CS2040 – Data Structures and Algorithms

Lecture 16 – Finding Shortest Way

from Here to There, Part II

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Outline

Four special cases of the classical SSSP problem

- Special Case 1: The graph is a **tree**
- Special Case 2: The graph is **unweighted**
- Special Case 3: The graph is **directed** and **acyclic** (DAG)
- Special Case 4ab: The graph has **no negative weight edge/cycle**
  - Introduce a new SSSP algo (Dijkstra's algorithm)

Basic Form and Variants of a Problem

In this lecture, we will continue on the same topic that we have seen in the previous lecture:

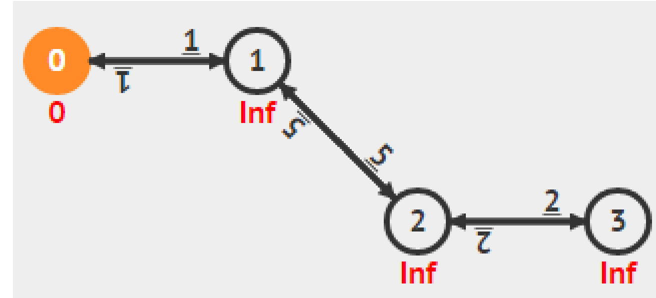
- The **Single-Source Shortest Path (SSSP)** problem

An idea from the previous lecture and this one is that a certain problem can be made **‘simpler’** if some assumptions are made

- These variants (special cases) may have better/more efficient algorithms

Special Case 1:

The weighted graph is a **Tree**



When the weighted graph is a tree, solving the SSSP problem becomes much easier as every path in a tree is a shortest path. **Q1: Why?**

There won't be any negative weight cycle. **Q2: Why?**

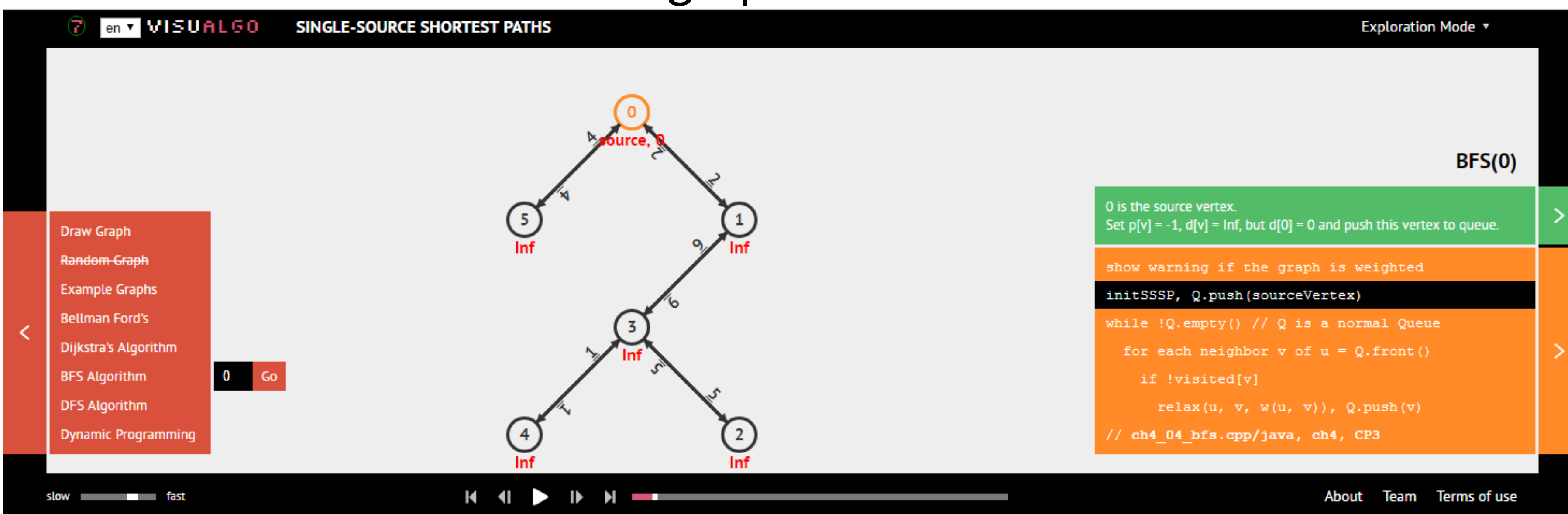
Thus, any **O(V)** graph traversal, i.e. **either DFS or BFS** can be used to solve this SSSP problem.
Q3: Why O(V) and not the standard O(V+E)?

Try in VisuAlgo!

(use DFS/BFS)

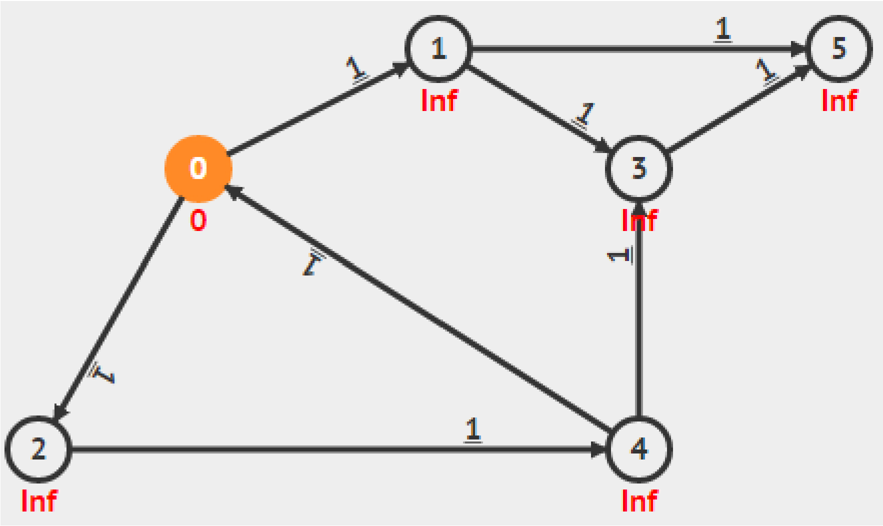
Try finding the shortest paths from source vertex 0 to other vertices in this weighted (undirected) tree

- Notice that you will always encounter unique (simple) path between those two vertices
- Try adding negative weight edges, it does not matter if the graph is a tree ☺



Special Case 2:

