Math140B - HW #3

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1. Q1

f is continuous on $[a, b]$ except at x_0, and is bounded by definition. α is continuous at x_0, so $R \in \mathcal{R}_a^b(\alpha)$ by Theorem 6.10. Let P be any partition of P. There is an interval in the partition such that $x_{i-1} \leq x_0 \leq x_i$ By definition of a partition, $x_{i-1} < x_i$, so another point besides x_0 exists in this interval; thus $m_i = 0$. Obviously, $m_j = 0$ for all $j \neq i$. This shows

$$L(P, f, \alpha) = 0$$

Because this holds for all P,

$$\int_a^b f d\alpha = \underline{\int_a^b} f d\alpha = 0$$

2. Q2

Suppose $\exists x \in [a, b]$ such that $f(x) > 0$. Let $\epsilon = f(x)$. Because f is continuous,

$$\exists \delta > 0 \text{ such that } \forall x' \in \mathrm{B}_\delta(x),\ |\epsilon - f(x')| < \epsilon/2$$

Namely, one can shrink δ such that $a < x - \delta < x + \delta < b$. Choose a partition P of $[a, b]$ such that

$$x - \delta < x_{i-1} < x_i < x + \delta$$

for some i. Then $\forall x' \in [x_{i-1}, x_i]$, $|\epsilon - f(x')| < \epsilon/2$. Particularly, $f(x') > \epsilon/2$. Because this holds for all $x' \in [x_{i-1}, x_i]$, $\epsilon/2$ is a lower bound for $f(t)$ on $[x_{i-1}, x_i]$, which shows $m_i \geq \epsilon/2 > 0$. Now,

$$L(P, f) \geq m_i \Delta x_i \geq \epsilon \Delta x_i/2 > 0$$

So $\underline{\int_a^b} f dx \geq \epsilon \Delta x_i/2 > 0$. Since $\int_a^b f dx$ exists, $\underline{\int_a^b} f dx = \int_a^b f dx$. This contradicts the assumption that $\int_a^b f dx = 0$. This shows that $f(t) = 0$ on all of $[a, b]$.

3. Q3

(a)

Given any partition P of $[-1,1]$, notice that $\Delta\beta_{1,i} \neq 0 \iff x_{i-1} \leq 0 < x_i$. Particularly, if k is the interval satisfying this condition, $\Delta\beta_{1,k} = 1$, and

$$U(P,f,\beta_1) - L(P,f,\beta_1) = M_k - m_k$$

Now, suppose $f(0+) \neq f(0)$. Then $\exists\epsilon > 0$ such that $\forall\delta > 0$, $\exists t \in (0,\delta)$ with $|f(t) - f(0)| \geq \epsilon$. Then for any partition P of $[-1,1]$, the partition k such that $x_{k-1} \leq 0 < x_k$ always contains a point with $|f(t) - f(0)| \geq \epsilon$. Suppose $f(t) > f(0)$. $M_k \geq f(t)$ and $-m_k > -f(0)$, so

$$M_k - m_k \geq f(t) - f(0) \geq \epsilon$$

Symmetric argument shows $f(0) < f(t) \implies M_k - m_k \geq \epsilon$. Consequently, $f \notin \mathcal{R}_{-1}^1(\beta_1)$.

Conversely, suppose $f(0+) = f(0)$ and fix $\epsilon > 0$. $\exists\delta > 0$ such that $\forall t \in (0,\delta)$, $|f(t) - f(0)| < \epsilon/2$. Create a partition P of $[-1,1]$ such that one of the partitions is of the form $[x_{k-1}, x_k] = [0, \delta/2]$. Then $M_k - f(0) < \epsilon/2$; supposing not, then $M_k \geq f(0) + \epsilon/2$, so $\exists t \in [0,\delta/2]$ such that $f(t) \geq f(0) + \epsilon/2$, i.e., $|f(t) - f(0)| \geq \epsilon$, contradicting the assumption on δ. Similar argument shows $f(0) - m_k < \epsilon/2$. This derives

$$M_k - m_k = M_k - f(0) + f(0) - m_k < \epsilon/2 + \epsilon/2 = \epsilon$$

This holds for arbitrary ϵ, so $f \in \mathcal{R}_{-1}^1(\beta_1)$.

