费米气体 RPA

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傅立叶变换系数

对平移不变的场而言，物理量计算往往在动量空间中完成。为避免混淆傅立叶变换系数，我们在此明确函数和场的傅立叶变换，这里的规定和 Altland & Simons, *Condensed Matter Field Theory* 书中的规定是一样的。

场的傅立叶变换系数规定为：

$$\psi_p = \frac{1}{\sqrt{\beta L^3}} \int d^4x \; e^{-ip\cdot x} \psi(x), \tag{1}$$

$$\psi(x) = \frac{1}{\sqrt{\beta L^3}} \sum_p e^{+ip\cdot x} \psi_p. \tag{2}$$

其中记 4 坐标和 4 动量为：

$$x \equiv (\boldsymbol{r}, \tau), \tag{3}$$

$$p \equiv (\boldsymbol{p}, \omega_n), \tag{4}$$

$$p\cdot x \equiv \boldsymbol{p}\cdot\boldsymbol{r} - \omega_n \tau. \tag{5}$$

坐标积分和动量求和为：

$$\int d^4x \equiv \int_0^\beta d\tau \int d^3r, \tag{6}$$

$$\sum_p \equiv \sum_{\omega_n} \sum_{\boldsymbol{p}} \tag{7}$$

函数的傅立叶变换采用另一种归一化：

$$f_{\boldsymbol{q}} = \int d^3r \; e^{-i\boldsymbol{q}\cdot\boldsymbol{r}} f(\boldsymbol{r}), \tag{8}$$

$$f(\boldsymbol{r}) = \frac{1}{L^3} \sum_{\boldsymbol{q}} e^{+i\boldsymbol{q}\cdot\boldsymbol{r}} f_{\boldsymbol{q}} \tag{9}$$

许多时候，我们会取热力学极限：$L \to \infty$，此时动量求和变为积分：

$$\frac{1}{L^3} \sum_{\boldsymbol{p}} \longrightarrow \int \frac{d^3p}{(2\pi)^3}, \tag{10}$$

作用量

相互作用费米气体的作用量为:

$$
\begin{aligned}
S &= S_0 + S_1, &(11)\\
S_0 &= \int d^4x \bar{\psi}_\sigma(x)\left(\partial_\tau - \frac{\nabla^2}{2m} - \mu\right)\psi_\sigma(x), &(12)\\
S_1 &= \frac{1}{2}\int d^4x_1 \int d^4x_2 \bar{\psi}_\sigma(x_1)\bar{\psi}_{\sigma'}(x_2) V(\boldsymbol{r}_1 - \boldsymbol{r}_2)\psi_{\sigma'}(x_2)\psi_\sigma(x_1). &(13)
\end{aligned}
$$

其中 $V(\boldsymbol{r})$ 是库伦相互作用:

$$
V(\boldsymbol{r}) = \frac{e^2}{r}. \tag{14}
$$

在动量空间中，作用量表达为求和式:

$$
\begin{aligned}
S_0 &= \sum_p \bar{\psi}_{p,\sigma}\left(-i\omega_n + \frac{\boldsymbol{p}^2}{2m} - \mu\right)\psi_{p,\sigma}, &(15)\\
S_1 &= \frac{T}{2L^3}\sum_{p_1,p_2,q}\bar{\psi}_{p_1+q,\sigma}\bar{\psi}_{p_2-q,\sigma'}V(\boldsymbol{q})\psi_{p_2,\sigma'}\psi_{p_1,\sigma}. &(16)
\end{aligned}
$$

其中 $V(\boldsymbol{q})$ 是库伦势的傅立叶变换:

