Math140B - HW #5

Jay Ser

2026.02.08

Ch. 6, Q11

I show the equivalent inequality

$$||f-h||_2^2 \le (||f-g||_2 + ||g-h||_2)^2$$

Starting from the left hand side,

$$\begin{aligned}
||f-h||_2^2 &= \int_a^b |f-g+g-h|^2\, d\alpha \\
&= \int_a^b (f-g)^2 d\alpha + \int_a^b (g-h)^2 d\alpha + 2\int_a^b (f-g)(g-h) d\alpha \\
&\le \int_a^b (f-g)^2 d\alpha + \int_a^b (g-h)^2 d\alpha + 2\left[\int_a^b |f-g|^2\, d\alpha\right]^{1/2}\left[\int_a^b |g-h|^2\, d\alpha\right]^{1/2} \quad (1)\\
&= ||f-g||_2^2 + ||g-h||_2^2 + 2\,||f-g||_2\,||g-h||_2 \quad (2)
\end{aligned}$$

where (1) is due to the Holder's inequality as described in Ch. 6, Q10 of Rudin. (2) is equal to $(||f-g||_2 + ||g-h||_2)^2$, the right hand side of the desired result. This proves the inequality.

Ch. 6, Q12

Defining $g(x)$ as given in the book, on $t \in [x_{i-1}, x_i]$, $g(t)$ is just a straight line from $f(x_{i-1})$ to $f(x_i)$ (e.g., put $g(t)$ in slope whatever form). So given any partition P of $[a,b]$,

$$\sup_{t\in[x_{i-1},x_i]} g(t) \le \sup_{t\in[x_{i-1},x_i]} f(t) = M_i$$

$$\inf_{t\in[x_{i-1},x_i]} g(t) \ge \inf_{t\in[x_{i-1},x_i]} f(t) = m_i$$

In turn,

$$\begin{aligned}
\sup_{t\in[x_{i-1},x_i]} |f(t)-g(t)| &\le \max\{\sup_{t\in[x_{i-1},x_i]} f(t) - \inf_{t\in[x_{i-1},x_i]} g(t), \sup_{t\in[x_{i-1},x_i]} g(t) - \inf_{t\in[x_{i-1},x_i]} f(t)\} \\
&\le \max\{M_i - m_i, M_i - m_i\} \\
&= M_i - m_i
\end{aligned}$$

Fix $\epsilon > 0$ and define $M := \sup_{t\in[a,b]} f(t)$ and $m := \inf_{t\in[a,b]} f(t)$, which are guaranteed to exist since $f \in \mathcal{R}_a^b(\alpha)$ and therefore f is bounded. To define a continuous function $g(x)$ such that $||f-g||_2 < \epsilon$, take a partition P of $[a,b]$ such that

$$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i < \frac{\epsilon^2}{M-m}$$

where $M_i = \sup_{t\in[x_{i-1},x_i]} f(t)$ and $m_i = \inf_{t\in[x_{i-1},x_i]} f(t)$ as usual. Then

$$\begin{aligned}
\int_a^b |f-g|^2 \, d\alpha &\leq U(P, |f-g|^2, \alpha) \\
&= \sum_{i=1}^{n} \sup_{t\in[x_{i-1},x_i]} |f(t)-g(t)|^2 \Delta\alpha_i \\
&\leq \sum_{i=1}^{n} (M_i - m_i)^2 \Delta\alpha_i && (1) \\
&\leq (M-m)\sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i \\
&< (M-m)\frac{\epsilon^2}{M-m} && (2) \\
&= \epsilon^2
\end{aligned}$$

where (1) by the analysis in the first paragraph, and (2) by assumption on P. This shows that $||f-g||_2 < \epsilon$.

Ch. 6, Q13

(a)

Performing u-sub with $\sqrt{u} = t$,

$$f(x) = \int_x^{x+1} \sin t^2 dt = \int_{x^2}^{(x+1)^2} \frac{\sin u}{2\sqrt{u}} du$$

Letting $F(u) = 1/\sqrt{u}$ and $G(u) = -\cos u$,

$$\begin{aligned}
f(x) &= -\frac{\cos u}{2\sqrt{u}}\bigg|_{x^2}^{(x+1)^2} - \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}} du \\
&= \frac{\cos x^2}{2x} - \frac{\cos(x+1)^2}{2(x+1)} - \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}} du
\end{aligned}$$

Then, assuming $x > 0$,

$$\begin{aligned}
|f(x)| &\leq \left|\frac{\cos x^2}{2x}\right| + \left|\frac{\cos(x+1)^2}{2(x+1)}\right| + \left|\int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}} du\right| \\
&\leq \frac{1}{2x} + \frac{1}{2(x+1)} + \int_{x^2}^{(x+1)^2} \frac{1}{4u^{3/2}} du \\
&= \frac{1}{2x} + \frac{1}{2(x+1)} - \frac{1}{2}(\frac{1}{x+1} - \frac{1}{x}) \\
&= \frac{1}{x}
\end{aligned}$$

as was to be shown.