The graph is **unweighted**



This has been discussed yesterday ☺

Solution: O(V+E) BFS

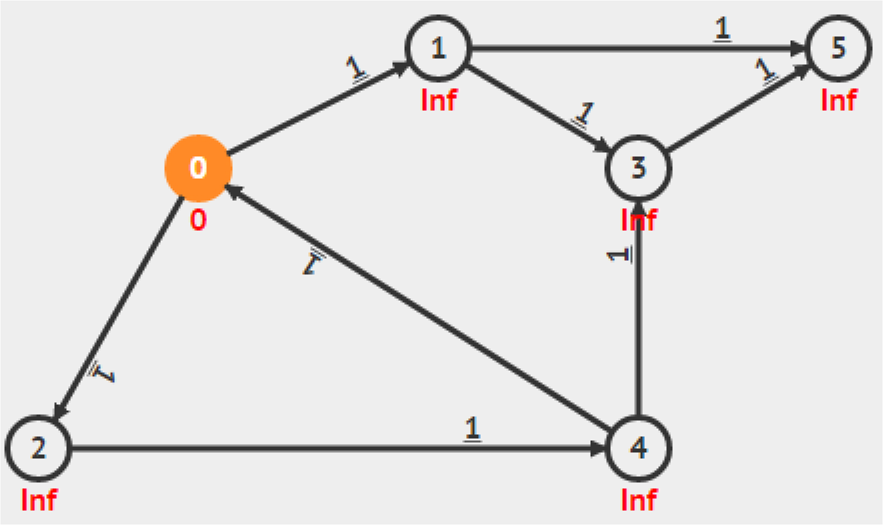
Important note:

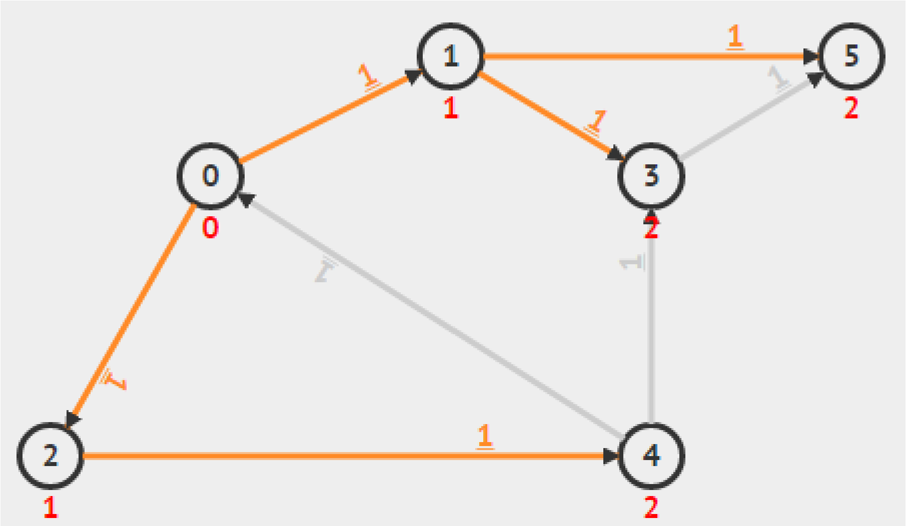
- For SSSP on unweighted graph, we can *only* use BFS
- For SSSP on tree, we can use *either* DFS/BFS

Try in VisuAlgo!

This graph is unweighted (i.e. all edge weight = 1)

Try finding the shortest paths from source vertex 0 to other vertices using **BFS**



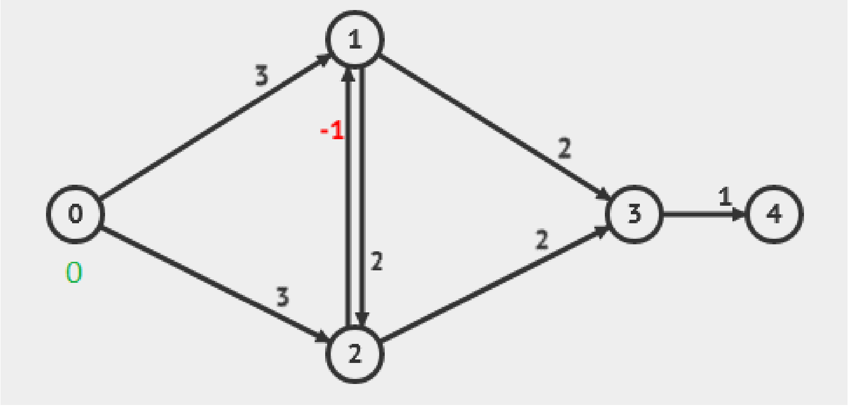


Special Case 3:

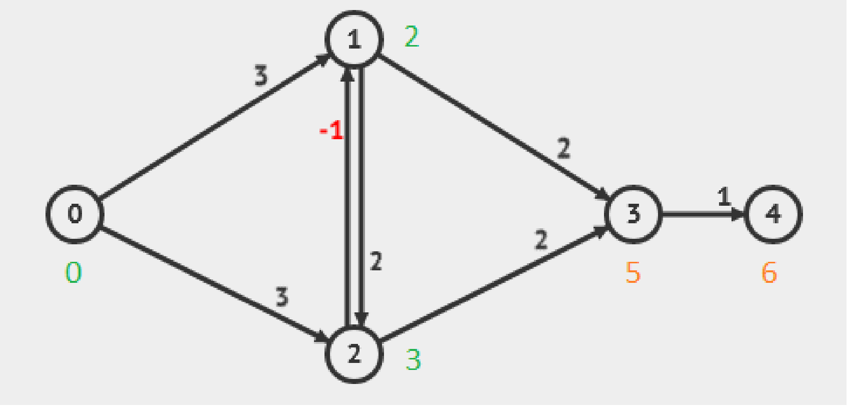
The weighted graph is **directed** & **acyclic** (DAG)

Cycle is a major issue in SSSP

- Can cause edges to be relaxed multiple times (depending on order of edge relaxation of the edges involved in the cycle)



Solve SSSP at source vertex 0



After one pass

Order of edge relaxation
0-1, 0-2, **1-2**, **1-3**, 2-3, **2-1**, 3-4

SP to vertex 3 and 4 not yet found because sequence of edge relaxation caused the longer paths (0,1,3) & (0,1,3,4) to be found first before the shorter paths (0,2,1,3) & (0,2,1,3,4)

Special Case 3:

The weighted graph is **directed** & **acyclic** (DAG)

When the graph is **acyclic** (has no cycle), a good ordering on the edges can be imposed → Topological ordering.

