Math140B - HW #6

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Q4

Let $f_n(x) := \frac{1}{1+n^2x}$.

(a) Absolute Convergence

For a fixed value of x, let $a_n := f(x)$. Suppose $x > 0$. Then

$$\left|\frac{1}{1+n^2x}\right| < \frac{1}{n^2x} = c\frac{1}{n^2}$$

where $c = \frac{1}{x}$. $\sum c\frac{1}{n^2}$ converges, so $\sum |a_n|$ converges by the comparison test when $x > 0$.

Suppose $x < 0$. Pick $N \in \mathbb{Z}_+$ such that $N^2|x| - 1 > 0$. Then $\forall n \geq N$,

$$\left|1+n^2x\right| = \left|n^2|x| - 1\right| = n^2|x| - 1 > 0$$

Using this inequality,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{|1+n^2x|} &= \sum_{n=1}^{N-1}\frac{1}{|1+n^2x|} + \sum_{n=N}^{\infty}\frac{1}{n^2|x|-1} \\ &< \sum_{n=1}^{N-1}\frac{1}{|1+n^2x|} + \frac{1}{|x|}\sum_{n=N}^{\infty}\frac{1}{n^2}\end{aligned}$$

so the series converges absolutely by the comparison test.

Of course, when $x = 0$, $f(x) = \sum_{n=1}^{\infty} 1$, which diverges.

(b) Uniform Convergence

Of course, the intervals for uniform convergence must avoid $x = 0$.

Take any $a > 0$. Then for any $x \in [a, \infty)$,

$$\sum_{n=1}^{\infty}\frac{1}{1+n^2x} < \frac{1}{a}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

which means any paramter that satisfies pointwise boundedness at a satisfies the boundedness at all $x \geq a$. Hence $f_n \to f$ uniformly on $[a, \infty)$ (and also (a, ∞)).

Next, fix any $b < 0$. As in (a) $\exists N \in \mathbb{Z}_+$ such that $|1 + N^2 x| \, N^2 |x| - 1 > 0$. Take any $x \in (-\infty, b]$. Using the fact that $N^2 |x| - 1 > N^2 |b| - 1 > 0$,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{|1+n^2 x|} &= \sum_{n=1}^{N-1} \frac{1}{|1+n^2 x|} + \sum_{n=N}^{\infty} \frac{1}{n^2 |x| - 1} \\ &< \sum_{n=1}^{N-1} 1 + \frac{1}{|b|} \sum_{n=N}^{\infty} \frac{1}{n^2}\end{aligned}$$

so $f_n \to f$ uniformly on $(-\infty, b]$ (and also $(-\infty, b)$).

The analysis above also illuminates that the f_n's do not uniformly converge to f on intervals $(0, a]$ or $[b, 0)$. As x gets closer and closer to 0, the greater the $N \in \mathbb{Z}_+$ required to ensure pointwise convergence to f since $\frac{1}{|x|}$ increases without bound as $x \to 0$.

(c) Continuous wherever the series converges?

Since absolute convergence implies convergence, $\sum \frac{1}{1+n^2 x}$ converges whenever $x \neq 0$. Also, when $x \neq 0$, x is always in some closed interval that excludes 0. Each individual f_n is continuous on that closed interval $(*)$ and the $f_n \to f$ on the closed interval, so f is continuous at x. In other words, f is continuous wherever the series converges.

$(*)$: technically, f_n where n satisfies $n^2 x = 1$ is not even defined at x.

(d) Is f bounded?

By (a), f is not bounded because $\sum_{n=1}^{\infty} \frac{1}{n^2 x + 1}$ diverges when $x = 0$.

Q9

Since the f_n's are continuous and converge uniformly to f, f is continuous. Take any $\forall \{x_n\}_n \subset E$ with $x_n \to x$. By the continuity of f,

$$\lim_{n \to \infty} f(x_n) = f(x)$$

Fix $\epsilon > 0$. $\exists N \in \mathbb{Z}_+$ such that $\forall n \geq N$,

$$|f_n(p) - f(p)| < \epsilon \text{ for any } p \in E \text{ and } |f(x_n) - f(x)| < \epsilon$$

By the triangle inequality, $\forall n \geq N$,

$$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < 2\epsilon$$

which shows that $f_n(x_n) \to f(x_n)$ as $n \to \infty$.

