Mini Project - 2

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1 Question 1

We have been given that x_n follows a Weibull distribution

$$f_{\theta,k}(x) = \begin{cases} \frac{k}{\theta}\left(\frac{x}{\theta}\right)^{k-1} e^{-\left(\frac{x}{\theta}\right)^k} & , x \geq 0 \\ 0 & , x < 0 \end{cases} \tag{1}$$

Here the true parameters θ= 2 and k=1

For which the likelihood function is:

$$\mathcal{L}_{\hat{x}}(\theta,k) = \prod_{i=1}^{N} f_{\theta,k}(x_i) = \prod_{i=1}^{N} \frac{k}{\theta}\left(\frac{x_i}{\theta}\right)^{k-1} e^{-\left(\frac{x_i}{\theta}\right)^k} = \frac{k^N}{\theta^{Nk}} e^{-\sum_{i=1}^{N}\left(\frac{x_i}{\theta}\right)^k} \prod_{i=1}^{N} x_i^{k-1}$$

$$\ell_{\hat{x}}(\theta,k) := \ln \mathcal{L}_{\hat{x}}(\theta,k) = N\ln k - Nk\ln\theta - \sum_{i=1}^{N}\left(\frac{x_i}{\theta}\right)^k + (k-1)\sum_{i=1}^{N}\ln x_i$$

Now since we know k only θ needs to be estimated which be done by taking the partial derivative of $\ell_{\hat{x}}(\theta,k)$ w.r.t θ as follows:

$$\frac{\partial l}{\partial \theta} = -Nk\frac{1}{\theta} + k\sum_{i=1}^{N} x_i^k \frac{1}{\theta^{k+1}} \overset{!}{=} 0$$

$$\frac{\partial l}{\partial k} = \frac{N}{k} - N\ln\theta - \sum_{i=1}^{N} \ln\left(\frac{x_i}{\theta}\right) e^{k\ln\left(\frac{x_i}{\theta}\right)} + \sum_{i=1}^{N}\ln x_i \overset{!}{=} 0$$

And taking k =1 It follows:

$$\boxed{\hat{\theta} = \left(\frac{1}{N}\sum_{i=1}^{N} x_i\right)}$$

Fischer information =

$$\boxed{I(\theta) = \frac{1}{\theta^2}}$$

1.1 Tabulating the E[σ] & Var(σ)

N	E[σ]	Var(σ)
1	1.94222	3.8309
10	2.04264	0.424264
100	1.96885	0.03530
1000	2.0015	0.00418
10000	1.9996	0.00037

It can be seen that the E[σ] converges to the true value i.e 2 implying that the estimator is unbiased.
Also, the variance decreases with an increase in sample size as the mle estimator is consistent and it converges to the true value of σ in probability

1.2 Plotting the pdf & CDF of the estimate $\hat{\theta}$

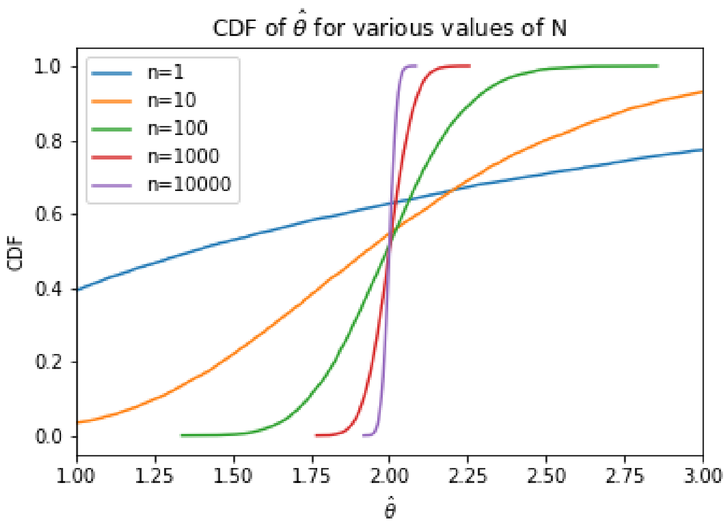


Figure 1: CDF of the estimate $\hat{\theta}$

It can be see that the CDF becomes narrower as the variance decreases with an increase in sample size

