Brannan's conjecture and trigonometric polynomials

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Abstract

For any integer $n \geq 1$, $\frac{1}{2} < r \leq 1$ and $0 \leq \theta < \pi$, we prove that:

$$\left(\sum_{k=1}^{2n-1} \frac{r^k}{k}(1-\cos k\theta)\right)^2 + \left(\sum_{k=1}^{2n-1} \frac{r^k}{k} \sin k\theta\right)^2 < 4\left(\sum_{k=1}^{n} \frac{r^{2k-1}}{2k-1}\right)^2,$$

which provides an affirmative answer to a related conjecture of Brannan's conjecture.

Keywords: Brannan's conjecture, trigonometric polynomials.

Introduction

Let

$$\frac{(1+zx)^{\alpha}}{(1-x)^{\beta}} = \sum_{n=0}^{\infty} A_n(\alpha,\beta,z)x^n,$$

where $\alpha, \beta > 0$ and $z = e^{i\theta}(\theta \in [0, 2\pi])$. In 1973, D.A. Brannan [2] conjectured that

$$|A_n(\alpha,\beta,z)| \leq A_n(\alpha,\beta,1) \tag{1}$$

for all $\alpha, \beta > 0$, all $z \in \mathbb{C}$ such that $|z| = 1$, and all odd integers n. Here, A_n refers to the coefficient of the n^{th} order term in the polynomial.

While the conjecture was proven for all $\alpha \geqslant 1, \beta > 1$ by D. Aharonov and S. Friedland [1], the case $0 < \alpha \leqslant 1$ and $0 < \beta \leqslant 1$ proved to be rather difficult. A recent attempt to prove the conjecture for $0 < \alpha < 1$ and $\beta = 1$ by R.W. Barnard et al. [3] reformulated inequality (1) into finding the largest r that satisfies

$$|A_n(\alpha,\beta,z)| \leqslant A_n(\alpha,\beta,r), \tag{2}$$

where z is generalised to $z = re^{i\theta}$ and A_n is treated as an analytic function. In the paper [3], the authors proved (2) holds for $0 < r \leqslant 1/2$ when n is odd. For the case where $1/2 < r \leqslant 1$ and n is odd, they showed (2) is equivalent to the following conjecture:

Conjecture 1. *For any integer $n \geq 1$, $\frac{1}{2} < r \leq 1$ and $0 \leq \theta < \pi$, the following inequality holds:*

$$\left(\sum_{k=1}^{2n-1} \frac{r^k}{k}(1-\cos k\theta)\right)^2 + \left(\sum_{k=1}^{2n-1} \frac{r^k}{k}\sin k\theta\right)^2 < 4\left(\sum_{k=1}^{n} \frac{r^{2k-1}}{2k-1}\right)^2. \tag{3}$$

In this project, we give an affirmative answer to the above conjecture.

After some algebra, we see that inequality (3) is equivalent to

$$\begin{aligned} &-\sum_{k=1}^{2n-1} \frac{r^k}{k}\cos k\theta + \frac{\left(\sum_{k=1}^{2n-1} \frac{r^k}{k}\cos k\theta\right)^2 + \left(\sum_{k=1}^{2n-1} \frac{r^k}{k}\sin k\theta\right)^2}{2\sum_{k=1}^{2n-1} \frac{r^k}{k}} \\ &< \sum_{k=1}^{2n-1} \frac{(-1)^{k-1}r^k}{k} + \frac{\left(\sum_{k=1}^{2n-1} \frac{(-1)^{k-1}r^k}{k}\right)^2}{2\sum_{k=1}^{2n-1} \frac{r^k}{k}}. \end{aligned} \tag{4}$$

In Chapter 1, we first address the case $r = 1$, where (4) is equivalent to

$$-\sum_{k=1}^{2n-1} \frac{\cos k\theta}{k} + \frac{\left(\sum_{k=1}^{2n-1} \frac{\cos k\theta}{k}\right)^2 + \left(\sum_{k=1}^{2n-1} \frac{\sin k\theta}{k}\right)^2}{2\sum_{k=1}^{2n-1} \frac{1}{k}} < \sum_{k=1}^{2n-1} \frac{(-1)^{k-1}}{k} + \frac{\left(\sum_{k=1}^{2n-1} \frac{(-1)^{k-1}}{k}\right)^2}{2\sum_{k=1}^{2n-1} \frac{1}{k}} \tag{5}$$

We show that inequality (5) is true $\theta \in [0, \frac{4n-3}{4n-1}\pi]$ where $n = 1, 2, 3$ and 4. Next, we use some results of Fong et al. [7] and Kim et al. [10] to establish

$$-\sum_{k=1}^{2n-1} \frac{\cos k\theta}{k} + \frac{\left(\sum_{k=1}^{2n-1} \frac{\sin k\theta}{k}\right)^2}{2\sum_{k=1}^{2n-1} \frac{1}{k}} < \sum_{k=1}^{2n-1} \frac{(-1)^{k-1}}{k} \tag{6}$$

for $n = 5, 6, \ldots$ and $0 \leqslant \theta < \frac{2\pi}{3}$. Then, we use integration by parts to derive some crucial estimates for $\sum_{k=1}^{2n-1} \frac{\cos k\theta}{k}$ and $\sum_{k=1}^{2n-1} \frac{\sin k\theta}{k}$ so that the left-hand side of inequality (6) is bounded above by an increasing functional upper bound on the interval $[\frac{2\pi}{3}, \frac{4n-3}{4n-1}\pi]$. Lastly, we prove that the left hand side of inequality (6) is an increasing function of θ on the interval $[\frac{4n-3}{4n-1}\pi, \pi]$.

As the above proof for the case $r = 1$ is insufficiently general to handle the case, $\frac{1}{2} < r < 1$, we first use some recent results of Fong et al. [8] and Kim et al. [10] to prove inequality (4) for $\theta \in [0, \frac{\pi}{3}]$ by establishing an important bound involving $\ln(1-r)$ and $\ln(1+r)$. Then, using [3], [8] and summation by parts, we establish in Lemma 2.8.1 that

$$\left|\sum_{k=1}^{2n-1} \frac{r^k}{k}\cos k\theta\right| \leqslant \sum_{k=1}^{2n-1} \frac{(-1)^{k-1}r^k}{k} \quad \text{for } \theta \in \left[\frac{\pi}{3}, \pi\right); \tag{7}$$

this allows for the following sufficient condition for (4):

$$-\sum_{k=1}^{2n-1}\frac{r^k}{k}\cos k\theta+\frac{\left(\sum_{k=1}^{2n-1}\frac{r^k}{k}\sin k\theta\right)^2}{2\sum_{k=1}^{2n-1}\frac{r^k}{k}}<\sum_{k=1}^{2n-1}\frac{(-1)^{k-1}r^k}{k}\quad(\frac{\pi}{3}\leqslant\theta<\pi). \tag{8}$$

We thus establish either inequality (4) or (8) for the cases $n=2,3$ and 4, with $\theta\in[\frac{\pi}{4},\pi-\frac{\pi}{2n}]$ via a more direction computational method. Next, we focus on constructing negative increasing functions, which are used to establish inequality (8) for the case $\theta\in[\frac{\pi}{4},\pi-\frac{\pi}{2n}]$, $5\leq n\leq 29$, $\theta\in[\pi-\frac{\pi}{2n},\pi)$, with $2\leq n\leq 29$ and $\theta\in[\frac{\pi}{3},\pi)$ for all $n\geq 30$. To do that, we use a combination of integration by parts and summation by parts to derive some crucial estimates for $\sum_{k=1}^{2n-1}\frac{r^k\cos k\theta}{k}$ and $\sum_{k=1}^{2n-1}\frac{r^k\sin k\theta}{k}$ so that the left-hand side of inequality (8) is bounded above by an increasing functional upper bound on the interval $[\frac{\pi}{3},\pi]$.

Finally in the end of Chapter 2, we combine the main results from Chapter 1 and Chapter 2 to provide an affirmative answer to Conjecture 1.

Chapter 1

The case $r = 1$

In this chapter, we give the following affirmative answer to Conjecture 1 when $r = 1$.

Theorem 1.0.1. If $n \in \mathbb{N}$ and $\theta \in [0, \pi)$, then

$$\left(\sum_{k=1}^{2n-1} \frac{1}{k}(1 - \cos k\theta)\right)^2 + \left(\sum_{k=1}^{2n-1} \frac{\sin k\theta}{k}\right)^2 < 4\left(\sum_{k=1}^{n} \frac{1}{2k-1}\right)^2. \tag{1.1}$$

The proof of Theorem 1.0.1 is given in Section 1.5 of this chapter.

1.1 The cases $n = 1, 2, 3, 4$

For each $n \in \mathbb{N}$ and $\theta \in [0, \pi]$ we set

$$S_n(\theta) := \sum_{k=1}^{n} \frac{\sin k\theta}{k} \quad \text{and} \quad C_n(\theta) := \sum_{k=1}^{n} \frac{\cos k\theta}{k}.$$

Then (1.1) is equivalent to

$$-C_{2n-1}(\theta) + \frac{(C_{2n-1}(\theta))^2 + (S_{2n-1}(\theta))^2}{2\sum\limits_{k=1}^{2n-1} \frac{1}{k}} < \frac{\left(\sum\limits_{k=1}^{n} \frac{2}{2k-1}\right)^2 - \left(\sum\limits_{k=1}^{2n-1} \frac{1}{k}\right)^2}{2\sum\limits_{k=1}^{2n-1} \frac{1}{k}}. \tag{1.2}$$

Moreover, (1.2) is equivalent to the inequality

$$-C_{2n-1}(\theta) + \frac{(C_{2n-1}(\theta))^2 + (S_{2n-1}(\theta))^2}{2\sum\limits_{k=1}^{2n-1} \frac{1}{k}} < -C_{2n-1}(\pi) + \frac{C_{2n-1}^2(\pi)}{2\sum\limits_{k=1}^{2n-1} \frac{1}{k}} \tag{1.3}$$

because

$$\begin{aligned}
\frac{\left(\sum\limits_{k=1}^{n} \frac{2}{2k-1}\right)^2 - \left(\sum\limits_{k=1}^{2n-1} \frac{1}{k}\right)^2}{2\sum\limits_{k=1}^{2n-1} \frac{1}{k}} &= \frac{1}{2\sum\limits_{k=1}^{2n-1} \frac{1}{k}} \left\{\sum_{k=1}^{n} \frac{2}{2k-1} - \sum_{k=1}^{2n-1} \frac{1}{k}\right\} \left\{\sum_{k=1}^{n} \frac{2}{2k-1} + \sum_{k=1}^{2n-1} \frac{1}{k}\right\} \\
&= \frac{1}{2\sum\limits_{k=1}^{2n-1} \frac{1}{k}} \left\{\sum_{k=1}^{2n-1} \frac{(-1)^{k-1}}{k}\right\} \left\{\sum_{k=1}^{2n-1} \frac{(-1)^{k-1}}{k} + 2\sum_{k=1}^{2n-1} \frac{1}{k}\right\} \\
&= \sum_{k=1}^{2n-1} \frac{(-1)^{k-1}}{k} + \frac{1}{2\sum\limits_{k=1}^{2n-1} \frac{1}{k}} \left(\sum_{k=1}^{2n-1} \frac{(-1)^{k-1}}{k}\right)^2.
\end{aligned}$$

Next, we use some known lemmas concerning trigonometric polynomials.

Lemma 1.1.1 (cf. [9, equation 6])**.** Let $n \in \mathbb{N}$ and let $\theta \in (0, \pi)$. Then

$$\frac{d}{d\theta}\left(S_{2n-1}(\theta)\right) = \frac{\cos n\theta \sin\left(n - \frac{1}{2}\right)\theta}{\sin \frac{\theta}{2}}.$$

Lemma 1.1.2 (cf. [11, equation 2])**.** Let $n \in \mathbb{N}$ and let $\theta \in (0, \pi)$. Then

$$\frac{d}{d\theta}\left(C_{2n-1}(\theta)\right) = -\frac{\sin n\theta \sin\left(n - \frac{1}{2}\right)\theta}{\sin\frac{\theta}{2}}.$$

The following computation involves the result of the above lemmas.

Theorem 1.1.3. Let $n \in \{1, 2, 3, 4\}$. If $\theta \in \left[0, \frac{4n-3}{4n-1}\pi\right]$, then

$$-C_{2n-1}(\theta) + \frac{\left(C_{2n-1}(\theta)\right)^2 + \left(S_{2n-1}(\theta)\right)^2}{2\sum\limits_{k=1}^{2n-1}\frac{1}{k}} < -C_{2n-1}(\pi) + \frac{C_{2n-1}^2(\pi)}{2\sum\limits_{k=1}^{2n-1}\frac{1}{k}}. \tag{1.4}$$

Proof. When $n = 1$, (1.4) holds since

$$\begin{aligned} -C_1(\theta) + \frac{\left(C_1(\theta)\right)^2 + \left(S_1(\theta)\right)^2}{2} &= -\cos\theta + \frac{1}{2} \\ &< -C_1(\pi) + \frac{C_1^2(\pi)}{2}. \end{aligned}$$

Next, we consider the remaining cases $n = 2, 3, 4$. Since a direct computation shows that the function $u \mapsto \frac{u^2}{2\sum_{k=1}^{2n-1}\frac{1}{k}} - u$ is decreasing on the closed interval $\left[\sum\limits_{k=1}^{2n-1}\frac{(-1)^k}{k}, \sum\limits_{k=1}^{2n-1}\frac{1}{k}\right]$, we conclude that

$$\max_{\theta \in I}\left\{\frac{\left(C_{2n-1}(\theta)\right)^2}{2\sum\limits_{k=1}^{2n-1}\frac{1}{k}} - C_{2n-1}(\theta)\right\} = \frac{\left(\max\limits_{\theta\in I}\{-C_{2n-1}(\theta)\}\right)^2}{2\sum\limits_{k=1}^{2n-1}\frac{1}{k}} + \max_{\theta \in I}\{-C_{2n-1}(\theta)\} \tag{1.5}$$

whenever I is a closed subinterval of $[0, \pi]$. Now we are ready to do some computation.

Using Lemma 1.1.1 and Lemma 1.1.2, we obtain

$$\max_{\theta\in[0,\frac{2\pi}{5}]}\{-C_3(\theta)\} = -C_3\left(\frac{2\pi}{5}\right) \text{ and } \max_{\theta\in[0,\frac{2\pi}{5}]}\left\{\frac{\left(S_3(\theta)\right)^2}{2\sum\limits_{k=1}^{3}\frac{1}{k}}\right\} = \frac{\left(S_3\left(\frac{\pi}{4}\right)\right)^2}{2\sum\limits_{k=1}^{3}\frac{1}{k}}$$

respectively. By combining the above absolute maxima and (1.5) with the observation $-C_3(\pi) + \frac{C_3^2(\pi)}{2\sum_{k=1}^{3}\frac{1}{k}} = 1.022\ldots$, we see that

$$\begin{aligned} \max_{\theta\in[0,\frac{2\pi}{5}]}\left\{-C_3(\theta) + \frac{\left(C_3(\theta)\right)^2 + \left(S_3(\theta)\right)^2}{2\sum\limits_{k=1}^{3}\frac{1}{k}}\right\} &\leqslant -C_3\left(\frac{2\pi}{5}\right) + \frac{\left(C_3\left(\frac{2\pi}{5}\right)\right)^2}{2\sum\limits_{k=1}^{3}\frac{1}{k}} + \frac{\left(S_3\left(\frac{\pi}{4}\right)\right)^2}{2\sum\limits_{k=1}^{3}\frac{1}{k}} \\ &= 0.969\ldots. \end{aligned}$$

Similarly,

$$\begin{aligned} \max_{\theta\in[\frac{2\pi}{5},\frac{5\pi}{7}]}\left\{-C_3(\theta) + \frac{\left(C_3(\theta)\right)^2 + \left(S_3(\theta)\right)^2}{2\sum\limits_{k=1}^{3}\frac{1}{k}}\right\} &\leqslant -C_3\left(\frac{\pi}{2}\right) + \frac{\left(C_3\left(\frac{\pi}{2}\right)\right)^2 + \left(S_3\left(\frac{2\pi}{5}\right)\right)^2}{2\sum\limits_{k=1}^{3}\frac{1}{k}} \\ &= 0.868\ldots. \end{aligned}$$

When $n = 3$ and $-C_5(\pi) + \frac{C_5^2(\pi)}{2\sum_{k=1}^{5}\frac{1}{k}} = 0.917\ldots$, we have

$$\max_{\theta\in[0,\frac{\pi}{2}]}\left\{-C_5(\theta) + \frac{\left(C_5(\theta)\right)^2 + \left(S_5(\theta)\right)^2}{2\sum\limits_{k=1}^{5}\frac{1}{k}}\right\} < -C_5\left(\frac{\pi}{2}\right) + \frac{\left(C_5\left(\frac{\pi}{2}\right)\right)^2 + \left(S_5\left(\frac{\pi}{6}\right)\right)^2}{2\sum\limits_{k=1}^{5}\frac{1}{k}} = 0.812\ldots$$

and

$$\max_{\theta\in[\frac{\pi}{2},\frac{9\pi}{11}]}\left\{-C_5(\theta)+\frac{(C_5(\theta))^2+(S_5(\theta))^2}{2\sum\limits_{k=1}^{5}\frac{1}{k}}\right\}<-C_5\left(\frac{2\pi}{3}\right)+\frac{\left(C_5\left(\frac{2\pi}{3}\right)\right)^2+\left(S_5\left(\frac{\pi}{2}\right)\right)^2}{2\sum\limits_{k=1}^{5}\frac{1}{k}}=0.896\ldots.$$

When $n=4$ and $-C_7(\pi)+\dfrac{C_7^2(\pi)}{2\sum\limits_{k=1}^{7}\frac{1}{k}}=0.870\ldots$, we have

$$\max_{\theta\in[0,\frac{3\pi}{8}]}\left\{-C_7(\theta)+\frac{(C_7(\theta))^2+(S_7(\theta))^2}{2\sum\limits_{k=1}^{7}\frac{1}{k}}\right\}<-C_7\left(\frac{3\pi}{8}\right)+\frac{\left(C_7\left(\frac{3\pi}{8}\right)\right)^2+\left(S_7\left(\frac{\pi}{8}\right)\right)^2}{2\sum\limits_{k=1}^{7}\frac{1}{k}}=0.557\ldots,$$

$$\max_{\theta\in[\frac{3\pi}{8},\frac{5\pi}{8}]}\left\{-C_7(\theta)+\frac{(C_7(\theta))^2+(S_7(\theta))^2}{2\sum\limits_{k=1}^{7}\frac{1}{k}}\right\}<-C_7\left(\frac{5\pi}{8}\right)+\frac{\left(C_7\left(\frac{5\pi}{8}\right)\right)^2+\left(S_7\left(\frac{3\pi}{8}\right)\right)^2}{2\sum\limits_{k=1}^{7}\frac{1}{k}}=0.700\ldots$$

and

$$\max_{\theta\in[\frac{5\pi}{8},\frac{13\pi}{15}]}\left\{-C_7(\theta)+\frac{(C_7(\theta))^2+(S_7(\theta))^2}{2\sum\limits_{k=1}^{7}\frac{1}{k}}\right\}<-C_7\left(\frac{3\pi}{4}\right)+\frac{\left(C_7\left(\frac{3\pi}{4}\right)\right)^2+\left(S_7\left(\frac{5\pi}{8}\right)\right)^2}{2\sum\limits_{k=1}^{7}\frac{1}{k}}=0.847\ldots.$$

The proof is complete. □

1.2 The case $n \geqslant 5$ and $\theta \in \left[0, \frac{2\pi}{3}\right]$

We begin with the following inequality involving $\left[0, \frac{\pi}{2}\right]$.

