Math140B - HW #2

Jay Ser

2026.01.18

1. Q6

Suppose g is not monotonic increasing on $(0,+\infty)$. Then $\exists t>0$ such that $g'(t)<0$. By the quotient rule for derivaties,

$$g'(x)=\frac{xf'(x)-f(x)}{x^2}$$

Thus

$$g'(t)<0 \iff tf'(t)-f(t)<0 \iff f'(t)<f(t)/t \iff f'(t)<\frac{f(t)-f(0)}{t-0}$$

By the Mean Value Theorem, $\exists c\in(0,t)$ such that $f'(c)=\frac{f(t)-f(0)}{t-0}$. But this equality is equivalent to saying $f'(c)>f'(t)$ for $c<t$. This contradicts the assumption that f' is monotonic increasing. Thus g is monotonic increasing.

2. Q11

I first show that

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

is an alternative definition for $f'(x)$, should it exist. More specifically, $f'(x)$ exists $\iff$ $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ exists. If so, $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$.

First, suppose $f'(x)$ exists and fix $\epsilon>0$. $\exists\delta>0$ such that $\forall x'\in \mathrm{B}_\delta(x)$, $\left|\frac{f(x')-f(x)}{x'-x}-f'(x)\right|<\epsilon$. Now, if $|h|<0$, then $x+h\in \mathrm{B}_\delta(x)$, so $\left|\frac{f(x+h)-f(x)}{h}-f'(x)\right|<\epsilon$. This shows $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$.

Conversely, suppose $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ exists; say the value of the limit is m. Fix $\epsilon>0$. $\exists\delta>0$ such that $\forall|h|<\delta$, $\left|\frac{f(x+h)-f(x)}{h}-m\right|<\epsilon$. Then $\forall x'$ satisfying $|x'-x|<\delta$, $\left|\frac{f(x')-f(x)}{x'-x}-m\right|<\epsilon$. This shows $f'(x)$ exists, and $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

An essentially identical proof shows that

$$f'(x)=\lim_{h\to 0}\frac{f(x-h)-f(x)}{-h}$$

is yet another alternative definition.

With these alternative definitions, solving the question is trivial. Because $f''(x)$ exists, the limits

$$\lim_{h\to 0} \frac{f'(x+h)-f'(x)}{h} \quad \text{and} \quad \lim_{h\to 0} \frac{f'(x-h)-f(x)}{-h}$$

exist. Thus

$$\lim_{h\to 0} \frac{f'(x+h)-f'(x-h)}{2h} = \frac{1}{2}\lim_{h\to 0} \frac{f'(x+h)-f'(x)}{h} + \frac{1}{2}\lim_{h\to 0} \frac{f'(x-h)-f'(x)}{-h} = f''(x)$$

Finally, L'Hospital's rule applies to the the initial limit of interest: $h^2 \to 0$ and $f(x+h)+f(x-h)-2f(x) \to 0$ as $h \to 0$, and the associated limit of the ratio of the derivatives (with respect to h) of the denominator and the numerator exists, yielding

$$\lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2} = \lim_{h\to 0} \frac{f'(x+h)-f'(x-h)}{2h} = f''(x)$$

as desired.

3. Q15

For any $x > a$ and $h > 0$, applying Taylor's Theorem with $n = 2$, $\alpha = x$ and $\beta = x + 2h$ yields a $\gamma \in (x, x+2h)$ such that

$$f(x+2h) = f(x) + f'(x)(x+2h-x) + \frac{f''(\gamma)}{2}(x+2h-x)^2$$

which can be reorganized as

$$f'(x) = \frac{1}{2h}(f(x+2h)-f(x)) - hf''(\gamma)$$

Euclidean norming both sides of the equation, then applying the triangle inequality to the right hand side,

$$|f'(x)| \leq \left|\frac{1}{2h}f(x+2h)\right| + \left|\frac{1}{2h}f(x)\right| + |hf''(\gamma)| \leq \underbrace{M_0/h + hM_2}_{(*)}$$