Assume that $f \in \mathcal{R}_{-1}^1(\beta_1)$. Given any partition P of $[-1,1]$, let k be the index such that $x_{k-1} \leq 0 < x_k$. Then $L(P,f,\beta_1) = m_k \leq f(0)$ and $U(P,f,\beta_1) = M_K \geq f(0)$. Hence $\underline{\int_{-1}^1} f d\beta_1 \leq f(0)$ and $\overline{\int_{-1}^1} f d\beta_1 \geq f(0)$, but $\int_{-1}^1 f d\beta_1 = \underline{\int_{-1}^1} f d\beta_1 = \overline{\int_{-1}^1} f d\beta_1$ by assumption. So $\int_{-1}^1 f d\beta_1 = f(0)$.

(b)

The similar result for β_2 is that

$$f \in \mathcal{R}_{-1}^1(\beta_2) \iff f(0-) = f(0); \text{ if so, } \int_{-1}^1 f d\beta_2 = f(0)$$

The assumptions are exactly symmetric to that of (a), so the result follows by symmetry. Note that $\beta_2(0) = 1$ vs $\beta_1(0) = 0$ is not a significant distinction since they have no effect on the function f itself.

(c)

Given a partition P of $[-1,1]$, two cases need be distinguished:

1. there is an index k such that $x_{k-1} < 0 < x_k$
2. there is an index k such that $x_{k-1} < x_k = 0 < x_{k+1}$

In case 1, $U(P, f, \beta_3) - L(P, f, \beta_3) = M_k - m_k$ as previous. In case 2, $\Delta\beta_{3,k} = \Delta\beta_{3,k+1} = 1/2$, so

$$U(P, f\beta_3) - L(P, f, \beta_3) = 1/2(M_k + M_{k+1} - m_k - m_{k+1})$$

Only this distinction need be adapted to the proof for part (a). Suppose f is not continuous at 0. Then $\exists\epsilon > 0$ such that $\forall\delta > 0$, $\exists t \in (-\delta, \delta)$ such that $|f(t) - f(0)| \geq \epsilon$. Given any partition P of $[-1, 1]$, if P satisfies case 1, then $U(P, f, \beta_3) - L(P, f, \beta_3) \geq \epsilon$ following the exact same logic as (a). Suppose P satisfies case 2. Then $\exists t_1, t_2$ such that $t_1 \in [x_{k-1}, x_k]$, $t_2 \in [x_k, x_{k+1}]$, and $|f(t_1) - f(0)| \geq \epsilon$, $|f(t_2) - f(0)| \geq \epsilon$. Then $M_k - m_k \geq \epsilon$ and $M_{k+1} - m_{k+1} \geq \epsilon$, so

$$U(P, f, \beta_3) - L(P, f, \beta_3) \geq 1/2(\epsilon + \epsilon) = \epsilon$$

Thus $f \notin \mathcal{R}_{-1}^1(\beta_3)$.

The converse is even more straightforward to show. The only difference is that now, once one fixes $\epsilon > 0$ and obtains the $\delta > 0$ satisfying continuity, one creates a partition P that contains an interval of the form $(-\delta/2, \delta/2)$. The rest of the proof follows directly as was shown in (a).