(b)

By the calculation in part (a),

$$\begin{aligned}
2xf(x) &= \cos x^2 - \frac{x\cos{(x+1)^2}}{x+1} - x\int_{x^2}^{(x+1)^2} \frac{\cos u}{2u^{3/2}} du \\
&= \cos x^2 - \cos(x+1)^2 + r(x)
\end{aligned}$$

where

$$r(x) = \frac{\cos{(x+1)^2}}{x+1} - x\int_{x^2}^{(x+1)^2} \frac{\cos u}{2u^{3/2}} du$$

Let $F(u) = u^{-3/2}$ and $G(u) = \sin u$ and integrate the last term of $r(x)$ by parts:

$$\begin{aligned}
x\int_{x^2}^{(x+1)^2} \frac{\cos u}{2u^{3/2}} du &= \frac{x}{2}\left[\frac{\sin u}{u^{3/2}}\bigg|_{x^2}^{(x+1)^2} + \frac{3}{2}\int_{x^2}^{(x+1)^2} \frac{\sin u}{u^{5/2}}\right] \\
&= \frac{x}{2}\left[\frac{\sin(x+1)^2}{(x+1)^3} - \frac{\sin x^2}{x^3} + \frac{3}{2}\int_{x^2}^{(x+1)^2} \frac{\sin u}{u^{5/2}}\right]
\end{aligned}$$

Thus, assuming $x > 0$,

$$\begin{aligned}
|r(x)| &\leq \frac{x}{2}\left[\frac{1}{(x+1)^3} + \frac{1}{x^3} + u^{-3/2}\Big|_{x^2}^{(x+1)^2}\right] + \frac{1}{x+1} \\
&= \frac{x}{(x+1)^3} + \frac{1}{x+1}
\end{aligned}$$

For some really big $c \in \mathbb{R}$, this final expression is indeed less than c/x

(d)

I couldn't tell you.

Ch. 6, Q19

I presume "a continuous 1-1 mapping of $[c, d]$ **onto** $[a, b]$" just means $\varphi : [c, d] \to [a, b]$ is bijective and continuous. Since $\varphi(c) = a$, it necessarily follows that $\varphi(d) = b$. If not, surjection of φ guarantees $\exists x \in (a, b)$ such that $\varphi(x) = b$, and because φ is continuous, $\varphi([c, x]) = [a, b]$. This contradicts the assumption that φ is injective.

It is trivial to see that γ_1 is an arc $\iff$ γ_2 is an arc since

$$\gamma_2 = \gamma_1 \circ \varphi, \; \gamma_2 \circ \varphi^{-1} = \gamma_1$$

and composition of injective maps are injective.

Similarly, γ_1 is a closed curve $\iff$ γ_2 is a closed curve since

$$\gamma_1(a) = \gamma_1(\varphi(c)) = \gamma_2(c), \; \gamma_1(b) = \gamma_1(\varphi(d)) = \gamma_2(d)$$

and thus $\gamma_1(a) = \gamma_1(b) \iff \gamma_2(c) = \gamma_2(d)$.

Finally, to see that $\Lambda(\gamma_1) = \Lambda(\gamma_2)$ and γ_1 is rectifiable $\iff$ γ_2 is rectifiable, notice that the bijection and continuity of φ gives the one-to-one correspondence

$$\text{partition } P = \{x_0, \ldots, x_n\} \text{ of } [a, b] \longleftrightarrow \text{partition } Q = \{y_0, \ldots, y_n\} \text{ of } [c, d]$$

where $x_i = \varphi(y_i)$ and $y_i = \varphi^{-1}(x_i)$. Thus for corresponding partitions P of $[a, b]$ and Q of $[c, d]$,

$$\sum_{i=1}^{n} |\gamma_1(x_i) - \gamma_1(x_{i-1})| = \sum_{i=1}^{n} |\gamma_2(y_i) - \gamma_2(y_{i-1})|$$

The desired results follow immediately.

Ch. 7, Q2

Suppose $f_n \to f$ uniformly on E and $g_n \to g$ uniformly on E. Fix $\epsilon > 0$. $\exists N \in \mathbb{Z}_+$ such that $\forall n \geq N$, $|f_n(x) - f(x)| < \epsilon$ and $|g_n(x) - g(x)| < \epsilon$. By triangle inequality,

$$|(f_n + g_n)(x) - (f + g)(x)| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)| = 2\epsilon$$

This shows that $f_n + g_n \to f + g$ uniformly on E.

