Math100B - HW #1

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1.

By the distributive law and cancellation law, $0r = (0+0)r = 0r + 0r \implies 0r = 0$

2.

Take any $a \in \mathbb{Q}$. Then a is the root of the linear polynomial $x - a$, hence a is an algebraic number.

3.

$(x - (7 + \sqrt{2}))(x - (7 - \sqrt{2})) = (x - 7)^2 - (\sqrt{2})^2 = x^2 - 14x + 47 \in \mathbb{Q}[x]$. By the above computation, $7 + \sqrt{2}$ is a root of the polynomial $x^2 - 14x + 47$, so it is an algebraic number over $\mathbb{Q}$.

Similarly,

$$
\begin{aligned}
&(x - (\sqrt{3} + \sqrt{-5}))(x - (\sqrt{3} - \sqrt{-5}))(x - (-\sqrt{3} + \sqrt{-5}))(x - (-\sqrt{3} - \sqrt{-5})) \\
&= ((x - \sqrt{3})^2 - (\sqrt{-5})^2)((x + \sqrt{3})^2 - (\sqrt{-5})^2) \\
&= (x^2 - 2\sqrt{3}x + 8)(x^2 + 2\sqrt{3}x + 8) \\
&= x^4 + 16x^2 + 64 - (2\sqrt{3}x)^2 \\
&= x^4 + 4x^2 + 64
\end{aligned}
$$

Clearly, $\sqrt{3} + \sqrt{-5}$ is a root of $x^4 + 4x^2 + 64 \in \mathbb{Q}[x]$, so $\sqrt{3} + \sqrt{-5}$ is an algebraic number over $\mathbb{Q}$.

4.

5.

6.

$a \in \mathbb{Z}_n$ is a unit $\iff \exists a' \in \mathbb{Z}_n$ such that $aa' = 1 \iff \exists a' \in \mathbb{Z}$ such that $aa' = 1 \mod n$. An elementary result from number theory is that the last statement is true if and only if $(a, n) = 1$. This shows that the units in $\mathbb{Z}_n$ are equivalence classes of $\mathbb{Z}$ that are prime to n.

7.

$f(x) := x^2 + x + 1$ is monic, so one can divide $g(x) := x^4 + 3x^3 + x^2 + 7$ with remainder by $f(x)$:

$$x^4 + 3x^3 + x^2 + 7x + 5 = (x^2 + 2x - 2)(x^2 + x + 1) + 7x + 7$$

where $r(x) := 7x + 7$ is the remainder.

Reducing modulo n, one sees $f(x) \mid g(x)$ in $\mathbb{Z}/n\mathbb{Z} \iff r(x) = 0$ in $\mathbb{Z}/n\mathbb{Z}$. Clearly, $7x + 7 = 0$ mod $n \iff n$ is a multiple of 7, i.e. $f(x) \mid g(x) \iff 7 \mid n$.

8.