We can "modify" the Bellman Ford's algorithm by replacing the outermost **V**-1 loop to just **one pass**

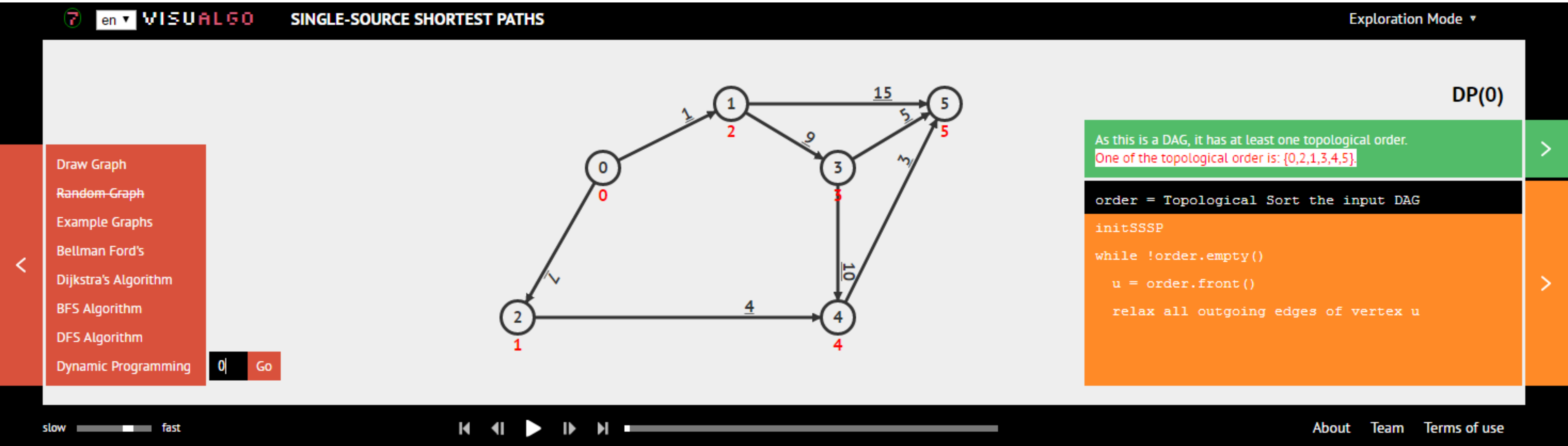
- i.e. we only run the relaxation across all edges once in topological order (recall topological sort in Lecture 13)

*Also known as "One-pass Bellman Ford"

Try in VisuAlgo!

One Topological Sort of the given DAG is {0, 2, 1, 3, 4, 5}

- Try relaxing the outgoing edges of vertices listed in the topological order above (starting from the source vertex, 0 in this case)
  - With just one pass, all vertices will have the correct D[v]



Special Case 4a:

The graph has **no negative weight edge**

Bellman Ford's algorithm works fine for all cases of SSSP on weighted graphs, but it runs in **O(VE)**… ☹

- For a **"reasonably sized"** weighted graphs with **V** ~ 1000 and **E** ~ 100000 (recall that **E** = O(**V**^{2}) in a complete simple graph), Bellman Ford's is (really) **"slow"**…

For many practical cases, the SSSP problem is performed on a graph where all its edges have **non-negative weight**

- Example: Traveling between two cities on a map (graph) usually takes **positive amount** of time units

Fortunately, there is a *faster* SSSP algorithm that exploits this property: The **Dijkstra's** algorithm

The ‘original version’

DIJKSTRA’S ALGORITHM

Key Ideas of (original) Dijkstra's Algorithm (1)

(for graphs with no negative weight edge)

Formal assumption:

- For each **edge(u, v)** ∈ **E**, we assume **w(u, v) ≥ 0 (non-negative)**

Key ideas of (the original) Dijkstra's algorithm:

- Maintain a set **Solved** of vertices whose **final shortest path weights** have been determined, initially **Solved** = **{s}**, source vertex **s** only
- Repeatedly select vertex **u** in {**V-Solved**} with the min shortest path *estimate* **D[u]**, add **u** to **Solved**, and relax all edges out of **u**
  - This entails the use of a kind of **"Priority Queue"**, **Q: Why?**
  - This choice of relaxation order is **"greedy"**: Select the "best so far"
    - Once added to **Solved** greedily, a vertex is never again enqueued in the PQ
    - But it eventually ends up with optimal result (see the proof later)

Note: Vertices are added to Solved in non-decreasing SP costs …

Key Ideas of (original) Dijkstra's Algorithm (2)

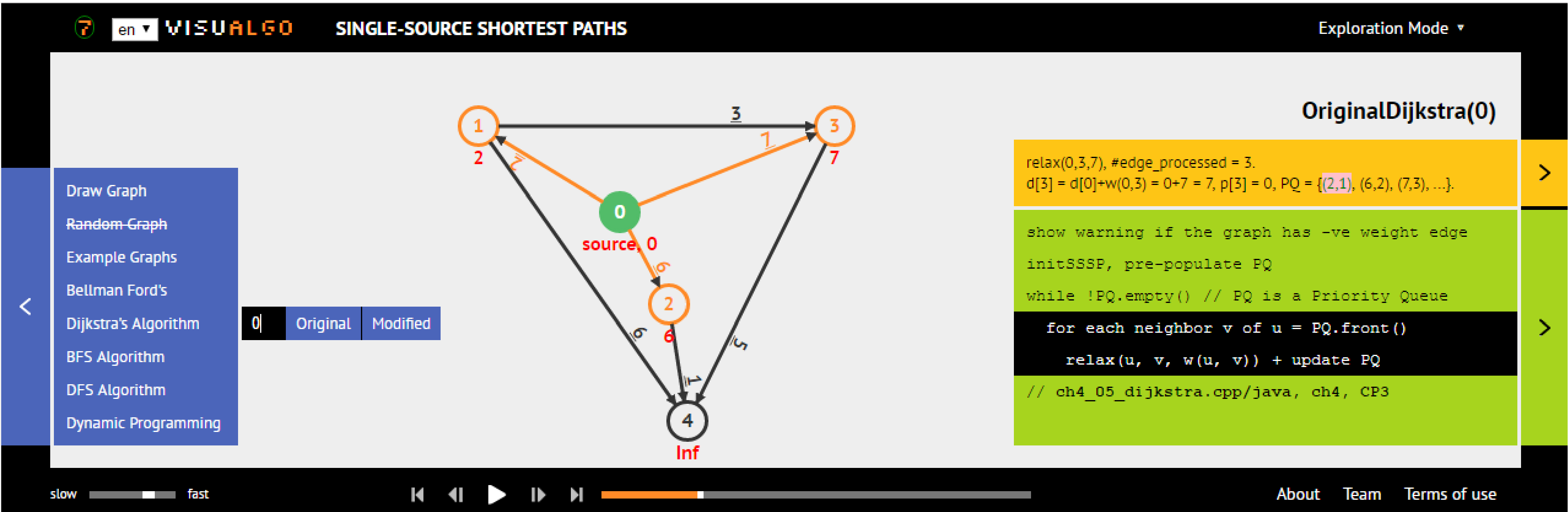
More details on key ideas of Dijkstra's algorithm:

1. PQ: Store the *shortest path estimate* for a vertex **v** as an IntegerPair (**d**, **v**) in the PQ, where **d** = **D[v]** (current shortest path estimate)
2. Initialization: Enqueue (∞,**v**) for all vertices **v** except for source **s** which will enqueue (0,**s**) into the PQ
   - PQ will store integer pair for all vertices at the start
3. Main loop: Keep removing vertex **u** with minimum **d** from the PQ, add **u** to **Solved** and relax all its outgoing edges (see point 4.) until the PQ is empty
   - When PQ is empty all the vertices will be in **Solved**
4. If an edge (**u**,**v**) is relaxed find the vertex **v** it is pointing to in the PQ and "update" the shortest path estimate
   - Need to find **v** quickly and perform PQ "DecreaseKey" operation (no Java PQ ☹)
   - Alternatively use bBST to implement the PQ (how?)

