Fermi Gas

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1 Fourier transformation convention

We use box normalization so that both momentums and Matsubara frequency are discrete. We combine the position with time, and similarly momentum with frequency to form a four-vectors

$$\begin{aligned} x &\coloneqq (\boldsymbol{x}, \tau), \\ p &\coloneqq (\boldsymbol{p}, \omega), \end{aligned}$$

where Matsubara frequency is:

$$\omega_n = \begin{cases} 2n\pi/\beta & boson \\ (2n+1)\pi/\beta & fermion \end{cases}.$$

For convenience, we define the inner product

$$p \cdot x \coloneqq \boldsymbol{p} \cdot \boldsymbol{x} - \omega\tau.$$

The Fourier transformation of fields is

$$\begin{aligned} \psi_p &= \frac{1}{\sqrt{\beta L^d}} \int_0^\beta d\tau \int d^d x e^{-ip\cdot x} \psi_x, \\ \psi_x &= \frac{1}{\sqrt{\beta L^d}} \sum_\omega \sum_{\boldsymbol{p}} e^{+ip\cdot x} \psi_p. \end{aligned}$$

Strictly speaking, the spatial integral should be written as $\int_V d^d x$. While generally, we consider it to be "semi-infinite" - while the integral in done on the infinite space, the system can still have a macroscopic volume denoted buy V.

A slightly different normalization is taken for Fourier transformation of functions:

$$\begin{aligned} f(\boldsymbol{k}) &= \frac{1}{L^d} \int d^d x e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{x}), \\ f(\boldsymbol{x}) &= \sum_{\boldsymbol{k}} e^{+i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{x}). \end{aligned}$$

Oftentimes, we are considering a semi-continuous system, where we can always make the substitution:

$$\sum_{\boldsymbol{k}} \longrightarrow L^d \int \frac{d^d k}{(2\pi)^d},$$

and in zero-temperature limit, we can replace frequency sum by integral:

$$\sum_l \longrightarrow \beta \int \frac{d\omega}{2\pi}$$

2 Action

The action of interacting Fermi gas is:

$$
\begin{aligned}
S &= S_0 + S_1, \\
S_0 &= \int d^4x \sum_{\sigma} \bar{\psi}^{\sigma}(x) \left(\partial_{\tau} - \frac{\nabla^2}{2m} - \mu \right) \psi^{\sigma}(x), \\
S_1 &= \frac{1}{2} \int d^4x_1 \int d^4x_2 \sum_{\sigma,\sigma'} \bar{\psi}^{\sigma}(x_1) \bar{\psi}^{\sigma'}(x_2) V(\boldsymbol{x}_1 - \boldsymbol{x}_2) \psi^{\sigma'}(x_2) \psi^{\sigma}(x_1).
\end{aligned}
$$

Where the $V(\boldsymbol{r})$ is the Coulomb interaction

$$
V(\boldsymbol{r}) = \frac{e^2}{r}.
$$

In momentum space:

$$
\psi_p^{\sigma} = \sqrt{\frac{T}{L^3}} \sum_p e^{-ip\cdot x} \psi^{\sigma}(x).
$$

The action is then

$$
\begin{aligned}
S_0 &= \sum_p \sum_{\sigma} \bar{\psi}_p^{\sigma} \left(-i\omega_l + \frac{\boldsymbol{p}^2}{2m} - \mu \right) \psi_p^{\sigma}, \\
S_1 &= \frac{T}{2L^3} \sum_{p_1,p_2,k} \sum_{\sigma,\sigma'} \bar{\psi}_{p_1+k}^{\sigma} \bar{\psi}_{p_2-k}^{\sigma'} V(\boldsymbol{k}) \psi_{p_2}^{\sigma'} \psi_{p_1}^{\sigma}.
\end{aligned}
$$