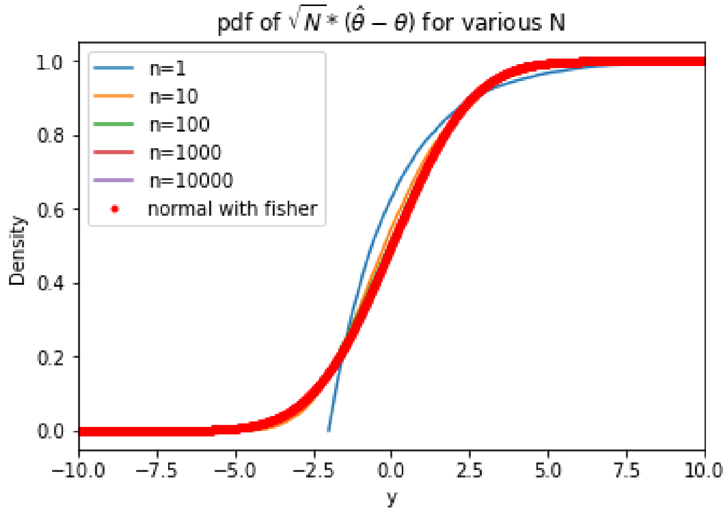


Figure 2: CDF of $\sqrt{N} * (\hat{\theta} - \theta)$

It can be see the CDF converges to the Normal distribution with 0 mean and variance given by the FI. This shows that the estimator is consistent and unbiased

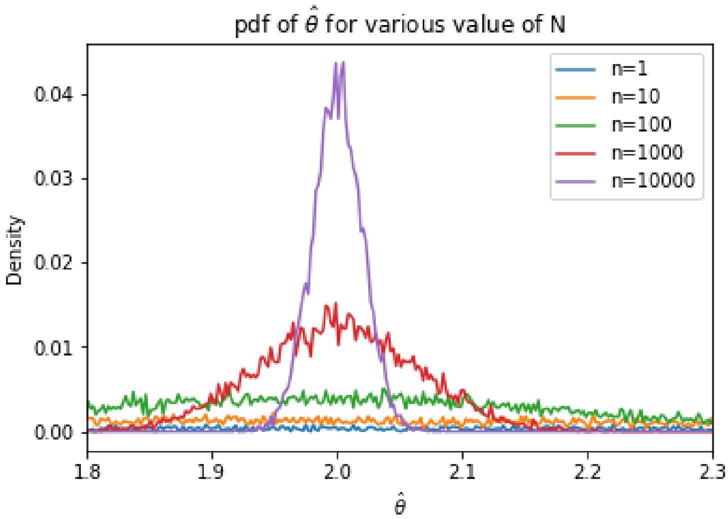


Figure 3: pdf of the estimate $\hat{\theta}$

The pdf also follows a Gaussian distribution with the width becoming narrower as sample size increases

2 Question 2

We have been given that Y_n follows a Gumbel (Type-1) Distribution with

$$f_y(y) = \frac{1}{\sigma} e^{-\frac{y-\mu}{\sigma}} e^{-e^{-\frac{y-\mu}{\sigma}}} \tag{2}$$

Here the true parameters μ= 14.9787 and σ_{gev}=5

For which the likelihood function is:

$$L = \prod_{i=1}^{n} \frac{1}{\sigma} e^{-\frac{y_i-\mu}{\sigma}} e^{-e^{-\frac{y_i-\mu}{\sigma}}}$$

$$L = \sigma^{-n} e^{-\sum_{i=1}^{n}(\frac{y_i-\mu}{\sigma} + e^{-\frac{y_i-\mu}{\sigma}})}$$

$$ln(L) = -nln(\sigma) - \sum_{i=1}^{n}(\frac{y_i-\mu}{\sigma} + e^{-\frac{y_i-\mu}{\sigma}})$$

Now since we know μ only σ needs to be estimated which be done by taking the partial derivative of $ln(L)$ w.r.t σ as follows:

$$\frac{\partial ln(L)}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i-\mu) - \frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i-\mu)e^{-\frac{y_i-\mu}{\sigma}} = 0$$

Since there is no easy closed form for this expression we use an iterative numerical technique called Newton-Raphson where we update σ as:

$$\sigma = \sigma - \frac{\frac{\partial ln(L)}{\partial \sigma}}{\frac{\partial^2 ln(L)}{\partial \sigma^2}} \tag{3}$$