Because this inequality holds for all $x > a$, $(*)$ is an upper bound for $|f'|$ on (a, ∞), which means

$$M_1 \leq M_0/h + hM_2 \tag{1}$$

But because h was chosen to be an arbitrary positive value, the bound on M_1 can further be minimized by finding $h > 0$ that minimizes $(*)$. To find such an h, define $\Delta(h) := M_0/h + hM_2$. Then

$$\Delta'(h) = M_2 - M_0/h^2, \quad \Delta''(h) = 2M_0/h^3$$

Because $h > 0$, $\Delta''(h) > 0$, which means any $h_0 > 0$ such that $\Delta'(h_0) = 0$ is a relative minima of Δ. Namely, the only h_0 with $\Delta'(h_0) = 0$ satisfies

$$M_2 - M_0/h_0^2 = 0 \iff M_2h_0^2 = M_0 \iff h^2 = M_0/M_2$$

Substituting $M_0 = M_2 h_0^2$ in equality (1) yields

$$M_1 \leq hM_2 + hM_2 = 2hM_2$$

Squaring both sides, then substituting $h^2 = M_0/M_2$,

$$M_1^2 \leq 4h^2 M_2^2 = 4M_0 M_2$$

which is the desired inequality.

To demonstrate the equality $M_1^2 = 4M_0M_2$ can occur, consider $f : (-1, \infty) \to \mathbb{R}$,

$$f(x) = \begin{cases} 2x^2 - 1 & (-1 < x < 0) \\ \frac{x^2-1}{x^2+1} & (0 \leq x < \infty) \end{cases}$$

On $(-1, 0)$, $0 < x^2 < 1 \implies -1 < 2x^2 - 1 < 1$. On $[0, \infty)$, $\left|\frac{x^2-1}{x^2+1}\right| \leq \frac{|x^2|+|1|}{x^2+1} = 1$. This shows $M_0 = 1$. Next, using the quotient rule and chain rule for derivatives, obtain

$$f'(x) = \begin{cases} 4x & (-1 < x < 0) \\ \frac{4x}{(x^2+1)^2} & (0 \leq x < \infty) \end{cases}, \quad f''(x) = \begin{cases} 4 & (-1 < x < 0) \\ 4(\frac{-3x^2+1}{(x^2+1)^2}) & (0 \leq x < \infty) \end{cases}$$

It is clear that $\sup |f'(x)| = 4$ and $\sup |f''(x)| = 4$ on $(-1, 0)$. On $[0, \infty)$, notice

1. $f'(x) < \frac{4x}{x^4} = \frac{4}{x^3}$
2. $f'(x) < \frac{4x}{1}$
3. $|f''(x)| \leq 4\left|\frac{-3x^2-3}{x^2+1} + \frac{4}{x^2+1}\right| \leq 4\left|-3 + \frac{4}{1}\right| = 4$

On $(1, \infty)$, inequality (1) shows $f'(x) \leq 4$. On $[0, 1]$, inequality (2) shows $f'(x) \leq 4$. Clearly $f'(x)$ is always positive on $[0, \infty)$, so $M_1 = 4$. Inequality (3) shows $M_2 = 4$. So $M_1^2 = 4M_0M_2$ is indeed possible.

4. Q16

Showing $\lim_{x\to\infty} f'(x) = 0$ is equivalent to showing

$$\forall \epsilon > 0, \exists x_0 \in \mathbb{R} \text{ such that } \forall x > x_0, |f'(x)| < \epsilon$$

Since f'' is bounded on $(0, \infty)$, let M_2 be the supremum of $|f''|$ on $(0, \infty)$. Now, fix $\epsilon > 0$. Since $\lim_{x\to\infty} f(x) = 0$, $\exists x_0 \in \mathbb{R}$ such that $\forall x > x_0$, $|f(x)| < \frac{\epsilon^2}{4M_2^2}$; without loss of generality, assume $x_0 > 0$. This shows that $|f|$ is bounded above on (x_0, ∞), so there is a least upper bound M_0' of $|f|$ on (x_0, ∞). Of course, by the initial assumption on f'', $|f''|$ is bounded above on (x_0, ∞); let M_2' be the supremum of $|f''|$ on (x_0, ∞). Now, the result from Q15 applies, giving

$$(M_1')^2 \leq 4M_0'M_2'$$

where M_1' is the least upper bound of $|f'|$ on (x_0, ∞). Since

$$M_2' \leq M_2 \text{ and } M_1' \leq \frac{\epsilon^2}{4M_2^2}$$

obtain

$$(M_1')^2 \leq 4M_2 \frac{\epsilon^2}{4M_2} = \epsilon^2 \implies M_1' \leq \epsilon$$