Q10

Let $f_n(x) := \frac{(nx)}{n^2}$. If $f(x) := \sum_{n=0}^{\infty} f_n(x)$, then clearly $\sum f_n \to f$ uniformly on $\mathbb{R}$ since for all $x \in \mathbb{R}$,

$$\left|\frac{(nx)}{n^2}\right| < \frac{1}{n^2}$$

and $\sum \frac{1}{n^2}$ converges.

Notice that for any n, (nx) is discontinuous on all integer multiples of $1/n$ and continuous everywhere else, i.e.,

$$(nx) \text{ is discontinuous on } x = x' \iff x' \in \left\{\frac{m}{n} \mid m \in \mathbb{Z}\right\} \tag{1}$$

Thus the individual f_n's are continuous on $\mathbb{R} \setminus \mathbb{Q} \implies \sum_{i=1}^{n} f_i$ are continuous on $\mathbb{R} \setminus \mathbb{Q}$ for any $n \implies f$ continuous on $\mathbb{R} \setminus \mathbb{Q}$ because $\sum f_n \to f$ uniformly.

On the other hand, f is discontinuous on $\mathbb{Q}$. Take any $p/q \in \mathbb{Q}$ where $\gcd(p,q) = 1$. By (1), f_n is discontinuous on $p/q \iff q \mid n$. Decompose f as follows:

$$f(x) = \underbrace{\sum_{q|n} f_n(x)}_{g(x)} + \underbrace{\sum_{q\nmid n} f_n(x)}_{h(x)}$$

where the n in both sums are positive integers, of course. It's easy to see that, by a similar reasoning as $\sum_{n=1}^{\infty} f_i$, the two series in the decomposition are uniformly convergent on $\mathbb{R}$. Furthermore, since each summand of h is continuous at p/q, h is continuous at p/q by uniform convergence. To show f is not continuous at p/q, it suffices to show g is not continuous at p/q. First, rewrite $g(x)$ as

$$g(x) = \sum_{k=1}^{\infty} f_{kq}(x) = \frac{1}{q^2}\sum_{k=1}^{\infty} \frac{(kqx)}{k^2}$$

Notice each f_{kq} is not only discontinuous at p/q, but also $f_{kq}(p/q) = 0$. So $g(p/q) = 0$ as well. However, the left-hand limit of g at p/q is not 0; instead,

$$\lim_{t\to(p/q)^-} \sum_{k=1}^{\infty} \frac{(kpt)}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

Fix $\epsilon > 0$. Since $\sum_k \frac{1}{k^2}$ converges, $\exists N \in \mathbb{Z}_+$ such that $\sum_{k=N+1}^{\infty} \frac{1}{k^2} < \epsilon/2$. Set

$$\delta := \min\{\frac{\epsilon}{2q\sum_{k=1}^{N} 1/k}, \frac{1}{Nq}\}$$

Take any $t \in (p/q - \delta, p/q)$ and rewrite $t = p/q - \delta'$ for some $\delta' < \delta$. Then

- $\sum_{k=N+1}^{\infty} \frac{1-(kqt)}{k^2} < \sum_{k=N+1}^{\infty} \frac{1}{k^2} < \epsilon/2$

- Notice that because $\delta' < \delta \leq \frac{1}{Nq}$, $(kq[p/q - \delta']) = (-kq\delta') = 1 - kq\delta'$ for all $k < N$. Hence

$$\begin{aligned}\sum_{k=1}^{N} \frac{1-(kpt)}{k^2} &= \sum_{k=1}^{N} \frac{kq\delta'}{k^2} \\ &= kq' \sum_{k=1}^{N} \frac{1}{k} \\ &< \epsilon/2\end{aligned}$$

  by choice of δ.

This shows $\left|1 - \sum_{k=1}^{\infty} \frac{(kqt)}{k^2}\right| < \epsilon$. In other words, $\lim_{t\to(p/q)^-} q^2 g(t) = \sum_{k=1}^{\infty} \frac{1}{k^2}$, the latter which does not agree with the value of $q^2 g(p/q) = 0$; g, and thus f, is discontinuous at p/q.

