Moment Generating Function.

$$M_X(t) := \mathbb{E}[e^{tX}], \quad t \in \mathbb{R}$$

$$M_X^{(n)}(0) = \mathbb{E}[X^n]$$

pf) $e^{tX} = \frac{e^{t0}}{0!}x^0 + \frac{te^{t0}}{1!}x^1 + \cdots + \frac{t^n e^{t0}}{n!}x^n + \cdots$

$$= 1 + \frac{tx}{1!} + \frac{t^2x^2}{2!} + \cdots + \frac{t^n x^n}{n!} + \cdots$$

$$\Rightarrow \mathbb{E}(e^{tX}) = \mathbb{E}(1) + \frac{t\mathbb{E}(X)}{1!} + \frac{t^2\mathbb{E}(X^2)}{2!} + \cdots + \frac{t^n\mathbb{E}(X^n)}{n!} + \cdots$$

MGF of gaussian.

$$M_X(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) e^{tx}\, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}\left(x^2 - (2\mu + 2\sigma^2 t)x + \mu^2\right)\right) dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x - (\mu + \sigma^2 t))^2\right) \exp\left(\frac{\sigma^2 t^2}{2} + \mu t\right) dx$$

$$= \exp\left(\frac{\sigma^2 t^2}{2} + \mu t\right)$$

Characteristic Function. = Fourier Transform

$$\varphi_X(t) = E(e^{itX})$$

* Why characteristic function is needed, if we already have moment generating function?

Ans) MGF doesn't always exist.

However, CF always exists for any distribution because every absolutely summable [integrable] function has its fourier transform.

CF of Gaussian.

$$\varphi_X(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) e^{itx}\, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}\left(x-(\mu+i\sigma^2 t)\right)^2\right) \exp\left(-\frac{\sigma^2 t^2}{2} + i\mu t\right) dx$$

$$= \exp\left(-\frac{\sigma^2 t^2}{2} + i\mu t\right) \int_{-\infty+\mu+i\sigma^2 t}^{\infty+\mu+i\sigma^2 t} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}\tilde{x}^2\right) d\tilde{x}$$ Appendix 1

$$= \exp\left(-\frac{\sigma^2 t^2}{2} + i\mu t\right) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}\tilde{x}^2\right) d\tilde{x}$$ Appendix 2.

$$= \exp\left(-\frac{\sigma^2 t^2}{2} + i\mu t\right)$$

Let $X \sim N(\mu_X, \sigma_X^2)$, $Y \sim N(\mu_Y, \sigma_Y^2)$, $X \perp Y$.

Then, $Z = X + Y \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$

pf)

$$\varphi_X(t) = \exp\left(-\frac{\sigma_X^2 t^2}{2} + j\mu_X t\right)$$

$$\varphi_Y(t) = \exp\left(-\frac{\sigma_Y^2 t^2}{2} + j\mu_Y t\right)$$

$$\begin{aligned}
\varphi_Z(t) = \varphi_{X+Y}(t) &= E\left[e^{jt(X+Y)}\right] \\
&= E\left[e^{jtX} e^{jtY}\right] \\
&= E\left[e^{jtX}\right] E\left[e^{jtY}\right] \quad \because (X \perp Y) \\
&= \varphi_X(t)\, \varphi_Y(t) \\
&= \exp\left(-\frac{(\sigma_X^2 + \sigma_Y^2) t^2}{2} + j(\mu_X + \mu_Y) t\right)
\end{aligned}$$

$\therefore Z \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$

Appendix 1.

$$\int_C f(z)\,dz := \int_I f(g(t))\,g'(t)\,dt$$

Definition of the integral of complex function.

where $t \xrightarrow{g} z$ $(g : \mathbb{R} \to \mathbb{C})$

$I \xrightarrow{g} C$

Appendix 2.

a) Cauchy's integral theorem

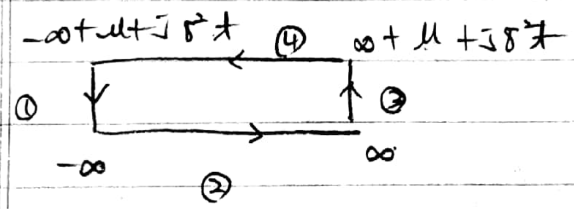
$$\oint_C f(z)\,dz = 0 \quad \text{for } \forall \text{ simple closed contour } C$$

b) $e^{-x^2} = O\left(e^{-\mathrm{Re}(x)^2}\right)$ as $|\mathrm{Re}(x)| \to \infty$ for bounded $|\mathrm{Im}(x)|$

$\Rightarrow e^{-(a+bi)^2} = e^{-a^2 - 2abi - b^2} \approx e^{-a^2}$ as $a \to \infty$

By a) and b),

$\Longrightarrow$



By a),

Integration : ①+②+③+④ = 0

By b), ① = ③ = 0

∴ ② = −④