GATE-EC2023

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Q65ST.2023: Let $\{X_n\}_{n\geq 1}$ be a sequence of independent and identically distributed random variables with mean 0 and variance 1, all of them defined on the same probability space. For n=1,2,3,..., let

$$Y_n = \frac{1}{n}(X_1X_2 + X_3X_4 + ... + X_{2n-1}X_{2n}) \tag{1}$$

Then which of the following statements is/are true?

(A) $\{\sqrt{n}Y_n\}_{n\geq 1}$ converges in distribution to a standard normal random variable.
(B) $\{Y_n\}_{n\geq 1}$ converges in 2nd mean to 0.
(C) $\{Y_n + \frac{1}{n}\}_{n\geq 1}$ converges in probability to 0.
(D) $\{X_n\}_{n\geq 1}$ converges almost surely to 0.

Solution: As X_i is a sequence of independent and identically distributed random variables,
Let $Z_i = X_iX_{i+1}$.

$$\begin{aligned} p_{Z_iZ_j}(x) &= p_{X_iX_{i+1}X_jX_{j+1}}(x) && (2)\\ &= p_{X_i}(x)p_{X_{i+1}}(x)p_{X_j}(x)p_{X_{j+1}}(x) && (3)\\ &= p_{X_iX_{i+1}}(x)p_{X_jX_{j+1}}(x) && (4)\\ &= p_{Z_i}(x)p_{Z_j}(x) && (5)\end{aligned}$$

Z_i is a independent R.V . Similarly,

$$\begin{aligned} F_{X_i}(x) = F_{X_{i+1}}(x) = F_{X_j}(x) &= F_{X_{j+1}}(x) && (6)\\ F_{X_i}(x)F_{X_{i+1}}(x) = F_{X_j}(x)&F_{X_{j+1}}(x) && (7)\\ F_{Z_i}(x) &= F_{Z_j}(x) && (8)\end{aligned}$$

Thus, Z_i is an identical distributed R.V.
Hence,

$$Z_n = \sum_{i=1}^{2n-1} Z_i \tag{9}$$

is an i.i.d

$$\begin{aligned} \mathrm{E}(Z_i) &= \mathrm{E}(X_iX_{i+1}) && (10)\\ &= \mathrm{E}(X_i)\,\mathrm{E}(X_{i+1}) = 0 && (11)\\ \mathrm{Var}(Z_i) &= \mathrm{E}(Z_i^2) - [\mathrm{E}(Z_i)]^2 && (12)\\ &= \mathrm{E}(Z_i^2) - 0 && (13)\\ &= \mathrm{E}(X_i^2X_{i+1}^2) && (14)\\ &= \mathrm{E}(X_i^2)\mathrm{E}(X_{i+1}^2) = 1 && (15)\end{aligned}$$

$$\begin{aligned} Y_n &= \frac{1}{n}\left(\sum_{i=1}^{2n-1} Z_i\right) && (16)\\ E(Y_n) &= E(X_i)E(X_{i+1}) = 0 && (17)\\ Var(Y_n) &= \mathrm{E}\left[\left(\frac{1}{n}\sum_{i=1}^{n} Z_i\right)^2\right] - \left(\mathrm{E}\left[\frac{1}{n}\sum_{i=1}^{n} Z_i\right]\right)^2 && (18)\\ &= \frac{1}{n^2}\left\{\mathrm{E}\left[\left(\sum_{i=1}^{n} Z_i\right)^2\right] - \left(\mathrm{E}\left[\sum_{i=1}^{n} Z_i\right]\right)^2\right\} && (19)\\ &= \frac{1}{n^2}\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\mathrm{E}\left(Z_iZ_j\right) - \mathrm{E}(Z_i)\,\mathrm{E}\left(Z_j\right)\right)\right\} && (20)\\ &= \frac{1}{n^2}\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{cov}\left(Z_i, Z_j\right)\right\} && (21)\end{aligned}$$

As all the variables are i.i.d's and are thus uncorrelated,

$$\mathrm{cov}\left(X_i, X_j\right) = \begin{cases} 0 & \text{if } i \neq j \\ \mathrm{var}(X_i) & \text{if } i = j \end{cases} \tag{22}$$

$$\begin{aligned} \mathrm{var}(Y_n) &= \frac{1}{n^2}\left(\sum_{i=1}^{n}\mathrm{cov}(Z_i, Z_i)\right) && (23)\\ &= \frac{1}{n^2}\left(\sum_{i=1}^{n}\mathrm{var}(Z_i)\right) && (24)\\ &= \frac{1}{n^2}\left(\sum_{i=1}^{n} 1\right) && (25)\\ &= \frac{1}{n} && (26)\end{aligned}$$

