Math188 - HW #2

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2026.01.20

1.

By definition, a partition of n is a strong composition of n. Namely, a partition of n into k parts is a strong composition of n into k parts. Thus, the number of partitions of n into k parts is at most the number of compositions of n into k parts, which is given by $\binom{n-1}{k-1}$, i.e.,

$$p(n,k) \leq \binom{n-1}{k-1}$$

Note that equality is indeed possible: $p(n,n) = 1 = \binom{n-1}{n-1}$, for example.

The lower bound for $p(n,k)$ is given by noting that at most $k!$ distinct k-part compositions of n correspond to a single k-part partition of n; namely, the k-part compositions of n that are rearrangements of each other correspond to a single k-part partition of n whose elements are the elements of the compositions in a weakly decreasing order. There are *at most* $k!$ distinct compositions corresponding to a single partition because if each element in the composition was unique, then $k!$ rearragments would be possible. However, the compositions may have repeating elements. Thus dividing the the total number of compositions of n into k parts by $k!$ potentially undercounts the number of distinct partitions of n into k, i.e.,

$$\frac{1}{k!}\binom{n-1}{k-1} \leq p(n,k)$$

Once again, equality is possible. For example, $p(n,1) = 1 = \frac{1}{1!}\binom{n-1}{1-1}$

2.

I couldn't reach a solution, but the work below may perhaps be relevant to a working solution.

Given a polynomial $f(x)$, let its degree be $m-1$ and its leading coefficient A. If $A < 0$, then $f(x) \to -\infty$ as $x \to \infty$, so $f(n) < p(n)$ is satisfied automatically for arbitrarily big n's.

Suppose $A > 0$. One knows from calculus that $g(x) = Bx^m +$ lower degree terms, where $B > 0$, grows faster than f as $x \to \infty$ and thus $g(n) > f(n)$ for arbitrarily big integers. Particularly, choose $g(x) = (x+1)^m$

Using the binomial theorem and the inequality from Q1,

$$\begin{aligned}
(n+1)^m &= \sum_{t=0}^{m} \binom{m}{t} n^t \\
&\leq \sum_{t=0}^{m} n^t (t+1)! p(m+1, t+1) \\
&\leq M_n \sum_{t=0}^{m} p(m+1, t+1) \\
&= M_n p(2m+2, m+1) \\
&< M_n p(2m+2)
\end{aligned}$$

where $M_n = n^m (m+1)!$.

Unfortunately, this is a poor bound. Here, the size of the partition, pertaining to the term $p(2m+2)$, is independent of n, and as far as I know, there is no way to relate the term M_n to the number of partitions of n. By definition, the growth of M_n as n increases scales with n^m, where m is fixed as initlaly defined. If I could count some partition of a quantity related to n, perhaps I could control M_n to make further progress.

3.

The equality to demonstrate is

$$c(n,k) = c(n-1, k-1) + c(n-1, k)$$

Starting from the definition of the left hand side,

$$\begin{aligned}
c(n,k) &= \underbrace{\text{\# of permutations in } S_n \text{ with } k \text{ cycles}}_{\text{—"—}} \\
&= (\text{—"— such that } n \mapsto n) + (\text{—"— such that } n \mapsto m,\ n \neq m) \\
&= (\text{\# of permutations in } S_{n-1} \text{ with } k-1 \text{ cycles}) && (1) \\
&\quad + (n-1)(\text{\# of permutations in } S_{n-1} \text{ with } k \text{ cycles}) && (2) \\
&= c(n-1, k-1) + c(n-1, k)
\end{aligned}$$

where the equality between (1) and (2) is interpreted as the following: the first term in (1) can be thought of as counting permutations in S_{n-1} with $k-1$ cycle since n forms a single cycle mapping to itself; for the second term in (1), consider $\sigma \in S_{n-1}$ with k cycles. Since $n \mapsto m$, $m \neq n$, "concatenating" n into any of the k cycles of σ yields a permutation in S_n with k cycles. namely, there are $n-1$ ways to create distinct permutations of S_n from σ since there are $n-1$ choices for m.

4.

I unfortunately couldn't find the combinatorial proof for the equality of the two polynomials, so I use induction instead. The equality to demonstrate is

$$\sum_{k=0}^{n} c(n,k) x^k = x(x+1)\cdots(x+n-1)$$

$n = 1$ is trivial because the left hand side is equal to $c(1,0) + c(1,1)x = x$, where $c(n,0) = 0$ for any n. Next, suppose $\sum_{k=0}^{n} c(n,k)x^k = x(x+1)\cdots(x+n-1)$; the identity must be demonstrated for $n+1$. By Q3),

$$\begin{aligned}\sum_{k=0}^{n+1} c(n+1,k)x^k &= \sum_{k=0}^{n+1}[c(n,k-1) + nc(n,k)]x^k \\ &= \sum_{k=-1}^{n} c(n,k)x^{k+1} + n\sum_{k=0}^{n+1} c(n,k)x^k \\ &= x \cdot x(x+1)\cdots(x+n-1) + nx(x+1)\cdots(x+n-1) \\ &= x(x+1)\cdots(x+n-1)(x+n)\end{aligned}$$

where the third equality follows from $c(n,k) = 0$ if $k \le 0$ or $k > n$. This proves the inductive step, so the identity follows.

