GATE-EC2023

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Q65ST.2023: Let $\{X_n\}_{n\geq 1}$ be a sequence of independent and identically distributed random variables with mean 0 and variance 1, all of them defined on the same probability space. For n=1,2,3,..., let

$$Y_n = \frac{1}{n}(X_1X_2 + X_3X_4 + ... + X_{2n-1}X_{2n}) \tag{1}$$

Then which of the following statements is/are true?

(A) $\{\sqrt{n}Y_n\}_{n\geq 1}$ converges in distribution to a standard normal random variable.
(B) $\{Y_n\}_{n\geq 1}$ converges in 2nd mean to 0.
(C) $\{Y_n + \frac{1}{n}\}_{n\geq 1}$ converges in probability to 0.
(D) $\{X_n\}_{n\geq 1}$ converges almost surely to 0.

Solution:

1) X_n converges in distribution to a standard normal random variable X, $X_n \xrightarrow{d} X$, then for all x,

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x) \tag{2}$$

where, $X \sim \mathcal{N}(0,1)$

2) X_n converges in p^{th} mean to X, then we have

$$\lim_{n\to\infty} E(|X_n - X|^p) = 0 \tag{3}$$

3) X_n converges in probability to X, $X_n \xrightarrow{p} X$, then for all $\epsilon > 0$,

$$\lim_{n\to\infty} \Pr(|X_n - X| > \epsilon) = 0 \tag{4}$$

4) X_n converges almost surely to X, $X_n \xrightarrow{a.s} X$, then for all $\epsilon > 0$,

$$\lim_{n\to\infty} \Pr(|X_n - X| \geq \epsilon) = 1 \tag{5}$$

5) Chebyshev's inequaltiy,

$$\Pr(|X - \mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2} \tag{6}$$

$$\begin{aligned}
\mathrm{E}(X) &= \int_{-\infty}^{\infty} xF_X x dx && (7)\\
&\because X \text{ is positive-valued} && (8)\\
&= \int_{0}^{\infty} xF_X x dx && (9)\\
&\geq \int_{\epsilon^2}^{\infty} xF_X x dx && (10)\\
&\geq \int_{\epsilon^2}^{\infty} kF_X x dx && (11)\\
&= \epsilon^2 \int_{\epsilon^2}^{\infty} F_X x dx && (12)\\
&= \epsilon^2 \Pr\left(X \geq \epsilon^2\right) && (13)
\end{aligned}$$

Now,

$$\begin{aligned}
\sigma^2 &= \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 && (14)\\
\sigma^2 &= \mathrm{E}(X^2) - \mu^2 && (15)\\
\sigma^2 &= \mathrm{E}(X^2) - \mathrm{E}(\mu^2) && (16)\\
\sigma^2 &= \mathrm{E}(X^2 - \mu^2) && (17)
\end{aligned}$$

using (**??**),

$$\begin{aligned}
\sigma^2 &= \epsilon^2 \Pr\left(X^2 - \mu^2 \geq \epsilon^2\right) && (18)\\
\frac{\sigma^2}{\epsilon^2} &= \Pr\left(X^2 - \mu^2 \geq \epsilon^2\right) && (19)\\
\Pr(|X - \mu| \geq \epsilon) &= \frac{\sigma^2}{\epsilon^2} && (20)
\end{aligned}$$

Now, as X_i is a sequence of independent and identically distributed random variables,
Let $Z_i = X_{2i-1}X_{2i}$.

$$\begin{aligned}
p_{Z_iZ_j}(x) &= p_{X_{2i-1}X_{2i}X_{2j-1}X_{2j}}(x) && (21)\\
&= p_{X_{2i-1}}(x)p_{X_{2i}}(x)p_{X_{2j-1}}(x)p_{X_{2j}}(x) && (22)\\
&= p_{X_{2i-1}X_{2i}}(x)p_{X_{2j-1}X_{2j}}(x) && (23)\\
&= p_{Z_i}(x)p_{Z_j}(x) && (24)
\end{aligned}$$

Z_i is a independent R.V . Similarly,

$$\begin{aligned}
F_{X_{2i-1}}(x) = F_{X_{2i}}(x) = F_{X_{2j-1}}(x) &= F_{X_{2j}}(x) && (25)\\
F_{X_{2i-1}}(x)F_{X_{2i}}(x) &= F_{X_{2j-1}}(x)F_{X_{2j}}(x) && (26)\\
F_{Z_i}(x) &= F_{Z_j}(x) && (27)
\end{aligned}$$

Thus, Z_i is an identical distributed R.V. Hence,

$$Z_n = \sum_{i=1}^{n} Z_i \tag{28}$$

is an i.i.d

$$\mathrm{E}\left(Z_i\right) = \mathrm{E}\left(X_{2i-1}X_{2i}\right) \tag{29}$$
$$= \mathrm{E}\left(X_{2i-1}\right)\mathrm{E}\left(X_{2i}\right) = 0 \tag{30}$$
$$\mathrm{Var}(Z_i) = \mathrm{E}(Z_i^2) - \left[\mathrm{E}\left(Z_i\right)\right]^2 \tag{31}$$
$$= \mathrm{E}(Z_i^2) - 0 \tag{32}$$
$$= \mathrm{E}(X_{2i-1}^2 X_{2i}^2) \tag{33}$$
$$= \mathrm{E}(X_{2i-1}^2)\mathrm{E}(X_{2i}^2) = 1 \tag{34}$$

