GATE-EC2023

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Q65ST.2023: Let $\{X_n\}_{n\geq1}$ be a sequence of independent and identically distributed random variables with mean 0 and variance 1, all of them defined on the same probability space. For n=1,2,3,..., let

$$Y_n = \frac{1}{n}(X_1X_2 + X_3X_4 + ... + X_{2n-1}X_{2n}) \tag{1}$$

Then which of the following statements is/are true?

(A) $\{\sqrt{n}Y_n\}_{n\geq1}$ converges in distribution to a standard normal random variable.
(B) $\{Y_n\}_{n\geq1}$ converges in 2nd mean to 0.
(C) $\{Y_n + \frac{1}{n}\}_{n\geq1}$ converges in probability to 0.
(D) $\{X_n\}_{n\geq1}$ converges almost surely to 0.

Solution:

1) X_n converges in distribution to a standard normal random variable X, $X_n \xrightarrow{d} X$, then for all x,

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x) \tag{2}$$

where, $X \sim \mathcal{N}(0,1)$

2) X_n converges in p^{th} mean to X, then we have

$$\lim_{n\to\infty} E(|X_n - X|^p) = 0 \tag{3}$$

3) X_n converges in probability to X, $X_n \xrightarrow{p} X$, then for all $\epsilon > 0$,

$$\lim_{n\to\infty} \Pr(|X_n - X| > \epsilon) = 0 \tag{4}$$

4) X_n converges almost surely to X, $X_n \xrightarrow{p} X$, then for all $\epsilon > 0$,

$$\lim_{n\to\infty} \Pr(|X_n - X| \geq \epsilon) = 1 \tag{5}$$

5) Chebyshev's inequaltiy,

$$\Pr(|X - \mu| \geq k\sigma) \leq \frac{\sigma^2}{(\sigma k)^2} \tag{6}$$

Now, as X_i is a sequence of independent and identically distributed random variables,

Let $Z_i = X_{2i-1}X_{2i}$.

$$\begin{aligned} p_{Z_iZ_j}(x) &= p_{X_{2i-1}X_{2i}X_{2j-1}X_{2j}}(x) && (7)\\ &= p_{X_{2i-1}}(x)p_{X_{2i}}(x)p_{X_{2j-1}}(x)p_{X_{2j}}(x) && (8)\\ &= p_{X_{2i-1}X_{2i}}(x)p_{X_{2j-1}X_{2j}}(x) && (9)\\ &= p_{Z_i}(x)p_{Z_j}(x) && (10)\end{aligned}$$

Z_i is a independent R.V . Similarly,

$$\begin{aligned} F_{X_{2i-1}}(x) = F_{X_{2i}}(x) = F_{X_{2j-1}}(x) = F_{X_{2j}}(x) && (11)\\ F_{X_{2i-1}}(x)F_{X_{2i}}(x) = F_{X_{2j-1}}(x)F_{X_{2j}}(x) && (12)\\ F_{Z_i}(x) = F_{Z_j}(x) && (13)\end{aligned}$$

Thus, Z_i is an identical distributed R.V.
Hence,

$$Z_n = \sum_{i=1}^{n} Z_i \tag{14}$$

is an i.i.d

$$\begin{aligned} \mathrm{E}(Z_i) &= \mathrm{E}(X_{2i-1}X_{2i}) && (15)\\ &= \mathrm{E}(X_{2i-1})\,\mathrm{E}(X_{2i}) = 0 && (16)\\ \mathrm{Var}(Z_i) &= \mathrm{E}(Z_i^2) - [\mathrm{E}(Z_i)]^2 && (17)\\ &= \mathrm{E}(Z_i^2) - 0 && (18)\\ &= \mathrm{E}(X_{2i-1}^2X_{2i}^2) && (19)\\ &= \mathrm{E}(X_{2i-1}^2)\mathrm{E}(X_{2i}^2) = 1 && (20)\end{aligned}$$

Now,

$$Y_n = \frac{1}{n}\left(\sum_{i=1}^{n} Z_i\right) \tag{21}$$

$$\begin{aligned} \mathrm{E}(Y_n) &= \mathrm{E}\left(\frac{1}{n}\sum_{i=1}^{n} Z_i\right) && (22)\\ &= \frac{1}{n}\sum_{i=1}^{n}\mathrm{E}(Z_i) = 0 && (23)\end{aligned}$$

