GATE-EC2023

Jay Vikrant EE22BTECH11025

Q65ST.2023: Let $\{X_n\}_{n\geq1}$ be a sequence of independent and identically distributed random variables with mean 0 and variance 1, all of them defined on the same probability space. For n=1,2,3,..., let

$$Y_n = \frac{1}{n}(X_1X_2 + X_3X_4 + ... + X_{2n-1}X_{2n}) \tag{1}$$

Then which of the following statements is/are true?

(A) $\{\sqrt{n}Y_n\}_{n\geq1}$ converges in distribution to a standard normal random variable.
(B) $\{Y_n\}_{n\geq1}$ converges in 2nd mean to 0.
(C) $\{Y_n + \frac{1}{n}\}_{n\geq1}$ converges in probability to 0.
(D) $\{X_n\}_{n\geq1}$ converges almost surely to 0.

Solution: As X_i is a sequence of independent and identically distributed random variables

$$E(X_iX_{i+1}) = E(X_i)E(X_{i+1}) = 0 \tag{2}$$
$$\implies E(Y_n) = 0 \tag{3}$$
$$Var(X_iX_{i+1}) = E(X_i^2X_{i+1}^2) - [E(X_iX_{i+1})]^2 = 1 \tag{4}$$
$$\implies Var(Y_n) = \frac{1}{n} \tag{5}$$
$$\tag{6}$$

(A) Using (3) and (5),

$$E(\sqrt{n}Y_n) = 0 \tag{7}$$
$$Var(\sqrt{n}Y_n) = \frac{1}{\sqrt{n}} \tag{8}$$

For some $\{Z_n\}_{n\geq1}$ converges in distribution to Z, $Z_n \xrightarrow{d} Z$, then for all z,

$$lim_{n\to\infty}F_{Z_n}(x) = F_Z(x) \tag{9}$$

where, $Z \sim \mathcal{N}(0,1)$

$$lim_{n\to\infty}F_{\sqrt{n}Y_n} = lim_{n\to\infty}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi\left(\frac{1}{\sqrt{n}}\right)}}e^{-\frac{(x-0)^2}{2\left(\frac{1}{\sqrt{n}}\right)}}\,dx \tag{10}$$
$$= lim_{n\to\infty}\int_{-\infty}^{+\infty}\frac{\sqrt[4]{n}}{\sqrt{2\pi}}e^{-\frac{\left(\sqrt[4]{n}x\right)^2}{2}}\,dx \tag{11}$$
$$= lim_{n\to\infty}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{(x)^2}{2}}\,dx \tag{12}$$
$$= F_Z(x) \tag{13}$$

where, $Z \sim \mathcal{N}(0,1)$
So, option (A) is correct

(B) For 2nd mean to be converging to 0,

$$lim_{n\to\infty}E(|Y_n - 0|^2) = lim_{n\to\infty}E(Y_n^2) \tag{14}$$
$$\because \frac{1}{n} = E(Y_n^2) - [E(Y_n)]^2 \tag{15}$$
$$= lim_{n\to\infty}\frac{1}{n} + [E(Y_n)]^2 \tag{16}$$
$$\because [E(Y_n)]^2 = 0 \tag{17}$$
$$= lim_{n\to\infty}\frac{1}{n} = 0 \tag{18}$$

Thus,$\{Y_n\}_{n\geq1}$ converges in 2nd mean to 0
Hence, option (B) is correct.

(C) For $\{Y_n + \frac{1}{n}\}_{n\geq1}$ to be converging $Y_n + \frac{1}{n} \xrightarrow{p} 0$

$$E(Y_n + \frac{1}{n}) = \frac{1}{n} \tag{19}$$
$$Var(Y_n + \frac{1}{n}) = \frac{1}{n} \tag{20}$$

$$lim_{n\to\infty}F_{Y_n+\frac{1}{n}} = lim_{n\to\infty}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi\left(\frac{1}{n}\right)}}e^{-\frac{\left(x-\frac{1}{n}\right)^2}{2\left(\frac{1}{n}\right)}}\,dx \tag{21}$$

$$= lim_{n\to\infty}\int_{-\infty}^{+\infty}\frac{n}{\sqrt{2\pi}}e^{-\frac{\left(n^2x-n\right)^2}{2}}\,dx \tag{22}$$

$$= lim_{n\to\infty}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{(nx-n)^2}{2}}\,dx \tag{23}$$

$$= 0 \tag{24}$$

Thus, $Y_n+\frac{1}{n}\xrightarrow{d}0 \implies Y_n+\frac{1}{n}\xrightarrow{p}0$
Note: This condition is only true for converges at real constants.
Hence, option (C) is correct.

(D) let $\{X_n\}_{n\geq 1}$ almost surely converges to 0. Then, $X_n\xrightarrow{p}1$.

$$lim_{n\to\infty}F_{X_n} = lim_{n\to\infty}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi(1)}}e^{\frac{-(x-0)^2}{2(1)^2}}\,dx \tag{25}$$

$$= 0 \tag{26}$$

$$X_n\xrightarrow{d}0 \tag{27}$$

$$\implies X_n\xrightarrow{p}0 \tag{28}$$

Hence,option (D) is incorrect