Math140B - HW #4

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Question 7

(a)

Let $S := \int_0^1 f(x)dx$ as in the canonical definition of the integral. Showing $S = \lim_{t\to 0} \int_t^1 f(x)dx$ is equivalent to showing

$$S - \int_t^1 f(x)dx \to 0$$

as $t \to 0$.

Fix some $t \in [0,1]$. Since $f \in \mathcal{R}_0^1$, $\int_0^t f(x)dx$ exists, and

$$S - \int_t^1 f(x)dx = \int_0^t f(x)dx$$

Let $M := \sup_{x\in[0,1]} |f(x)|$. Then $M \geq \sup_{x\in[0,t]} |f(x)|$ and thus

$$\left|\int_0^t f(x)dx\right| \leq M(t-0) = Mt$$

$Mt \to 0$ as $t \to 0$ (from the positive, since the space of consideration is $[0,1]$), and $\left|\int_0^t f(x)dx\right| \geq 0$, so

$$\left|\int_0^t f(x)dx\right| \to 0$$

as $t \to 0$. This implies $\int_0^t f(x)dx = 0$, hence $\lim_{t\to 0} \int_t^1 f(x)dx = S$.

(b)

Consider the two limits

$$\int_1^\infty \frac{\sin x}{x} dx \qquad \text{and} \qquad \int_1^\infty \left|\frac{\sin x}{x}\right| dx$$

The former converges while the latter diverges. Integrating the first limit by parts with $F(x) = \frac{1}{x}$ and $G(x) = -\cos x$,

$$\int_1^b \frac{\sin x}{x} dx = -\frac{\cos x}{x}\Big|_1^b - \int_1^b \frac{\cos x}{x^2} dx$$

Both terms in the sum converge as $b \to \infty$. The first term converges because $-\frac{1}{b} \le \frac{\cos b}{b} \le \frac{1}{b}$ and the lower and upper bounds both tend to 0 as $b \to \infty$. The second term converges by the Integral Test described in Q8 of Rudin: $\int_1^\infty f(x)dx$ converges if and only if $\sum_{n=1}^\infty f(n)$ converges. Here, $\left|\frac{\cos n}{n^2}\right| \le \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ converges, so $\sum_{n=1}^\infty \frac{\cos n}{n^2}$ converges by the Comparison Test. This demonstrates that $\int_1^\infty \frac{\sin x}{x}dx$ converges.

To see that $\int_1^\infty \left|\frac{\sin x}{x}\right| dx$ does not converge, observe the following inequalities arising from basic trigonometry:

$$\begin{aligned}\left|\frac{\sin x}{x}\right| = \frac{|\sin x|}{x} &\ge \frac{\sin^2 x}{x} \\ &= \frac{\cos 2x - 1}{2x}\end{aligned}$$

assuming $x \ge 1$. Now, the Integral Test shows that $\sum_{n=1}^\infty \frac{\cos 2n}{2n}$ converges (e.g., $\int_1^\infty \frac{\cos x}{x}dx$ converges for the same reason that $\int_1^\infty \frac{\sin x}{x}$ converges), and it is already known that $\sum_{n=1}^\infty \frac{1}{2n}$ does not converge. Hence $\sum_{n=1}^\infty \frac{\cos 2n-1}{2n}$ diverges. By the Comparison Test, $\sum_{n=1}^\infty \left|\frac{\sin x}{x}\right|$ diverges as well, and it follows that that $\int_1^\infty \left|\frac{\sin x}{x}\right| dx$ diverges as well.

By the previous analysis, it is immediate that for $c \in (0,1]$, $\int_1^{1/c} \frac{\sin x}{x}dx$ converges while $\int_1^{1/c} \left|\frac{\sin x}{x}\right| dx$ diverges as $c \to 0$ (for any $c \in (0,1]$, both integrals exist because $\frac{\sin x}{x}$ and $\left|\frac{\sin x}{x}\right|$ are continuous on $[c,1]$). Since $\phi(x) := \frac{1}{x}$ is a strictly decreasing function that maps $[c,1]$ to $[1,1/c]$, it is possible to perform change of variables, yielding

$$\int_c^1 x \sin\frac{1}{x} dx = \int_1^{1/c} \frac{\sin x}{x}dx \quad \text{and} \quad \int_c^1 x\left|\sin\frac{1}{x}\right| dx = \int_1^{1/c} \left|\frac{\sin x}{x}\right| dx$$

Letting $f(x) = x\sin\frac{1}{x}$, one sees $\int_c^1 f(x)dx$ converges while $\int_c^1 |f(x)|\,dx$ diverges as $c \to 0$, as was to be demonstrated.

Question 10

(a)

Here, I use some facts from previous calculus classes without proof; I hope they will be excused...

