ES202 - Assignment Solutions

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Contents

1 Introduction

These are my solutions to the assignment given in the course ES202.

1.1 AI Policy of This Paper

Large-Language-Model's (LLM) are only used in the formatting of this file. At no point, LLM's are used to solve the questions in the assignment, unlike other students taking the course who like to ask the help of LLMs even during the examinations. The reason this is written in LaTeX rather than by hand, is only because I have no time to do it by hand, and wanted to improve my LaTeX skills. The git commit history can be found in the GitHub repository jayshozie/es202-assignment, also as a proof of the fact that this entire document was written by hand.

2 Question 1

Problem An airplane is monitored at coordinates $(5, 7, 4)$ relative to the airport (South, East, Up). Find the directional angles of the plane.

Solution: Let the position vector of the plane be $\vec{r}$. We define the axes such that $x =$ South, $y =$ East, and $z =$ Up.

$$\begin{aligned}\vec{r} &= 5\hat{\imath} + 7\hat{\jmath} + 4\hat{k}\\ \|\vec{r}\| &= \sqrt{5^2 + 7^2 + 4^2}\\ &= \sqrt{25 + 49 + 16}\\ &= \sqrt{90}\\ &\approx 9.4868\end{aligned}$$

The directional angles α, β, γ are given by the direction cosines:

$$\begin{aligned}\alpha &= \cos^{-1}\left(\frac{r_x}{\|\vec{r}\|}\right) & \beta &= \cos^{-1}\left(\frac{r_y}{\|\vec{r}\|}\right) & \gamma &= \cos^{-1}\left(\frac{r_z}{\|\vec{r}\|}\right)\\ &= \cos^{-1}\left(\frac{5}{\sqrt{90}}\right) & &= \cos^{-1}\left(\frac{7}{\sqrt{90}}\right) & &= \cos^{-1}\left(\frac{4}{\sqrt{90}}\right)\\ &\approx 58.19^\circ & &\approx 42.45^\circ & &\approx 64.06^\circ\end{aligned}$$

3 Question 2

Problem Prove that $\|\vec{a} \cdot \vec{b}\| \leq \|\vec{a}\| \cdot \|\vec{b}\|$

Solution By the geometric definition of the dot product, the angle θ between the vectors $\|\vec{a}\|$ and $\|\vec{b}\|$ is given by:

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \, \|\vec{b}\| \, \cos\theta$$

We know that for any real angle θ, the cosine function is bounded:

$$-1 \leq \cos\theta \leq 1 \implies \|\cos\theta\| \leq 1$$

Substituting this inequality back into our original equation:

$$\|\vec{a} \cdot \vec{b}\| = \|\vec{a}\| \cdot \|\vec{b}\| \underbrace{\|\cos\theta\|}_{\leq 1}$$

Therefore proving:

$$\|\vec{a} \cdot \vec{b}\| \leq \|\vec{a}\| \cdot \|\vec{b}\| \tag{1}$$

4 Question 3

Problem Prove $\|\vec{a}+\vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$

Solution Since magnitudes are non-negative by definition, proving the inequality is equivalent to proving it for the squares of the magnitudes. Consider the square of the sum:

$$\begin{aligned}
\|\vec{a}+\vec{b}\|^2 &= (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) \\
&= \vec{a}\cdot\vec{a} + 2(\vec{a}\cdot\vec{b}) + \vec{b}\cdot\vec{b} \\
&= \|\vec{a}\|^2 + 2(\vec{a}\cdot\vec{b}) + \|\vec{b}\|^2
\end{aligned}$$

From (1) (Cauchy-Schwartz Inequality), we established that

$$\vec{a}\cdot\vec{b} \leq \|\vec{a}\cdot\vec{b}\| \leq \|\vec{a}\|\,\|\vec{b}\|$$

We substitute this upper bound into the equation:

$$\|\vec{a}+\vec{b}\|^2 \leq \|\vec{a}\|^2 + 2\|\vec{a}\|\,\|\vec{b}\| + \|\vec{b}\|^2$$