$$
\begin{aligned}
V(\boldsymbol{q}) &= \lim_{\alpha\to 0} e^2 \int d^3r \frac{e^{-i\boldsymbol{q}\cdot\boldsymbol{r}-\alpha r}}{r} &(17)\\
&= \lim_{\alpha\to 0} e^2 \int_0^{+\infty} r^2 dr \int_{-1}^{+1} d(\cos\theta) \int_0^{2\pi} d\theta \frac{e^{-iqr\cos\theta-\alpha r}}{r} &(18)\\
&= \lim_{\alpha\to 0} 2\pi e^2 \int_0^{+\infty} dr \cdot re^{-\alpha r}\frac{e^{-iqr} - e^{iqr}}{-iqr} &(19)\\
&= \lim_{\alpha\to 0} \frac{2\pi e^2}{iq}\int_0^{+\infty} dr\left(e^{iqr-\alpha r} - e^{-iqr-\alpha r}\right) &(20)\\
&= \lim_{\alpha\to 0} \frac{2\pi e^2}{iq}\left[\frac{-1}{iq-\alpha} - \frac{-1}{-iq-\alpha}\right] &(21)\\
&= \lim_{\alpha\to 0} \frac{4\pi e^2}{q^2+\alpha^2} &(22)\\
&= \frac{4\pi e^2}{q^2}. &(23)
\end{aligned}
$$

作用量的指数做级数展开为:

$$
e^{-S} = e^{-S_0}e^{-S_1} = e^{-S_0}\cdot\sum_{n=1}^{+\infty}\frac{(-1)^n}{n!}S_1^n. \tag{24}
$$

高斯积分与频率求和

Grassmann 代数

Grassmann 数满足性质：

$$\begin{aligned}
\psi_1\psi_2 &= -\psi_2\psi_1, &(25)\\
f(\psi) &= 1+f'(0)\psi, &(26)\\
\int d\psi &= 0, &(27)\\
\int \psi d\psi &= 1. &(28)
\end{aligned}$$

由此，Grassmann 高斯型实际上是二次型：

$$e^{-a\bar{\psi}\psi} = 1 - a\bar{\psi}\psi. \tag{29}$$

因此高斯积分为：

$$\begin{aligned}
\int d\bar{\psi}d\psi e^{-\bar{\psi}a\psi} &= \int d\bar{\psi}d\psi\left(1-a\bar{\psi}\psi\right) &(30)\\
&= a\int d\bar{\psi}d\psi\psi\bar{\psi} &(31)\\
&= a, &(32)
\end{aligned}$$

多变量情形为：

$$\int d\bar{\psi}d\psi e^{-\bar{\psi}a\psi+\bar{u}\psi+\bar{\psi}v} = ae^{\bar{u}v}, \tag{33}$$

$$\int d\bar{\boldsymbol{\psi}}d\boldsymbol{\psi}e^{-\bar{\boldsymbol{\psi}}^T\boldsymbol{A}\boldsymbol{\psi}+\bar{\boldsymbol{u}}^T\cdot\boldsymbol{\psi}+\bar{\boldsymbol{\psi}}^T\cdot\boldsymbol{v}} = \det\boldsymbol{A}e^{\bar{\boldsymbol{u}}^T\boldsymbol{A}^{-1}\boldsymbol{v}}. \tag{34}$$

Wick 定理

泛函平均值 $\langle\cdots\rangle$ 定义为：

$$\langle\cdots\rangle \equiv \frac{\int D\left[\bar{\psi},\psi\right](\cdots)\exp\left(-\bar{\boldsymbol{\psi}}^T\boldsymbol{A}\boldsymbol{\psi}\right)}{\int D\left[\bar{\psi},\psi\right]\exp\left(-\bar{\boldsymbol{\psi}}^T\boldsymbol{A}\boldsymbol{\psi}\right)}, \tag{35}$$

其中积分测度为：

$$D\left[\psi^\dagger,\psi\right] \equiv \prod_i d\psi_i^* d\psi_i. \tag{36}$$

Wick 定理可用于计算 $2n$ 点关联函数：

$$\left\langle\psi_{i_1}\cdots\psi_{i_n}\bar{\psi}_{j_1}\cdots\bar{\psi}_{j_n}\right\rangle. \tag{37}$$

为计算此关联函数，引入生成泛函：

$$\begin{aligned}
Z\left[\boldsymbol{u},\boldsymbol{v}\right] &\equiv \frac{\int D\left[\bar{\psi},\psi\right]\exp\left(-\bar{\boldsymbol{\psi}}^T\boldsymbol{A}\boldsymbol{\psi}+\bar{\boldsymbol{u}}^T\cdot\boldsymbol{\psi}+\bar{\boldsymbol{\psi}}^T\cdot\boldsymbol{v}\right)}{\int D\left[\bar{\psi},\psi\right]\exp\left(-\bar{\boldsymbol{\psi}}^T\boldsymbol{A}\boldsymbol{\psi}\right)} &(38)\\
&= \exp\left(\bar{\boldsymbol{u}}^T\boldsymbol{A}^{-1}\boldsymbol{v}\right) &(39)\\
&= \prod_{i,j}\exp\left[\bar{u}_i\left(\boldsymbol{A}^{-1}\right)_{ij}v_j\right]. &(40)
\end{aligned}$$

