GATE-EC2023

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Q65ST.2023: Let $\{X_n\}_{n\geq1}$ be a sequence of independent and identically distributed random variables with mean 0 and variance 1, all of them defined on the same probability space. For n=1,2,3,..., let

$$Y_n = \frac{1}{n}(X_1X_2 + X_3X_4 + ... + X_{2n-1}X_{2n}) \tag{1}$$

Then which of the following statements is/are true?

(A) $\{\sqrt{n}Y_n\}_{n\geq1}$ converges in distribution to a standard normal random variable.
(B) $\{Y_n\}_{n\geq1}$ converges in 2nd mean to 0.
(C) $\{Y_n + \frac{1}{n}\}_{n\geq1}$ converges in probability to 0.
(D) $\{X_n\}_{n\geq1}$ converges almost surely to 0.

Solution: As X_i is a sequence of independent and identically distributed random variables,
Since X_1, X_2, X_3, X_4 are independent R.V ,

$$\Pr(X_1X_2X_3X_4) = \Pr(X_1)\Pr(X_2)\Pr(X_3)\Pr(X_4) \tag{2}$$
$$= \Pr(X_1X_2)\Pr(X_3X_4) \tag{3}$$
$$= \Pr((X_1X_2)(X_3X_4)) \tag{4}$$

X_iX_{i+1} is a independent R.V
Also, X_1, X_2, X_3, X_4 are identical distributed R.V
Then, X_1X_2 and X_3X_4 are also identical distributed R.V
Thus,X_iX_{i+1} is a identical distributed R.V.
Hence $\sum_{i=1}^{2n-1} X_iX_{i+1}$ is iid.

$$E(X_iX_{i+1}) = E(X_i)E(X_{i+1}) = 0 \tag{5}$$
$$\implies E(Y_n) = 0 \tag{6}$$
$$Var(X_iX_{i+1}) = E(X_i^2X_{i+1}^2) - [E(X_iX_{i+1})]^2 = 1 \tag{7}$$
$$\implies Var(Y_n) = \frac{1}{n} \tag{8}$$
$$\tag{9}$$

(A) Using (6) and (8),

$$E(\sqrt{n}Y_n) = 0 \tag{10}$$
$$Var(\sqrt{n}Y_n) = \frac{1}{\sqrt{n}} \tag{11}$$

For some $\{Z_n\}_{n\geq1}$ converges in distribution to Z, $Z_n \xrightarrow{d} Z$, then for all z,

$$lim_{n\to\infty}F_{Z_n}(x) = F_Z(x) \tag{12}$$

where, $Z \sim \mathcal{N}(0,1)$

$$lim_{n\to\infty}F_{\sqrt{n}Y_n} = lim_{n\to\infty}\int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\left(\frac{1}{\sqrt{n}}\right)}}e^{-\frac{(x-0)^2}{2\left(\frac{1}{\sqrt{n}}\right)}}\,dx \tag{13}$$
$$= lim_{n\to\infty}\int_{-\infty}^{x} \frac{\sqrt[4]{n}}{\sqrt{2\pi}}e^{-\frac{\left(\sqrt[4]{n}x\right)^2}{2}}\,dx \tag{14}$$
$$= lim_{n\to\infty}\int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}e^{-\frac{(x)^2}{2}}\,dx \tag{15}$$
$$= F_Z(x) \tag{16}$$

where, $Z \sim \mathcal{N}(0,1)$
So, option (A) is correct

(B) For 2nd mean to be converging to 0,

$$lim_{n\to\infty}E(|Y_n - 0|^2) = lim_{n\to\infty}E(Y_n^2) \tag{17}$$
$$\because \frac{1}{n} = E(Y_n^2) - [E(Y_n)]^2 \tag{18}$$
$$= lim_{n\to\infty}\frac{1}{n} + [E(Y_n)]^2 \tag{19}$$
$$\because [E(Y_n)]^2 = 0 \tag{20}$$
$$= lim_{n\to\infty}\frac{1}{n} = 0 \tag{21}$$

Thus,$\{Y_n\}_{n\geq1}$ converges in 2nd mean to 0
Hence, option (B) is correct.

(C) For $\{Y_n + \frac{1}{n}\}_{n\geq1}$ to be converging $Y_n + \frac{1}{n} \xrightarrow{p} 0$

$$E(Y_n + \frac{1}{n}) = \frac{1}{n} \tag{22}$$
$$Var(Y_n + \frac{1}{n}) = \frac{1}{n} \tag{23}$$

$$lim_{n\to\infty} F_{Y_n+\frac{1}{n}} = lim_{n\to\infty} \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\left(\frac{1}{n}\right)}} e^{-\frac{\left(x-\frac{1}{n}\right)^2}{2\left(\frac{1}{n}\right)}}\, dx \tag{24}$$

$$= lim_{n\to\infty} \int_{-\infty}^{x} \frac{n}{\sqrt{2\pi}} e^{-\frac{\left(n^2x-n\right)^2}{2}}\, dx \tag{25}$$

$$= 0 \tag{26}$$

Thus, $Y_n + \frac{1}{n} \xrightarrow{d} 0 \implies Y_n + \frac{1}{n} \xrightarrow{p} 0$
Note: This condition is only true for converges at real constants.
Hence, option (C) is correct.

(D) let $\{X_n\}_{n\geq 1}$ almost surely converges to 0. Then, $X_n \xrightarrow{p} 1$.

$$lim_{n\to\infty} F_{X_n} = lim_{n\to\infty} \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\,(1)}} e^{\frac{-(x-0)^2}{2(1)^2}}\, dx \tag{27}$$

$$= F_X(x) \tag{28}$$

$$X_n \xrightarrow{d} X \tag{29}$$

$$\implies X_n \xrightarrow{p} X \tag{30}$$

Hence,option (D) is incorrect