Math188 - HW #5

Jay Ser

2026.02.17

1.

Suppose $f(x) = \sum a_n x^n$, $f(x) \neq 0$, has a square root, say $g(x) = \sum b_n x^n$. By multiplication in $\mathbb{C}[[x]]$,

$$a_n = \sum_{i=0}^{n} b_i b_{n-i} \tag{1}$$

Assume $a_0 = a_1 = \ldots = a_{2n} = 0$. To prove that $\text{mdeg}(f)$ is even, it suffices to show that $a_{2n+1} = 0$. By (1) and the fact that $2n+1$ is odd,

$$a_{2n+1} = 2\sum_{i=0}^{n} b_i b_{2n+1-i} \tag{2}$$

One can induct on i to show that each $b_i = 0$ for $0 \leq i \leq n$. When $i = 0$, $a_0 = b_0 b_0 = 0 \implies b_0 = 0$. Suppose for $0 \leq i < n$ that $b_0 = b_1 = \ldots = b_i = 0$. Since $2i + 2 \leq 2(n-1) + 2 = 2n$,

$$a_{2i+2} = 0 = 2\sum_{j=0}^{i} b_j b_{2i+2-j} + b_{i+1}^2$$

Because $b_j = 0$ for each summand, $a_{2i+2} = 0 = b_{i+1}^2 = 0 \implies b_{i+1} = 0$, as was to be shown. Since $b_0 = b_1 = \ldots = b_n = 0$, it is clear that (2) is equal to 0.

Conversely, let $f(x) = \sum a_n x^n$, $f(x) \neq 0$ with $\text{mdeg}(f) = 2k$. Let $g(x) = \sum b_n x^n$ with b_n defined as such:

- $b_0 = b_1 = b_{k-1} := 0$.
- $b_k := \sqrt{a_{2k}}$, where $b_k \neq 0$ since $a_{2k} \neq 0$ by definition of f.
- Suppose $b_0, b_1, \ldots, b_n$ is defined for $n \geq k$. Then let $b_{n+1} := \sqrt{a_{2(n+1)} - 2\sum_{i=0}^{n} b_i b_{2(n+1)-i}}$

Then one can check that $g(x)^2 = f(x)$ (e.g., via calculation demonstrated in the previous paragraph).

2.

(a)

$$\begin{aligned}
\sum_{n\geq 0} a_n x^n &= \sum_{n\geq 0} n^2 x^n \\
&= \sum_{n\geq 0} \left[x^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2}(x^n) + nx^n\right] \\
&= x^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2}(\sum_{n\geq 0} x^n) + \sum_{n\geq 0} nx^n \\
&= \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2} \\
&= \frac{x^2+x}{(1-x)^3}
\end{aligned}$$

(b)

$$\begin{aligned}
\sum_{n\geq 0} a_n x^n &= \sum_{n\geq 0} \binom{n}{k} x^n \\
&= \sum_{n\geq k} \frac{n!}{k!(n-k)!} x^n \\
&= \frac{1}{k!} \sum_{n\geq k} n(n-1)\cdots(n-k+1)x^n \\
&= \frac{1}{k!} \sum_{n\geq k} x^k \frac{\mathrm{d}^k}{\mathrm{d}x^k}(x^n) \\
&= \frac{x^k}{k!} \frac{\mathrm{d}^k}{\mathrm{d}x^k}(\sum_{n\geq k} x^n) \\
&= \frac{x^k}{k!} \frac{k!}{(1-x)^k} \\
&= \frac{x^k}{(1-x)^k}
\end{aligned}$$

(c)

Let $A(x) := \sum_{n\geq 0} a_n x^n$.

$$\begin{aligned} A(x) - a_0 &= \sum_{n\geq 0} a_{n+1} x^{n+1} \\ &= \sum_{n\geq 0} (\frac{a_n}{3} + 1) x^{n+1} \\ &= \frac{x}{3} A(x) + \frac{x}{1-x} \end{aligned}$$

which gives

$$\begin{aligned} A(x)(1 - \frac{x}{3}) &= \frac{x}{1-x} + \frac{1-x}{1-x} \\ \implies A(x) &= \frac{1}{1-x} \cdot \frac{3}{3-x} \\ &= \frac{3}{2(1-x)} - \frac{3}{2(3-x)} \end{aligned}$$

3.