关联函数可由对生成泛函求若干偏导得到：

$$\left\langle \psi_{i_1}\cdots\psi_{i_n}\bar{\psi}_{j_1}\cdots\bar{\psi}_{j_n}\right\rangle = \left.\frac{\partial^{2n}F\left[\bar{\boldsymbol{u}},\boldsymbol{v}\right]}{\partial\bar{u}_{i_1}\cdots\partial\bar{u}_{i_n}\partial v_{j_1}\cdots\partial v_{j_n}}\right|_{\bar{\boldsymbol{u}}=\boldsymbol{v}=0} \tag{41}$$

$$= \left.\frac{\partial^{n}}{\partial\bar{u}_{i_1}\cdots\partial\bar{u}_{i_n}}\prod_{mn}\bar{u}_m\left(\boldsymbol{A}^{-1}\right)_{mj_n}\right|_{\boldsymbol{u}=0} \tag{42}$$

$$= \sum_P signP\cdot\left(\boldsymbol{A}^{-1}\right)_{i_1j_{P1}}\cdots\left(\boldsymbol{A}^{-1}\right)_{i_nj_{pn}} \tag{43}$$

松原频率求和

利用费米分布函数

$$n_F\left(z\right)\equiv\frac{1}{\exp\left(\beta z\right)+1}, \tag{44}$$

频率求和 $\frac{1}{\beta}\sum_l f\left(\omega_l\right)$ 可以转化为复平面内的回路积分：

$$\frac{1}{\beta}\sum_l f\left(\omega_l\right)=-\oint\frac{dz}{2\pi i}n_F\left(z\right)f\left(i\omega_l\to z\right). \tag{45}$$

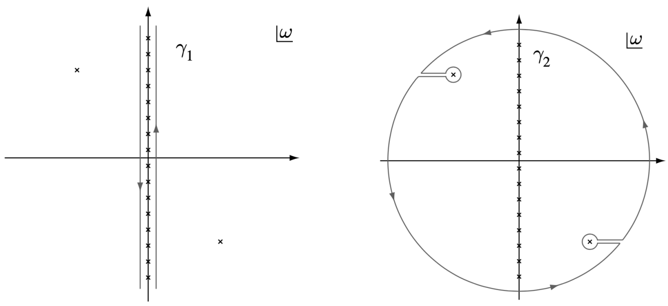


图 1: 积分回路

如果函数 $f\left(i\omega_l\to z\right)$ 在 $z\to\infty$ 快速趋于 0(快于 z^{-1})，且在复平面有极点，则利用留数定理，回路积分进一步化为：

$$\frac{1}{\beta}\sum_l f\left(\omega_l\right)=\sum_{z_k}\mathrm{Res}\left[n_F(z)f\left(i\omega_l\to z\right)\right]_{z=z_k}. \tag{46}$$

自由费米气体

自由场作用量为：

$$S_0=\sum_p\sum_\sigma\bar{\psi}_p^\sigma\left(-i\omega_l+\frac{\boldsymbol{p}^2}{2m}-\mu\right)\psi_p^\sigma. \tag{47}$$

自由场的泛函平均为：

$$\left\langle\cdots\right\rangle_0=\frac{\int D\left[\bar{\psi},\psi\right]\left(\cdots\right)\exp\left(-S_0\right)}{\int D\left[\bar{\psi},\psi\right]\exp\left(-S_0\right)}. \tag{48}$$

格林函数

自由费米气体的格林函数为

$$G_p=\left\langle\psi_p\bar{\psi}_p\right\rangle=\frac{1}{-i\omega_n+\xi_{\boldsymbol{p}}}, \tag{49}$$