Where

$$\frac{\partial^2 ln(L)}{\partial \sigma^2} = \frac{n}{\sigma^2} - \frac{2}{\sigma^3}\sum_{i=1}^{n}(y_i-\mu) + \frac{2}{\sigma^3}\sum_{i=1}^{n}(y_i-\mu)e^{-\frac{y_i-\mu}{\sigma}} - \frac{2}{\sigma^4}\sum_{i=1}^{n}(y_i-\mu)^2 e^{-\frac{y_i-\mu}{\sigma}}$$

And the Fischer n is given by the integral which is implemented using numerical packages:

$$\int_0^\infty \frac{1}{\sigma^2} - \frac{2}{\sigma^3}\sum_{i=1}^{n}(y_i-\mu) + \frac{2}{\sigma^3}\sum_{i=1}^{n}(y_i-\mu)e^{-\frac{y_i-\mu}{\sigma}} - \frac{2}{\sigma^4}\sum_{i=1}^{n}(y_i-\mu)^2 e^{-\frac{y_i-\mu}{\sigma}} \; f_y(y:\sigma)\, dy$$

2.1 Tabulating the $E[\sigma]$ & $Var(\sigma)$

N	$E[\sigma]$	$Var(\sigma)$
1	4.2046	12.1047
10	4.8867	1.03424
100	4.8076	0.14665
1000	4.8310	0.012605
10000	4.81147	0.001196

It can be seen that the $E[\sigma]$ converges to 4.8 instead of 5 implying that the estimator is biased.
Nevertheless, the variance decreases with an increase in sample size as the mle estimator is consistent and it converges to the true value of σ in probability

2.2 Plotting the CDF of the estimate $\hat{\sigma}_{gev}$

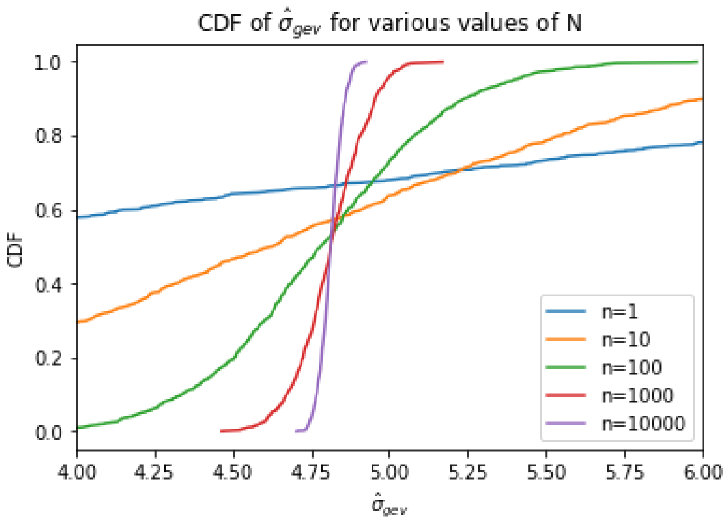
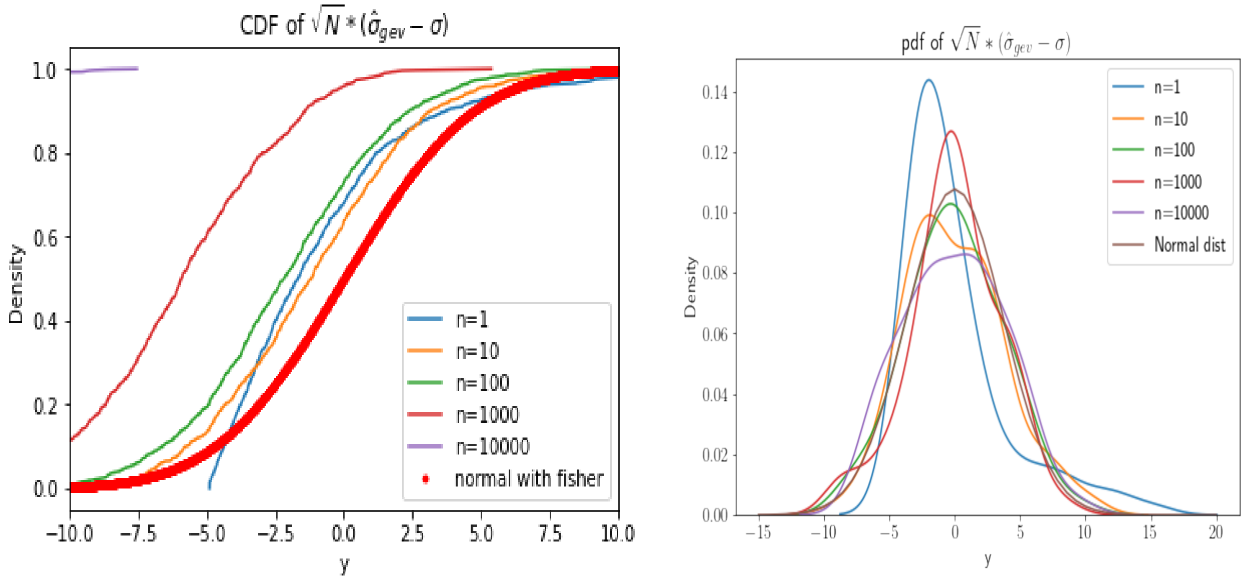