The $V(\boldsymbol{q})$ is the Fourier transformed interaction:

$$
\begin{aligned}
V(\boldsymbol{k}) &= \lim_{\alpha\to 0} e^2 \int d^3r \frac{e^{-i\boldsymbol{k}\cdot\boldsymbol{r}-\alpha r}}{r} \\
&= \lim_{\alpha\to 0} e^2 \int_0^{+\infty} r^2 dr \int_{-1}^{+1} d(\cos\theta) \int_0^{2\pi} d\theta \frac{e^{-ikr\cos\theta-\alpha r}}{r} \\
&= \lim_{\alpha\to 0} 2\pi e^2 \int_0^{+\infty} dr \cdot r e^{-\alpha r} \frac{e^{-ikr} - e^{ikr}}{-iqr} \\
&= \lim_{\alpha\to 0} \frac{2\pi e^2}{ik} \int_0^{+\infty} dr \left(e^{(ik-\alpha)r} - e^{(-ikr-\alpha)} \right) \\
&= \lim_{\alpha\to 0} \frac{2\pi e^2}{ik} \left[\frac{-1}{ik-\alpha} - \frac{-1}{-ik-\alpha} \right] \\
&= \lim_{\alpha\to 0} \frac{4\pi e^2}{k^2+\alpha^2} \\
&= \frac{4\pi e^2}{k^2}.
\end{aligned}
$$

The exponential of action can then be expanded in perturbative series:

$$
\begin{aligned}
e^{-S} &= e^{-S_0} e^{-S_1} \\
&= e^{-S_0} \cdot \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} S_1^n.
\end{aligned}
$$

3 Gaussian integral and frequency summation

3.1 Grassmann variable

Grassmann number has the defining properties:

$$\psi_1\psi_2 = -\psi_2\psi_1,$$

$$f(\psi) = 1 + f^{'}(0)\,\psi,$$
$$\int d\psi = 0,$$
$$\int d\psi\psi = \partial_\psi \psi = 1.$$

With those relations, the Gaussian form is actually a quadratic form:

$$e^{-a\bar{\psi}\psi} = 1 - a\bar{\psi}\psi,$$

so the Gaussian integral is

$$\begin{aligned}
\int d\bar{\psi}d\psi e^{-\bar{\psi}a\psi} &= \int d\bar{\psi}d\psi\left(1 - a\bar{\psi}\psi\right)\\
&= a\int d\bar{\psi}d\psi\psi\bar{\psi}\\
&= a,
\end{aligned}$$

$$\int d\bar{\psi}d\psi e^{-\bar{\psi}a\psi + \bar{u}\psi + \bar{\psi}v} = ae^{\bar{u}v},$$

$$\int d\bar{\boldsymbol{\psi}}d\boldsymbol{\psi}e^{-\bar{\boldsymbol{\psi}}^T\boldsymbol{A}\boldsymbol{\psi} + \bar{\boldsymbol{u}}^T\cdot\boldsymbol{\psi} + \bar{\boldsymbol{\psi}}^T\cdot\boldsymbol{v}} = \det\boldsymbol{A}e^{\bar{\boldsymbol{u}}^T\boldsymbol{A}^{-1}\boldsymbol{v}}.$$

3.2 Wick's theorem

We define an averaging by Gaussian integral:

$$\langle\cdots\rangle \coloneqq \frac{\int D\left[\bar{\psi},\psi\right](\cdots)\exp\left(-\bar{\boldsymbol{\psi}}^T\boldsymbol{A}\boldsymbol{\psi}\right)}{\int D\left[\bar{\psi},\psi\right]\exp\left(-\bar{\boldsymbol{\psi}}^T\boldsymbol{A}\boldsymbol{\psi}\right)},$$

where the measure is

$$D\left[\psi^\dagger,\psi\right] \coloneqq \prod_i d\psi_i^* d\psi_i$$

Wick's theorem is used to calculate the $2n$ point correlation function:

$$\left\langle\psi_{i_1}\cdots\psi_{i_n}\bar{\psi}_{j_1}\cdots\bar{\psi}_{j_n}\right\rangle$$