Lemma 1.2.1. If $\theta \in \left[0, \frac{\pi}{2}\right]$, then

$$-(1+\cos\theta)^2\sum_{k=1}^{5}\frac{1}{2k-1}+\frac{1}{16}(1+\cos\theta)^4+\left(\frac{\pi-\theta}{2}+\frac{3}{10}\right)^2<0.$$

Proof. In view of the following observations

$$\begin{aligned}-u\sum_{k=1}^{5}\frac{1}{2k-1}+\frac{u^2}{16} &= -\frac{563}{315}u+\frac{u^2}{16}\\ &= \frac{u}{16}(u-4)-\frac{1937}{1260}u\end{aligned}$$

and $\frac{1937}{1260}>\frac{36}{25}$, it suffices to show that

$$\frac{\pi-\theta}{2}+\frac{3}{10}<\frac{6}{5}(1+\cos\theta)\ \text{ for } \theta\in\left[0,\frac{\pi}{2}\right]. \tag{1.6}$$

Let us now consider

$$h(\theta)=\frac{\pi-\theta}{2}+\frac{3}{10}-\frac{6}{5}(1+\cos\theta)\ \text{ for } \theta\in\left[0,\frac{\pi}{2}\right].$$

Since $\theta_0=\sin^{-1}\left(\frac{5}{12}\right)$ is the only zero of h' and

$$\max_{\theta\in\left[0,\frac{\pi}{2}\right]}h(\theta)=\max\left\{h(0),h(\theta_0),h\left(\frac{\pi}{2}\right)\right\}<-0.546<0,$$

(1.6) follows and the proof is complete. □

In order to establish a similar lemma involving the interval $\left[\frac{\pi}{2},\frac{2\pi}{3}\right]$, we need the following result.

Lemma 1.2.2. If $x \in \left[-\frac{1}{2}, 0\right]$, then

$$-\frac{2929}{1260} + \frac{6079}{1260}x + \frac{8063}{1260}x^2 - \frac{9}{4}x^3 - x^4 < 0.$$

Proof. We let

$$f(x) = -\frac{2929}{1260} + \frac{6079}{1260}x + \frac{8063}{1260}x^2 - \frac{9}{4}x^3 - x^4.$$

After differentiating, we get

$$f'(x) = \frac{6079}{1260} + \frac{8063}{630}x - \frac{27}{4}x^2 - 4x^3$$

and

$$\begin{aligned} f''(x) &= \frac{8063}{630} - \frac{27}{2}x - 12x^2 \\ &= \frac{8063}{630} + \frac{243}{64} - 12\left(x + \frac{27}{48}\right)^2 \quad \text{for } x \in \left(-\frac{1}{2}, 0\right). \end{aligned}$$

Since $-\frac{27}{48} < -\frac{1}{2}$ and $\lim_{x \to 0^-} f''(x) = \frac{8063}{630}$, we conclude that

$$f \text{ is strictly convex on } \left(-\frac{1}{2}, 0\right).$$

Hence, for $x \in [-\frac{1}{2}, 0]$, we have

$$\begin{aligned} f(x) &\leqslant f\left(-\frac{1}{2}\right) + 2\left(f(0) - f\left(-\frac{1}{2}\right)\right)\left(x + \frac{1}{2}\right) \\ &\leqslant f(0) < 0. \end{aligned}$$

□

Lemma 1.2.3. If $\theta \in \left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$, then

$$-(1+\cos\theta)^2 \sum_{k=1}^{5} \frac{1}{2k-1} + \frac{1}{16}(1+\cos\theta)^4 + \left(\frac{\pi - \theta}{2} + \frac{1}{8}\right)^2 < 0.$$

Proof. For each $\theta \in \left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$ we set

$$g(\theta) = -(1+\cos\theta)^2 \sum_{k=1}^{5} \frac{1}{2k-1} + \frac{1}{16}(1+\cos\theta)^4 + \left(\frac{\pi - \theta}{2} + \frac{1}{8}\right)^2.$$

Since the substitution $x = \cos\theta$ yields

$$\begin{aligned} g''(\theta) &= \left(\frac{3\sin^2\theta}{4} + \frac{811}{315}\right)\cos^2\theta + \left(\frac{9\sin^2\theta}{4} + \frac{811}{315}\right)\cos\theta - \frac{\sin^4\theta}{4} - \frac{811\sin^2\theta}{315} + \frac{1}{2} \\ &= -x^4 - \frac{9}{4}x^3 + \frac{8063}{1260}x^2 + \frac{6079}{1260}x - \frac{2929}{1260}, \end{aligned}$$

an application of Lemma 1.2.2 shows that g' is decreasing on $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$. Hence $\lim_{\theta \to \frac{\pi}{2}^+} g'(\theta) = -1.024\cdots < 0$, $g\left(\frac{\pi}{2}\right) < 0$ and the continuity of g on $\left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$ yield the desired conclusion. □

In order to proceed further, we need some recent results established by Fong et al. [7] and Kim et al. [10].

Theorem 1.2.4 (cf. [7, Theorem 1.3])**.** Let $n \in \mathbb{N}$ and let $\theta \in (0, \pi)$. Then

$$\sum_{k=1}^{2\lfloor \frac{n}{2} \rfloor + 1} \frac{(-1)^{k-1}}{k} + \sum_{k=1}^{n} \frac{\cos k\theta}{k} \geqslant \frac{1}{4}(1+\cos\theta)^2, \tag{1.7}$$

where equality holds if and only if $n = 2$ and $\theta = \pi - \cos^{-1}\frac{1}{3}$.

Lemma 1.2.5 (cf. [10, Lemma 2.2])**.** Let $n \in \mathbb{N}$. If $q \in \{1, 2, ..., \lfloor \frac{n+1}{2} \rfloor\}$, then

$$\max_{\theta \in \left[\frac{(4q-2)\pi}{2n+1}, \pi\right]} \left\{ \sum_{k=1}^{n} \frac{\sin k\theta}{k} - \frac{\pi - \theta}{2} \right\} = \sum_{k=1}^{n} \frac{\sin k \frac{(4q-2)\pi}{2n+1}}{k} - \frac{\pi - \frac{(4q-2)\pi}{2n+1}}{2}. \tag{1.8}$$

Theorem 1.2.6 (cf. [10, Theorem 2.5])**.** Let $n \in \mathbb{N}$. If $p \in \mathbb{N}$, then the sequence

$$\left\{ (-1)^{p-1} \left(\sum_{k=1}^{n} \frac{\sin k \frac{2p\pi}{2n+1}}{k} - \frac{\pi - \frac{2p\pi}{2n+1}}{2} \right) \right\}_{n=p}^{\infty} \tag{1.9}$$

is decreasing.

We are now ready to state and prove the main result of this section.

Theorem 1.2.7. If $n \geqslant 5$, $n \in \mathbb{N}$ and $\theta \in \left[0, \frac{2\pi}{3}\right]$, then

$$\left(\sum_{k=1}^{2n-1} \frac{1}{k}(1 - \cos k\theta) \right)^2 + \left(\sum_{k=1}^{2n-1} \frac{\sin k\theta}{k} \right)^2 < 4 \left(\sum_{k=1}^{n} \frac{1}{2k-1} \right)^2.$$

Proof. According to Theorem 1.2.4,

$$\sum_{k=1}^{2n-1} \frac{1 - \cos k\theta}{k} \leqslant \sum_{k=1}^{n} \frac{2}{2k-1} - \frac{1}{4} (1 + \cos\theta)^2.$$

Thus, it is sufficient to show that

$$-(1 + \cos\theta)^2 \sum_{k=1}^{5} \frac{1}{2k-1} + \frac{1}{16}(1 + \cos\theta)^4 + \left(\sum_{k=1}^{2n-1} \frac{\sin k\theta}{k} \right)^2 < 0. \tag{1.10}$$

Next, we infer from Lemma 1.2.5 and Theorem 1.2.6 that

$$\max_{\theta \in \left[0, \frac{\pi}{2}\right]} \left\{ S_{2n-1}(\theta) - \frac{\pi - \theta}{2} \right\} \leqslant S_9\left(\frac{2\pi}{19}\right) - \frac{\pi - \frac{6\pi}{19}}{2} = 0.282\ldots < \frac{3}{10}. \tag{1.11}$$

Hence, (1.13) and the Fejér-Jackson inequality $\sum\limits_{k=1}^{n} \frac{\sin k\theta}{k} > 0$ (see, for example, [10]) yields

$$S_{2n-1}^2(\theta) < \left(\frac{\pi - \theta}{2} + \frac{3}{10} \right)^2. \tag{1.12}$$

Finally, we combine (1.12) and Lemma 1.2.1 to establish (1.12) for the case $\theta \in \left[0, \frac{\pi}{2}\right]$. A similar reasoning yields (1.12) for the case $\theta \in \left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$. □

1.3 Further estimates involving $n \geqslant 5$

The main aim of this section is to show that (1.5) holds if $\theta \in \left[\frac{2\pi}{3}, \frac{(4n-3)\pi}{4n-1}\right]$ and $n = 5, 6, 7, \ldots$.

Lemma 1.3.1. Let $n \in \mathbb{N}$ and let $\theta \in (0, \pi)$. Then

$$S_{2n-1}(\theta) - \frac{\pi - \theta}{2} < \frac{1}{4n-1} \left(2 \csc \frac{\theta}{2} - 1 \right).$$

Proof. First we obtain a simplified formula for $S_{2n-1}'(\theta) + \frac{1}{2}$:

$$S_{2n-1}'(\theta) + \frac{1}{2} \;=\; \frac{\sin{(2n - \frac{1}{2})\theta} - \sin\frac{\theta}{2}}{2 \sin\frac{\theta}{2}} + \frac{1}{2} \;=\; \frac{\sin{(2n - \frac{1}{2})\theta}}{2 \sin\frac{\theta}{2}}.$$

Hence an integration by parts yields

$$S_{2n-1}(\theta) - \frac{\pi - \theta}{2} \;=\; S_{2n-1}(\pi) - \frac{\pi - \pi}{2} - \int_{\theta}^{\pi} \frac{\sin{(2n - \frac{1}{2})x}}{2 \sin\frac{x}{2}} \, dx$$

$$
\begin{aligned}
&= -\frac{\csc\frac{\theta}{2}}{4n-1}\cos\left(2n-\frac{1}{2}\right)\theta+\int_\theta^\pi \frac{\cos\left(2n-\frac{1}{2}\right)x}{2(4n-1)}\frac{\cos\frac{x}{2}}{\sin^2\frac{x}{2}}\,dx\\
&< \frac{\csc\frac{\theta}{2}}{4n-1}+\frac{1}{2(4n-1)}\int_\theta^\pi \frac{\cos\frac{x}{2}}{\sin^2\frac{x}{2}}\,dx\\
&= \frac{1}{4n-1}\left(2\csc\frac{\theta}{2}-1\right).
\end{aligned}
$$

□

Lemma 1.3.2. Let $n\in\mathbb{N}$ and $\theta\in(0,\pi)$. Then

$$C_{2n-1}(\theta)+\ln\left(\sin\frac{\theta}{2}\right) > -\frac{2}{4n-1}\left(\csc\frac{\theta}{2}-1\right)+C_{2n-1}(\pi).$$

Proof. Following the proof of Lemma 1.3.1, we have

$$C'_{2n-1}(\theta)+\frac{\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}} = \frac{\cos\left(2n-\frac{1}{2}\right)\theta-\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}}+\frac{\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}} = \frac{\cos\left(2n-\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}}.$$

Hence an integration by parts yields

$$
\begin{aligned}
C_{2n-1}(\theta)+\ln\left(\sin\frac{\theta}{2}\right) &= C_{2n-1}(\pi)+\ln\left(\sin\frac{\pi}{2}\right)-\int_\theta^\pi \frac{\cos\left(2n-\frac{1}{2}\right)x}{2\sin\frac{x}{2}}\,dx\\
&= C_{2n-1}(\pi)+\frac{\csc\frac{\theta}{2}}{4n-1}\sin\left(2n-\frac{1}{2}\right)\theta+\frac{1}{4n-1}-\int_\theta^\pi \frac{\sin\left(2n-\frac{1}{2}\right)x}{2(4n-1)}\frac{\cos\frac{x}{2}}{\sin^2\frac{x}{2}}\,dx\\
&> C_{2n-1}(\pi)-\frac{\csc\frac{\theta}{2}}{4n-1}+\frac{1}{4n-1}-\frac{1}{2(4n-1)}\int_\theta^\pi \frac{\cos\frac{x}{2}}{\sin^2\frac{x}{2}}\,dx\\
&= C_{2n-1}(\pi)-\frac{2}{4n-1}\left(\csc\frac{\theta}{2}-1\right).
\end{aligned}
$$

□

In order to proceed further, for each $n\in\mathbb{N}$ and $\theta\in(0,\pi)$, we set

$$F_n(\theta)=\frac{1}{4n-1}\left(2\csc\frac{\theta}{2}-1\right)+\frac{\pi-\theta}{2}$$

and

$$G_n(\theta)=-\frac{2}{4n-1}\left(\csc\frac{\theta}{2}-1\right)-\ln\left(\sin\frac{\theta}{2}\right)+C_{2n-1}(\pi).$$

We are now ready to establish the following crucial lemma.

Lemma 1.3.3. Let $n\in\mathbb{N}$ and suppose $n\geqslant 5$. Then both functions $-G_n$ and $F_n^2-2G_n\sum\limits_{k=1}^{2n-1}\frac{1}{k}$ are increasing on $\left[\frac{2\pi}{3},\frac{(4n-3)\pi}{4n-1}\right]$.

Proof. For each $\theta\in\left(\frac{2\pi}{3},\pi\right)$ and $n\geqslant 5$, we obtain

$$
\begin{aligned}
G'_n(\theta) &= \frac{1}{4n-1}\csc\frac{\theta}{2}\cot\frac{\theta}{2}-\frac{1}{2}\cot\frac{\theta}{2}\\
&< \left(\frac{1}{19}\csc\frac{\pi}{3}-\frac{1}{2}\right)\cot\frac{\theta}{2}\\
&< -\frac{5}{12}\cot\frac{\theta}{2}.
\end{aligned}
\tag{1.13}
$$

Since $-G_n$ is continuous on $\left[\frac{2\pi}{3},\pi\right]$, we infer from (1.13) that $-G_n$ is increasing on $\left[\frac{2\pi}{3},\pi\right]$.

For each integer $n\geqslant 5$ and $\theta\in\left(\frac{2\pi}{3},\frac{4n-3}{4n-1}\pi\right)$

$$\frac{d}{d\theta}\left((F_n(\theta))^2-2G_n(\theta)\left(\sum_{k=1}^{2n-1}\frac{1}{k}\right)\right)$$

$$\begin{aligned}
&= 2F_n(\theta)F_n'(\theta) - 2G_n'(\theta)\left(\sum_{k=1}^{2n-1}\frac{1}{k}\right)\\
&> 2\left\{\frac{1}{4n-1}\left(2\csc\frac{\theta}{2}-1\right)+\frac{\pi-\theta}{2}\right\}\left\{-\frac{1}{4n-1}\cot\frac{\theta}{2}\csc\frac{\theta}{2}-\frac{1}{2}\right\}+\frac{5}{6}\left(\cot\frac{\theta}{2}\right)\left(\sum_{k=1}^{2n-1}\frac{1}{k}\right) \text{(by (1.13))}\\
&> 2\left\{\frac{1}{4n-1}\left(2\csc\frac{\pi}{3}-1\right)+\frac{\pi-\theta}{2}\right\}\left\{-\frac{1}{18}\cot\frac{\pi}{3}\csc\frac{\pi}{3}-\frac{1}{2}\right\}+\frac{5}{6}\left(\frac{\pi-\theta}{2}\right)\left(\sum_{k=1}^{9}\frac{1}{k}\right)
\end{aligned}$$

$$\begin{aligned}
&> -\frac{29}{27}\left(\frac{1}{4n-1}\left(\frac{12}{5}-1\right)+\frac{\pi-\theta}{2}\right)+\frac{5}{6}\left(\frac{\pi-\theta}{2}\right)\left(\sum_{k=1}^{9}\frac{1}{k}\right)\\
&> \left(\frac{5}{6}\left(\sum_{k=1}^{9}\frac{1}{k}\right)-\frac{29}{27}\right)\frac{\pi-\theta}{2}-\frac{29}{27}\left(\frac{1}{4n-1}\right)\left(\frac{12}{5}-1\right)\\
&\geqslant \left(\frac{250}{108}-\frac{29}{27}\right)\frac{3}{4n-1}-\frac{29}{27}\left(\frac{1}{4n-1}\right)\left(\frac{12}{5}-1\right)\\
&> \frac{2}{4n-1}\\
&> 0.
\end{aligned}$$

Therefore a standard argument shows that $F_n^2 - 2G_n\sum\limits_{k=1}^{2n-1}\frac{1}{k}$ is increasing on $\left[\frac{2\pi}{3}, \frac{(4n-3)\pi}{4n-1}\right]$. □

The main result of this section is the following theorem.

Theorem 1.3.4. Let n be any integer satisfying $n \geqslant 5$. If $\theta \in \left[\frac{2\pi}{3}, \frac{(4n-3)\pi}{4n-1}\right]$, then

$$-\sum_{k=1}^{2n-1}\frac{\cos k\theta}{k}+\frac{\left(\sum\limits_{k=1}^{2n-1}\frac{\sin k\theta}{k}\right)^2}{2\sum\limits_{k=1}^{2n-1}\frac{1}{k}} < \sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}. \tag{1.14}$$

Proof. By taking into account of Lemma 1.3.3, we need to establish the following inequality

$$-G_n\left(\frac{(4n-3)\pi}{4n-1}\right)+\frac{\left(F_n\left(\frac{(4n-3)\pi}{4n-1}\right)\right)^2}{2\sum\limits_{k=1}^{2n-1}\frac{1}{k}} < \sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}.$$

For each integer $n \geqslant 5$ we have

$$\begin{aligned}
&-G_n\left(\frac{(4n-3)\pi}{4n-1}\right)+\frac{\left(F_n\left(\frac{4n-3}{4n-1}\pi\right)\right)^2}{2\sum_{k=1}^{2n-1}\frac{1}{k}}\\
&= \frac{2}{4n-1}\left(\csc\frac{4n-3}{8n-2}\pi-1\right)+\ln\left(\sin\frac{4n-3}{8n-2}\pi\right)-g_n(\pi)+\frac{\left(\frac{1}{4n-1}\left(2\csc\frac{4n-3}{8n-2}\pi-1\right)+\frac{\pi-\frac{4n-3}{8n-2}\pi}{2}\right)^2}{2\sum\limits_{k=1}^{2n-1}\frac{1}{k}}\\
&= \frac{2}{4n-1}\left(\sec\frac{\pi}{4n-1}-1\right)+\ln\left(\cos\frac{\pi}{4n-1}\right)+\sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}+\frac{\left(\frac{1}{4n-1}\left(2\sec\frac{\pi}{8n-2}-1\right)+\frac{\pi}{4n-1}\right)^2}{2\sum\limits_{k=1}^{2n-1}\frac{1}{k}}\\
&< \frac{2}{4n-1}\left(\frac{3}{5}\right)\left(\frac{\pi}{4n-1}\right)^2-\frac{1}{2}\left(\frac{\pi}{4n-1}\right)^2+\sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}+\frac{\left(\frac{1}{4n-1}\left(1+\frac{6}{5}\left(\frac{\pi}{4n-1}\right)^2\right)+\frac{\pi}{4n-1}\right)^2}{2\sum\limits_{k=1}^{2n-1}\frac{1}{k}}\\
&< \left(\frac{6}{5(4n-1)}-\frac{1}{2}\right)\frac{\pi^2}{(4n-1)^2}+\sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}+\frac{1}{(4n-1)^2\cdot 2\sum\limits_{k=1}^{9}\frac{1}{k}}\left(1+\frac{6\pi^2}{5(4n-1)^2}+\pi\right)
\end{aligned}$$

$$\leqslant \left(\frac{6}{5(19)}-\frac{1}{2}\right)\frac{\pi^2}{(4n-1)^2}+\frac{1}{(4n-1)^2\cdot 2\sum\limits_{k=1}^{9}\frac{1}{k}}\left(1+\frac{6\pi^2}{5(19^2)}+\pi\right)+\sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}$$

$$\leqslant \frac{1}{(4n-1)^2}\left(-\frac{83}{190}\pi^2+\frac{\left(\pi+1+\frac{6}{1805}\pi^2\right)^2}{2\sum\limits_{k=1}^{9}\frac{1}{k}}\right)+\sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}$$

$$< \sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}$$

because

$$-\frac{83}{190}\pi^2+\frac{\left(\pi+1+\frac{6\pi^2}{1805}\right)^2}{2\sum\limits_{k=1}^{9}\frac{1}{k}}=-1.231\ldots<0.$$

The proof is complete. □

1.4 Some estimates involving the interval $\left[\frac{(4n-3)\pi}{4n-1},\pi\right]$

The main goal of this section is to show that the function

$$L_{2n-1}:\theta\mapsto -C_{2n-1}(\theta)+\frac{\left(C_{2n-1}(\theta)\right)^2+\left(S_{2n-1}(\theta)\right)^2}{2\sum\limits_{k=1}^{2n-1}\frac{1}{k}}$$

is strictly increasing on the interval $\left[\frac{4n-3}{4n-1}\pi,\pi\right]$. To do so, we need a few known results.