SSSP: Dijkstra's (Original)

Ask VisuAlgo to perform Dijkstra's (Original) algorithm *from various sources* on the sample Graph (CP4 4.16)

The screen shot below shows the *initial stage* of **Dijkstra(0)** (the original algorithm)



Why Does This Greedy Strategy Works? (1)

i.e. why is it sufficient to only process each vertex just once?

Loop invariant = *Every vertex v in set* ***Solved*** *has correct shortest path distance from source, i.e D[v] =* **δ(s, v)**

- This is true initially, **Solved = {s}** and **D[s] = δ(s, s) = 0**

Dijkstra's algorithm iteratively adds the next vertex **u** with the lowest **D[u]** into set **Solved**

- Is the loop invariant always valid?
- Let's see a short lemma first which will be used to proof the loop invariant holds

Lemma 1: Subpaths of a shortest path are shortest paths

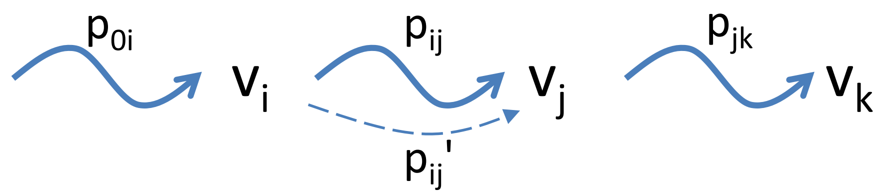
Let **p** be the shortest path: $p = \langle v_0, v_1, v_2, \ldots, v_k \rangle$

Let $\mathbf{p_{ij}}$ be the subpath of **p**: $p_{ij} = \langle v_i, v_{i+1}, \ldots, v_j \rangle, 0 \le i \le j \le k$

Then $\mathbf{p_{ij}}$ is a shortest path (from **i** to **j**)

Proof by contradiction:

- Let the shortest path **p** = v_0



- If $\mathbf{p_{ij}}$ is not the shortest path, the we have another $\mathbf{p_{ij}}$**'** that is shorter than $\mathbf{p_{ij}}$. We can then cut out $\mathbf{p_{ij}}$ and replace it with $\mathbf{p_{ij}}$**'**, which result in a shorter path from $\mathbf{v_0}$ to $\mathbf{v_k}$
- But **p** is the shortest path from v_0 to v_k → contradiction!
- Thus $\mathbf{p_{ij}}$ must be a shortest path between v_i and v_j

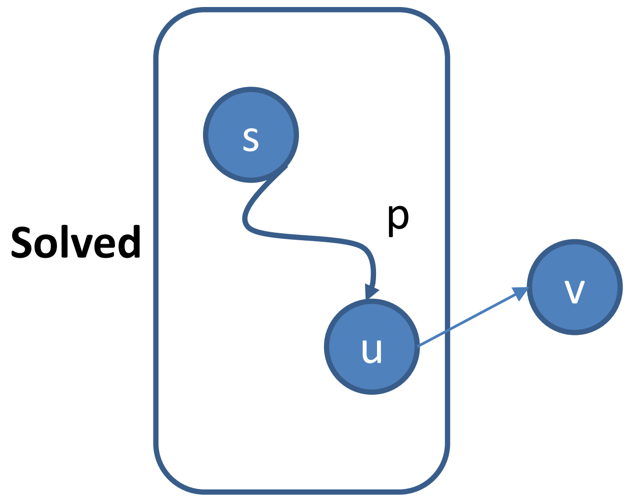
Lemma 2: After a vertex v is added to **Solved**, SP from s to v has been found (1)

Proof by contradiction:

- Let v be the 1st vertex added to **Solved** where SP from s to v has not be found when it was added

- Let p be path from s to v when v was added to **Solved**

$p = s \rightsquigarrow u \rightarrow v$



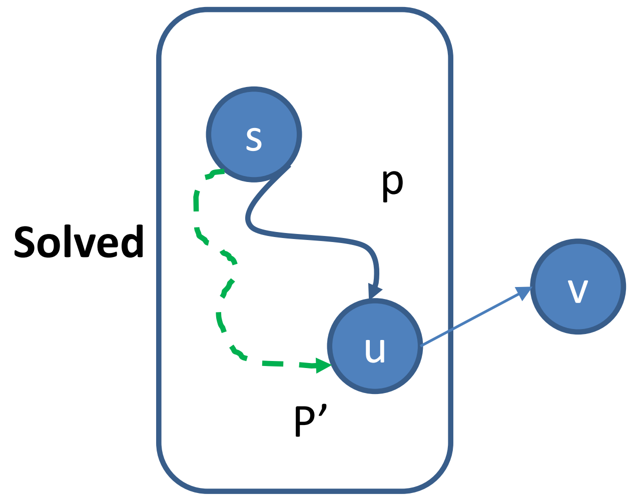
Observations:

1. All vertices in $s \rightsquigarrow u$ must be in **Solved**
2. $s \rightsquigarrow u$ must be the SP from s to u since v is the 1st one added wrongly

Lemma 2: After a vertex v is added to **Solved**, SP from s to v has been found (2)

- There are then only 3 possibilities for the correct SP p’

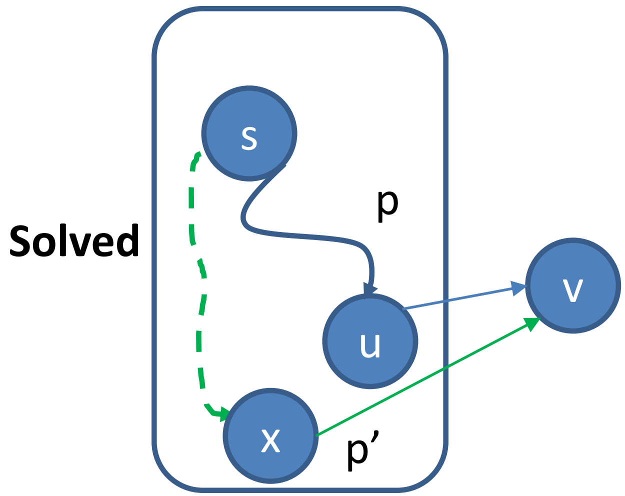
- Possibility 1: Predecessor of v in the correct SP is still u but the path from s to u is not the same