Figure 4: CDF of the estimate $\hat{\sigma}_{gev}$

It can be see that the CDF becomes narrower as the variance decreases with an increase in sample size



(a) CDF

(b) pdf

Figure 5: CDF and pdf of $\sqrt{N} * (\hat{\sigma}_{gev} - \sigma)$

Here we can clearly see the effect of the biasedness of the estimator. The CDF and pdf are both shifted.

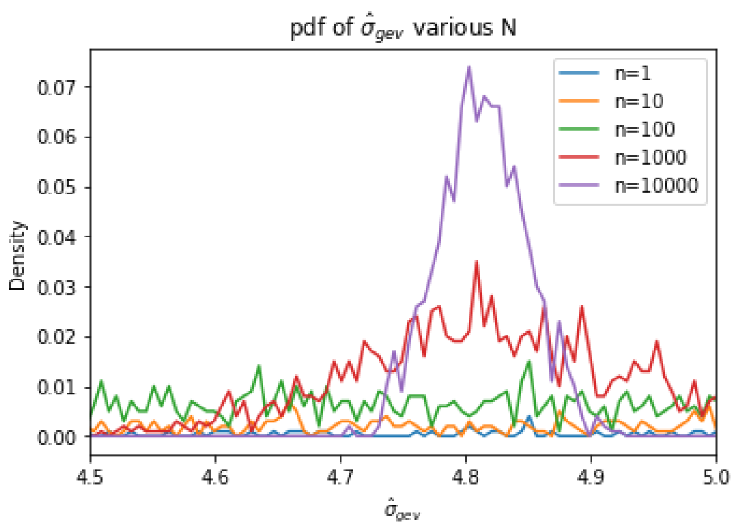


Figure 6: pdf of the estimate $\hat{\sigma}_{gev}$

The pdf also follows a Gaussian distribution with the width becoming narrower as sample size increases .However it is heavy tailed towards the right as the distribution is extreme valued

3 Question 3

Using the generalized Pareto distribution (GPD)
We get the MLE as:

$$\boxed{\hat{\sigma}_{gpd} = \left(\frac{1}{L} \sum_{i=1}^{n} S_i \right)}$$

And the Fischer information =

$$\boxed{I(\theta) = \frac{1}{\sigma_{gpd}^2}}$$

3.1 Tabulating the E[$\hat{\sigma}_{gev}$] & Var($\hat{\sigma}_{gev}$)

N	E[σ]	Var(σ)
1	4.93320	21.4661
10	5.0411	15.2535
100	5.03381	1.29202
1000	5.00518	0.12399
10000	4.9981	0.01247

It can be seen that the E[σ_{gev}] converges to the true value i.e 5 implying that the estimator is unbiased.
Also, the variance decreases with an increase in sample size as the mle estimator is consistent and it converges to the true value of σ_{gev} in probability