(d)

Suppose f is continuous at 0. Then $f(0-)$ and $f(0+)$ both exist and are equal to $f(0)$. $\int_{-1}^1 f d\beta_1 = \int_{-1}^1 f d\beta_2 = f(0)$ have both been shown, so only $\int_{-1}^1 f d\beta_3 = 0$ need be shown. Well, once again, take any partition P of $[-1, 1]$. If P satisfies case 1, then $L(P, f, \beta_3) = m_k \leq f(0)$ and $U(P, f\beta_3) = M_k \geq f(0)$ as (a). If P satisfies case 2, then $f(0)$ occurs in both intervals k and $k + 1$, so

$$L(P, f, \beta_3) = 1/2(m_k + m_{k+1}) \leq 1/2(f(0) + f(0)) = f(0)$$

and $U(P, f, \beta_3) \geq f(0)$ similarly. Hence $\underline{\int}_{-1}^1 f d\beta_3 \leq f(0)$ and $\overline{\int}_{-1}^1 f d\beta_3 \geq f(0)$, but $\int_{-1}^1 f d\beta_3 = \underline{\int}_{-1}^1 f d\beta_3 = \overline{\int}_{-1}^1 f d\beta_3$ by assumption. So $\int_{-1}^1 f d\beta_3 = f(0)$.

4. Q4

See Q5(a); the exact same analysis applies, except now every $m_i = 0$ and hence $L(P, f) = 0$. But $U(P, f) = b - a$ still; this holds for every partition P of $[a, b]$, so the lower and upper limits are not equal.

5. Q5

(a) $f^2 \in \mathcal{R}$ does not imply $f \in \mathcal{R}$

For example, take

$$f : [a, b] \to \mathbb{R};\ f(x) = \begin{cases} 1 & (x \in \mathbb{Q}_{[a,b]}) \\ -1 & (x \notin \mathbb{Q}_{[a,b]}) \end{cases}$$

Then $f^2(x) = 1$ for all $x \in \mathbb{R}$, so $f^2 \in \mathcal{R}_a^b$. However, $f \notin \mathcal{R}_a^b$: for any partition P of $[a, b]$, every interval $[x_{i-1}, x_i]$ of P contains an irrational point and a rational point, hence every $m_i = -1$ and every $M_i = 1$. Then $L(P, f) = -(b - a)$, $U(P, f) = b - a$. Because this holds for every P, the lower and upper limits of f are not equal.

(b) $f^3 \in \mathcal{R} \implies f \in \mathcal{R}$

Unlike $\sqrt{x}$, $\sqrt[3]{x}$ is a continuous function that is defined on all of $\mathbb{R}$. So $\sqrt[3]{f^3} = f$ is defined on the entire image of f^3 (which is bounded because $f^3 \in \mathcal{R}$), so $f \in \mathcal{R}$ by Theorem 6.11.

6. Q6

Let E be the Cantor set with E_k a union of 2^k intervals of length 3^{-k} as defined in pg. 41 of Rudin. Fix $\epsilon > 0$ and pick $N \in \mathbb{Z}_+$ such that $(2/3)^N < \epsilon$. Now, create 2^N intervals (u_j, v_j) in the following way: if a and b are the left and right endpoints of the jth segment in E_N, pick $\eta > 0$ such that $(\frac{2}{3+2\eta})^N < \epsilon$ (such an η exists due to choice of N). Then define

$$u_j = a - \eta,\ v_j = b + \eta$$

Namely, one can shrink η such that the intervals (u_j, v_j) are mutually disjoint. Then $K := \bigcup(u_j, v_j) \supset E_N \supset E$ and the lengths of the intervals, given by $\sum_{i=1}^{2^N} 3^{-N} = (\frac{2}{3})^N$, is less than ϵ.

$[0,1] \setminus K$ is closed and bounded, so it is compact. f is continuous on $[0,1] \setminus K$, so $\exists \delta > 0$ such that

$$\forall s, t \in [0,1] \setminus K,\ |s - t| \implies |f(s) - f(t)|$$

Now, form a partition $P = \{x_0, \ldots, x_n\}$ of $[0,1]$ such that:

1. every u_j and v_j is in P.
2. no point of (u_j, v_j) is in P.
3. for every partition index i such that $x_{i-1} \neq u_j$ for some j, $\Delta x_i < \delta$.