- Cannot be the case by way of Lemma 1 and the fact that s $\rightsquigarrow$ u is SP from s to u by observation 2

Lemma 2: After a vertex v is added to **Solved**, SP from s to v has been found (3)

- Possibility 2: Predecessor of v in the correct SP p' is another vertex x

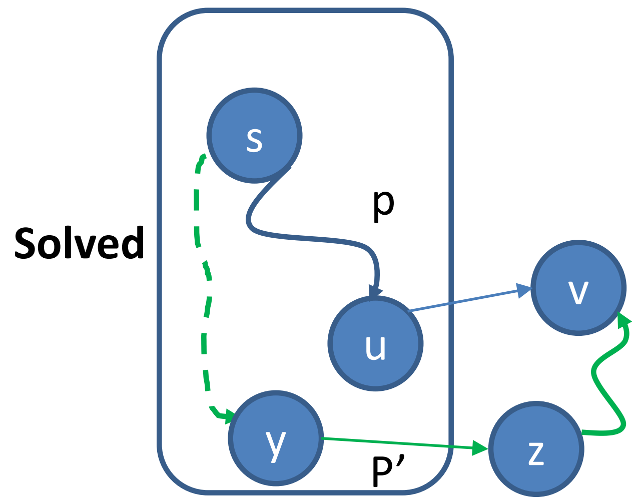


- Cannot be the case since v had the lowest cost in the PQ through relaxation of (u,v) and not (x,v), therefore

cost(p') = cost(SP(s,x))+w(x,v) > cost(p) = cost(SP(s,u))+w(u,v)

Lemma 2: After a vertex v is added to **Solved**, SP from s to v has been found (4)

- Possibility 3: There exist at least one vertex along correct SP p' from s to v which is not in **Solved**. Let z be the first such vertex.



- Since y and u in **Solved**, their SP is correct and they will have correctly relaxed their neighbor z and v respectively
- Since v was added to Solved instead of z, we have

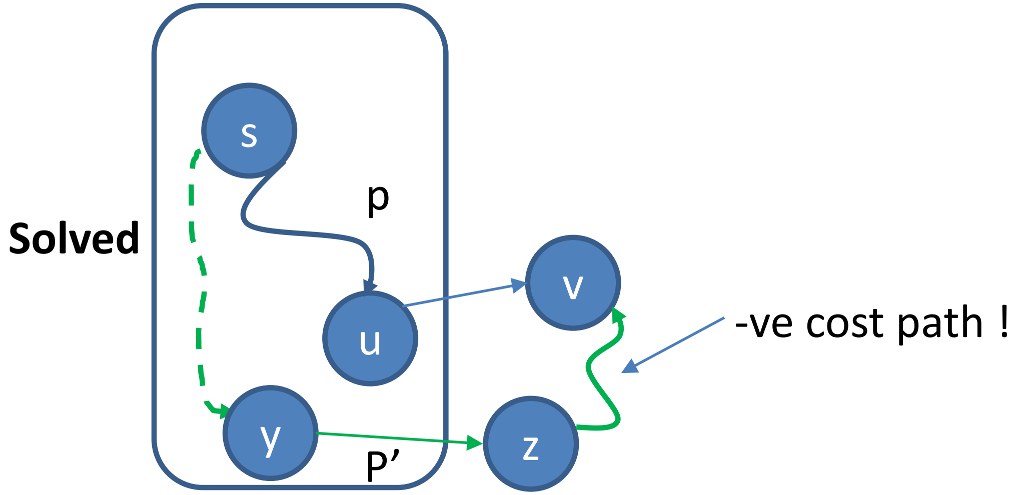
$$\color{purple}{\text{cost(SP(s,z))}} = \text{cost(SP(s,y))+w(y,z)} > \text{cost(p)}$$

Lemma 2: After a vertex v is added to **Solved**, SP from s to v has been found (5)

- Now for

cost(p') = cost(SP(s,z))+cost(SP(z,v)) < cost(p)

cost(SP(z,v)) must be < 0 which means there are –ve edge weights which is a contradiction that the graph only has +ve edge weights.



- Since there is no 1st vertex which is added wrongly, the algorithm is correct

Why Does This Greedy Strategy Works? (2)

i.e. why is it sufficient to only process each vertex just once?

- Therefore by lemma 2, since SP to v has been found once it is put into Solved, we will never need to revisit it again, thus greedy works

Original Dijkstra's – Analysis (1)

In the original Dijkstra's, each vertex will only be inserted and extracted from the priority queue **once**

- As there are **V** vertices, we will do this at most O(**V**) times
- Each insert/extract min runs in O(log **V**) (since at most V items in the PQ) if implemented using **binary min heap (insert/extractMin)** or **bBST (insert/deleteMin)**

Therefore this part is O(**V** log **V**)

Original Dijkstra’s – Analysis (2)

Every time a vertex is processed, we relax its neighbors

- In total, all O(**E**) edges are processed (and only once for each edge)
- If by relaxing edge(**u**, **v**), we have to decrease **D[v]**, we call the O(log **V**) **DecreaseKey() in binary min heap** (harder to implement) or simply **delete old entry and then re-insert new entry in balanced BST** (which also runs in O(log **V**), but this is much easier to implement)
  - **The easiest implementation is to use **Java TreeSet** as the PQ

