GATE-EC2023

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Q65ST.2023: Let $\{X_n\}_{n\geq 1}$ be a sequence of independent and identically distributed random variables with mean 0 and variance 1, all of them defined on the same probability space. For n=1,2,3,..., let

$$Y_n = \frac{1}{n}(X_1X_2 + X_3X_4 + ... + X_{2n-1}X_{2n}) \quad (1)$$

Then which of the following statements is/are true?

(A) $\{\sqrt{n}Y_n\}_{n\geq 1}$ converges in distribution to a standard normal random variable.

(B) $\{Y_n\}_{n\geq 1}$ converges in 2nd mean to 0.

(C) $\{Y_n + \frac{1}{n}\}_{n\geq 1}$ converges in probability to 0.

(D) $\{X_n\}_{n\geq 1}$ converges almost surely to 0.

Solution: As X_i is a sequence of independent and identically distributed random variables,
Let $Z_i = X_iX_{i+1}$.

$$p_{Z_iZ_j}(x) = p_{X_iX_{i+1}X_jX_{j+1}}(x) \quad (2)$$
$$= p_{X_i}(x)p_{X_{i+1}}(x)p_{X_j}(x)p_{X_{j+1}}(x) \quad (3)$$
$$= p_{X_iX_{i+1}}(x)p_{X_jX_{j+1}}(x) \quad (4)$$
$$= p_{Z_i}(x)p_{Z_j}(x) \quad (5)$$

Z_i is a independent R.V . Similarly,

$$F_{X_i}(x) = F_{X_{i+1}}(x) = F_{X_j}(x) = F_{X_{j+1}}(x) \quad (6)$$
$$F_{X_i}(x)F_{X_{i+1}}(x) = F_{X_j}(x)F_{X_{j+1}}(x) \quad (7)$$
$$F_{Z_i}(x) = F_{Z_j}(x) \quad (8)$$

Thus, Z_i is an identical distributed R.V.
Hence,

$$Z_n = \sum_{i=1}^{2n-1} Z_i \quad (9)$$

is an i.i.d

$$E(Z_i) = E(X_iX_{i+1}) \quad (10)$$
$$= E(X_i)E(X_{i+1}) = 0 \quad (11)$$
$$\text{Var}(Z_i) = E(Z_i^2) - [E(Z_i)]^2 \quad (12)$$
$$= E(Z_i^2) - 0 \quad (13)$$
$$= E(X_i^2X_{i+1}^2) \quad (14)$$
$$= E(X_i^2)E(X_{i+1}^2) = 1 \quad (15)$$

$$Y_n = \frac{1}{n}\left(\sum_{i=1}^{2n-1} Z_i\right) \quad (16)$$

$$E(Y_n) = E(\frac{1}{n}\sum_{i=1}^{2n-1} Z_i) \quad (17)$$

$$= \frac{1}{n}\sum_{i=1}^{2n-1} E(Z_i) = 0 \quad (18)$$

As all the variables are i.i.d's and are thus uncorrelated,

$$\text{cov}\left(X_i, X_j\right) = \begin{cases} 0 & \text{if } i \neq j \\ \text{var}(X_i) & \text{if } i = j \end{cases} \quad (19)$$

$$\text{var}(Y_n) = \frac{1}{n^2}\left(\sum_{i=1}^{2n-1} \text{cov}(Z_i, Z_i)\right) \quad (20)$$

$$= \frac{1}{n^2}\left(\sum_{i=1}^{2n-1} \text{var}(Z_i)\right) \quad (21)$$

$$= \frac{1}{n^2}\left(\sum_{i=1}^{2n-1} 1\right) \quad (22)$$

$$= \frac{2n-1}{n^2} \quad (23)$$

(A) For some $\{P_n\}_{n\geq 1}$ converges in distribution to P, $P_n \overset{d}{\to} P$, then for all p,

$$lim_{n\to\infty} F_{P_n}(x) = F_P(x) \quad (24)$$

since it is true for all p, so it is also be true for normal distribution

$$E\left(\sqrt{n}Y_n\right) = \sqrt{n}E(Y_n) = 0 \quad (25)$$

$$\text{Var}\left(\sqrt{n}Y_n\right) = \left(\sqrt{n}\right)^2 \text{Var}(Y_n) = \frac{2n-1}{n} \quad (26)$$