Recognizing the right-hand side as a perfect expansion $(x+y)^2 = x^2+2xy+y^2$:

$$\|\vec{a}+\vec{b}\|^2 \leq \left(\|\vec{a}\| + \|\vec{b}\|\right)^2$$

Taking the square root of both sides, which is valid since magnitudes are non-negative:

$$\|\vec{a}+\vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\| \tag{2}$$

5 Question 4

Problem Prove $\|\vec{a} \times \vec{b}\|^2 = \|\vec{a}\|^2 \|\vec{b}\|^2 - (\vec{a} \cdot \vec{b})^2$

Solution Magnitude of the vector-product of two vectors $\vec{a}$ and $\vec{b}$, separated by an angle θ, is defined as:

$$\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \, \|\vec{b}\| \, \sin\theta \tag{3}$$

Square both sides:

$$\begin{aligned} \|\vec{a} \times \vec{b}\|^2 &= (\|\vec{a}\| \, \|\vec{b}\| \, \sin\theta)^2 \\ &= \|\vec{a}\|^2 \|\vec{b}\|^2 \sin^2\theta \end{aligned}$$

Since,

$$\begin{aligned} \cos^2\theta + \sin^2\theta &= 1 \\ \sin^2\theta &= 1 - \cos^2\theta \end{aligned}$$

By substituting that to our original equality's right-hand side, we get:

$$\|\vec{a} \times \vec{b}\|^2 = \|\vec{a}\|^2 \|\vec{b}\|^2 (1 - \cos^2\theta)$$

Then, by distributing $\|\vec{a}\|^2 \|\vec{b}\|^2$, we get:

$$= \|\vec{a}\|^2 \|\vec{b}\|^2 - \|\vec{a}\|^2 \|\vec{b}\|^2 \cos^2\theta$$

Observe that,

$$\|\vec{a}\|^2 \|\vec{b}\|^2 \cos^2\theta = (\vec{a} \cdot \vec{b})^2$$

Thus, by substituting that, we complete our proof:

$$\|\vec{a} \times \vec{b}\|^2 = \|\vec{a}\|^2 \|\vec{b}\|^2 - (\vec{a} \cdot \vec{b})^2 \tag{4}$$

6 Question 5

Problem Let vectors $\vec{u}_1 = (1,0,0)$, $\vec{u}_2 = (1,1,0)$, and $\vec{u}_3 = (1,1,1)$ form a basis for the vector space $\mathbb{R}^3$. Show that these vectors are linearly independent and express vector $\vec{a} = (3,-4,8)$ as a linear combination of them.

Solution We will divide our solution to two parts. In the first part, we'll prove that the given vectors $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$ are linearly independent, thus forming a basis for $\mathbb{R}^3$; then we'll find a linear combination for the vector $\vec{a}$.

Part 1: Linear Independence We form a matrix A with the vectors $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$ as columns. The vectors are linearly independent if $\det(A) \neq 0$.

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

Since this is an upper-triangular matrix, the determinant is the product of the diagonal entries:

$$\det(A) = 1 \cdot 1 \cdot 1 = 1 \neq 0$$

Therefore, the vectors are linearly independent and form a basis for $\mathbb{R}^3$.

Part 2: Linear Combination We wish to find coefficients c_1, c_2, and c_3 such that:

$$c_1 \cdot \vec{u}_1 + c_2 \cdot \vec{u}_2 + c_3 \cdot \vec{u}_3 = \vec{a}$$

This corresponds to the linear system:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 1 & 8 \end{array}\right]$$

Using back-substitution:

1. $c_3 = 8$
2. $c_2 + 8 = -4 \implies c_2 = -12$
3. $c_1 + (-12) + 8 = 3 \implies c_1 - 4 = 3 \implies c_1 = 7$

Using those coefficients, we can say that:

$$\vec{a} = 7\vec{u}_1 - 12\vec{u}_2 + 8\vec{u}_3$$

7 Question 6

Problem Obtain an orthonormal set from the given set of vectors using Gram-Schmidt Orthogonalization Process:

$$\vec{B} = \left\{ \left(\frac{1}{2}, \frac{1}{2}, 1\right) \left(-1, 1, -\frac{1}{2}\right) \left(-1, \frac{1}{2}, 1\right) \right\}$$

Solution Let the given vectors be $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$. We will generate an orthogonal set $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ and then normalize them to get the orthonormal set $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$.