As a bonus, a similar analysis shows that the right-hand limit of g at p/q is 0. Recapitulating, f is continuous on the irrationals and discontinuous on the rationals, which is a dense, countable subset of $\mathbb{R}$.

Turning to integrability, take any bounded interval $[a, b]$ in $\mathbb{R}$. For any n, (1) implies that f_n has finitely many discontinuities. Hence $\sum_{i=1}^{n} f_i$ has finitely many discontinuities for any n, which means

$$\sum_{i=1}^{n} f_i \in \mathscr{R}_a^b$$

for every n. By uniform convergence, $f \in \mathscr{R}_a^b$ as well.

Q12

By Theorem 4.2, the following Cauchy criterion for real valued functions can be given:

Claim 1. *For any real-valued function h, $\lim_{t\to x}$ exists (and is finite) $\iff$ $\forall\epsilon < 0$, $\exists\delta > 0$ such that*

$$\forall t, t' \in B_\delta(x),\ |h(t) - h(t')| < \epsilon$$

Similarly, $\lim_{t\to\infty}$ exists and is finite $\iff$ $\forall\epsilon > 0$, $\exists B \in \mathbb{R}$ such that

$$\forall b, b' \geq B,\ |h(b) - h(b')| < \epsilon$$

The Cauchy criterion can be used to prove that $\int_0^\infty f_n dx$ is bounded for all n. First, fix $t > 0$. Then, for any n, $\left|\int_t^\infty f_n dx\right|$ converges and is finite: Fix $\epsilon > 0$. Since $\int_t^\infty g dx$ converges and is finite, $\exists T \in \mathbb{R}$ such that $\forall T' > T'' \geq T$, $\left|\int_t^{T'} g dx - \int_t^{T''} g dx\right| = \left|\int_{T'}^{T''} g dx\right| < \epsilon$. But

$$\left|\int_{T'}^{T''} f_n dx\right| \leq \int_{T'}^{T''} |f_n|\, dx \leq \int_{T}^{T''} g dx < \epsilon$$

so the T that satisfies the Cauchy criterion for g satisfies the Cauchy criterion for the f_n's as well. This shows that $\int_t^\infty f_n dx$ converges and is finite. A similar proof can be used to show that for any fixed $T > 0$, $\int_0^T f_n$ converges and is finite. Hence

$$\int_0^\infty f_n dx \text{ exists and is finite}$$

Also, because $f_n \to f$ uniformly, $|f| \leq g$. Suppose not; $\exists x \in (0, \infty)$ such that $|f(x)| > g(x)$. By the convergence of f_n, $\exists N \in \mathbb{Z}_+$ such that $\forall n \geq N$, $|f(x) - f_n(x)| < |f(x)| - g(x)$. But the reverse triangle inequality says $|f(x)| - |f_n(x)| \leq |f(x) - f_n(x)| < |f(x)| - g(x) \implies |f_n(x)| > g(x)$ for all $n \geq N$, a contradiction. So $|f| \leq g$, and by the same reasoning as for f_n,

$$\int_0^\infty f dx \text{ exists and is finite}$$

Now, to show

$$\lim_{n\to\infty} \int_0^\infty f_n dx = \int_0^\infty f dx$$

fix $\epsilon > 0$. Since $\int_0^T g dx \to \int_0^\infty g dx$ as $T \to \infty$, $\exists T > 0$ such that $\int_0^\infty g dx - \int_0^T g dx = \int_T^\infty g dx < \epsilon$. Similarly, $\exists t > 0$ such that $\int_0^t g dx < \epsilon$. Also, because $f_n \to f$ uniformly, $\exists N \in \mathbb{Z}_+$ such that $\forall n \geq N, \forall x \in (0, \infty)$, $|f_n(x) - f(x)| < \frac{\epsilon}{T-t}$. Then $\left|\int_t^T (f_n - f) dx\right| \leq \int_t^T |f_n - f| \, dx < \epsilon$. Putting this together, for all $n \geq N$,

$$\begin{aligned}\left|\int_0^\infty (f_n - f) dx\right| &\leq \left|\int_0^t f dx\right| + \left|\int_0^t f_n dx\right| + \int_t^T |f_n - f| \, dx + \left|\int_T^\infty f_n dx\right| + \left|\int_T^\infty f dx\right| \\ &< 5\epsilon\end{aligned}$$

as was to be shown.

