Math200A - HW #9

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2025.12.04

1.

(a)

Composing the homomorphisms $\pi : R \to R/I$ and $\iota : R/I \to (R/I)[x]$ yields a homomorphism $\varphi : R \to (R/I)[x]$. By the substition principle, there is a unique homomorphism

$$\mathit{\Phi} : R[x] \to (R/I)[x] \quad \text{satisfying } \mathit{\Phi}|_R = \varphi, \quad x \mapsto x$$

$\mathit{\Phi}$ is surjective because π is surjective. $f(x) \in \ker \mathit{\Phi} \iff$ every coefficient of f is in $I \iff f(x) \in I[x]$, so $\ker \mathit{\Phi} = I[x]$.

(b)

Suppose $P \in \operatorname{Spec} R$, $f(x)g(x) \in P[x]$, and $f(x) \notin P[x]$. Write

$$f(x) = \sum_{i=0}^{n} a_i x^i, \quad g(x) = \sum_{i=0}^{m} b_i x^i, \quad f(x)g(x) = \sum_{i=0}^{n+m} c_i x^i = \sum_{i=0}^{n+m} \sum_{j=0}^{i} a_j b_{i-j} x^i$$

and let k be the smallest integer such that $a_k \notin P$. By assumption, $a_0, \ldots, a_{k-1} \in P$. Now, induct on $0 \le l \le m$ to show all $b_l \in P$. Since $f(x)g(x) \in P[x]$, $c_k = \sum_{j=0}^{k} a_j b_{k-j} \in P$. The terms $a_0 b_k, a_1 b_{k-1}, \ldots, a_{k-1} b_1$ of the sum are in P, so the only remaining term of the sum $a_k b_0 \in P$ as well. $a_k \notin P \implies b_0 \in P$, which verifies the base case.

Suppose $b_0, \ldots, b_{l-1} \in P$ for $l - 1 < m$. Then $c_{k+l} = \sum_{j=0}^{k+l} a_j b_{k+l-j} \in P$, and the terms of the sum

$$a_0 b_{k+l}, a_1 b_{k+l-1}, \ldots, a_{k-1} b_{l+1}, a_{k+1} b_{l-1}, a_{k+2} b_{l-2}, \ldots, a_{k+l} b_0$$

are all either 0 (i.e. if $k + 1 > n$) or in P, so the only term of the sum not in this list $a_k b_l \in P$ as well. $a_k \neq P \implies b_l \in P$.

This shows every coefficient of $g(x) \in P$, i.e. $g(x) \in P[x]$, so $P[x] \in \operatorname{Spec} R[x]$

(c)

I love vim! (I love new things)