Step 1. Process the first vector To use the Gram-Schmidt Orthogonalization Process, we need to pick a vector. For convention, we'll pick $\vec{v}_1$ as our first vector.

$$\vec{u}_1 = \vec{v}_1 = \left(\frac{1}{2}, \frac{1}{2}, 1\right)$$

Calculating the magnitude of $\vec{u}_1$ gives us:

$$\begin{aligned} \|\vec{u}_1\| &= \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + 1^2} \\ &= \sqrt{\frac{3}{2}} \end{aligned}$$

So, our first orthonormal vector $\vec{e}_1$ is:

$$\begin{aligned} \vec{e}_1 &= \frac{\vec{u}_1}{\|\vec{u}_1\|} \\ &= \sqrt{\frac{2}{3}} \left(\frac{1}{2}, \frac{1}{2}, 1\right) \end{aligned}$$

Step 2. Orthogonalize the second vector We calculate the projection of $\vec{u}_2$ onto $\vec{u}_1$.

$$\begin{aligned} \vec{v}_2 \cdot \vec{u}_1 &= (-1)\left(\frac{1}{2}\right) + (1)\left(\frac{1}{2}\right) + \left(-\frac{1}{2}\right)(1) = -\frac{1}{2} \\ \vec{u}_2 &= \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{u}_1}{\|\vec{u}_1\|^2}\vec{u}_1 = \left(-1, 1, -\frac{1}{2}\right) - \frac{-1/2}{3/2}\left(\frac{1}{2}, \frac{1}{2}, 1\right) \\ &= \left(-1, 1, -\frac{1}{2}\right) + \frac{1}{3}\left(\frac{1}{2}, \frac{1}{2}, 1\right) = \left(-\frac{5}{6}, \frac{7}{6}, -\frac{1}{6}\right) \end{aligned}$$

We, then, normalize $\vec{u}_2$ by:

$$\|\vec{u}_2\|^2 = \frac{25}{36} + \frac{49}{36} + \frac{1}{36} = \frac{75}{36} = \frac{25}{12} \implies \|\vec{u}_2\| = \frac{5}{2\sqrt{3}}$$
$$\vec{e}_2 = \frac{\vec{u}_2}{\|\vec{u}_2\|} = \frac{2\sqrt{3}}{5}\left(-\frac{5}{6}, \frac{7}{6}, -\frac{1}{6}\right) = \left(-\frac{\sqrt{3}}{3}, \frac{7\sqrt{3}}{15}, -\frac{\sqrt{3}}{15}\right)$$

Step 3: Orthogonalize the third vector Formula: $\vec{u}_3 = \vec{v}_3 - \text{proj}_{\vec{u}_1}(\vec{v}_3) - \text{proj}_{\vec{u}_2}(\vec{v}_3)$. First, we compute the projection coefficients:

$$\frac{\vec{v}_3 \cdot \vec{u}_1}{\|\vec{u}_1\|^2} = \frac{(-1)(\frac{1}{2}) + (\frac{1}{2})(\frac{1}{2}) + (1)(1)}{3/2} = \frac{3/4}{3/2} = \frac{1}{2}$$
$$\frac{\vec{v}_3 \cdot \vec{u}_2}{\|\vec{u}_2\|^2} = \frac{(-1)(-\frac{5}{6}) + (\frac{1}{2})(\frac{7}{6}) + (1)(-\frac{1}{6})}{25/12} = \frac{\frac{5}{6} + \frac{7}{12} - \frac{2}{12}}{25/12} = \frac{15/12}{25/12} = \frac{3}{5}$$

