Brannan's conjecture and trigonometric polynomials

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June 5, 2025

Abstract

For any integer $n \geqslant 1$ and $0 \leqslant \theta < \pi$, we prove that

$$\left(\sum_{k=1}^{2n-1} \frac{1}{k}(1-\cos k\theta)\right)^2 + \left(\sum_{k=1}^{2n-1} \frac{\sin k\theta}{k}\right)^2 < 4\left(\sum_{k=1}^{n} \frac{1}{2k-1}\right)^2.$$

This gives a partial answer to a conjecture related to Brannan's conjecture.

1 Introduction

In 1973, D. A. Brannan [3] conjectured that if we let

$$\frac{(1+zx)^{\alpha}}{(1-x)^{\beta}} = \sum_{m=0}^{\infty} A_m(\alpha,\beta,z)\, x^n$$

where $\alpha > 0, \beta > 0$, and $z = e^{i\theta}, 0 \leqslant \theta \leqslant 2\pi$, then

$$|A_m(\alpha,\beta,z)| \leqslant A_m(\alpha,\beta,1) \tag{1.1}$$

for all odd integer m. Here, A_m refers to the coefficient of the mth order term in the polynomial.

While the conjecture was proven for all $\alpha \geqslant 1, \beta > 1$ by D. Aharonov and S. Friedland [1], the case $0 < \alpha \leqslant 1$ and $0 < \beta \leqslant 1$ turned out to be rather difficult. Recently, in an attempt to prove the conjecture for $0 < \alpha < 1$ and $\beta = 1$, R.W. Barnard et al.[2] reformulates inequality (1.1) into finding the largest r satisfying

$$|A_m(\alpha,\beta,z)| \leqslant A_m(\alpha,\beta,r) \tag{1.2}$$

where they have generalized $z = re^{i\theta}$ and treat A_m as analytic.

In the paper [2], the authors try to show that $r \leqslant 1$ for all odd m. In particular, they successfully show that inequality (1.2) holds for $0 < r \leqslant 1/2$. Furthermore, they reduce the case $1/2 < r \leqslant 1$ into proving the following conjecture, expressed in term of the trigonometric form as

$$\left(\sum_{k=1}^{2n-1} \frac{r^k}{k}(1-\cos k\theta)\right)^2 + \left(\sum_{k=1}^{2n-1} \frac{r^k}{k}\sin k\theta\right)^2 < 4\left(\sum_{k=1}^{n} \frac{r^{2k-1}}{2k-1}\right)^2 \tag{1.3}$$

for $1/2 < r \leqslant 1, n \in \mathbb{N}$ and notably $\theta \in [0,\pi)$.

[0]Key words and phrases: Brannan Conjecture, Trigonometric polynomials, Inequalities

In this project, we give a partial answer to the above conjecture.

Theorem 1.1. If $n \in \mathbb{N}$ and $\theta \in [0, \pi)$, then

$$\left(\sum_{k=1}^{2n-1} \frac{1}{k}(1-\cos k\theta)\right)^2 + \left(\sum_{k=1}^{2n-1} \frac{\sin k\theta}{k}\right)^2 < 4\left(\sum_{k=1}^{n} \frac{1}{2k-1}\right)^2. \tag{1.4}$$

The report is organised as follows. In Section 2, we show that (1.4) is equivalent to

$$-\sum_{k=1}^{2n-1} \frac{\cos k\theta}{k} + \frac{(\sum_{k=1}^{2n-1} \frac{\cos k\theta}{k})^2 + (\sum_{k=1}^{2n-1} \frac{\sin k\theta}{k})^2}{2\sum_{k=1}^{2n-1} \frac{1}{k}} < \sum_{k=1}^{2n-1} \frac{(-1)^{k-1}}{k} + \frac{(\sum_{k=1}^{2n-1} \frac{(-1)^{k-1}}{k})^2}{2\sum_{k=1}^{2n-1} \frac{1}{k}} \tag{1.5}$$

and prove that (1.5) is true for $n = 1, 2, 3$ and 4. In Section 3, we use some result of Fong et al. [5] and Kim et al. [7] to establish

$$-\sum_{k=1}^{2n-1} \frac{\cos k\theta}{k} + \frac{(\sum_{k=1}^{2n-1} \frac{\sin k\theta}{k})^2}{2\sum_{k=1}^{2n-1} \frac{1}{k}} < \sum_{k=1}^{2n-1} \frac{(-1)^{k-1}}{k} \tag{1.6}$$

for $n = 5, 6, \ldots$ and $\frac{\pi}{3} \leqslant \theta < \pi$, observing that

$$\left|\sum_{k=1}^{2n-1} \frac{\cos k\theta}{k}\right| \leqslant \sum_{k=1}^{2n-1} \frac{(-1)^{k-1}}{k} \text{ for } \theta \in \left[\frac{\pi}{3}, \pi\right). \tag{1.7}$$

In Section 4, we use integration by parts to derive some crucial estimates for $\sum_{k=1}^{2n-1} \frac{\cos k\theta}{k}$ and $\sum_{k=1}^{2n-1} \frac{\sin k\theta}{k}$ so that the left-hand side of (1.6) is bounded above by an increasing functional upper bound on the interval $[\frac{2\pi}{3}, \frac{4n-3}{4n-1}\pi]$. In Section 5, we prove that the left hand side of (1.6) is an increasing function of θ on the interval $[\frac{4n-3}{4n-1}\pi, \pi]$. Finally, the last section is devoted to the proof of Theorem 1.1.