Finally, because M_1' is the supremum of $|f'|$ on (x_0, ∞), the inequality above shows $|f'(x)| < \epsilon$ on (x_0, ∞). This proves that $\lim_{x\to\infty} f'(x) = 0$.

5. Q17

Let $\alpha := 0$ and $n = 3$. Per notation in Theorem 5.15 and substituting the given values,

$$P(t) := \sum_{k=0}^{2} \frac{f^{(k)}(s)}{k!}(t-\alpha)^k = \frac{1}{2}f''(0)t^2$$

Taylor's Theorem applies both when $\beta = +1$ and when $\beta = -1$. When $\beta = +1$, $\exists t \in (-1, 0)$ such that

$$f(-1) = \frac{1}{2}f''(0)(-1)^2 + \frac{1}{6}f^{(3)}(t)(-1)^3 \iff 0 = \frac{1}{2}f''(0) - \frac{1}{6}f^{(3)}(t)$$

When $\beta = -1$, $\exists s \in (0, 1)$ such that

$$f(1) = \frac{1}{2}f''(0)(1)^2 + \frac{1}{6}f^{(3)}(s)(1)^3 \iff 1 = \frac{1}{2}f''(0) + \frac{1}{6}f^{(3)}(s)$$

Combining these two equations, multiplying both sides of the equation by 6, and rearranging yields

$$6 = f^{(3)}(s) + f^{(3)}(t)$$

If $f^{(3)}(s) \geq 3$, there is nothing more to be shown. Otherwise, $f^{(3)}(s) < 3$, which implies $6 - f^{(3)}(t) < 3 \iff f^{(3)}(t) \geq 3$, as was to be shown.

6. Q19

(a)

Fix $\epsilon > 0$. Since $f'(0)$ exists, $\exists \delta > 0$ such that

$$\forall t \in \mathrm{B}_\delta(0), \left| \frac{f(t) - f(0)}{t} - f'(0) \right| < \epsilon/2$$

Also, because $\alpha_n, \beta_n \to 0$ as $n \to \infty$, $\exists N \in \mathbb{Z}_+$ such that

$$\forall n \geq N,\ |\alpha_n| < \delta,\ |\beta_n| < \delta$$

Thus for any $n \geq N$,

$$\left| \frac{f(\alpha_n) - f(0)}{\alpha_n} - f'(0) \right| < \epsilon/2, \quad \left| \frac{f(\beta_n) - f(0)}{\beta_n} - f'(0) \right| < \epsilon/2$$

Then the following inequalities show that $\lim_{n\to\infty} D_n = f'(0)$:

$$\begin{aligned}
|D_n - f'(0)| &= \frac{1}{|\beta_n - \alpha_n|} \left| f(\beta_n) - f(\alpha_n) - (\beta_n - \alpha_n) f'(0) \right| \\
&= \frac{1}{|\beta_n - \alpha_n|} \left| f(\beta_n) - f(0) - \beta_n f'(0) - (f(\alpha_n) - f(0) - \alpha_n f'(0)) \right| \\
&\leq \frac{|\beta_n|}{|\beta_n - \alpha_n|} \left| \frac{f(\beta_n) - f(0)}{\beta_n} - f'(0) \right| + \frac{|\alpha_n|}{|\beta_n - \alpha_n|} \left| \frac{f(\alpha_n) - f(0)}{\alpha_n} - f'(0) \right| \\
&< \epsilon/2 + \epsilon/2 \\
&= \epsilon
\end{aligned}$$

where the final inequality holds because $\alpha_n < 0 < \beta_n \implies |\beta_n - \alpha_n| > |\beta_n|$ and $|\beta_n - \alpha_n| > |\alpha_n|$.

