motivation. Weinberg (Peskin)

$\eta_{\mu\nu} = diag\,(1,-1,-1,-1)$ ()

$$\Lambda^T \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \Lambda = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}. \tag{1}$$

() $(\delta\omega)^{\mu}{}_{\nu}$:

$$\Lambda^{\mu}{}_{\nu} \approx \delta^{\mu}{}_{\nu} + (\delta\omega)^{\mu}{}_{\nu}, \tag{2}$$
$$\left(\Lambda^T\right)_{\mu}{}^{\nu} \approx \delta_{\mu}{}^{\mu} + \left(\delta\omega^T\right)_{\mu}{}^{\nu}. \tag{3}$$

$$\begin{aligned} \eta_{\mu\nu} &= \left(\delta_{\mu}{}^{\rho} + (\delta\omega)^{\mu}{}_{\nu}\right) \eta_{\rho\sigma} \left(\delta^{\sigma}{}_{\nu} + \left(\delta\omega^T\right)_{\mu}{}^{\nu}\right) \\ &\approx \eta_{\mu\nu} + (\delta\omega)^{\mu}{}_{\nu} \eta_{\rho\sigma} \delta^{\sigma}{}_{\nu} + \delta_{\mu}{}^{\rho} \eta_{\rho\sigma} \left(\delta\omega^T\right)_{\mu}{}^{\nu} \\ &= \eta_{\mu\nu} + (\delta\omega)_{\mu\nu} + \left(\delta\omega^T\right)_{\mu\nu}. \end{aligned} \tag{4}$$

$\delta\omega_{\mu\nu} = -\delta\omega_{\nu\mu}$.

$$(\delta\omega)^{\mu}{}_{\nu} = \eta^{\mu\sigma} (\delta\omega)_{\sigma\nu}. \tag{5}$$

4 6

$$L^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \; L^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \; L^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6}$$

$$K^1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; K^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; K^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{7}$$

3 3 3 3 3 boost $(\theta_1, \theta_2, \theta_3)$ boost $(\beta_1, \beta_2, \beta_3)$.

$$\Lambda(\boldsymbol{\theta}, \boldsymbol{\beta}) = \exp(\boldsymbol{\theta} \cdot \boldsymbol{J} + \boldsymbol{\beta} \cdot \boldsymbol{K}). \tag{8}$$

6

$$\left[L^i, L^j\right] = \epsilon^{ijk} L^k, \tag{9}$$
$$\left[K^i, K^j\right] = -\epsilon^{ijk} L^k, \tag{10}$$
$$\left[L^i, K^j\right] = \epsilon^{ijk} K^k. \tag{11}$$

$$J_+^i = \frac{1}{2}\left(L^i + iK^i\right), \tag{12}$$
$$J_-^i = \frac{1}{2}\left(L^i - iK^i\right). \tag{13}$$

0 :

$$
\begin{aligned}
\left[J_+^i, J_-^j\right] =& \frac{1}{4}\left[L^i + iK^i, L^j - iK^j\right] \\
=& \frac{1}{4}\left(i\epsilon^{ijk}L^k + \epsilon^{ijk}K^k - \epsilon^{ijk}K^k - i\epsilon^{ijk}L^k\right) \\
=& 0.
\end{aligned} \tag{14}
$$

$SU(2)$

$$
\begin{aligned}
\left[J_+^i, J_+^j\right] =& \frac{1}{4}\left[L^i + iK^i, L^j + iK^j\right] \\
=& \frac{1}{4}\left(i\epsilon^{ijk}L^k - \epsilon^{ijk}K^k - \epsilon^{ijk}K^k + i\epsilon^{ijk}L^k\right) \\
=& \frac{i}{2}\epsilon^{ijk}\left(L^k + iK^k\right) \\
=& \frac{i}{2}\epsilon^{ijk}J_+^k.
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
\left[J_-^i, J_-^j\right] =& \frac{1}{4}\left[L^i - iK^i, L^j - iK^j\right] \\
=& \frac{1}{4}\left(i\epsilon^{ijk}L^k + \epsilon^{ijk}K^k + \epsilon^{ijk}K^k + i\epsilon^{ijk}L^k\right) \\
=& \frac{i}{2}\epsilon^{ijk}\left(L^k - iK^k\right) \\
=& \frac{i}{2}\epsilon^{ijk}J_-^k.
\end{aligned} \tag{16}
$$