Q14

For all $t \in I$, $|x(t)| \leq \sum_{n=1}^\infty 2^{-n}$, the latter being a convergent geometric series. By the Weierstrass M-test, $x(t)$ converges uniformly on I. Since for each n, $f(3^{2n-1}t)$ is a composition of continuous functions, $x(t)$ is continuous by uniform convergence. By the same reasoning as $x(t)$, $y(t)$ converges uniformly on I and is continuous; $\Phi(t)$ is thus continuous.

Let E be the Cantor set. To see $\Phi(E) = I^2$, first notice $\forall z \in I$, there is a binary sequence $\{a_n\}$, where each $a_n \in \{0, 1\}$, such that $\sum_{n=1}^\infty 2^{-n} a_n = z$: essentially, the a_n's "binary search" for z. The precise construction of the a_n's is as follows: let $s_0 = 0$ and for $n \geq 1$, $s_n = s_{n-1} + a_n$. Then inductively define

$$a_n = \begin{cases} 1, & z - s_{n-1} \geq 2^{-n} \\ 0, & \text{otherwise} \end{cases}$$

From this construction, one can see $0 \leq z - s^n < 2^{-n}$ for each n, and since $2^{-n} \to 0$, $z - s^n \to 0$. Of course, $s_n = \sum_{k=1}^n 2^{-k} a_k$, so $\sum_{k=1}^n 2^{-k} a_k = z$.

So for all $x_0, y_0 \in I^2$, one can build a binary sequence $\{a_n\}$ such that $x_0 = \sum_{n=1}^\infty 2^{-n} a_{2n-1}$, $y_0 = \sum_{n=1}^\infty 2^{-n} a_{2n}$. Now, let

$$t := \sum_{n=1}^\infty 3^{-n-1} 2 a_n$$

where $t \in E$ by Ch. 3, Q19. Now, for any integer k,

$$\begin{aligned}
f(3^k t) &= f(\sum_{n=1}^{\infty} 2a_n 3^{k-n-1}) \\
&= f(2\sum_{n=1}^{k-1} a_n 3^{k-n-1} + \sum_{n=k}^{\infty} 2a_n 3^{k-n-1}) \\
&= f(\sum_{n=k}^{\infty} 2a_n 3^{k-n-1}) && (1) \\
&= f(\frac{2}{3}a_k + \underbrace{\sum_{n=k+1} 2a_n 3^{k-n-1}}_{\delta}) && (2)
\end{aligned}$$

where (1) follows because f has a periodicity of 2. Notice that due the indexing of the sum, $\delta < 1/3$. Consequently, if $a_k = 0$, then $0 \leq \frac{2}{3}a_k + \delta < 1/3$, which means by (2), $f(3^k t) = 0$. Similarly, if $a_k = 1$, then $\frac{2}{3} \leq \frac{2}{3}a_k + \delta$, so $f(3^k t) = 1$. Either way, $f(3^k t) = a_k$.

Using this, one easily sees

$$x(t) = \sum_{n=1}^{\infty} 2^{-n} a^{2n-1} = x_0 \text{ and } y(t) = \sum_{n=1}^{\infty} 2^{-n} a^{2n} = y_0$$

This shows that $f(E) = I^2$.

Q15

f is uniformly continuous on $[0, \infty)$. Fix $\epsilon > 0$. Because the f_n's are equicontinuous,

$$\exists \delta > 0 \text{ such that } \forall n \in \mathbb{Z}_+, \forall x, y \in [0,1] \text{ with } |x-y| < \delta, |f_n(x) - f_n(y)| < \epsilon$$

Take any $x, y > 0$ such that $|x - y| < \delta$. Let $n := \max\{\lceil x \rceil, \lceil y \rceil\}$. Then

$$\frac{x}{n}, \frac{y}{n} \in [0,1] \text{ and } \left|\frac{x}{n} - \frac{y}{n}\right| = \frac{1}{n}|x-y| < \delta$$

By equicontinuity, $|f_n(x/n) - f_n(y/n)| = |f(x) - f(y)| < \epsilon$, which shows that f is uniformly continuous on $[0, \infty))$.