2 The cases $n = 1, 2, 3, 4$.

For each $n \in \mathbb{N}$ and $\theta \in [0, \pi]$ we set

$$S_n(\theta) := \sum_{k=1}^{n} \frac{\sin k\theta}{k} \quad \text{and} \quad C_n(\theta) := \sum_{k=1}^{n} \frac{\cos k\theta}{k}.$$

Then (1.4) is equivalent to

$$-C_{2n-1}(\theta) + \frac{(C_{2n-1}(\theta))^2 + (S_{2n-1}(\theta))^2}{2\sum\limits_{k=1}^{2n-1} \frac{1}{k}} < \frac{\left(\sum\limits_{k=1}^{n} \frac{2}{2k-1}\right)^2 - \left(\sum\limits_{k=1}^{2n-1} \frac{1}{k}\right)^2}{2\sum\limits_{k=1}^{2n-1} \frac{1}{k}}. \tag{2.1}$$

Moreover, (2.1) is equivalent to the inequality

$$-C_{2n-1}(\theta) + \frac{(C_{2n-1}(\theta))^2 + (S_{2n-1}(\theta))^2}{2\sum\limits_{k=1}^{2n-1} \frac{1}{k}} < -C_{2n-1}(\pi) + \frac{C_{2n-1}^2(\pi)}{2\sum\limits_{k=1}^{2n-1} \frac{1}{k}} \tag{2.2}$$

because

$$\begin{aligned}\frac{\left(\sum\limits_{k=1}^{n}\frac{2}{2k-1}\right)^2-\left(\sum\limits_{k=1}^{2n-1}\frac{1}{k}\right)^2}{2\sum\limits_{k=1}^{2n-1}\frac{1}{k}} &= \frac{1}{2\sum\limits_{k=1}^{2n-1}\frac{1}{k}}\left\{\sum_{k=1}^{n}\frac{2}{2k-1}-\sum_{k=1}^{2n-1}\frac{1}{k}\right\}\left\{\sum_{k=1}^{n}\frac{2}{2k-1}+\sum_{k=1}^{2n-1}\frac{1}{k}\right\}\\ &= \frac{1}{2\sum\limits_{k=1}^{2n-1}\frac{1}{k}}\left\{\sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}\right\}\left\{\sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}+2\sum_{k=1}^{2n-1}\frac{1}{k}\right\}\\ &= \sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}+\frac{1}{2\sum\limits_{k=1}^{2n-1}\frac{1}{k}}\left(\sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}\right)^2.\end{aligned}$$

Next, we use some known lemmas concerning trigonometric polynomials.

Lemma 2.1 (cf. [6, equation 6])**.** Let $n\in\mathbb{N}$ and let $\theta\in(0,\pi)$. Then

$$\frac{d}{d\theta}\left(S_{2n-1}(\theta)\right)=\frac{\cos n\theta\sin\left(n-\frac{1}{2}\right)\theta}{\sin\frac{\theta}{2}}.$$

Lemma 2.2 (cf. [8, equation 2])**.** Let $n\in\mathbb{N}$ and let $\theta\in(0,\pi)$. Then

$$\frac{d}{d\theta}\left(C_{2n-1}(\theta)\right)=-\frac{\sin n\theta\sin\left(n-\frac{1}{2}\right)\theta}{\sin\frac{\theta}{2}}.$$

The following computation involves the result of the above lemmas.

Theorem 2.3. Let $n\in\{1,2,3,4\}$. If $\theta\in\left[0,\frac{4n-3}{4n-1}\pi\right]$, then

$$-C_{2n-1}(\theta)+\frac{\left(C_{2n-1}(\theta)\right)^2+\left(S_{2n-1}(\theta)\right)^2}{2\sum\limits_{k=1}^{2n-1}\frac{1}{k}}<-C_{2n-1}(\pi)+\frac{C_{2n-1}^2(\pi)}{2\sum\limits_{k=1}^{2n-1}\frac{1}{k}}. \tag{2.3}$$

Proof. When $n=1$, (2.3) holds since

$$\begin{aligned}-C_1(\theta)+\frac{\left(C_1(\theta)\right)^2+\left(S_1(\theta)\right)^2}{2} &= -\cos\theta+\frac{1}{2}\\ &< -C_1(\pi)+\frac{C_1^2(\pi)}{2}.\end{aligned}$$

Next, we consider the remaining cases $n=2,3,4$. Since a direct computation shows that the function $u\mapsto\frac{u^2}{2\sum_{k=1}^{2n-1}\frac{1}{k}}-u$ is decreasing on the closed interval $\left[\sum\limits_{k=1}^{2n-1}\frac{(-1)^k}{k},\sum\limits_{k=1}^{2n-1}\frac{1}{k}\right]$, we conclude that

$$\max_{\theta\in I}\left\{\frac{\left(C_{2n-1}(\theta)\right)^2}{2\sum\limits_{k=1}^{2n-1}\frac{1}{k}}-C_{2n-1}(\theta)\right\}=\frac{\left(\max\limits_{\theta\in I}\{-C_{2n-1}(\theta)\}\right)^2}{2\sum\limits_{k=1}^{2n-1}\frac{1}{k}}+\max_{\theta\in I}\{-C_{2n-1}(\theta)\} \tag{2.4}$$

whenever I is a closed subinterval of $[0,\pi]$. Now we are ready to do some computation.

Using Lemma 2.1 and Lemma 2.2, we obtain

$$\max_{\theta\in[0,\frac{2\pi}{5}]}\{-C_3(\theta)\}=-C_3\left(\frac{2\pi}{5}\right)\text{ and }\max_{\theta\in[0,\frac{2\pi}{5}]}\left\{\frac{\left(S_3(\theta)\right)^2}{2\sum\limits_{k=1}^{3}\frac{1}{k}}\right\}=\frac{\left(S_3\left(\frac{\pi}{4}\right)\right)^2}{2\sum\limits_{k=1}^{3}\frac{1}{k}}$$

respectively. By combining the above absolute maxima and (2.4) with the observation $-C_3(\pi)+\dfrac{C_3^2(\pi)}{2\sum\limits_{k=1}^{3}\frac{1}{k}}=1.022\ldots$, we see that

$$\begin{aligned}\max_{\theta\in[0,\frac{2\pi}{5}]}\left\{-C_3\left(\theta\right)+\frac{\left(C_3(\theta)\right)^2+\left(S_3(\theta)\right)^2}{2\sum\limits_{k=1}^{3}\frac{1}{k}}\right\} &\leqslant -C_3\left(\frac{2\pi}{5}\right)+\frac{\left(C_3\left(\frac{2\pi}{5}\right)\right)^2}{2\sum\limits_{k=1}^{3}\frac{1}{k}}+\frac{\left(S_3\left(\frac{\pi}{4}\right)\right)^2}{2\sum\limits_{k=1}^{3}\frac{1}{k}}\\ &= 0.969\ldots.\end{aligned}$$

