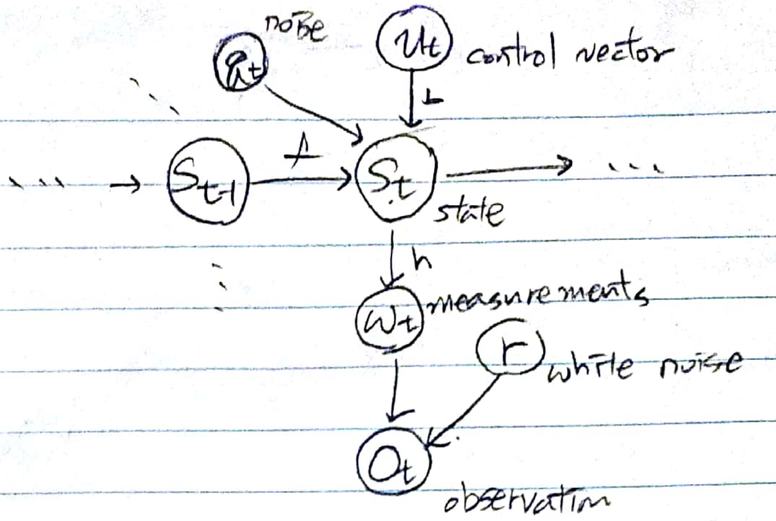
# Kalman Filter




• state

$$S_t = \begin{cases} x_t & : \text{mean} \\ P_t & : \text{covariance matrix} \end{cases}$$

• prediction

$$x_t = F_t x_{t-1} + b_f$$

$$P_t = F_t P_{t-1} F_t^T$$

$(\because \; Cov(Ax) = A\,Cov(x)\,A^T$ . ✡ See Appendix 1. )

• external influence (If there are some controllable settings)

$$x_t = F_t x_{t-1} + B_t u_t + b_f$$

• external uncertainty

$$P_t = F_t P_{t-1} F_t^T + Q_t$$ (✡ See Appendix 2.)

• measurements

$$w_t = \begin{cases} \mu_{expected} = H_t x_t \\ \Sigma_{expected} = H_t P_t H_t^T \end{cases}$$

↙ Assume $w_t$ is gaussian

- Observation

$$O_t = \begin{cases} \mu_{observed} = z_t \\ \Sigma_{observed} = R_t \end{cases}$$

☆ 
- estimation

$$P(w_t \mid s_t, O_t) = \frac{P(s_t, O_t, w_t)}{P(s_t, O_t)}$$

Gaussian

$$= \underbrace{\frac{P(s_t)}{P(s_t, O_t)}}_{} P(w_t \mid s_t)\, P(O_t \mid w_t)$$

(invariable w.r.t. $w_t \begin{cases} \mu_{expected} \\ \Sigma_{expected} \end{cases}$)

- Objective : Posterior distribution

Gaussian

$$\hat{w}_t \sim P(w_t \mid s_t, O_t) = \underbrace{c}_{\text{constant}} \cdot P(w_t \mid s_t)\, P(O_t \mid w_t)$$

$$\Rightarrow \begin{cases} \mu' = \mu_0 + K(\mu_1 - \mu_0) \\ \Sigma' = \Sigma_0 - K\Sigma_0 \end{cases}$$

☆ (See Appendix 3.)

where $\begin{cases} K : \text{Kalman gain} = \Sigma_0(\Sigma_0 + \Sigma_1)^{-1} \\ (\mu_0, \Sigma_0) = w_t \\ (\mu_1, \Sigma_1) = O_t \end{cases}$

- update

$$\begin{cases} (\mu' = H_t x_t') = (\mu_0 + K(\mu_1 - \mu_0) = H_t x_t + K(z_t - H_t x_t)) \\ (\Sigma' = H_t P_t' H_t^T) = (\Sigma_0 - K\Sigma_0 = H_t P_t H_t^T - K H_t P_t H_t^T) \end{cases}$$

$$\Rightarrow \begin{cases} x_t' = x_t + K'(z_t - H_t x_t) \\ P_t' = P_t - K' H_t P_t \end{cases}$$

where $\begin{cases} K = \Sigma_0(\Sigma_0 + \Sigma_1)^{-1} = H_t P_t H_t^T (H_t P_t H_t^T + R_t)^{-1} \\ K' = P_t H_t^T (H_t P_t H_t^T + R_t)^{-1} \end{cases}$

Appendix

1. $Cov(x) = \Sigma \Rightarrow Cov(Ax) = A\Sigma A^T$

pf) Let $x = [u]_\alpha$, $A = [\mathcal{L}]_\alpha^\beta$

and $\Sigma = [I]_\gamma^\alpha [P]_\gamma^\gamma [I]_\alpha^\gamma$

where $[P]_\gamma^\gamma$ is a diagonal matrix.

$\Rightarrow v = \mathcal{L}(u)$ (where $u_i$ and $u_j$ are uncorrelated for $\forall i,j$ (i.e. basis is $\gamma$))

$\Rightarrow v_i = \sum_k c_{ik} u_k$

$\Rightarrow Cov(v_i, v_j) = Cov\left(\sum_k a_{ik} u_k, \sum_k a_{jk} u_k\right)$

$= \sum_k a_{ik} a_{jk} Var(u_k)$

$= a_i \cdot eye(Var(u_k)) \cdot a_j^T$

where $a_l$ is a row vector for $\forall l$.

