Math188 - HW #3

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2026.01.26

1.

The equality to demonstrate is

$$A_n = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} B(n-k)$$

Let $C_n(k) := \{\text{set partition of } [n] \mid k, k+1 \text{ are in the same block}\}$ for $k \in [n-1]$. Then given any $I \subset [n-1]$,

$$\begin{aligned}\#\bigcap_{k\in I} C_n(k) &= \#\{\text{set partition of } [n] \mid k, k+1 \text{ in the same block for all } k \in I\}\\ &= B(n - \#I)\end{aligned}$$

where the final equality is interpreted as follows: each $k \in I$ "removes" a single distinct equivalence class of $[n]$; if k is already blocked with $k-1$ because $k-1 \in I$, then $k+1$ blocks with k and $k-1$; if $k \in I$ is not blocked with $k-1$, then the distinct equivalence class of $k+1$ is absorbed into equivalence class of k. So after blocking each $k \in I$, then there are $n - \#I$ equivalence classes that can be freely partitioned; there are $B(n - \#I)$ ways to partition these equivalence classes.

Then, using the principle of inclusion-exclusion,

$$\begin{aligned}\#A_n &= B(n) - \#\bigcup_{k=1}^{n-1} C_n(k)\\ &= B(n) - \sum_{I \neq \emptyset \subset [n-1]} (-1)^{\#I+1} \# \bigcap_{k\in I} C_n(k)\\ &= B(n) + \sum_{k=1}^{n-1} (-1)^k \binom{n-1}{k} B(n-k)\\ &= \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} B(n-k)\end{aligned}$$

where the final equality follows by considering the lone $B(n)$ term as the $k = 0$ summand.

2.

3.

4.

Consider $n \times m$ matrices. Without loss of generality, assume $n \geq m$, i.e., more rows than cols (if $m > n$, then consider the transpose of the matrix).

$$A_k := \{\text{matrices with no column of 0's and row } k \text{ is } 0\}$$

Then by the principle of inclusion and exclusion,

$$\begin{aligned} M(n,m) &= (2^n - 1)^m - \#\bigcup_{k=1}^{n} A_k \\ &= (2^n - 1)^m - \sum_{I \neq \emptyset \subset [n]} (-1)^{\#I+1} \#(\bigcap_{k \in I} A_k) \end{aligned}$$

where the first term is the number of matrices without any column of 0's: in each column, there are 2^n different configurations and only a single configuration, the row of column of 0's, must be avoided. By a similar counting argument,

$$\#(\bigcap_{k \in I} A_k) = \#\{\text{matrices with no column of 0's, } k\text{th row is 0 for each } k \in I\} = (2^{n-\#I} - 1)^m$$

where the $2^{n-\#I}$ accounts for the fact that $\#I$ rows are fixed. Thus

$$\begin{aligned} M(n,m) &= (2^n - 1)^m + \sum_{k=1}^{n} (-1)^k \binom{n}{k} (2^{n-k} - 1)^m \\ &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} (2^{n-k} - 1)^m \end{aligned}$$

5.