5.

The equality to demonstrate is

$$C_{n+1} = \sum_{k=0}^{n} C_k \cdot C_{n-k}$$

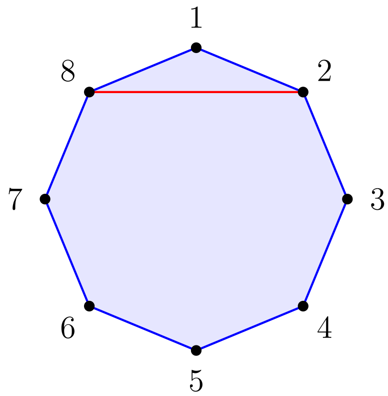
Any Dyck path of size $n+1$ visits a coordinate (k,k) at least once, where $0 \le k \le n$. Thus, denoting $C_{n+1,k}$ to be the number of Dyck paths of size $n+1$ where k denotes the most Northeast (k,k) coordinate $(0 \le k \le n)$, visited by the path,

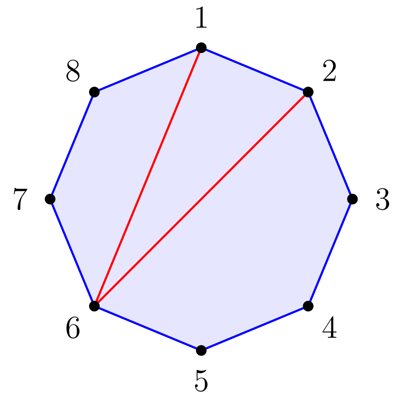
$$C_{n+1} = \sum_{k=0}^{n} C_{n+1,k}$$

Let p be one of the Dyck paths belonging to $C_{n+1,k}$. Then p's path from $(0,0)$ to (k,k), then from (k,k) to $(n+1,n+1)$ uniquely describes p amongst all Dyck paths of size $n+1$. Namely, there are C_k paths from $(0,0)$ to (k,k); after reaching the (k,k) coordinate, the path never goes strictly below the line $y = x+1$ unless p reaches $y = n+1$. The number of such paths from (k,k) to (n,n) is equivalent to the number of modified Dyck paths from $(k,k+1)$ to $(n,n+1)$ that never goes strictly below the shifted diagonal $y = x+1$; this is equivalent to the number of regular Dyck paths from $(0,0)$ to $(n-k,n-k)$. So, $C_{n+1,k} = C_k \cdot C_{n-k}$, giving the desired identity

$$C_{n+1} = \sum_{k=0}^{n} C_k \cdot C_{n-k}$$

6.





Let t_n be the number of triangulations of an n-gon, where $n \geq 2$. Trivially, $t_2 = t_3 = 1$. Next, consider an arbitrary $n \geq 2$. Fix any vertex as vertex 1, then label the vertices clockwise. First, note that in any triangulation of the n-gon, vertex 1 can either have indegree 0 or indegree ≥ 1. If the indegree of vertex 1 is 0, then edges (12) and $(1n)$ are necessarily part of a single triangle (else, the triangulation of the n-gon fails). This further implies that the triangulation includes an edge $(2n)$ such that $\Delta(12n)$ exists. This construction is demonstrated in the top left figure for $n = 8$. Now, the remaining $n-1$ vertices, vertices $2, \ldots, n$, must be triangulated; the number of ways to do so is exactly t_{n-1} because vertex 1 is completely irrelevant now, while the remaining $n-1$ vertices remain unconstrained.

Next, let's count the number of ways to triangluate the n-gon given that vertex 1 is connectd to at least one other vertex (other than the adjacent vertices 2 and n). This is given by

$$\sum_{k=3}^{n-1} (\text{\# of ways to triangulate the } n\text{-gon where } k \text{ is the lowest-valued vertex connected to vertex } 1)$$

Suppose $k \in \{3, \ldots, n-1\}$ is the lowest-valued vertex connected to vertex 1. Because vertex 1 is not connected to any of the vertices $3, \ldots, k-1$, the triangulation includes a triangle $\Delta(12k)$. This construction is demonstrated in the top right figure for $n = 8$, $k = 6$. This leaves the sub-polygons $(23 \cdots k)$ and $(k(k+1) \cdots n1)$ to be freely triangulated; furthermore, the triangulation of the sub-polygons are independent of each other. Thus, for a fixed k, the number of ways to triangulate the n-gon is given by $t_{k-1} \cdot t_{n-k+2}$.

Putting this all together,

$$\begin{aligned} t_n &= t_{n-1} + \sum_{k=3}^{n-1} t_{k-1} \cdot t_{n-k+2} \\ &= \sum_{k=3}^{n} t_{k-1} \cdot t_{n-k+2} \\ &= \sum_{k=2}^{n-1} t_k \cdot t_{n-k+1} \end{aligned}$$

where the second equality is obtained by using the fact that $t_2 = 1$ and absorbing the initial term $t_{n-1} = t_{n-1} \cdot t_2$ into the sum.

