# Math140B - HW #1

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## 1. Q2

Suppose $f$ is not strictly increasing on $(a, b)$, that is, $\exists c, d \in \mathbb{R}$ such that $a < c < d < b$ and $f(c) \geq f(d)$. $f$ is differentiable on $(a, b)$ with $(c, d) \subset (a, b)$, so there is an open neighborhood containing $[c, d]$ such that $f$ is continuous on the neighborhood. Of course, $f$ is differentiable on $(c, d)$, thus satisfying the hypothesis of the Mean Value Theorem. Applying the Mean Value Theorem to $f$ on points $c$ and $d$, $\exists x \in (c, d)$ such that $f(d) - f(c) = (d - c)f'(x)$. Namely, $c < d$ and $f(c) \geq f(d)$, so $f'(x) \leq 0$, which contradicts the fact that $f$ has a positive derivative on $(a, b)$.

Since $f$ is strictly increasing on $(a, b)$, i.e. $f(x_1) \neq f(x_2)\, \forall x_1 \neq x_2 \in (a, b)$, $f$ is injective on $(a, b)$; $f$ is bijection from $(a, b)$ to the image $f((a, b))$. Denote $f^{-1}$ as $g : f((a, b)) \to (a, b)$ and let $h(t) := g(f(t)) = t$.

First, assume $g$ is differentiable. By the chain rule, $h'(x) = g'(f(x))f'(x)$. Simultaneously, $h'(x) = 1$. Combining these equalities,

$$g'(f(x)) = \frac{1}{f'(x)}$$

Symmetric reasoning shows that if $g$ is differentiable, then necessarily $g'(y) = \frac{1}{f'g(y)}$ for any $y \in f((a, b)$ Indeed, this calculation can be used to confirm that $g$ is indeed differentiable. Notice that because $f$ is differentiable on any $x \in (a, b)$ and $f'(x) \neq 0$,

$$\lim_{t \to x} \frac{1}{\frac{f(x) - f(t)}{x - t}} = \lim_{t \to x} \frac{x - t}{f(x) - f(t)}$$

exists, and the value of the limit is $1/f'(x)$. Now, fix $\epsilon > 0$ and pick any $y \in f((a, b))$. Since $f$ is injective, $\exists! x \in (a, b)$ such that $f(x) = y$. $\lim_{t \to x} \frac{x-t}{f(x)-f(t)} = 1/f'(x)$, so $\exists \eta > 0$ such that $\forall t \in \mathrm{B}_\eta(x)$, $\left|\frac{x-t}{f(x)-f(t)} - 1/f'(x)\right| < \epsilon$.

Now, $g$ is continuous, so $\exists \delta > 0$ such that $\forall s \in \mathrm{B}_\delta(y)$, $|g(y) - g(s)| < \eta$. Denote $t = g(s)$. Since $g(y) = x$, $|x - t| < \eta$ and thus the following holds:

$$\left| \frac{g(y) - g(s)}{y - s} - \frac{1}{f'(g(y))} \right| = \left| \frac{x - t}{f(x) - f(t)} - \frac{1}{f'(x)} \right| < \epsilon$$

This shows that $g$ is differentiable on its entire domain with $g'(y) = 1/f'(g(y))$.

# 2. Q4

Define

$$g(x) := c_0 x + \frac{1}{2}c_1 x^2 + \ldots + \frac{1}{n}c_{n-1}x^n + \frac{1}{n+1}c_n x^{n+1}$$

Notice

- $g(1) = 0$ by the hypothesis, and also $g(0) = 0$.
- $g'(x) = c_0 + c_1 x + \ldots + c_{n-1}x^{n-1} + c_n x^n = 0$, which is the polynomial of interest in this question.

$g$ is a polynomial, so it is infinitely differentiable everywhere. By the Mean Value Theorem, $\exists x_0 \in (0,1)$ such that $g'(x_0)(1-0) = g(1) - g(0)$, i.e., $g'(x_0) = 0$, as was to be shown.

# 3. Q8

Fix $\epsilon > 0$. By the definition of the derivative and the continuity of $f'$, $\forall x \in [a,b], \exists \delta_x > 0$ such that $\forall t \in [a,b]$ satisfying $|x-t| < \delta_x$,

$$\left| f'(x) - \frac{f(x)-f(t)}{x-t} \right| < \epsilon/2, \quad |f'(x) - f'(t)| < \epsilon/2$$

$\bigcup_{x\in[a,b]} \mathrm{B}_{\delta_x/2}(x)$ form an open cover for $[a,b]$ and $[a,b]$ is compact, so $\exists x_1, \ldots, x_n \in [a,b]$ with associated neighborhoods $\mathrm{B}_{\delta_1/2}(x_1), \ldots, \mathrm{B}_{\delta_n/2}(x_n)$ whose union is an open subcover of $[a,b]$. Let

$$\delta := \min\{\delta_1/2, \ldots, \delta_n/2\}$$

Then $\forall x, t \in [a,b]$ with $|x-t| < \delta$,

$$|x_k - t| \le |x_k - x| + |x - t| \le \delta_k$$

for some $k \in \{1, \ldots, n\}$ satisfying $|x - x_k| < \delta_k/2$, which is guaranteed to exist since the neighborhoods of the $x_i$'s form an open cover for $[a,b]$. Finally, because both $x, t \in \mathrm{B}_{\gamma_k}(x_k)$,

$$\left| f'(x) - \frac{f(x)-f(t)}{x-t} \right| \le |f'(x) - f'(x_k)| + \left| f'(x_k) - \frac{f(x_k)-f(t)}{x_k - t} \right| < \epsilon/2 + \epsilon/2 = \epsilon$$

as was to be shown.

Under the same hypothesis but with $f : [a,b] \to \mathbb{R}^n$, the result holds as previous. The easiest way to convince oneself of this fact is to use the box metric on $\mathbb{R}^n$. Then the previous calculations apply more or less immediately to the current case, with nothing more to be checked.

# 4. Q9

By the L'Hospital's Rule,

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0}$$

exists: the continuity of $f$ ensures $f(x)-f(0) \to 0$ as $x \to 0$. Of course, the denominator vanishes as $x \to 0$ as well. The derivative of the numerator with respect to $x$ is just $f'(x)$, while the derivative of the denominator is 1, so the L'Hospital's rule applies, and

$$f'(0) := \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{f'(x)}{1} = 3$$