Lemma 1.4.1 (cf. [4, Lemma 3.5])**.** If $n\in\mathbb{N}$ and $\theta\in\left[\frac{\pi}{2},\pi\right]$, then

$$S_{2n-1}(\theta)\leqslant\sin\theta.$$

Lemma 1.4.2. Let $n\in\mathbb{N}$. If $\theta\in\left[\frac{\pi}{2},\pi\right]$, then

$$C_{2n-1}(\theta)\leqslant 0.$$

Proof. When $n=1$, we have $C_1(\theta)=\cos\theta\leqslant 0$ whenever $\theta\in\left[\frac{\pi}{2},\pi\right]$.

When $n\geqslant 2$, we follow the proof of Lemma 1.3.2 to obtain

$$\begin{aligned}
C_{2n-1}(\theta)+\ln\left(\sin\frac{\theta}{2}\right) &= C_{2n-1}(\pi)+\ln\left(\sin\frac{\pi}{2}\right)-\int_\theta^\pi\frac{\cos\left(2n-\frac{1}{2}\right)x}{2\sin\frac{x}{2}}\,dx\\
&= C_{2n-1}(\pi)+\frac{\csc\frac{\theta}{2}}{4n-1}\sin\left(2n-\frac{1}{2}\right)\theta+\frac{1}{4n-1}-\int_\theta^\pi\frac{\sin\left(2n-\frac{1}{2}\right)x}{2(4n-1)}\frac{\cos\frac{x}{2}}{\sin^2\frac{x}{2}}\,dx\\
&< C_{2n-1}(\pi)+\frac{\csc\frac{\theta}{2}}{4n-1}+\frac{1}{4n-1}+\frac{1}{2(4n-1)}\int_\theta^\pi\frac{\cos\frac{x}{2}}{\sin^2\frac{x}{2}}\,dx\\
&= C_{2n-1}(\pi)+\frac{2}{4n-1}\csc\frac{\theta}{2}.
\end{aligned}$$

Since the function $\theta\mapsto\csc\frac{\theta}{2}$ is decreasing on the interval $\left[\frac{\pi}{2},\pi\right]$, it is sufficient to show that

$$C_{2n-1}(\pi)+\ln\sqrt{2}+\frac{2\sqrt{2}}{4n-1}<0\text{ for }n=2,3,\ldots. \tag{1.15}$$

When $n=2$, a direct computation yields

$$C_3(\pi)+\ln\sqrt{2}+\frac{2\sqrt{2}}{4(2)-1}=-0.0826\ldots<0.$$

Since a standard argument reveals that the sequence $(C_{2n-1}(\pi))_{n=3}^{\infty}$ is increasing, we conclude that if $n \geqslant 3$, then

$$\begin{aligned} C_{2n-1}(\pi) + \ln\sqrt{2} + \frac{2\sqrt{2}}{4n-1} &< \lim_{n\to\infty} C_{2n-1}(\pi) + \ln\sqrt{2} + \frac{2\sqrt{2}}{11} \\ &= -0.089\ldots < 0. \end{aligned}$$

Therefore (1.15) holds. The proof is complete. □

Lemma 1.4.3. If $\theta \in \left(\frac{4n-3}{4n-1}\pi, \pi\right)$, then $\sin n\theta \sin\left(n-\frac{1}{2}\right)\theta > 0$.

Proof. We have

$$\sin n\theta \sin\left(n-\frac{1}{2}\right)\theta = \frac{\cos\frac{\theta}{2} - \cos\left(2n-\frac{1}{2}\right)\theta}{2}.$$

Since $\left(2n-\frac{1}{2}\right)\theta \in \left(\left(2n-\frac{3}{2}\right)\pi, \left(2n-\frac{1}{2}\right)\pi\right)$, we conclude that the function $\theta \mapsto \cos\left(2n-\frac{1}{2}\right)\theta$ is negative on the interval $\left(\frac{4n-3}{4n-1}\pi, \pi\right)$ and the lemma follows. □

Lemma 1.4.4. Let $n \in \mathbb{N}$. If $\theta \in \left(\frac{4n-3}{4n-1}\pi, \pi\right)$, then

$$\frac{\sin n\theta \sin\left(n-\frac{1}{2}\right)\theta}{\sin\frac{\theta}{2}\sum\limits_{k=1}^{2n-1}\frac{1}{k}}\left\{\sum_{k=1}^{2n-1}\frac{1-\cos k\theta}{k} + \cot n\theta \sum_{k=1}^{2n-1}\frac{\sin k\theta}{k}\right\} > 0.$$

Proof. In view of Lemma 1.4.3, it suffices to prove that

$$\sum_{k=1}^{2n-1}\frac{1-\cos k\theta}{k} + \cot n\theta \sum_{k=1}^{2n-1}\frac{\sin k\theta}{k} > 0.$$

We consider 2 cases.

Case 1: $\theta \in \left(\frac{4n-3}{4n-1}\pi, \pi - \frac{\pi}{2n}\right]$.

Since

$$\sum_{k=1}^{2n-1}\frac{1-\cos k\theta}{k} > 0$$

and we have the celebrated Fejér-Jackson inequality $\sum\limits_{k=1}^{n}\frac{\sin k\theta}{k} > 0$ (see, for example, [10]), it is sufficient to check that $\cot n\theta \geqslant 0$. Indeed, we have $n\theta \in \left((n-1)\pi, n\pi - \frac{\pi}{2}\right]$ and so $\cot n\theta \geqslant 0$.

Case 2: $\theta \in \left[\pi - \frac{\pi}{2n}, \pi\right)$.

We have $n\theta \in \left[n\pi - \frac{\pi}{2}, n\pi\right)$ and so $\cot n\theta \leqslant 0$. In view of Lemmas 1.4.1 and 1.4.2, it remains to check that

$$1 + \cot n\theta \sin\theta > 0 \quad (\theta \in \left(\pi - \frac{\pi}{2n}, \pi\right)),$$

which is equivalent to

$$\sin\theta < -\tan n\theta \quad (\theta \in \left(\pi - \frac{\pi}{2n}, \pi\right)). \tag{1.16}$$

Using the substitution $\tau = \pi - \theta$, we see that (1.16) is equivalent to

$$\sin\tau < \tan n\tau \quad (0 < \tau < \frac{\pi}{2n}). \tag{1.17}$$

Finally, since both functions $\tau \mapsto \sec\tau$ and $\tau \mapsto \tan\tau$ are increasing on the open interval $\left(0, \frac{\pi}{2}\right)$, we obtain (1.17):

$$\tan n\tau > \tan\tau > \sin\tau.$$

The proof is complete. □

We are now ready to state and prove the main result of this section.

Theorem 1.4.5. Let $n \in \mathbb{N}$. Then the function L_{2n-1} is increasing on the closed interval $\left[\frac{(4n-3)\pi}{4n-1}, \pi\right]$.

Proof. For each $\theta \in \left(\frac{(4n-3)\pi}{4n-1}, \pi\right)$ we have

$$\frac{d}{d\theta}\left\{\frac{(S_{2n-1}(\theta))^2}{2\sum\limits_{k=1}^{2n-1}\frac{1}{k}}\right\} = \frac{\cos n\theta \sin\left(n-\frac{1}{2}\right)\theta}{\sin\frac{\theta}{2}\sum\limits_{k=1}^{2n-1}\frac{1}{k}}\left\{\sum_{k=1}^{2n-1}\frac{\sin k\theta}{k}\right\}$$

and

$$\begin{aligned}\frac{d}{d\theta}\left\{-C_{2n-1}(\theta)+\frac{(C_{2n-1}(\theta))^2}{2\sum\limits_{k=1}^{2n-1}\frac{1}{k}}\right\} &= \frac{\sin n\theta \sin\left(n-\frac{1}{2}\right)\theta}{\sin\frac{\theta}{2}}\left\{1-\frac{C_{2n-1}(\theta)}{\sum\limits_{k=1}^{2n-1}\frac{1}{k}}\right\} \\ &= \frac{\sin n\theta \sin\left(n-\frac{1}{2}\right)\theta}{\sin\frac{\theta}{2}\sum\limits_{k=1}^{2n-1}\frac{1}{k}}\left\{\sum_{k=1}^{2n-1}\frac{1-\cos k\theta}{k}\right\}; \qquad (1.18)\end{aligned}$$

that is

$$\frac{d}{d\theta}\left\{-C_{2n-1}(\theta)+\frac{(S_{2n-1}(\theta))^2+(C_{2n-1}(\theta))^2}{2\sum\limits_{k=1}^{2n-1}\frac{1}{k}}\right\} = \frac{\sin n\theta \sin\left(n-\frac{1}{2}\right)\theta}{\sin\frac{\theta}{2}\sum\limits_{k=1}^{2n-1}\frac{1}{k}}\left\{\sum_{k=1}^{2n-1}\frac{1-\cos k\theta}{k}+\cot n\theta\sum_{k=1}^{2n-1}\frac{\sin k\theta}{k}\right\}.$$

Finally, an application of Lemma 1.4.4 yields the desired result. □

We are now ready to proof Theorem 1.0.1.

1.5 Proof of Theorem 1.0.1

Let $n \in \{1,2,3,4\}$. If $\theta \in \left[0, \frac{4n-3}{4n-1}\pi\right]$, then we apply Theorem 1.1.3 to obtain (1.3). On the other hand, we invoke Theorem 1.4.5 to show that (1.3) is valid whenever $\theta \in \left[\frac{4n-3}{4n-1}\pi, \pi\right]$.

Next, we consider $n \geq 5$. There are 3 cases to consider.

Case 1: $\theta \in \left[0, \frac{2\pi}{3}\right]$.

In this case, (1.1) is a consequence of Theorem 1.2.7.

Case 2: $\theta \in \left[\frac{2\pi}{3}, \frac{4n-3}{4n-1}\pi\right]$.

In this case, (1.3) follows from Theorem 1.3.4 and the following corollary of Theorem 1.2.4:

$$\sum_{k=1}^{2n-1}\frac{\cos k\theta}{k} < \sum_{k=1}^{2n-1}\frac{(-1)^k}{k} = C_{2n-1}(\pi).$$

Case 3: $\theta \in \left[\frac{4n-3}{4n-1}\pi, \pi\right]$.

In the third case, an application of Theorem 1.4.5 yields (1.3).

The proof is complete.

Chapter 2

The case $\frac{1}{2} < r < 1$

In this chapter, we give the following affirmative answer to Conjecture 1 when $\frac{1}{2} < r < 1$.

Theorem 2.0.1. *If $n \in \mathbb{N}$, $r \in (\frac{1}{2}, 1)$ and $\theta \in [0, \pi)$, then*

$$\left(\sum_{k=1}^{2n-1} \frac{r^k}{k}(1-\cos k\theta)\right)^2 + \left(\sum_{k=1}^{2n-1} \frac{r^k}{k}\sin k\theta\right)^2 < 4\left(\sum_{k=1}^{n} \frac{r^{2k-1}}{2k-1}\right)^2.$$

The proof of Theorem 2.0.1 is included in the end of this chapter. The proof of Conjecture 1 is also included in this chapter.

2.1 Bounds on the interval $[0, \frac{\pi}{3}]$

In this section, we will prove the conjecture on the interval $[0, \frac{\pi}{4}]$ for $n \in \mathbb{N}$ and $[\frac{\pi}{4}, \frac{\pi}{3}]$ for $n \geqslant 3$. The case of $n = 2$ on the interval $[\frac{\pi}{4}, \frac{\pi}{3}]$ will be covered in section 2.2. In this section, we require the following result of Fong et al.

Theorem 2.1.1 (cf. [8, Theorem 1.1])**.** *If $p \in \mathbb{N}$, then the following sequence*

$$\left((-1)^p\left\{\sum_{k=1}^{n} \frac{\cos\frac{(2p-1)k\pi}{2n+1}}{k} + \ln\left(2\sin\frac{(2p-1)\pi}{4n+2}\right)\right\}\right)_{n=p}^{\infty} \tag{2.1}$$

is increasing.

Using Theorem 2.1.1, we derive several inequalities for Young's cosine polynomial $\sum_{k=1}^{n} \frac{\cos k\theta}{k}$.

Lemma 2.1.2. *Let $n \in \mathbb{N}$. Then*

$$\min_{\theta\in[0,\frac{\pi}{6}]} \sum_{k=1}^{n} \frac{\cos k\theta}{k} > 0.4565, \tag{2.2}$$

$$\min_{\theta\in[\frac{\pi}{6},\frac{\pi}{5}]} \sum_{k=1}^{n} \frac{\cos k\theta}{k} > \frac{1}{4}, \tag{2.3}$$

$$\min_{\theta\in[\frac{\pi}{5},\frac{\pi}{4}]} \sum_{k=1}^{n} \frac{\cos k\theta}{k} > 0.065 \tag{2.4}$$

and

$$\min_{\theta\in[\frac{\pi}{4},\frac{\pi}{3}]} \sum_{k=1}^{n} \frac{\cos k\theta}{k} > -0.21. \tag{2.5}$$

Proof. We first establish the inequality (2.2). Using the following identity (cf. [11])

$$\frac{d}{d\theta}\left(\sum_{k=1}^{n} \frac{\cos k\theta}{k}\right) = -\sin\frac{(n+1)\theta}{2}\sin\frac{n\theta}{2}\csc\frac{\theta}{2}, \tag{2.6}$$

we have

$$\min_{\theta\in[0,\frac{\pi}{6}]}\sum_{k=1}^{n}\frac{\cos k\theta}{k}>\frac{9}{20}\quad(n=1,2,...,6). \tag{2.7}$$

Since $\cos t>0$ for $t\in\left(0,\frac{\pi}{2}\right)$ and $\cos\frac{\pi}{3}=\frac{1}{2}$, we conclude that

$$\min_{\theta\in[0,\frac{\pi}{2n+1}]}\sum_{k=1}^{n}\frac{\cos k\theta}{k}>\frac{9}{20}\quad(n=1,2,...) \tag{2.8}$$

Thus, it remains to consider the case $\theta\in\left[\frac{\pi}{2n+1},\pi\right]$, where $n\geqslant 7$.

According to Theorem 2.1.1, the sequence

$$\left(\sum_{k=1}^{n}\frac{\cos\frac{3k\pi}{2n+1}}{k}+\ln\left(2\sin\frac{3\pi}{4n+2}\right)\right)_{n=1}^{\infty}$$

is increasing. Hence, for any integer $n\geqslant 7$, we deduce that

$$\begin{aligned}
&\min_{\theta\in[\frac{\pi}{2n+1},\frac{\pi}{6}]}\sum_{k=1}^{n}\frac{\cos k\theta}{k}\\
\geqslant\;&\min_{\theta\in[\frac{\pi}{2n+1},\pi]}\left\{\sum_{k=1}^{n}\frac{\cos k\theta}{k}+\ln\left(2\sin\frac{\theta}{2}\right)\right\}-\ln\left(2\sin\frac{\pi}{12}\right)\\
=\;&\left\{\sum_{k=1}^{n}\frac{\cos\frac{3k\pi}{2n+1}}{k}+\ln\left(2\sin\frac{3\pi}{4n+2}\right)\right\}-\ln\left(2\sin\frac{\pi}{12}\right)\\
\geqslant\;&\left\{\sum_{k=1}^{7}\frac{\cos\frac{3k\pi}{2(7)+1}}{k}+\ln\left(2\sin\frac{3\pi}{4(7)+2}\right)\right\}-\ln\left(2\sin\frac{\pi}{12}\right)\\
\geqslant\;&-0.2019\ldots-\ln\left(2\sin\frac{\pi}{12}\right)\\
=\;&0.4565\ldots.
\end{aligned} \tag{2.9}$$

Thus, (2.2) follows from (2.7), (2.8) and (2.9).

The proof of (2.3) is similar to that of (2.2). Indeed, since we have

$$\min_{\theta\in\left[\frac{\pi}{6},\frac{\pi}{5}\right]}\sum_{k=1}^{n}\frac{\cos k\theta}{k}>\frac{1}{4}\quad(n=1,2,\ldots,8), \tag{2.10}$$

$\left[\frac{\pi}{6},\frac{\pi}{5}\right]\subset\left[\frac{3\pi}{19},\pi\right]$,

$$\sum_{k=1}^{n}\frac{\cos\frac{3k\pi}{2n+1}}{k}+\ln\left(2\sin\frac{3\pi}{4n+2}\right)\geqslant -0.2005946\ldots\quad(n=9,10,11,\ldots) \tag{2.11}$$

and

$$-\ln\left(2\sin\frac{\pi}{10}\right)-0.2006>\frac{1}{4}, \tag{2.12}$$

(2.3) holds. Finally, we modify the proof of (2.3) to show that (2.4) is a consequence of the following inequalities

$$\min_{\theta\in\left[\frac{\pi}{5},\frac{\pi}{4}\right]}\sum_{k=1}^{n}\frac{\cos k\theta}{k}>0.076\quad(n=1,2,\ldots,6), \tag{2.13}$$

$$\sum_{k=1}^{n}\frac{\cos\frac{3k\pi}{2n+1}}{k}+\ln\left(2\sin\frac{3\pi}{4n+2}\right)\geqslant -0.20193\ldots\quad(n=7,8,9,\ldots) \tag{2.14}$$

and

$$-\ln\left(2\sin\frac{\pi}{8}\right)-0.20193>0.065. \tag{2.15}$$

The proof of (2.5) is similar to that of (2.2), (2.3) and (2.4). This completes the proof. □

In order to establish a sine counterpart of Lemma 2.1.2, we need the following result of Kim et al.

