

GATE-EC2023

Jay Vikrant EE22BTECH11025

Q65ST.2023: Let $\{X_n\}_{n \geq 1}$ be a sequence of independent and identically distributed random variables with mean 0 and variance 1, all of them defined on the same probability space. For $n=1,2,3,\dots$, let

$$Y_n = \frac{1}{n}(X_1X_2 + X_3X_4 + \dots + X_{2n-1}X_{2n}) \quad (1)$$

Then which of the following statements is/are true?

- (A) $\{\sqrt{n}Y_n\}_{n \geq 1}$ converges in distribution to a standard normal random variable.
- (B) $\{Y_n\}_{n \geq 1}$ converges in 2nd mean to 0.
- (C) $\{Y_n + \frac{1}{n}\}_{n \geq 1}$ converges in probability to 0.
- (D) $\{X_n\}_{n \geq 1}$ converges almost surely to 0.

Solution: As X_i is a sequence of independent and identically distributed random variables, Let $Z_i = X_iX_{i+1}$.

$$p_{Z_iZ_j}(x) = p_{X_iX_{i+1}X_jX_{j+1}}(x) \quad (2)$$

$$= p_{X_i}(x)p_{X_{i+1}}(x)p_{X_j}(x)p_{X_{j+1}}(x) \quad (3)$$

$$= p_{X_iX_{i+1}}(x)p_{X_jX_{j+1}}(x) \quad (4)$$

$$= p_{Z_i}(x)p_{Z_j}(x) \quad (5)$$

Z_i is a independent R.V . Similarly,

$$F_{X_i}(x) = F_{X_{i+1}}(x) = F_{X_j}(x) = F_{X_{j+1}}(x) \quad (6)$$

$$F_{X_i}(x)F_{X_{i+1}}(x) = F_{X_j}(x)F_{X_{j+1}}(x) \quad (7)$$

$$F_{Z_i}(x) = F_{Z_j}(x) \quad (8)$$

Thus, Z_i is an identical distributed R.V.

Hence,

$$Z_n = \sum_{i=1}^{2n-1} Z_i \quad (9)$$

is an i.i.d

$$E(Z_i) = E(X_iX_{i+1}) \quad (10)$$

$$= E(X_i)E(X_{i+1}) = 0 \quad (11)$$

$$\text{Var}(Z_i) = E(Z_i^2) - [E(Z_i)]^2 \quad (12)$$

$$= E(Z_i^2) - 0 \quad (13)$$

$$= E(X_i^2X_{i+1}^2) \quad (14)$$

$$= E(X_i^2)E(X_{i+1}^2) = 1 \quad (15)$$

$$Y_n = \frac{1}{n} \left(\sum_{i=1}^{2n-1} Z_i \right) \quad (16)$$

$$E(Y_n) = E\left(\frac{1}{n} \sum_{i=1}^{2n-1} Z_i\right) \quad (17)$$

$$= \frac{1}{n} \sum_{i=1}^{2n-1} E(Z_i) = 0 \quad (18)$$

As all the variables are i.i.d's and are thus uncorrelated,

$$\text{cov}(X_i, X_j) = \begin{cases} 0 & \text{if } i \neq j \\ \text{var}(X_i) & \text{if } i = j \end{cases} \quad (19)$$

$$\text{var}(Y_n) = \frac{1}{n^2} \left(\sum_{i=1}^{2n-1} \text{cov}(Z_i, Z_i) \right) \quad (20)$$

$$= \frac{1}{n^2} \left(\sum_{i=1}^{2n-1} \text{var}(Z_i) \right) \quad (21)$$

$$= \frac{1}{n^2} \left(\sum_{i=1}^{2n-1} 1 \right) \quad (22)$$

$$= \frac{2n-1}{n^2} \quad (23)$$

- (A) For some $\{P_n\}_{n \geq 1}$ converges in distribution to P , $P_n \xrightarrow{d} P$, then for all p,

$$\lim_{n \rightarrow \infty} F_{P_n}(x) = F_P(x) \quad (24)$$

since it is true for all p, so it is also be true for normal distribution

$$E(\sqrt{n}Y_n) = \sqrt{n}E(Y_n) = 0 \quad (25)$$

$$\text{Var}(\sqrt{n}Y_n) = (\sqrt{n})^2 \text{Var}(Y_n) = \frac{2n-1}{n} \quad (26)$$