Similarly,

$$\begin{aligned}\max_{\theta\in[\frac{2\pi}{5},\frac{5\pi}{7}]}\left\{-C_3\left(\theta\right)+\frac{\left(C_3(\theta)\right)^2+\left(S_3(\theta)\right)^2}{2\sum\limits_{k=1}^{3}\frac{1}{k}}\right\} &\leqslant -C_3\left(\frac{\pi}{2}\right)+\frac{\left(C_3\left(\frac{\pi}{2}\right)\right)^2+\left(S_3\left(\frac{2\pi}{5}\right)\right)^2}{2\sum\limits_{k=1}^{3}\frac{1}{k}}\\ &= 0.868\ldots.\end{aligned}$$

When $n=3$ and $-C_5(\pi)+\dfrac{C_5^2(\pi)}{2\sum\limits_{k=1}^{5}\frac{1}{k}}=0.917\ldots$, we have

$$\max_{\theta\in[0,\frac{\pi}{2}]}\left\{-C_5\left(\theta\right)+\frac{\left(C_5(\theta)\right)^2+\left(S_5(\theta)\right)^2}{2\sum\limits_{k=1}^{5}\frac{1}{k}}\right\}<-C_5\left(\frac{\pi}{2}\right)+\frac{\left(C_5\left(\frac{\pi}{2}\right)\right)^2+\left(S_5\left(\frac{\pi}{6}\right)\right)^2}{2\sum\limits_{k=1}^{5}\frac{1}{k}}=0.812\ldots$$

and

$$\max_{\theta\in[\frac{\pi}{2},\frac{9\pi}{11}]}\left\{-C_5\left(\theta\right)+\frac{\left(C_5(\theta)\right)^2+\left(S_5(\theta)\right)^2}{2\sum\limits_{k=1}^{5}\frac{1}{k}}\right\}<-C_5\left(\frac{2\pi}{3}\right)+\frac{\left(C_5\left(\frac{2\pi}{3}\right)\right)^2+\left(S_5\left(\frac{\pi}{2}\right)\right)^2}{2\sum\limits_{k=1}^{5}\frac{1}{k}}=0.896\ldots.$$

When $n=4$ and $-C_7(\pi)+\dfrac{C_7^2(\pi)}{2\sum\limits_{k=1}^{7}\frac{1}{k}}=0.870\ldots$, we have

$$\max_{\theta\in[0,\frac{3\pi}{8}]}\left\{-C_7\left(\theta\right)+\frac{\left(C_7(\theta)\right)^2+\left(S_7(\theta)\right)^2}{2\sum\limits_{k=1}^{7}\frac{1}{k}}\right\}<-C_7\left(\frac{3\pi}{8}\right)+\frac{\left(C_7\left(\frac{3\pi}{8}\right)\right)^2+\left(S_7\left(\frac{\pi}{8}\right)\right)^2}{2\sum\limits_{k=1}^{7}\frac{1}{k}}=0.557\ldots,$$

$$\max_{\theta\in[\frac{3\pi}{8},\frac{5\pi}{8}]}\left\{-C_7\left(\theta\right)+\frac{\left(C_7(\theta)\right)^2+\left(S_7(\theta)\right)^2}{2\sum\limits_{k=1}^{7}\frac{1}{k}}\right\}<-C_7\left(\frac{5\pi}{8}\right)+\frac{\left(C_7\left(\frac{5\pi}{8}\right)\right)^2+\left(S_7\left(\frac{3\pi}{8}\right)\right)^2}{2\sum\limits_{k=1}^{7}\frac{1}{k}}=0.700\ldots$$

and

$$\max_{\theta\in[\frac{5\pi}{8},\frac{13\pi}{15}]}\left\{-C_7\left(\theta\right)+\frac{\left(C_7(\theta)\right)^2+\left(S_7(\theta)\right)^2}{2\sum\limits_{k=1}^{7}\frac{1}{k}}\right\}<-C_7\left(\frac{3\pi}{4}\right)+\frac{\left(C_7\left(\frac{3\pi}{4}\right)\right)^2+\left(S_7\left(\frac{5\pi}{8}\right)\right)^2}{2\sum\limits_{k=1}^{7}\frac{1}{k}}=0.847\ldots.$$

The proof is complete. □

3 The case $n \geqslant 5$ and $\theta \in \left[0, \frac{2\pi}{3}\right]$.

We begin with the following inequality involving $\left[0, \frac{\pi}{2}\right]$.

Lemma 3.1. If $\theta \in \left[0, \frac{\pi}{2}\right]$, then

$$-(1+\cos\theta)^2 \sum_{k=1}^{5} \frac{1}{2k-1} + \frac{1}{16}(1+\cos\theta)^4 + \left(\frac{\pi-\theta}{2} + \frac{3}{10}\right)^2 < 0.$$

Proof. In view of the following observations

$$\begin{aligned} -u\sum_{k=1}^{5} \frac{1}{2k-1} + \frac{u^2}{16} &= -\frac{563}{315}u + \frac{u^2}{16} \\ &= \frac{u}{16}(u-4) - \frac{1937}{1260}u \end{aligned}$$

and $\frac{1937}{1260} > \frac{36}{25}$, it suffices to show that

$$\frac{\pi-\theta}{2} + \frac{3}{10} < \frac{6}{5}\,(1+\cos\theta) \ \text{ for } \theta \in \left[0, \frac{\pi}{2}\right]. \tag{3.1}$$

Let us now consider

$$h(\theta) = \frac{\pi-\theta}{2} + \frac{3}{10} - \frac{6}{5}\,(1+\cos\theta) \ \text{ for } \theta \in \left[0, \frac{\pi}{2}\right].$$

Since $\theta_0 = \sin^{-1}\left(\frac{5}{12}\right)$ is the only zero of h' and

$$\max_{\theta\in\left[0,\frac{\pi}{2}\right]} h(\theta) = \max\left\{h(0), h(\theta_0), h\left(\frac{\pi}{2}\right)\right\} < -0.546 < 0,$$

(3.1) follows and the proof is complete. □

In order to establish a similar lemma involving the interval $\left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$, we need the following result.