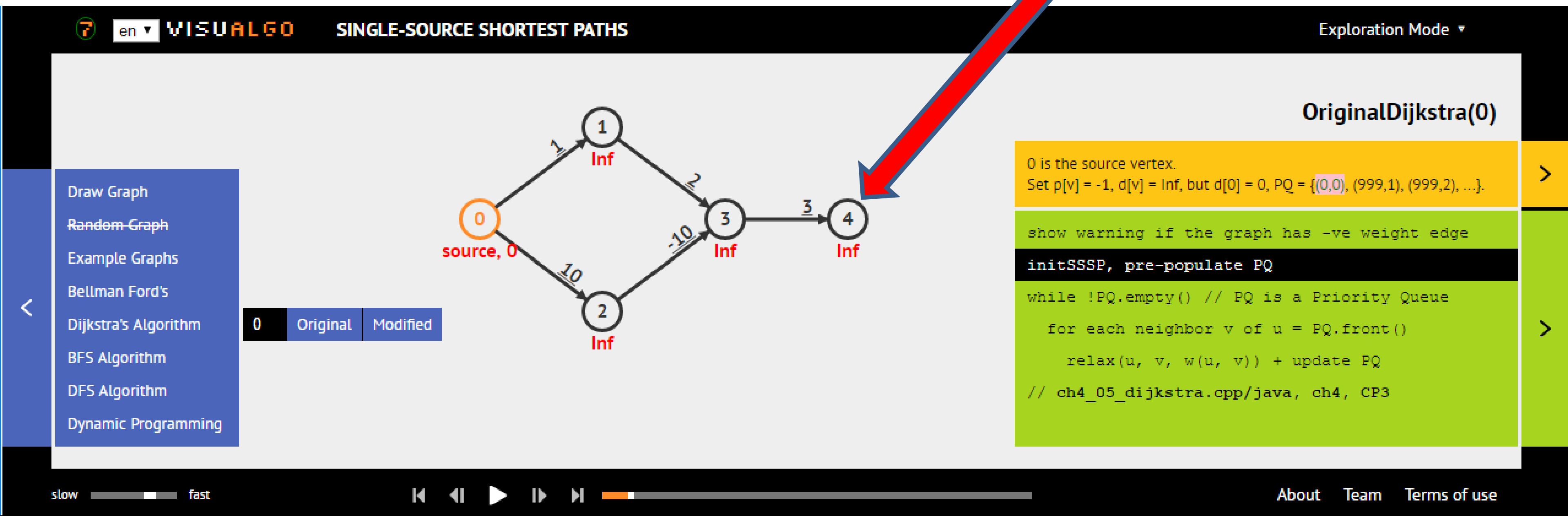
This part is O(**E** log **V**)

Thus overall, Dijkstra’s runs in O(**V** log **V** + **E** log **V**), or more well known as an **O((V+E)** log **V)** algorithm

Wait… Let's try this!

Ask VisuAlgo to perform Dijkstra's (Original) algorithm from source = 0 on the sample Graph (CP4 4.20)

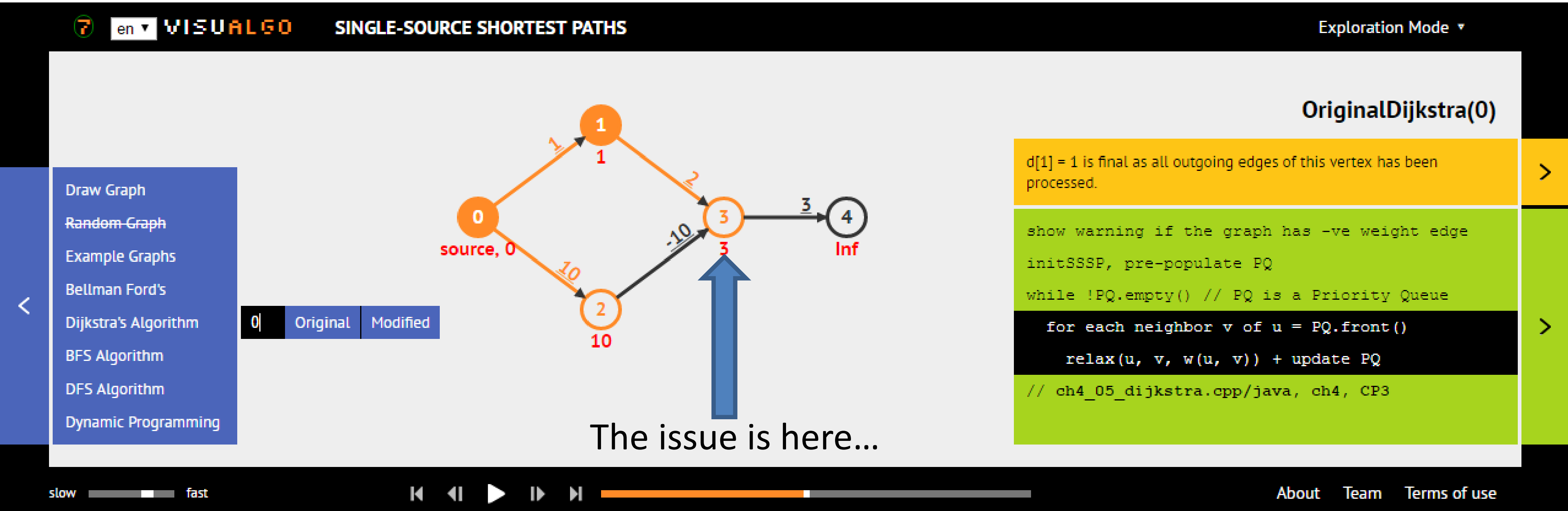
Do you get correct answer at vertex 4?



Why Does This Greedy Strategy Not Work This Time ☹?

The presence of negative-weight edge can cause the vertices "greedily" chosen first eventually not to have the true shortest path from the source!

- It happens to vertex 3 in this example



The ‘modified’ implementation

DIJKSTRA’S ALGORITHM

Special Case 4b:

The graph has **no negative weight cycle**

For many practical cases, the SSSP problem is performed on a graph where its edges may have **negative weight but it has no negative cycle**

We have another version of Dijkstra's algorithm that can handle this case: The **Modified Dijkstra's** algorithm

Implementation of Modified Dijkstra's Algorithm (1)

Formal assumption (different from the original one):

- The graph has **no negative weight cycle** (but can have negative weight edges)

Key ideas:

- Allow a vertex to be possibly processed multiple times as detailed below and in the next slide
- Use a **built-in** priority queue in **Java Collections** to order the next vertex **u** to be processed based on its **D[u]**
  - This vertex information is stored as IntegerPair **(d, u)** where **d** = **D[u]** (the current shortest path estimate)
- But with modification: We use **"Lazy Data Structure"** strategy
  - **Main idea:** No need to maintain just one IntegerPair (shortest path estimate) for each vertex **v** in the PQ
  - Can have multiple shortest path estimates to exist in the PQ for a vertex **v**

Implementation of Modified Dijkstra's Algorithm (2)

Lazy DS: Extract pair **(d, u)** in **front of the priority queue PQ** with the minimum shortest path estimate ***so far***

- if **d = D[u]**, we relax all edges out of **u**,
  else if **d > D[u]**, we discard this inferior **(d, u)** pair
  - Since there can be multiple copies of **(d, u)** pair we only want the most up to date copy
  - See below to understand how we get multiple copies !
- If during edge relaxation, **D[v]** of a neighbor **v** of **u** *decreases*, enqueue a new **(D[v], v)** pair for *future propagation* of shortest path estimate
  - No need to find the **v** in the **PQ** and update it!
  - Thus no need to implement **DecreaseKey** (which you don't have in Java PriorityQueue class) or need bBST implementation of PQ!