$$\lim_{n \rightarrow \infty} F_{\sqrt{n}Y_n} = \lim_{n \rightarrow \infty} \int_{-\infty}^x \frac{1}{\sqrt{2\pi \left(\frac{2n-1}{n}\right)}} e^{-\frac{(x-0)^2}{2 \left(\frac{2n-1}{n}\right)}} dx \quad (27)$$

$$= \int_{-\infty}^x \frac{1}{\sqrt{4\pi}} e^{-\frac{x^2}{4}} dx \quad (28)$$

$$= F_P(x) \quad (29)$$

where, $P \sim \mathcal{N}(0, \sqrt{2})$

So, option (A) is correct

(B) For 2nd mean to be converging to 0,

$$\lim_{n \rightarrow \infty} E(|Y_n - 0|^2) = \lim_{n \rightarrow \infty} E(Y_n^2) \quad (30)$$

$$\because \frac{2n-1}{n^2} = E(Y_n^2) - [E(Y_n)]^2 \quad (31)$$

$$= \lim_{n \rightarrow \infty} \frac{2n-1}{n^2} + [E(Y_n)]^2 \quad (32)$$

$$\because [E(Y_n)]^2 = 0 \quad (33)$$

$$= \lim_{n \rightarrow \infty} \frac{2n-1}{n^2} = 0 \quad (34)$$

Thus, $\{Y_n\}_{n \geq 1}$ converges in 2nd mean to 0

Hence, option (B) is correct.

(C) For $\{Y_n + \frac{1}{n}\}_{n \geq 1}$ to be converging $Y_n + \frac{1}{n} \xrightarrow{P} 0$

$$\lim_{n \rightarrow \infty} \Pr\left(|Y_n + \frac{1}{n} - Y|\geq \epsilon\right) = \lim_{n \rightarrow \infty} \Pr\left(|Y_n + \frac{1}{n}|\geq \epsilon\right) \quad (35)$$

$$\leq \lim_{n \rightarrow \infty} \frac{\text{Var}(Y_n + \frac{1}{n})}{\epsilon^2} \quad (36)$$

$$\forall \epsilon > 0 \quad (37)$$

$$= \lim_{n \rightarrow \infty} \frac{\text{Var}(Y_n) + \text{Var}(\frac{1}{n})}{\epsilon^2} \quad (38)$$

$$= \lim_{n \rightarrow \infty} \frac{\frac{2n-1}{n^2} + 0}{\epsilon^2} \quad (39)$$

$$= 0 \quad (40)$$

(D) For $X_n = \sum_{i=1}^n X_i$ converges almost surely to 0.

$X_n \xrightarrow{P} X$, where $X = 0$. and,

$$\lim_{n \rightarrow \infty} \Pr(|X_n - X| \geq \epsilon) = 1 \quad (41)$$

uncorrelated,

$$\text{cov}(X_i, X_j) = \begin{cases} 0 & \text{if } i \neq j \\ \text{var}(X_i) & \text{if } i = j \end{cases} \quad (42)$$

$$\text{var}(X_n) = \sum_{i=1}^n \text{cov}(X_i, X_i) \quad (43)$$

$$= \sum_{i=1}^n \text{var}(X_i) \quad (44)$$

$$= \sum_{i=1}^n 1 \quad (45)$$

$$= n \quad (46)$$

$$\lim_{n \rightarrow \infty} \Pr(|X_n| \geq \epsilon) \leq \lim_{n \rightarrow \infty} \frac{\text{Var}(X_n)}{\epsilon^2} \quad (47)$$

$$\forall \epsilon > 0 \quad (48)$$

$$= \lim_{n \rightarrow \infty} \frac{\text{Var}(X_n)}{\epsilon^2} \quad (49)$$

$$= \lim_{n \rightarrow \infty} \frac{n}{\epsilon^2} = \infty \quad (50)$$

Hence, option (D) is incorrect

As all the variables are i.i.d's and are thus