(b)

By assumption on α_n and β_n, $\exists M \in \mathbb{R}$ such that

$$\left| \frac{\alpha_n}{\beta_n - \alpha_n} \right| < \left| \frac{\beta_n}{\beta_n - \alpha_n} \right| \leq M$$

for all $n \in \mathbb{Z}_{\geq 0}$.

Now, the proof follows identically to (a), except that, given $\epsilon > 0$, one needs to set $\delta > 0$ to be the value such that

$$\forall t \in \mathrm{B}_\delta(0),\ \left| \frac{f(t) - f(0)}{t} - f'(0) \right| < \frac{\epsilon}{2M}$$

(c)

Define the sequence $\{\gamma_n\}_{n \in \mathbb{Z}_+}$ as such: Given $n \in \mathbb{Z}_+$, $-1 < \alpha_n < \beta_n < 1$, so f' is continuous on $[\alpha_n, \beta_n]$. The Mean Value Theorem applies, giving $\gamma_n \in (\alpha_n, \beta_n)$ such that

$$f'(\gamma_n) = \frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n} = D_n$$

Since $\alpha_n \to 0$, $\beta_n \to 0$, and $\alpha_n < \gamma_n < \beta_n$,

$$\gamma_n \to 0$$

as well. Now, fix $\epsilon > 0$. Because f' is continuous on $x = 0$, $\exists \delta > 0$ such that

$$\forall t \in \mathrm{B}_\delta(0),\ |f'(t) - f'(0)| < \epsilon$$

Furthermore, $\exists N \in \mathbb{Z}_+$ such that $\forall n \geq N$, $|\gamma_n| < \delta$. Thus, for $n \geq N$,

$$|f'(\gamma_n) - f'(0)| = |D_n - f'(0)| < \epsilon$$

which shows $D_n \to f'(0)$.

7. Q22

(a)

Suppose $\exists x_1 < x_2$ such that $f(x_1) = x_1$ and $f(x_2) = x_2$. By assumption on f, the Mean Value Theorem applies: $\exists c \in (x_1, x_2)$ such that $(x_1 - x_2)f'(c) = f(x_1) - f(x_2)$. Since $x_2 - x_1 > 0$,

$$\begin{aligned} f'(c) &= \frac{f(x_1) - f(x_2)}{x_1 - x_2} \\ &= \frac{x_1 - x_2}{x_1 - x_2} \\ &= 1 \end{aligned}$$

which contradicts the assumption on f'. So f has at most one fixed point.

(b)

Suppose

$$f(t) = t + (1 + e^t)^{-1}$$

has a fixed point, say $f(x) = x = x + (1 + e^x)^{-1}$, which is equivalent to

$$0 = (1 + e^x)^{-1}$$

Of course, $(1 + e^x)^{-1}$ is never equal to zero. So f has no fixed point.

Next, recall from calculus that

$$\frac{d}{dt}(e^t) = e^t, \quad \frac{d}{dt}(\frac{1}{t}) = -\frac{1}{t^2}$$

Using the chain rule, calculate

$$f'(t) = 1 - \frac{e^t}{(1 + e^t)^2}$$

$e^t > 0$ implies both

$$\frac{e^t}{(1 + e^t)^2} > 0, \quad e^t < 1 + e^t < (1 + e^t)^2$$

So $0 < \frac{e^t}{(1+e^t)^2} < 1$, which means

$$0 < f'(t) < 1$$

(c)

Suppose $f(t) \neq t$ for all $t \in \mathbb{R}$. Then, because f is differentiable and thus continuous everywhere, $\forall t \in \mathbb{R}$, $f(t) > t$ or $\forall t \in \mathbb{R}$, $f(t) < t$: supposing not, and $\exists x_1, x_2 \in \mathbb{R}$ such that $f(x_1) > x_1$ and $f(x_2) < x_2$, then the Intermediate Value Theorem (applied to $f(t) - t$) says $\exists c$ between x_1 and x_2 such that $f(c) = c$.