Modified Dijkstra's Algorithm

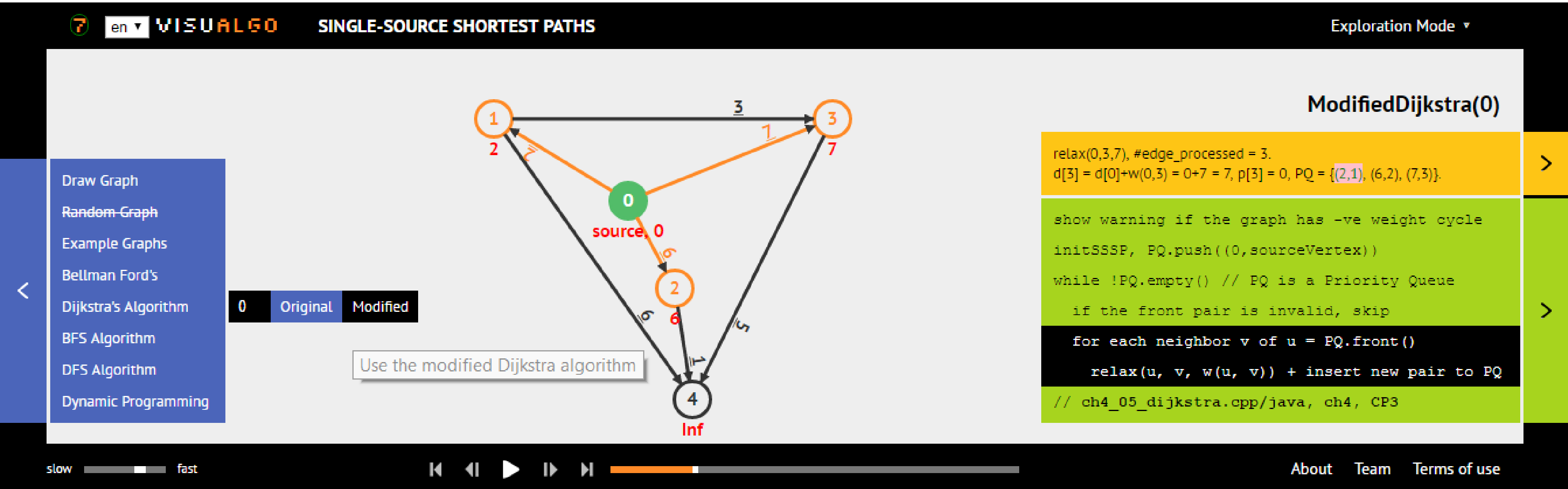
```
initSSSP(s)

PQ.enqueue((0, s)) // store pair of (dist[u], u)
while PQ is not empty // order: increasing dist[u]
  (d, u) ← PQ.dequeue()
  if d == D[u] // important check, lazy DS
    for each vertex v adjacent to u
      if D[v] > D[u] + w(u, v) // can relax
        D[v] = D[u] + w(u, v) // relax
        PQ.enqueue((D[v], v)) // (re)enqueue this
```

SSSP: Dijkstra's (Modified)

Ask VisuAlgo to perform Dijkstra's (Modified) algorithm *__from various sources__* on the sample Graph (CP4 4.16)

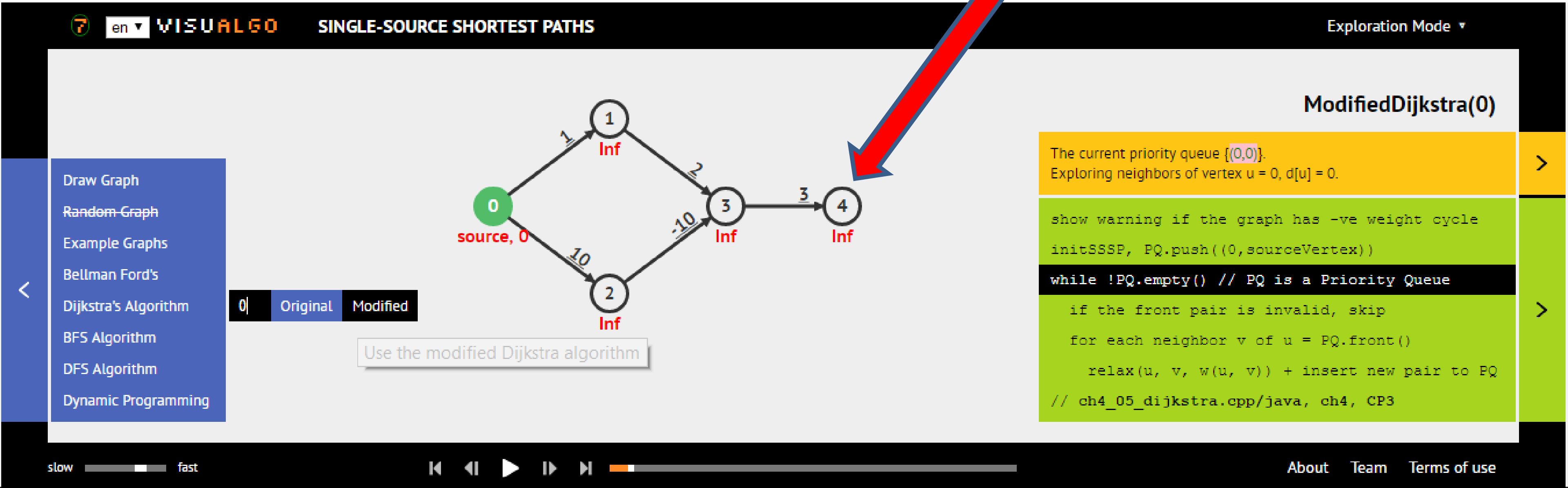
The screen shot below shows the *initial stage* of **Dijkstra(0)** (the modified algorithm)



Try!

Ask VisuAlgo to perform Dijkstra's (**modified**) algorithm from source = 0 on the sample Graph (CP4 4.20)

Do you get correct answer at vertex 4?



