# 1 Algorithm for simulating Multi-Population Mean Field Game (MPMFG) Social Network Problem

This document serves to discuss details of the attempted re-creation of the MPMFG simulation for the social network problem as presented by Banez et al [1], since they did not release their code.

First we present the finite difference algorithm, then we proceed by explaining how to initialize the system, each update rule, and how we reached the final form used in our Matlab simulation. Anytime that there were pieces missing from their description we explain the assumptions we made.

In the finite difference method, state and time is divided into a finite grid, and each place in the grid is updated each time step based on its neighbors. The algorithm from Banez for solving a MPMFG numerically is as follows:

**Algorithm 1:** Solving a Multiple-Population Mean Field Game Problem.

```
Initialize: m^0_{p,j}, v^M_{p,j}, u^0_{p,j},
while it ≤ I or error > ε do
  for p = 1, ..., P, j = 0, ..., L, k = 0, ..., M − 1 do
    Solve the mean field m^{k+1}_{p,j} using (47).
  end for
  for p = 1, ..., P, j = 0, ..., L, k = M, ..., 2 do
    Solve the value v^{k−1}_{p,j} using (48).
  end for
  for p = 1, ..., P, j = 0, ..., L, k = 0, ..., M do
    Update u^k_{p,j} using (49).
    Compute error^k_{p,j} = |∂r^k_{p,j}/∂u^k_{p,j}|.
  end for
end while
```

## 1.1 Initialization

$m^0_{p,j}$ is initialized according to the Gaussian distribution for each population as described in the paper, but there is no explanation as to how to $v^M_{p,j}$ and $u^0_{p,j}$ should be initialized. $u^0_{p,j}$ represents the control effort at time zero, and this is solved for using gradient descent (equation 49), so we initialize it to a small value, 0.01. $v^k_{p,j}$ is defined in equation 26:

$$v^k_{p,j} = v_p(x_j, t_k) = \min_{u_p \in \mathcal{U}_p} J_p(u_p, \boldsymbol{m}). \tag{26}$$

where $J_p(u_p, \boldsymbol{m})$ is the cost function. Ideally at the end $(t = M)$, there would be zero cost, so we make the choice to initialize $v^M_{p,j}$ to zero.

## 1.2 Mean Field Update Rule

The rule for updating the mean field is equation 47:

$$m^{k+1}_{p,j} = 0.5(m^k_{p,j+1} + m^k_{p,j-1}) + \Delta t \left( -\frac{\phi^k_{p,j+1} - \phi^k_{p,j-1}}{2\Delta x} + \varepsilon \frac{m^k_{p,j+1} - 2m^k_{p,j} + m^k_{p,j-1}}{(\Delta x)^2} \right) \tag{47}$$

where $\phi^k_{p,j}$ is defined as $\phi^k_{p,j} = f^k_{p,j} m^k_{p,j}$, and $\varepsilon = \sigma_p^2/2$ ($\sigma_p$ is a population parameter).

The opinion drift, $f^k_{p,j}$, comes from equation 25, the opinion dynamics equation:

$$
\begin{aligned}
dx(t) =& \left( \lambda_p \sum_{l=1}^{P} \eta_l \bar{a}_{pl} \bar{x}_l + (1-\lambda_p) \bar{a}_{pp} \bar{x}_{p0} + a_p x + b_p u_p \right) dt \\
& + \sigma_p \, dw_p(t), \\
=& f_p(x, u_p, \boldsymbol{m}) \, dt + \sigma_p \, dw_p(t),
\end{aligned} \tag{25}
$$

So $f_{p,j}^k = \lambda_p \sum_{l=1}^{P} \eta_l \bar{a}_{pl} \bar{x}_l^k + (1-\lambda_p) \bar{a}_{pp} \bar{x}_p^0 + a_p x_j + b_p u_{p,j}^k$, where $\lambda_p$, $\eta_l$, $\bar{a}_{pl}$, $\bar{a}_{pp}$, $a_p$, and $b_p$ are all constants. $\bar{x}_l^k$ is the average opinion of population $l$ at time $t_k$, while $\bar{x}_p^0$ is the average initial opinion and is given as a constant in the simulation parameters. The paper doesn't specify how to calculate $\bar{x}_l^k$. Since the mean field denotes the probability distribution of the players with state $\boldsymbol{x}$ at time $t$, we calculate the average population opinion as the expected value of the population opinion under the mean field distribution: $\bar{x}_l^k = \sum_{j=0}^{L} x_j m_{l,j}^k$

