Brannan's conjecture and trignometric polynomials

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Abstract

For any integer $n \geq 1$, $\frac{1}{2} < r \leq 1$ and $0 \leq \theta < \pi$, we prove that:

$$\left(\sum_{k=1}^{2n-1} \frac{r^k}{k}(1-\cos k\theta)\right)^2 + \left(\sum_{k=1}^{2n-1} \frac{r^k}{k}\sin k\theta\right)^2 < 4\left(\sum_{k=1}^{n} \frac{r^{2k-1}}{2k-1}\right)^2,$$

which provides an affirmative answer to a related conjecture of Brannan's conjecture.

Brannan's conjecture

Let

$$\frac{(1+zx)^\alpha}{(1-x)^\beta} = \sum_{n=0}^{\infty} A_n(\alpha, \beta, z) x^n,$$

where $\alpha, \beta > 0$ and $z = e^{i\theta}$ $(\theta \in [0, 2\pi])$.

Brannan's conjecture

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where $\alpha,\beta > 0$ and $z = e^{i\theta}$ ($\theta \in [0, 2\pi]$).

In 1973, D.A. Brannan [2] conjectured that

$$|A_n(\alpha,\beta,z)| \leq A_n(\alpha,\beta,1) \tag{1}$$

for all $\alpha,\beta > 0$, $z \in \mathbb{C}$ such that $|z| = 1$, and all odd integers n.
(A_n refers to coefficient of the n
th*orderterm*)

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While the conjecture was proven for $\alpha \geqslant 1, \beta > 1$ by Aharonov & Friedland [1], the case $0 < \alpha \leqslant 1$ and $0 < \beta \leqslant 1$ proved to be difficult.

R.W. Barnard et al. [3]

Barnard et al. [3] attempted to prove the conjecture for $0 < \alpha < 1$ and $\beta = 1$ by reformulating inequality (1) into finding the largest r that satisfies

$$|A_n(\alpha, \beta, z)| \leqslant A_n(\alpha, \beta, r), \tag{2}$$

where z is generalised to $z = re^{i\theta}$ and A_n is treated as an analytic function.

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The authors proved (2) holds for $0 < r \leqslant 1/2$ when n is odd.

R.W. Barnard et al. [3]

For the case $1/2 < r \leqslant 1$ and n is odd, they showed (2) is equivalent to:

Conjecture 1.

For any integer $n \geq 1$, $\frac{1}{2} < r \leq 1$ and $0 \leq \theta < \pi$, the following inequality holds:

$$\left(\sum_{k=1}^{2n-1} \frac{r^k}{k}(1-\cos k\theta)\right)^2 + \left(\sum_{k=1}^{2n-1} \frac{r^k}{k}\sin k\theta\right)^2 < 4\left(\sum_{k=1}^{n} \frac{r^{2k-1}}{2k-1}\right)^2.$$

We give an affirmative answer to this conjecture.

Rearrangement

The inequality is equivalent to

$$-\sum_{k=1}^{2n-1}\frac{r^k}{k}\cos k\theta+\frac{\left(\sum_{k=1}^{2n-1}\frac{r^k}{k}\cos k\theta\right)^2+\left(\sum_{k=1}^{2n-1}\frac{r^k}{k}\sin k\theta\right)^2}{2\sum_{k=1}^{2n-1}\frac{r^k}{k}}$$
$$<\sum_{k=1}^{2n-1}\frac{(-1)^{k-1}r^k}{k}+\frac{\left(\sum_{k=1}^{2n-1}\frac{(-1)^{k-1}r^k}{k}\right)^2}{2\sum_{k=1}^{2n-1}\frac{r^k}{k}}. \qquad (4)$$

Presentation flow

We split into two cases $\rightarrow r = 1$ & $\frac{1}{2} < r < 1$. For each, we split into further intervals.

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$r = 1$:

- $\theta \in [0, \frac{4n-3}{4n-1}\pi]$, $n = 1, 2, 3, 4$
- $\theta \in [0, \frac{2\pi}{3})$, $n \geqslant 5$
- $\theta \in [\frac{2\pi}{3}, \frac{4n-3}{4n-1}\pi]$, $n \geqslant 5$
- $\theta \in [\frac{4n-3}{4n-1}\pi, \pi)$, $n \in \mathbb{N}$

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$\frac{1}{2} < r < 1$:

- $\theta \in [0, \frac{\pi}{3}]$, $n \geqslant 2$
- $\theta \in [\frac{\pi}{4}, \pi - \frac{\pi}{2n}]$, $n = 2, 3, 4$
- $\theta \in [\frac{\pi}{4}, \pi - \frac{\pi}{2n}]$, $5 \leqslant n \leqslant 29$
- $\theta \in [\pi - \frac{\pi}{2n}\pi, \pi)$, $2 \leqslant n \leqslant 29$
- $\theta \in [\frac{\pi}{3}, \pi)$, $n \geqslant 30$

The case $r = 1$

Proof for the case $r = 1$

We consider the following functions:

$$L_{2n-1} : \theta \mapsto -\sum_{k=1}^{2n-1} \frac{\cos k\theta}{k} + \frac{\left(\sum\limits_{k=1}^{2n-1} \frac{\cos k\theta}{k}\right)^2 + \left(\sum\limits_{k=1}^{2n-1} \frac{\sin k\theta}{k}\right)^2}{2\sum\limits_{k=1}^{2n-1} \frac{1}{k}}$$

$$\text{and} \quad R : n \mapsto \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} + \frac{\left(\sum\limits_{k=1}^{n} \frac{(-1)^{k-1}}{k}\right)^2}{2\sum\limits_{k=1}^{n} \frac{1}{k}},$$

where $n \in \mathbb{N}$ and $\theta \in [0, \pi]$. Then, the conjecture for the case $r = 1$ is equivalent to the following inequality

$$L_{2n-1}(\theta) < R(2n-1).$$

$\theta \in \left[0, \frac{4n-3}{4n-1}\pi\right]$ and $n = 1, 2, 3, 4.$

Let

$$S_n(\theta) := \sum_{k=1}^{n} \frac{\sin k\theta}{k} \quad \text{and} \quad C_n(\theta) := \sum_{k=1}^{n} \frac{\cos k\theta}{k} \quad (n \in \mathbb{N}\ ;\ \theta \in [0, \pi])\,.$$

Then,

$$S'_{2n-1}(\theta) = \frac{\cos n\theta \sin\left(n - \frac{1}{2}\right)\theta}{\sin\frac{\theta}{2}} \quad \text{and} \quad C'_{2n-1}(\theta) = -\frac{\sin n\theta \sin\left(n - \frac{1}{2}\right)\theta}{\sin\frac{\theta}{2}}$$

$$(n \in \mathbb{N}\ ;\ \theta \in (0, \pi))\,.$$

$\theta \in \left[0, \frac{4n-3}{4n-1}\pi\right]$ and $n = 1, 2, 3, 4$ continued.

Using (10) we show, by computing the stationary points, that

$$\begin{aligned} \max_{\theta \in \left[0, \frac{(4n-3)}{4n-1}\pi\right]} L_{2n-1}(\theta) &= \max_{\theta \in \left[0, \frac{(4n-3)}{4n-1}\pi\right]} \left\{ -C_{2n-1}(\theta) + \frac{(C_{2n-1}(\theta))^2 + (S_{2n-1} }{2 \sum_{k=1}^{2n-1} \frac{1}{k}} \right. \\ &\leqslant \max_{\theta \in \left[0, \frac{(4n-3)}{4n-1}\pi\right]} \left\{ -C_{2n-1}(\theta) + \frac{(C_{2n-1}(\theta))^2}{2 \sum_{k=1}^{2n-1} \frac{1}{k}} \right\} + \quad {}_{\theta \in [0} \\ &< R(2n-1). \text{ for } n = 1, 2, 3, 4. \end{aligned}$$

The proof is complete.

$\theta \in \left[0, \frac{2\pi}{3}\right]$ and $n = 5, 6, 7 \ldots$.

In this section, we evoke a theorem by Fong et al.[7].

Theorem 1.2.4(cf. [7, Theorem 1.3]).

