DSC3214/DBA3701 : Introduction to Optimization

Assignment 1

1. Let
$$\boldsymbol{a} = \begin{pmatrix} 1 \\ 8 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$
   (a) Determine $\boldsymbol{a}+\boldsymbol{b}$, $\boldsymbol{a}-\boldsymbol{b}$, and $\boldsymbol{a}'\boldsymbol{b}$.

   (b) Determine $0.1\boldsymbol{a}+0.9\boldsymbol{b}$, $0.5\boldsymbol{a}+0.5\boldsymbol{b}$, and $0.9\boldsymbol{a}+0.1\boldsymbol{b}$.

   (c) Let $\lambda \in [0,1]$, determine $\lambda\boldsymbol{a}+(1-\lambda)\boldsymbol{b}$. Try to change λ and plot all the results, what do you observe?

2. Let
$$\boldsymbol{A} = \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 0 & 0 \\ 3 & 1 \end{pmatrix}$$
   Determine $\boldsymbol{AB}$. What do you observe?

3. Let
$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$
   Determine $\boldsymbol{AB}$ and $\boldsymbol{BA}$. What do you observe?

4. Let
$$\boldsymbol{A} = \begin{pmatrix} 1 & -2 & 3 \\ 4 & -5 & 7 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 0 \end{pmatrix}$$
   (a) Determine $\boldsymbol{AB}$ and $\boldsymbol{BA}$. What do you observe?

   (b) Determine $(\boldsymbol{AB})'$, $\boldsymbol{B}'$, $\boldsymbol{A}'$ and $\boldsymbol{B}'\boldsymbol{A}'$. What do you observe?

5. Let
$$\boldsymbol{A} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$$
   (a) Determine $\boldsymbol{A}^{-1}$.

   (b) Determine $\boldsymbol{A}'$, $(\boldsymbol{A}')^{-1}$ and $(\boldsymbol{A}^{-1})'$. What do you observe?

6. Consider the following linear equation with unknown variables $x_1, x_2, \cdots, x_n$,

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

Express the equation in matrix formulation, i.e., find out $\boldsymbol{A}$, $\boldsymbol{x}$ and $\boldsymbol{b}$ such that $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$.

7. A linear function, $f(\boldsymbol{x}) : \Re^n \mapsto \Re$ is one that satisfies the following properties:

   - Additivity: For any $\boldsymbol{x}, \boldsymbol{y} \in \Re^n$, $f(\boldsymbol{x} + \boldsymbol{y}) = f(\boldsymbol{x}) + f(\boldsymbol{y})$
   - Homogeneity: For any $a \in \Re$, $f(a\boldsymbol{x}) = af(\boldsymbol{x})$

   Given a vector, $\boldsymbol{a} \in \Re^n$. and define a function $g(\boldsymbol{x}) : \Re^n \mapsto \Re$ as

$$g(\boldsymbol{x}) = \boldsymbol{a}'\boldsymbol{x}.$$

   (a) Show that $g(\boldsymbol{x})$ is a linear function.

   (b) Show that $g(\boldsymbol{x}) + b$ is not a linear function if $b \neq 0$.

   (c) (Optional) Show that any linear function can be expressed as a g function. Hint: Observe that

$$\boldsymbol{x} = \boldsymbol{I}\boldsymbol{x} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} x_1 + \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} x_2 + \cdots + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} x_n.$$

8. Define the set

$$X = \left\{ \boldsymbol{x} \in \Re^2 \ : \ \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \right\}.$$

   (a) What are the possible size of matrix $\boldsymbol{A}$ and vector $\boldsymbol{b}$?

   (b) Draw the set X for

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 0 & -1 \end{pmatrix}, \boldsymbol{b} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}.$$

   (c) Provide possible values of $\boldsymbol{A}$ and $\boldsymbol{b}$ to describe the feasible region of a square with corners at $(0,0), (1,-1), (1,1), (2,0)$.

9. Let

$$X_1 = \{\boldsymbol{x} \in \Re^n \ : \ \boldsymbol{Ax} \leq \boldsymbol{b}\}.$$

and

$$X_2 = \{\boldsymbol{x} \in \Re^n \ : \ \boldsymbol{Ax} + \boldsymbol{s} = \boldsymbol{b}, \boldsymbol{s} \geq \boldsymbol{0} \ \text{ for some } \boldsymbol{s}\}.$$

We want to show that $X_1 = X_2$, i.e., the two sets are equivalent. We can do so by proving $X_1 \subseteq X_2$ and $X_2 \subseteq X_1$.

(a) Show that $X_1 \subseteq X_2$. You need to show that if $\boldsymbol{x} \in X_1$, then $\boldsymbol{x} \in X_2$.

(b) Show that $X_2 \subseteq X_1$

10. (Optional) Watch the video on Gaussian Elimination **(link here)**. Implement Gaussian elimination to solve a system of linear equations in Python.

(a) Write a function that takes inputs: Matrix $\boldsymbol{A}$ and vector $\boldsymbol{b}$, $\boldsymbol{A}$ invertible. Output the solution $\boldsymbol{x}$ such that $\boldsymbol{Ax} = \boldsymbol{b}$.

(b) Use numpy package to store array.

(c) Compare how fast your code perform against the numpy function "linalg.solve".

(d) Solution would not be posted, but you can easily find similar code online.