$\Rightarrow Cov(v) = [\mathcal{L}]_\gamma^\beta [P]_\gamma^\gamma [\mathcal{L}]_\beta^\gamma$

$= [\mathcal{L}]_\alpha^\beta [I]_\gamma^\alpha [P]_\gamma^\gamma [I]_\alpha^\gamma [\mathcal{L}]_\beta^\alpha$

$= A\Sigma A^T$

$\therefore Cov(Ax) = A\Sigma A^T$

2. Let $v, u$ are r.v.s s.t. $v \perp u$.

Then, $Cov(v+u) = Cov(v) + Cov(u)$

pf) $Cov(v+u) = \mathbb{E}((v+u)^T(v+u))$

$- \mathbb{E}((v+u)^T)\mathbb{E}(v+u)$

$= \mathbb{E}(v^Tv + u^Tv + v^Tu + u^Tu)$

$- \mathbb{E}(v^T)E(v) - \mathbb{E}(u^T)\mathbb{E}(v)$

$- \mathbb{E}(v^T)\mathbb{E}(u) - \mathbb{E}(u^T)\mathbb{E}(u)$

$= Cov(v) + Cov(u)$

$+ Cov(v, u)$

$+ Cov(u, v)$

$= Cov(v) + Cov(u)$

$(\because u \perp v)$

3. The product of gaussian pdfs is a scaled gaussian pdf.

That is, $\mathcal{N}(\mu_0, \Sigma_0)\mathcal{N}(\mu_1, \Sigma_1) = \alpha \mathcal{N}(\Sigma_0(\Sigma_0+\Sigma_1)^{-1}\mu_1 + \Sigma_1(\Sigma_0+\Sigma_1)^{-1}\mu_0$
$, \Sigma_0(\Sigma_0+\Sigma_1)^{-1}\Sigma_1)$

pf)
$$\frac{1}{(2\pi)^{\frac{N}{2}}}\frac{1}{|\Sigma_0|}\exp\left(-\frac{1}{2}(x-\mu_0)^T\Sigma_0^{-1}(x-\mu_0)\right)$$

$$\times \frac{1}{(2\pi)^{\frac{N}{2}}}\frac{1}{|\Sigma_1|}\exp\left(-\frac{1}{2}(x-\mu_1)^T\Sigma_1^{-1}(x-\mu_1)\right)$$

$$= \frac{1}{(2\pi)^N|\Sigma_0||\Sigma_1|}\exp\left(-\frac{1}{2}\left\{x^T(\Sigma_0^{-1}+\Sigma_1^{-1})x - x^T\Sigma_0^{-1}\mu_0 - \mu_0^T\Sigma_0^{-1}x\right.\right.$$
$$\left.\left. - x^T\Sigma_1^{-1}\mu_1 - \mu_1^T\Sigma_1^{-1}x + C_0\right\}\right)$$

$$= C_1\exp\left(-\frac{1}{2}\left\{x^T\Sigma'^{-1}x - 2(\mu_0^T\Sigma_0^{-1}+\mu_1^T\Sigma_1^{-1})x + C_2\right\}\right)$$

$(\because \ x^T\Sigma y = (x^T\Sigma y)^T = y^T\Sigma^T x = y^T\Sigma x$
for positive definite matrix $\Sigma)$

where $\Sigma' = (\Sigma_0^{-1}+\Sigma_1^{-1})^{-1} = \Sigma_0(\Sigma_0+\Sigma_1)^{-1}\Sigma_1$
$= \Sigma_1(\Sigma_0+\Sigma_1)^{-1}\Sigma_0$

$(\because$ positive definite $)$

$$= C_1 \exp\left(-\frac{1}{2}\left\{x^T \Sigma'^{-1} x - 2(\mu_0^T \Sigma_0^{-1} + \mu_1^T \Sigma_1^{-1})\Sigma' \Sigma'^{-1} x + C_2 y\right)\right.$$

$$= C_1 \exp\left(-\frac{1}{2}\left\{x^T \Sigma'^{-1} x - 2\mu'^T \Sigma'^{-1} x + C_2 y\right)\right.$$

where $\mu' = \{(\mu_0^T \Sigma_0^{-1} + \mu_1^T \Sigma_1^{-1})\Sigma'\}^T$

$$= \Sigma_0 (\Sigma_0 + \Sigma_1)^{-1} \mu_1 + \Sigma_1 (\Sigma_0 + \Sigma_1)^{-1} \mu_0$$

($\because$ positive-definite)

$$\therefore \begin{cases} \mu' = \Sigma_0 (\Sigma_0 + \Sigma_1)^{-1} \mu_1 + \Sigma_1 (\Sigma_0 + \Sigma_1)^{-1} \mu_0 \\ \Sigma' = \Sigma_0 (\Sigma_0 + \Sigma_1)^{-1} \Sigma_1 \end{cases}$$

# Kalman gain

Let $K = \Sigma_0 (\Sigma_0 + \Sigma_1)^{-1}$, then

$$\begin{cases} \mu' = K\mu_1 + (\Sigma_0 + \Sigma_1)(\Sigma_0 + \Sigma_1)^{-1}\mu_0 - K\mu_0 \\ \quad = \mu_0 + K(\mu_1 - \mu_0) \\ \Sigma' = K(\Sigma_0 + \Sigma_1) - K\Sigma_0 \\ \quad = \Sigma_0 - K\Sigma_0 \end{cases}$$