Let $n \in \mathbb{N}$ and let $\theta \in (0, \pi)$. Then

$$\sum_{k=1}^{2\left\lfloor \frac{n}{2} \right\rfloor + 1} \frac{(-1)^{k-1}}{k} + \sum_{k=1}^{n} \frac{\cos k\theta}{k} \geqslant \frac{1}{4}(1 + \cos\theta)^2,$$

where equality holds if and only if $n = 2$ and $\theta = \pi - \cos^{-1}\frac{1}{3}$.

An application of Theorem 1.2.4 shows that the conjecture is equivalent to

$$-(1 + \cos\theta)^2 \sum_{k=1}^{2n-1} \frac{1}{2k-1} + \frac{1}{16}(1 + \cos\theta)^4 + \left(\sum_{k=1}^{2n-1} \frac{\sin k\theta}{k} \right)^2 < 0. \quad (5)$$

$\theta \in \left[0, \frac{2\pi}{3}\right]$ and $n = 5, 6, 7 \ldots$ continued.

We next consider a few results from Kim et al. [10].

Lemma 1.2.5(cf. [10, Lemma 2.2]).

Let $n \in \mathbb{N}$. If $q \in \{1, 2, ..., \lfloor \frac{n+1}{2} \rfloor\}$, then

$$\max_{\theta \in \left[\frac{(4q-2)\pi}{2n+1}, \pi\right]} \left\{ \sum_{k=1}^{n} \frac{\sin k\theta}{k} - \frac{\pi - \theta}{2} \right\} = \sum_{k=1}^{n} \frac{\sin k \frac{(4q-2)\pi}{2n+1}}{k} - \frac{\pi - \frac{(4q-2)\pi}{2n+1}}{2}.$$

Theorem 1.2.6(cf. [10, Theorem 2.5]).

Let $n \in \mathbb{N}$. If $p \in \mathbb{N}$, then the sequence

$$\left((-1)^{p-1} \left(\sum_{k=1}^{n} \frac{\sin k \frac{2p\pi}{2n+1}}{k} - \frac{\pi - \frac{2p\pi}{2n+1}}{2} \right) \right)_{n=p}^{\infty}$$

is decreasing.

$\theta \in \left[0, \frac{2\pi}{3}\right]$ and $n = 5, 6, 7 \ldots$ continued.

Using Lemma 1.2.5 and Theorem 1.2.6, we get

$$\max_{\theta \in \left[0, \frac{\pi}{2}\right]} \left\{ \sum_{k=1}^{2n-1} \frac{\sin k\theta}{k} - \frac{\pi - \theta}{2} \right\} \leqslant \sum_{k=1}^{9} \frac{\sin k\frac{2\pi}{19}}{k} - \frac{\pi - \frac{2\pi}{19}}{2} = 0.282\ldots < \frac{3}{10}. \quad ($$

Combining (5) and (6), we have

$$-(1+\cos\theta)^2 \sum_{k=1}^{2n-1} \frac{1}{2k-1} + \frac{1}{16}(1+\cos\theta)^4 + \left(\frac{\pi - \theta}{2} + \frac{3}{10} \right)^2 < 0. \quad (7)$$

A similar reasoning shows that (5) is also true for $\theta \in \left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$. The proof is complete.

$\theta \in \left[\frac{2\pi}{3}, \frac{4n-3}{4n-1}\pi\right]$ and $n = 5, 6, 7, \ldots$

First, we have

Lemma 1.3.1

Let $n \in \mathbb{N}$ and let $\theta \in (0, \pi)$. Then

$$S_{2n-1}(\theta) \;<\; F_n(\theta) := \frac{1}{4n-1}\left(2\csc\frac{\theta}{2} - 1\right) + \frac{\pi - \theta}{2}.$$

Lemma 1.3.2

Let $n \in \mathbb{N}$ and let $\theta \in (0, \pi)$. Then

$$C_{2n-1}(\theta) \;>\; G_n(\theta) := -\frac{2}{4n-1}\left(\csc\frac{\theta}{2} - 1\right) + C_{2n-1}(\pi) - \ln\left(\sin\frac{\theta}{2}\right).$$

$\theta \in \left[\frac{2\pi}{3}, \frac{4n-3}{4n-1}\pi\right]$ and $n = 5, 6, 7, \ldots$ continued.

Illustration of figures.

Consider the following two graphs.

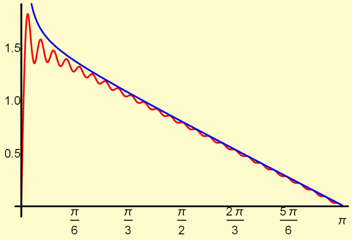

Figure: Graphs of S_{49} and F_{25}.

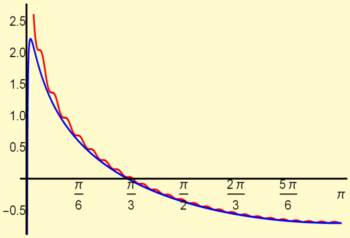

Figure: Graphs of C_{49} and G_{25}.

$\theta \in \left[\frac{2\pi}{3}, \frac{4n-3}{4n-1}\pi\right]$ and $n = 5, 6, 7, \dots$ continued.

Since Lemma 1.3.1 and Lemma 1.3.2 yields

$$L_{2n-1}(\theta) = -C_{2n-1}(\theta) + \frac{C_{2n-1}^2(\theta) + S_{2n-1}^2(\theta)}{2\sum\limits_{k=1}^{2n-1}\frac{1}{k}} < -G_n(\theta) + \frac{G_n^2(\theta) + F_n^2(\theta)}{2\sum\limits_{k=1}^{2n-1}\frac{1}{k}},$$

we need the following result.

$\theta \in \left[\frac{2\pi}{3}, \frac{4n-3}{4n-1}\pi\right]$ and $n = 5, 6, 7, \ldots$ continued.

Lemma 1.3.3

Let $n \in \mathbb{N}$ and $n \geqslant 5$. Then

$$-G_n + \frac{G_n^2 + F_n^2}{2\sum_{k=1}^{2n-1} \frac{1}{k}} \quad \text{is increasing on} \left[\frac{2\pi}{3}, \frac{(4n-3)\pi}{4n-1}\right].$$

Theorem 1.3.4

Let $n \in \mathbb{N}$ and $n \geqslant 5$. Then

$$-G_n\left(\frac{(4n-3)\pi}{4n-1}\right) + \frac{G_n^2\left(\frac{(4n-3)\pi}{4n-1}\right) + F_n^2\left(\frac{(4n-3)\pi}{4n-1}\right)}{2\sum_{k=1}^{2n-1} \frac{1}{k}} < R(2n-1).$$

The proof is complete.

$\theta \in \left[\frac{4n-3}{4n-1}\pi, \pi\right)$ and $n = 1, 2, 3, \ldots$.

$$L'_{2n-1}(\theta) = \frac{\sin n\theta \sin\left(n - \frac{1}{2}\right)\theta}{\sin\frac{\theta}{2}\sum_{k=1}^{2n-1}\frac{1}{k}}\left\{\sum_{k=1}^{2n-1}\frac{1-\cos k\theta}{k} + \cot n\theta \sum_{k=1}^{2n-1}\frac{\sin k\theta}{k}\right\}. \quad (9)$$

Last but not least, since $L_{2n-1}(\pi) = R(2n-1)$, we show that

Theorem 1.4.5

Let $n \in \mathbb{N}$. Then L_{2n-1} is increasing on the closed interval $\left[\frac{(4n-3)\pi}{4n-1}, \pi\right]$.

The proof is complete.

proof of 1.0.1

Theorem 1.1.3

Let $n \in \{1, 2, 3, 4\}$. If $\theta \in \left[0, \frac{4n-3}{4n-1}\pi\right]$, then

$$-C_{2n-1}(\theta) + \frac{(C_{2n-1}(\theta))^2 + (S_{2n-1}(\theta))^2}{2\sum\limits_{k=1}^{2n-1} \frac{1}{k}} < -C_{2n-1}(\pi) + \frac{C_{2n-1}^2(\pi)}{2\sum\limits_{k=1}^{2n-1} \frac{1}{k}}. \quad (10)$$

Theorem 1.2.7

If $n \geqslant 5$, $n \in \mathbb{N}$ and $\theta \in \left[0, \frac{2\pi}{3}\right]$, then

$$\left(\sum_{k=1}^{2n-1} \frac{1}{k}(1 - \cos k\theta)\right)^2 + \left(\sum_{k=1}^{2n-1} \frac{\sin k\theta}{k}\right)^2 < 4\left(\sum_{k=1}^{n} \frac{1}{2k-1}\right)^2.$$

Proof of 1.0.1 continued.