Now substitute back to find $\vec{u}_3$:

$$\begin{aligned}
\vec{u}_3 &= \vec{v}_3 - \frac{1}{2}\vec{u}_1 - \frac{3}{5}\vec{u}_2 \\
&= \left(-1, \frac{1}{2}, 1\right) - \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right) - \left(-\frac{1}{2}, \frac{7}{10}, -\frac{1}{10}\right) \\
&= (-1 - 0.25 + 0.5, \quad 0.5 - 0.25 - 0.7, \quad 1 - 0.5 + 0.1) \\
&= (-0.75, -0.45, 0.6) = \left(-\frac{3}{4}, -\frac{9}{20}, \frac{3}{5}\right)
\end{aligned}$$

Normalize $\vec{u}_3$:

$$\|\vec{u}_3\|^2 = \frac{9}{16} + \frac{81}{400} + \frac{9}{25} = \frac{225 + 81 + 144}{400} = \frac{450}{400} = \frac{9}{8}$$
$$\vec{e}_3 = \frac{\vec{u}_3}{\sqrt{9/8}} = \frac{2\sqrt{2}}{3}\left(-\frac{3}{4}, -\frac{9}{20}, \frac{3}{5}\right) = \left(-\frac{\sqrt{2}}{2}, -\frac{3\sqrt{2}}{10}, \frac{2\sqrt{2}}{5}\right)$$

Final Answer: The orthonormal set is $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$, where:

$$\vec{e}_1 = \left(\frac{\sqrt{2}}{2\sqrt{3}}, \frac{\sqrt{2}}{2\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}\right)$$
$$\vec{e}_2 = \left(-\frac{\sqrt{3}}{3}, \frac{7\sqrt{3}}{15}, -\frac{\sqrt{3}}{15}\right)$$
$$\vec{e}_3 = \left(-\frac{\sqrt{2}}{2}, -\frac{3\sqrt{2}}{10}, \frac{2\sqrt{2}}{5}\right)$$

8 Question 7

Problem Verify that the matrix A satisfies its own characteristic equation

$$A = \begin{bmatrix} 1 & -2 \\ 4 & 5 \end{bmatrix}$$

Solution
Step 1. Find the Characteristic Equation The characteristic equation of a matrix is given by $\det(A - \lambda I) = 0$.

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & -2 \\ 4 & 5-\lambda \end{vmatrix} \\ &= (1-\lambda)(5-\lambda) - (-2)(4) \\ &= (5 - \lambda - 5\lambda + \lambda^2) - (-8) \\ &= \lambda^2 - 6\lambda + 5 + 8 \\ &= \lambda^2 - 6\lambda + 13 \end{aligned}$$

Thus, the characteristic equation is $\lambda^2 - 6\lambda + 13 = 0$.
Step 2. Verify for Matrix A According to the Cayley-Hamilton theorem, the matrix A should satisfy:

$$A^2 - 6A + 13I = 0$$

First, we calculate A^2:

$$\begin{aligned} A^2 &= \begin{bmatrix} 1 & -2 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 4 & 5 \end{bmatrix} \\ &= \begin{bmatrix} (1)(1) + (-2)(4) & (1)(-2) + (-2)(5) \\ (4)(1) + (5)(4) & (4)(-2) + (5)(5) \end{bmatrix} \\ &= \begin{bmatrix} 1-8 & -2-10 \\ 4+20 & -8+25 \end{bmatrix} = \begin{bmatrix} -7 & -12 \\ 24 & 17 \end{bmatrix} \end{aligned}$$