Theorem 2.1.3 (cf. [10, Theorem 2.5])**.** *If $r \in \mathbb{N}$, then the following sequence*

$$\left(\sum_{k=1}^{n} \frac{\sin \frac{(4r-2)k\pi}{2n+1}}{k} - \frac{\pi - \frac{(4r-2)\pi}{2n+1}}{2} \right)_{n=2r-1}^{\infty} \tag{2.16}$$

is decreasing.

Lemma 2.1.4. *Let $n \in \mathbb{N}$. Then*

$$\max_{\theta \in [0, \frac{\pi}{6}]} \sum_{k=1}^{n} \frac{\sin k\theta}{k} < \mathrm{Si}\,(\pi) := \int_0^{\pi} \frac{\sin \tau}{\tau}\, d\tau, \tag{2.17}$$

$$\max_{\theta \in [\frac{\pi}{6}, \frac{\pi}{5}]} \sum_{k=1}^{n} \frac{\sin k\theta}{k} < \frac{8}{5}, \tag{2.18}$$

$$\max_{\theta \in [\frac{\pi}{5}, \frac{\pi}{4}]} \sum_{k=1}^{n} \frac{\sin k\theta}{k} < 1.55 \tag{2.19}$$

and

$$\max_{\theta \in [\frac{\pi}{4}, \frac{\pi}{3}]} \sum_{k=1}^{n} \frac{\sin k\theta}{k} < \frac{13}{9}. \tag{2.20}$$

Proof. The proof of (2.17) can be found in [4]. Since the proofs of inequalities (2.18), (2.19) and (2.20) are similar, we give the proof of (2.20).

Using the following identity (cf. [9])

$$\frac{d}{d\theta} \left(\sum_{k=1}^{n} \frac{\sin k\theta}{k} \right) = \frac{\sin \frac{n\theta}{2} \cos \frac{(n+1)\theta}{2}}{\sin \frac{\theta}{2}},$$

we find that

$$\max_{\theta \in [\frac{\pi}{4}, \frac{\pi}{3}]} \sum_{k=1}^{n} \frac{\sin k\theta}{k} < \frac{13}{9} \quad (n = 1, 2, ..., 11). \tag{2.21}$$

Now we apply Theorem 2.1.3 to deduce that the sequence

$$\left(\sum_{k=1}^{n} \frac{\sin \frac{6k\pi}{2n+1}}{k} - \frac{\pi - \frac{6\pi}{2n+1}}{2} \right)_{n=12}^{\infty}$$

is decreasing; therefore for any integer $n \geqslant 12$, we get

$$\begin{aligned} \max_{\theta \in [\frac{\pi}{4}, \frac{\pi}{3}]} \sum_{k=1}^{n} \frac{\sin k\theta}{k} &\leqslant \max_{\theta \in [\frac{6\pi}{2n+1}, \pi]} \left\{ \sum_{k=1}^{n} \frac{\sin k\theta}{k} - \frac{\pi - \theta}{2} \right\} + \frac{\pi - \frac{\pi}{4}}{2} \\ &= \sum_{k=1}^{n} \frac{\sin \frac{6k\pi}{2n+1}}{k} - \frac{\pi - \frac{6\pi}{2n+1}}{2} + \frac{3\pi}{8} \\ &\leqslant \sum_{k=1}^{12} \frac{\sin \frac{6k\pi}{25}}{k} - \frac{\pi - \frac{6\pi}{25}}{2} + \frac{3\pi}{8} \\ &< \frac{13}{9}. \end{aligned} \tag{2.22}$$

Finally, we combine (2.21) and (2.22) to obtain (2.20). The proof is complete. $\square$

Our next goal is to provide several useful lower bounds for the function $r \mapsto \sum_{k=1}^{2n-1} \frac{(-1)^{k-1} r^k}{k} + \frac{\left(\sum_{k=1}^{2n-1} \frac{(-1)^{k-1} r^k}{k} \right)^2}{2 \sum_{k=1}^{2n-1} \frac{r^k}{k}}$.

Lemma 2.1.5. *Let α and β be real numbers with $\beta > 0$. Then the function $u \mapsto -u + \frac{u^2}{2\beta}$ is decreasing on $[\alpha, \beta]$.*

Proof. The derivative of the above function is

$$-1 + \frac{u}{\beta}$$

which is negative when $\beta > u$. Hence, since the polynomial is continuous on $[\alpha, \beta]$, and $\frac{d}{du}\left(-u + \frac{u^2}{2\beta}\right) < 0$ for $u \in (\alpha, \beta)$, the lemma is proven. $\square$

Lemma 2.1.6. *If $r \in \left(\frac{1}{2}, 1\right)$, then*

$$-0.4565r + \frac{(0.4565r)^2 + (r\mathrm{Si}\,(\pi))^2}{2\sum_{k=1}^{3}\frac{r^k}{k}} < \ln(1+r) - \frac{(\ln(1+r))^2}{2\ln(1-r)}, \tag{2.23}$$

$$-0.25r + \frac{(0.25r)^2 + (1.6r)^2}{2\sum_{k=1}^{3}\frac{r^k}{k}} < \ln(1+r) - \frac{(\ln(1+r))^2}{2\ln(1-r)}, \tag{2.24}$$

$$-0.065r + \frac{(0.065r)^2 + (1.55r)^2}{2\sum_{k=1}^{3}\frac{r^k}{k}} < \ln(1+r) - \frac{(\ln(1+r))^2}{2\ln(1-r)} \tag{2.25}$$

and

$$0.21r + \frac{(-0.21r)^2 + \left(\frac{13r}{9}\right)^2}{2\sum_{k=1}^{5}\frac{r^k}{k}} < \ln(1+r) - \frac{(\ln(1+r))^2}{2\ln(1-r)}. \tag{2.26}$$

Proof. Since the proofs of (2.23), (2.24), (2.25) and (2.26) are similar, we provide the derivation of (2.23). Using differentiation, we obtain

$$-0.4565r + \frac{(0.4565r)^2 + (r\mathrm{Si}\,(\pi))^2}{2\sum_{k=1}^{3}\frac{r^k}{k}} < \frac{8}{7}\ln(1+r) \quad \left(\frac{1}{2} \leqslant r \leqslant \frac{3}{4}\right) \tag{2.27}$$

and

$$-0.4565r + \frac{(0.4565r)^2 + (r\mathrm{Si}\,(\pi))^2}{2\sum_{k=1}^{3}\frac{r^k}{k}} < \ln(1+r) \quad \left(\frac{3}{4} \leqslant r \leqslant 1\right). \tag{2.28}$$

Then the desired inequality (2.23) follows from (2.27), the inequality

$$-\frac{\ln 1.5}{2\ln 0.25} = 0.146\ldots > \frac{1}{7}$$

and (2.28). $\square$

We are now ready to state and prove the main result of this section.

Theorem 2.1.7. *If $n \in \mathbb{N}$, $r \in \left(\frac{1}{2}, 1\right)$ and $\theta \in \left[0, \frac{\pi}{3}\right]$, then*

$$-\sum_{k=1}^{2n-1}\frac{r^k}{k}\cos k\theta + \frac{\left(\sum_{k=1}^{2n-1}\frac{r^k}{k}\cos k\theta\right)^2 + \left(\sum_{k=1}^{2n-1}\frac{r^k}{k}\sin k\theta\right)^2}{2\sum_{k=1}^{2n-1}\frac{r^k}{k}}$$

$$< \sum_{k=1}^{2n-1}\frac{(-1)^{k-1}r^k}{k} + \frac{\left(\sum_{k=1}^{2n-1}\frac{(-1)^{k-1}r^k}{k}\right)^2}{2\sum_{k=1}^{2n-1}\frac{r^k}{k}}.$$

Proof. Using summation by parts and (2.2), we first note that for $\theta \in [0, \frac{\pi}{6}]$,

$$\begin{aligned}
-\sum_{k=1}^{2n-1} \frac{r^k}{k} \cos k\theta &= -\left\{ r^{2n-1} \left(\sum_{k=1}^{2n-1} \frac{\cos k\theta}{k} \right) + \sum_{k=1}^{2n-2} \left(r^k - r^{k+1}\right) \sum_{j=1}^{k} \frac{\cos j\theta}{j} \right\} \\
&\leqslant -\left\{ r^{2n-1} \left(\min_{\theta \in [0, \frac{\pi}{6}]} \sum_{k=1}^{2n-1} \frac{\cos k\theta}{k} \right) + \left(r - r^{2n-1}\right) \left(\min_{\theta \in [0, \frac{\pi}{6}]} \sum_{k=1}^{2n-1} \frac{\cos k\theta}{k} \right) \right\} \\
&< -0.4565r.
\end{aligned}$$

Similarly, using summation by parts and (2.17), we have for $\theta \in [0, \frac{\pi}{6}]$,

$$\sum_{k=1}^{2n-1} \frac{r^k}{k} \sin k\theta < r\mathrm{Si}\,(\pi).$$

Therefore, we infer from Lemma 2.1.5 and (2.23) that for $\theta \in [0, \frac{\pi}{6}]$,

$$\begin{aligned}
&-\sum_{k=1}^{2n-1} \frac{r^k}{k} \cos k\theta + \frac{\left(\sum_{k=1}^{2n-1} \frac{r^k}{k} \cos k\theta\right)^2 + \left(\sum_{k=1}^{2n-1} \frac{r^k}{k} \sin k\theta\right)^2}{2\sum_{k=1}^{2n-1} \frac{r^k}{k}} \\
&< -0.4565r + \frac{(0.4565r)^2 + (r\mathrm{Si}\,(\pi))^2}{2\sum_{k=1}^{3} \frac{r^k}{k}} \\
&< \ln(1+r) - \frac{(\ln(1+r))^2}{2\ln(1-r)}. \qquad (2.29)
\end{aligned}$$

A similar argument applies for the remaining cases, where all inequalities stated allow us to conclude that on $\theta \in I$, where $I \in \{[0, \frac{\pi}{6}], [\frac{\pi}{6}, \frac{\pi}{5}], [\frac{\pi}{5}, \frac{\pi}{4}], [\frac{\pi}{4}, \frac{\pi}{3}]\}$,

$$-\sum_{k=1}^{2n-1} \frac{r^k}{k} \cos k\theta + \frac{\left(\sum_{k=1}^{2n-1} \frac{r^k}{k} \cos k\theta\right)^2 + \left(\sum_{k=1}^{2n-1} \frac{r^k}{k} \sin k\theta\right)^2}{2\sum_{k=1}^{2n-1} \frac{r^k}{k}} < \ln(1+r) - \frac{(\ln(1+r))^2}{2\ln(1-r)}. \qquad (2.30)$$

Since $\sum_{k=1}^{2n-1} \frac{(-1)^{k-1}r^k}{k}$ is an alternating sum ending on a positive term, it is an overestimate of the infinite sum. Hence, we have

$$\sum_{k=1}^{2n-1} \frac{(-1)^{k-1}r^k}{k} > \sum_{k=1}^{\infty} \frac{(-1)^{k-1}r^k}{k} = \ln(1+r).$$

Next, since $\sum_{k=1}^{\infty} \frac{r^k}{k}$ is a Maclaurin series with all positive terms, we have

$$\sum_{k=1}^{\infty} \frac{r^k}{k} = -\ln(1-r) > \sum_{k=1}^{2n-1} \frac{r^k}{k}.$$

Therefore, (2.30) yields

$$\begin{aligned}
&-\sum_{k=1}^{2n-1} \frac{r^k}{k} \cos k\theta + \frac{\left(\displaystyle\sum_{k=1}^{2n-1} \frac{r^k}{k} \cos k\theta\right)^2 + \left(\displaystyle\sum_{k=1}^{2n-1} \frac{r^k}{k} \sin k\theta\right)^2}{2\displaystyle\sum_{k=1}^{2n-1} \frac{r^k}{k}} \\
&< \quad \ln(1+r) - \frac{(\ln(1+r))^2}{2\ln(1-r)} \\
&< \quad \sum_{k=1}^{2n-1} \frac{(-1)^{k-1}r^k}{k} + \frac{\left(\displaystyle\sum_{k=1}^{2n-1} \frac{(-1)^{k-1}r^k}{k}\right)^2}{2\displaystyle\sum_{k=1}^{2n-1} \frac{r^k}{k}}.
\end{aligned}$$

This concludes the proof. ☐

2.2 Bounds for $[\frac{\pi}{4}, \pi - \frac{\pi}{2n}]$, $n = 2, 3, 4$

Lemma 2.2.1. *Let $r \in (\frac{1}{2}, 1)$, $\theta \in [\frac{\pi}{4}, \frac{\pi}{3}]$ and $n = 2, 3, 4$. Then*

$$\sum_{k=1}^{2n-1} \frac{(-1)^{k-1} r^k}{k} > -\sum_{k=1}^{2n-1} \frac{r^k}{k} \cos k\theta + \frac{\left(\sum_{k=1}^{2n-1} \frac{r^k}{k} \cos k\theta\right)^2 + \left(\sum_{k=1}^{2n-1} \frac{r^k}{k} \sin k\theta\right)^2}{2 \sum_{k=1}^{2n-1} \frac{r^k}{k}}.$$

Proof. Using summation by parts,

$$\begin{aligned}
& -\sum_{k=1}^{2n-1} \frac{r^k}{k} \cos k\theta + \frac{\left(\sum_{k=1}^{2n-1} \frac{r^k}{k} \cos k\theta\right)^2 + \left(\sum_{k=1}^{2n-1} \frac{r^k}{k} \sin k\theta\right)^2}{2 \sum_{k=1}^{2n-1} \frac{r^k}{k}} \\
&= r^{2n-1} \left(-\sum_{k=1}^{2n-1} \frac{\cos k\theta}{k}\right) + \sum_{k=1}^{2n-2} (r^k - r^{k+1}) \left(-\sum_{j=1}^{k} \frac{\cos j\theta}{j}\right) \\
&+ \frac{1}{2\sum_{k=1}^{2n-1} \frac{r^k}{k}} \times \\
&\left(r^{2n-1} \left(-\sum_{k=1}^{2n-1} \frac{\cos k\theta}{k}\right) + \sum_{k=1}^{2n-2} (r^k - r^{k+1}) \left(-\sum_{j=1}^{k} \frac{\cos j\theta}{j}\right)\right)^2 \\
&+ \frac{1}{2\sum_{k=1}^{2n-1} \frac{r^k}{k}} \times \\
&\left(r^{2n-1} \left(\sum_{k=1}^{2n-1} \frac{\sin k\theta}{k}\right) + \sum_{k=1}^{2n-2} (r^k - r^{k+1}) \left(\sum_{j=1}^{k} \frac{\sin j\theta}{j}\right)\right)^2 \\
&< r^{2n-1} \max_{\theta \in [\frac{\pi}{4}, \frac{\pi}{3}]} \left\{-\sum_{k=1}^{2n-1} \frac{\cos k\theta}{k}\right\} + \sum_{k=1}^{2n-2} (r^k - r^{k+1}) \max_{\theta \in [\frac{\pi}{4}, \frac{\pi}{3}]} \left\{-\sum_{j=1}^{k} \frac{\cos j\theta}{j}\right\} \\
&+ \frac{1}{2\sum_{k=1}^{2n-1} \frac{r^k}{k}} \times \\
&\left(r^{2n-1} \max_{\theta \in [\frac{\pi}{4}, \frac{\pi}{3}]} \left\{-\sum_{k=1}^{2n-1} \frac{\cos k\theta}{k}\right\} + \sum_{k=1}^{2n-2} (r^k - r^{k+1}) \max_{\theta \in [\frac{\pi}{4}, \frac{\pi}{3}]} \left\{-\sum_{j=1}^{k} \frac{\cos j\theta}{j}\right\}\right)^2 \\
&+ \frac{1}{2\sum_{k=1}^{2n-1} \frac{r^k}{k}} \times \\
&\left(r^{2n-1} \max_{\theta \in [\frac{\pi}{4}, \frac{\pi}{3}]} \left\{\sum_{k=1}^{2n-1} \frac{\sin k\theta}{k}\right\} + \sum_{k=1}^{2n-2} (r^k - r^{k+1}) \max_{\theta \in [\frac{\pi}{4}, \frac{\pi}{3}]} \left\{\sum_{j=1}^{k} \frac{\sin j\theta}{j}\right\}\right)^2.
\end{aligned}$$

It is hence sufficient to prove that

$$\begin{aligned}
\sum_{k=1}^{2n-1} \frac{(-1)^{k-1} r^k}{k} &> r^{2n-1} \max_{\theta \in [\frac{\pi}{4}, \frac{\pi}{3}]} \left\{-\sum_{k=1}^{2n-1} \frac{\cos k\theta}{k}\right\} + \sum_{k=1}^{2n-2} (r^k - r^{k+1}) \max_{\theta \in [\frac{\pi}{4}, \frac{\pi}{3}]} \left\{-\sum_{j=1}^{k} \frac{\cos j\theta}{j}\right\} \\
&+ \frac{1}{2\sum_{k=1}^{2n-1} \frac{r^k}{k}} \times \\
&\left(r^{2n-1} \max_{\theta \in [\frac{\pi}{4}, \frac{\pi}{3}]} \left\{-\sum_{k=1}^{2n-1} \frac{\cos k\theta}{k}\right\} + \sum_{k=1}^{2n-2} (r^k - r^{k+1}) \max_{\theta \in [\frac{\pi}{4}, \frac{\pi}{3}]} \left\{-\sum_{j=1}^{k} \frac{\cos j\theta}{j}\right\}\right)^2 \\
&+ \frac{1}{2\sum_{k=1}^{2n-1} \frac{r^k}{k}} \times
\end{aligned}$$

$$\left(r^{2n-1}\max_{\theta\in\left[\frac{\pi}{4},\frac{\pi}{3}\right]}\left\{\sum_{k=1}^{2n-1}\frac{\sin k\theta}{k}\right\}+\sum_{k=1}^{2n-2}\left(r^k-r^{k+1}\right)\max_{\theta\in\left[\frac{\pi}{4},\frac{\pi}{3}\right]}\left\{\sum_{j=1}^{k}\frac{\sin j\theta}{j}\right\}\right)^2.$$

Solving this computationally on Mathematica 12.1, we show that this is true:

lemma3.1-2.png

lemma3.1-3.png

lemma3.1-4.png

Figure 2.1: The case $n = 2$

Figure 2.2: The case $n = 3$

Figure 2.3: The case $n = 4$

The yellow line represents the LHS of the above inequality, with the blue line representing the bound on the LHS.