Let $n \in \{1, 2, 3, 4\}$. If $\theta \in \left[0, \frac{4n-3}{4n-1}\pi\right]$, then we apply Theorem 1.1.3 to obtain

$$-C_{2n-1}(\theta) + \frac{(C_{2n-1}(\theta))^2 + (S_{2n-1}(\theta))^2}{2\sum_{k=1}^{2n-1}\frac{1}{k}} < -C_{2n-1}(\pi) + \frac{C_{2n-1}^2(\pi)}{2\sum_{k=1}^{2n-1}\frac{1}{k}} \quad (11)$$

. On the other hand, we invoke Theorem 1.4.5 to show that (11) is valid whenever $\theta \in \left[\frac{4n-3}{4n-1}\pi, \pi\right]$.

Next, we consider $n \geq 5$. There are 3 cases to consider.

Case 1: $\theta \in \left[0, \frac{2\pi}{3}\right]$.

In this case, the conjecture for the case r=1 is a consequence of Theorem 1.2.7.

proof of 1.0.1 continued

Case 2: $\theta \in \left[\frac{2\pi}{3}, \frac{4n-3}{4n-1}\pi\right]$.

In this case, (11) follows from Theorem 1.3.4 and the following corollary of Theorem 1.2.4:

$$\sum_{k=1}^{2n-1} \frac{\cos k\theta}{k} < \sum_{k=1}^{2n-1} \frac{(-1)^k}{k} = C_{2n-1}(\pi).$$

Case 3: $\theta \in \left[\frac{4n-3}{4n-1}\pi, \pi\right]$.

In the third case, an application of Theorem 1.4.5 yields (11).

The proof is complete.

The case $\frac{1}{2} < r < 1$

Bounds on $[0, \frac{\pi}{3}]$

The case of $n = 2$ will be addressed later. We use the following result of Fong et al.:

Theorem 2.1.1 [cf. [8, Theorem 1.1]]

If $p \in \mathbb{N}$, then the following sequence

$$\left((-1)^p \left\{\sum_{k=1}^{n} \frac{\cos \frac{(2p-1)k\pi}{2n+1}}{k} + \ln\left(2\sin\frac{(2p-1)\pi}{4n+2}\right)\right\}\right)_{n=p}^{\infty}$$

is increasing.

Bounding $\sum_{k=1}^{n} \frac{\cos k\theta}{k}$

We use Theorem 2.1.1 to establish:

Lemma 2.1.2.

Let $n \in \mathbb{N}$. Then

$$\min_{\theta \in [0, \frac{\pi}{6}]} \sum_{k=1}^{n} \frac{\cos k\theta}{k} > 0.4565, \tag{13}$$

$$\min_{\theta \in [\frac{\pi}{6}, \frac{\pi}{5}]} \sum_{k=1}^{n} \frac{\cos k\theta}{k} > \frac{1}{4}, \tag{14}$$

$$\min_{\theta \in [\frac{\pi}{5}, \frac{\pi}{4}]} \sum_{k=1}^{n} \frac{\cos k\theta}{k} > 0.065 \tag{15}$$

and

$\theta \in [\frac{\pi}{4}, \frac{\pi}{3}]$

Cases of small n – requires manual handling

Lemma 2.2.1.

Let $r \in (\frac{1}{2}, 1)$, $\theta \in [\frac{\pi}{4}, \frac{\pi}{3}]$ and $n = 2, 3, 4$. Then

$$\sum_{k=1}^{2n-1} \frac{(-1)^{k-1} r^k}{k} > -\sum_{k=1}^{2n-1} \frac{r^k}{k} \cos k\theta + \frac{\left(\sum_{k=1}^{2n-1} \frac{r^k}{k} \cos k\theta\right)^2 + \left(\sum_{k=1}^{2n-1} \frac{r^k}{k} \sin k\theta\right)^2}{2 \sum_{k=1}^{2n-1} \frac{r^k}{k}}.$$

Required SBP identity: $\sum_{i=1}^{n} a_i b_i = b_n \sum_{i=1}^{n} a_i + \sum_{i=1}^{n-1} (b_i - b_{i+1}) \sum_{j=1}^{i} a_j.$

Proof.

$$-\sum_{k=1}^{2n-1} \frac{r^k}{k} \cos k\theta + \frac{\left(\sum_{k=1}^{2n-1} \frac{r^k}{k} \cos k\theta\right)^2 + \left(\sum_{k=1}^{2n-1} \frac{r^k}{k} \sin k\theta\right)^2}{2 \sum_{k=1}^{2n-1} \frac{r^k}{k}}$$

$$< r^{2n-1} \max_{\theta \in \left[\frac{\pi}{4}, \frac{\pi}{3}\right]} \left\{ -\sum_{k=1}^{2n-1} \frac{\cos k\theta}{k} \right\} + \sum_{k=1}^{2n-2} (r^k - r^{k+1}) \max_{\theta \in \left[\frac{\pi}{4}, \frac{\pi}{3}\right]} \left\{ -\sum_{j=1}^{k} \frac{\cos j\theta}{j} \right\}$$

$$+ \frac{1}{2 \sum_{k=1}^{2n-1} \frac{r^k}{k}} \times$$

$$\left(r^{2n-1} \max_{\theta \in \left[\frac{\pi}{4}, \frac{\pi}{3}\right]} \left\{ -\sum_{k=1}^{2n-1} \frac{\cos k\theta}{k} \right\} + \sum_{k=1}^{2n-2} (r^k - r^{k+1}) \max_{\theta \in \left[\frac{\pi}{4}, \frac{\pi}{3}\right]} \left\{ -\sum_{j=1}^{k} \frac{\cos j\theta}{j} \right\} \right)^2$$

$$+ \frac{1}{2 \sum_{k=1}^{2n-1} \frac{r^k}{k}} \times$$

$$\left(r^{2n-1} \max_{\theta \in \left[\frac{\pi}{4}, \frac{\pi}{3}\right]} \left\{ \sum_{k=1}^{2n-1} \frac{\sin k\theta}{k} \right\} + \sum_{k=1}^{2n-2} (r^k - r^{k+1}) \max_{\theta \in \left[\frac{\pi}{4}, \frac{\pi}{3}\right]} \left\{ \sum_{j=1}^{k} \frac{\sin j\theta}{j} \right\} \right)^2 .$$

□

Bounding trigonometric sums

We require the following results:

Lemma 2.3.1. [cf. [5]]

Let $r \in (\frac{1}{2}, 1)$, $\theta \in [0, \pi]$ and $n \in \mathbb{N}$. Then

$$\sum_{k=1}^{n} r^k \sin k\theta = \frac{r \sin\theta}{r^2 - 2r\cos\theta + 1} + \frac{r^{n+2}\sin n\theta - r^{n+1}\sin(n+1)\theta}{r^2 - 2r\cos\theta + 1}.$$

Lemma 2.3.6. [cf. [5]]

Let $r \in (\frac{1}{2}, 1)$, $\theta \in [0, \pi]$ and $n \in \mathbb{N}$. Then

$$\sum_{k=1}^{n} r^k \cos k\theta = \frac{r \cos\theta - r^2}{r^2 - 2r\cos\theta + 1} + \frac{r^{n+2}\cos n\theta - r^{n+1}\cos(n+1)\theta}{r^2 - 2r\cos\theta + 1}.$$

Bounding trigonometric sums

We apply Squeeze Theorem to find the limit at infinity:

Lemma 2.3.2. [cf. [5]]

Let $r \in (\frac{1}{2}, 1)$ and $\theta \in [0, \pi]$. Then

$$\sum_{k=1}^{\infty} r^k \sin k\theta = \frac{r \sin\theta}{r^2 - 2r\cos\theta + 1}.$$

Lemma 2.3.7. [cf. [5]]

Let $r \in (\frac{1}{2}, 1)$ and $\theta \in [0, \pi]$. Then

$$\sum_{k=1}^{\infty} r^k \cos k\theta = \frac{r \cos\theta - r^2}{r^2 - 2r\cos\theta + 1}.$$

Bounding trigonometric sums

Integrate previous results to obtain desired sums:

Lemma 2.3.3. [cf. [5]]

Let $r \in (\frac{1}{2}, 1)$ and $\theta \in [0, \pi]$. Then

$$\sum_{k=1}^{\infty} \frac{r^k}{k} \cos k\theta = -\frac{1}{2} \ln\left(r^2 - 2r\cos\theta + 1\right).$$

Lemma 2.3.8. [cf. [5]]

Let $r \in (\frac{1}{2}, 1)$ and $\theta \in [0, \pi]$. Then

$$\sum_{k=1}^{\infty} \frac{r^k}{k} \sin k\theta = \tan^{-1}\left(\frac{r\sin\theta}{1 - r\cos\theta}\right).$$

Bounding trigonometric sums

We hence have the following estimate for our cosine polynomial:

Lemma 2.3.4.