The corresponding generating function is

$$\begin{aligned}
Z\left[\boldsymbol{u},\boldsymbol{v}\right] &\coloneqq \frac{\int D\left[\bar{\psi},\psi\right]\exp\left(-\bar{\boldsymbol{\psi}}^T\boldsymbol{A}\boldsymbol{\psi} + \bar{\boldsymbol{u}}^T\cdot\boldsymbol{\psi} + \bar{\boldsymbol{\psi}}^T\cdot\boldsymbol{v}\right)}{\int D\left[\bar{\psi},\psi\right]\exp\left(-\bar{\boldsymbol{\psi}}^T\boldsymbol{A}\boldsymbol{\psi}\right)}\\
&= \exp\left(\bar{\boldsymbol{u}}^T\boldsymbol{A}^{-1}\boldsymbol{v}\right)\\
&= \prod_{i,j}\exp\left[\bar{u}_i\left(\boldsymbol{A}^{-1}\right)_{ij}v_j\right].
\end{aligned}$$

The correlation function can be derived by differentiating the generating function:

$$\begin{aligned}\left\langle \psi_{i_1}\cdots\psi_{i_n}\bar{\psi}_{j_1}\cdots\bar{\psi}_{j_n}\right\rangle &= \left.\frac{\partial^{2n}F\left[\bar{\boldsymbol{u}},\boldsymbol{v}\right]}{\partial\bar{u}_{i_1}\cdots\partial\bar{u}_{i_n}\partial v_{j_1}\cdots\partial v_{j_n}}\right|_{\bar{\boldsymbol{u}}=\boldsymbol{v}=0}\\ &= \left.\frac{\partial^{n}}{\partial\bar{u}_{i_1}\cdots\partial\bar{u}_{i_n}}\prod_{mn}\bar{u}_m\left(\boldsymbol{A}^{-1}\right)_{mj_n}\right|_{\boldsymbol{u}=0}\\ &= \sum_P signP\cdot\left(\boldsymbol{A}^{-1}\right)_{i_1 j_{P1}}\cdots\left(\boldsymbol{A}^{-1}\right)_{i_n j_{pn}}\end{aligned}$$

3.3 Matsubara summation

Use fermion distribution function

$$n_F\left(z\right)\coloneqq\frac{1}{\exp\left(\beta z\right)+1}$$

as auxiliary function. The summation $\sum_l f\left(\omega_l\right)$ could be evaluated by a loop integral:

$$\sum_l f\left(\omega_l\right)=-\beta\oint\frac{dz}{2\pi i}n_F\left(z\right)f\left(i\omega_l\to z\right).$$

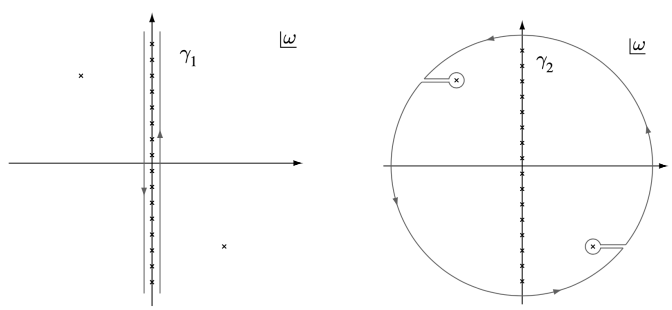


Figure 3.1: Integration contour

If function $f\left(i\omega_l\to z\right)$ has simple poles, the integral can be evaluated using residue theorem:

$$\sum_l f\left(\omega_l\right)=\beta\sum_{z_k}n_F\left(z_k\right)\,Res\left[f\left(i\omega_l\to z\right)\right]\big|_{z=z_k}.$$

4 Free Fermi gas

The unperturbed action is:

$$S_0=\sum_p\sum_\sigma\bar{\psi}_p^\sigma\left(-i\omega_l+\frac{\boldsymbol{p}^2}{2m}-\mu\right)\psi_p^\sigma.$$

The free field average is:

$$\left\langle\cdots\right\rangle_0=\frac{\int D\left[\bar{\psi},\psi\right]\left(\cdots\right)\exp\left(-S_0\right)}{\int D\left[\bar{\psi},\psi\right]\exp\left(-S_0\right)}.$$

4.1 Green function

The Green function of the quadratic field action is:

$$G^{(0)}_{\sigma\sigma'}(p) = \frac{\delta_{\sigma\sigma'}}{i\omega_l - \frac{\boldsymbol{p}^2}{2m} + \mu}.$$