$SU(2)$

$$so(1,3) \approx su(2) \otimes su(2). \tag{17}$$

$su(2)$ ()() (j_+, j_-) $(2j_+ + 1)(2j_- + 1)$,

(j_+, j_-)		
$(0,0)$	1	
$\left(\frac{1}{2},0\right)$	2	weyl
$\left(0,\frac{1}{2}\right)$	2	Weyl
$\left(\frac{1}{2},\frac{1}{2}\right)$	4	4-
$\left(\frac{1}{2},0\right) \oplus \left(0,\frac{1}{2}\right)$	4	Dirac ()

Weyl
$\left(\frac{1}{2},0\right)$, Weyl

$$\psi_L = \begin{pmatrix} \psi_L^1 \\ \psi_L^2 \end{pmatrix}. \tag{18}$$

$$\psi_L^i \to \left(\Lambda_L\right)^i{}_j \psi_L^j. \tag{19}$$

Λ_L 2×2 $\left(\frac{1}{2},0\right)$ $j_- = 0$, 0

$$J_-^i \psi = 0, \ \forall i. \tag{20}$$

$$J_-^i = \frac{1}{2}\left(J^i - iK^i\right) = 0. \tag{21}$$

$$J_+^i \psi = \frac{1}{2}\left(J^i + iK^i\right)\psi = J_-^i \psi. \tag{22}$$

J_-^i $su(2)$ (i)

$$iJ^i = iJ_-^i = \sigma^i; \tag{23}$$

$$iK^i = J_-^i = -i\sigma^i. \tag{24}$$

$\left(\frac{1}{2}, 0\right)$

$$\Lambda_L = e^{\frac{1}{2}(-i\boldsymbol{\theta}-\boldsymbol{\beta})\cdot\boldsymbol{\sigma}}. \tag{25}$$

$SU(2)$, boost $\left(0, \frac{1}{2}\right)$

$$\Lambda_R = e^{\frac{1}{2}(-i\boldsymbol{\theta}+\boldsymbol{\beta})\cdot\boldsymbol{\sigma}}. \tag{26}$$

Weyl
boost trick σ^2 ()

$$\sigma^2\boldsymbol{\sigma}\sigma^2 = -\boldsymbol{\sigma}^*. \tag{27}$$

$$\sigma^2\Lambda_L^*\sigma^2 = \Lambda_R, \tag{28}$$
$$\sigma^2\Lambda_R^*\sigma^2 = \Lambda_L. \tag{29}$$

$$\psi_L \to -i\sigma^2\psi_L^*. \tag{30}$$

$$\begin{aligned} -i\sigma^2\psi_L^* &\to -i\sigma^2\Lambda_L^*\psi_L^* \\ &= -i\sigma^2\Lambda_L^*\sigma^2\sigma^2\psi_L^* \\ &= \Lambda_R\left(-i\sigma^2\psi_L^*\right). \end{aligned} \tag{31}$$

$$\psi_R \eqsim -i\sigma^2\psi_L^*, \tag{32}$$
$$\psi_L \eqsim -i\sigma^2\psi_R^*. \tag{33}$$

$$\psi_{L,i} = \epsilon_{ij}\psi_L^j = \begin{pmatrix} \psi_L^2 \\ -\psi_L^1 \end{pmatrix}. \tag{34}$$

$$\begin{aligned} \psi_{L,i}\psi_L^i &= \epsilon_{ij}\psi_L^i\psi_L^j \\ &\to \epsilon_{ij}(\Lambda_L)^i{}_k(\Lambda_L)^j{}_l\psi_L^k\psi_L^l \\ &= \det(\Lambda_L)\,\epsilon_{kl}\psi_L^k\psi_L^l \\ &= \psi_{L,i}\psi_L^i. \end{aligned} \tag{35}$$

$\epsilon_{ij} = \left(i\sigma^2\right)_{ij}$,

$$\begin{aligned} \psi_{L,i}\psi_L^i &= i\psi_L^T\sigma^2\psi_L \\ &= \left(-i\sigma^2\psi_L^*\right)^\dagger\psi_L \\ &\eqsim \psi_R^\dagger\psi_L. \end{aligned} \tag{36}$$