Let $r \in (\frac{1}{2}, 1)$, $\theta \in [0, \pi)$ and $n \in \mathbb{N}$. Then

$$\sum_{k=1}^{2n-1} \frac{r^k \cos k\theta}{k} > \sum_{k=1}^{2n-1} \frac{r^k \cos k\pi}{k} - \frac{1}{2}\ln(1 - 2r\cos\theta + r^2) + \ln(1+r) + \frac{2\left(\frac{r^{2n}}{2n} + \frac{r^{2n+1}}{2n-1}\right)}{(1+r)^2} + \frac{\frac{r^{2n+1}}{2n-1}(\cos(2n-1)\theta - 1) - \frac{r^{2n}}{2n}(\cos 2n\theta + 1)}{1 - 2r\cos\theta + r^2}.$$

Bounding trigonometric sums

This can be used to derive:

Lemma 2.3.5.

Let $r \in (\frac{1}{2}, 1)$, $\theta \in [\frac{\pi}{3}, \pi)$ and $n \in \mathbb{N}$. Then

$$\sum_{k=1}^{2n-1} \frac{r^k \cos k\theta}{k} > \sum_{k=1}^{2n-1} \frac{r^k \cos k\pi}{k} - \frac{1}{2}\ln(1 - 2r\cos\theta + r^2) + \ln(1+r) + 2\left(\frac{r^{2n}}{2n} + \frac{r^{2n+1}}{2n-1}\right)\left(\frac{1}{(1+r)^2} - \frac{1}{1 - 2r\cos\theta + r^2}\right).$$

Bounding trigonometric sums

Similar results can be established for sine polynomials:

Lemma 2.3.10.

Let $r \in (\frac{1}{2}, 1)$, $\theta \in [\frac{\pi}{3}, \pi)$ and $n \in \mathbb{N}$. Then

$$\sum_{k=1}^{2n-1} \frac{r^k \sin k\theta}{k} < \tan^{-1}\left(\frac{r \sin\theta}{1 - r\cos\theta}\right) + \left(\frac{r^{2n}}{2n} + \frac{r^{2n+1}}{2n-1}\right)\left(\frac{2}{1 - 2r\cos\theta + r^2} - \frac{1}{(1+r)^2}\right).$$

Increasing functions on $[\frac{\pi}{3}, \pi - \frac{\pi}{2n}], 5 \leqslant n \leqslant 29$

Using Lemmas 2.3.5 and 2.3.10, we can hence define the function:
Let $[\alpha, \beta] \subset \left[\frac{\pi}{3}, \pi\right)$ and let

$$\begin{aligned} F_n(\theta) \quad := \quad & \frac{1}{2}\ln\left(1 - 2r\cos\theta + r^2\right) - \ln(1+r) \\ & + 2\left(\frac{r^{2n}}{2n} + \frac{r^{2n+1}}{2n-1}\right)\left(\frac{1}{1 - 2r\cos\alpha + r^2} - \frac{1}{(1+r)^2}\right) \\ & + \frac{1}{2\sum_{k=1}^{2n-1}\frac{r^k}{k}}\left(\tan^{-1}\left(\frac{r\sin\theta}{1 - r\cos\theta}\right)\right. \\ & \left. + \left(\frac{r^{2n}}{2n} + \frac{r^{2n+1}}{2n-1}\right)\left(\frac{2}{1 - 2r\cos\alpha + r^2} - \frac{1}{(1+r)^2}\right)\right)^2 \end{aligned}$$

(17)

for $\theta \in [\alpha, \beta]$.

Increasing functions on $[\frac{\pi}{3}, \pi - \frac{\pi}{2n}], 5 \leqslant n \leqslant 29$

which leads us to the following theorem:

Theorem 2.3.11

Let n be any integer satisfying $n \geqslant 5$. If $[\alpha, \beta] \subseteq \left[\frac{\pi}{3}, \frac{\pi}{2}\right]$, $[\alpha, \beta] \subseteq \left[\frac{\pi}{2}, \frac{3\pi}{4}\right]$ or $[\alpha, \beta] \subseteq \left[\frac{3\pi}{4}, \pi - \frac{\pi}{2n}\right]$, then $F_n'(\theta) > 0$ for $\theta \in (\alpha, \beta)$.

Increasing functions on $[\frac{\pi}{3}, \pi - \frac{\pi}{2n}], 5 \leqslant n \leqslant 29$

Proof.

Let $\theta \in (\alpha, \beta)$. Then we have

$$F_n'(\theta) > \frac{r\sin\theta}{1 - 2r\cos\theta + r^2}\left(1 - \frac{r^2 - r\cos\theta}{(1 - r\cos\theta)\sum_{k=1}^{2n-1}\frac{r^k}{k}}\right) - \frac{\left(\frac{2}{1-2r\cos\alpha+r^2} - \frac{1}{(1+r)^2}\right)\left(\frac{r^{2n}}{2n} + \frac{r^{2n+1}}{2n-1}\right)}{\sum_{k=1}^{2n-1}\frac{r^k}{k}} \cdot \frac{r^2 - r\cos\theta}{1 - 2r\cos\theta + r^2}.$$

We consider three cases:

Case 1: $\frac{\pi}{3} \leq \alpha < \beta \leq \frac{\pi}{2}$.
Case 2: $\frac{\pi}{2} \leq \alpha < \beta \leq \frac{3\pi}{4}$.
Case 3: $\frac{3\pi}{4} \leq \alpha < \beta \leq \pi - \frac{\pi}{2n}$.

$\square$

Increasing functions on $[\frac{\pi}{3}, \pi - \frac{\pi}{2n}], 5 \leqslant n \leqslant 29$

Proof.

Case 1: $\frac{\pi}{3} \leq \alpha < \beta \leq \frac{\pi}{2}$.
On this interval, we utilize the increasing nature of the function $\theta \mapsto \sin\theta$ (where $\sin\theta \geq 0$ on this interval) and the decreasing nature of the function $\theta \mapsto \cos\theta$ on $[\frac{\pi}{3}, \frac{\pi}{2}]$ to show that for any integer $n \geq 5$,

$$\begin{aligned}
F_n'(\theta) &> \frac{r\sin\frac{\pi}{3}}{1 - 2r\cos\frac{\pi}{2} + r^2}\left(1 - \frac{r^2 - r\cos\frac{\pi}{2}}{\left(1 - r\cos\frac{\pi}{2}\right)\sum_{k=1}^{9}\frac{r^k}{k}}\right) \\
&\quad - \frac{\left(\frac{2}{1-2r\cos\frac{\pi}{3}+r^2} - \frac{1}{(1+r)^2}\right)\left(\frac{r^{10}}{10} + \frac{r^{11}}{9}\right)}{\sum_{k=1}^{9}\frac{r^k}{k}} \cdot \frac{r^2 - r\cos\frac{\pi}{2}}{1 - 2r\cos\frac{\pi}{2} + r^2} \\
&= \frac{r}{1+r^2}\left(\frac{\sqrt{3}}{2}\left(1 - \frac{r^2}{\sum_{k=1}^{9}\frac{r^k}{k}}\right) - \frac{r\left(\frac{2}{1-r+r^2} - \frac{1}{(1+r)^2}\right)\left(\frac{r^{10}}{10} + \frac{r^{11}}{9}\right)}{\sum_{k=1}^{9}\frac{r^k}{k}}\right) \\
&> 0
\end{aligned}$$

Increasing functions on $[\frac{\pi}{3}, \pi - \frac{\pi}{2n}], 5 \leqslant n \leqslant 29$

Proof.

since

$$\frac{\sqrt{3}}{2}\left(1 - \frac{r^2}{\sum_{k=1}^{9} \frac{r^k}{k}}\right) - \frac{r\left(\frac{2}{1-r+r^2} - \frac{1}{(1+r)^2}\right)\left(\frac{r^{10}}{10} + \frac{r^{11}}{9}\right)}{\sum_{k=1}^{9} \frac{r^k}{k}} > 0$$

whenever $r \in \left(\frac{1}{2}, 1\right)$ as verified using Sturm's Theorem.
The proofs of Case 2 and Case 3 are similar. □

With Theorem 2.3.11, we are now ready to construct a negative increasing function on $[\frac{\pi}{3}, \pi - \frac{\pi}{2n}$ for $5 \leqslant n \leqslant 29$.