Sum the Matsubara frequency:

$$\begin{aligned}
G^{(0)}_{\sigma\sigma'}(\boldsymbol{p}) &= \sum_l \frac{\delta_{\sigma\sigma'}}{i\omega_l - \frac{\boldsymbol{p}^2}{2m} + \mu} \\
&= \sum_{z_k} \frac{\beta\delta_{\sigma\sigma'}}{\exp(\beta z_k) + 1}\, Res\left[\frac{1}{z - \frac{\boldsymbol{p}^2}{2m} + \mu}\right]\Bigg|_{z=z_k} \\
&= \frac{\beta\delta_{\sigma\sigma'}}{e^{\beta(\epsilon_{\boldsymbol{p}} - \mu)} + 1} \\
&= \delta_{\sigma\sigma'}\beta n_F(\epsilon_{\boldsymbol{p}} - \mu).
\end{aligned}$$

4.2 Free energy

The partition function

$$\mathcal{Z}^{(0)} = \prod_{p,\sigma}\left(-i\omega_l + \frac{\boldsymbol{p}^2}{2m} - \mu\right)$$

gives the free energy:

$$\begin{aligned}
\mathcal{F}^{(0)} &= -T\ln\mathcal{Z} \\
&= -2T\sum_{\boldsymbol{p}}\sum_l \ln\left(-i\omega_l + \frac{\boldsymbol{p}^2}{2m} - \mu\right).
\end{aligned}$$

The Matsubara summation is done by integrating over a path circumventing the branch cut:

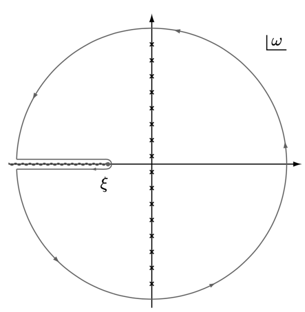


Figure 4.1: Integration contour

$$\begin{aligned}
\sum_l \ln\left(-i\omega_l + \frac{\boldsymbol{p}^2}{2m} - \mu\right) &= -\beta\int_{-\infty}^{+\infty}\frac{dz}{2\pi i} n_F(z)\ln\left(\frac{z^+ - \frac{\boldsymbol{p}^2}{2m} + \mu}{z^- - \frac{\boldsymbol{k}^2}{2m} + \mu}\right) \\
&\overset{b.p.}{\Longrightarrow} -\int_{-\infty}^{+\infty}\frac{dz}{2\pi i}\ln\left(1 + e^{-\beta z}\right)\left[\frac{1}{z - \frac{\boldsymbol{p}^2}{2m} + \mu + i\eta} - \frac{1}{z - \frac{\boldsymbol{p}^2}{2m} + \mu - i\eta}\right] \\
&= \ln\left(1 + e^{-\beta(\epsilon_{\boldsymbol{p}} - \mu)}\right),
\end{aligned}$$

where we have used the identity:

$$\lim_{\eta\to 0^+}\frac{1}{x\pm i\eta}=\mp i\pi\delta\left(x\right)+\mathcal{P}\left(\frac{1}{x}\right).$$

So the free energy is:

$$\mathcal{F}^{(0)}=2T\sum_{\boldsymbol{p}}\ln\left(1+e^{-\beta\left(\epsilon_{\boldsymbol{p}}-\mu\right)}\right).$$

In low-T limit:

$$\begin{aligned}
\mathcal{F}^{(0)} &\approx 2\sum_{p<k_F}\left(\frac{\boldsymbol{p}^2}{2m}-\mu\right)\\
&= \frac{2L^3}{\left(2\pi\right)^3}\cdot 4\pi\cdot\int_0^{k_F}dp\cdot p^2\cdot\frac{p^2-k_F^2}{2m}\\
&= -\frac{2L^3}{\left(2\pi\right)^3}\cdot 4\pi\cdot\frac{k_F^5}{15m}\\
&= -\frac{L^3k_F^5}{15\pi^2 m}
\end{aligned}$$

Note that:

$$\begin{aligned}
N &= \sum_{p<k_F}\sum_{\sigma}1\\
&= \frac{2L^3}{\left(2\pi\right)^3}\cdot\frac{4\pi}{3}k_F^3.
\end{aligned}$$