Lemma 3.2. If $x \in \left[-\frac{1}{2}, 0\right]$, then

$$-\frac{2929}{1260} + \frac{6079}{1260}x + \frac{8063}{1260}x^2 - \frac{9}{4}x^3 - x^4 < 0.$$

Proof. We let

$$f(x) = -\frac{2929}{1260} + \frac{6079}{1260}x + \frac{8063}{1260}x^2 - \frac{9}{4}x^3 - x^4.$$

After differentiating, we get

$$f'(x) = \frac{6079}{1260} + \frac{8063}{630}x - \frac{27}{4}x^2 - 4x^3$$

and

$$\begin{aligned} f''(x) &= \frac{8063}{630} - \frac{27}{2}x - 12x^2 \\ &= \frac{8063}{630} + \frac{243}{64} - 12\left(x + \frac{27}{48}\right)^2 \ \text{ for } x \in \left(-\frac{1}{2}, 0\right). \end{aligned}$$

Since $-\frac{27}{48} < -\frac{1}{2}$ and $\lim_{x\to 0^-} f''(x) = \frac{8063}{630}$, we conclude that

$$f \text{ is strictly convex on } \left(-\frac{1}{2}, 0\right).$$

Hence, for $x \in [-\frac{1}{2}, 0]$, we have

$$\begin{aligned} f(x) &\leqslant f\left(-\frac{1}{2}\right) + 2\left(f(0) - f\left(-\frac{1}{2}\right)\right)\left(x + \frac{1}{2}\right) \\ &\leqslant f(0) < 0. \end{aligned}$$

$\square$

Lemma 3.3. If $\theta \in \left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$, then

$$-(1+\cos\theta)^2 \sum_{k=1}^{5} \frac{1}{2k-1} + \frac{1}{16}(1+\cos\theta)^4 + \left(\frac{\pi-\theta}{2} + \frac{1}{8}\right)^2 < 0.$$

Proof. For each $\theta \in \left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$ we set

$$g(\theta) = -(1+\cos\theta)^2 \sum_{k=1}^{5} \frac{1}{2k-1} + \frac{1}{16}(1+\cos\theta)^4 + \left(\frac{\pi-\theta}{2} + \frac{1}{8}\right)^2.$$

Since the substitution $x = \cos\theta$ yields

$$\begin{aligned} g''(\theta) &= \left(\frac{3\sin^2\theta}{4} + \frac{811}{315}\right)\cos^2\theta + \left(\frac{9\sin^2\theta}{4} + \frac{811}{315}\right)\cos\theta - \frac{\sin^4\theta}{4} - \frac{811\sin^2\theta}{315} + \frac{1}{2} \\ &= -x^4 - \frac{9}{4}x^3 + \frac{8063}{1260}x^2 + \frac{6079}{1260}x - \frac{2929}{1260}, \end{aligned}$$

an application of Lemma 3.2 shows that g' is decreasing on $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$. Hence $\lim\limits_{\theta \to \frac{\pi}{2}^+} g'(\theta) = -1.024\cdots < 0$, $g\left(\frac{\pi}{2}\right) < 0$ and the continuity of g on $\left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$ yield the desired conclusion. $\square$

In order to proceed further, we need some recent results established by Fong et al. [5] and Kim et al. [7].

Theorem 3.4 (cf. [5, Theorem 1.3])**.** Let $n \in \mathbb{N}$ and let $\theta \in (0, \pi)$. Then

$$\sum_{k=1}^{2\lfloor \frac{n}{2} \rfloor + 1} \frac{(-1)^{k-1}}{k} + \sum_{k=1}^{n} \frac{\cos k\theta}{k} \geqslant \frac{1}{4}(1+\cos\theta)^2, \tag{3.2}$$

where equality holds if and only if $n = 2$ and $\theta = \pi - \cos^{-1}\frac{1}{3}$.

Lemma 3.5 (cf. [7, Lemma 2.2])**.** Let $n \in \mathbb{N}$. If $q \in \{1, 2, ..., \lfloor \frac{n+1}{2} \rfloor\}$, then

$$\max_{\theta \in \left[\frac{(4q-2)\pi}{2n+1}, \pi\right]} \left\{ \sum_{k=1}^{n} \frac{\sin k\theta}{k} - \frac{\pi-\theta}{2} \right\} = \sum_{k=1}^{n} \frac{\sin k\frac{(4q-2)\pi}{2n+1}}{k} - \frac{\pi - \frac{(4q-2)\pi}{2n+1}}{2}. \tag{3.3}$$

Theorem 3.6 (cf. [7, Theorem 2.5])**.** Let $n \in \mathbb{N}$. If $p \in \mathbb{N}$, then the sequence

$$\left\{ (-1)^{p-1} \left(\sum_{k=1}^{n} \frac{\sin k\frac{2p\pi}{2n+1}}{k} - \frac{\pi - \frac{2p\pi}{2n+1}}{2} \right) \right\}_{n=p}^{\infty} \tag{3.4}$$

is decreasing.

We are now ready to state and prove the main result of this section.

Theorem 3.7. If $n \geqslant 5$, $n \in \mathbb{N}$ and $\theta \in \left[0, \frac{2\pi}{3}\right]$, then

$$\left(\sum_{k=1}^{2n-1} \frac{1}{k}(1-\cos k\theta)\right)^2 + \left(\sum_{k=1}^{2n-1} \frac{\sin k\theta}{k}\right)^2 < 4\left(\sum_{k=1}^{n} \frac{1}{2k-1}\right)^2.$$

Proof. According to Theorem 3.4,

$$\sum_{k=1}^{2n-1} \frac{1-\cos k\theta}{k} \leqslant \sum_{k=1}^{n} \frac{2}{2k-1} - \frac{1}{4}(1+\cos\theta)^2.$$

Thus, it is sufficient to show that

$$-(1+\cos\theta)^2 \sum_{k=1}^{5} \frac{1}{2k-1} + \frac{1}{16}(1+\cos\theta)^4 + \left(\sum_{k=1}^{2n-1} \frac{\sin k\theta}{k}\right)^2 < 0. \tag{3.5}$$