Negative increasing function on $[\frac{\pi}{3}, \pi - \frac{\pi}{2n}]$, $5 \leqslant n \leqslant 29$

Lemma 2.4.1

Let $r \in (\frac{1}{2}, 1)$, $\frac{\pi}{3} \leq \alpha < \beta \leq \pi - \frac{\pi}{2n}$ and $5 \leq n \leq 29$. If

$$F_n(\theta) = \frac{1}{2}\ln(1 - 2r\cos\theta + r^2) - \ln(1+r) - 2\left(\frac{r^{2n}}{2n} + \frac{r^{2n+1}}{2n-1}\right)\left(\frac{1}{(1+r)^2} -\right.$$

$$+ \frac{\left(\tan^{-1}\left(\frac{r\sin\theta}{1-r\cos\theta}\right) + \left(\frac{r^{2n}}{2n} + \frac{r^{2n+1}}{2n-1}\right)\left(\frac{2}{1-2r\cos\alpha+r^2} - \frac{1}{(1+r)^2}\right)\right)^2}{2\sum_{k=1}^{2n-1}\frac{r^k}{k}},$$

then $F_n(\theta) < 0$ for $\theta \in [\alpha, \beta]$.

Negative increasing function on $[\frac{\pi}{3}, \pi - \frac{\pi}{2n}]$, $5 \leqslant n \leqslant 29$

Proof.

When $n = 5$ and $(\alpha, \beta) = \left(\frac{\pi}{3}, \frac{34\pi}{100}\right)$ we have

$$F_5(\theta) < 0 \text{ whenever } \theta \in [\alpha, \beta]\,;$$

a verification of this inequality is given in Figure 2.4. Similarly, we have

$$F_5(\theta) < 0$$

for $\theta \in [\alpha, \beta]$ and

$$[\alpha, \beta] \in \left\{ \left[\frac{34\pi}{100}, \frac{35\pi}{100}\right], \left[\frac{35\pi}{100}, \frac{3\pi}{8}\right], \left[\frac{3\pi}{8}, \frac{2\pi}{5}\right], \left[\frac{2\pi}{5}, \frac{3\pi}{7}\right], \left[\frac{3\pi}{7}, \frac{\pi}{2}\right] \right\}.$$

$\square$

Negative increasing function on $[\frac{\pi}{3}, \pi - \frac{\pi}{2n}]$, $5 \leqslant n \leqslant 29$

Proof.

Hence, an application of Theorem 2.3.11 reveals that $F_n(\theta) < 0$ whenever $\theta \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right]$ and $n = 5, 6, \dots, 29$. Likewise, $F_n(\theta) < 0$ holds whenever $\theta \in \left[\frac{\pi}{2}, \frac{3\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \frac{9\pi}{10}\right]$ and $n = 5, 6, \dots, 29$.
Now we let $N \in \{6, \dots, 29\}$. Since

$$F_n(\theta) < 0$$

whenever $\theta \in \left[\pi - \frac{\pi}{2N-2}, \pi - \frac{\pi}{2N}\right]$, we infer from Theorem 2.3.11 that $F_n(\theta) < 0$ if $\theta \in \left[\pi - \frac{\pi}{2N-2}, \pi - \frac{\pi}{2N}\right]$ and $n = N, \dots, 29$. □

Negative increasing function on $[\pi - \frac{\pi}{2n}, \pi)$, $3 \leqslant n \leqslant 29$

To construct a negative increasing function on this interval, we first need a new estimate for the sine polynomial. To obtain the new estimate, we would need the following lemma:

Lemma 2.5.1

Let $r \in (\frac{1}{2}, 1)$, $n \in \mathbb{N}$ and $\theta \in (0, \pi)$. Then

$$\left| \sum_{k=1}^{2n-1} \frac{r^k \sin k\theta}{k} - \tan^{-1}\left(\frac{r \sin\theta}{1 - r\cos\theta}\right) \right| < \frac{r^{2n}}{2n} \csc\frac{\theta}{2}.$$

The proof of Lemma 2.5.1 involves summation by parts and the use of Lemma 2.3.8.

Negative increasing function on $[\pi - \frac{\pi}{2n}, \pi)$, $3 \leqslant n \leqslant 29$

Using Lemma 2.4.1, we obtain the following estimate for sine polynomial.

Lemma 2.5.2

Let $r \in (\frac{1}{2}, 1)$, $n \in \mathbb{N}$ and $\theta \in [\frac{\pi}{3}, \pi)$. Then

$$\sum_{k=1}^{2n-1} \frac{r^k \sin k\theta}{k} < (1 + r^{2n-1}) \tan^{-1}\left(\frac{r \sin\theta}{1 - r\cos\theta}\right).$$

Negative increasing function on $[\pi - \frac{\pi}{2n}, \pi)$, $3 \leqslant n \leqslant 29$

Proof.

In view of Lemma 2.5.1, it is sufficient to show that

$$\tan^{-1}\left(\frac{r\sin\theta}{1 - r\cos\theta}\right) > \frac{r}{2n}\csc\frac{\theta}{2}$$

when $n \in \mathbb{N}$ and $\theta \in [\frac{\pi}{3}, \pi - \frac{\pi}{2n})$. We consider 4 cases:

Case 1: $\theta \in [\frac{\pi}{3}, \frac{2\pi}{3}]$
Case 2: $\theta \in [\frac{2\pi}{3}, \frac{3\pi}{4}]$
Case 3: $\theta \in [\frac{3\pi}{4}, \pi - \frac{\pi}{2n}]$
Case 4: $\theta \in [\pi - \frac{\pi}{2n}, \pi)$ $\square$

Negative increasing function on $[\pi - \frac{\pi}{2n}, \pi)$, $3 \leqslant n \leqslant 29$

Proof.

Case 1: $\theta \in [\frac{\pi}{3}, \frac{2\pi}{3}]$
On this interval, we have

$$\tan^{-1}\left(\frac{r\sin\frac{2\pi}{3}}{1 - r\cos\frac{2\pi}{3}}\right) - \frac{r}{2n}\csc\frac{\pi}{6} > 0$$

for $r \in (\frac{1}{2}, 1)$.
Case 2: $\theta \in [\frac{2\pi}{3}, \frac{3\pi}{4}]$
The proof is similar to Case 1. $\square$

Negative increasing function on $[\pi - \frac{\pi}{2n}, \pi)$, $3 \leqslant n \leqslant 29$

Proof.

Case 3:$\theta \in [\frac{3\pi}{4}, \pi - \frac{\pi}{2n}]$
On this interval, we have

$$\begin{aligned}
&\tan^{-1}\left(\frac{r\sin\frac{\pi}{2n}}{1 + r\cos\frac{\pi}{2n}}\right) - \frac{r}{2n}\csc\frac{3\pi}{8} \\
&> \tan^{-1}\left(r\tan\frac{\pi}{4n}\right) - \frac{r}{2n}\csc\frac{3\pi}{8} \\
&> \frac{r}{n}\left(\frac{\pi}{4} - \frac{r^2\pi^3}{192n^2} - \frac{1}{2}\csc\frac{3\pi}{8}\right) \\
&> 0
\end{aligned}$$

for $r \in (\frac{1}{2}, 1)$ and $n > 2$. □

Negative increasing function on $[\pi - \frac{\pi}{2n}, \pi)$, $3 \leqslant n \leqslant 29$

Proof.