First, notice that since $1/p + 1/q = 1$ and $p, q \ge 0$,

$$p = \frac{q}{q-1},\ q = \frac{p}{p-1},\ \text{which implies } p,q > 1$$

Now, take any $v \ge 0$ and define

$$f(u) = \frac{u^p}{p} - \frac{v^q}{q} - uv$$

Regardless of the specific value of p, u^p is continuous on $[0,\infty)$, hence $f(u)$ is continuous on $[0,\infty)$. Also, because $p > 1$, u^p grows faster than $-uv$, hence $f(u) \to \infty$ as $u \to \infty$. Next,

$$f'(u) = u^{p-1} - v$$

Note $f'(0) = -v < 0$. Because $f(u) \to \infty$ and f' is continuous on $[0, \infty)$, there must be a point $u_0 \in [0, \infty)$ such that $f'(u_0) = 0$. But because $p > 1$, f' is monotonic increasing, hence u_0 is a relative minimum of f'. Furthermore, u_0 is the unique value satisfying $f'(u_0) = 0$:

$$f'(u) = 0 \iff u^{p-1} - v = 0 \iff u^{p-1} = v \tag{1}$$

Since $u > 0$, only a single value for u satisfies $u^{p-1} - v = 0$, namely $u = u_0$. The final equation of (1), gives two identities satisfied by u_0:

$$u_0^{(p-1)q} = v^q \iff u_0^p = v^q$$
$$u_0^{p-1}u_0 = u_0 v \iff u_0^p = u_0 v$$

Namely, the if and only if signifies that u_0 is the unique value that satisfies these identities.

Recapitulating, the above shows that u_0 is the only relative extreme of f. Particularly, f obtains a relative minimum on u_0. Using the relations satisfied by u_0,

$$\begin{aligned} f(u_0) &= \frac{u_0^p}{p} + \frac{v^q}{q} - u_0 v \\ &= u_0^p(\frac{1}{p} + \frac{1}{q} - 1) \\ &= 0 \end{aligned}$$

which means $f(u) \geq 0$. This yields the following conclusions, as desired:

1. $uv \leq \frac{u^p}{p} + \frac{v^q}{q}$ for any $u, v \geq 0$.
2. Equality holds if and only if $u^p = v^q$.

(b)

By the inequality derived in (a), so $fg \leq \frac{f^p}{p} + \frac{g^q}{q}$. By Theorem 6.12(b),

$$\begin{aligned} \int_a^b fg d\alpha &= \int_a^b (\frac{f^p}{p} + \frac{g^p}{p}) d\alpha \\ &= \frac{1}{p}\int_a^b f^p d\alpha + \frac{1}{q}\int_a^b g^q d\alpha \\ &= 1/p + 1/q = 1 \end{aligned}$$

where all the integrals are guaranteed to exist due to Theorems 6.12 and 6.13.

(c)

Let

$$F := \left[\int_a^b |f|^p \, d\alpha\right]^{1/p}, \; G := \left[\int_a^b |g|^q \, d\alpha\right]^{1/q}$$

The desired inequality is equivalent to

$$\frac{\left|\int_a^b fg d\alpha\right|}{FG} \leq 1$$

Let $h(x) := \frac{|f(x)|}{F}$ and $q(x) := \frac{|g(x)|}{G}$. Notice

$$\begin{aligned}\int_a^b h^p d\alpha &= \int_a^b \frac{|f|^p}{F^p} d\alpha \\ &= \frac{1}{F^p}\int_a^b |f|^p \, d\alpha \\ &= \frac{1}{F^p} \cdot F^p \\ &= 1\end{aligned}$$

Similarly, $\int_a^b q^p d\alpha = 1$. By definition, $h, q \geq 0$; $|h| = h$ and $|q| = q$. Then, by Theorem 6.25 and the result from part (b),

$$\begin{aligned}\frac{\left|\int_a^b fg d\alpha\right|}{FG} &\leq \int_a^b \frac{|f|}{F}\frac{|g|}{G} d\alpha \\ &= \int_a^b hq d\alpha \\ &\leq 1\end{aligned}$$

as was to be shown.

Question 15

(a)

First, I show the equality

$$\int_a^b xf(x)f'(x)dx = -1/2$$

Define $H(x) := xf(x)$ and $G(x) = f(x)$. Then $h(x) = f(x) + xf'(x)$ and $g(x) = f'(x)$. Integrating by parts,

$$\begin{aligned}\int_a^b xf(x)f'(x)dx &= \int_a^b H(x)g(x)dx \\ &= H(x)G(x)\big|_a^b - \int_a^b h(x)G(x)dx \\ &= xf^2(x)\big|_a^b - \int_a^b f^2(x)dx - \int_a^b xf(x)f'(x)dx\end{aligned}$$

Algebraically manipulating the first expression and last expression of the equalities above and substituting known values,

$$2\int_a^b xf(x)f'(x)dx = -1$$

which gives the desired equality.

(b)

Next, I show the inequality

$$\int_a^b [f'(x)]^2\, dx \cdot \int_a^b x^2 f^2(x) dx \geq 1/4$$

Let $p = q = 2$. Then $1/p + 1/q = 1$. Define the functions $r(x) := xf(x)$ and $t(x) = f'(x)$. By Q10(c),

$$\left| \int_a^b r(x)t(x)dx \right| \leq \left[\int_a^b |r(x)|^2\, dx \right]^{1/2} \left[\int_a^b |t(x)|^2\, dx \right]^{1/2} \tag{1}$$

Because $r(x)$ and $t(x)$ are real-valued functions, (1) substitutes to

$$\left| \int_a^b xf(x)f'(x)dx \right| = 1/2 \leq \left[\int_a^b [f'(x)]^2\, dx \right]^{1/2} \cdot \left[\int_a^b x^2 f^2(x) dx \right]^{1/2}$$

and squaring both sides gives the desired inequality.

*** one should be able to show the strict equality by checking the criterion for equality outlined in Q10, but that is very tedious.