其中 $\xi_{\boldsymbol{p}}=\frac{\boldsymbol{p}^2}{2m}-\mu$. 对频率求和的结果为:

$$
\begin{aligned}
\frac{1}{\beta}\sum_n G_p &= \frac{1}{\beta}\sum_n \frac{1}{-i\omega_n+\xi_{\boldsymbol{p}}} && (50)\\
&= -\sum_{z_k} \mathrm{Res}\left[\frac{1}{e^{\beta z}+1}\frac{1}{z-\xi_{\boldsymbol{p}}}\right]\Bigg|_{z=z_k} && (51)\\
&= -n_F(\xi_{\boldsymbol{p}}). && (52)
\end{aligned}
$$

自由能

配分函数为:

$$
Z^{(0)}=\prod_{p,\sigma}\left[\beta(-i\omega_l+\xi_{\boldsymbol{p}})\right]. \tag{53}
$$

自由能为

$$
\begin{aligned}
F^{(0)} &= -T\ln Z^{(0)} && (54)\\
&= -2T\sum_p \ln\left[\beta(-i\omega_l+\xi_{\boldsymbol{p}})\right]. && (55)
\end{aligned}
$$

松原频率求和可转换为复平面内回路积分。ln 函数在复平面上有割线，回路转换为绕割线的积分，也就是实轴上下部分的积分:

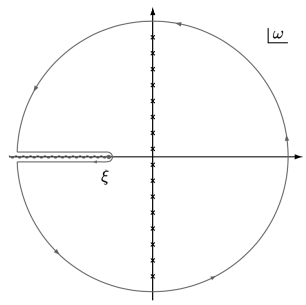


图 2: 含有割线的积分回路

$$
\begin{aligned}
\sum_n \ln\left[\beta(-i\omega_n+\xi_{\boldsymbol{p}})\right] &= -\beta\int_{-\infty}^{+\infty}\frac{dz}{2\pi i}n_F(z)\ln\left(\frac{z^+-\frac{\boldsymbol{p}^2}{2m}+\mu}{z^--\frac{\boldsymbol{k}^2}{2m}+\mu}\right) && (56)\\
&\overset{b.p.}{\Longrightarrow} -\int_{-\infty}^{+\infty}\frac{dz}{2\pi i}\ln\left(1+e^{-\beta z}\right)\left[\frac{1}{z-\frac{\boldsymbol{p}^2}{2m}+\mu+i\eta}-\frac{1}{z-\frac{\boldsymbol{p}^2}{2m}+\mu-i\eta}\right] && (57)\\
&= \ln\left(1+e^{-\beta(\epsilon_{\boldsymbol{p}}-\mu)}\right), && (58)
\end{aligned}
$$

其中我们利用了等式:

$$
\lim_{\eta\to0^+}\frac{1}{x\pm i\eta}=\mp i\pi\delta(x)+\mathcal{P}\left(\frac{1}{x}\right). \tag{59}
$$

因此自由能为:

$$
F^{(0)}=-2T\sum_{\boldsymbol{p}}\ln\left(1+e^{-\beta(\epsilon_{\boldsymbol{p}}-\mu)}\right). \tag{60}
$$

低温极限下：

$$F^{(0)} \approx 2\sum_{p<p_F}\left(\frac{\boldsymbol{p}^2}{2m}-\mu\right) \tag{61}$$
$$= \frac{2L^3}{(2\pi)^3}\cdot 4\pi\cdot\int_0^{p_F} dp\cdot p^2\cdot\frac{p^2-p_F^2}{2m} \tag{62}$$
$$= -\frac{2L^3}{(2\pi)^3}\cdot 4\pi\cdot\frac{p_F^5}{15m} \tag{63}$$
$$= -\frac{L^3 p_F^5}{15\pi^2 m} \tag{64}$$

注意到总粒子数为：

$$N = \sum_{p<p_F}\sum_{\sigma} 1 \tag{65}$$
$$= \frac{2L^3}{(2\pi)^3}\cdot\frac{4\pi}{3}p_F^3. \tag{66}$$

所以自由能可进一步化简为：

$$F^{(0)} \approx -\frac{2L^3}{(2\pi)^3}\cdot\frac{4\pi}{3}p_F^3\cdot\frac{p_F^2}{5m} \tag{67}$$
$$= -\frac{2}{5}N\mu. \tag{68}$$