Case 4: $\theta \in [\pi - \frac{\pi}{2n}, \pi)$
On this interval, we have

$$\frac{d}{d\theta}\left(\left(1 + r^{2n-1}\right)\tan^{-1}\left(\frac{r\sin\theta}{1 - r\cos\theta}\right) - \sum_{k=1}^{2n-1}\frac{r^k \sin k\theta}{k}\right)$$
$$= \frac{r^{2n}\left(\cos\theta + \cos 2n\theta\right) - r^{2n+1}\left(1 + \cos(2n-1)\theta\right)}{1 - 2r\cos\theta + r^2}.$$

We now let $\alpha = \pi - \theta$, then

$$\begin{aligned}\cos\theta + \cos 2n\theta &= -\cos\alpha + \cos 2n\alpha \\ &< 0\end{aligned}$$

provided that $\alpha \in (0, \frac{\pi}{2n}]$. □

Negative increasing function on $[\pi - \frac{\pi}{2n}, \pi)$, $3 \leqslant n \leqslant 29$

Proof.

We hence conclude that the function $\theta \mapsto \left(1 + r^{2n-1}\right) \tan^{-1}\left(\frac{r \sin\theta}{1 - r\cos\theta}\right) - \sum_{k=1}^{2n-1} \frac{r^k \sin k\theta}{k}$ is decreasing on the interval $[\pi - \frac{\pi}{2n}, \pi)$, and is equal to 0 at $\theta = \pi$. Hence,

$$\left(1 + r^{2n-1}\right) \tan^{-1}\left(\frac{r \sin\theta}{1 - r\cos\theta}\right) - \sum_{k=1}^{2n-1} \frac{r^k \sin k\theta}{k} > 0$$

for $\theta \in (\pi - \frac{\pi}{2n}, \pi)$. This completes the proof. □

Negative increasing function on $[\pi - \frac{\pi}{2n}, \pi)$, $3 \leqslant n \leqslant 29$

Using Lemma Lemma 2.5.2, we can hence define the function:
Let $\theta \in [\pi - \frac{\pi}{2n}, \pi)$ and let

$$g_n(\theta) = -\sum_{k=1}^{2n-1} \frac{r^k \cos k\theta}{k} + \frac{\left(\left(1 + r^{2n-1}\right) \tan^{-1}\left(\frac{r \sin\theta}{1 - r\cos\theta}\right)\right)^2}{2\sum_{k=1}^{2n-1} \frac{r^k}{k}},$$

we then have the following theorem.

Negative increasing function on $[\pi - \frac{\pi}{2n}, \pi)$, $3 \leqslant n \leqslant 29$

Theorem 2.5.3

Let n be any integer satisfying $3 \leqslant n \leqslant 29$. If $\theta \in (\pi - \frac{\pi}{2n}, \pi)$, then $g_n'(\theta) > 0$.

Negative increasing function on $[\pi - \frac{\pi}{2n}, \pi)$, $3 \leqslant n \leqslant 29$

Proof.

We apply Lemma 2.3.1 to show that for each $\theta \in (\pi - \frac{\pi}{2n}, \pi)$,

$$g_n'(\theta) = \sum_{k=1}^{2n-1} r^k \sin k\theta + \frac{\left(1 + r^{2n-1}\right)^2 \left(\tan^{-1}\left(\frac{r\sin\theta}{1-r\cos\theta}\right)\right) \cdot \frac{r\cos\theta - r^2}{1-2r\cos\theta+r^2}}{\sum_{k=1}^{2n-1} \frac{r^k}{k}}$$

$$> \frac{r\sin\theta}{1-2r\cos\theta+r^2}\left\{\frac{r\sin\theta - r^{2n}\sin 2n\theta + r^{2n+1}\sin(2n-1)\theta}{r\sin\theta} - \frac{r(1+}{\sum_{k}^{2}}\right.$$

$\square$

Negative increasing function on $[\pi - \frac{\pi}{2n}, \pi)$, $3 \leqslant n \leqslant 29$

Proof.

We now consider the substitution $\alpha = \pi - \theta$:

$$\frac{r\sin\theta - r^{2n}\sin 2n\theta + r^{2n+1}\sin(2n-1)\theta}{r\sin\theta} = 1 + \frac{r^{2n}\sin(2n\alpha) + r^{2n+1}\sin(2n -}{r\sin\alpha}$$

Thus,

$$g_n'(\theta) > \frac{r\sin\theta}{1 - 2r\cos\theta + r^2}\left\{\lim_{\theta\mapsto(\pi-\frac{\pi}{2n})^+}\frac{r\sin\theta - r^{2n}\sin 2n\theta + r^{2n+1}\sin(2n-1)}{r\sin\theta}\right.$$

$$= \frac{r\sin\theta}{1 - 2r\cos\theta + r^2}\left\{1 + r^{2n} - \frac{r(1 + r^{2n-1})^2}{\sum_{k=1}^{2n-1}\frac{r^k}{k}}\right\}.$$

□

Negative increasing function on $[\pi - \frac{\pi}{2n}, \pi)$, $3 \leqslant n \leqslant 29$

Proof.

Since

$$1 + r^{2n} - \frac{r(1 + r^{2n-1})^2}{\sum_{k=1}^{2n-1} \frac{r^k}{k}} > 0 \text{ for } n = 3, 4, 5, \ldots, 29,$$

and for $r \in \left(\frac{1}{2}, 1\right)$, the proof is complete. $\square$

Negative increasing function on $[\pi - \frac{\pi}{2n}, \pi)$, $3 \leqslant n \leqslant 29$

With Theorem 2.5.3, we can now establish the following Theorem:

Theorem 2.5.4

Let n be any integer satisfying $3 \leqslant n \leqslant 29$. Then

$$g_n(\theta) + \sum_{k=1}^{2n-1} \frac{r^k \cos k\pi}{k} < 0 \tag{18}$$

for $\theta \in (\pi - \frac{\pi}{2n}, \pi)$.

Proof.

Since $g_n(\pi) = \sum_{k=1}^{2n-1} \frac{(-1)^{k-1} r^k}{k}$, the result is an immediate consequence of Theorem 2.5.3. □

Negative increasing function on $[\frac{3\pi}{4}, \pi)$, $n = 2$

Firstly, we require the following lemma:

Lemma 2.6.1

Let $r \in [\frac{1}{2}, 1)$. Then the function $\theta \mapsto \sum_{k=1}^{3} r^k \cos k\theta$ is decreasing on $[\frac{3\pi}{4}, \pi)$.

Next, let $r \in (\frac{1}{2}, 1)$ and $\theta \in (\frac{3\pi}{4}, \pi)$ and let

$$h(\theta) = -\sum_{k=1}^{3} \frac{r^k \cos k\theta}{k} + \frac{\left(\sum_{k=1}^{3} \frac{r^k \sin k\theta}{k}\right)^2}{\sum_{k=1}^{3} \frac{r^k}{k}}.$$

We now use Lemma 2.6.1 to prove the following theorem.

Negative increasing function on $[\frac{3\pi}{4}, \pi)$, $n = 2$

Theorem 2.6.2

If $\theta \in (\frac{3\pi}{4}, \pi)$, then $h'(\theta) > 0$.

Proof.

For each $\theta \in (\frac{3\pi}{4}, \pi)$, we infer from Lemma 2.6.1 that

$$\begin{aligned} h'(\theta) &= \sum_{k=1}^{3} r^k \sin k\theta + \frac{\left(\sum_{k=1}^{3} \frac{r^k \sin k\theta}{k}\right)\left(\sum_{k=1}^{3} r^k \cos k\theta\right)}{\sum_{k=1}^{3} \frac{r^k}{k}} \\ &\geq \sum_{k=1}^{3} r^k \sin k\theta + \frac{\left(\sum_{k=1}^{3} \frac{r^k \sin k\theta}{k}\right)\left(\sum_{k=1}^{3} r^k \cos k\pi\right)}{\sum_{k=1}^{3} \frac{r^k}{k}} \\ &= \sin\theta \left\{ \left(r + \frac{r \sum_{k=1}^{3} (-1)^k r^k}{\sum_{k=1}^{3} \frac{r^k}{k}} \right) + 2\cos\theta \left(r^2 + \frac{r^2 \sum_{k=1}^{3} (-1)^k r^k}{2 \sum_{k=1}^{3} \frac{r^k}{k}} \right) + \right. \end{aligned}$$

Negative increasing function on $[\frac{3\pi}{4}, \pi)$, $n = 2$

Proof.

