小变换下，作用量积分在任何区域内不变，即：

$$
\int_{\Omega'} d^4x' \mathcal{L}'(x') = \int_{\Omega} d^4x \mathcal{L}(x), \tag{3}
$$

其中：

$$
\mathcal{L}'(x') = \mathcal{L}\left(\phi'_r(x'), \partial\phi'_r/\partial x'^\mu\right), \tag{4}
$$

上式中 Ω' 是原区域 Ω 经过坐标变换后对应的区域。

诺特定理

诺特定理告诉我们一个连续对称性对应一个守恒量。我们在这一节中将证明这点。为了证明的方便，我们对场定义一个总变分：

$$
\tilde{\delta}\phi_r(x) := \phi'_r(x) - \phi_r(x), \tag{5}
$$

这个变分实际上包含了场的内秉变化和坐标变换，它和场内秉变化的关系为 (保留至一阶意义下)：

$$
\begin{aligned}
\tilde{\delta}\phi_r(x) &= \phi'_r(x) - \phi'_r(x') + \phi'_r(x') - \phi_r(x)\\
&= \delta\phi_r(x) - \delta x^\mu \frac{\partial}{\partial x^\mu}\phi'(x)\\
&\approx \delta\phi_r(x) - \delta x^\mu \frac{\partial}{\partial x^\mu}\phi(x), &(6)
\end{aligned}
$$

这个变分定义在坐标的同一点上，因此可以和微分算符对易，这使得数学处理更加方便。接下来我们写出作用力积分：

$$\begin{aligned}\delta S &= \int_{\Omega'} d^4x'\mathcal{L}'(x') - \int_{\Omega} d^4x\mathcal{L}(x) \\ &= \int_{\Omega} d^4x\delta\mathcal{L}(x) + \int_{\Omega'} d^4x'\mathcal{L}(x) - \int_{\Omega} d^4x\mathcal{L}(x).\end{aligned} \tag{7}$$

为计算第二第三项，我们需要首先计算雅可比矩阵 (保留至一阶意义下)：

$$\left|\frac{\partial x'^{\mu}}{\partial x^{\nu}}\right| = \left|\delta^{\mu}_{\nu} + \frac{\partial \delta x^{\mu}}{\partial x^{\nu}}\right| = \left(1 + \frac{\partial \delta x^{\mu}}{\partial x^{\mu}}\right). \tag{8}$$

因此第二第三项可以化简为:

$$\begin{aligned}\int_{\Omega'} d^4x'\mathcal{L}(x) - \int_{\Omega} d^4x\mathcal{L}(x) &= \int_{\Omega} d^4x\left(1 + \frac{\partial \delta x^{\mu}}{\partial x^{\mu}}\right)\mathcal{L}(x) - \int_{\Omega} d^4x\mathcal{L}(x) \\ &= \int_{\Omega} d^4x\frac{\partial \delta x^{\mu}}{\partial x^{\mu}}\mathcal{L}(x).\end{aligned} \tag{9}$$

我们将上述变分式写成包含总变分的形式:

$$\begin{aligned}\delta S &= \int_{\Omega} d^4x\left[\delta\mathcal{L}(x) + \frac{\partial \delta x^{\mu}}{\partial x^{\mu}}\mathcal{L}(x)\right] \\ &= \int_{\Omega} d^4x\left[\tilde{\delta}\mathcal{L}(x) + \delta x^{\mu}\frac{\partial}{\partial x^{\mu}}\mathcal{L}(x) + \frac{\partial \delta x^{\mu}}{\partial x^{\mu}}\mathcal{L}(x)\right] \\ &= \int_{\Omega} d^4x\left[\tilde{\delta}\mathcal{L}(x) + \frac{\partial}{\partial x^{\mu}}\left(\mathcal{L}(x)\delta x^{\mu}\right)\right].\end{aligned} \tag{10}$$

上面定义的总变分表现地像一个“真正的”变分，将它作用于拉氏量密度上得到:

$$\begin{aligned}\tilde{\delta}\mathcal{L}(x) &= \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi_r)}\tilde{\delta}(\partial_{\mu}\phi_r) + \frac{\partial\mathcal{L}}{\partial\phi_r}\tilde{\delta}\phi_r(x) \\ &= \partial_{\mu}\left[\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi_r)}\tilde{\delta}\phi_r(x)\right] + \left[\frac{\partial\mathcal{L}}{\partial\phi_r} - \partial_{\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi_r)}\right)\right]\tilde{\delta}\phi_r(x).\end{aligned}$$

代入上式，并利用欧拉-拉格朗日方程化简:

$$\begin{aligned}\delta S &= \int_{\Omega} d^4x\partial_{\mu}\left[\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi_r)}\tilde{\delta}\phi_r(x) + \mathcal{L}(x)\delta x^{\mu}\right] \\ &= \int_{\Omega} d^4x\partial_{\mu}\left[\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi_r)}\delta\phi_r(x) - \left(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi_r)}\frac{\partial\phi_r}{\partial x^{\nu}}\delta x^{\nu} - \mathcal{L}(x)\delta x^{\mu}\right)\right] \\ &= \int_{\Omega} d^4x\partial_{\mu}\left[\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi_r)}\delta\phi_r(x) - \left(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi_r)}\frac{\partial\phi_r}{\partial x^{\nu}} - \delta^{\mu}{}_{\nu}\mathcal{L}(x)\right)\delta x^{\nu}\right].\end{aligned} \tag{11}$$

