GATE-EC2023

Jay Vikrant EE22BTECH11025

Q65ST.2023: Let $\{X_n\}_{n\geq 1}$ be a sequence of independent and identically distributed random variables with mean 0 and variance 1, all of them defined on the same probability space. For n=1,2,3,..., let

$$Y_n = \frac{1}{n}(X_1X_2 + X_3X_4 + ... + X_{2n-1}X_{2n}) \tag{1}$$

Then which of the following statements is/are true?

(A) $\{\sqrt{n}Y_n\}_{n\geq 1}$ converges in distribution to a standard normal random variable.
(B) $\{Y_n\}_{n\geq 1}$ converges in 2nd mean to 0.
(C) $\{Y_n + \frac{1}{n}\}_{n\geq 1}$ converges in probability to 0.
(D) $\{X_n\}_{n\geq 1}$ converges almost surely to 0.

Solution: As X_i is a sequence of independent and identically distributed random variables,
Let $Z_i = X_iX_{i+1}$.

$$p_{Z_iZ_j}(x) = p_{X_iX_{i+1}X_jX_{j+1}}(x) \tag{2}$$
$$= p_{X_i}(x)p_{X_{i+1}}(x)p_{X_j}(x)p_{X_{j+1}}(x) \tag{3}$$
$$= p_{X_iX_{i+1}}(x)p_{X_jX_{j+1}}(x) \tag{4}$$
$$= p_{Z_i}(x)p_{Z_j}(x) \tag{5}$$

Z_i is a independent R.V . Similarly,

$$F_{X_i}(x) = F_{X_{i+1}}(x) = F_{X_j}(x) = F_{X_{j+1}}(x) \tag{6}$$
$$F_{X_i}(x)F_{X_{i+1}}(x) = F_{X_j}(x)F_{X_{j+1}}(x) \tag{7}$$
$$F_{Z_i}(x) = F_{Z_j}(x) \tag{8}$$

Thus, Z_i is an identical distributed R.V.
Hence,

$$Z_n = \sum_{i=1}^{2n-1} Z_i \tag{9}$$

is an i.i.d

$$\mathrm{E}(Z_i) = \mathrm{E}(X_iX_{i+1}) \tag{10}$$
$$= \mathrm{E}(X_i)\,\mathrm{E}(X_{i+1}) = 0 \tag{11}$$
$$\mathrm{Var}(Z_i) = \mathrm{E}(Z_i^2) - [\mathrm{E}(Z_i)]^2 \tag{12}$$
$$= \mathrm{E}(Z_i^2) - 0 \tag{13}$$
$$= \mathrm{E}(X_i^2X_{i+1}^2) \tag{14}$$
$$= \mathrm{E}(X_i^2)\mathrm{E}(X_{i+1}^2) = 1 \tag{15}$$

$$Y_n = \frac{1}{n}\left(\sum_{i=1}^{2n-1} Z_i\right) \tag{16}$$
$$E(Y_n) = E(X_i)E(X_{i+1}) = 0 \tag{17}$$
$$Var(Y_n) = \mathrm{E}\left[\left(\frac{1}{n}\sum_{i=1}^{n} Z_i\right)^2\right] - \left(\mathrm{E}\left[\frac{1}{n}\sum_{i=1}^{n} Z_i\right]\right)^2 \tag{18}$$
$$= \frac{1}{n^2}\left\{\mathrm{E}\left[\left(\sum_{i=1}^{n} Z_i\right)^2\right] - \left(\mathrm{E}\left[\sum_{i=1}^{n} Z_i\right]\right)^2\right\} \tag{19}$$
$$= \frac{1}{n^2}\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\mathrm{E}\left(Z_iZ_j\right) - \mathrm{E}(Z_i)\,\mathrm{E}\left(Z_j\right)\right)\right\} \tag{20}$$
$$= \frac{1}{n^2}\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{cov}\left(Z_i, Z_j\right)\right\} \tag{21}$$