So the free energy can be further simplified to:

$$\begin{aligned}
\mathcal{F}^{(0)} &\approx -\frac{2L^3}{\left(2\pi\right)^3}\cdot\frac{4\pi}{3}k_F^3\cdot\frac{k_F^2}{5m}\\
&= -\frac{2}{5}N\mu.
\end{aligned}$$

5 Free energy RPA

The functional integral to the first order is:

$$\int D\left[\bar{\psi},\psi\right]e^{-S}\overset{1^{st}}{\Longrightarrow}\int D\left[\bar{\psi},\psi\right]\left(-S_1\right)e^{-S_0}.$$

Now consider the Feynman rule for free energy

$$\mathcal{F}=-T\ln\mathcal{Z}$$

By some combinatory relation, the sum of all bubble graphs can be decomposed into the product of the summation of connected bubble graphs, divided by permutation number $n!$:

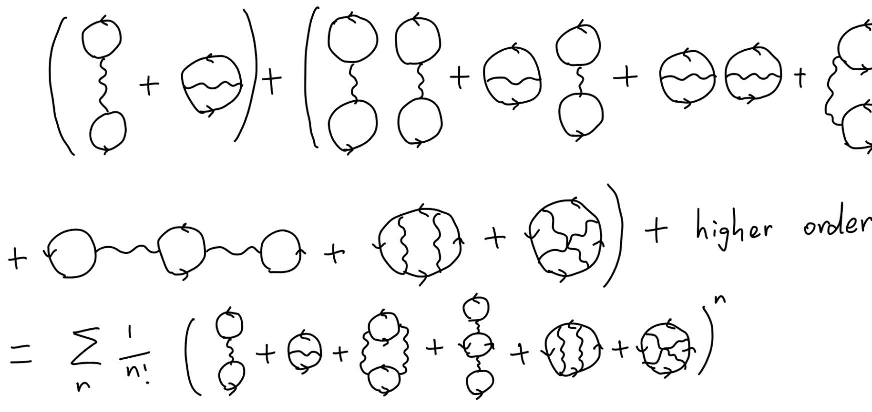


Figure 5.1: Cluster decomposition

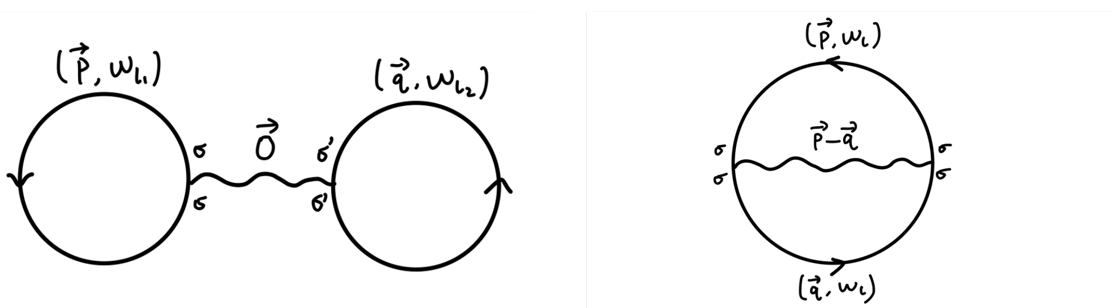
$$
\begin{aligned}
\mathcal{Z} &= \mathcal{Z}_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \langle S_1^n \rangle_0 \\
&= \mathcal{Z}_0 \prod_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \langle S_1^m \rangle_0^c \right)^n \\
&= \mathcal{Z}_0 \exp \left(\sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \langle S_1^m \rangle_0^c \right).
\end{aligned}
$$

The perturbative expansion of free energy is then:

$$
\mathcal{F} = \mathcal{F}^{(0)} - T \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \langle S_1^m \rangle_0^c .
$$

5.1 First order

To the first order:



(a) Hatree term.