Firstly, for $\theta \in (\frac{3\pi}{4}, \frac{4\pi}{5})$,

$$h'(\theta) \geq \sin\theta \left\{ \left(r + \frac{r \sum_{k=1}^{3} (-1)^k r^k}{\sum_{k=1}^{3} \frac{r^k}{k}} \right) + 2\cos\left(\frac{4\pi}{5}\right) \left(r^2 + \frac{r^2 \sum_{k=1}^{3} (-1)^k r}{2 \sum_{k=1}^{3} \frac{r^k}{k}} \right.\right.$$
$$> 0.$$

Next, for $\theta \in (\frac{4\pi}{5}, \pi)$,

$$h'(\theta) \geq \sin\theta \left\{ \left(r + \frac{r \sum_{k=1}^{3} (-1)^k r^k}{\sum_{k=1}^{3} \frac{r^k}{k}} \right) + 2\cos(\pi) \left(r^2 + \frac{r^2 \sum_{k=1}^{3} (-1)^k r^k}{2 \sum_{k=1}^{3} \frac{r^k}{k}} \right) \right.$$
$$> 0.$$

This completes the proof. □

Negative increasing function on $[\frac{3\pi}{4}, \pi)$, $n = 2$

With Theorem 2.6.2, we can now establish the following Theorem:

Theorem 2.6.3

If $\theta \in [\frac{3\pi}{4}, \pi)$, then

$$h(\theta) + \sum_{k=1}^{3} \frac{r^k \cos k\pi}{k} < 0.$$

Proof.

Since $h(\pi) = \sum_{k=1}^{3} \frac{(-1)^{k-1} r^k}{k}$, the result follows from Theorem 2.6.2. $\square$

Negative increasing function on $[\frac{\pi}{3}, \pi)$, $n \geqslant 30$

For this case, we let $r \in (\frac{1}{2}, 1)$, $\theta \in [\frac{\pi}{3}, \pi)$ and let

$$p_n(\theta) = \frac{1}{2}\ln(1 - 2r\cos\theta + r^2) - \ln(1+r) - 2\left(\frac{r^{2n}}{2n} + \frac{r^{2n+1}}{2n-1}\right)\left(\frac{1}{(1+r)^2} - \right.$$
$$+ \frac{(1 + r^{2n-1})^2 \left(\tan^{-1}\left(\frac{r\sin\theta}{1 - r\cos\theta}\right)\right)^2}{2\sum_{k=1}^{2n-1} \frac{r^k}{k}},$$

then, we have the following Theorem:

Theorem 2.7.1

Let n be any integer satisfying $n \geqslant 30$. If $\theta \in (\frac{\pi}{3}, \pi)$, then $p_n'(\theta) > 0$.

Negative increasing function on $[\frac{\pi}{3}, \pi)$, $n \geqslant 30$

Proof.

For each $\theta \in (\frac{\pi}{3}, \frac{\pi}{2})$, we have

$$p_n'(\theta) = \frac{r\sin\theta}{1-2r\cos\theta+r^2} - 4\left(\frac{r^{2n}}{2n}+\frac{r^{2n+1}}{2n-1}\right)\frac{r\sin\theta}{(1-2r\cos\theta+r^2)^2} + \frac{(1+r}{\sum_{k=}^{2n}}$$

$$> \frac{r\sin\theta}{(1-2r\cos\theta+r^2)^2}\left\{1-2r\cos\theta+r^2-4\left(\frac{r^{2n}}{2n}+\frac{r^{2n+1}}{2n-1}\right)+\frac{(1+}{\sum_{k}^{2}}\right.$$

Thus, it remains to show that

$$1-2r\cos\theta+r^2-4\left(\frac{r^{2n}}{2n}+\frac{r^{2n+1}}{2n-1}\right)+\frac{\left(1+r^{2n-1}\right)^2}{\sum_{k=1}^{2n-1}\frac{r^k}{k}}\cdot\frac{r\cos\theta-r^2}{1-r\cos\theta}\cdot(1-2r$$

□

Negative increasing function on $[\frac{\pi}{3}, \pi)$, $n \geqslant 30$

Proof.

$$
\begin{aligned}
& 1 - 2r\cos\theta + r^2 - 4\left(\frac{r^{2n}}{2n} + \frac{r^{2n+1}}{2n-1}\right) + \frac{\left(1 + r^{2n-1}\right)^2}{\sum_{k=1}^{2n-1} \frac{r^k}{k}} \cdot \frac{r\cos\theta - r^2}{1 - r\cos\theta} \cdot (\\
> \; & \left(1 - \frac{r^2\left(1 + r^{2n-1}\right)^2}{\sum_{k=1}^{2n-1} \frac{r^k}{k}}\right)\left(1 - 2r\cos\frac{\pi}{3} + r^2\right) - 4\left(\frac{r^{2n}}{2n} + \frac{r^{2n+1}}{2n-1}\right) \\
> \; & 0,
\end{aligned}
$$

where the last inequality holds because the sequence

$$
\left(\left(1 - \frac{r^2\left(1 + r^{2n-1}\right)^2}{\sum_{k=1}^{2n-1} \frac{r^k}{k}}\right)\left(1 - r + r^2\right) - 4\left(\frac{r^{2n}}{2n} + \frac{r^{2n+1}}{2n-1}\right)\right)_{n=30}^{\infty}
$$

is increasing whenever $r \in \left(\frac{1}{2}, 1\right)$. $\square$

Negative increasing function on $[\frac{\pi}{3}, \pi)$, $n \geqslant 30$

With Theorem 2.7.1, we can now establish the following Theorem:

Theorem 2.7.2

Let n be any integer satisfying $n \geqslant 30$. Then $p_n(\theta) < 0$ for $\theta \in (\frac{\pi}{3}, \pi)$.

Proof.

Since $p_n(\pi) = 0$, the theorem is a consequence of Theorem 2.7.1. $\square$

We are now ready to prove the Conjecture for this case.

The case $\frac{1}{2} < r < 1$

Theorem 2.8.1

If $n \in \mathbb{N}$, $r \in (\frac{1}{2}, 1)$ and $\theta \in [0, \pi)$, then

$$\left(\sum_{k=1}^{2n-1} \frac{r^k}{k}(1 - \cos k\theta)\right)^2 + \left(\sum_{k=1}^{2n-1} \frac{r^k}{k} \sin k\theta\right)^2 < 4\left(\sum_{k=1}^{n} \frac{r^{2k-1}}{2k-1}\right)^2.$$

To recap, the above inequality is equivalent to

$$\begin{aligned} & -\sum_{k=1}^{2n-1} \frac{r^k}{k} \cos k\theta + \frac{\left(\sum_{k=1}^{2n-1} \frac{r^k}{k} \cos k\theta\right)^2 + \left(\sum_{k=1}^{2n-1} \frac{r^k}{k} \sin k\theta\right)^2}{2\sum_{k=1}^{2n-1} \frac{r^k}{k}} \\ < \; & \sum_{k=1}^{2n-1} \frac{(-1)^{k-1} r^k}{k} + \frac{\left(\sum_{k=1}^{2n-1} \frac{(-1)^{k-1} r^k}{k}\right)^2}{2\sum_{k=1}^{2n-1} \frac{r^k}{k}}. \end{aligned} \tag{19}$$

Proof of Theorem 2.8.1

Firstly, the case $n = 1$ can be directly shown to be true, as it is equivalent to

$$r^2(1 - \cos\theta)^2 + r^2 \sin^2\theta < 4r^2,$$

which is true if and only if $\cos\theta > -1$ for $\theta \in [0, \pi)$.
Next, we apply Theorem 2.1.7 to show that (4) is valid when $n \geqslant 2$ and $\theta \in [0, \frac{\pi}{4}]$ and when $n \geqslant 3$ and $\theta \in [\frac{\pi}{3}, \frac{\pi}{4}]$. The case $n = 2$ whenever $\theta \in [\frac{\pi}{4}, \frac{\pi}{3}]$ is proven in view of Lemma 2.2.1.
We would now need the following lemma for the proof of Theorem 2.8.1 when $\frac{1}{2} < r < 1$ and $\theta \in [\frac{\pi}{3}, \pi)$:

Proof of Theorem 2.8.1

Lemma 2.8.2

Let $n \in \mathbb{N}$, $r \in (\frac{1}{2}, 1)$ and $\theta \in [\frac{\pi}{3}, \pi)$. Then

$$\left| \sum_{k=1}^{2n-1} \frac{r^k}{k} \cos k\theta \right| \leqslant \sum_{k=1}^{2n-1} \frac{(-1)^{k-1} r^k}{k}.$$

Proof of Theorem 2.8.1

Proof.

