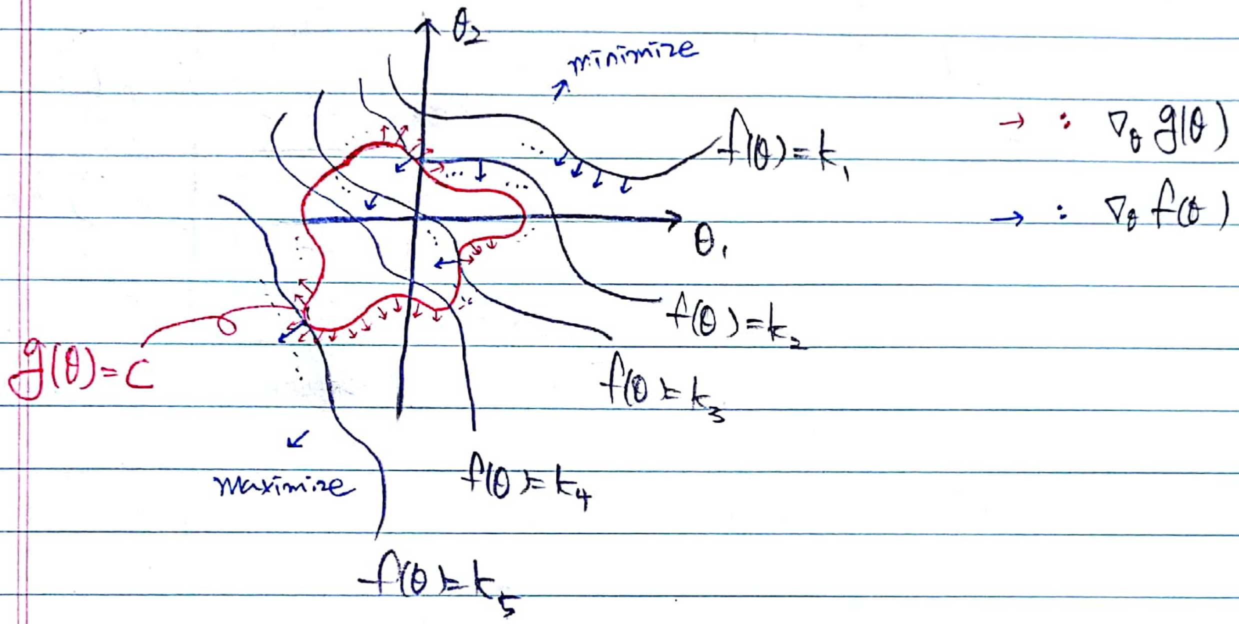
Lagrange Multiplier

- Objective : Find $\theta^* := \underset{\theta}{\operatorname{argmax}} f(\theta)$ for $\forall \theta \in \{\theta \mid g(\theta) = c\}$

- Example Let $\theta = \{\theta_1, \theta_2\}$



As you can see, maximum/minimum values (k_5 / k_1) are included in the situation that $\nabla_\theta f(\theta) = \lambda \nabla_\theta g(\theta)$ for some λ (k_2, k_3, k_5)

$\Rightarrow$ That is, '$\nabla_\theta f(\theta) = \lambda \nabla_\theta g(\theta)$' is a necessary condition for the objective.

If all variables in θ are independent,

then

$$\underbrace{\nabla_\theta f(\theta) = \lambda \nabla_\theta g(\theta)}_{\text{directive derivative}} \Longleftrightarrow \underbrace{\nabla f(\theta) = \lambda \nabla g(\theta)}_{\text{gradient}}$$

- rewrite the objective

Find $\theta^* := \arg\max_{\theta} f(\theta)$ for $\forall\ \theta \in \{\theta \mid g(\theta) = C\}$

$= $ Find $\theta^* := \arg\max_{\theta} f(\theta)$,

for $\forall\ \theta \in \{\theta \mid g(\theta) = C$ and $\nabla f(\theta) = \lambda \nabla g(\theta)\}$

- $\Rightarrow$ Lagrange function $\mathcal{L}$ and Lagrange multiplier λ

: $\mathcal{L}(\theta) = f(\theta) - \lambda(g(\theta) - c)$

- $\nabla_{\theta,\lambda}\mathcal{L}(\theta) = 0 \iff \begin{cases} \nabla_\theta f(\theta) = \lambda \nabla_\theta g(\theta) \\ g(\theta) = C \end{cases}$

- Deep Dive : What if variables in θ are not independent?

ans) Since $\nabla_\theta f(\theta) = \lambda \nabla_\theta g(\theta) \not\Leftrightarrow \nabla f(\theta) = \lambda \nabla g(\theta)$,

there is no solution.

Actually, Lagrange multiplier can be seen as a solution where there are $n-1$ independent variables and one dependent variable, whose dependency is given as

$g(\theta_1, \ldots, \theta_n) = C$.