(b) Fock term.

$$
\begin{aligned}
\mathcal{F}^{(1)} &= \frac{T^2}{2L^3} \sum_{p_1,p_2,k} \sum_{\sigma,\sigma'} \left\langle \bar{\psi}^{\sigma}_{p_1+k} \bar{\psi}^{\sigma'}_{p_2-k} V(\boldsymbol{k}) \psi^{\sigma'}_{p_2} \psi^{\sigma}_{p_1} \right\rangle_0^c \\
&= \mathcal{F}^{(1)}_{Hatree} + \mathcal{F}^{(1)}_{Fock},
\end{aligned}
$$

where the Hartree and Fock term is:

$$\begin{aligned}\mathcal{F}^{(1)}_{Hatree} &= \frac{2T^2}{L^3}\sum_{p_1,p_2} G^{(0)}_{p_1}G^{(0)}_{p_2}V\left(\mathbf{0}\right)\\ &= 0,\end{aligned}$$

$$\begin{aligned}\mathcal{F}^{(1)}_{Fock} &= -\frac{T^2}{L^3}\sum_{p_1,p_2} G^{(0)}_{p_2}G^{(0)}_{p_1}V\left(\boldsymbol{p}_1-\boldsymbol{p}_2\right)\\ &= -\frac{1}{L^3}\sum_{\boldsymbol{p}_1,\boldsymbol{p}_2} n_F\left(\epsilon_{\boldsymbol{p}_1}-\mu\right)n_F\left(\epsilon_{\boldsymbol{p}_2}-\mu\right)\frac{4\pi e^2}{\left|\boldsymbol{p}_1-\boldsymbol{p}_2\right|^2}\\ &= -L^3\int\frac{d^3p_1}{(2\pi)^3}\int\frac{d^3p_2}{(2\pi)^3}n_F\left(\epsilon_{\boldsymbol{p}_1}-\mu\right)n_F\left(\epsilon_{\boldsymbol{p}_2}-\mu\right)\frac{4\pi e^2}{\left|\boldsymbol{p}_1-\boldsymbol{p}_2\right|^2}.\end{aligned}$$

In low-T limit:

$$\mathcal{F}^{(1)}_{Fock}\approx -\frac{2e^2L^3}{(2\pi)^5}\int_{p,q<k_F}\frac{d^3pd^3q}{\left|\boldsymbol{p}-\boldsymbol{q}\right|^2}.$$

The integral is:

$$\begin{aligned}I &= \int_{p,q<k_F} d^3\boldsymbol{p}d^3\boldsymbol{q}\frac{1}{\left|\boldsymbol{p}-\boldsymbol{q}\right|^2}\\ &= 4\pi\int_0^{k_F}p^2dp\int_0^{k_F}q^2dq\int_{-1}^{+1}d\left(\cos\theta\right)\int_0^{2\pi}d\theta\frac{1}{p^2+q^2-2pq\cos\theta}\\ &= 8\pi^2\int_0^{k_F}p^2dp\int_0^{k_F}q^2dq\int_{-1}^{+1}\frac{d\left(p^2+q^2-2pq\cos\theta\right)}{-2pq}\frac{1}{p^2+q^2-2pq\cos\theta}\\ &= 8\pi^2\int_0^{k_F}p^2dp\int_0^{k_F}q^2dq\frac{1}{pq}\ln\left|\frac{p+q}{p-q}\right|\\ &= 16\pi^2\int_0^{k_F}p^3dp\int_0^1 dt\left[t\ln\left(\frac{1+t}{1-t}\right)\right]\\ &= 16\pi^2\int_0^{k_F}p^3dp\left[\int_0^1 dx\left(x-1\right)\ln x+\int_1^2 dx\left(x-1\right)\ln x\right]\\ &= 16\pi^2\int_0^{k_F}p^3dp\left[\frac{t^2\ln t}{2}-\frac{t^2}{4}+t-t\ln t\right]\Big|_0^2\\ &= 16\pi^2\int_0^{k_F}p^3dp\\ &= (2\pi)^2k_F^4.\end{aligned}$$

So the free energy to the first order is:

$$\mathcal{F}^{(1)}=\mathcal{F}^{(1)}_{Fock}=-\frac{2e^2L^3k_F^4}{(2\pi)^3}.$$

5.2 Random phase approximation

The largest perturbative contribution is the ring graphs:

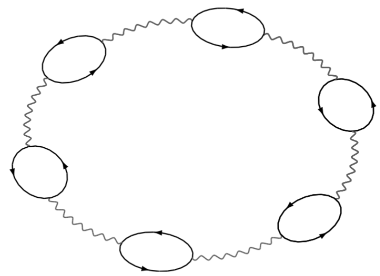


Figure 5.2: Ring graph

$$F^{(n)}_{RPA} = -\frac{T}{2n}\sum_q \left(\frac{2T}{L^3}\sum_p G^{(0)}_p G^{(0)}_{p+q}\right)^n,$$

where $2n$ is the symmetry factor. To evaluate the n^{th} order free energy, we first calculate the polarization operator:

$$\begin{aligned}
\Pi_q &:= \frac{2T}{L^3}\sum_{\boldsymbol{p}}\sum_{\omega_p} G^{(0)}_p G^{(0)}_{p+q} \\
&= \frac{2T}{L^3}\sum_{\boldsymbol{p}}\sum_{\omega_p} \frac{1}{i\omega_p - \epsilon_{\boldsymbol{p}} + \mu}\cdot\frac{1}{i\left(\omega_p+\omega_q\right) - \epsilon_{\boldsymbol{p}+\boldsymbol{q}} + \mu} \\
&= \frac{2}{L^3}\sum_{\boldsymbol{p}} \frac{n_F\left(\epsilon_{\boldsymbol{p}+\boldsymbol{q}}-\mu\right) - n_F\left(\epsilon_{\boldsymbol{p}}-\mu\right)}{-i\omega_q + \epsilon_{\boldsymbol{p}+\boldsymbol{q}} - \epsilon_{\boldsymbol{p}}} \\
&= \frac{2}{\left(2\pi\right)^3}\int d^3p \frac{n_F\left(\epsilon_{\boldsymbol{p}+\boldsymbol{q}}-\mu\right) - n_F\left(\epsilon_{\boldsymbol{p}}-\mu\right)}{-i\omega_q + \epsilon_{\boldsymbol{p}+\boldsymbol{q}} - \epsilon_{\boldsymbol{p}}}.
\end{aligned}$$

To evaluate the integration, first note that for small $\boldsymbol{q}$ and low temperature, we can linearize the energy and distribution difference:

$$\begin{aligned}
\epsilon_{\boldsymbol{p}+\boldsymbol{q}} - \epsilon_{\boldsymbol{p}} &\approx \frac{\boldsymbol{p}\cdot\boldsymbol{q}}{m}, \\
n_F\left(\epsilon_{\boldsymbol{p}+\boldsymbol{q}}-\mu\right) - n_F\left(\epsilon_{\boldsymbol{p}}-\mu\right) &\approx \frac{\partial n_F}{\partial \epsilon}\left(\epsilon_{\boldsymbol{p}}-\mu\right)\cdot\frac{\boldsymbol{p}\cdot\boldsymbol{q}}{m} \\
&\approx -\delta^{(3)}\left(\epsilon_{\boldsymbol{p}}-\mu\right)\cdot\frac{\boldsymbol{p}\cdot\boldsymbol{q}}{m}.
\end{aligned}$$

So the integration becomes:

$$\begin{aligned}
\Pi_q &\approx \frac{-2}{\left(2\pi\right)^3}\int p_F^2 dp\,\delta\left(\frac{p^2}{2m}-\mu\right)\int_{-1}^{+1} d\left(\cos\theta\right)\int_0^{2\pi} d\phi \frac{p_F q\cos\theta/m}{-i\omega_q + p_F q\cos\theta/m} \\
&= -\frac{1}{2\pi^2}\int p_F^2 dp\,\delta\left(\frac{p^2}{2m}-\mu\right)\int_{-1}^{+1} d\left(\cos\theta\right)\frac{v_F q\cos\theta}{-i\omega_q + v_F q\cos\theta} \\
&= -\frac{1}{2\pi^2}\int p_F^2 dp\,\delta\left(\frac{p^2}{2m}-\mu\right)\left[1 + \frac{i\omega_q}{2v_F q}\ln\left(\frac{i\omega_q + v_F q}{i\omega_q - v_F q}\right)\right] \\
&= -\frac{m p_F}{\pi^2}\left[1 - \frac{i\omega_q}{2v_F q}\ln\left(\frac{i\omega_q + v_F q}{i\omega_q - v_F q}\right)\right].
\end{aligned}$$

where we have used

$$\delta\left(f\left(z\right)\right)=\sum_{z_k}\frac{\delta\left(z-z_k\right)}{\left|f'\left(z_k\right)\right|}.$$