First, assume $f(0) = b > 0$. Let

$$x_0 := \frac{b}{1 - A}$$

where $x_0 > 0$. By the previous paragraph, $f(x_0) > x_0$. Now, by the Mean Value Theorem, $\exists c \in (0, x_0)$ satisfying

$$\begin{aligned} f'(c) &= \frac{f(x_0) - f(0)}{x_0 - 0} \\ &> \frac{x_0 - b}{x_0} \\ &= A \end{aligned}$$

which contradicts the assumption on f'.

Symmetrically, suppose $f(0) = b < 0$. Let

$$x_0 := \frac{b}{1 - A}$$

where $x_0 < 0$. This time, $f(x_0) < x_0$ by the argument in the first paragraph. Applying the Mean Value Theorem, $\exists c \in (x_0, 0)$ such that

$$\begin{aligned} f'(c) &= \frac{f(x_0) - f(0)}{x_0 - 0} \\ &> \frac{x_0 - b}{x_0} \\ &= A \end{aligned}$$

where the inequality "flips" because x_0 in the numerator is negative. Once again, this violates the initial assumption on f'. Of course, $f(0) = 0$ contradicts the assumption that $f(x) \neq x$ on all of $\mathbb{R}$. This shows that $\exists x \in \mathbb{R}$ such that $f(x) = x$.

To show that $\lim_{n\to\infty} x_n = x$ for an arbitrary $x_1 \in \mathbb{R}$, notice the following:

1. By part (a), $t = x$ is the unique value in $\mathbb{R}$ satisfying $f(t) = t$. As such, if $x_1 \neq x$, then $f(x_n) \neq x_n$ for all $n \in \mathbb{Z}_+$.

2. If $x_1 = x$, then $x_n = x$ for all $n \in \mathbb{Z}_+$.

Now, assuming $x_1 \neq x$, let $\Delta := |x_1 - x|$. Then the following induction shows that $|x_n - x| < A^{n-1}\Delta$ for $n \geq 2$. For $n = 2$, suppose $|x_n - x| \geq A\Delta$, which is equivalent to saying $|f(x_1) - f(x)| \geq A\Delta$. By the Mean Value Theorem, $\exists c$ between x_1 and x such that

$$\begin{aligned} f'(c) &= \frac{f(x_1) - f(x)}{x_1 - x} \\ \implies |f'(c)| &= \frac{|f(x_1) - f(x)|}{|x_1 - x|} \\ &\geq \frac{A\Delta}{\Delta} = A \end{aligned}$$

This contradicts the assumption that $|f'(t)| \leq A < 1$ for all $t \in \mathbb{R}$.

For $n > 2$, assume $|x_n - x| < A^{n-1}\Delta$. If $|x_{n+1} - x| \geq A^n\Delta$, then $|f(x_n) - f(x)| \geq A^n\Delta$. By the Mean Value Theorem, $\exists c$ in between x_n and x such that

$$\begin{aligned} f'(c) &= \frac{f(x_n) - f(x)}{x_n - x} \\ \implies |f'(c)| &= \frac{|f(x_n) - f(x)|}{|x_n - x|} \\ &\geq \frac{A^n\Delta}{A^{n-1}\Delta} = A \end{aligned}$$

Once again, this contradicts the assumption on f'. This proves my claim.

Finally, fix $\delta > 0$. Then find $N \in \mathbb{Z}_+$ such that $A^{N-1}\Delta < \delta$. Then for all $n \geq N$,

$$|x_n - x| < |x_N - x| < \delta$$

since $|A| < 1$. This shows that $\lim_{n\to\infty} x_n = x$.

(d)

The path

$$(x_1, x_2) \to (x_2, x_2) \to (x_2, x_3) \to (x_3, x_3) \to (x_3, x_4) \to \dots$$

is indeed a zig zag tending to (x, x) because as implied by the induction proof, $\{|x - x_n|\}$ is a strictly decreasing sequence that tends to 0.