3.2 Plotting the pdf & CDF of the estimate $\hat{\theta}$

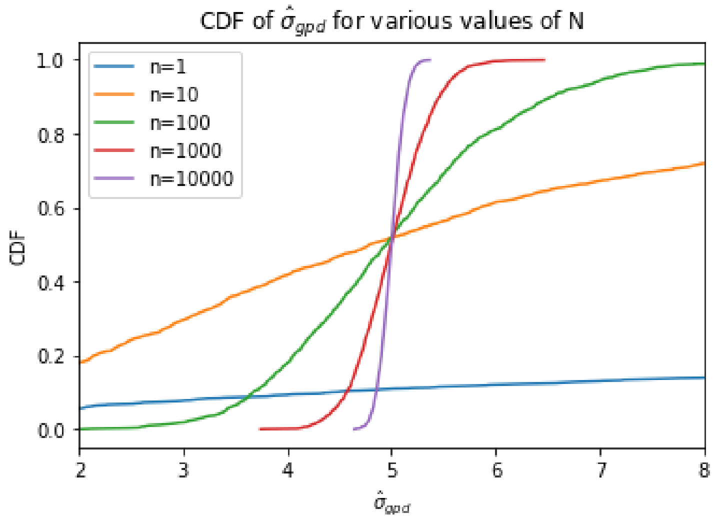


Figure 7: CDF of the estimate $\hat{\sigma}_{gpd}$

It can be see that the CDF becomes narrower as the variance decreases with an increase in sample size

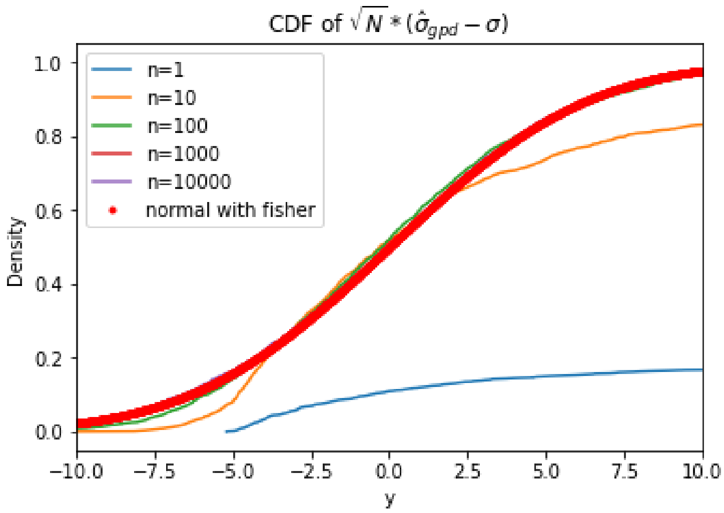


Figure 8: CDF of $\sqrt{N} * (\hat{\sigma}_{gpd} - \sigma)$

It can be see the CDF converges to the Normal distribution with 0 mean and variance given by the FI. This shows that the estimator is consistent and unbiased.

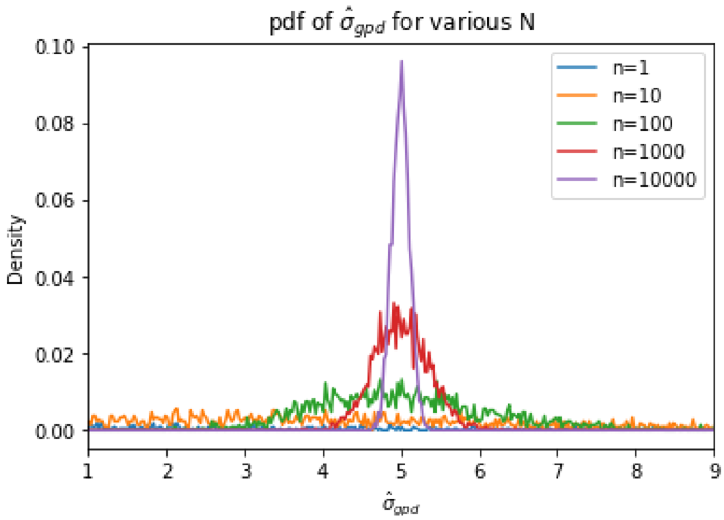


Figure 9: pdf of the estimate $\hat{\sigma}_{gpd}$

The pdf also follows a Gaussian distribution with the width becoming narrower as sample size increases. And it is centered around 5 as the estimator is unbiased