Now, we substitute A^2 and A into the equation:

$$\begin{aligned} A^2 - 6A + 13I &= \begin{bmatrix} -7 & -12 \\ 24 & 17 \end{bmatrix} - 6 \begin{bmatrix} 1 & -2 \\ 4 & 5 \end{bmatrix} + 13 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -7 & -12 \\ 24 & 17 \end{bmatrix} - \begin{bmatrix} 6 & -12 \\ 24 & 30 \end{bmatrix} + \begin{bmatrix} 13 & 0 \\ 0 & 13 \end{bmatrix} \\ &= \begin{bmatrix} -7-6+13 & -12-(-12)+0 \\ 24-24+0 & 17-30+13 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

The result is the zero matrix, verifying that A satisfies its own characteristic equation.

9 Question 8

Problem Compute $A^m : A = \begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix}$, $m = 6$.

Solution We will compute A^6 by diagonalizing the matrix. We find matrices P and D such that $A = PDP^{-1}$, which implies $A^6 = PD^6P^{-1}$.

Step 1. Find Eigenvalues Since A is an upper triangular matrix, its eigenvalues are the diagonal entries:

$$\lambda_1 = -1, \quad \lambda_2 = -3$$

Step 2. Find Eigenvectors

For $\lambda_1 = -1$:

$$(A - (-1)I)\vec{v}_1 = \begin{bmatrix} 0 & 2 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From row 1:

$$2y = 0 \implies y = 0$$

x is a free variable, so we choose

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

For $\lambda_2 = -3$:

$$(A - (-3)I)\vec{v}_2 = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From row 1:

$$2x + 2y = 0 \implies x = -y$$

Let $y = 1$, then $x = -1$. We choose

$$\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Step 3: Construct Matrices P and D The matrix P consists of the eigenvectors, and D contains the eigenvalues:

$$P = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$$

We need to find to have the final diagonalization formula P^{-1}. Observe that the determinant of P is $(1)(1) - (-1)(0) = 1$.

$$P^{-1} = \frac{1}{1}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Step 4: Compute A^6 Using the diagonalization formula:

$$\begin{aligned}
A^6 &= PD^6P^{-1} \\
&= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (-1)^6 & 0 \\ 0 & (-3)^6 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 729 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -729 \\ 0 & 729 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1(1) + (-729)(0) & 1(1) + (-729)(1) \\ 0(1) + 729(0) & 0(1) + 729(1) \end{bmatrix} \\
&= \begin{bmatrix} 1 & -728 \\ 0 & 729 \end{bmatrix}
\end{aligned}$$

10 Question 9

Problem Determine whether the given matrix A is diagonalizable. If so, find the matrix P that diagonalizes A, and the diagonal matrix D such that $D = P^{-1}AP$.

Solution

Step 1. Find Eigenvalues We solve the characteristic equation $\det(A - \lambda I) = 0$:

$$\begin{aligned} \begin{vmatrix} -\lambda & 5 \\ 1 & -\lambda \end{vmatrix} &= 0 \\ (-\lambda)(-\lambda) - (5)(1) &= 0 \\ \lambda^2 - 5 &= 0 \implies \lambda = \pm\sqrt{5} \end{aligned}$$

Since there are two distinct real eigenvalues, the matrix is diagonalizable. Let $\lambda_1 = \sqrt{5}$ and $\lambda_2 = -\sqrt{5}$.

Step 2. Find Eigenvectors

For $\lambda_1 = \sqrt{5}$:

$$(A - \sqrt{5}I)\vec{v}_1 = \begin{bmatrix} -\sqrt{5} & 5 \\ 1 & -\sqrt{5} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From the second row:

$$1x - \sqrt{5}y = 0 \implies x = \sqrt{5}y$$

Let $y = 1$, then $x = \sqrt{5}$. We choose the eigenvector $\vec{v}_1$ corresponding to the eigenvalue λ_1 as:

$$\vec{v}_1 = \begin{bmatrix} \sqrt{5} \\ 1 \end{bmatrix}$$

For $\lambda_2 = -\sqrt{5}$:

$$(A - (-\sqrt{5})I)\vec{v}_2 = \begin{bmatrix} \sqrt{5} & 5 \\ 1 & \sqrt{5} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From the second row:

$$1x + \sqrt{5}y = 0 \implies x = -\sqrt{5}y$$

Let $y = 1$, then $x = -\sqrt{5}$. We choose the eigenvector $\vec{v}_2$ corresponding to the eigenvalue λ_2 as:

$$\vec{v}_2 = \begin{bmatrix} -\sqrt{5} \\ 1 \end{bmatrix}$$

Step 3. Construct Matrices P and D The diagonal matrix D contains the eigenvalues, and P contains the corresponding eigenvectors as columns.

$$D = \begin{bmatrix} \sqrt{5} & 0 \\ 0 & -\sqrt{5} \end{bmatrix}, \quad P = \begin{bmatrix} \sqrt{5} & -\sqrt{5} \\ 1 & 1 \end{bmatrix}$$

So, the matrix A is diagonalizable with the matrices P and D given above.

11 Question 10

Problem Find a basis for i) column space, ii) row space, iii) null space of matrix A:

$$A = \begin{bmatrix} 0 & 6 & 6 & 0 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & -3 & 4 \\ 1 & 0 & 2 & 0 \end{bmatrix}$$

Solution To find the bases, we perform Gaussian Elimination to reduce matrix A to Row Echelon Form (REF).

Step 1. Row Reduction Swap R_1 and R_2 to get a pivot in the first column:

$$\xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 6 & 6 & 0 \\ 0 & 1 & -3 & 4 \\ 1 & 0 & 2 & 0 \end{bmatrix}$$

Eliminate the entry in R_4 using R_1 ($R_4 \to R_4 - R_1$):

$$\xrightarrow{R_4 - R_1} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 6 & 6 & 0 \\ 0 & 1 & -3 & 4 \\ 0 & -2 & 1 & -1 \end{bmatrix}$$

Simplify R_2 by dividing by 6 ($R_2 \to \frac{1}{6}R_2$):

$$\xrightarrow{\frac{1}{6}R_2} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -3 & 4 \\ 0 & -2 & 1 & -1 \end{bmatrix}$$

Eliminate entries below the second pivot ($R_3 \to R_3 - R_2$ and $R_4 \to R_4 + 2R_2$):

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -4 & 4 \\ 0 & 0 & 3 & -1 \end{bmatrix}$$

Simplify R_3 ($R_3 \to -\frac{1}{4}R_3$) to get pivot 1:

$$\xrightarrow{-\frac{1}{4}R_3} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 3 & -1 \end{bmatrix}$$

Eliminate the entry in R_4 ($R_4 \to R_4 - 3R_3$):

$$\xrightarrow{R_4 - 3R_3} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix} : \text{REF}$$

Now that the matrix is in **Row Echelon Form**, we have pivots in columns 1, 2, 3, and 4. Since there are 4 pivots for a 4×4 matrix, the matrix has Full Rank (Rank = 4).

i) Basis for Column Space The basis for the column space consists of the pivot columns from the **original** matrix A. Since all columns have pivots:

$$\text{Basis}_{Col} = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 6 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 6 \\ 1 \\ -3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 4 \\ 0 \end{pmatrix} \right\}$$

ii) Basis for Row Space The basis for the row space consists of the non-zero rows of the **Row Echelon Form**:

$$\text{Basis}_{Row} = \{(1, 2, 1, 1), \quad (0, 1, 1, 0), \quad (0, 0, 1, -1), \quad (0, 0, 0, 1)\}$$

iii) Basis for Null Space The null space is found by solving $A\vec{x} = \vec{0}$. Since the matrix is full rank, there are no free variables. The only solution is the trivial solution $\vec{x} = \vec{0}$.

$$\text{Null Space} = \{\vec{0}\}$$

The dimension of the null space is 0, so the basis is the empty set $\emptyset$.