The proof is hence complete. □

Lemma 2.2.2. *Let $r \in (\frac{1}{2}, 1)$, $\theta \in J \in \{\left[\frac{\pi}{3}, \frac{2\pi}{5}\right], \left[\frac{2\pi}{5}, \frac{\pi}{2}\right]\}$ and $n = 2, 3, 4$. Then*

$$\sum_{k=1}^{2n-1}\frac{(-1)^{k-1}r^k}{k} > -\sum_{k=1}^{2n-1}\frac{r^k}{k}\cos k\theta + \frac{\left(\sum_{k=1}^{2n-1}\frac{r^k}{k}\sin k\theta\right)^2}{2\sum_{k=1}^{2n-1}\frac{r^k}{k}}.$$

Proof. The proof is similar to Lemma 2.2.1. □

Lemma 2.2.3. *Let $r \in (\frac{1}{2}, 1)$, $\theta \in J \in \{\left[\frac{\pi}{2}, \frac{3\pi}{4}\right], \left[\frac{3\pi}{4}, \pi - \frac{\pi}{2n}\right]\}$ and $n = 2, 3, 4$. Then*

$$\sum_{k=1}^{2n-1}\frac{(-1)^{k-1}r^k}{k} > -\sum_{k=1}^{2n-1}\frac{r^k}{k}\cos k\theta + \frac{\left(\sum_{k=1}^{2n-1}\frac{r^k}{k}\sin k\theta\right)^2}{2\sum_{k=1}^{2n-1}\frac{r^k}{k}}.$$

Proof. The proof is similar to Lemmas 2.2.1 and 2.2.2. □

2.3 An increasing function on $\left[\frac{\pi}{3}, \pi - \frac{\pi}{2n}\right]$, $5 \le n \le 29$

Lemma 2.3.1 (cf. [5])**.** *Let $r \in (\frac{1}{2}, 1)$, $\theta \in [0, \pi]$ and $n \in \mathbb{N}$. Then*

$$\sum_{k=1}^{n} r^k \sin k\theta = \frac{r\sin\theta}{r^2 - 2r\cos\theta + 1} + \frac{r^{n+2}\sin n\theta - r^{n+1}\sin(n+1)\theta}{r^2 - 2r\cos\theta + 1}.$$

Lemma 2.3.2 (cf. [5])**.** *Let $r \in (\frac{1}{2}, 1)$ and $\theta \in [0, \pi]$. Then*

$$\sum_{k=1}^{\infty} r^k \sin k\theta = \frac{r\sin\theta}{r^2 - 2r\cos\theta + 1}.$$

Proof. We apply Lemma 2.3.1 and Squeeze Theorem to obtain

$$\sum_{k=1}^{\infty} r^k \sin k\theta = \lim_{n\to\infty}\left(\frac{r\sin\theta}{r^2 - 2r\cos\theta + 1} + \frac{r^{n+2}\sin n\theta - r^{n+1}\sin(n+1)\theta}{r^2 - 2r\cos\theta + 1}\right)$$

$$
\begin{aligned}
&= \frac{r\sin\theta}{r^2-2r\cos\theta+1} + \frac{1}{r^2-2r\cos\theta+1}\left(\lim_{n\to\infty}\left(r^{n+2}\sin n\theta - r^{n+1}\sin(n+1)\theta\right)\right)\\
&= \frac{r\sin\theta}{r^2-2r\cos\theta+1}.
\end{aligned}
$$

□

Lemma 2.3.3 (cf. [5])**.** *Let $r \in (\frac{1}{2}, 1)$ and $\theta \in [0,\pi]$. Then*

$$\sum_{k=1}^{\infty} \frac{r^k}{k}\cos k\theta = -\frac{1}{2}\ln\left(r^2-2r\cos\theta+1\right).$$

Proof. Using our hypothesis on r, the series $\sum_{k=1}^{\infty} r^k \sin k\theta$ converges uniformly on $[0,\pi]$. Thus

$$
\begin{aligned}
\sum_{k=1}^{\infty}\int_0^{\theta} r^k \sin kt\, dt &= \int_0^{\theta}\sum_{k=1}^{\infty} r^k \sin kt\, dt; \quad \text{that is,}\\
\sum_{k=1}^{\infty}\left(-\frac{r^k}{k}\cos k\theta + \frac{r^k}{k}\right) &= \int_0^{\theta}\frac{r\sin t}{r^2-2r\cos t+1}\, dt \quad \text{or}\\
\sum_{k=1}^{\infty}\frac{r^k}{k}\cos k\theta &= -\frac{1}{2}\left[\ln\left|r^2-2r\cos t+1\right|\right]_0^{\theta} - \left(-\sum_{k=1}^{\infty}\frac{r^k}{k}\right),
\end{aligned}
$$

which gives the required identity. □

We are now ready to state and prove a crucial estimate for the sum $\sum_{k=1}^{2n-1}\frac{r^k}{k}\cos k\theta$.

Lemma 2.3.4. *Let $r \in (\frac{1}{2}, 1)$, $\theta \in [0,\pi)$ and $n \in \mathbb{N}$. Then*

$$
\begin{aligned}
\sum_{k=1}^{2n-1}\frac{r^k\cos k\theta}{k} > \sum_{k=1}^{2n-1}\frac{r^k\cos k\pi}{k} &- \frac{1}{2}\ln(1-2r\cos\theta+r^2) + \ln(1+r) + \frac{2\left(\frac{r^{2n}}{2n}+\frac{r^{2n+1}}{2n-1}\right)}{(1+r)^2}\\
&+ \frac{\frac{r^{2n+1}}{2n-1}\left(\cos(2n-1)\theta - 1\right) - \frac{r^{2n}}{2n}\left(\cos 2n\theta + 1\right)}{1-2r\cos\theta+r^2}.
\end{aligned}
$$

Proof. For each $x \in [0,\pi)$, we set

$$f(x) = \sum_{k=1}^{2n-1}\frac{r^k\cos kx}{k} - \sum_{k=1}^{2n-1}\frac{r^k\cos k\pi}{k} + \ln(1-2r\cos x+r^2) - \ln(1+r).$$

Then, for each $x \in [0,\pi)$, we use Lemma 2.3.1 to obtain $f'(x)$:

$$
\begin{aligned}
f'(x) &= -\sum_{k=1}^{2n-1} r^k \sin kx + \frac{r\sin x}{1-2r\cos x+r^2}\\
&= -\frac{r^{2n}(r\sin(2n-1)x - \sin 2nx)}{1-2r\cos x+r^2}.
\end{aligned}
$$

Using integration by parts with

$$
\begin{aligned}
u &= -\frac{1}{1-2r\cos x+r^2} & dv &= (r\sin(2n-1)x - \sin 2nx)\, dx\\
du &= \frac{2r\sin x}{\left(1-2r\cos x+r^2\right)^2}\, dx & v &= -\frac{r\cos(2n-1)x}{2n-1} + \frac{\cos 2nx}{2n},
\end{aligned}
$$

$$
\begin{aligned}
&f(\theta)\\
&= -\int_{\theta}^{\pi} -\frac{r^{2n}(r\sin(2n-1)x - \sin 2nx)}{1-2r\cos x+r^2}\, dx^{\dagger}\\
&= -r^{2n}\left\{\left[\frac{\left(\frac{r}{2n-1}\cos(2n-1)x - \frac{1}{2n}\cos 2nx\right)}{1-2r\cos x+r^2}\right]_{\theta}^{\pi} - \int_{\theta}^{\pi}\frac{2r\sin x}{\left(1-2r\cos x+r^2\right)^2}\left(\frac{\cos 2nx}{2n} - \frac{r\cos(2n-1)x}{2n-1}\right) dx\right\}
\end{aligned}
$$

$$
\begin{aligned}
&> - r^{2n}\left\{\frac{-\frac{1}{2n}-\frac{r}{2n-1}}{(1+r)^2}-\left(\frac{\frac{r}{2n-1}\cos(2n-1)\theta-\frac{1}{2n}\cos 2n\theta}{1-2r\cos\theta+r^2}\right)+\left(\frac{1}{2n}+\frac{r}{2n-1}\right)\int_\theta^\pi \frac{2r\sin x}{(1-2r\cos x+r^2)^2}dx\right\}\\
&= - r^{2n}\left\{\frac{-\frac{1}{2n}-\frac{r}{2n-1}}{(1+r)^2}-\left(\frac{\frac{r}{2n-1}\cos(2n-1)\theta-\frac{1}{2n}\cos 2n\theta}{1-2r\cos\theta+r^2}\right)+\left(\frac{1}{2n}+\frac{r}{2n-1}\right)\left(-\frac{1}{(1+r)^2}+\frac{1}{1-2r\cos\theta+r^2}\right)\right\}\\
&=\frac{2\left(\frac{r^{2n}}{2n}+\frac{r^{2n+1}}{2n-1}\right)}{(1+r)^2}+\frac{\frac{r^{2n+1}}{2n-1}\left(\cos(2n-1)\theta-1\right)-\frac{r^{2n}}{2n}\left(\cos 2n\theta+1\right)}{1-2r\cos\theta+r^2}\ ,
\end{aligned}
$$

which is equivalent to the required inequality. □

Lemma 2.3.5. *Let $r \in (\frac{1}{2},1)$, $\theta \in [\frac{\pi}{3},\pi)$ and $n \in \mathbb{N}$. Then*

$$
\sum_{k=1}^{2n-1}\frac{r^k\cos k\theta}{k} > \sum_{k=1}^{2n-1}\frac{r^k\cos k\pi}{k}-\frac{1}{2}\ln(1-2r\cos\theta+r^2)+\ln(1+r)+2\left(\frac{r^{2n}}{2n}+\frac{r^{2n+1}}{2n-1}\right)\left(\frac{1}{(1+r)^2}-\frac{1}{1-2r\cos\theta+r^2}\right).
$$

Proof. Since $\cos(2n-1)\theta \geq -1$ and $-\cos 2n\theta \geq -1$, an application of Lemma 2.3.4 yields the desired inequality. □

Lemma 2.3.6 (cf. [5]). *Let $r \in (\frac{1}{2},1)$, $\theta \in [0,\pi]$ and $n \in \mathbb{N}$. Then*

$$
\sum_{k=1}^{n} r^k\cos k\theta = \frac{r\cos\theta-r^2}{r^2-2r\cos\theta+1}+\frac{r^{n+2}\cos n\theta-r^{n+1}\cos(n+1)\theta}{r^2-2r\cos\theta+1}.
$$

Lemma 2.3.7 (cf. [5]). *Let $r \in (\frac{1}{2},1)$ and $\theta \in [0,\pi]$. Then*

$$
\sum_{k=1}^{\infty} r^k\cos k\theta = \frac{r\cos\theta-r^2}{r^2-2r\cos\theta+1}.
$$

Proof. We apply Lemma 2.3.6 and Squeeze Theorem to obtain

$$
\begin{aligned}
\sum_{k=1}^{\infty} r^k\cos k\theta &= \lim_{n\to\infty}\left(\frac{r\cos\theta-r^2}{r^2-2r\cos\theta+1}+\frac{r^{n+2}\cos n\theta-r^{n+1}\cos(n+1)\theta}{r^2-2r\cos\theta+1}\right)\\
&= \frac{r\cos\theta-r^2}{r^2-2r\cos\theta+1}+\frac{1}{r^2-2r\cos\theta+1}\left(\lim_{n\to\infty} r^n\left(r^2\cos n\theta-r\cos(n+1)\theta\right)\right)\\
&= \frac{r\cos\theta-r^2}{r^2-2r\cos\theta+1}.
\end{aligned}
$$

□

Lemma 2.3.8 (cf. [5]). *Let $r \in (\frac{1}{2},1)$ and $\theta \in [0,\pi]$. Then*

$$
\sum_{k=1}^{\infty}\frac{r^k}{k}\sin k\theta = \tan^{-1}\left(\frac{r\sin\theta}{1-r\cos\theta}\right).
$$

Proof. Using our hypothesis on r, the series $\sum_{k=1}^{\infty} r^k\cos k\theta$ converges uniformly on $[0,\pi]$. Thus

$$
\sum_{k=1}^{\infty}\int_0^\theta r^k\cos kt\,dt = \int_0^\theta\sum_{k=1}^{\infty} r^k\cos kt\,dt.
$$

Hence, Lemma 2.3.7 and integration by substitution yields the desired conclusion:

$$
\begin{aligned}
\sum_{k=1}^{\infty}\frac{r^k}{k}\sin k\theta &= \int_0^\theta \frac{r\cos t-r^2}{r^2-2r\cos t+1}\,dt\\
&= \int_0^{\frac{r\sin\theta}{1-r\cos\theta}}\frac{1}{1+u^2}\,du\\
&= \tan^{-1}\left(\frac{r\sin\theta}{1-r\cos\theta}\right).
\end{aligned}
$$

□

The following result yields a crucial inequality for the sum $\sum_{k=1}^{2n-1}\frac{r^k\sin k\theta}{k}$.

†Integration from 0 to θ was also attempted; however, this does not provide as good a bound.

Lemma 2.3.9. *Let $r \in (\frac{1}{2}, 1)$, $\theta \in [0, \pi)$ and $n \in \mathbb{N}$. Then*

$$\sum_{k=1}^{2n-1} \frac{r^k \sin k\theta}{k} < \tan^{-1}\left(\frac{r\sin\theta}{1-r\cos\theta}\right) - \frac{\left(\frac{r^{2n}}{2n}+\frac{r^{2n+1}}{2n-1}\right)}{(1+r)^2} + \frac{\frac{r^{2n+1}}{2n-1}\left(1+\sin(2n-1)\theta\right)+\frac{r^{2n}}{2n}\left(1-\sin(2n\theta)\right)}{1-2r\cos\theta+r^2}.$$

Proof. For each $x \in [0, \pi)$,we set

$$g(x) = \sum_{k=1}^{2n-1} \frac{r^k \sin kx}{k} - \tan^{-1}\left(\frac{r\sin x}{1-r\cos x}\right).$$

Then, for each $x \in [0, \pi)$, we infer from Lemma 4.6 that

$$\begin{aligned} g'(x) &= \sum_{k=1}^{2n-1} r^k \cos kx + \frac{r^2 - r\cos x}{1-2r\cos x + r^2} \\ &= \frac{r^{2n}(r\cos(2n-1)x - \cos 2nx)}{1-2r\cos x+r^2}. \end{aligned}$$

Using integration by parts with

$$\begin{aligned} u &= \frac{1}{1-2r\cos x+r^2} & dv &= (r\cos(2n-1)x - \cos 2nx)\, dx \\ du &= -\frac{2r\sin x}{(1-2r\cos x+r^2)^2}\, dx & v &= \frac{r\sin(2n-1)x}{2n-1} - \frac{\sin 2nx}{2n}\ , \end{aligned}$$

$$\begin{aligned} & g(\theta) \\ =& -\int_\theta^\pi \frac{r^{2n}\left(r\cos(2n-1)x + \cos 2nx\right)}{1-2r\cos x+r^2} dx \\ =& -r^{2n}\left\{\left[\frac{\left(\frac{r}{2n-1}\sin(2n-1)x - \frac{1}{2n}\sin 2nx\right)}{1-2r\cos x+r^2}\right]_\theta^\pi + \int_\theta^\pi \frac{2r\sin x}{(1-2r\cos x+r^2)^2}\left(\frac{r\sin(2n-1)x}{2n-1} - \frac{\sin 2nx}{2n}\right) dx\right\} \\ <& -r^{2n}\left\{-\left(\frac{\frac{r}{2n-1}\sin(2n-1)\theta - \frac{1}{2n}\sin 2n\theta}{1-2r\cos\theta+r^2}\right) - \left(\frac{1}{2n}+\frac{r}{2n-1}\right)\int_\theta^\pi \frac{2r\sin x}{(1-2r\cos x+r^2)^2} dx\right\} \\ =& -r^{2n}\left\{-\left(\frac{\frac{r}{2n-1}\sin(2n-1)\theta - \frac{1}{2n}\sin 2n\theta}{1-2r\cos\theta+r^2}\right) - \left(\frac{1}{2n}+\frac{r}{2n-1}\right)\left(-\frac{1}{(1+r)^2} + \frac{1}{1-2r\cos\theta+r^2}\right)\right\} \\ =& -\frac{\left(\frac{r^{2n}}{2n}+\frac{r^{2n+1}}{2n-1}\right)}{(1+r)^2} + \frac{\frac{r^{2n+1}}{2n-1}\sin(2n-1)\theta - \frac{r^{2n}}{2n}\sin 2n\theta + \frac{r^{2n+1}}{2n-1} + \frac{r^{2n}}{2n}}{1-2r\cos\theta+r^2} \\ =& -\frac{\left(\frac{r^{2n}}{2n}+\frac{r^{2n+1}}{2n-1}\right)}{(1+r)^2} + \frac{\frac{r^{2n+1}}{2n-1}\left(1+\sin(2n-1)\theta\right)+\frac{r^{2n}}{2n}\left(1-\sin(2n\theta)\right)}{1-2r\cos\theta+r^2}. \end{aligned}$$

The proof is complete. ☐

Lemma 2.3.10. *Let $r \in (\frac{1}{2}, 1)$, $\theta \in [\frac{\pi}{3}, \pi)$ and $n \in \mathbb{N}$. Then*

$$\sum_{k=1}^{2n-1} \frac{r^k \sin k\theta}{k} < \tan^{-1}\left(\frac{r\sin\theta}{1-r\cos\theta}\right) + \left(\frac{r^{2n}}{2n}+\frac{r^{2n+1}}{2n-1}\right)\left(\frac{2}{1-2r\cos\theta+r^2} - \frac{1}{(1+r)^2}\right).$$

Proof. Since $\sin(2n-1)\theta \le 1$ and $-\cos 2n\theta \le 1$, an application of Lemma 2.3.9 completes the proof. ☐

Lemma 2.3.11. *Let $r \in (\frac{1}{2}, 1)$. Then $\theta \mapsto \frac{r^2 - r\cos\theta}{1-r\cos\theta}$ is increasing on $[0, \pi)$.*

Proof. For $\theta \in (0, \pi)$ and $r \in (\frac{1}{2}, 1)$, we have

$$\begin{aligned} \frac{d}{d\theta}\left(\frac{r^2 - r\cos\theta}{1-r\cos\theta}\right) &= \frac{r-r^3}{(r\cos\theta-1)^2}\sin\theta \\ &> 0. \end{aligned} \tag{2.31}$$

Since the function $\theta \mapsto \frac{r^2 - r\cos\theta}{1-r\cos\theta}$ is continuous on $[0, \pi)$ and (2.31) holds, the lemma is proven. ☐

Lemma 2.3.12. *Let $r \in (\frac{1}{2}, 1)$,$\theta \in [0, \pi)$. Then $\theta \mapsto \frac{r^2 - r\cos\theta}{1-2r\cos\theta+r^2}$ is increasing on $[0, \pi)$.*

Proof. For $\theta \in (0, \pi)$, we have

$$\begin{aligned}\frac{d}{d\theta}\left(\frac{r^2 - r\cos\theta}{1-2r\cos\theta+r^2}\right) &= \frac{r - r^3}{\left(1-2r\cos\theta+r^2\right)^2}\sin\theta \\ &> 0\end{aligned}$$

whenever $\theta \in (0, \pi)$ and $r \in (\frac{1}{2}, 1)$. Since the function is continuous on $[0, \pi)$, and the derivative is positive on $(0, \pi)$, the lemma is proven. ☐

Using Lemmas 2.3.5 and 2.3.10, we are ready to construct an increasing function defined on $\left[\frac{\pi}{3}, \pi - \frac{\pi}{2n}\right]$.

Let $[\alpha, \beta] \subset \left[\frac{\pi}{3}, \pi\right)$ and let

$$\begin{aligned}&F_n(\theta) \\ := \; &\frac{1}{2}\ln\left(1-2r\cos\theta+r^2\right) - \ln(1+r) \\ &+2\left(\frac{r^{2n}}{2n} + \frac{r^{2n+1}}{2n-1}\right)\left(\frac{1}{1-2r\cos\alpha+r^2} - \frac{1}{(1+r)^2}\right) \\ &+\frac{1}{2\sum_{k=1}^{2n-1}\frac{r^k}{k}}\left(\tan^{-1}\left(\frac{r\sin\theta}{1-r\cos\theta}\right) + \left(\frac{r^{2n}}{2n} + \frac{r^{2n+1}}{2n-1}\right)\left(\frac{2}{1-2r\cos\alpha+r^2} - \frac{1}{(1+r)^2}\right)\right)^2\end{aligned}$$

for $\theta \in [\alpha, \beta]$.