Next, we infer from Lemma 3.5 and Theorem 3.6 that

$$\max_{\theta\in\left[0,\frac{\pi}{2}\right]} \left\{ S_{2n-1}(\theta) - \frac{\pi-\theta}{2} \right\} \leqslant S_9\left(\frac{2\pi}{19}\right) - \frac{\pi - \frac{6\pi}{19}}{2} = 0.282\ldots < \frac{3}{10}. \tag{3.6}$$

Hence, (3.6) and the Fejér-Jackson inequality $\sum\limits_{k=1}^{n} \frac{\sin k\theta}{k} > 0$ (see, for example, [7]) yields

$$S_{2n-1}^2(\theta) < \left(\frac{\pi-\theta}{2} + \frac{3}{10}\right)^2. \tag{3.7}$$

Finally, we combine (3.7) and Lemma 3.1 to establish (3.5) for the case $\theta \in \left[0, \frac{\pi}{2}\right]$. A similar reasoning yields (3.5) for the case $\theta \in \left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$. □

4 Further estimates involving $n \geqslant 5$.

The main aim of this section is to show that (2.2) holds if $\theta \in \left[\frac{2\pi}{3}, \frac{(4n-3)\pi}{4n-1}\right]$ and $n = 5, 6, 7, \ldots$.

Lemma 4.1. Let $n \in \mathbb{N}$ and let $\theta \in (0, \pi)$. Then

$$S_{2n-1}(\theta) - \frac{\pi-\theta}{2} < \frac{1}{4n-1}\left(2\csc\frac{\theta}{2} - 1\right).$$

Proof. First we obtain a simplified formula for $S'_{2n-1}(\theta) + \frac{1}{2}$:

$$S'_{2n-1}(\theta) + \frac{1}{2} = \frac{\sin(2n-\frac{1}{2})\theta - \sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}} + \frac{1}{2} = \frac{\sin(2n-\frac{1}{2})\theta}{2\sin\frac{\theta}{2}}.$$

Hence an integration by parts yields

$$\begin{aligned}
S_{2n-1}(\theta) - \frac{\pi-\theta}{2} &= S_{2n-1}(\pi) - \frac{\pi-\pi}{2} - \int_\theta^\pi \frac{\sin(2n-\frac{1}{2})x}{2\sin\frac{x}{2}}\,dx \\
&= -\frac{\csc\frac{\theta}{2}}{4n-1}\cos\left(2n-\frac{1}{2}\right)\theta + \int_\theta^\pi \frac{\cos\left(2n-\frac{1}{2}\right)x}{2(4n-1)} \frac{\cos\frac{x}{2}}{\sin^2\frac{x}{2}}\,dx \\
&< \frac{\csc\frac{\theta}{2}}{4n-1} + \frac{1}{2(4n-1)}\int_\theta^\pi \frac{\cos\frac{x}{2}}{\sin^2\frac{x}{2}}\,dx \\
&= \frac{1}{4n-1}\left(2\csc\frac{\theta}{2} - 1\right).
\end{aligned}$$

□

Lemma 4.2. Let $n \in \mathbb{N}$ and $\theta \in (0, \pi)$. Then

$$C_{2n-1}(\theta) + \ln\left(\sin\frac{\theta}{2}\right) > -\frac{2}{4n-1}\left(\csc\frac{\theta}{2} - 1\right) + C_{2n-1}(\pi).$$

Proof. Following the proof of Lemma 4.1, we have

$$C'_{2n-1}(\theta) + \frac{\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}} = \frac{\cos\left(2n-\frac{1}{2}\right)\theta - \cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}} + \frac{\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}} = \frac{\cos\left(2n-\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}}.$$

Hence an integration by parts yields

$$\begin{aligned}
C_{2n-1}(\theta) + \ln\left(\sin\frac{\theta}{2}\right) &= C_{2n-1}(\pi) + \ln\left(\sin\frac{\pi}{2}\right) - \int_\theta^\pi \frac{\cos\left(2n-\frac{1}{2}\right)x}{2\sin\frac{x}{2}}\,dx \\
&= C_{2n-1}(\pi) + \frac{\csc\frac{\theta}{2}}{4n-1}\sin\left(2n-\frac{1}{2}\right)\theta + \frac{1}{4n-1} - \int_\theta^\pi \frac{\sin\left(2n-\frac{1}{2}\right)x}{2(4n-1)}\frac{\cos\frac{x}{2}}{\sin^2\frac{x}{2}}\,dx \\
&> C_{2n-1}(\pi) - \frac{\csc\frac{\theta}{2}}{4n-1} + \frac{1}{4n-1} - \frac{1}{2(4n-1)}\int_\theta^\pi \frac{\cos\frac{x}{2}}{\sin^2\frac{x}{2}}\,dx \\
&= C_{2n-1}(\pi) - \frac{2}{4n-1}\left(\csc\frac{\theta}{2} - 1\right).
\end{aligned}$$

□

In order to proceed further, for each $n \in \mathbb{N}$ and $\theta \in (0, \pi)$, we set

$$F_n(\theta) = \frac{1}{4n-1}\left(2\csc\frac{\theta}{2} - 1\right) + \frac{\pi - \theta}{2}$$

and

$$G_n(\theta) = -\frac{2}{4n-1}\left(\csc\frac{\theta}{2} - 1\right) - \ln\left(\sin\frac{\theta}{2}\right) + C_{2n-1}(\pi).$$

We are now ready to establish the following crucial lemma.

Lemma 4.3. Let $n \in \mathbb{N}$ and suppose $n \geqslant 5$. Then both functions $-G_n$ and $F_n^2 - 2G_n \sum\limits_{k=1}^{2n-1} \frac{1}{k}$ are increasing on $\left[\frac{2\pi}{3}, \frac{(4n-3)\pi}{4n-1}\right]$.

