Brannan's conjecture and trigonometric polynomials

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Introduction

Trigonometric inequalities have been an important area of research in mathematics, physics and engineering. In particular, R.W. Barnard et al. [2] have shown that the famous Brannan Conjecture is equivalent to the following open problem.

Conjecture 1 (cf. [2, Conjecture 2]). If $n \in \mathbb{N}$, $0 < r \leqslant 1$ and $\theta \in [0, \pi)$, then

$$\left(\sum_{k=1}^{2n-1} \frac{r^k}{k}(1-\cos k\theta)\right)^2 + \left(\sum_{k=1}^{2n-1} \frac{r^k}{k}\sin k\theta\right)^2 < 4\left(\sum_{k=1}^{n} \frac{r^{2k-1}}{2k-1}\right)^2.$$

In this project, we give a partial answer to the above conjecture.

Theorem 2 If $n \in \mathbb{N}$ and $\theta \in [0, \pi)$, then

$$\left(\sum_{k=1}^{2n-1} \frac{1}{k}(1-\cos k\theta)\right)^2 + \left(\sum_{k=1}^{2n-1} \frac{\sin k\theta}{k}\right)^2 < 4\left(\sum_{k=1}^{n} \frac{1}{2k-1}\right)^2. \tag{1}$$

Proof of Theorem 2

We consider the following functions

$$L_{2n-1} : \theta \mapsto -\sum_{k=1}^{2n-1} \frac{\cos k\theta}{k} + \frac{\left(\sum_{k=1}^{2n-1} \frac{\cos k\theta}{k}\right)^2 + \left(\sum_{k=1}^{2n-1} \frac{\sin k\theta}{k}\right)^2}{2\sum_{k=1}^{2n-1} \frac{1}{k}} \quad \text{and} \quad R : n \mapsto \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} + \frac{\left(\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}\right)^2}{2\sum_{k=1}^{n} \frac{1}{k}},$$

where $n \in \mathbb{N}$ and $\theta \in [0, \pi]$. Then, (1) is equivalent to the following inequality

$$L_{2n-1}(\theta) < R(2n-1).$$

Outline of proof of Theorem 2. We have the following assertions:

(A) $L_{2n-1}(\theta) < R(2n-1)$ for $\theta \in \left[0, \frac{4n-3}{4n-1}\pi\right]$ and $n = 1, 2, 3, 4$.

(B) (1) holds whenever $\theta \in \left[0, \frac{2\pi}{3}\right]$ and $n = 5, 6, 7 \ldots$.

(C) $L_{2n-1}(\theta) < R(2n-1)$ for $\theta \in \left[\frac{2\pi}{3}, \frac{4n-3}{4n-1}\pi\right]$ and $n = 5, 6, 7, \ldots$.

(D) $L_{2n-1}(\theta) < R(2n-1)$ for $\theta \in \left[\frac{4n-3}{4n-1}\pi, \pi\right)$ and $n = 1, 2, 3, \ldots$.

Proof of assertion (A).

Let

$$S_n(\theta) := \sum_{k=1}^{n} \frac{\sin k\theta}{k} \quad \text{and} \quad C_n(\theta) := \sum_{k=1}^{n} \frac{\cos k\theta}{k} \quad (n \in \mathbb{N}\,;\ \theta \in [0, \pi]).$$

Then,

$$S'_{2n-1}(\theta) = \frac{\cos n\theta \sin\left(n-\frac{1}{2}\right)\theta}{\sin\frac{\theta}{2}} \quad \text{and} \quad C'_{2n-1}(\theta) = -\frac{\sin n\theta \sin\left(n-\frac{1}{2}\right)\theta}{\sin\frac{\theta}{2}} \quad (n \in \mathbb{N}\,;\ \theta \in (0, \pi)). \tag{2}$$

Using (2) we show, by computing the stationary points, that

$$\begin{aligned}
\max_{\theta \in [0, \frac{4n-3}{4n-1}\pi]} L_{2n-1}(\theta) &= \max_{\theta \in [0, \frac{4n-3}{4n-1}\pi]} \left\{-C_{2n-1}(\theta) + \frac{(C_{2n-1}(\theta))^2 + (S_{2n-1}(\theta))^2}{2\sum_{k=1}^{2n-1}\frac{1}{k}}\right\} \\
&\leqslant \max_{\theta \in [0, \frac{4n-3}{4n-1}\pi]} \left\{-C_{2n-1}(\theta) + \frac{(C_{2n-1}(\theta))^2}{2\sum_{k=1}^{2n-1}\frac{1}{k}}\right\} + \max_{\theta \in [0, \frac{4n-3}{4n-1}\pi]} \left\{\frac{(S_{2n-1}(\theta))^2}{2\sum_{k=1}^{2n-1}\frac{1}{k}}\right\} \\
&< R(2n-1). \text{ for } n = 1, 2, 3, 4.
\end{aligned}$$

The proof is complete.