自由能 RPA

考虑配分函数各阶微扰展开：

$$Z = \int D[\bar{\psi},\psi]e^{-S_0}\cdot\sum_{n=1}^{+\infty}\frac{(-1)^n}{n!}S_1^n \tag{69}$$
$$= Z^{(0)}\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\langle S_1^n\rangle_0. \tag{70}$$

其中，泛函平均 $\langle S_1^n\rangle_0$ 可通过计算 n 条相互作用线的泡泡图得到：

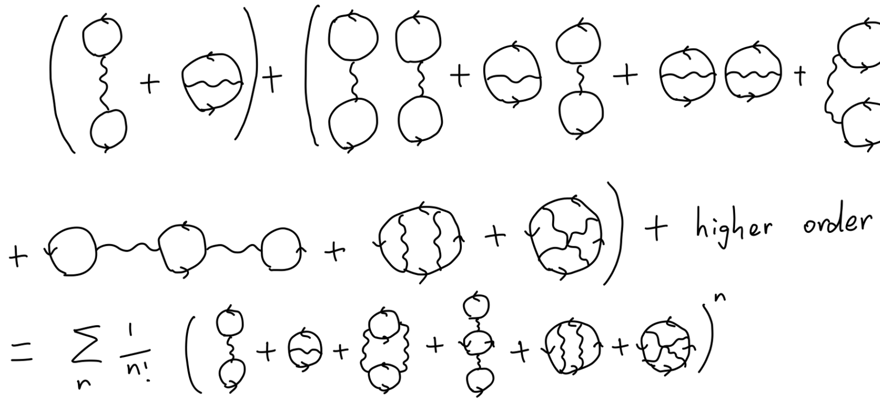


图 3: 集团分解

利用简单的排列组合技巧，这些泡泡图又可以按照联通泡泡图分组：

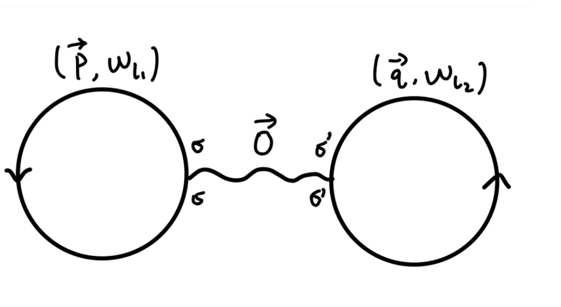
$$
\begin{aligned}
Z &= Z^{(0)} \prod_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \left\langle S_1^m \right\rangle_0^c \right)^n &(71)\\
&= Z^{(0)} \exp \left(\sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \left\langle S_1^m \right\rangle_0^c \right). &(72)
\end{aligned}
$$

对上式取对数，就得到了自由能的各阶微扰展开项：

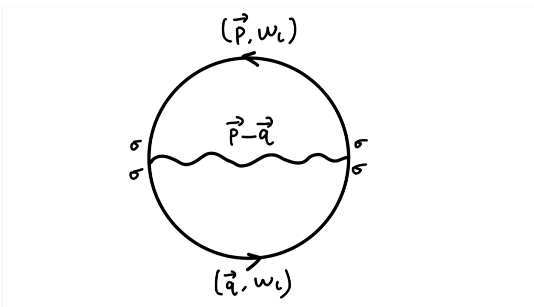
$$
F = F^{(0)} - T \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \left\langle S_1^m \right\rangle_0^c . \tag{73}
$$

一阶微扰

一阶微扰对应两张图：



(a) Hatree



(b) Fock

$$
\begin{aligned}
F^{(1)} &= \frac{T^2}{2L^3} \sum_{p_1,p_2,k} \sum_{\sigma,\sigma'} \left\langle \bar{\psi}^{\sigma}_{p_1+k} \bar{\psi}^{\sigma'}_{p_2-k} V(\boldsymbol{k})\, \psi^{\sigma'}_{p_2} \psi^{\sigma}_{p_1} \right\rangle_0^c &(74)\\
&= F^{(1)}_{Hatree} + F^{(1)}_{Fock}, &(75)
\end{aligned}
$$