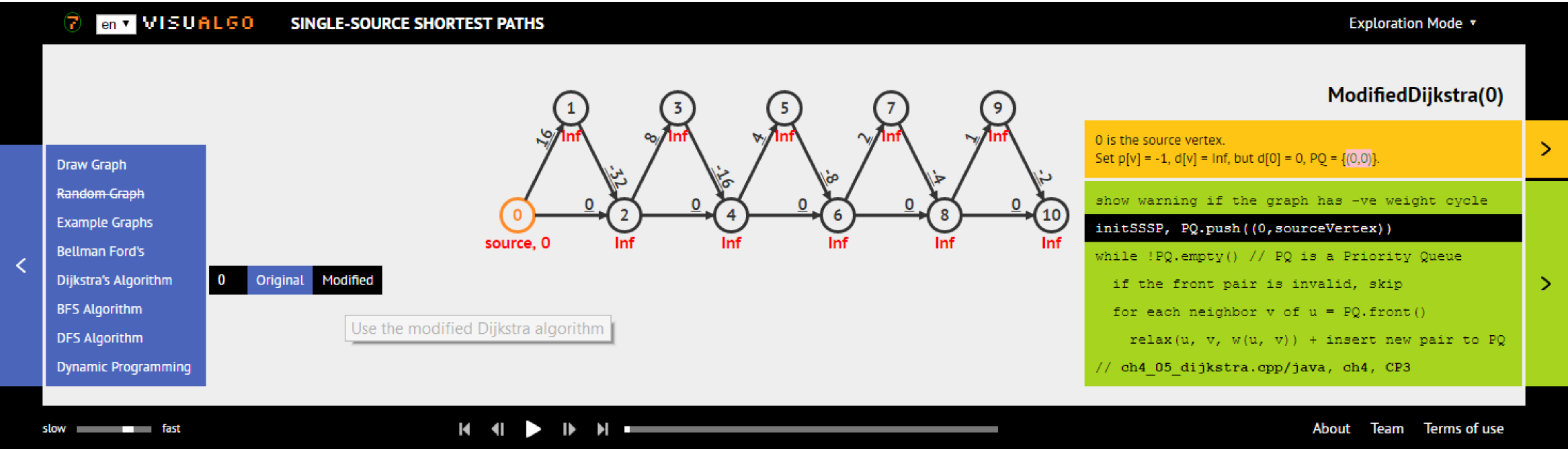
Modified Dijkstra's – Analysis
for graphs with no negative weight edge

If there is **no-negative weight edge**, there will never be another path that can decrease **D[u]** once **u** is dequeued from the PQ and processed **(Original Dijkstra's proof)**

- Thus each vertex will still be dequeued from the PQ and processed once;
  - Even though a vertex v can have multiple copies in the PQ outdated copies are not processed due to the (d > D[v]) check
- Each processed vertex can at most relax all its neighbors thus making as many insertions into the PQ as there are neighbors
- In total the number of insertions into the PQ is **O(E)** meaning the size of the PQ is at most **O(E)**
- At the end, the PQ is empty so we have made **O(E)** insertions and extractMin, each taking at most **O(logE)** time, thus total time is **O(ElogE)**. This is the same as **O((V+E)logV)** except when **E < O(V),** then **O(ElogE)** < (**O((V+E)logV) = O(VlogV)**)

Not an all-conquering algorithm… Check this

If there are negative weight edges without negative cycle, then there exist some (extreme) cases where the modified Dijkstra's re-process the same vertices several/many/crazy amount of times…



About that Extreme Test Case

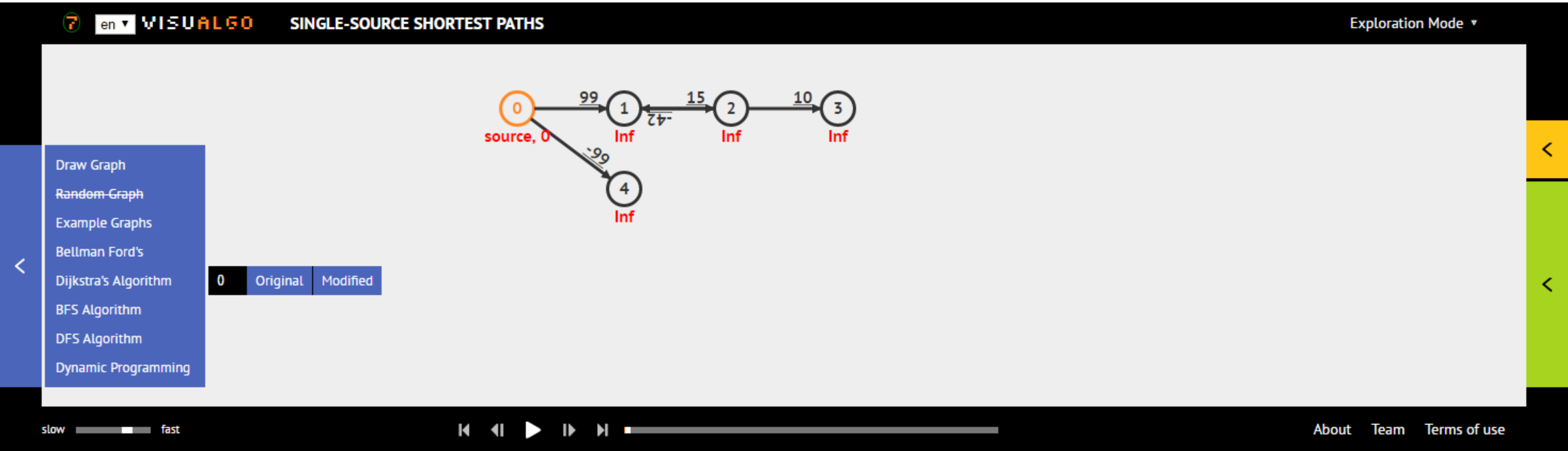
Such extreme cases that causes *exponential time complexity* are *rare* and thus in practice, the modified Dijkstra's implementation runs <u>much faster</u> than the Bellman Ford's algorithm ☺

- If you know your graph has only a few (or no) negative weight edge, this version is probably one of the best current implementation of Dijkstra's algorithm
- But, if you know for sure that your graph has a high probability of having a negative weight cycle, use the tighter (and also simpler) O(**VE**) Bellman Ford's algorithm as this modified Dijkstra's implementation can be <u>*trapped in an infinite loop*</u>

Try Sample Graph, CP4 4.22!

Find the shortest paths from s = 0 to the rest

- Which one **can terminate**?
  The original or the modified Dijkstra's algorithm?
- Which one is **correct when it terminates**?
  The original or the modified Dijkstra's algorithm?



Summary of Various SSSP Algorithms

- General case: weighted graph
  - Use O(**VE**) Bellman Ford's algorithm (the previous lecture)
- Special case 1: Tree
  - Use O(**V**) BFS or DFS ☺
- Special case 2: unweighted graph
  - Use O(**V**+**E**) BFS ☺
- Special case 3: DAG
  - Use O(**V**+**E**) DFS to get the topological sort,
    then relax the vertices using this topological order
- Special case 4ab: graph has no negative weight/negative cycle
  - Use O((**V**+**E**) log **V**) original/O(**E**log**E**) modified Dijkstra's, respectively