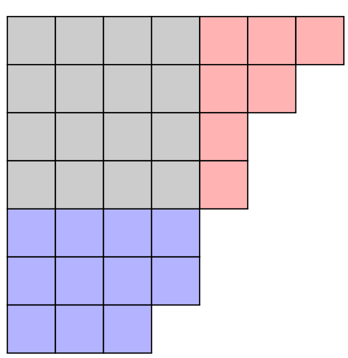
The equality to demonstrate is

$$\prod_{i\geq 1} \frac{1}{1-q^i} = \sum_{k\geq 0} \frac{q^{k^2}}{\prod_{i=1}^{k} (1-q^i)^2} \tag{1}$$

The left hand side of (1) is the generating function

$$\sum_{\text{partitions } \lambda} q^{|\lambda|}$$

The only partition that doesn't have a Durfee square (i.e., a Durfee square of size 0) is the empty partition. Any nonempty partition λ has a Durfee square of size at least 1; suppose the Durfee square of λ is of size k. For example, if $k = 4$, then λ looks something like this:



Notice that for any partition with a Durfee square of size k, the pink blocks form a Young diagram with at most k parts, while the blue blocks form a Young diamgram with the first part of size at most k. Conjugating the pink partition, the pink young diagram is a partition whose first part has size at most k. Finally, observe that the Durfee square contributes k^2 blocks to λ. This gives

$$\begin{aligned}\sum_{\text{partitions } \lambda} q^{|\lambda|} &= \sum_{k\geq 0} \sum_{\substack{\text{partitions } \lambda, \\ \text{ds}(\lambda)=k}} q^{|\lambda|} \\ &= \sum_{k\geq 0} q^{k^2}(1+q+q^2+\ldots)^2(1+q^2+q^4+\ldots)^2\cdots(1+q^k+q^{2k}+\ldots)^2 \\ &= \sum_{k\geq 0} \frac{q^{k^2}}{\prod_{i=1}^{k}(1-q^i)^2}\end{aligned}$$

where $\text{ds}(\lambda)$ denotes the size of the Durfee squaure of λ. This proves the desired identity.

4.

The equality to demonstrate is

$$\prod_{i\geq 0}(1+q^{2i+1}) = \sum_{k\geq 0} \frac{q^{k^2}}{\prod_{i=1}^{k}(1-q^{2i})} \tag{1}$$

Notice that the left hand side of (1) is the generating function

$$\sum_{\substack{\text{partitions } \lambda \\ \text{with distinct odd parts}}} q^{|\lambda|} = (1+q)(1+q^3)(1+q^5)\cdots$$

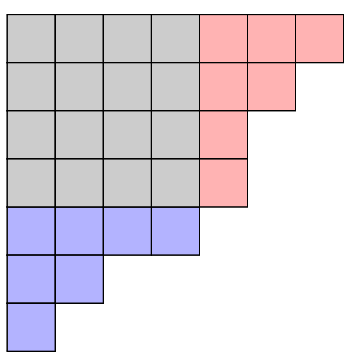
Recall the bijection $\{\lambda \vdash n \mid \lambda = \lambda'\} \longleftrightarrow \{\lambda \vdash n \mid \lambda \text{ has distinct odd parts}\}$. So in general, there is a bijection

$$\{\text{partition } \lambda \mid \lambda = \lambda'\} \longleftrightarrow \{\text{partition } \lambda \mid \lambda \text{ has distinct odd parts}\}$$

where both the forward map and the reverse map preserve the size of the partition. Hence the left hand size of (1) is

$$\sum_{\substack{\text{partitions } \lambda, \\ \lambda=\lambda'}} q^{|\lambda|}$$

Notice that any partition λ with $\lambda = \lambda'$ has a Young diagram that looks like this:



To verbally describe it, if λ has a Durfee square of size k, the pink blocks form a Young diagram with at most k parts, the blue blocks form a Young diagram whose first part has size at most k, and the pink Young diagram and blue Young diagram are conjugates of each other. Thus $|\lambda| = k^2 + 2\,|\lambda_{\text{pink}}|$. This gives

$$\begin{aligned}\sum_{\substack{\text{partitions } \lambda,\\ \lambda=\lambda'}} q^{|\lambda|} &= \sum_{k\geq 0} q^{k^2}(1+q^2+q^4+\ldots)(1+q^4+q^8+\ldots)\cdots(1+q^{2k}+q^{4k}+\ldots) \\ &= \sum_{k\geq 0} \frac{q^{k^2}}{\prod_{i=1}^{k}(1-q^{2i})}\end{aligned}$$

as was to be shown.

5.

The whatever derivative rule gives

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}(1+x)^a = a(a-1)\cdots(a-k+1)(1+x)^{a-k}$$

Thus the Taylor expansion of $f(x) = (1+x)^a$ is

$$\begin{aligned}(1+x)^a &= \sum_{k\geq 0} \frac{f^{(k)}(0)}{k!}x^k \\ &= \sum_{k\geq 0} \frac{a(a-1)\cdots(a-k+1)}{k!}x^k \\ &= \sum_{k\geq 0} \binom{a}{k} x^k\end{aligned}$$