Proof. For each $\theta \in \left(\frac{2\pi}{3}, \pi\right)$ and $n \geqslant 5$, we obtain

$$\begin{aligned}
G'_n(\theta) &= \frac{1}{4n-1}\csc\frac{\theta}{2}\cot\frac{\theta}{2} - \frac{1}{2}\cot\frac{\theta}{2} \\
&< \left(\frac{1}{19}\csc\frac{\pi}{3} - \frac{1}{2}\right)\cot\frac{\theta}{2} \\
&< -\frac{5}{12}\cot\frac{\theta}{2}.
\end{aligned} \tag{4.1}$$

Since $-G_n$ is continuous on $\left[\frac{2\pi}{3}, \pi\right]$, we infer from (4.1) that $-G_n$ is increasing on $\left[\frac{2\pi}{3}, \pi\right]$.

For each integer $n \geqslant 5$ and $\theta \in \left(\frac{2\pi}{3}, \frac{4n-3}{4n-1}\pi\right)$

$$\begin{aligned}
&\frac{d}{d\theta}\left((F_n(\theta))^2 - 2G_n(\theta)\left(\sum_{k=1}^{2n-1}\frac{1}{k}\right)\right)\\
= \;& 2F_n(\theta)F_n'(\theta) - 2G_n'(\theta)\left(\sum_{k=1}^{2n-1}\frac{1}{k}\right)\\
> \;& 2\left\{\frac{1}{4n-1}\left(2\csc\frac{\theta}{2}-1\right)+\frac{\pi-\theta}{2}\right\}\left\{-\frac{1}{4n-1}\cot\frac{\theta}{2}\csc\frac{\theta}{2}-\frac{1}{2}\right\}+\ \frac{5}{6}\left(\cot\frac{\theta}{2}\right)\left(\sum_{k=1}^{2n-1}\frac{1}{k}\right) \text{(by (4.1))}\\
> \;& 2\left\{\frac{1}{4n-1}\left(2\csc\frac{\pi}{3}-1\right)+\frac{\pi-\theta}{2}\right\}\left\{-\frac{1}{18}\cot\frac{\pi}{3}\csc\frac{\pi}{3}-\frac{1}{2}\right\}+\ \frac{5}{6}\left(\frac{\pi-\theta}{2}\right)\left(\sum_{k=1}^{9}\frac{1}{k}\right)
\end{aligned}$$

$$\begin{aligned}
> \;& -\frac{29}{27}\left(\frac{1}{4n-1}\left(\frac{12}{5}-1\right)+\frac{\pi-\theta}{2}\right)+\frac{5}{6}\left(\frac{\pi-\theta}{2}\right)\left(\sum_{k=1}^{9}\frac{1}{k}\right)\\
> \;& \left(\frac{5}{6}\left(\sum_{k=1}^{9}\frac{1}{k}\right)-\frac{29}{27}\right)\frac{\pi-\theta}{2}-\frac{29}{27}\left(\frac{1}{4n-1}\right)\left(\frac{12}{5}-1\right)\\
\geqslant \;& \left(\frac{250}{108}-\frac{29}{27}\right)\frac{3}{4n-1}-\frac{29}{27}\left(\frac{1}{4n-1}\right)\left(\frac{12}{5}-1\right)\\
> \;& \frac{2}{4n-1}\\
> \;& 0.
\end{aligned}$$

Therefore a standard argument shows that $F_n^2 - 2G_n\sum_{k=1}^{2n-1}\frac{1}{k}$ is increasing on $\left[\frac{2\pi}{3}, \frac{(4n-3)\pi}{4n-1}\right]$. $\square$

The main result of this section is the following theorem.

Theorem 4.4. Let n be any integer satisfying $n \geqslant 5$. If $\theta \in \left[\frac{2\pi}{3}, \frac{(4n-3)\pi}{4n-1}\right]$, then

$$-\sum_{k=1}^{2n-1}\frac{\cos k\theta}{k} + \frac{\left(\sum_{k=1}^{2n-1}\frac{\sin k\theta}{k}\right)^2}{2\sum_{k=1}^{2n-1}\frac{1}{k}} < \sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}. \tag{4.2}$$

Proof. By taking into account of Lemma 4.3, we need to establish the following inequality

$$-G_n\left(\frac{(4n-3)\pi}{4n-1}\right) + \frac{\left(F_n\left(\frac{(4n-3)\pi}{4n-1}\right)\right)^2}{2\sum_{k=1}^{2n-1}\frac{1}{k}} < \sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}.$$

For each integer $n \geqslant 5$ we have

$$
\begin{aligned}
& -G_n\left(\frac{(4n-3)\pi}{4n-1}\right)+\frac{\left(F_n\left(\frac{4n-3}{4n-1}\pi\right)\right)^2}{2\sum_{k=1}^{2n-1}\frac{1}{k}} \\
= \quad & \frac{2}{4n-1}\left(\csc\frac{4n-3}{8n-2}\pi-1\right)+\ln\left(\sin\frac{4n-3}{8n-2}\pi\right)-g_n(\pi)+\frac{\left(\frac{1}{4n-1}\left(2\csc\frac{4n-3}{8n-2}\pi-1\right)+\frac{\pi-\frac{4n-3}{8n-2}\pi}{2}\right)^2}{2\sum\limits_{k=1}^{2n-1}\frac{1}{k}} \\
= \quad & \frac{2}{4n-1}\left(\sec\frac{\pi}{4n-1}-1\right)+\ln\left(\cos\frac{\pi}{4n-1}\right)+\sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}+\frac{\left(\frac{1}{4n-1}\left(2\sec\frac{\pi}{8n-2}-1\right)+\frac{\pi}{4n-1}\right)^2}{2\sum\limits_{k=1}^{2n-1}\frac{1}{k}} \\
< \quad & \frac{2}{4n-1}\left(\frac{3}{5}\right)\left(\frac{\pi}{4n-1}\right)^2-\frac{1}{2}\left(\frac{\pi}{4n-1}\right)^2+\sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}+\frac{\left(\frac{1}{4n-1}\left(1+\frac{6}{5}\left(\frac{\pi}{4n-1}\right)^2\right)+\frac{\pi}{4n-1}\right)^2}{2\sum\limits_{k=1}^{2n-1}\frac{1}{k}} \\
< \quad & \left(\frac{6}{5(4n-1)}-\frac{1}{2}\right)\frac{\pi^2}{(4n-1)^2}+\sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}+\frac{1}{(4n-1)^2\cdot 2\sum\limits_{k=1}^{9}\frac{1}{k}}\left(1+\frac{6\pi^2}{5(4n-1)^2}+\pi\right) \\
\leqslant \quad & \left(\frac{6}{5(19)}-\frac{1}{2}\right)\frac{\pi^2}{(4n-1)^2}+\frac{1}{(4n-1)^2\cdot 2\sum\limits_{k=1}^{9}\frac{1}{k}}\left(1+\frac{6\pi^2}{5(19^2)}+\pi\right)+\sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}
\end{aligned}
$$