8. Q26

I first follow the derivation outlined in the book. Given a fixed $x_0 \in (a, b]$, let

$$M_0 := \sup_{x\in[a,x_0]} |f(x)|, \quad M_1 := \sup_{x\in[a,x_0]} |f'(x)|$$

Now, pick any $x \in (a, x_0]$. By the Mean Value Theorem, $\exists c \in (a, x)$ such that

$$f'(c) = \frac{f(x) - f(a)}{x - a} = \frac{f(x)}{x - a}$$

Applying absolute value and rearranging,

$$\begin{aligned} |f(x)| &= |f'(c)|\,|x - a| \\ &\leq |f'(c)|\,(x_0 - a) \\ &\leq M_1(x_0 - a) \end{aligned}$$

where the first inequality holds because $a < x \leq x_0$. Combining the above inequality with the assumption on $|f|$ and $|f'|$,

$$|f(x)| \leq M_1(x_0 - a) \leq A(x_0 - a)M_0 \tag{2}$$

where the second inequality, equivalent to $M_1 \leq AM_0$, holds because AM_0 is an upper bound for $|f'(t)|$ on $[a, x_0]$.

Now, notice that if $A(x_0 - a) < 1$, then $M_0 = 0$ necessarily; otherwise, $A(x_0 - a)M_0 < M_0$, and because the latter is the least upper bound for $|f(t)|$ on $[a, x_0]$, $\exists t \in [a, x_0]$ such that $|f(t)| > A(x_0 - a)M_0$, which contradicts the fact that inequality (2) holds for all $x \in [a, x_0]$.

Any $x_0 \in (a, a + 1/A)$ satisfies $A(x_0 - a) < 1$ as desired, forcing $M_0 = 0$, where $M_0 = \sup_{x\in[a,x_0]} |f(x)|$. If $a + 1/A > b$, then $f(x) = 0$ on $[a, b]$ automatically. Denote $a' := a + \frac{1}{a}$ and consider the case $b \geq a'$. By the argument above, $f'(x) = 0$ for $x \in (a, a')$. Let $S = \{x \in [a', b] \mid f(x) \neq 0\}$. To solve the problem, $S = \emptyset$ must be proven. Suppose $S \neq \emptyset$. S is bounded below by a, so there is some

$$\alpha := \inf S$$

The following shows that $f(\alpha) = 0$. If $f(\alpha) \neq 0$, then $\exists \gamma \in (a, \alpha)$ such that

$$f'(\gamma)(\alpha - a) = f(\alpha) - f(a) = f(\alpha)$$

i.e., $f'(\gamma) \neq 0$. But because $\gamma < \alpha = \inf S$, $f(\gamma) = 0$. By the initial assumption, $|f'(\gamma)| \leq |f(\gamma)| = 0$, which is a contradiction.

Now, since f is continuous on the compact set $[a, b]$, $\exists K > 0$ such that $|f(x)| \leq K$ on $[a, b]$. Next, once again by the definition of α,

$$\forall \delta > 0,\ \exists c \in (\alpha, \alpha + \delta \frac{b - a}{K}) \text{ such that } f(c) \neq 0$$

By the Mean Value Theorem, $\exists \gamma \in (\alpha, c)$ such that $f'(\gamma)(c - \alpha) = f(c) - f(\alpha) = f(c)$. Applying absolute value on both sides,

$$\begin{aligned} |f(c)| &= |f'(\gamma)|\,|c - \alpha| \\ &\leq A\,|f'(\gamma)|\,(c - \alpha) \\ &< K \frac{1}{b - a}(c - \alpha) \qquad (3) \\ &< \frac{K}{b - a}\delta \frac{b - a}{K} = \delta \qquad (4) \end{aligned}$$

where inequality (3) is due to the assumption that $b > a + 1/A$, which is equivalent to $A > \frac{1}{b-a}$; and inequality (4) holds because $c \in (\alpha, \alpha + \delta \frac{b-a}{K})$. Since $|f(c)| < \delta$ for arbitrary $\delta > 0$, $|f(c)| = 0$. This contradicts the assumption that $f(c) \neq 0$. Hence, $S = \emptyset$, and I am free!!!