$\left(\frac{1}{2},\frac{1}{2}\right)$ 4 4-
$\left(\frac{1}{2},\frac{1}{2}\right)$ Weyl Weyl

$$\left(\frac{1}{2},\frac{1}{2}\right)=\left(\frac{1}{2},0\right)\otimes\left(0,\frac{1}{2}\right). \tag{37}$$

$$\psi=\psi_L\otimes\psi_R=\begin{pmatrix}\psi_L^1\psi_R^1\\ \psi_L^1\psi_R^2\\ \psi_L^2\psi_R^1\\ \psi_L^2\psi_R^2\end{pmatrix}. \tag{38}$$

$$\Lambda_{\left(\frac{1}{2},\frac{1}{2}\right)}\psi=\left(\Lambda_L\otimes\Lambda_R\right)\psi. \tag{39}$$

4-trick ψ_R

$$M=\psi_L\cdot\psi_R^T=\begin{pmatrix}\psi_L^1\psi_R^1\psi_L^1\psi_R^2\\ \psi_L^2\psi_R^1\psi_L^2\psi_R^2\end{pmatrix}.$$

4 $\left(\frac{1}{2},\frac{1}{2}\right)$

$$M\to\Lambda_L M\Lambda_R^T. \tag{40}$$

$\boldsymbol{\sigma}\to\boldsymbol{\sigma}^*$. σ^2

$$\begin{aligned}M\sigma^2\to&\Lambda_L M\sigma^2\left(\sigma^2\Lambda_R^T\sigma^2\right)\\ =&\Lambda_L\left(M\sigma^2\right)\bar{\Lambda}_R.\end{aligned} \tag{41}$$

$$\bar{\Lambda}_R=\Lambda_R=e^{\frac{1}{2}(i\boldsymbol{\theta}-\boldsymbol{\beta})\cdot\boldsymbol{\sigma}}. \tag{42}$$

$$\begin{aligned}M\sigma^2=&\psi_L\cdot\psi_R^T\sigma^2\\ =&\psi_L\cdot\left(\sigma^2\psi_R^*\right)^\dagger\\ =&i\psi_L\cdot\psi_L^\dagger.\end{aligned} \tag{43}$$

$M\sigma^2$

$$M\sigma^2=\begin{pmatrix}V^0-V^3 & -V^1+iV^2\\ -V^1-iV^2 & V^0+V^3\end{pmatrix}. \tag{44}$$

4- V^μ 4-

$$\begin{aligned}T_LMT_R=&\left(1-\frac{i}{2}\theta_i\sigma^i-\frac{1}{2}\beta_i\sigma^i\right)\left(V^0-V^i\sigma^i\right)\left(1+\frac{i}{2}\theta_i\sigma^i-\frac{1}{2}\beta_i\sigma^i\right)\\ =&V^0-V^i\sigma^i-\beta_i\sigma^iV^0+\frac{i}{2}\theta_iV^j\left[\sigma^i,\sigma^j\right]+\frac{1}{2}\beta_iV^j\left\{\sigma^i,\sigma^j\right\}\\ =&V^0-V^i\sigma^i-\beta_i\sigma^iV^0-\epsilon^{ijk}\theta_iV^j\sigma^k+\beta_iV^i\\ =&\left(V^0+\beta_iV^i\right)-\left(V^i+\beta_iV^0+\epsilon^{ijk}\theta_jV^k\right)\sigma^i.\end{aligned} \tag{45}$$

$$\begin{pmatrix} 1 & \beta_1 & \beta_2 & \beta_3 \\ \beta_1 & 1 & -\theta_3 & \theta_2 \\ \beta_2 & \theta_3 & 1 & -\theta_1 \\ \beta_3 & -\theta_2 & \theta_1 & 1 \end{pmatrix} \begin{pmatrix} V^0 \\ V^1 \\ V^2 \\ V^3 \end{pmatrix} = \begin{pmatrix} V^0 + \beta_1 V^1 + \beta_2 V^2 + \beta_3 V^3 \\ V^1 + \beta_1 V^0 + \theta_2 V^3 - \theta_3 V^2 \\ V^2 + \beta_2 V^0 + \theta_3 V^1 - \theta_1 V^3 \\ V^3 + \beta_3 V^0 + \theta_1 V^2 - \theta_2 V^1 \end{pmatrix}. \tag{46}$$