Theorem 2.3.13. *Let n be any integer satisfying $n \geqslant 5$. If $[\alpha, \beta] \subseteq \left[\frac{\pi}{3}, \frac{\pi}{2}\right]$, $[\alpha, \beta] \subseteq \left[\frac{\pi}{2}, \frac{3\pi}{4}\right]$ or $[\alpha, \beta] \subseteq \left[\frac{3\pi}{4}, \pi - \frac{\pi}{2n}\right]$, then $F_n'(\theta) > 0$ for $\theta \in (\alpha, \beta)$.*

Proof. Let $\theta \in (\alpha, \beta)$. Then we have

$$\begin{aligned}&F_n'(\theta) \\ &= \frac{r\sin\theta}{1-2r\cos\theta+r^2} + \frac{\tan^{-1}\left(\frac{r\sin\theta}{1-r\cos\theta}\right) + \left(\frac{2}{1-2r\cos\alpha+r^2} - \frac{1}{(1+r)^2}\right)\left(\frac{r^{2n}}{2n} + \frac{r^{2n+1}}{2n-1}\right)}{\sum_{k=1}^{2n-1}\frac{r^k}{k}} \cdot \frac{r\cos\theta - r^2}{1-2r\cos\theta+r^2} \\ &> \frac{r\sin\theta}{1-2r\cos\theta+r^2} + \frac{\left(\frac{r\sin\theta}{1-r\cos\theta}\right) + \left(\frac{2}{1-2r\cos\alpha+r^2} - \frac{1}{(1+r)^2}\right)\left(\frac{r^{2n}}{2n} + \frac{r^{2n+1}}{2n-1}\right)}{\sum_{k=1}^{2n-1}\frac{r^k}{k}} \cdot \frac{r\cos\theta - r^2}{1-2r\cos\theta+r^2} \quad (\tan^{-1} x < x \text{ and } r\cos\theta < r^2) \\ &= \frac{r\sin\theta}{1-2r\cos\theta+r^2} + \frac{(r\sin\theta)(r\cos\theta - r^2)}{(1-r\cos\theta)(1-2r\cos\theta+r^2)\sum_{k=1}^{2n-1}\frac{r^k}{k}} \\ &+ \frac{\left(\frac{2}{1-2r\cos\alpha+r^2} - \frac{1}{(1+r)^2}\right)\left(\frac{r^{2n}}{2n} + \frac{r^{2n+1}}{2n-1}\right)}{\sum_{k=1}^{2n-1}\frac{r^k}{k}} \cdot \frac{r\cos\theta - r^2}{1-2r\cos\theta+r^2} \\ &= \frac{r\sin\theta}{1-2r\cos\theta+r^2}\left(1 + \frac{r\cos\theta - r^2}{(1-r\cos\theta)\sum_{k=1}^{2n-1}\frac{r^k}{k}}\right) + \frac{\left(\frac{2}{1-2r\cos\alpha+r^2} - \frac{1}{(1+r)^2}\right)\left(\frac{r^{2n}}{2n} + \frac{r^{2n+1}}{2n-1}\right)}{\sum_{k=1}^{2n-1}\frac{r^k}{k}} \cdot \frac{r\cos\theta - r^2}{1-2r\cos\theta+r^2} \\ &= \frac{r\sin\theta}{1-2r\cos\theta+r^2}\left(1 - \frac{r^2 - r\cos\theta}{(1-r\cos\theta)\sum_{k=1}^{2n-1}\frac{r^k}{k}}\right) - \frac{\left(\frac{2}{1-2r\cos\alpha+r^2} - \frac{1}{(1+r)^2}\right)\left(\frac{r^{2n}}{2n} + \frac{r^{2n+1}}{2n-1}\right)}{\sum_{k=1}^{2n-1}\frac{r^k}{k}} \cdot \frac{r^2 - r\cos\theta}{1-2r\cos\theta+r^2}.\end{aligned}$$

We consider three cases.

Case 1: $\frac{\pi}{3} \leq \alpha < \beta \leq \frac{\pi}{2}$. We apply Lemmas 2.3.11 and 2.3.12 in conjunction with the increasing nature of the function $\theta \mapsto \sin\theta$ (where $\sin\theta \geq 0$ on this interval) and the decreasing nature of the function $\theta \mapsto \cos\theta$ on $\left[\frac{\pi}{3}, \frac{\pi}{2}\right]$ to show that for any integer $n \geq 5$,

$$\begin{aligned}F_n'(\theta) &> \frac{r\sin\alpha}{1-2r\cos\beta+r^2}\left(1 - \frac{r^2 - r\cos\beta}{(1-r\cos\beta)\sum_{k=1}^{9}\frac{r^k}{k}}\right) - \frac{\left(\frac{2}{1-2r\cos\alpha+r^2} - \frac{1}{(1+r)^2}\right)\left(\frac{r^{10}}{10} + \frac{r^{11}}{9}\right)}{\sum_{k=1}^{9}\frac{r^k}{k}} \cdot \frac{r^2 - r\cos\beta}{1-2r\cos\beta+r^2} \\ &\geq \frac{r\sin\frac{\pi}{3}}{1-2r\cos\frac{\pi}{2}+r^2}\left(1 - \frac{r^2 - r\cos\frac{\pi}{2}}{\left(1-r\cos\frac{\pi}{2}\right)\sum_{k=1}^{9}\frac{r^k}{k}}\right) - \frac{\left(\frac{2}{1-2r\cos\frac{\pi}{3}+r^2} - \frac{1}{(1+r)^2}\right)\left(\frac{r^{10}}{10} + \frac{r^{11}}{9}\right)}{\sum_{k=1}^{9}\frac{r^k}{k}} \cdot \frac{r^2 - r\cos\frac{\pi}{2}}{1-2r\cos\frac{\pi}{2}+r^2}\end{aligned}$$

$$
\begin{aligned}
&= \frac{r}{1+r^2}\left(\sin\frac{\pi}{3}\left(1-\frac{r^2-r\cos\frac{\pi}{2}}{\left(1-r\cos\frac{\pi}{2}\right)\sum_{k=1}^{9}\frac{r^k}{k}}\right)-\frac{\left(\frac{2}{1-2r\cos\frac{\pi}{3}+r^2}-\frac{1}{(1+r)^2}\right)\left(\frac{r^{10}}{10}+\frac{r^{11}}{9}\right)}{\sum_{k=1}^{9}\frac{r^k}{k}}\cdot\left(r-\cos\frac{\pi}{2}\right)\right)\\
&= \frac{r}{1+r^2}\left(\frac{\sqrt{3}}{2}\left(1-\frac{r^2}{\sum_{k=1}^{9}}\frac{r^k}{k}\right)-\frac{r\left(\frac{2}{1-r+r^2}-\frac{1}{(1+r)^2}\right)\left(\frac{r^{10}}{10}+\frac{r^{11}}{9}\right)}{\sum_{k=1}^{9}\frac{r^k}{k}}\right)\\
&> 0
\end{aligned}
$$

whenever $r \in \left(\frac{1}{2},1\right)$ as verified using Sturm's Theorem.right)-r$\left(\frac{2}{1-r+r^2}-\frac{1}{(1+r)^2}\right)\left(\frac{r^{10}}{10}+\frac{r^{11}}{9}\right)_{\overline{\sum_{k=1}^{9}\frac{r^k}{k}>0}}$

since We hence conclude that for any integer $n \geq 5$ and $r \in \left(\frac{1}{2},1\right)$ the function F_n is increasing on $[\alpha,\beta]$ for $[\alpha,\beta] \subseteq [\frac{\pi}{3},\frac{\pi}{2}]$.

Case 2: $\frac{\pi}{2} \leq \alpha < \beta \leq \frac{3\pi}{4}$. We apply Lemmas 2.3.11 and 2.3.12 in conjunction with the decreasing nature of the functions $\theta \mapsto \sin\theta$ and $\theta \mapsto \cos\theta$ on $[\frac{\pi}{2},\frac{3\pi}{4}]$ (where $\sin\theta \geq 0$ on this interval) to show that for any integer $n \geq 5$,

$$
\begin{aligned}
F_n'(\theta) &> \frac{r\sin\beta}{1-2r\cos\beta+r^2}\left(1-\frac{r^2-r\cos\beta}{(1-r\cos\beta)\sum_{k=1}^{2n-1}\frac{r^k}{k}}\right)-\frac{\left(\frac{2}{1-2r\cos\alpha+r^2}-\frac{1}{(1+r)^2}\right)\left(\frac{r^{2n}}{2n}+\frac{r^{2n+1}}{2n-1}\right)}{\sum_{k=1}^{2n-1}\frac{r^k}{k}}\cdot\frac{r^2-r\cos\beta}{1-2r\cos\beta+r^2}\\
&\geq \frac{r\sin\frac{3\pi}{4}}{1-2r\cos\frac{3\pi}{4}+r^2}\left(1-\frac{r^2-r\cos\frac{3\pi}{4}}{\left(1-r\cos\frac{3\pi}{4}\right)\sum_{k=1}^{9}\frac{r^k}{k}}\right)-\frac{\left(\frac{2}{1-2r\cos\frac{\pi}{2}+r^2}-\frac{1}{(1+r)^2}\right)\left(\frac{r^{10}}{10}+\frac{r^{11}}{9}\right)}{\sum_{k=1}^{9}\frac{r^k}{k}}\cdot\frac{r^2-r\cos\frac{3\pi}{4}}{1-2r\cos\frac{3\pi}{4}+r^2}\\
&= \frac{r}{1+\sqrt{2}r+r^2}\left(\sin\frac{3\pi}{4}\left(1-\frac{r^2-r\cos\frac{3\pi}{4}}{\left(1-r\cos\frac{3\pi}{4}\right)\sum_{k=1}^{9}\frac{r^k}{k}}\right)-\frac{\left(\frac{2}{1+r^2}-\frac{1}{(1+r)^2}\right)\left(\frac{r^{10}}{10}+\frac{r^{11}}{9}\right)}{\sum_{k=1}^{9}\frac{r^k}{k}}\cdot\left(r-\cos\frac{3\pi}{4}\right)\right)\\
&> 0
\end{aligned}
$$

since

$$
\left(\sin\frac{3\pi}{4}\right)\left(1-\frac{r^2-r\cos\frac{3\pi}{4}}{\left(1-r\cos\frac{3\pi}{4}\right)\sum_{k=1}^{9}\frac{r^k}{k}}\right)-\frac{\left(\frac{2}{1+r^2}-\frac{1}{(1+r)^2}\right)\left(\frac{r^{10}}{10}+\frac{r^{11}}{9}\right)}{\sum_{k=1}^{9}\frac{r^k}{k}}\cdot\left(r-\cos\frac{3\pi}{4}\right)>0
$$

whenever $r \in \left(\frac{1}{2},1\right)$ as verified using Sturm's Theorem. We hence conclude that for any integer $n \geq 5$ and $r \in \left(\frac{1}{2},1\right)$ the function F_n is increasing on $[\alpha,\beta]$ for $[\alpha,\beta] \subseteq [\frac{\pi}{2},\frac{3\pi}{4}]$.

Case 3: $\frac{3\pi}{4} \leq \alpha < \beta \leq \pi - \frac{\pi}{2n}$. We apply Lemmas 2.3.11 and 2.3.12 in conjunction with the decreasing nature of the functions $\theta \mapsto \sin\theta$ and $\theta \mapsto \cos\theta$ on $[\frac{3\pi}{4},\pi-\frac{\pi}{2n}]$ (where $\sin\theta \geq 0$ on this interval) to show that for any integer $n \geq 5$,

$$
\begin{aligned}
F_n'(\theta) &> \frac{r\sin\beta}{1-2r\cos\beta+r^2}\left(1-\frac{r^2-r\cos\beta}{(1-r\cos\beta)\sum_{k=1}^{2n-1}\frac{r^k}{k}}\right)-\frac{\left(\frac{2}{1-2r\cos\alpha+r^2}-\frac{1}{(1+r)^2}\right)\left(\frac{r^{2n}}{2n}+\frac{r^{2n+1}}{2n-1}\right)}{\sum_{k=1}^{2n-1}\frac{r^k}{k}}\cdot\frac{r^2-r\cos\beta}{1-2r\cos\beta+r^2}\\
&\geq \frac{1}{1+2r\cos\frac{\pi}{2n}+r^2}\\
&\left(\left(r\sin\frac{\pi}{2n}\right)\left(1-\frac{r^2+r\cos\frac{\pi}{2n}}{\left(1+r\cos\frac{\pi}{2n}\right)\sum_{k=1}^{2n-1}\frac{r^k}{k}}\right)-\frac{\left(\frac{2}{1-2r\cos\frac{3\pi}{4}+r^2}-\frac{1}{(1+r)^2}\right)\left(\frac{r^{2n}}{2n}+\frac{r^{2n+1}}{2n-1}\right)}{\sum_{k=1}^{2n-1}\frac{r^k}{k}}\cdot\left(r^2+r\cos\frac{\pi}{2n}\right)\right).
\end{aligned}
$$

Hence, it suffices to show that

$$
\left(r\sin\frac{\pi}{2n}\right)\left(1-\frac{r^2+r\cos\frac{\pi}{2n}}{\left(1+r\cos\frac{\pi}{2n}\right)\sum_{k=1}^{2n-1}\frac{r^k}{k}}\right)-\frac{\left(\frac{2}{1-2r\cos\frac{3\pi}{4}+r^2}-\frac{1}{(1+r)^2}\right)\left(\frac{r^{2n}}{2n}+\frac{r^{2n+1}}{2n-1}\right)}{\sum_{k=1}^{2n-1}\frac{r^k}{k}}\cdot\left(r^2+r\cos\frac{\pi}{2n}\right)>0.
$$

Since

$$
\begin{aligned}
&\left(r\sin\frac{\pi}{2n}\right)\left(1-\frac{r^2+r\cos\frac{\pi}{2n}}{\left(1+r\cos\frac{\pi}{2n}\right)\sum_{k=1}^{2n-1}\frac{r^k}{k}}\right)-\frac{\left(\frac{2}{1-2r\cos\frac{3\pi}{4}+r^2}-\frac{1}{(1+r)^2}\right)\left(\frac{r^{2n}}{2n}+\frac{r^{2n+1}}{2n-1}\right)}{\sum_{k=1}^{2n-1}\frac{r^k}{k}}\cdot\left(r^2+r\cos\frac{\pi}{2n}\right)\\
&\geq \left(r\sin\frac{\pi}{2n}\right)\left(1-\frac{r}{\sum_{k=1}^{2n-1}\frac{r^k}{k}}\right)-\frac{\left(\frac{2}{1-2r\cos\frac{3\pi}{4}+r^2}-\frac{1}{(1+r)^2}\right)\left(\frac{r^{2n}}{2n}+\frac{r^{2n+1}}{2n-1}\right)}{\sum_{k=1}^{2n-1}\frac{r^k}{k}}(r^2+r)
\end{aligned}
$$

$$
\begin{aligned}
&\geq \left(\frac{r\pi}{2n}-\frac{r\pi^3}{48n^3}\right)\left(1-\frac{r}{\sum_{k=1}^{9}\frac{r^k}{k}}\right)-\left(\frac{2}{1-2r\cos\frac{3\pi}{4}+r^2}-\frac{1}{(1+r)^2}\right)\left(\frac{r^{2n}}{2n}+\frac{r^{2n+1}}{2n-1}\right)(r+1)\\
&\geq \left(\frac{3r\pi}{8n}\right)\left(1-\frac{r}{\sum_{k=1}^{9}\frac{r^k}{k}}\right)-\left(\frac{2}{1-2r\cos\frac{3\pi}{4}+r^2}-\frac{1}{(1+r)^2}\right)\left(\frac{r^{2n}}{2n}+\frac{r^{2n+1}}{2n-1}\right)(r+1)\\
&= \frac{r}{n}\left\{\left(\frac{3\pi}{8}\right)\left(1-\frac{r}{\sum_{k=1}^{9}\frac{r^k}{k}}\right)-\left(\frac{2}{1-2r\cos\frac{3\pi}{4}+r^2}-\frac{1}{(1+r)^2}\right)\left(\frac{r^{2n}}{2}+\frac{nr^{2n+1}}{2n-1}\right)\left(\frac{r+1}{r}\right)\right\}\\
&\geq \frac{r}{n}\left\{\left(\frac{3\pi}{8}\right)\left(1-\frac{r}{\sum_{k=1}^{9}\frac{r^k}{k}}\right)-\left(\frac{2}{1-2r\cos\frac{3\pi}{4}+r^2}-\frac{1}{(1+r)^2}\right)\left(\frac{r^{10}}{2}+\frac{4r^{11}}{9}\right)\left(\frac{r+1}{r}\right)\right\}\\
&> 0,
\end{aligned}
$$

which may be verified using Sturm's theorem. We hence conclude that for any integer $n \geq 5$ and $r \in \left(\frac{1}{2}, 1\right)$ the function F_n is increasing on $[\alpha, \beta]$ for $[\alpha, \beta] \subseteq [\frac{3\pi}{4}, \pi - \frac{\pi}{2n}]$ This completes the proof.

□

2.4 Bounds for $[\frac{\pi}{3}, \pi - \frac{\pi}{2n}]$, $5 \leq n \leq 29$

Lemma 2.4.1. *Let $r \in (\frac{1}{2}, 1)$, $\frac{\pi}{3} \leq \alpha < \beta \leq \pi - \frac{\pi}{2n}$ and $5 \leq n \leq 29$. If*

$$
\begin{aligned}
F_n(\theta) = {}& \frac{1}{2}\ln(1-2r\cos\theta+r^2)-\ln(1+r)-2\left(\frac{r^{2n}}{2n}+\frac{r^{2n+1}}{2n-1}\right)\left(\frac{1}{(1+r)^2}-\frac{1}{1-2r\cos\alpha+r^2}\right)\\
&+\frac{\left(\tan^{-1}\left(\frac{r\sin\theta}{1-r\cos\theta}\right)+\left(\frac{r^{2n}}{2n}+\frac{r^{2n+1}}{2n-1}\right)\left(\frac{2}{1-2r\cos\alpha+r^2}-\frac{1}{(1+r)^2}\right)\right)^2}{2\sum_{k=1}^{2n-1}\frac{r^k}{k}},
\end{aligned}
$$

then $F_n(\theta) < 0$ for $\theta \in [\alpha, \beta]$.

Proof. When $n = 5$ and $(\alpha, \beta) = \left(\frac{\pi}{3}, \frac{34\pi}{100}\right)$ we have

$$F_5(\theta) < 0 \text{ whenever } \theta \in [\alpha, \beta]\,;$$

a verification of this inequality is given in Figure 2.4. Similarly, we have

$$F_5(\theta) < 0$$

for $\theta \in [\alpha, \beta]$ and

$$[\alpha, \beta] \in \left\{\left[\frac{34\pi}{100}, \frac{35\pi}{100}\right], \left[\frac{35\pi}{100}, \frac{3\pi}{8}\right], \left[\frac{3\pi}{8}, \frac{2\pi}{5}\right], \left[\frac{2\pi}{5}, \frac{3\pi}{7}\right], \left[\frac{3\pi}{7}, \frac{\pi}{2}\right]\right\}.$$

Hence an application of Theorem 2.3.11 reveals that

$$
\begin{aligned}
&\frac{1}{2}\ln\left(1-2r\cos\theta+r^2\right)-\ln(1+r)\\
&+2\left(\frac{r^{2n}}{2n}+\frac{r^{2n+1}}{2n-1}\right)\left(\frac{1}{1-2r\cos\theta+r^2}-\frac{1}{(1+r)^2}\right)\\
&+\frac{\left(\tan^{-1}\left(\frac{r\sin\theta}{1-r\cos\theta}\right)+\left(\frac{2}{1-2r\cos\theta+r^2}-\frac{1}{(1+r)^2}\right)\left(\frac{r^{2n}}{2n}+\frac{r^{2n+1}}{2n-1}\right)\right)^2}{2\sum_{k=1}^{2n-1}\frac{r^k}{k}}\\
<\;& 0
\end{aligned}
\tag{2.32}
$$

whenever $\theta \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right]$ and $n = 5, 6, \ldots, 29$. Likewise, (2.32) holds whenever $\theta \in \left[\frac{\pi}{2}, \frac{3\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \frac{9\pi}{10}\right]$ and $n = 5, 6, \ldots, 29$.