12 Question 11

Problem Obtain an orthonormal set from the given set of vectors using Gram-Schmidt Orthogonalization Process.

$$\vec{V}_1 = (1,0,1), \quad \vec{V}_2 = (1,1,0), \quad \vec{V}_3 = (1,-2,-3)$$

Solution We generate an orthogonal set $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ and then normalize to get $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$.

Step 1. Process first vector
Set $\vec{u}_1 = \vec{V}_1 = (1,0,1)$.

$$\|\vec{u}_1\|^2 = 1^2 + 0^2 + 1^2 = 2 \implies \|\vec{u}_1\| = \sqrt{2}$$

The first orthonormal vector is:

$$\vec{e}_1 = \frac{\vec{u}_1}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

Step 2. Orthogonalize second vector
Calculate projection of $\vec{V}_2$ onto $\vec{u}_1$:

$$\begin{aligned}
\vec{V}_2 \cdot \vec{u}_1 &= (1)(1) + (1)(0) + (0)(1) = 1 \\
\vec{u}_2 &= \vec{V}_2 - \frac{\vec{V}_2 \cdot \vec{u}_1}{\|\vec{u}_1\|^2}\vec{u}_1 \\
&= (1,1,0) - \frac{1}{2}(1,0,1) \\
&= (1,1,0) - (0.5,0,0.5) \\
&= (0.5,1,-0.5) = \left(\frac{1}{2}, 1, -\frac{1}{2}\right)
\end{aligned}$$

Normalize $\vec{u}_2$:

$$\begin{aligned}
\|\vec{u}_2\|^2 &= \left(\frac{1}{2}\right)^2 + 1^2 + \left(-\frac{1}{2}\right)^2 = \frac{1}{4} + 1 + \frac{1}{4} = \frac{3}{2} \\
\|\vec{u}_2\| &= \sqrt{\frac{3}{2}} \\
\vec{e}_2 &= \frac{\vec{u}_2}{\sqrt{3/2}} = \sqrt{\frac{2}{3}}\left(\frac{1}{2}, 1, -\frac{1}{2}\right) = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)
\end{aligned}$$

Step 3. Orthogonalize third vector
Formula: $\vec{u}_3 = \vec{V}_3 - \text{proj}_{\vec{u}_1}(\vec{V}_3) - \text{proj}_{\vec{u}_2}(\vec{V}_3)$. First, compute the dot products:

$$\begin{aligned}
\vec{V}_3 \cdot \vec{u}_1 &= (1)(1) + (-2)(0) + (-3)(1) = 1 - 3 = -2 \\
\vec{V}_3 \cdot \vec{u}_2 &= (1)(0.5) + (-2)(1) + (-3)(-0.5) = 0.5 - 2 + 1.5 = 0
\end{aligned}$$

Since $\vec{V}_3 \cdot \vec{u}_2 = 0$, the vector $\vec{V}_3$ is already orthogonal to $\vec{u}_2$, so the second projection term is zero.

$$\begin{aligned}
\vec{u}_3 &= \vec{V}_3 - \frac{-2}{2}\vec{u}_1 - 0 \\
&= (1, -2, -3) - (-1)(1, 0, 1) \\
&= (1, -2, -3) + (1, 0, 1) \\
&= (2, -2, -2)
\end{aligned}$$

Normalize $\vec{u}_3$:

$$\begin{aligned}
\|\vec{u}_3\| &= \sqrt{2^2 + (-2)^2 + (-2)^2} = \sqrt{4 + 4 + 4} = \sqrt{12} = 2\sqrt{3} \\
\vec{e}_3 &= \frac{(2, -2, -2)}{2\sqrt{3}} = \frac{(1, -1, -1)}{\sqrt{3}} = \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)
\end{aligned}$$

The orthonormal set is $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$, where

$$\vec{e}_1 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \quad \vec{e}_2 = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right), \quad \vec{e}_3 = \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$$