Proof of assertion (B).

In this section, we evoke a theorem by Fong et al. [5].

Theorem 3. (cf. [5, Theorem 1.3]). Let $n \in \mathbb{N}$ and let $\theta \in (0, \pi)$. Then

$$\sum_{k=1}^{2\left\lfloor\frac{n}{2}\right\rfloor+1} \frac{(-1)^{k-1}}{k} + \sum_{k=1}^{n} \frac{\cos k\theta}{k} \geqslant \frac{1}{4}(1+\cos\theta)^2,$$

where equality holds if and only if $n = 2$ and $\theta = \pi - \cos^{-1}\frac{1}{3}$.

An application of Theorem 3 shows that (1) is equivalent to

$$-(1+\cos\theta)^2 \sum_{k=1}^{2n-1} \frac{1}{2k-1} + \frac{1}{16}(1+\cos\theta)^4 + \left(\sum_{k=1}^{2n-1} \frac{\sin k\theta}{k}\right)^2 < 0. \tag{3}$$

We next consider a few results from Kim et al. [7].

Lemma 4. (cf. [7, Lemma 2.2]). Let $n \in \mathbb{N}$. If $q \in \{1, 2, \ldots, \lfloor\frac{n+1}{2}\rfloor\}$, then

$$\max_{\theta \in \left[\frac{(4q-2)\pi}{2n+1}, \pi\right]} \left\{\sum_{k=1}^{n} \frac{\sin k\theta}{k} - \frac{\pi-\theta}{2}\right\} = \sum_{k=1}^{n} \frac{\sin k\frac{(4q-2)\pi}{2n+1}}{k} - \frac{\pi - \frac{(4q-2)\pi}{2n+1}}{2}.$$

Theorem 5. (cf. [7, Theorem 2.5]). Let $n \in \mathbb{N}$. If $p \in \mathbb{N}$, then the sequence

$$\left((-1)^{p-1}\left(\sum_{k=1}^{n} \frac{\sin k\frac{2p\pi}{2n+1}}{k} - \frac{\pi - \frac{2p\pi}{2n+1}}{2}\right)\right)_{n=p}^{\infty}$$

is decreasing.

Using Lemma 4 and Theorem 5, we get

$$\max_{\theta \in [0, \frac{\pi}{2}]} \left\{\sum_{k=1}^{2n-1} \frac{\sin k\theta}{k} - \frac{\pi-\theta}{2}\right\} \leqslant \sum_{k=1}^{9} \frac{\sin k\frac{2\pi}{19}}{k} - \frac{\pi - \frac{2\pi}{19}}{2} = 0.282\ldots < \frac{3}{10}. \tag{4}$$

Combining (3) and (4), we have

$$-(1+\cos\theta)^2 \sum_{k=1}^{2n-1} \frac{1}{2k-1} + \frac{1}{16}(1+\cos\theta)^4 + \left(\frac{\pi-\theta}{2} + \frac{3}{10}\right)^2 < 0. \tag{5}$$

A similar reasoning shows that (3) is also true for $\theta \in \left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$. The proof is complete.

Proof of assertion (C).

First, we have

Lemma 6. Let $n \in \mathbb{N}$ and let $\theta \in (0, \pi)$. Then

$$S_{2n-1}(\theta) \;<\; F_n(\theta) := \frac{1}{4n-1}\left(2\csc\frac{\theta}{2} - 1\right) + \frac{\pi-\theta}{2}$$

and

$$C_{2n-1}(\theta) \;>\; G_n(\theta) := -\frac{2}{4n-1}\left(\csc\frac{\theta}{2} - 1\right) + C_{2n-1}(\pi) - \ln\left(\sin\frac{\theta}{2}\right).$$

Illustration of figures. Consider the following two graphs.