分别对应 Hartree 项贡献和 Fock 项贡献:

$$F_{Hatree}^{(1)} = \frac{2T^2}{L^3} \sum_{p_1, p_2} G_{p_1} G_{p_2} V(\mathbf{0}) = 0, \tag{76}$$

$$\begin{aligned} F_{Fock}^{(1)} &= -\frac{T^2}{L^3} \sum_{p_1, p_2} G_{p_2} G_{p_1} V(\boldsymbol{p}_1 - \boldsymbol{p}_2) && (77) \\ &= -\frac{1}{L^3} \sum_{\boldsymbol{p}_1, \boldsymbol{p}_2} n_F(\xi_{\boldsymbol{p}_1}) n_F(\xi_{\boldsymbol{p}_2}) \frac{4\pi e^2}{|\boldsymbol{p}_1 - \boldsymbol{p}_2|^2} && (78) \\ &= -L^3 \int \frac{d^3 p_1}{(2\pi)^3} \int \frac{d^3 p_2}{(2\pi)^3} n_F(\xi_{\boldsymbol{p}_1}) n_F(\xi_{\boldsymbol{p}_2}) \frac{4\pi e^2}{|\boldsymbol{p}_1 - \boldsymbol{p}_2|^2}. && (79) \end{aligned}$$

在低温极限下:

$$F_{Fock}^{(1)} \approx -\frac{2e^2 L^3}{(2\pi)^5} \int_{p,q<p_F} \frac{d^3 p d^3 q}{|\boldsymbol{p} - \boldsymbol{q}|^2}. \tag{80}$$

其中积分为:

$$\begin{aligned} I &= \int_{p,q<p_F} d^3\boldsymbol{p} d^3\boldsymbol{q} \frac{1}{|\boldsymbol{p} - \boldsymbol{q}|^2} && (81) \\ &= 4\pi \int_0^{p_F} p^2 dp \int_0^{p_F} q^2 dq \int_{-1}^{+1} d(\cos\theta) \int_0^{2\pi} d\theta \frac{1}{p^2 + q^2 - 2pq\cos\theta} && (82) \\ &= 8\pi^2 \int_0^{p_F} p^2 dp \int_0^{p_F} q^2 dq \int_{-1}^{+1} \frac{d\left(p^2 + q^2 - 2pq\cos\theta\right)}{-2pq} \frac{1}{p^2 + q^2 - 2pq\cos\theta} && (83) \\ &= 8\pi^2 \int_0^{p_F} p^2 dp \int_0^{p_F} q^2 dq \frac{1}{pq} \ln\left|\frac{p+q}{p-q}\right| && (84) \\ &= 16\pi^2 \int_0^{p_F} p^3 dp \int_0^1 dt \left[t \ln\left(\frac{1+t}{1-t}\right)\right] && (85) \\ &= 16\pi^2 \int_0^{p_F} p^3 dp \left[\int_0^1 dx\,(x-1)\ln x + \int_1^2 dx\,(x-1)\ln x\right] && (86) \\ &= 16\pi^2 \int_0^{p_F} p^3 dp \left[\frac{t^2 \ln t}{2} - \frac{t^2}{4} + t - t\ln t\right]\Bigg|_0^2 && (87) \\ &= 16\pi^2 \int_0^{p_F} p^3 dp && (88) \\ &= (2\pi)^2 p_F F^4. && (89) \end{aligned}$$

因此自由能的一阶微扰为:

$$F^{(1)} = F_{Fock}^{(1)} = -\frac{2e^2 L^3 p_F^4}{(2\pi)^3}. \tag{90}$$

无规相近似

各阶微扰中最大贡献来源于项链图:

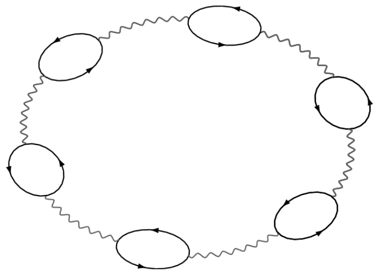


图 4: RPA 项链图

其大小为:

$$F_{RPA}^{(n)} = -\frac{T}{2n}\sum_{q}\left(\frac{2T}{L^3}\sum_{p}G_pG_{p+q}\right)^n, \tag{91}$$