$$
\begin{aligned}
\leqslant \quad & \frac{1}{(4n-1)^2}\left(-\frac{83}{190}\pi^2+\frac{\left(\pi+1+\frac{6}{1805}\pi^2\right)^2}{2\sum\limits_{k=1}^{9}\frac{1}{k}}\right)+\sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k} \\
< \quad & \sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}
\end{aligned}
$$

because

$$
-\frac{83}{190}\pi^2+\frac{\left(\pi+1+\frac{6\pi^2}{1805}\right)^2}{2\sum\limits_{k=1}^{9}\frac{1}{k}}=-1.231\ldots<0.
$$

The proof is complete. □

5 Some estimates involving the interval $\left[\frac{(4n-3)\pi}{4n-1},\pi\right]$.

The main goal of this section is to show that the function

$$
L_{2n-1}:\theta\mapsto -C_{2n-1}(\theta)+\frac{\left(C_{2n-1}(\theta)\right)^2+\left(S_{2n-1}(\theta)\right)^2}{2\sum\limits_{k=1}^{2n-1}\frac{1}{k}}
$$

is strictly increasing on the interval $\left[\frac{4n-3}{4n-1}\pi,\pi\right]$. To do so, we need a few known results.

Lemma 5.1 (cf. [4, Lemma 3.5])**.** If $n \in \mathbb{N}$ and $\theta \in \left[\frac{\pi}{2}, \pi\right]$, then

$$S_{2n-1}(\theta) \leqslant \sin\theta.$$

Lemma 5.2. Let $n \in \mathbb{N}$. If $\theta \in \left[\frac{\pi}{2}, \pi\right]$, then

$$C_{2n-1}(\theta) \leqslant 0.$$

Proof. When $n = 1$, we have $C_1(\theta) = \cos\theta \leqslant 0$ whenever $\theta \in \left[\frac{\pi}{2}, \pi\right]$.

When $n \geqslant 2$, we follow the proof of Lemma 4.2 to obtain

$$\begin{aligned}
C_{2n-1}(\theta) + \ln\left(\sin\frac{\theta}{2}\right) &= C_{2n-1}(\pi) + \ln\left(\sin\frac{\pi}{2}\right) - \int_\theta^\pi \frac{\cos\left(2n-\frac{1}{2}\right)x}{2\sin\frac{x}{2}}\, dx \\
&= C_{2n-1}(\pi) + \frac{\csc\frac{\theta}{2}}{4n-1}\sin\left(2n-\frac{1}{2}\right)\theta + \frac{1}{4n-1} - \int_\theta^\pi \frac{\sin\left(2n-\frac{1}{2}\right)x}{2(4n-1)}\frac{\cos\frac{x}{2}}{\sin^2\frac{x}{2}}\, dx \\
&< C_{2n-1}(\pi) + \frac{\csc\frac{\theta}{2}}{4n-1} + \frac{1}{4n-1} + \frac{1}{2(4n-1)}\int_\theta^\pi \frac{\cos\frac{x}{2}}{\sin^2\frac{x}{2}}\, dx \\
&= C_{2n-1}(\pi) + \frac{2}{4n-1}\csc\frac{\theta}{2}.
\end{aligned}$$

Since the function $\theta \mapsto \csc\frac{\theta}{2}$ is decreasing on the interval $\left[\frac{\pi}{2}, \pi\right]$, it is sufficient to show that

$$C_{2n-1}(\pi) + \ln\sqrt{2} + \frac{2\sqrt{2}}{4n-1} < 0 \text{ for } n = 2, 3, \ldots. \tag{5.1}$$

When $n = 2$, a direct computation yields

$$C_3(\pi) + \ln\sqrt{2} + \frac{2\sqrt{2}}{4(2)-1} = -0.0826\ldots < 0.$$

Since a standard argument reveals that the sequence $\left(C_{2n-1}(\pi)\right)_{n=3}^\infty$ is increasing, we conclude that if $n \geqslant 3$, then

$$\begin{aligned}
C_{2n-1}(\pi) + \ln\sqrt{2} + \frac{2\sqrt{2}}{4n-1} &< \lim_{n\to\infty} C_{2n-1}(\pi) + \ln\sqrt{2} + \frac{2\sqrt{2}}{11} \\
&= -0.089\ldots < 0.
\end{aligned}$$

Therefore (5.1) holds. The proof is complete. ☐

Lemma 5.3. If $\theta \in \left(\frac{4n-3}{4n-1}\pi, \pi\right)$, then $\sin n\theta \sin\left(n-\frac{1}{2}\right)\theta > 0$.

Proof. We have

$$\sin n\theta \sin\left(n-\frac{1}{2}\right)\theta = \frac{\cos\frac{\theta}{2} - \cos\left(2n-\frac{1}{2}\right)\theta}{2}.$$

Since $\left(2n-\frac{1}{2}\right)\theta \in \left(\left(2n-\frac{3}{2}\right)\pi, \left(2n-\frac{1}{2}\right)\pi\right)$, we conclude that the function $\theta \mapsto \cos\left(2n-\frac{1}{2}\right)\theta$ is negative on the interval $\left(\frac{4n-3}{4n-1}\pi, \pi\right)$ and the lemma follows. ☐

Lemma 5.4. Let $n \in \mathbb{N}$. If $\theta \in \left(\frac{4n-3}{4n-1}\pi, \pi\right)$, then

$$\frac{\sin n\theta \sin\left(n-\frac{1}{2}\right)\theta}{\sin\frac{\theta}{2}\sum_{k=1}^{2n-1}\frac{1}{k}}\left\{\sum_{k=1}^{2n-1}\frac{1-\cos k\theta}{k} + \cot n\theta \sum_{k=1}^{2n-1}\frac{\sin k\theta}{k}\right\} > 0.$$

Proof. In view of Lemma 5.3, it suffices to prove that

$$\sum_{k=1}^{2n-1} \frac{1-\cos k\theta}{k} + \cot n\theta \sum_{k=1}^{2n-1} \frac{\sin k\theta}{k} > 0.$$

We consider 2 cases.

