GATE-EC2023

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Q65EC.2023: Let $\{X_n\}_{n\geq 1}$ be a sequence of independent and identically distributed random variables with mean 0 and variance 1, all of them defined on the same probability space. For n=1,2,3,..., let

$$Y_n = \frac{1}{n}(X_1X_2 + X_3X_4 + ... + X_{2n-1}X_{2n}) \tag{1}$$

Then which of the following statements is/are true?

(A) $\{\sqrt{n}Y_n\}_{n\geq 1}$ converges in distribution to a standard normal random variable.
(B) $\{Y_n\}_{n\geq 1}$ converges in 2nd mean to 0.
(C) $\{Y_n + \frac{1}{n}\}_{n\geq 1}$ converges in probability to 0.
(D) $\{X_n\}_{n\geq 1}$ converges almost surely to 0.

Solution: As X_i is a sequence of independent and identically distributed random variables

$$E(X_iX_{i+1}) = E(X_i)E(X_{i+1}) = 0 \tag{2}$$
$$\implies E(Y_n) = 0 \tag{3}$$
$$Var(X_iX_{i+1}) = E(X_i^2X_{i+1}^2) - [E(X_iX_{i+1})]^2 = 1 \tag{4}$$
$$\implies Var(Y_n) = 1 \tag{5}$$

(A) Using (3) and (5),

$$E(\sqrt{n}Y_n) = \sqrt{n}\left(\frac{1}{n}\right)E(X_1X_2 + X_3X_4...) \tag{6}$$
$$= \sqrt{n}\left(\frac{1}{n}\right)(E(X_1)E(X_2)E(X_3)E(X_4)...) \tag{7}$$
$$= 0 \tag{8}$$
$$Var(\sqrt{n}Y_n) = \sqrt{n}(Var(X_1X_2) + Var(X_3X_4)...) \tag{9}$$
$$= n^2\left(\frac{1}{n}\right)(1 + 1 + ...) \tag{10}$$
$$= 1 \tag{11}$$
$$\{\sqrt{n}Y_n\} \sim \mathcal{N}(0, 1) \tag{12}$$

So, option (A) is correct

(B) For 2nd mean to be converging to 0,

$$lim_{n\to\infty}E(|Y_n - 0|^2) = lim_{n\to\infty}E(Y_n^2) \tag{13}$$
$$= lim_{n\to\infty}\frac{1}{n} + [E(Y_n)^2] \tag{14}$$
$$\because [E(Y_n)]^2 = 0 \tag{15}$$
$$= lim_{n\to\infty}\frac{1}{n} = 0 \tag{16}$$

Thus, $\{Y_n\}_{n\geq 1}$ converges in 2nd mean to 0
Hence, option (B) is correct.

(C) For $\{Y_n + \frac{1}{n}\}_{n\geq 1}$ to be converging $Y_n + \frac{1}{n} \xrightarrow{p} 0$

$$lim_{n\to\infty}E(Y_n + \frac{1}{n}) = lim_{n\to\infty}E(Y_n) + \frac{1}{n} \tag{17}$$
$$\because E(Y_n) = 0 \tag{18}$$
$$= lim_{n\to\infty}\frac{1}{n} = 0 \tag{19}$$

Hence, option (C) is correct.

(D) let $\{X_n\}_{n\geq 1}$ almost surely converges to 0. Then, $\{X_n\}_{n\geq 1}$ in probability must converge to 0. $lim_{n\to\infty}\Pr(X_n)$ may or may not be zero.
so, our initial assumption is incorrect
Hence, option (D) is incorrect.