其中 $2n$ 是费曼图的的对称因子。要计算第 n 阶微扰，首先考虑极化算符:

$$\begin{aligned}
\Pi_q &\equiv \frac{2T}{L^3}\sum_{\boldsymbol{p}}\sum_{\omega_p}G_pG_{p+q} && (92)\\
&= \frac{2T}{L^3}\sum_{\boldsymbol{p}}\sum_{\omega_p}\frac{1}{i\omega_p-\epsilon_{\boldsymbol{p}}+\mu}\cdot\frac{1}{i\left(\omega_p+\omega_q\right)-\epsilon_{\boldsymbol{p}+\boldsymbol{q}}+\mu} && (93)\\
&= \frac{2}{L^3}\sum_{\boldsymbol{p}}\frac{n_F\left(\xi_{\boldsymbol{p}+\boldsymbol{q}}\right)-n_F\left(\xi_{\boldsymbol{p}}\right)}{-i\omega_q+\xi_{\boldsymbol{p}+\boldsymbol{q}}-\xi_{\boldsymbol{p}}} && (94)\\
&= \frac{2}{\left(2\pi\right)^3}\int d^3p\frac{n_F\left(\xi_{\boldsymbol{p}+\boldsymbol{q}}\right)-n_F\left(\xi_{\boldsymbol{p}}\right)}{-i\omega_q+\xi_{\boldsymbol{p}+\boldsymbol{q}}-\xi_{\boldsymbol{p}}}. && (95)
\end{aligned}$$

为进一步化简上式，考虑能量关于小 $\boldsymbol{q}$ 改变量的微元:

$$\xi_{\boldsymbol{p}+\boldsymbol{q}}-\xi_{\boldsymbol{p}} \approx \frac{\boldsymbol{p}\cdot\boldsymbol{q}}{m} \tag{96}$$

由此得到费米分布

$$\begin{aligned}
n_F\left(\xi_{\boldsymbol{p}+\boldsymbol{q}}\right)-n_F\left(\xi_{\boldsymbol{p}}\right) &\approx \frac{\partial n_F}{\partial\epsilon}\left(\xi_{\boldsymbol{p}}\right)\cdot\frac{\boldsymbol{p}\cdot\boldsymbol{q}}{m} && (97)\\
&\approx -\delta^{(3)}\left(\xi_{\boldsymbol{p}}\right)\cdot\frac{\boldsymbol{p}\cdot\boldsymbol{q}}{m}. && (98)
\end{aligned}$$

因此，上述积分可化简为:

$$\begin{aligned}
\Pi_q &\approx \frac{-2}{\left(2\pi\right)^3}\int p_F^2dp\delta^{(3)}(\xi_{\boldsymbol{p}})\int_{-1}^{+1}d\left(\cos\theta\right)\int_0^{2\pi}d\phi\frac{p_Fq\cos\theta/m}{-i\omega_q+p_Fq\cos\theta/m} && (99)\\
&= -\frac{1}{2\pi^2}\int p_F^2dp\delta^{(3)}(\xi_{\boldsymbol{p}})\int_{-1}^{+1}d\left(\cos\theta\right)\frac{v_Fq\cos\theta}{-i\omega_q+v_Fq\cos\theta} && (100)\\
&= -\frac{1}{2\pi^2}\int p_F^2dp\delta^{(3)}(\xi_{\boldsymbol{p}})\left[1+\frac{i\omega_q}{2v_Fq}\ln\left(\frac{i\omega_q+v_Fq}{i\omega_q-v_Fq}\right)\right] && (101)\\
&= -\frac{mp_F}{\pi^2}\left[1-\frac{i\omega_q}{2v_Fq}\ln\left(\frac{i\omega_q+v_Fq}{i\omega_q-v_Fq}\right)\right]. && (102)
\end{aligned}$$

其中我们利用了关系：

$$\delta\left(f\left(z\right)\right)=\sum_{z_k}\frac{\delta\left(z-z_k\right)}{\left|f'\left(z_k\right)\right|}. \tag{103}$$