Case 1: $\theta \in \left(\frac{4n-3}{4n-1}\pi, \pi - \frac{\pi}{2n}\right]$.

Since

$$\sum_{k=1}^{2n-1} \frac{1-\cos k\theta}{k} > 0$$

and we have the celebrated Fejér-Jackson inequality $\sum\limits_{k=1}^{n} \frac{\sin k\theta}{k} > 0$ (see, for example, [7]), it is sufficient to check that $\cot n\theta \geqslant 0$. Indeed, we have $n\theta \in \left((n-1)\pi, n\pi - \frac{\pi}{2}\right]$ and so $\cot n\theta \geqslant 0$.

Case 2: $\theta \in \left[\pi - \frac{\pi}{2n}, \pi\right)$.

We have $n\theta \in \left[n\pi - \frac{\pi}{2}, n\pi\right)$ and so $\cot n\theta \leqslant 0$. In view of Lemmas 5.1 and 5.2, it remains to check that

$$1 + \cot n\theta \sin\theta > 0 \quad (\theta \in \left(\pi - \frac{\pi}{2n}, \pi\right)),$$

which is equivalent to

$$\sin\theta < -\tan n\theta \quad (\theta \in \left(\pi - \frac{\pi}{2n}, \pi\right)). \tag{5.2}$$

Using the substitution $\tau = \pi - \theta$, we see that (5.2) is equivalent to

$$\sin\tau < \tan n\tau \quad (0 < \tau < \frac{\pi}{2n}). \tag{5.3}$$

Finally, since both functions $\tau \mapsto \sec\tau$ and $\tau \mapsto \tan\tau$ are increasing on the open interval $\left(0, \frac{\pi}{2}\right)$, we obtain (5.3):

$$\tan n\tau > \tan\tau > \sin\tau.$$

The proof is complete. □

We are now ready to state and prove the main result of this section.

Theorem 5.5. Let $n \in \mathbb{N}$. Then the function L_{2n-1} is increasing on the closed interval $\left[\frac{(4n-3)\pi}{4n-1}, \pi\right]$.

Proof. For each $\theta \in \left(\frac{(4n-3)\pi}{4n-1}, \pi\right)$ we have

$$\frac{d}{d\theta}\left\{\frac{\left(S_{2n-1}(\theta)\right)^2}{2\sum\limits_{k=1}^{2n-1}\frac{1}{k}}\right\} = \frac{\cos n\theta \sin\left(n-\frac{1}{2}\right)\theta}{\sin\frac{\theta}{2}\sum\limits_{k=1}^{2n-1}\frac{1}{k}}\left\{\sum_{k=1}^{2n-1}\frac{\sin k\theta}{k}\right\}$$

and

$$\begin{aligned}\frac{d}{d\theta}\left\{-C_{2n-1}(\theta) + \frac{\left(C_{2n-1}(\theta)\right)^2}{2\sum\limits_{k=1}^{2n-1}\frac{1}{k}}\right\} &= \frac{\sin n\theta \sin\left(n-\frac{1}{2}\right)\theta}{\sin\frac{\theta}{2}}\left\{1 - \frac{C_{2n-1}(\theta)}{\sum\limits_{k=1}^{2n-1}\frac{1}{k}}\right\} \\ &= \frac{\sin n\theta \sin\left(n-\frac{1}{2}\right)\theta}{\sin\frac{\theta}{2}\sum\limits_{k=1}^{2n-1}\frac{1}{k}}\left\{\sum_{k=1}^{2n-1}\frac{1-\cos k\theta}{k}\right\}; \end{aligned} \tag{5.4}$$

that is

$$\frac{d}{d\theta}\left\{-C_{2n-1}(\theta)+\frac{\left(S_{2n-1}(\theta)\right)^2+\left(C_{2n-1}(\theta)\right)^2}{2\sum\limits_{k=1}^{2n-1}\frac{1}{k}}\right\}=\frac{\sin n\theta\sin\left(n-\frac{1}{2}\right)\theta}{\sin\frac{\theta}{2}\sum\limits_{k=1}^{2n-1}\frac{1}{k}}\left\{\sum_{k=1}^{2n-1}\frac{1-\cos k\theta}{k}+\cot n\theta\sum_{k=1}^{2n-1}\frac{\sin k\theta}{k}\right\}.$$

Finally, an application of Lemma 5.4 yields the desired result. □

6 Proof of Theorem 1.1

Let $n\in\{1,2,3,4\}$. If $\theta\in\left[0,\frac{4n-3}{4n-1}\pi\right]$, then we apply Theorem 2.3 to obtain (2.2). On the other hand, we invoke Theorem 5.5 to show that (2.2) is valid whenever $\theta\in\left[\frac{4n-3}{4n-1}\pi,\pi\right]$.

Next, we consider $n\geq 5$. There are 3 cases to consider.

Case 1: $\theta\in\left[0,\frac{2\pi}{3}\right]$.

In this case, (1.4) is a consequence of Theorem 3.7.

Case 2: $\theta\in\left[\frac{2\pi}{3},\frac{4n-3}{4n-1}\pi\right]$.

In this case, (2.2) follows from Theorem 4.4 and the following corollary of Theorem 3.4:

$$\sum_{k=1}^{2n-1}\frac{\cos k\theta}{k}<\sum_{k=1}^{2n-1}\frac{(-1)^k}{k}=C_{2n-1}(\pi).$$

Case 3: $\theta\in\left[\frac{4n-3}{4n-1}\pi,\pi\right]$.

In the third case, an application of Theorem 5.5 yields (2.2).

The proof is complete.

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