If $\{f_n\}$ and $\{g_n\}$ are sequences of bounded and uniformly converging sequences, then Ch. 7, Q1 says $\{f_n\}$ and $\{g_n\}$ are both uniformly bounded; say $|f_n(x)| \leq C_1$ and $|g_n(x)| \leq C_2$ for all $x \in E$ and $n \in \mathbb{Z}_+$. Fix $\epsilon > 0$. $\exists N \in \mathbb{Z}_+$ such that $\forall m \geq n \geq N$, $|f_n(x) - f_m(x)| < \epsilon$ and $|g_n(x) - g_m(x)| < \epsilon$. Then, for all $m \geq n \geq N$,

$$\begin{aligned}
|f_n(x)g_n(x) - f_m(x)g_m(x)| &= |f_n(x)g_n(x) - f_n(x)g_m(x) + f_n(x)g_m(x) - f_m(x)g_m(x)| \\
&\leq |f_n(x)(g_n(x) - g_m(x))| + |g_m(x)(f_n(x) - f_m(x))| \\
&< \epsilon \, |f_n(x)| + \epsilon \, |g_m(x)| \\
&\leq \epsilon(C_1 + C_2)
\end{aligned}$$

Since C_1 and C_2 are fixed, the inequality above shows that $\{f_n g_n\}$ converges uniformly by the Cauchy criterion.

Ch. 7, Q3

Let $E = (0, \infty)$ and consider $f_n(x) = x + 1/n$ and $g_n = 1/x$. $\{g_n\}$ is a constant sequence; namely, $g_n \to 1/x$ uniformly. To see that $f_n \to x$ uniformly, fix $\epsilon > 0$ and pick $N \in \mathbb{Z}_+$ such that $1/N < \epsilon$. Then $\forall n \geq N$, $|f_n(x) - x| = |1/n| < \epsilon$.

Now, let

$$h_n(x) := (f_n g_n)(x) = 1 + \frac{1}{nx}$$

It is clear that $h_n \to 1$ pointwise. However, it does not uniformly converge:

$$\lim_{n\to\infty} \lim_{x\to 0} h_n(x) = \infty \neq 0 = \lim_{x\to 0} \lim_{n\to\infty} h_n(x)$$

Theorem 7.11 says that the above cannot happen if $\{h_n\}$ uniformly converges.

Ch. 7, Q7

Let's first analyze $f_n(x) = \frac{x}{1+nx^2}$ for an arbitrary $n \in \mathbb{Z}_+$. Using the quotient rule for derivatives, one calculates

$$f_n'(x) = \frac{1 - nx^2}{(1+nx^2)^2}, \; f_n''(x) = \frac{2nx(nx^2 - 3)}{(nx^2+1)^3}$$

Note that both these f' and f'' are defined everywhere since the denominator of f and f' is never zero.

$f'(x) = 0 \iff 1 - nx^2 = 0 \iff x = \pm 1/\sqrt{n}$ gives the critical points of f. $f''(1/\sqrt{n}) = -\sqrt{n}/2 < 0$ and $f''(-1/\sqrt{n}) = \sqrt{n}/2 > 0$. This shows that f attains a relative maximum at $1/\sqrt{n}$ and relative minimum at $-1/\sqrt{n}$. Notice that because f is continuous on all of $\mathbb{R}$ (its numerator and denominator are continuous everywhere, and the denominator is always positive), these the relative mimimum and maximum are in fact global minimum and maximum. Finally, it's clear that $f_n(x) = -f_n(-x)$ for all $x \in \mathbb{R}$, which means

$$|f_n(x)| \leq f_n(1/\sqrt{n}) = \frac{1}{2\sqrt{n}}$$

Now, let $M_n = \frac{1}{2\sqrt{n}}$ for all $n \in \mathbb{Z}_+$. By the paragraph above, $M_n = \sup_{x\in R} |f_n - 0|$. Clearly $M_n \to 0$, so $f_n \to 0$ uniformly.

Next, fix any $t \in \mathbb{R} \setminus 0$. Since $|1 - nt^2| \leq |1| + |nt^2| = 1 + nt^2$ for any $n \in \mathbb{Z}_+$,

$$|f_n'(t)| = \left| \frac{1 - nt^2}{(1+nt^2)^2} \right| \leq \frac{|1 + nt^2|}{|1+nt^2|^2} = \frac{1}{1+nt^2}$$

The final term goes 0 as $n \to \infty$, so $f_n'(t) \to 0$ as well.

But if $t = 0$, then $f_n'(t) = 1$ for any n. This shows that $\lim_{n\to\infty} f_n'(x) = f'(x) = 0 \iff x \neq 0$.