Now,

$$Y_n = \frac{1}{n}\left(\sum_{i=1}^{n} Z_i\right) \tag{35}$$
$$\mathrm{E}(Y_n) = \mathrm{E}\left(\frac{1}{n}\sum_{i=1}^{n} Z_i\right) \tag{36}$$
$$= \frac{1}{n}\sum_{i=1}^{n}\mathrm{E}(Z_i) = 0 \tag{37}$$

As all the variables are i.i.d's and are thus uncorrelated,

$$\mathrm{cov}\left(X_i, X_j\right) = \begin{cases} 0 & \text{if } i \neq j \\ \mathrm{Var}\left(X_i\right) & \text{if } i = j \end{cases} \tag{38}$$
$$\mathrm{Var}\left(Y_n\right) = \frac{1}{n^2}\left(\sum_{i=1}^{n}\mathrm{cov}\left(Z_i, Z_i\right)\right) \tag{39}$$
$$= \frac{1}{n^2}\left(\sum_{i=1}^{n}\mathrm{var}\left(Z_i\right)\right) \tag{40}$$
$$= \frac{1}{n^2}\left(\sum_{i=1}^{n} 1\right) \tag{41}$$
$$= \frac{1}{n} \tag{42}$$

(A) by central limit theorem,

$$\frac{Y_n - \mu}{\sigma} \sim \mathcal{N}(0,1) \tag{43}$$
$$= \frac{Y_n - 0}{\sqrt{\frac{1}{n}}} \sim \mathcal{N}(0,1) \tag{44}$$
$$= \sqrt{n}Y_n \sim \mathcal{N}(0,1) \tag{45}$$

Thus,

$$\lim_{n\to\infty} F_{\sqrt{n}Y_n} = F_P(x) \tag{46}$$

where, $P \sim \mathcal{N}(0,1)$
So, option (A) is correct

(B) For 2nd mean to be converging to 0,

$$\lim_{n\to\infty} E(|Y_n - 0|^2) = \lim_{n\to\infty} E(Y_n^2) \tag{47}$$
$$\because \frac{1}{n} = E(Y_n^2) - [E(Y_n)]^2 \tag{48}$$
$$= \lim_{n\to\infty}\frac{1}{n} + [E(Y_n)]^2 \tag{49}$$
$$\because [E(Y_n)]^2 = 0 \tag{50}$$
$$= \lim_{n\to\infty}\frac{1}{n} = 0 \tag{51}$$

Thus,$\{Y_n\}_{n\geq 1}$ converges in 2nd mean to 0
Hence, option (B) is correct.

(C) For $\{Y_n + \frac{1}{n}\}_{n\geq 1}$ to be converging $Y_n + \frac{1}{n} \xrightarrow{p} 0$

$$\lim_{n\to\infty}\Pr\left(\left|Y_n + \frac{1}{n} - Y\right| \geq \epsilon\right) = \lim_{n\to\infty}\Pr\left(\left|Y_n + \frac{1}{n}\right| \geq \epsilon\right) \tag{52}$$
$$\leq \lim_{n\to\infty}\frac{\mathrm{Var}(Y_n + \frac{1}{n})}{\epsilon^2} \tag{53}$$
$$\forall \epsilon > 0 \tag{54}$$
$$= \lim_{n\to\infty}\frac{\mathrm{Var}(Y_n) + \mathrm{Var}(\frac{1}{n})}{\epsilon^2} \tag{55}$$
$$= \lim_{n\to\infty}\frac{\frac{1}{n} + 0}{\epsilon^2} \tag{56}$$
$$= 0 \tag{57}$$

hence, option (C) is correct

(D) As all the variables are i.i.d's and are thus uncorrelated,

$$\mathrm{Var}\left(X_n\right) = \sum_{i=1}^{n}\mathrm{cov}\left(X_i, X_i\right) \tag{58}$$
$$= \sum_{i=1}^{n}\mathrm{var}\left(X_i\right) \tag{59}$$
$$= \sum_{i=1}^{n} 1 \tag{60}$$
$$= n \tag{61}$$

$$\lim_{n\to\infty} \Pr\left(|X_n - 0| \geq \epsilon\right) \leq \lim_{n\to\infty} \frac{Var(X_n)}{\epsilon^2} \tag{62}$$

$$\forall \epsilon > 0 \tag{63}$$

$$= \lim_{n\to\infty} \frac{Var(X_n)}{\epsilon^2} \tag{64}$$

$$= \lim_{n\to\infty} \frac{n}{\epsilon^2} = \infty \tag{65}$$

Hence,option (D) is incorrect