(A) For some $\{P_n\}_{n\geq 1}$ converges in distribution to P, $P_n \xrightarrow{d} P$, then for all p,

$$lim_{n\to\infty}F_{P_n}(x) = F_P(x) \tag{27}$$

since it is true for all p, so it is also be true for normal distribution

$$\mathrm{E}\left(\sqrt{n}Y_n\right) = \sqrt{n}\mathrm{E}(Y_n) = 0 \tag{28}$$

$$\mathrm{Var}\left(\sqrt{n}Y_n\right) = \left(\sqrt{n}\right)^2\mathrm{Var}(Y_n) = 1 \tag{29}$$

$$lim_{n\to\infty} F_{\sqrt{n}Y_n} = lim_{n\to\infty} \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi(1)}} e^{-\frac{(x-0)^2}{2(1)^2}}\, dx \tag{30}$$

$$= lim_{n\to\infty} \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x)^2}{2}}\, dx \tag{31}$$

$$= F_P(x) \tag{32}$$

where, $P \sim \mathcal{N}(0,1)$
So, option (A) is correct

(B) For 2nd mean to be converging to 0,

$$lim_{n\to\infty} E(|Y_n - 0|^2) = lim_{n\to\infty} E(Y_n^2) \tag{33}$$

$$\because \frac{1}{n} = E(Y_n^2) - [E(Y_n)]^2 \tag{34}$$

$$= lim_{n\to\infty} \frac{1}{n} + [E(Y_n)]^2 \tag{35}$$

$$\because [E(Y_n)]^2 = 0 \tag{36}$$

$$= lim_{n\to\infty} \frac{1}{n} = 0 \tag{37}$$

Thus,$\{Y_n\}_{n\geq 1}$ converges in 2nd mean to 0
Hence, option (B) is correct.

(C) For $\{Y_n + \frac{1}{n}\}_{n\geq 1}$ to be converging $Y_n + \frac{1}{n} \xrightarrow{p} 0$

$$\lim_{n\to\infty} \Pr\left(|Y_n + \frac{1}{n} - Y| \geq \epsilon\right) = \lim_{n\to\infty} \Pr\left(|Y_n + \frac{1}{n}| \geq \epsilon\right) \tag{38}$$

$$\leq \lim_{n\to\infty} \frac{Var(Y_n + \frac{1}{n})}{\epsilon^2} \tag{39}$$

$$\forall \epsilon > 0 \tag{40}$$

$$= \lim_{n\to\infty} \frac{Var(Y_n) + Var(\frac{1}{n})}{\epsilon^2} \tag{41}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n} + 0}{\epsilon^2} \tag{42}$$

$$= 0 \tag{43}$$

(D) For $X_n = \sum_{i=1}^{n} X_i$ converges almost surely to 0.
$X_n \xrightarrow{p} X$,where $X = 0$. and,

$$\lim_{n\to\infty} \Pr(|X_n - X| \geq \epsilon) = 1 \tag{44}$$

As all the variables are i.i.d's and are thus uncorrelated,

$$\operatorname{cov}\left(X_i, X_j\right) = \begin{cases} 0 & \text{if } i \neq j \\ \operatorname{var}(X_i) & \text{if } i = j \end{cases} \tag{45}$$

$$\operatorname{var}(X_n) = \sum_{i=1}^{n} \operatorname{cov}(X_i, X_i) \tag{46}$$

$$= \sum_{i=1}^{n} \operatorname{var}(X_i) \tag{47}$$

$$= \sum_{i=1}^{n} 1 \tag{48}$$

$$= n \tag{49}$$

$$\lim_{n\to\infty} \Pr(|X_n| \geq \epsilon) \leq \lim_{n\to\infty} \frac{Var(X_n)}{\epsilon^2} \tag{50}$$

$$\forall \epsilon > 0 \tag{51}$$

$$= \lim_{n\to\infty} \frac{Var(X_n)}{\epsilon^2} \tag{52}$$

$$= \lim_{n\to\infty} \frac{n}{\epsilon^2} = \infty \tag{53}$$

Hence,option (D) is incorrect