$$lim_{n\to\infty} F_{\sqrt{n}Y_n} = lim_{n\to\infty} \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\left(\frac{2n-1}{n}\right)}} e^{-\frac{(x-0)^2}{2\left(\frac{2n-1}{n}\right)}} dx \tag{27}$$

$$= \int_{-\infty}^{x} \frac{1}{\sqrt{4\pi}} e^{-\frac{(x)^2}{4}} dx \tag{28}$$

$$= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{\left(\frac{x}{\sqrt{2}}\right)^2}{2}} d\left(\frac{x}{\sqrt{2}}\right) \tag{29}$$

$$= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x)^2}{2}} dx \tag{30}$$

$$= F_P(x) \tag{31}$$

where, $P \sim \mathcal{N}(0,1)$

So, option (A) is correct

(B) For 2nd mean to be converging to 0,

$$lim_{n\to\infty} E(|Y_n - 0|^2) = lim_{n\to\infty} E(Y_n^2) \tag{32}$$

$$\because \frac{2n-1}{n^2} = E(Y_n^2) - [E(Y_n)]^2 \tag{33}$$

$$= lim_{n\to\infty} \frac{2n-1}{n^2} + [E(Y_n)]^2 \tag{34}$$

$$\because [E(Y_n)]^2 = 0 \tag{35}$$

$$= lim_{n\to\infty} \frac{2n-1}{n^2} = 0 \tag{36}$$

Thus, $\{Y_n\}_{n\geq 1}$ converges in 2nd mean to 0
Hence, option (B) is correct.

(C) For $\{Y_n + \frac{1}{n}\}_{n\geq 1}$ to be converging $Y_n + \frac{1}{n} \xrightarrow{p} 0$

$$\lim_{n\to\infty} \Pr\left(|Y_n + \frac{1}{n} - Y| \geq \epsilon\right) = \lim_{n\to\infty} \Pr\left(|Y_n + \frac{1}{n}| \geq \epsilon\right) \tag{37}$$

$$\leq \lim_{n\to\infty} \frac{Var(Y_n + \frac{1}{n})}{\epsilon^2} \tag{38}$$

$$\forall \epsilon > 0 \tag{39}$$

$$= \lim_{n\to\infty} \frac{Var(Y_n) + Var(\frac{1}{n})}{\epsilon^2} \tag{40}$$

$$= \lim_{n\to\infty} \frac{\frac{2n-1}{n^2} + 0}{\epsilon^2} \tag{41}$$

$$= 0 \tag{42}$$

(D) For $X_n = \sum_{i=1}^{n} X_i$ converges almost surely to 0.
$X_n \xrightarrow{p} X$,where $X = 0$. and,

$$\lim_{n\to\infty} \Pr(|X_n - X| \geq \epsilon) = 1 \tag{43}$$

As all the variables are i.i.d's and are thus uncorrelated,

$$\text{cov}\left(X_i, X_j\right) = \begin{cases} 0 & \text{if } i \neq j \\ \text{var}(X_i) & \text{if } i = j \end{cases} \tag{44}$$

$$\text{var}(X_n) = \sum_{i=1}^{n} \text{cov}(X_i, X_i) \tag{45}$$

$$= \sum_{i=1}^{n} \text{var}(X_i) \tag{46}$$

$$= \sum_{i=1}^{n} 1 \tag{47}$$

$$= n \tag{48}$$

$$\lim_{n\to\infty} \Pr(|X_n| \geq \epsilon) \leq \lim_{n\to\infty} \frac{Var(X_n)}{\epsilon^2} \tag{49}$$

$$\forall \epsilon > 0 \tag{50}$$

$$= \lim_{n\to\infty} \frac{Var(X_n)}{\epsilon^2} \tag{51}$$

$$= \lim_{n\to\infty} \frac{n}{\epsilon^2} = \infty \tag{52}$$

Hence, option (D) is incorrect