V^μ 4- $\psi_L \cdot \psi_L^\dagger$

$$\begin{pmatrix} \psi_L^1 \psi_L^{*1} \psi_L^1 \psi_L^{*2} \\ \psi_L^2 \psi_L^{*1} \psi_L^2 \psi_L^{*2} \end{pmatrix} = \begin{pmatrix} V^0 - V^3 & -V^1 + iV^2 \\ -V^1 - iV^2 & V^0 + V^3 \end{pmatrix}. \tag{47}$$

$$V^\mu = i\psi_L^\dagger \bar{\sigma}^\mu \psi_L. \tag{48}$$

$$\bar{\sigma}^\mu := \left(1, -\sigma^1, -\sigma^2, -\sigma^3\right). \tag{49}$$

4-

$$V^\mu = i\psi_R^\dagger \sigma^\mu \psi_R. \tag{50}$$

$$\sigma^\mu := \left(1, \sigma^1, \sigma^2, \sigma^3\right). \tag{51}$$

Dirac ()
weyl σ^μ

$$i\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L. \tag{52}$$

Weyl Klein-Gordon $\psi_L^\dagger m \psi_L$

$$\psi_i := \epsilon_{ij} \psi^j = \left(i\sigma^2\right)_{ij} \psi^i \tag{53}$$

$$\psi_{Li} \psi_L^i = i\psi_L^T \sigma^2 \psi_L. \tag{54}$$

$$L = i\psi_L^\dagger \sigma^\mu \partial_\mu \psi_L + im\psi_L^T \sigma^2 \psi_L. \tag{55}$$

$U(1)$ $\psi_L \to e^{i\theta} \psi_L$

$$\begin{aligned} \mathcal{L}' &= \frac{1}{2}\left(\mathcal{L} + \mathcal{L}^*\right) \\ &= i\chi^\dagger \bar{\sigma}^\mu \partial_\mu \chi + \frac{i}{2} m \left(\chi^T \sigma^2 \chi - \chi^\dagger \sigma^2 \chi^*\right). \end{aligned} \tag{56}$$

ψ_L χ, Majorana Majorana Dirac 4 4

$$\mathcal{L}_L = i\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L + \frac{i}{2} m \left(\psi_L^T \sigma^2 \psi_L - \psi_L^\dagger \sigma^2 \psi_L^*\right), \tag{57}$$

$$\mathcal{L}_R = i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R + \frac{i}{2} m \left(\psi_R^T \sigma^2 \psi_R - \psi_R^\dagger \sigma^2 \psi_R^*\right). \tag{58}$$

$\psi_R \eqsim -i\sigma^2 \psi_L^*$, $\psi_L \eqsim -i\sigma^2 \psi_R^*$:

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_L + \mathcal{L}_R \\ &\eqsim i\psi_L^\dagger \sigma^\mu \partial_\mu \psi_L + i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R - m\left(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L\right). \end{aligned} \tag{59}$$

$\psi := (\psi_L, \psi_R)^T$:

$$\mathcal{L} = \psi^\dagger \begin{pmatrix} i\sigma^\mu \partial_\mu & -m \\ -m & i\sigma^\mu \partial_\mu \end{pmatrix} \psi. \tag{60}$$

Dirac

Dirac

$\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$ Weyl

$$\psi = \begin{pmatrix} \psi_L^1 \\ \psi_L^2 \\ \psi_R^1 \\ \psi_R^2 \end{pmatrix}. \tag{61}$$

$$\Lambda_{Dirac} = \begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{pmatrix}. \tag{62}$$

$\psi_R^\dagger \psi_L, \psi_L^\dagger \psi_R$

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}. \tag{63}$$

$$\bar{\psi} := \psi^\dagger \gamma^0. \tag{64}$$

$\bar{\psi}\psi$. 4-

$$\begin{aligned} V^\mu &= (\psi_L^\dagger \psi_R^\dagger) \begin{pmatrix} \bar{\sigma}^\mu & 0 \\ 0 & \sigma^\mu \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \\ &= (\psi_L^\dagger \psi_R^\dagger) \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \begin{pmatrix} \bar{\sigma}^\mu & 0 \\ 0 & \sigma^\mu \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \\ &= \bar{\psi} \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \psi. \end{aligned} \tag{65}$$

gamma

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \tag{66}$$

γ^μ 4- Dirac

$$\mathcal{L} = \bar{\psi} \left(\gamma^\mu \partial_\mu - m \right) \psi. \tag{67}$$