对称性要求对所有 Ω ，$\delta S = 0$ ，所以被积函数应为 0 ，这样，我们就得到了一个无散度的流：

$$f^{\mu} := \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi_r)}\delta\phi_r(x) - \left(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi_r)}\frac{\partial\phi_r}{\partial x^{\nu}} - \delta^{\mu}{}_{\nu}\mathcal{L}(x)\right)\delta x^{\nu}, \tag{12}$$

$$\partial_{\mu}f^{\mu} = \partial_t f^0 + \nabla\cdot\mathbf{f} = 0. \tag{13}$$

这个无散流给出了系统守恒量:

$$Q := \int d^3 x f^0. \tag{14}$$

能量-动量张量

时空平移不变性

第一个例子是我们提到过的时空对称性，这个对称性只由时空平移不变形生成，无穷小变换可以写为:

$$\begin{cases} \delta x^\mu & = \epsilon^{\mu\nu}\alpha_\nu \\ \delta\phi_r & = 0 \end{cases}. \tag{15}$$

考虑一个沿着 x^ν 方向的无穷小平移，对称性给出的守恒流为:

$$f^\mu = -\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi_r\right)}\frac{\partial\phi_r}{\partial x^\nu} - \delta^\mu{}_\nu\mathcal{L}\left(x\right)\right)\alpha_\nu. \tag{16}$$

我们据此定义能量-动量张量:

$$\Theta^\mu{}_\nu := \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi_r\right)}\frac{\partial\phi_r}{\partial x^\nu} - \delta^\mu{}_\nu\mathcal{L}\left(x\right), \tag{17}$$

能量量张量可以直接给出守恒量，时间方向的平移不变性给出守恒量:

$$E = \int d^3 x\Theta^0{}_0 = \int d^3 x\left[\pi_r\left(x\right)\dot{\phi}_r\left(x\right) - \mathcal{L}\left(x\right)\right]. \tag{18}$$

这正是哈密顿量密度在空间积分，也即能量守恒。而空间方向平移不变性给出另外 3 个守恒量:

$$P_\nu = \int d^3 x\Theta^0{}_\nu = \int d^3 x\pi_r\left(x\right)\partial_\nu\phi_r\left(x\right). \tag{19}$$

这是场的动量密度积分，即动量守恒。

电磁场中的能动张量

电磁场 (无源) 的拉氏量密度定义为:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{20}$$

其中电磁场张量

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{21}$$

$$= \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix} \tag{22}$$

以 A^μ 为正则坐标，经过同样的计算可以得到电磁场的能动张量:

$$\Theta^{\mu\nu}=\frac{1}{4}g^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}-F^{\mu\sigma}\partial^{\nu}A_{\sigma}. \tag{23}$$

该能动张量给出了电磁场的能量和动量:

$$\begin{aligned} E &= \int d^3x\Theta^{00}=\int d^3x\frac{\boldsymbol{E}^2+\boldsymbol{B}^2}{2}, && (24)\\ P^i &= \int d^3x\Theta^{0i}=\int d^3x\left(\boldsymbol{E}\times\boldsymbol{B}\right)^i. && (25)\end{aligned}$$

守恒荷

内秉对称性

第二个例子是场本身的内秉对称性。考虑场本身的无穷小变换:

$$\phi\left(x\right)\to\phi'\left(x'\right)=\phi\left(x\right)+i\epsilon\lambda_{rs}\phi_s\left(x\right). \tag{26}$$

该变换对应的守恒流为:

$$f^{\mu}=\frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu}\phi_r\right)}\delta\phi_r=i\epsilon\frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu}\phi_r\right)}\lambda_{rs}\phi_s\left(x\right). \tag{27}$$

相应的守恒量为:

$$Q=\int d^3x\pi_r\left(x\right)\lambda_{rs}\phi_s\left(x\right). \tag{28}$$

U(1) 守恒荷

其中一个重要的情况就是一些复场中的 $U\left(1\right)$ 对称性:

$$\begin{aligned} \phi' &= e^{-i\epsilon}\phi, && (29)\\ \phi'^{*} &= e^{i\epsilon}\phi^{*}. && (30)\end{aligned}$$

这种变换对应的守恒量称为场携带的守恒荷:

$$Q=\left(-i\right)\int d^3x\left[\pi\left(x\right)\phi\left(x\right)-\pi^{*}\left(x\right)\phi^{*}\left(x\right)\right]. \tag{31}$$