只考虑这种项链图的微扰展开称为无规相近似，在此近似下的自由能为：

$$F_{RPA}=\frac{T}{2}\sum_q \ln(1-V(\boldsymbol{q})\Pi_q). \tag{104}$$

自能图

相互作用线的自能费曼图为：

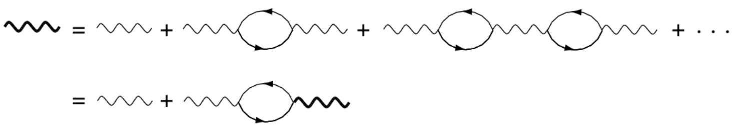


图 5: 相互作用自能展开

由此得到的有效作用势为：

$$\begin{aligned} V_{eff}\left(q\right) &= V\left(\boldsymbol{q}\right)+V\left(\boldsymbol{q}\right)\Pi_q V_{eff}\left(q\right) && (105)\\ &= \frac{V\left(\boldsymbol{q}\right)}{1-V\left(\boldsymbol{q}\right)\Pi_q} && (106)\\ &\equiv \frac{V\left(\boldsymbol{q}\right)}{\epsilon\left(q\right)}, && (107)\end{aligned}$$

其中定义介电常数 $\epsilon(q)$：

$$\epsilon\left(q\right)\equiv 1-V\left(\boldsymbol{q}\right)\Pi_q. \tag{108}$$

对小动量 $\boldsymbol{q}$ 和小频率 $\omega_n \ll qv_F$：

$$\Pi\left(\boldsymbol{q},\omega_q\right)\approx-\frac{mp_F}{\pi^2}\equiv-\nu_0. \tag{109}$$

在此极限下：

$$V\left(q\right)\approx\frac{1}{V^{-1}\left(\boldsymbol{q}\right)+\nu_0}=\frac{4\pi e^2}{q^2+4\pi e^2\nu_0}\equiv\frac{4\pi e^2}{q^2+\lambda^{-2}}. \tag{110}$$

其中 $\lambda\equiv\left(4\pi e^2\nu_0\right)^{-1/2}$ 称为 Thomas-Fermi 屏蔽长度。为明确其物理意义，将相互作用势变回实空间：

$$\begin{aligned} V\left(r\right) &= \int\frac{d^3q}{\left(2\pi\right)^3}e^{i\boldsymbol{q}\cdot r}\frac{4\pi e^2}{q^2+\lambda^{-2}} && (111)\\ &= \frac{e^2}{\pi}\int_0^{+\infty}q^2dq\int_{-1}^{+1}d\left(\cos\theta\right)\frac{e^{iqr\cos\theta}}{q^2+\lambda^{-2}} && (112)\\ &= \frac{e^2}{i\pi r}\int_0^{+\infty}dq\frac{q}{q^2+\lambda^{-2}}\left(e^{+iqr}-e^{-iqr}\right) && (113)\\ &= \frac{e^2}{i\pi r}\int_{-\infty}^{+\infty}dq\frac{q}{q^2+\lambda^{-2}}e^{+iqr}. && (114)\end{aligned}$$

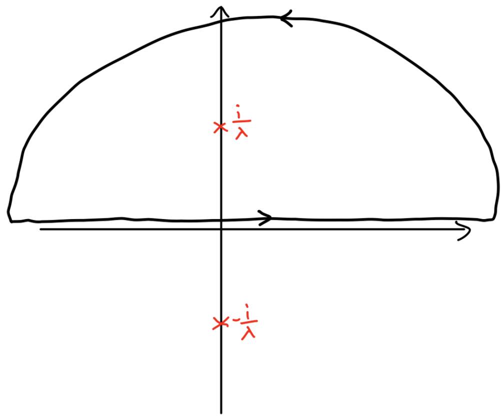


图 6: 积分回路

利用留数定理:

$$Res\left[\frac{q}{q^2+\lambda^{-2}}e^{+iqr}\right]\Big|_{q=\frac{i}{\lambda}}=\frac{1}{2}e^{-\frac{r}{\lambda}}, \tag{115}$$

得到屏蔽库伦势为:

$$V\left(r\right)=\frac{e^2}{r}e^{-\frac{r}{\lambda}}. \tag{116}$$

有效势在实空间指数衰减，衰减特征长度就是 Thomas-Fermi 屏蔽长度 λ.