This is exactly the same recursion rule for the catalan numbers.

$$c_n = t_{n+2}$$

Verifying this identity, first note that the $n = 0, 1$ cases are trivial since $c_0 = c_1 = t_2 = t_3 = 0$ by definition. Next, using strong induction for n where $c_k = t_{k+2}$ for $k \leq n$,

$$\begin{aligned} t_{n+3} &= \sum_{k=2}^{n+2} t_k \cdot (t_{n-k+4}) \\ &= \sum_{k=0}^{n} t_{k+2} \cdot t_{n-k+2} \\ &= \sum_{k=0}^{n} c_n \cdot c_{n-k} \\ &= c_{n+1} \end{aligned}$$

where the final equality is the result shown in Q5. This shows that the Catalan number C_n counts the number of triangulations of a $n + 2$-gon.

7.

First, I define a parallel definition of **ascents**: given a fixed $winS_n$, an index $1 \leq i \leq n - 1$ is an ascent if $w(i) < w(i + 1)$.

(a)

Suppose $w \in S_n$ has k descents; let $\{i_1, \ldots, i_k\} \subset [n-1]$ be the indices satisfying the descent condition. Then for any index j in the complement $[n - 1] \setminus \{i_1, \ldots, i_k\}$,

$$w(j) < w(j + 1)$$

because $w(j) > w(j + 1)$ is ruled out by definition of j and $w(j) = w(j + 1)$ is ruled out by the fact that w is a permutation (bijection) of $[n]$. Hence there are $n - k - 1$ ascents in w, and it is clear that

$$w = [w(1), w(2), \ldots, w(n)] \mapsto [w(n), w(n - 1), \ldots, w(2), w(1)]$$

sends a permutation of $[n]$ with k descents to a permutation of $[n]$ with $n - k - 1$ descents since the map essentially flips descents into ascents and vice versa.

The reverse map follows the exact same logic. If $w \in S_n$ has $n - k - 1$ descents, send

$$w = [w(1), w(2), \ldots, w(n)] \mapsto [w(n), w(n - 1), \ldots, w(2), w(1)]$$

to obtain a permutation of $[n]$ with k descents. This establishes the bijection that proves $A(n, k) = A(n, n - k - 1)$.

(b)

It is not difficult to see that one can obtain all elements of S_n in the following way: if $w = [w(1), \dots, w(n-1)] \in S_{n-1}$, then then pick any $i \in [n]$. Then define

$$w' = \begin{cases} [w(1), \dots, w(i-1), n, w(i), \dots, w(n-1)] & (1 < i < n) \\ [n, w(1), \dots, w(n-1)] & (i = 1) \\ [w(1), \dots, w(n-1), n] & (i = n) \end{cases} \tag{3}$$

Spanning i over all of $[n]$ and w over all of S_{n-1} produces all the elements of S_n.

Now, I demonstrate the identity

$$A(n,k) = (n-k)A(n-1,k-1) + (k+1)A(n-1,k)$$

The left hand side, by definition, counts the number of $w' \in S_n$ with exactly k descents. Consider all possible $w \in S_{n-1}$ that produces a $w \in S_n$ with k descents via the method described above. Well, if w' is defined as (3), with $1 < i < n$, then index i is a descent since $i \mapsto n$ and n is guaranteed to be the unique maximum value of w' due to $w' \in S_n$. By similar reasoning, index $i-1$ is guaranteed to be an ascent of w'. Of course, if $i = 1$, then index i is again a descent, while if $i = n$, then $i-1$ is an ascent while i itself is neither an ascent nor descent by definition. Thus, "inserting" n to w at position i to form w' either adds one additional descending point if $i = 1$ or $i-1$ is an ascent of w, or it preserves the number of descents if $i = n$ or $i-1$ is already a descent of w.

By the analysis above, the only way to get $w' \in S_n$ with k descents is to form w' from $w \in S_{n-1}$, where either

1. w has $k-1$ descents and n is inserted at some position i such that $i = 1$ or if $i-1$ is an ascent
2. w has k descents and n is inserted at some position i such that $i = n$ or $i-1$ is a descent.

For case 1, there are $A(n-1,k-1)$ ways to choose w and $(n-1)-(k-1)-1+1 = n-k$ ways to choose i since w having $k-1$ descents means it has $(n-1)-(k-1)-1 = n-k-1$ ascents; one of the ascents can be chosen, or position can be chosen. For case 2, there are $A(n-1,k)$ was to choose w and $k+1$ ways to choose i since either one of the k ascents or position 1 can be chosen. Cases 1 and 2 are mutually exclusive, so we get the desired identity

$$A(n,k) = (n-k)A(n-1,k-1) + (k+1)A(n-1,k)$$

Collaboration Disclosure

I used Claude.ai to help with the figures in Q6.