角动量

洛伦兹对称性

一个洛伦兹协变的理论要求作用量积分在洛伦兹变换下不变。洛伦兹变换对应的无穷小变换为:

$$\begin{aligned} x'^{\mu} &= x^{\mu}+\delta\omega^{\mu\nu}x_{\nu}. && (32)\\ \phi_r'\left(x'\right) &= \left(1-\frac{i}{2}\delta\omega_{\mu\nu}\left(J^{\mu\nu}\right)_{rs}\right)\phi_s\left(x\right), && (33)\end{aligned}$$

注意其中既有坐标的变换，又有场的变换。对坐标的洛伦兹变换保持内积不变，因此：

$$
\begin{aligned}
x'^{\mu}x'_{\mu} &= \left(x^{\mu}+\delta\omega^{\mu\nu}x_{\nu}\right)\left(x_{\mu}+\delta\omega_{\mu}{}^{\sigma}x_{\sigma}\right) \\
&= x^{\mu}x_{\mu}+\left(\delta\omega_{\mu\nu}+\delta\omega_{\nu\mu}\right)x_{\mu}x_{\nu}+O\left(\delta^{2}\right), \qquad (34)
\end{aligned}
$$

这说明 $\delta\omega_{\mu\nu}$ 是反对称的。相应的，场变换的生成元 $\mathcal{J}^{\mu\nu}$ 也是反对称的。因此洛伦兹群只包含 6 个生成元，对应 3 个空间转动和三个方向的 boost. 代入诺特流表达式中：

$$
\begin{aligned}
f^{\mu} &= -\frac{i}{2}\frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu}\phi_{r}\right)}\delta\omega_{\nu\lambda}\left(J^{\nu\lambda}\right)_{rs}\phi_{s}\left(x\right)-\Theta^{\mu\nu}\delta\omega_{\nu\lambda}x^{\lambda} \\
&= \frac{1}{2}\delta\omega_{\nu\lambda}\left[-i\frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu}\phi_{r}\right)}\left(J^{\nu\lambda}\right)_{rs}\phi_{s}\left(x\right)-\Theta^{\mu\nu}x^{\lambda}+\Theta^{\mu\nu}x^{\lambda}\right] \\
&:= \frac{1}{2}\delta\omega_{\nu\lambda}M^{\mu\nu\lambda}, \qquad (35)
\end{aligned}
$$

我们得到了一个守恒流：

$$
M^{\mu\nu\lambda}:=-i\frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu}\phi_{r}\right)}\left(J^{\nu\lambda}\right)_{rs}\phi_{s}\left(x\right)-\Theta^{\mu\nu}x^{\lambda}+\Theta^{\mu\nu}x^{\lambda}. \qquad (36)
$$

相应的守恒量为：

$$
\begin{aligned}
M^{\nu\lambda} &= \int d^{3}xM^{0\nu\lambda} \\
&= \int d^{3}x\left[-i\pi_{r}\left(J^{\nu\lambda}\right)_{rs}\phi_{s}-\Theta^{0\nu}x^{\lambda}+\Theta^{0\nu}x^{\lambda}\right] \\
&= \int d^{3}x\left[-i\pi_{r}\left(J^{\nu\lambda}\right)_{rs}\phi_{s}-\left(\pi_{r}x^{\lambda}\partial^{\nu}\phi_{r}-x^{\lambda}g^{0\nu}\mathcal{L}\right)+\left(\pi_{r}x^{\nu}\partial^{\lambda}\phi_{r}-x^{\nu}g^{0\lambda}\mathcal{L}\right)\right] \\
&= \int d^{3}x\left[-i\pi_{r}\left(J^{\nu\lambda}\right)_{rs}\phi_{s}+\pi_{r}\left(x^{\nu}\partial^{\lambda}-x^{\lambda}\partial^{\nu}\right)\phi_{r}+\left(x^{\lambda}g^{0\nu}-x^{\lambda}g^{0\mu}\right)\mathcal{L}\right]. \qquad (37)
\end{aligned}
$$

考虑空间分量，此守恒量对应场的角动量，其中我们还可以将角动量拆成坐标部分和内秉部分，分别对应轨道角动量和自旋角动量：

$$
\begin{aligned}
L^{ij} &:= \int d^{3}x\left[\pi_{r}\left(x\right)\left(x^{i}\partial^{j}-x^{j}\partial^{i}\right)\phi_{r}\left(x\right)\right], \qquad (38) \\
S^{ij} &:= \left(-i\right)\int d^{3}x\left[\pi_{r}\left(x\right)\left(J^{ij}\right)_{rs}\phi_{s}\left(x\right)\right]. \qquad (39)
\end{aligned}
$$

电磁场的自旋角动量

电磁场拉氏量密度代入角动量表达式后得到：

$$
M^{ij}=\Theta^{0i}x^{j}-\Theta^{0j}x^{i}+\left(F^{0j}A^{i}-F^{0i}A^{j}\right). \qquad (40)
$$

其中内秉自旋部分角动量为：

$$
\begin{aligned}
S^{ij} &= \int d^{3}x\left(E^{i}A^{j}-E^{j}A^{i}\right), \qquad (41) \\
\boldsymbol{S} &= \int d^{3}\boldsymbol{E}\times\boldsymbol{A}. \qquad (42)
\end{aligned}
$$