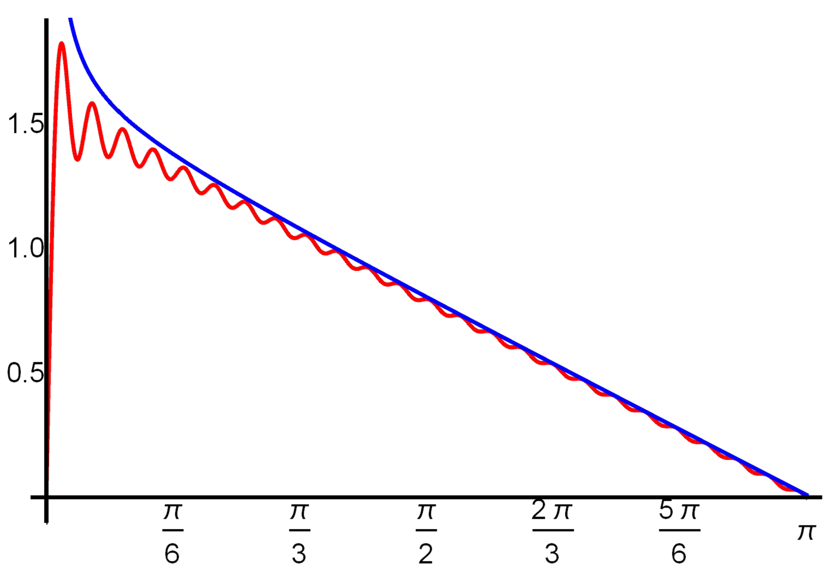

Figure 1: Graphs of S_{49} and F_{25}.

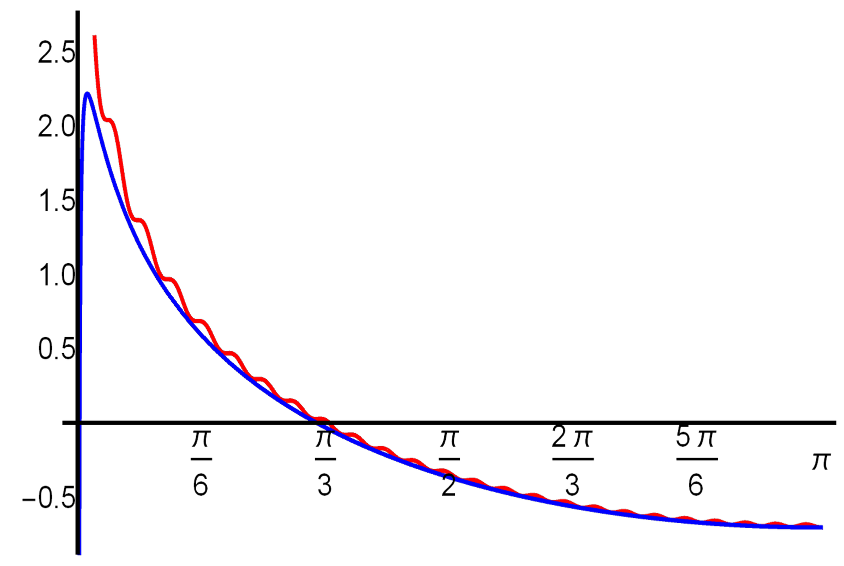

Figure 2: Graphs of C_{49} and G_{25}.

Since Lemma 6 yields

$$L_{2n-1}(\theta) = -C_{2n-1}(\theta) + \frac{C_{2n-1}^2(\theta) + S_{2n-1}^2(\theta)}{2\sum_{k=1}^{2n-1}\frac{1}{k}} < -G_n(\theta) + \frac{G_n^2(\theta) + F_n^2(\theta)}{2\sum_{k=1}^{2n-1}\frac{1}{k}}, \tag{6}$$

we need the following result.

Theorem 7. Let $n \in \mathbb{N}$ and $n \geqslant 5$. Then

$$-G_n + \frac{G_n^2 + F_n^2}{2\sum_{k=1}^{2n-1}\frac{1}{k}} \quad \text{is increasing on} \left[\frac{2\pi}{3}, \frac{(4n-3)\pi}{4n-1}\right]$$

and

$$-G_n\left(\frac{(4n-3)\pi}{4n-1}\right) + \frac{G_n^2\left(\frac{(4n-3)\pi}{4n-1}\right) + F_n^2\left(\frac{(4n-3)\pi}{4n-1}\right)}{2\sum_{k=1}^{2n-1}\frac{1}{k}} < R(2n-1).$$

The proof is complete.

Proof of assertion (D).

We have

$$L'_{2n-1}(\theta) = \frac{\sin n\theta \sin\left(n-\frac{1}{2}\right)\theta}{\sin\frac{\theta}{2}\sum_{k=1}^{2n-1}\frac{1}{k}} \left\{\sum_{k=1}^{2n-1} \frac{1-\cos k\theta}{k} + \cot n\theta \sum_{k=1}^{2n-1} \frac{\sin k\theta}{k}\right\}. \tag{7}$$

Last but not least, since $L_{2n-1}(\pi) = R(2n-1)$, we show that

Theorem 8. Let $n \in \mathbb{N}$. Then L_{2n-1} is increasing on the closed interval $\left[\frac{(4n-3)\pi}{4n-1}, \pi\right]$.

The proof is complete.

Conclusion

When $r = 1$, we give an affirmative answer to Conjecture 1. Moreover, our Theorem 3, Lemma 4, Theorem 5 and Lemma 6 are useful for proving other trigonometric inequalities.

References

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