Then for every i such that $x_{i-1} \neq u_j$ for some j, $M_i - m_i < \epsilon$ by choice of P and δ. Also, denoting $M := \sup_{x \in [0,1]} |f(x)|$, $M_i - m_i < 2M$ for all i. Then

$$\begin{aligned} U(P,f) - L(P,f) &= \sum_{i=1}^{n} (M_i - m_i)\Delta x_i \\ &\leq \underbrace{\epsilon(1-0)}_{(*)} + \underbrace{2M(\frac{2}{3+2\eta})^N}_{(**)} \\ &< \epsilon + 2M\epsilon \end{aligned}$$

where $(*)$ accounts for i such that $x_{i-1} \neq u_j$ and $(**)$ accounts for i such that $x_{i-1} = u_j$ for some j. Namely, in $(**)$, there are 2^N such intervals in P, which are just some (u_j, v_j), each of length $\frac{1}{3+2\eta}$. Because this holds for arbitrary ϵ, $f \in \mathcal{R}_0^1$.

7. Q8

Suppose $\int_1^\infty f dx$ converges to L. For each $k \in \mathbb{Z}_+$, define

$$a_k = \int_1^k f dx$$

Then $\lim_{k\to\infty} a_k = L$: fix $\epsilon > 0$. Since $\int_1^\infty f dx$ converges, $\exists B \geq 1$ such that $\forall b \geq B$, $\left|\int_1^b f dx - L\right| < \epsilon$. Setting $N = \lceil B \rceil$, it is clear that for every integer $n \geq N$, $|a_k - L| < \epsilon$.

With this setup, one can prove that the Cauchy Criterion for series applies to $\sum_{n=1}^\infty f(n)$. Fix $\epsilon > 0$. Because $\{a_k\}$ converges, it is Cauchy. So $\exists N \in \mathbb{Z}_+$ such that $\forall n \geq m \geq N$,

$$|a_n - a_m| = \left|\int_m^n f dx\right| = \int_m^n f dx < \epsilon$$

where the second inequality is due to f being nonnegative. Let P be a partition of $[m, n]$ whose points are exactly the integers between m and n, inclusive:

$$P = \{m, m+1, \dots, n-1, n\}$$

For notational simplicity, I will start indexing the x_k's of P from m and end at n, i.e., $x_m = m, \dots, x_n = n$, and denote interval k as the one ranging $[x_{k-1}, x_k] = [k-1, k]$. Because f is monotonic decreasing,

$$m_k = f(k)$$

Of course, $\Delta x_k = 1$ for every k. Thus,

$$L(P, f) = \sum_{k=m+1}^{n} f(k)$$

Finally, $|L(P, f)| = L(P, f) \leq \int_m^n f dx < \epsilon$, thus satisfying the Cauchy Criterion for convergent series. This holds for arbitrary $\epsilon > 0$, so $\sum_{n=1}^\infty f(n)$ converges.

Conversely, suppose $\int_1^\infty f dx$ diverges. Looking at the proof of the convergence of $\{a_k\}_k$ in the previous paragraph, it is easy to see that, in fact, $\{a_k\}_k$ converges $\iff$ $\int_1^\infty f dx$ converges. So $\{a_k\}$ diverges under this assumption, and because it is a sequence in $\mathbb{R}$, $\{a_k\}$ is not a Cauchy Sequence. Thus, $\exists \epsilon > 0$, $\exists n \geq m \geq N$ such that

$$|a_n - a_m| = \left|\int_m^n f dx\right| = \int_m^n f dx \geq \epsilon$$

where N is any positive integer. Define the same partition $P = \{m, m+1, \dots, n-1, n\}$ as before. Because f is monotonic decreasing, $M_k = f(k-1)$, so

$$U(P, f) = \sum_{k=m+1}^{n} f(k-1) = \sum_{k=m}^{n-1} f(k)$$

But $U(P, f) \geq \int_m^n f dx \geq \epsilon$. In other words, the Cauchy Criterion for the series $\sum_{n=1}^\infty f(n)$ fails, so the series diverges.