Thus we can expand the given opinion drift into:

$$
f_{p,j}^k = \lambda_p \sum_{l=1}^{P} \eta_l \bar{a}_{pl} \left( \sum_{j=0}^{L} x_j m_{l,j}^k \right) + (1-\lambda_p) \bar{a}_{pp} \bar{x}_p^0 + a_p x_j + b_p u_{p,j}^k
$$

### 1.3 Adjoint Variable / Value Function Update Rule

The paper lists the rule for updating the value function in equation 48:

$$
\begin{aligned}
v_{p,j}^{k-1} = 0.5(v_{p,j-1}^k + v_{p,j+1}^k) + \Delta t \Bigg( & r_{p,j}^k + m_{p,j}^k \frac{\partial r_{p,j}^k}{m_{p,j}^k} - f_{p,j}^k \frac{v_{p,j-1}^k - v_{p,j+1}^k}{2\Delta x} \\
& - \frac{\partial f_p}{m_{p,j}^k} \sum_{l=1}^{P} m_{l,j}^k \frac{v_{l,j-1}^k - v_{l,j+1}^k}{2\Delta x} \\
& + \varepsilon \frac{v_{p,j-1}^k - 2v_{p,j}^k + v_{p,j+1}^k}{(\Delta x)^2} \Bigg)
\end{aligned} \tag{48}
$$

It appears to be missing the partial derivative symbol on two terms, $\frac{\partial r_{p,j}^k}{m_{p,j}^k}$ and $\frac{\partial f_p}{m_{p,j}^k}$, so we add that back in:

$$
\begin{aligned}
v_{p,j}^{k-1} = 0.5(v_{p,j-1}^k + v_{p,j+1}^k) + \Delta t \Bigg( & r_{p,j}^k + m_{p,j}^k \frac{\partial r_{p,j}^k}{\partial m_{p,j}^k} - f_{p,j}^k \frac{v_{p,j-1}^k - v_{p,j+1}^k}{2\Delta x} \\
& - \frac{\partial f_p}{\partial m_{p,j}^k} \sum_{l=1}^{P} m_{l,j}^k \frac{v_{l,j-1}^k - v_{l,j+1}^k}{2\Delta x} \\
& + \varepsilon \frac{v_{p,j-1}^k - 2v_{p,j}^k + v_{p,j+1}^k}{(\Delta x)^2} \Bigg)
\end{aligned}
$$

$\frac{\partial r_{p,j}^k}{\partial m_{p,j}^k}$ and $\frac{\partial f_p}{\partial m_{p,j}^k}$ aren't given, so we derive them. First, $\frac{\partial f_p}{\partial m_{p,j}^k}$:

$$
\frac{\partial f_p}{\partial m_{p,j}^k} = \lambda_p \eta_p \bar{a}_{pp} \frac{\partial \bar{x}_p^k}{\partial m_{p,j}^k} = \lambda_p \eta_p \bar{a}_{pp} x_j
$$

$r_{p,j}^k$ is the running cost and comes from equation 23:

$$r_{p,j}^k = r_p(x_j, u_p, \boldsymbol{m}) = c_1 u_p^2 + \sum_{l=1}^{P} c_{2pl} m_l + c_3 \lambda_p |x - x_{pd}| + c_4(1 - \lambda_p)|x - \bar{x}_{p0}|, \tag{23}$$

Where $c_1$, $c_2$, $c_3$, $c_4$, and $x_{pd}$ are constants. So we can derive $\frac{\partial r_{p,j}^k}{\partial m_{p,j}^k}$:

$$\frac{\partial r_{p,j}^k}{\partial m_{p,j}^k} = c_{2pp}$$

### 1.4 Gradient Descent for Control Effort

Equation 49 defines the gradient descent used to calculate the control effort:

$$u_{p,j}^k \leftarrow \frac{w}{1+w} u_{p,j}^k - \frac{1}{1+w} \frac{\partial r_{p,j}^k}{\partial u_{p,j}^k} \tag{49}$$