6 Self energy

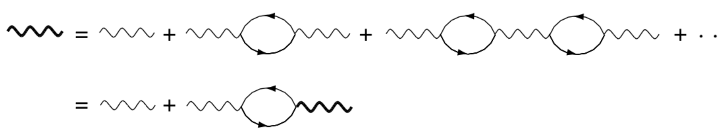


Figure 6.1: Photon self energy

$$\begin{aligned}V_{eff}\left(q\right) &= V\left(\boldsymbol{q}\right)+V\left(\boldsymbol{q}\right)\Pi_q V_{eff}\left(q\right)\\ &= \frac{V\left(\boldsymbol{q}\right)}{1-V\left(\boldsymbol{q}\right)\Pi_q}\\ &=: \frac{V\left(\boldsymbol{q}\right)}{\epsilon\left(q\right)},\end{aligned}$$

where we have defined a dielectric function:

$$\epsilon\left(q\right):=1-V\left(\boldsymbol{q}\right)\Pi_q.$$

For small $\boldsymbol{q}$ and ω_q :

$$\omega_q\ll qv_F,$$

$$\Pi\left(\boldsymbol{q},\omega_q\right)\approx-\frac{mp_F}{\pi^2}=:-\nu_0.$$

In this limit:

$$V\left(q\right)\approx\frac{1}{V^{-1}\left(\boldsymbol{q}\right)+\nu_0}=\frac{4\pi e^2}{q^2+4\pi e^2\nu_0}=:\frac{4\pi e^2}{q^2+\lambda^{-2}}.$$

The $\lambda:=\left(4\pi e^2\nu_0\right)^{-1/2}$ is the Thomas-Fermi screening length. To see the physical meaning, do the inverse Fourier transformation:

$$\begin{aligned}V\left(r\right) &= \int\frac{d^3q}{\left(2\pi\right)^3}e^{i\boldsymbol{q}\cdot\boldsymbol{r}}\frac{4\pi e^2}{q^2+\lambda^{-2}}\\ &= \frac{e^2}{\pi}\int_0^{+\infty}q^2dq\int_{-1}^{+1}d\left(\cos\theta\right)\frac{e^{iqr\cos\theta}}{q^2+\lambda^{-2}}\\ &= \frac{e^2}{i\pi r}\int_0^{+\infty}dq\frac{q}{q^2+\lambda^{-2}}\left(e^{+iqr}-e^{-iqr}\right)\\ &= \frac{e^2}{i\pi r}\int_{-\infty}^{+\infty}dq\frac{q}{q^2+\lambda^{-2}}e^{+iqr}.\end{aligned}$$

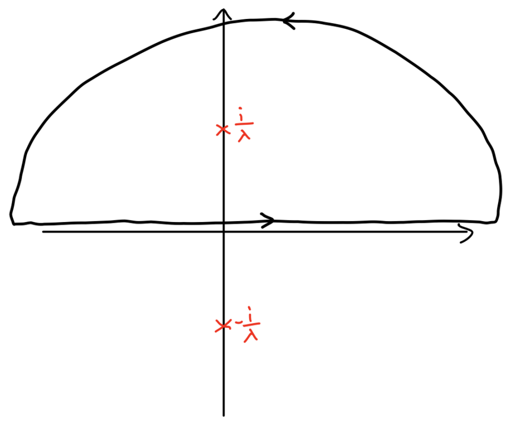


Figure 6.2: Integration contour

Apply residue theorem to get the result:

$$Res\left[\frac{q}{q^2+\lambda^{-2}}e^{+iqr}\right]\bigg|_{q=\frac{i}{\lambda}}=\frac{1}{2}e^{-\frac{r}{\lambda}},$$

$$V\left(r\right)=\frac{e^2}{r}e^{-\frac{r}{\lambda}}.$$