As all the variables are i.i.d's and are thus uncorrelated,

$$\mathrm{cov}\left(X_i, X_j\right) = \begin{cases} 0 & \text{if } i \neq j \\ \mathrm{var}(X_i) & \text{if } i = j \end{cases} \tag{22}$$
$$\mathrm{var}(Y_n) = \frac{1}{n^2}\left(\sum_{i=1}^{n}\mathrm{cov}(Z_i, Z_i)\right) \tag{23}$$
$$= \frac{1}{n^2}\left(\sum_{i=1}^{n}\mathrm{var}(Z_i)\right) \tag{24}$$
$$= \frac{1}{n^2}\left(\sum_{i=1}^{n} 1\right) \tag{25}$$
$$= \frac{1}{n} \tag{26}$$

(A) For some $\{P_n\}_{n\geq 1}$ converges in distribution to P, $P_n \xrightarrow{d} P$, then for all p,

$$lim_{n\to\infty}F_{P_n}(x) = F_P(x) \tag{27}$$

since it is true for all p, so it is also be true for normal distribution

$$lim_{n\to\infty}F_{\sqrt{n}Y_n} = lim_{n\to\infty}\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi(1)}}e^{-\frac{(x-0)^2}{2(1)^2}}\,dx \tag{28}$$

$$= lim_{n\to\infty}\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}e^{-\frac{(x)^2}{2}}\,dx \tag{29}$$

$$= F_P(x) \tag{30}$$

where, $P \sim \mathcal{N}(0,1)$
So, option (A) is correct

(B) For 2nd mean to be converging to 0,

$$lim_{n\to\infty}E(|Y_n - 0|^2) = lim_{n\to\infty}E(Y_n^2) \tag{31}$$

$$\because \frac{1}{n} = E(Y_n^2) - [E(Y_n)]^2 \tag{32}$$

$$= lim_{n\to\infty}\frac{1}{n} + [E(Y_n)]^2 \tag{33}$$

$$\because [E(Y_n)]^2 = 0 \tag{34}$$

$$= lim_{n\to\infty}\frac{1}{n} = 0 \tag{35}$$

Thus,$\{Y_n\}_{n\geq 1}$ converges in 2nd mean to 0
Hence, option (B) is correct.

(C) For $\{Y_n + \frac{1}{n}\}_{n\geq 1}$ to be converging $Y_n + \frac{1}{n} \xrightarrow{p} 0$

$$\lim_{n\to\infty}\Pr\left(|Y_n + \frac{1}{n} - Y| \geq \epsilon\right) = \lim_{n\to\infty}\Pr\left(|Y_n + \frac{1}{n}| \geq \epsilon\right) \tag{36}$$

$$\leq \lim_{n\to\infty}\frac{Var(Y_n + \frac{1}{n})}{\epsilon^2} \tag{37}$$

$$\forall \epsilon > 0 \tag{38}$$

$$= \lim_{n\to\infty}\frac{Var(Y_n) + Var(\frac{1}{n})}{\epsilon^2} \tag{39}$$

$$= \lim_{n\to\infty}\frac{\frac{1}{n} + 0}{\epsilon^2} \tag{40}$$

$$= 0 \tag{41}$$

(D) For $X_n = \sum_{i=1}^{n} X_i$ converges almost surely to 0.
$X_n \xrightarrow{p} X$,where $X = 0$. and,

$$\lim_{n\to\infty}\Pr(|X_n - X| \geq \epsilon) = 1 \tag{42}$$

As all the variables are i.i.d's and are thus uncorrelated,

$$\mathrm{cov}\left(X_i, X_j\right) = \begin{cases} 0 & \text{if } i \neq j \\ \mathrm{var}(X_i) & \text{if } i = j \end{cases} \tag{43}$$

$$\mathrm{var}(X_n) = \sum_{i=1}^{n}\mathrm{cov}(X_i, X_i) \tag{44}$$

$$= \sum_{i=1}^{n}\mathrm{var}(X_i) \tag{45}$$

$$= \sum_{i=1}^{n} 1 \tag{46}$$

$$= n \tag{47}$$

$$\lim_{n\to\infty}\Pr(|X_n| \geq \epsilon) \leq \lim_{n\to\infty}\frac{Var(X_n)}{\epsilon^2} \tag{48}$$

$$\forall \epsilon > 0 \tag{49}$$

$$= \lim_{n\to\infty}\frac{Var(X_n)}{\epsilon^2} \tag{50}$$

$$= \lim_{n\to\infty}\frac{n}{\epsilon^2} = \infty \tag{51}$$

Hence,option (D) is incorrect