$$\left(\sum_{k=1}^{2n-1}\frac{1}{k}(1-\cos k\theta)\right)^2+\left(\sum_{k=1}^{2n-1}\frac{\sin k\theta}{k}\right)^2<4\left(\sum_{k=1}^{n}\frac{1}{2k-1}\right)^2$$

A SHARP TRIGONOMETRIC DOUBLE INEQUALITY AND BRANNAN'S CONJECTURE: THE CASE r=1

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CONTENT PAGE

01. BACKGROUND

In 1973, *D. A. Brannan*

conjectured that if we let

$$\frac{(1+zx)^{\alpha}}{(1-x)^{\beta}} = \sum_{m=0}^{\infty} A_m(\alpha, \beta, z)\, x^n$$

Where α >0, β >0,and z=e^iθ, 0≤θ≤2π,
then

$$|A_m(\alpha, \beta, z)| \leqslant A_m(\alpha, \beta, 1)$$

for all odd integer m. Here, A_m refers to the coefficient of the mth order term in the polynomial.

01. BACKGROUND

Recently, in an attempt to prove the conjecture for 0< α <1 and β= 1, *R.W. Barnard et al*. reformulates inequality into finding the largest r satisfying

$$|A_m(\alpha, \beta, z)| \leqslant A_m(\alpha, \beta, r)$$

In the paper, the authors try to show that $r \leq 1$ for all old m. In particular, they successfully show that the inequality above holds for $0 < r \leq 1/2$.

01. BACKGROUND

R.W. Barnard et al. reduce the case where $1/2<r\leq 1$ into proving the following conjecture, expressed in term of the trigonometric form as

$$\left(\sum_{k=1}^{2n-1}\frac{r^k}{k}(1-\cos k\theta)\right)^2+\left(\sum_{k=1}^{2n-1}\frac{r^k}{k}\sin k\theta\right)^2<4\left(\sum_{k=1}^{n}\frac{1}{2k-1}\right)^2$$

for $1/2<r\leq 1$, $n\in\mathbb{N}$ and notably $\theta\in[0,\pi)$

Therefore in this paper, we give a partial answer to the above conjecture for $r=1$

02. METHODOLOGY

$$\left(\sum_{k=1}^{2n-1} \frac{1}{k}(1 - \cos k\theta)\right)^2 + \left(\sum_{k=1}^{2n-1} \frac{\sin k\theta}{k}\right)^2 < 4\left(\sum_{k=1}^{n} \frac{1}{2k-1}\right)^2$$

for $n \geq 5$ and $\theta \in [0, 2\pi/3]$

METHODOLOGY

LEMMA 3.1

$$-\frac{2929}{1260}+\frac{6079}{1260}x+\frac{8063}{1260}x^2-\frac{9}{4}x^3-x^4<0.$$

LEMMA 3.2

$$-(1+\cos\theta)^2\sum_{k=1}^{5}\frac{1}{2k-1}+\frac{1}{16}(1+\cos\theta)^4+\left(\frac{\pi-\theta}{2}+\frac{1}{8}\right)^2<0.$$

$$\theta\in\left[\frac{\pi}{2},\frac{2\pi}{3}\right]$$

LEMMA 3.3

$$-(1+\cos\theta)^2\sum_{k=1}^{5}\frac{1}{2k-1}+\frac{1}{16}(1+\cos\theta)^4+\left(\frac{\pi-\theta}{2}+\frac{3}{10}\right)^2<0.$$

$$\theta\in\left[0,\frac{\pi}{2}\right]$$

LEMMA 3.4

$$\left(\sum_{k=1}^{2n-1}\frac{1}{k}(1-\cos k\theta)\right)^2+\left(\sum_{k=1}^{2n-1}\frac{\sin k\theta}{k}\right)^2<4\left(\sum_{k=1}^{n}\frac{1}{2k-1}\right)^2.$$

$$\theta\in[0,\frac{2\pi}{3}]$$

03.
APPLICATIONS

- Engineering
- Signal processing
- Digital filters

THANK YOU