

# GATE-EC2023

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**Q65ST.2023:** Let  $\{X_n\}_{n \geq 1}$  be a sequence of independent and identically distributed random variables with mean 0 and variance 1, all of them defined on the same probability space. For  $n=1,2,3,\dots$ , let

$$Y_n = \frac{1}{n}(X_1X_2 + X_3X_4 + \dots + X_{2n-1}X_{2n}) \quad (1)$$

Then which of the following statements is/are true?

- (A)  $\{\sqrt{n}Y_n\}_{n \geq 1}$  converges in distribution to a standard normal random variable.
- (B)  $\{Y_n\}_{n \geq 1}$  converges in 2nd mean to 0.
- (C)  $\{Y_n + \frac{1}{n}\}_{n \geq 1}$  converges in probability to 0.
- (D)  $\{X_n\}_{n \geq 1}$  converges almost surely to 0.

**Solution:** As  $X_i$  is a sequence of independent and identically distributed random variables,

Since  $X_1, X_2, X_3, X_4$  are independent R.V ,

$$P_{X_1X_2X_3X_4}(x) = p_{X_1}(x)p_{X_2}(x)p_{X_3}(x)p_{X_4}(x) \quad (2)$$

$$= p_{X_1X_2}(x)p_{X_3X_4}(x) \quad (3)$$

$$= p_{(X_1X_2)(X_3X_4)}(x) \quad (4)$$

$X_iX_{i+1}$  is a independent R.V

Also,  $X_1, X_2, X_3, X_4$  are identical distributed R.V

Then,  $X_1X_2$  and  $X_3X_4$  are also identical distributed R.V

Thus,  $X_iX_{i+1}$  is a identical distributed R.V.

Hence  $\sum_{i=1}^{2n-1} X_iX_{i+1}$  is iid.

Now, let  $Z_i = X_iX_{i+1}$  then,

$$E(Z_i) = E(X_i)E(X_{i+1}) = 0 \quad (5)$$

$$Var(Z_i) = E(Z_i^2) - [E(Z_i)]^2 = 1 \quad (6)$$

$$\implies Z \sim \mathcal{N}(0, 1) \quad (7)$$

$$Y_n = \frac{1}{n} \left( \sum_{i=1}^{2n-1} Z_i \right) \quad (8)$$

$$\implies E(Y_n) = 0 \quad (9)$$

$$\implies Var(Y_n) = \frac{1}{n} \quad (10)$$

$$\implies Y \sim \mathcal{N}\left(0, \frac{1}{\sqrt{n}}\right) \quad (11)$$

(A) Using (11),

$$\sqrt{n}Y \sim \mathcal{N}(0, 1) \quad (12)$$

For some  $\{P_n\}_{n \geq 1}$  converges in distribution to  $P$ ,  $P_n \xrightarrow{d} P$ , then for all  $p$ ,

$$\lim_{n \rightarrow \infty} F_{P_n}(x) = F_P(x) \quad (13)$$

where,  $P \sim \mathcal{N}(0, 1)$

$$\lim_{n \rightarrow \infty} F_{\sqrt{n}Y_n} = \lim_{n \rightarrow \infty} \int_{-\infty}^x \frac{1}{\sqrt{2\pi}(1)} e^{-\frac{(x-0)^2}{2(1)^2}} dx \quad (14)$$

$$= \lim_{n \rightarrow \infty} \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{(x)^2}{2}} dx \quad (15)$$

$$= F_P(x) \quad (16)$$

where,  $P \sim \mathcal{N}(0, 1)$

So, option (A) is correct

(B) For 2nd mean to be converging to 0,

$$\lim_{n \rightarrow \infty} E(|Y_n - 0|^2) = \lim_{n \rightarrow \infty} E(Y_n^2) \quad (17)$$

$$\because \frac{1}{n} = E(Y_n^2) - [E(Y_n)]^2 \quad (18)$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} + [E(Y_n)]^2 \quad (19)$$

$$\because [E(Y_n)]^2 = 0 \quad (20)$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} = 0 \quad (21)$$

Thus,  $\{Y_n\}_{n \geq 1}$  converges in 2nd mean to 0  
Hence, option (B) is correct.

(C) For  $\{Y_n + \frac{1}{n}\}_{n \geq 1}$  to be converging  $Y_n + \frac{1}{n} \xrightarrow{p} 0$   
Using (11) ,

$$E(Y_n + \frac{1}{n}) = \frac{1}{n} \quad (22)$$

$$Var(Y_n + \frac{1}{n}) = \frac{1}{n} \quad (23)$$

$$(24)$$

$$\lim_{n \rightarrow \infty} F_{Y_n + \frac{1}{n}} = \lim_{n \rightarrow \infty} \int_{-\infty}^x \frac{1}{\sqrt{2\pi} \left(\frac{1}{n}\right)} e^{-\frac{\left(x - \frac{1}{n}\right)^2}{2\left(\frac{1}{n}\right)}} dx \quad (25)$$

$$= \lim_{n \rightarrow \infty} \int_{-\infty}^x \frac{n}{\sqrt{2\pi}} e^{-\frac{(n^2 x - n)^2}{2}} dx \quad (26)$$

$$= 0 \quad (27)$$

Thus,  $Y_n + \frac{1}{n} \xrightarrow{d} 0 \implies Y_n + \frac{1}{n} \xrightarrow{p} 0$

Note: This condition is only true for converges at real constants.

Hence, option (C) is correct.

(D) let  $\{X_n\}_{n \geq 1}$  almost surely converges to 0. Then,  $X_n \xrightarrow{p} 1$ .

$$\lim_{n \rightarrow \infty} F_{X_n} = \lim_{n \rightarrow \infty} \int_{-\infty}^x \frac{1}{\sqrt{2\pi} (1)} e^{-\frac{(x-0)^2}{2(1)^2}} dx \quad (28)$$

$$= F_X(x) \quad (29)$$

$$X_n \xrightarrow{d} X \quad (30)$$

$$\implies X_n \xrightarrow{p} X \quad (31)$$

Hence, option (D) is incorrect