Now we let $N \in \{6, \ldots, 29\}$. Since

$$F_n(\theta) < 0$$

whenever $\theta \in \left[\pi - \frac{\pi}{2N-2}, \pi - \frac{\pi}{2N}\right]$, we infer from Theorem 2.3.11 that (2.32) holds if $\theta \in \left[\pi - \frac{\pi}{2N-2}, \pi - \frac{\pi}{2N}\right]$ and $n = N, \ldots, 29$. □

func1.png

Figure 2.4: The case $n = 5$, $\theta \in [\frac{\pi}{3}, \frac{34\pi}{100}]$

func2.png

Figure 2.5: The case $n = 6$, $\theta \in [\frac{9\pi}{10}, \frac{11\pi}{12}]$

2.5 Bounds for $[\pi - \frac{\pi}{2n}, \pi)$, $3 \leq n \leq 29$

Following the proof of [8, Lemma 2.4], we obtain the following result:

Lemma 2.5.1. *Let $r \in (\frac{1}{2}, 1)$, $n \in \mathbb{N}$ and $\theta \in (0, \pi)$. Then*

$$\left| \sum_{k=1}^{2n-1} \frac{r^k \sin k\theta}{k} - \tan^{-1}\left(\frac{r \sin\theta}{1 - r\cos\theta}\right) \right| < \frac{r^{2n}}{2n} \csc\frac{\theta}{2}.$$

Proof. By Lemma 2.3.8 and summation by parts,

$$\begin{aligned}
&\left| \sum_{k=1}^{2n-1} \frac{r^k \sin k\theta}{k} - \tan^{-1}\left(\frac{r \sin\theta}{1 - r\cos\theta}\right) \right| \\
&= \left| \sum_{k=1}^{2n-1} \frac{r^k \sin k\theta}{k} - \sum_{k=1}^{\infty} \frac{r^k \sin k\theta}{k} \right| \\
&= \left| - \sum_{k=2n}^{\infty} \frac{r^k \sin k\theta}{k} \right| \\
&= \left| \sum_{k=2n}^{\infty} \left(\frac{r^k}{k} - \frac{r^{k+1}}{k+1}\right) \sum_{j=2n}^{k} \sin j\theta \right| \\
&= \left| \sum_{k=2n}^{\infty} \left(\frac{r^k}{k} - \frac{r^{k+1}}{k+1}\right) \frac{\sin\frac{n\theta}{2} \sin\frac{(n+1)\theta}{2}}{\sin\frac{\theta}{2}} \right| \\
&< \sum_{k=2n}^{\infty} \left(\frac{r^k}{k} - \frac{r^{k+1}}{k+1}\right) \csc\frac{\theta}{2} \\
&= \left(\frac{r^{2n}}{2n}\right) \csc\frac{\theta}{2}.
\end{aligned}$$

□

Lemma 2.5.2. *Let $r \in (\frac{1}{2}, 1)$, $n \in \mathbb{N}$ and $\theta \in [\frac{\pi}{3}, \pi)$. Then*

$$\sum_{k=1}^{2n-1} \frac{r^k \sin k\theta}{k} < (1 + r^{2n-1}) \tan^{-1}\left(\frac{r \sin\theta}{1 - r\cos\theta}\right).$$

Proof. In view of Lemma 2.5.1, it is sufficient to show that

$$\tan^{-1}\left(\frac{r \sin\theta}{1 - r\cos\theta}\right) > \frac{r}{2n} \csc\frac{\theta}{2}$$

when $n \in \mathbb{N}$ and $\theta \in [\frac{\pi}{3}, \pi - \frac{\pi}{2n})$.

When $\theta \in (\frac{\pi}{3}, \pi)$,

$$\frac{d}{d\theta}\left(\tan^{-1}\left(\frac{r\sin\theta}{1-r\cos\theta}\right)\right) = -\frac{r(r-\cos\theta)}{1-2r\cos\theta+r^2} < 0$$

and

$$\frac{d}{d\theta}\left(\frac{r}{2n}\csc\frac{\theta}{2}\right) = -\frac{r\left(\csc\frac{\theta}{2}\right)^3\sin\theta}{8n} < 0$$

for $r \in (\frac{1}{2}, 1)$. Hence, we conclude that $\theta \mapsto \tan^{-1}\left(\frac{r\sin\theta}{1-r\cos\theta}\right)$ and $\theta \mapsto \frac{r}{2n}\csc\frac{\theta}{2}$ are decreasing on $[\frac{\pi}{3}, \pi)$.

We consider 4 cases.

Firstly, for $\theta \in [\frac{\pi}{3}, \frac{2\pi}{3}]$, we have

$$\tan^{-1}\left(\frac{r\sin\frac{2\pi}{3}}{1-r\cos\frac{2\pi}{3}}\right) - \frac{r}{2n}\csc\frac{\pi}{6} > 0$$

for $r \in (\frac{1}{2}, 1)$. Hence, we conclude that $\tan^{-1}\left(\frac{r\sin\theta}{1-r\cos\theta}\right) > \frac{r}{2n}\csc\frac{\theta}{2}$ when $\theta \in [\frac{\pi}{3}, \frac{2\pi}{3}]$.

Similarly, for $\theta \in [\frac{2\pi}{3}, \frac{3\pi}{4}]$, we have

$$\tan^{-1}\left(\frac{r\sin\frac{3\pi}{4}}{1-r\cos\frac{3\pi}{4}}\right) - \frac{r}{2n}\csc\frac{\pi}{3} > 0$$

for $r \in (\frac{1}{2}, 1)$. Hence, we conclude that $\tan^{-1}\left(\frac{r\sin\theta}{1-r\cos\theta}\right) > \frac{r}{2n}\csc\frac{\theta}{2}$ when $\theta \in [\frac{2\pi}{3}, \frac{3\pi}{4}]$.

Next, for $\theta \in [\frac{3\pi}{4}, \pi - \frac{\pi}{2n}]$, we have

$$\begin{aligned}
&\tan^{-1}\left(\frac{r\sin\frac{\pi}{2n}}{1+r\cos\frac{\pi}{2n}}\right) - \frac{r}{2n}\csc\frac{3\pi}{8} \\
&> \tan^{-1}\left(r\tan\frac{\pi}{4n}\right) - \frac{r}{2n}\csc\frac{3\pi}{8} \\
&> r\tan\frac{\pi}{4n} - \frac{r^3\left(\tan\frac{\pi}{4n}\right)^3}{3} - \frac{r}{2n}\csc\frac{3\pi}{8} \\
&> r\left(\frac{\pi}{4n}\right) - \frac{r^3\left(\frac{\pi}{4n}\right)^3}{3} - \frac{r}{2n}\csc\frac{3\pi}{8} \\
&= \frac{r}{n}\left(\frac{\pi}{4} - \frac{r^2\pi^3}{192n^2} - \frac{1}{2}\csc\frac{3\pi}{8}\right) \\
&> 0
\end{aligned}$$

for $r \in (\frac{1}{2}, 1)$ and $n > 2$.

Finally, on the interval $(\pi - \frac{\pi}{2n}, \pi)$, we have

$$\begin{aligned}
&\frac{d}{d\theta}\left(\left(1+r^{2n-1}\right)\tan^{-1}\left(\frac{r\sin\theta}{1-r\cos\theta}\right) - \sum_{k=1}^{2n-1}\frac{r^k\sin k\theta}{k}\right) \\
=&r^{2n-1}\left(\frac{r\cos\theta - r^2}{1-2r\cos\theta+r^2}\right) - \frac{r^{2n}\cos 2n\theta - r^{2n+1}\cos(2n-1)\theta}{1-2r\cos\theta+r^2} \\
=&\frac{r^{2n}\left(\cos\theta+\cos 2n\theta\right) - r^{2n+1}\left(1+\cos(2n-1)\theta\right)}{1-2r\cos\theta+r^2}.
\end{aligned}$$

We now let $\alpha = \pi - \theta$, then

$$\begin{aligned}
\cos\theta + \cos 2n\theta &= -\cos\alpha + \cos 2n\alpha \\
&< 0
\end{aligned}$$

provided that $\alpha \in (0, \frac{\pi}{2n}]$. We hence conclude that the function $\theta \mapsto \left(1+r^{2n-1}\right)\tan^{-1}\left(\frac{r\sin\theta}{1-r\cos\theta}\right) - \sum_{k=1}^{2n-1}\frac{r^k \sin k\theta}{k}$ is decreasing on the interval $[\pi - \frac{\pi}{2n}, \pi)$, and is equal to 0 at $\theta = \pi$. Hence,

$$\left(1+r^{2n-1}\right)\tan^{-1}\left(\frac{r\sin\theta}{1-r\cos\theta}\right) - \sum_{k=1}^{2n-1}\frac{r^k \sin k\theta}{k} > 0$$

for $\theta \in (\pi - \frac{\pi}{2n}, \pi)$. This completes the proof. $\square$

Now, let $\theta \in [\pi - \frac{\pi}{2n}, \pi)$ and let

$$g_n(\theta) = -\sum_{k=1}^{2n-1}\frac{r^k \cos k\theta}{k} + \frac{\left(\left(1+r^{2n-1}\right)\tan^{-1}\left(\frac{r\sin\theta}{1-r\cos\theta}\right)\right)^2}{2\sum_{k=1}^{2n-1}\frac{r^k}{k}}.$$

Theorem 2.5.3. *Let n be any integer satisfying $3 \leqslant n \leqslant 29$. If $\theta \in (\pi - \frac{\pi}{2n}, \pi)$, then $g_n'(\theta) > 0$.*

Proof. We apply Lemma 2.3.1 to show that for each $\theta \in (\pi - \frac{\pi}{2n}, \pi)$,

$$\begin{aligned}
&g_n'(\theta)\\
&= \sum_{k=1}^{2n-1} r^k \sin k\theta + \frac{\left(1+r^{2n-1}\right)^2\left(\tan^{-1}\left(\frac{r\sin\theta}{1-r\cos\theta}\right)\right)\cdot\frac{r\cos\theta - r^2}{1-2r\cos\theta+r^2}}{\sum_{k=1}^{2n-1}\frac{r^k}{k}}\\
&= \frac{r\sin\theta - r^{2n}\sin(2n\theta) + r^{2n+1}\sin(2n-1)\theta}{r^2 - 2r\cos\theta + 1} + \frac{\left(1+r^{2n-1}\right)^2\left(\tan^{-1}\left(\frac{r\sin\theta}{1-r\cos\theta}\right)\right)\cdot\frac{r\cos\theta - r^2}{1-2r\cos\theta+r^2}}{\sum_{k=1}^{2n-1}\frac{r^k}{k}}\\
&> \frac{r\sin\theta - r^{2n}\sin(2n\theta) + r^{2n+1}\sin(2n-1)\theta}{r^2 - 2r\cos\theta + 1} + \frac{\left(1+r^{2n-1}\right)^2\left(\frac{r\sin\theta}{1-r\cos\theta}\right)\cdot\frac{r\cos\theta - r^2}{1-2r\cos\theta+r^2}}{\sum_{k=1}^{2n-1}\frac{r^k}{k}} \text{ (since } \tan^{-1}x < x \text{ and } r\cos\theta < r^2)\\
&= \frac{r\sin\theta}{1-2r\cos\theta+r^2}\left\{\frac{r\sin\theta - r^{2n}\sin 2n\theta + r^{2n+1}\sin(2n-1)\theta}{r\sin\theta} + \frac{(1+r^{2n-1})^2}{\sum_{k=1}^{2n-1}\frac{r^k}{k}}\cdot\frac{r\cos\theta - r^2}{1-r\cos\theta}\right\}\\
&\geq \frac{r\sin\theta}{1-2r\cos\theta+r^2}\left\{\frac{r\sin\theta - r^{2n}\sin 2n\theta + r^{2n+1}\sin(2n-1)\theta}{r\sin\theta} - \frac{r(1+r^{2n-1})^2}{\sum_{k=1}^{2n-1}\frac{r^k}{k}}\right\},
\end{aligned}$$

as the function $\theta \mapsto \frac{r\cos\theta - r^2}{1-r\cos\theta}$ is minimised at its right endpoint due to its decreasing nature as established by Lemma 2.3.11. We now consider the substitution $\alpha = \pi - \theta$:

$$\begin{aligned}
\frac{r\sin\theta - r^{2n}\sin 2n\theta + r^{2n+1}\sin(2n-1)\theta}{r\sin\theta} &= 1 + \frac{-r^{2n}\sin(2n\pi - 2n\alpha) + r^{2n+1}\sin((2n-1)\pi - (2n-1)\alpha)}{r\sin(\pi-\alpha)}\\
&= 1 + \frac{-r^{2n}\sin(-2n\alpha) - r^{2n+1}\sin(-(2n-1)\alpha)}{r\sin\alpha}\\
&= 1 + \frac{r^{2n}\sin(2n\alpha) + r^{2n+1}\sin(2n-1)\alpha}{r\sin\alpha}.
\end{aligned}$$

It is a well known result that for $k \in \mathbb{N}\backslash\{1\}$, the function $x \mapsto \frac{\sin kx}{\sin x}$ is decreasing on $(0, \frac{\pi}{k})$, hence the function $\theta \mapsto \frac{r\sin\theta - r^{2n}\sin 2n\theta + r^{2n+1}\sin(2n-1)\theta}{r\sin\theta}$ is increasing on $(\pi - \frac{\pi}{2n}, \pi)$.

Thus,

$$\begin{aligned}
g_n'(\theta) &> \frac{r\sin\theta}{1-2r\cos\theta+r^2}\left\{\lim_{\theta\mapsto(\pi-\frac{\pi}{2n})^+}\frac{r\sin\theta - r^{2n}\sin 2n\theta + r^{2n+1}\sin(2n-1)\theta}{r\sin\theta} - \frac{r(1+r^{2n-1})^2}{\sum_{k=1}^{2n-1}\frac{r^k}{k}}\right\}\\
&= \frac{r\sin\theta}{1-2r\cos\theta+r^2}\left\{1 + r^{2n} - \frac{r(1+r^{2n-1})^2}{\sum_{k=1}^{2n-1}\frac{r^k}{k}}\right\}.
\end{aligned}$$

Since

$$1 + r^{2n} - \frac{r(1+r^{2n-1})^2}{\sum_{k=1}^{2n-1}\frac{r^k}{k}} > 0 \text{ for } n = 3, 4, 5, \ldots, 29,$$

and for $r \in \left(\frac{1}{2}, 1\right)$, the proof is complete.

$\square$

Theorem 2.5.4. *Let n be any integer satisfying $3 \leqslant n \leqslant 29$. Then*

$$g_n(\theta) + \sum_{k=1}^{2n-1} \frac{r^k \cos k\pi}{k} < 0 \tag{2.33}$$

for $\theta \in (\pi - \frac{\pi}{2n}, \pi)$.

Proof. Since $g_n(\pi) = \sum_{k=1}^{2n-1} \frac{(-1)^{k-1} r^k}{k}$, the result is an immediate consequence of Theorem 2.5.3. $\square$

2.6 Bound for $[\frac{3\pi}{4}, \pi), n = 2$

Lemma 2.6.1. *Let $r \in [\frac{1}{2}, 1)$. Then the function $\theta \mapsto \sum_{k=1}^{3} r^k \cos k\theta$ is decreasing on $[\frac{3\pi}{4}, \pi)$.*

Proof. For each $\theta \in (\frac{3\pi}{4}, \pi)$,

$$\begin{aligned}
\frac{d}{d\theta}\left(\sum_{k=1}^{3} r^k \cos k\theta\right) &= \frac{d}{d\theta}\left(r\cos\theta + r^2\cos 2\theta + r^3 \cos 3\theta\right) \\
&= -r\sin\theta\left(1 + 2r\cos\theta + r^2\frac{\sin 3\theta}{\sin\theta}\right) \\
&\leq -r\sin\theta\left(1 + 2r\cos(\pi) + r^2\frac{\sin\left(3\left(\frac{3\pi}{4}\right)\right)}{\sin\left(\frac{3\pi}{4}\right)}\right) \\
&= -r\sin\theta\left(1 - 2r + r^2\right) \\
&= -r\sin\theta\left(1 - r\right)^2 \\
&< 0.
\end{aligned}$$

Hence, since the function $\theta \mapsto \sum_{k=1}^{3} r^k \cos k\theta$ is continuous on $[\frac{3\pi}{4}, \pi)$, we conclude that $\theta \mapsto \sum_{k=1}^{3} r^k \cos k\theta$ is decreasing on $[\frac{3\pi}{4}, \pi)$. This completes the proof. $\square$

Next, let $r \in (\frac{1}{2}, 1)$ and $\theta \in (\frac{3\pi}{4}, \pi)$ and let

$$h(\theta) = -\sum_{k=1}^{3} \frac{r^k \cos k\theta}{k} + \frac{\left(\sum_{k=1}^{3} \frac{r^k \sin k\theta}{k}\right)^2}{\sum_{k=1}^{3} \frac{r^k}{k}}.$$

We now use Lemma 2.6.1 to prove the following theorem.