As all the variables are i.i.d's and are thus uncorrelated,

$$\operatorname{cov}\left(X_i, X_j\right) = \begin{cases} 0 & \text{if } i \neq j \\ \operatorname{Var}(X_i) & \text{if } i = j \end{cases} \tag{24}$$

$$\operatorname{Var}(Y_n) = \frac{1}{n^2}\left(\sum_{i=1}^{n} \operatorname{cov}(Z_i, Z_i)\right) \tag{25}$$

$$= \frac{1}{n^2}\left(\sum_{i=1}^{n} \operatorname{var}(Z_i)\right) \tag{26}$$

$$= \frac{1}{n^2}\left(\sum_{i=1}^{n} 1\right) \tag{27}$$

$$= \frac{1}{n} \tag{28}$$

(A) by central limit theorem,

$$\frac{Y_n - \mu}{\sigma} \sim \mathcal{N}(0,1) \tag{29}$$

$$= \frac{Y_n - 0}{\sqrt{\frac{1}{n}}} \sim \mathcal{N}(0,1) \tag{30}$$

$$= \sqrt{n} Y_n \sim \mathcal{N}(0,1) \tag{31}$$

Thus,

$$\lim_{n\to\infty} F_{\sqrt{n}Y_n} = F_P(x) \tag{32}$$

where, $P \sim \mathcal{N}(0,1)$
So, option (A) is correct

(B) For 2nd mean to be converging to 0,

$$\lim_{n\to\infty} E(|Y_n - 0|^2) = \lim_{n\to\infty} E(Y_n^2) \tag{33}$$

$$\because \frac{1}{n} = E(Y_n^2) - [E(Y_n)]^2 \tag{34}$$

$$= \lim_{n\to\infty} \frac{1}{n} + [E(Y_n)]^2 \tag{35}$$

$$\because [E(Y_n)]^2 = 0 \tag{36}$$

$$= \lim_{n\to\infty} \frac{1}{n} = 0 \tag{37}$$

Thus,$\{Y_n\}_{n\geq 1}$ converges in 2nd mean to 0
Hence, option (B) is correct.

(C) For $\{Y_n + \frac{1}{n}\}_{n\geq 1}$ to be converging $Y_n + \frac{1}{n} \xrightarrow{p} 0$

$$\lim_{n\to\infty} \Pr\left(\left|Y_n + \frac{1}{n} - Y\right| \geq \epsilon\right) = \lim_{n\to\infty} \Pr\left(\left|Y_n + \frac{1}{n}\right| \geq \epsilon\right) \tag{38}$$

$$\leq \lim_{n\to\infty} \frac{\operatorname{Var}(Y_n + \frac{1}{n})}{\epsilon^2} \tag{39}$$

$$\forall \epsilon > 0 \tag{40}$$

$$= \lim_{n\to\infty} \frac{\operatorname{Var}(Y_n) + \operatorname{Var}(\frac{1}{n})}{\epsilon^2} \tag{41}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n} + 0}{\epsilon^2} \tag{42}$$

$$= 0 \tag{43}$$

hence, option (C) is correct

(D) As all the variables are i.i.d's and are thus uncorrelated,

$$\operatorname{Var}(X_n) = \sum_{i=1}^{n} \operatorname{cov}(X_i, X_i) \tag{44}$$

$$= \sum_{i=1}^{n} \operatorname{var}(X_i) \tag{45}$$

$$= \sum_{i=1}^{n} 1 \tag{46}$$

$$= n \tag{47}$$

$$\lim_{n\to\infty} \Pr(|X_n - 0| \geq \epsilon) \leq \lim_{n\to\infty} \frac{Var(X_n)}{\epsilon^2} \tag{48}$$

$$\forall \epsilon > 0 \tag{49}$$

$$= \lim_{n\to\infty} \frac{Var(X_n)}{\epsilon^2} \tag{50}$$

$$= \lim_{n\to\infty} \frac{n}{\epsilon^2} = \infty \tag{51}$$

Hence,option (D) is incorrect