Next, I show that $(1+x)^a(1+x)^b = (1+x)^{a+b}$ as power series. Rewriting the left-hand side as power series,

$$\Big(\sum_{k\geq 0}\binom{a}{k}x^k\Big)\cdot\Big(\sum_{k\geq 0}\binom{b}{k}x^k\Big) = \sum_{k\geq 0} c_k x^k \tag{1}$$

where

$$c_k = \sum_{i+j=k}\binom{a}{i}\binom{b}{j}$$

View c_k as a polynomial in a. If a is any integer, then

$$c_k := \sum_{i+j=k}\binom{a}{i}\binom{b}{j} = \binom{a+b}{k}$$

by the following combinatorial proof: The right hand side counts the number of ways to choose k elements from $[a+b]$. The total number of ways to do so is the sum of the number of ways to choose $0 \leq i \leq k$ elements from $[a]$, then choose the remaining $k-i$ elements from $[b]$, which is the left hand side.

Uh technically the above proof only holds if b is also an integer, and for two polynomials in $\mathbb{C}[a, b]$, showing that they agree on infinitely many pairs (a_0, b_0) doens't prove that the two polynomials are identical, but oh well, I tried my best. Assuming $c_k = \binom{a+b}{k}$ identically, (1) is equal to

$$(x+1)^{a+b}$$

as was to be shown.

6.

Let

$$b_n := \begin{cases} (n-1)!, & \text{if } n \in \{1, 2, 3, 6\} \\ 0, & \text{else} \end{cases}$$

In other words, if $n \in \{1, 2, 3, 6\}$, b_n is the number of ways to form an n-cycle in $[n]$; if n is not a divisor of 6, then forcibly set the number of ways to structure $[n]$ to be zero.

Now, since a_n counts the number of $w \in \mathfrak{S}_n$ such that $w^6 = 1$, which is equivalently the number of $w \in \mathfrak{S}_n$ whose cycle decomposition only consists of 1, 2, 3, and 6-cycles, a_n can be interpreted as the number of ways to:

1. Partition $[n]$ into $B_1, B_2, \ldots, B_k$ such that the size of each $B_i \in \{1, 2, 3, 6\}$.
2. For each B_i, put the elements of B_i into a single cycle.

As such, the following exponentiation makes sense:

$$\begin{aligned} A(x) := \sum_{n \geq 0} \frac{a_n}{n!} x^n &= \exp(\sum_{m \geq 0} \frac{b_m}{m!} x^m) \\ &= \exp(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^6}{6}) \end{aligned}$$

7.

c_n can be interpreted as the number of ways to:

1. Partition $[n]$ into $B_1, \ldots, B_k$.
2. For each B_i, single out a single element, then form a single cycle out of the rest of the elements.

Namely, if $\#B_i = m > 1$, then the number of ways to achieve (2) is $m(m-2)!$. If $\#B_i = 1$, then of course the number of ways to achieve (2) is just 1. Thus

$$\begin{aligned}
C(x) := \sum_{n\geq 0} \frac{c_n}{n!}x^n &= \exp(x + \sum_{m>1} \frac{m(m-2)!}{m!}x^m) \\
&= \exp(x + \sum_{m\geq 1} \frac{x^{m+1}}{m}) \\
&= \exp(x - x\log(1-x)) \\
&= e^x e^{-x\log(1-x)} \\
&= e^x \left(\frac{1}{1-x}\right)^{-x}
\end{aligned}$$

8.

***** I accidently made $c_n = \#$ of decorated permutations in $\mathfrak{S}_n$ instead of b_n!!!**

Define

$$\begin{aligned}
a_n &:= \#\text{ of ways to form a single cycle from } [n] \\
&= (n-1)!, \\
b_n &:= \#\text{ of ways to pick a single element from } [n] \\
&= n
\end{aligned}$$

and let

$$\begin{aligned}
A(x) &:= \sum_{n\geq 1} a_n \frac{x^n}{n!} \\
&= \sum_{n\geq 1} \frac{x^n}{n} \\
&= -\log(1-x)
\end{aligned}$$

and

$$\begin{aligned}
B(x) &:= \sum_{n\geq 0} b_n \frac{x^n}{n!} \\
&= \sum_{n\geq 0} n\frac{x^n}{n!} \\
&= xe^x
\end{aligned}$$

Now, note that c_n counts the number of ways to:

1. partition $[n]$ into $B_1, \ldots, B_k$.
2. For each B_i, form a single cycle from the elements of B_i.

3. Pick one block from $B_1, \ldots, B_k$ (to decorate it)

Particularly, for each B_i, if $\#B_i = m$, (2) is given by a_m. If $[n]$ is partitioned into k blocks $B_1, \ldots, B_k$, then (3) is given by b_k. This means

$$\begin{aligned} C(x) := \sum_{n \geq 0} c_n \frac{x^n}{n!} &= B(A(x)) \\ &= (-\log(1-x))e^{-\log(1-x)} \\ &= \frac{-\log(1-x)}{1-x} \end{aligned}$$