Theorem 2.6.2. *If $\theta \in (\frac{3\pi}{4}, \pi)$, then $h'(\theta) > 0$.*

Proof. For each $\theta \in (\frac{3\pi}{4}, \pi)$, we infer from Lemma 2.6.1 that

$$\begin{aligned}
h'(\theta) &= \sum_{k=1}^{3} r^k \sin k\theta + \frac{\left(\sum_{k=1}^{3} \frac{r^k \sin k\theta}{k}\right)\left(\sum_{k=1}^{3} r^k \cos k\theta\right)}{\sum_{k=1}^{3} \frac{r^k}{k}} \\
&\geq \sum_{k=1}^{3} r^k \sin k\theta + \frac{\left(\sum_{k=1}^{3} \frac{r^k \sin k\theta}{k}\right)\left(\sum_{k=1}^{3} r^k \cos k\pi\right)}{\sum_{k=1}^{3} \frac{r^k}{k}} \\
&= \sum_{k=1}^{3} r^k \sin k\theta + \frac{\left(\sum_{k=1}^{3} \frac{r^k \sin k\theta}{k}\right)\left(\sum_{k=1}^{3} (-1)^k r^k\right)}{\sum_{k=1}^{3} \frac{r^k}{k}} \\
&= \sin\theta\left(r + \frac{r\sum_{k=1}^{3}(-1)^k r^k}{\sum_{k=1}^{3}\frac{r^k}{k}}\right) + \sin 2\theta\left(r^2 + \frac{r^2\sum_{k=1}^{3}(-1)^k r^k}{2\sum_{k=1}^{3}\frac{r^k}{k}}\right) + \sin 3\theta\left(r^3 + \frac{r^3\sum_{k=1}^{3}(-1)^k r^k}{3\sum_{k=1}^{3}\frac{r^k}{k}}\right) \\
&= \sin\theta\left\{\left(r + \frac{r\sum_{k=1}^{3}(-1)^k r^k}{\sum_{k=1}^{3}\frac{r^k}{k}}\right) + 2\cos\theta\left(r^2 + \frac{r^2\sum_{k=1}^{3}(-1)^k r^k}{2\sum_{k=1}^{3}\frac{r^k}{k}}\right) + \frac{\sin 3\theta}{\sin\theta}\left(r^3 + \frac{r^3\sum_{k=1}^{3}(-1)^k r^k}{3\sum_{k=1}^{3}\frac{r^k}{k}}\right)\right\}.
\end{aligned}$$

Firstly, for $\theta \in (\frac{3\pi}{4}, \frac{4\pi}{5})$,

$$\begin{aligned}
h'(\theta) &\geq \sin\theta\left\{\left(r + \frac{r\sum_{k=1}^{3}(-1)^k r^k}{\sum_{k=1}^{3}\frac{r^k}{k}}\right) + 2\cos\left(\frac{4\pi}{5}\right)\left(r^2 + \frac{r^2\sum_{k=1}^{3}(-1)^k r^k}{2\sum_{k=1}^{3}\frac{r^k}{k}}\right) + \frac{\sin\left(3\left(\frac{3\pi}{4}\right)\right)}{\sin\left(\frac{3\pi}{4}\right)}\left(r^3 + \frac{r^3\sum_{k=1}^{3}(-1)^k r^k}{3\sum_{k=1}^{3}\frac{r^k}{k}}\right)\right\} \\
&> 0.
\end{aligned}$$

Next, for $\theta \in (\frac{4\pi}{5}, \pi)$,

$$h'(\theta) \geq \sin\theta \left\{ \left(r + \frac{r\sum_{k=1}^{3}(-1)^k r^k}{\sum_{k=1}^{3}\frac{r^k}{k}} \right) + 2\cos(\pi)\left(r^2 + \frac{r^2\sum_{k=1}^{3}(-1)^k r^k}{2\sum_{k=1}^{3}\frac{r^k}{k}} \right) + \frac{\sin\left(3\left(\frac{4\pi}{5}\right)\right)}{\sin\left(\frac{4\pi}{5}\right)} \left(r^3 + \frac{r^3\sum_{k=1}^{3}(-1)^k r^k}{3\sum_{k=1}^{3}\frac{r^k}{k}} \right) \right\}$$
$$> 0.$$

This completes the proof.

□

Theorem 2.6.3. *If $\theta \in [\frac{3\pi}{4}, \pi)$, then*

$$h(\theta) + \sum_{k=1}^{3} \frac{r^k \cos k\pi}{k} < 0.$$

Proof. Since $h(\pi) = 0$, the result follows from Theorem 2.6.2. □

2.7 Bounds for $[\frac{\pi}{3}, \pi)$, $n \geqslant 30$

Let $r \in (\frac{1}{2}, 1)$, $\theta \in [\frac{\pi}{3}, \pi)$ and let

$$\begin{aligned} p_n(\theta) =& \frac{1}{2}\ln(1 - 2r\cos\theta + r^2) - \ln(1+r) - 2\left(\frac{r^{2n}}{2n} + \frac{r^{2n+1}}{2n-1}\right)\left(\frac{1}{(1+r)^2} - \frac{1}{1-2r\cos\theta + r^2}\right) \\ &+ \frac{\left(1 + r^{2n-1}\right)^2 \left(\tan^{-1}\left(\frac{r\sin\theta}{1-r\cos\theta}\right)\right)^2}{2\sum_{k=1}^{2n-1}\frac{r^k}{k}}. \end{aligned}$$

Theorem 2.7.1. *Let n be any integer satisfying $n \geqslant 30$. If $\theta \in (\frac{\pi}{3}, \pi)$, then $p_n'(\theta) > 0$.*

Proof. For each $\theta \in (\frac{\pi}{3}, \frac{\pi}{2})$, we have

$$\begin{aligned} p_n'(\theta) &= \frac{d}{d\theta}\left\{ \frac{1}{2}\ln\left(1 - 2r\cos\theta + r^2\right) + \frac{2\left(\frac{r^{2n}}{2n} + \frac{r^{2n+1}}{2n-1}\right)}{1 - 2r\cos\theta + r^2} + \frac{\left(1 + r^{2n-1}\right)^2}{2\sum_{k=1}^{2n-1}\frac{r^k}{k}}\left(\tan^{-1}\left(\frac{r\sin\theta}{1 - r\cos\theta}\right)\right)^2 \right\} \\ &= \frac{r\sin\theta}{1 - 2r\cos\theta + r^2} - 4\left(\frac{r^{2n}}{2n} + \frac{r^{2n+1}}{2n-1}\right)\frac{r\sin\theta}{(1 - 2r\cos\theta + r^2)^2} + \frac{\left(1 + r^{2n-1}\right)^2}{\sum_{k=1}^{2n-1}\frac{r^k}{k}} \cdot \frac{r\cos\theta - r^2}{1 - 2r\cos\theta + r^2} \cdot \tan^{-1}\left(\frac{r\sin\theta}{1 - r\cos\theta}\right) \\ &> \frac{r\sin\theta}{(1 - 2r\cos\theta + r^2)^2}\left\{ 1 - 2r\cos\theta + r^2 - 4\left(\frac{r^{2n}}{2n} + \frac{r^{2n+1}}{2n-1}\right) + \frac{\left(1 + r^{2n-1}\right)^2}{\sum_{k=1}^{2n-1}\frac{r^k}{k}} \cdot \frac{r\cos\theta - r^2}{1 - r\cos\theta} \cdot (1 - 2r\cos\theta + r^2) \right\}. \end{aligned}$$

Thus, it remains to show that

$$1 - 2r\cos\theta + r^2 - 4\left(\frac{r^{2n}}{2n} + \frac{r^{2n+1}}{2n-1}\right) + \frac{\left(1 + r^{2n-1}\right)^2}{\sum_{k=1}^{2n-1}\frac{r^k}{k}} \cdot \frac{r\cos\theta - r^2}{1 - r\cos\theta} \cdot (1 - 2r\cos\theta + r^2) > 0.$$

By Lemma 2.3.11,

$$\begin{aligned} & 1 - 2r\cos\theta + r^2 - 4\left(\frac{r^{2n}}{2n} + \frac{r^{2n+1}}{2n-1}\right) + \frac{\left(1 + r^{2n-1}\right)^2}{\sum_{k=1}^{2n-1}\frac{r^k}{k}} \cdot \frac{r\cos\theta - r^2}{1 - r\cos\theta} \cdot (1 - 2r\cos\theta + r^2) \\ > \ & 1 - 2r\cos\theta + r^2 - 4\left(\frac{r^{2n}}{2n} + \frac{r^{2n+1}}{2n-1}\right) + \frac{\left(1 + r^{2n-1}\right)^2}{\sum_{k=1}^{2n-1}\frac{r^k}{k}} \cdot \frac{r\cos\frac{\pi}{2} - r^2}{1 - r\cos\frac{\pi}{2}} \cdot (1 - 2r\cos\theta + r^2) \\ > \ & \left(1 - \frac{r^2\left(1 + r^{2n-1}\right)^2}{\sum_{k=1}^{2n-1}\frac{r^k}{k}}\right)\left(1 - 2r\cos\frac{\pi}{3} + r^2\right) - 4\left(\frac{r^{2n}}{2n} + \frac{r^{2n+1}}{2n-1}\right) \\ > \ & 0, \end{aligned}$$

where the last inequality holds because the sequence

$$\left(\left(1 - \frac{r^2\left(1 + r^{2n-1}\right)^2}{\sum_{k=1}^{2n-1}\frac{r^k}{k}}\right)\left(1 - r + r^2\right) - 4\left(\frac{r^{2n}}{2n} + \frac{r^{2n+1}}{2n-1}\right)\right)_{n=30}^{\infty}$$

is increasing whenever $r \in \left(\frac{1}{2}, 1\right)$, we have the following result:

$$\left(1 - \frac{r\left(1+r^{59}\right)^2}{\sum_{k=1}^{59} \frac{r^k}{k}}\right)\left(1 - r + r^2\right) - 4\left(\frac{r^{60}}{60} + \frac{r^{61}}{59}\right) > 0.$$

This completes the proof.

□

Similarly, the result holds if $\theta \in (\frac{\pi}{2}, \pi)$.

Theorem 2.7.2. *Let n be any integer satisfying $n \geqslant 30$. Then $p_n(\theta) < 0$ for $\theta \in (\frac{\pi}{3}, \pi)$.*

Proof. Since $p_n(\pi) = 0$, the theorem is a consequence of Theorem 2.7.1. □

We are now ready to prove Theorem 2.0.1.

2.8 Proof of Theorem 2.0.1

Firstly, the case $n = 1$ can be directly shown to be true, as it is equivalent to

$$r^2(1-\cos\theta)^2 + r^2\sin^2\theta < 4r^2,$$

which is true if and only if $\cos\theta > -1$ for $\theta \in [0, \pi)$.

Next, we apply Theorem 2.1.7 to show that (4) is valid when $n \geqslant 2$ and $\theta \in [0, \frac{\pi}{4}]$ and when $n \geqslant 3$ and $\theta \in [\frac{\pi}{3}, \frac{\pi}{4}]$. The case $n = 2$ whenever $\theta \in [\frac{\pi}{4}, \frac{\pi}{3}]$ is proven in view of Lemma 2.2.1.

We would now need the following lemma for the proof of Theorem 2.0.1 when $\frac{1}{2} < r < 1$ and $\theta \in [\frac{\pi}{3}, \pi)$:

Lemma 2.8.1. *Let $n \in \mathbb{N}, r \in (\frac{1}{2}, 1)$ and $\theta \in [\frac{\pi}{3}, \pi)$. Then*

$$\left|\sum_{k=1}^{2n-1} \frac{r^k}{k}\cos k\theta\right| \leqslant \sum_{k=1}^{2n-1} \frac{(-1)^{k-1} r^k}{k}.$$

Proof. We apply Theorem 2.1.1 to show that for any integer $n \geqslant 7$,

$$\begin{aligned}
\max_{\theta\in\left[\frac{\pi}{3},\pi\right]}\left\{\sum_{k=1}^{n}\frac{\cos k\theta}{k} + \ln\left(2\sin\frac{\theta}{2}\right)\right\} &\leqslant \max_{\theta\in\left[\frac{5\pi}{2n+1},\pi\right]}\left\{\sum_{k=1}^{n}\frac{\cos k\theta}{k} + \ln\left(2\sin\frac{\theta}{2}\right)\right\} \\
&= \sum_{k=1}^{n}\frac{\cos\frac{5k\pi}{2n+1}}{k} + \ln\left(2\sin\frac{5\pi}{4n+2}\right) \\
&\leqslant \sum_{k=1}^{7}\frac{\cos\frac{5k\pi}{15}}{k} \\
&< \frac{1}{2};
\end{aligned}$$

that is,

$$\begin{aligned}
\max_{\theta\in\left[\frac{\pi}{3},\pi\right]}\sum_{k=1}^{n}\frac{\cos k\theta}{k} &< \frac{1}{2} - \ln\left(2\sin\frac{\pi}{6}\right) \\
&= \frac{1}{2}.
\end{aligned}$$

When $n = 1, 2, \ldots, 6$, a direct computation shows that

$$\max_{\theta\in\left[\frac{\pi}{3},\pi\right]}\sum_{k=1}^{n}\frac{\cos k\theta}{k} \leqslant \frac{1}{2}.$$

Therefore, using summation by parts, we have

$$\sum_{k=1}^{2n-1}\frac{r^k}{k}\cos k\theta = r^{2n-1}\left(\sum_{k=1}^{2n-1}\frac{\cos k\theta}{k}\right) + \sum_{k=1}^{2n-2}\left(r^k - r^{k+1}\right)\sum_{j=1}^{k}\frac{\cos j\theta}{j}$$

$$
\begin{aligned}
&< r^{2n-1}\left(\frac{1}{2}\right) + \left(r - r^{2n-1}\right)\left(\frac{1}{2}\right) \\
&= \frac{r}{2} \\
&< r - \frac{r^2}{2} \\
&< \sum_{k=1}^{2n-1} \frac{(-1)^{k-1} r^k}{k},
\end{aligned} \tag{2.34}
$$

using a known result of the error bound of alternating sums.

Next, using the result (cf. [3, (6.2)]), for $n \in \mathbb{N}$, $r \in (0, 1]$, $\theta \in [0, \pi)$, we have

$$
\sum_{k=1}^{2n-1} \frac{r^k}{k} \cos k\theta > \sum_{k=1}^{2n-1} \frac{(-1)^k r^k}{k}. \tag{2.35}
$$

Therefore, a combination of (2.34) and (2.35) completes the proof. □

In view of Lemma 2.8.1, the following theorem gives an affirmative answer to Theorem 2.0.1 under the additional assumption that $\theta \in [\frac{\pi}{3}, \pi)$:

Theorem 2.8.2. *For any integer $n \geq 1$, $\frac{1}{2} < r < 1$ and $\frac{\pi}{3} \leq \theta < \pi$, the following inequality holds:*

$$
-\sum_{k=1}^{2n-1} \frac{r^k}{k} \cos k\theta + \frac{\left(\sum_{k=1}^{2n-1} \frac{r^k}{k} \sin k\theta\right)^2}{2\sum_{k=1}^{2n-1} \frac{r^k}{k}} < \sum_{k=1}^{2n-1} \frac{(-1)^{k-1} r^k}{k}.
$$

Proof. For $\theta \in [\frac{\pi}{3}, \pi - \frac{\pi}{2n})$ and $n = 5, \ldots, 29$, we use Lemmas 2.3.5, 2.3.10 and 2.4.1 to obtain the following inequality:

$$
\sum_{k=1}^{2n-1} \frac{(-1)^k r^k}{k} - \sum_{k=1}^{2n-1} \frac{r^k}{k} \cos k\theta + \frac{(\sum_{k=1}^{2n-1} \frac{r^k}{k} \sin k\theta)^2}{2\sum_{k=1}^{2n-1} \frac{r^k}{k}} < F_n(\theta) < 0.
$$

Next, for each $\theta \in \left[\frac{\pi}{3}, \pi - \frac{\pi}{2n}\right)$ and $n = 2, 3, 4$, we infer from Lemmas 2.2.2 and Lemma 2.2.3 to obtain the following inequality:

$$
\sum_{k=1}^{2n-1} \frac{(-1)^k r^k}{k} - \sum_{k=1}^{2n-1} \frac{r^k}{k} \cos k\theta + \frac{\left(\sum_{k=1}^{2n-1} \frac{r^k}{k} \sin k\theta\right)^2}{2\sum_{k=1}^{2n-1} \frac{r^k}{k}} \quad < \quad 0.
$$

Using this reasoning, for each $\theta \in \left[\pi - \frac{\pi}{2n}, \pi\right)$ and $3 \leqslant n \leqslant 29$ we use Lemmas 2.3.5, 2.3.10 and Theorem 2.5.4 to obtain the following inequality:

$$
\sum_{k=1}^{2n-1} \frac{(-1)^k r^k}{k} - \sum_{k=1}^{2n-1} \frac{r^k}{k} \cos k\theta + \frac{\left(\sum_{k=1}^{2n-1} \frac{r^k}{k} \sin k\theta\right)^2}{2\sum_{k=1}^{2n-1} \frac{r^k}{k}} < g_n(\theta) < 0.
$$

Similarly, for each $\theta \in \left[\pi - \frac{\pi}{2n}, \pi\right)$ and $n = 2$ we infer from Theorem 2.6.3 to obtain the following inequality:

$$
\sum_{k=1}^{2n-1} \frac{(-1)^k r^k}{k} - \sum_{k=1}^{2n-1} \frac{r^k}{k} \cos k\theta + \frac{\left(\sum_{k=1}^{2n-1} \frac{r^k}{k} \sin k\theta\right)^2}{2\sum_{k=1}^{2n-1} \frac{r^k}{k}} \quad < \quad 0.
$$

Lastly, for each $\theta \in \left[\frac{\pi}{3}, \pi\right)$ and integer $n \geqslant 30$, we use Lemmas 2.3.5, 2.3.10 and Theorem 2.7.2 to obtain the following inequality:

$$
\sum_{k=1}^{2n-1} \frac{(-1)^k r^k}{k} - \sum_{k=1}^{2n-1} \frac{r^k}{k} \cos k\theta + \frac{\left(\sum_{k=1}^{2n-1} \frac{r^k}{k} \sin k\theta\right)^2}{2\sum_{k=1}^{2n-1} \frac{r^k}{k}} < p_n(\theta) < 0.
$$

This completes the proof. □

Hence, the proof of Theorem 2.0.1 is complete.

2.9 Proof of Conjecture 1

In view of Theorems 1.0.1 and 2.0.1, we have proven Conjecture 1 for $\frac{1}{2} < r \leqslant 1$.

Theorem 2.9.1. *For any integer $n \geq 1$, $\frac{1}{2} < r \leq 1$ and $0 \leq \theta < \pi$, the following inequality holds:*

$$\left(\sum_{k=1}^{2n-1} \frac{r^k}{k}(1-\cos k\theta)\right)^2 + \left(\sum_{k=1}^{2n-1} \frac{r^k}{k} \sin k\theta\right)^2 < 4\left(\sum_{k=1}^{n} \frac{r^{2k-1}}{2k-1}\right)^2.$$

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Bibliography

[1] D. Aharonov, S. Friedland, On an inequality connected with the coefficient conjecture for functions of bounded boundary rotation, Ann. Acad. Sci. Fenn., Ser. A 1 Math. 524 (1972), 14 p.

[2] D.A. Brannan, On coefficient problems for certain power series. Proceedings of the Symposium on Complex Analysis (Univ. Kent, Canterbury, 1973), pp. 1727. London Math. Soc. Lecture Note Ser., No. 12, Cambridge Univ. Press, London, 1974.

[3] R.W. Barnard, U.C. Jayatilake and A.Yu. Solynin, Brannan's conjecture and trigonometric sums, Proc. Amer. Math. Soc. 143(5) (2015) 2117−2128.

[4] J.Q. Chong, X.C. Huang, T.Y. Lee, J.T. Li, H.X. Sim, J.R. Soh, G.J. Tan, J.K.H. Tai, Some functional upper bounds for Fejér's sine polynomial, to appear in Studia Scientiarum Mathematicarum Hungarica.

[5] J.Q. Chong, X.C. Huang and J.T. Li, A real variable method for summing certain trigonometric series, preprint.

[6] J.Z.Y. Fong, T.Y Lee and P.X. Wong, A functional bound for Young's cosine polynomial, *Acta Math. Hungar.* **160** (2020), 337−342.

[7] J.Z.Y. Fong, T.Y. Lee, R.N. Rao and P.X. Wong, A functional bound for Young's cosine polynomial II, Publ. Math. Debrecen 96 (2020), 445−457.

[8] J.Z.Y. Fong, T.Y. Lee and P.X. Wong, On partial sums of a Fourier series, preprint.

[9] T. H. Gronwall, Über die Gibbssche Erscheinung und die trigonometrischen Summen $\sin x + \frac{1}{2}\sin 2x + \ldots + \frac{1}{n}\sin nx$, Math. Ann. **72** (1912), 288−243.

[10] Y.B. Kim, T.Y. Lee, V. S, H.X. Sim and J.K.H. Tai, A Sharp Trigonometric Double Inequality, Publ. Math. Debrecen **98** (2021), 231−242.

[11] W.H. Young, On a certain series of Fourier, *Proc. London Math. Soc.* **11** (1913), 357−366.