We calculate $\frac{\partial r_{p,j}^k}{\partial u_{p,j}^k}$ from the running cost:

$$\frac{\partial r_{p,j}^k}{\partial u_{p,j}^k} = 2 c_1 u_{p,j}^k$$

It seems strange that the mean field or value function do not appear in the gradient descent term here. If only $u_{p,j}^0$ is initialized, how will the control term be updated at other values of $k$?

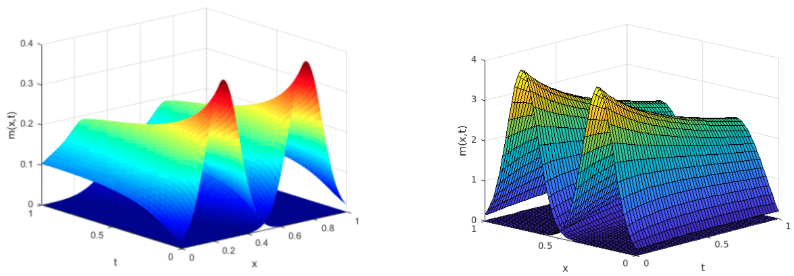
### 1.5 Final Assumptions

In the original paper, this is how they describe the simulation parameters:

> Consider a social network consisting of users that can be divided into two separate populations, $p = 1, 2$. Let the state or opinion of a user be $x \in [0, 1]$ and time be $t \in [0, 1]$. Let the initial mean fields $m_p(x, 0)$ be $\mathcal{N}(0.25, 0.1)$ and $\mathcal{N}(0.75, 0.1)$ for populations 1 and 2, respectively. Also, assume a 50%-50% ratio between the two populations (i.e., $\eta_1 = \eta_2 = 0.5$). The opinion dynamics equation parameters are set at $\bar{a}_p = 0.001$, $a_p = -0.001$, and $\sigma_p = 0.01$. Meanwhile, the cost function parameters are $c_1 = 0.5, c_2 = 1, c_3 = 2, c_4 = 1, x_{0,1} = 0.25, x_{0,2} = 0.75$. Finally, $w = 1,000$, $K = 2,000$, and $\epsilon = 1 \times 10^{-6}$.

With only two populations, the adjacency between them is the same, so $\bar{a}_{pl} = \bar{a}_p = 0.001$. We assume that $\bar{a}_{pp} = \bar{a}_p$. They also don't mention the value of $b_p$ they use, so we assume $b_p = 1$ for all $p$, a choice the authors mention in an earlier version of this paper [2]. $K$ in their description likely means the maximum iterations, but we use 200, not 2,000. The desired population is also omitted, and we assume that is the same as the initial average population opinion.

Also, any time we index into the mean field, the value function, or the control and the index would be out of bounds of the finite grid (in either the opinion state or time), we simply substitute zero in for that value. The paper doesn't specify how to deal with this issue. The size of the grid is not specified, so we use $L = 50$, $M = 50$.




(a) Banez et al. mean field

(b) Our mean field

Figure 1: Banez's vs our mean field

## 2 Simulation Results

We were pleasantly surprised to produce a mean field similar to what the original authors show, despite all the ambiguity. Above are the plots.

## References


[1] Reginald A. Banez, Hao Gao, Lixin Li, Chungang Yang, Zhu Han, and H. Vincent Poor. Modeling and analysis of opinion dynamics in social networks using multiple-population mean field games. *IEEE Transactions on Signal and Information Processing over Networks*, 8:301–316, 2022.

[2] Reginald A. Banez, Hao Gao, Lixin Li, Chungang Yang, Zhu Han, and H. Vincent Poor. Belief and opinion evolution in social networks based on a multi-population mean field game approach. In *ICC 2020 - 2020 IEEE International Conference on Communications (ICC)*, pages 1–6, 2020.