We apply Theorem 2.1.1 to show that for any integer $n \geqslant 7$,

$$\max_{\theta \in [\frac{\pi}{3}, \pi]} \left\{ \sum_{k=1}^{n} \frac{\cos k\theta}{k} + \ln\left(2 \sin \frac{\theta}{2}\right) \right\} \leqslant \max_{\theta \in [\frac{5\pi}{2n+1}, \pi]} \left\{ \sum_{k=1}^{n} \frac{\cos k\theta}{k} + \ln\left(2 \sin \frac{\theta}{2}\right)\right.$$
$$< \frac{1}{2};$$

that is,

$$\max_{\theta \in [\frac{\pi}{3}, \pi]} \sum_{k=1}^{n} \frac{\cos k\theta}{k} < \frac{1}{2} - \ln\left(2 \sin \frac{\pi}{6}\right) = \frac{1}{2}.$$

When $n = 1, 2, \ldots, 6$, a direct computation shows that

$$\max \sum^{n} \frac{\cos k\theta}{} \leqslant \frac{1}{}.$$

Proof of Theorem 2.8.1

Proof.

Using summation by parts and a known result of the error bound of alternating sums, we have

$$\sum_{k=1}^{2n-1} \frac{r^k}{k} \cos k\theta < \sum_{k=1}^{2n-1} \frac{(-1)^{k-1} r^k}{k}, \tag{20}$$

Next, using the result (cf. [3, (6.2)]), for $n \in \mathbb{N}$, $r \in (0,1]$, $\theta \in [0,\pi)$, we have

$$\sum_{k=1}^{2n-1} \frac{r^k}{k} \cos k\theta > \sum_{k=1}^{2n-1} \frac{(-1)^k r^k}{k}. \tag{21}$$

Therefore, a combination of (20) and (21) completes the proof. □

Proof of Theorem 2.8.1

With Lemma 2.8.2 and the additional assumption that $\theta \in [\frac{\pi}{3}, \pi)$, (4) can be condensed to

$$-\sum_{k=1}^{2n-1} \frac{r^k}{k} \cos k\theta + \frac{\left(\sum_{k=1}^{2n-1} \frac{r^k}{k} \sin k\theta\right)^2}{2\sum_{k=1}^{2n-1} \frac{r^k}{k}} < \sum_{k=1}^{2n-1} \frac{(-1)^{k-1} r^k}{k} \tag{22}$$

Hence, the following theorem gives an affirmative answer to Theorem 2.8.1 when $\theta \in [\frac{\pi}{3}, \pi)$:

Theorem 2.8.3

For any integer $n \geq 1$, $\frac{1}{2} < r < 1$ and $\frac{\pi}{3} \leq \theta < \pi$, the following inequality holds:

$$-\sum_{k=1}^{2n-1} \frac{r^k}{k} \cos k\theta + \frac{\left(\sum_{k=1}^{2n-1} \frac{r^k}{k} \sin k\theta\right)^2}{2\sum_{k=1}^{2n-1} \frac{r^k}{k}} < \sum_{k=1}^{2n-1} \frac{(-1)^{k-1} r^k}{k}.$$

Proof of Theorem 2.8.1

Proof.

For $\theta \in [\frac{\pi}{3}, \pi - \frac{\pi}{2n})$ and $n = 5, \ldots, 29$, we use Lemmas 2.3.5, 2.3.10 and 2.4.1 to obtain the following inequality:

$$\sum_{k=1}^{2n-1} \frac{(-1)^k r^k}{k} - \sum_{k=1}^{2n-1} \frac{r^k}{k} \cos k\theta + \frac{(\sum_{k=1}^{2n-1} \frac{r^k}{k} \sin k\theta)^2}{2\sum_{k=1}^{2n-1} \frac{r^k}{k}} < F_n(\theta) < 0.$$

Next, for each $\theta \in \left[\frac{\pi}{3}, \pi - \frac{\pi}{2n}\right)$ and $n = 2, 3, 4$, we infer from Lemmas 2.2.2 and Lemma 2.2.3 to obtain the following inequality:

$$\sum_{k=1}^{2n-1} \frac{(-1)^k r^k}{k} - \sum_{k=1}^{2n-1} \frac{r^k}{k} \cos k\theta + \frac{\left(\sum_{k=1}^{2n-1} \frac{r^k}{k} \sin k\theta\right)^2}{2\sum_{k=1}^{2n-1} \frac{r^k}{k}} < 0.$$

$\square$

Proof of Theorem 2.8.1

Proof.

Using this reasoning, for each $\theta \in \left[\pi - \frac{\pi}{2n}, \pi\right)$ and $3 \leqslant n \leqslant 29$ we use Lemmas 2.3.5, 2.3.10 and Theorem 2.5.4 to obtain the following inequality:

$$\sum_{k=1}^{2n-1} \frac{(-1)^k r^k}{k} - \sum_{k=1}^{2n-1} \frac{r^k}{k} \cos k\theta + \frac{\left(\sum_{k=1}^{2n-1} \frac{r^k}{k} \sin k\theta\right)^2}{2\sum_{k=1}^{2n-1} \frac{r^k}{k}} < g_n(\theta) < 0.$$

Similarly, for each $\theta \in \left[\pi - \frac{\pi}{2n}, \pi\right)$ and $n = 2$ we infer from Theorem 2.6.3 to obtain the following inequality:

$$\sum_{k=1}^{2n-1} \frac{(-1)^k r^k}{k} - \sum_{k=1}^{2n-1} \frac{r^k}{k} \cos k\theta + \frac{\left(\sum_{k=1}^{2n-1} \frac{r^k}{k} \sin k\theta\right)^2}{2\sum_{k=1}^{2n-1} \frac{r^k}{k}} < 0.$$

□

Proof of Theorem 2.8.1

Proof.

Lastly, for each $\theta \in \left[\frac{\pi}{3}, \pi\right)$ and integer $n \geqslant 30$, we use Lemmas 2.3.5, 2.3.10 and Theorem 2.7.2 to obtain the following inequality:

$$\sum_{k=1}^{2n-1} \frac{(-1)^k r^k}{k} - \sum_{k=1}^{2n-1} \frac{r^k}{k} \cos k\theta + \frac{\left(\sum_{k=1}^{2n-1} \frac{r^k}{k} \sin k\theta\right)^2}{2\sum_{k=1}^{2n-1} \frac{r^k}{k}} < p_n(\theta) < 0.$$

This completes the proof. □

With Theorem 2.8.3, the proof of Theorem 2.8.1 is complete.

Proof of Conjecture 1

In view of Theorem 1.XX and Theorem 2.8.1, we have proven Conjecture 1 for $\frac{1}{2} < r \leqslant 1$.

Theorem 2.9.1

For any integer $n \geq 1$, $\frac{1}{2} < r \leq 1$ and $0 \leq \theta < \pi$, the following inequality holds:

$$\left(\sum_{k=1}^{2n-1} \frac{r^k}{k}(1 - \cos k\theta)\right)^2 + \left(\sum_{k=1}^{2n-1} \frac{r^k}{k} \sin k\theta\right)^2 < 4\left(\sum_{k=1}^{n} \frac{r^{2k-1}}{2k-1}\right)^2.$$

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Brannan's conjecture and trigonometric polynomials

Thank you!

Appendix

Bounding trigonometric sums

Similar results can be established for sine polynomials:

Lemma 2.3.9.

Let $r \in (\frac{1}{2}, 1)$, $\theta \in [0, \pi)$ and $n \in \mathbb{N}$. Then

$$\sum_{k=1}^{2n-1} \frac{r^k \sin k\theta}{k} < \tan^{-1}\left(\frac{r \sin\theta}{1 - r\cos\theta}\right) - \frac{\left(\frac{r^{2n}}{2n} + \frac{r^{2n+1}}{2n-1}\right)}{(1+r)^2} + \frac{\frac{r^{2n+1}}{2n-1}\left(1 + \sin(2n-1)\theta\right) + \frac{r^{2n}}{2n}\left(1 - \sin(2n\theta)\right)}{1 - 2r\cos\theta + r^2}.$$