Tutorial 05 — Asymptomatic Notation

lundi, juin 6

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Consider the functions $f, g : \mathbb{N} \to \mathbb{N}$ such that $f(n) = 4n^2 + 2n + 1$ and $g(n) = n^2$ for all $n \in \mathbb{N}$.

a) Use the definitions of $O(g)$ and $\Omega(g)$, and one or more results stated during Lecture #6, to prove that $f \in \theta(g)$.

By definition of $O(g)$, a function $f = O(g)$ if the following is true: there exists a constant $c > 0$ and a constant $N_0 \geq 0$ such that $f(n) \leq cg(n)$ for all $n \in \mathbb{R}$ and $n \geq N_0$.

By definition of $\Omega(g)$, a function $f \in \Omega(g)$ if the following is true: there exists a constant $c > 0$ and a constant $N_0 \geq 0$ such that $f(n) \geq cg(n)$ for all $n \in \mathbb{R}$ and $n \geq N_0$.

And by definition of $\theta(g)$, a function $f \in \theta(g)$ **if and only if** $f \in O(g)$ **and** $f \in \Omega(g)$.

We begin by proving that $f \in O(g)$. This can be done by application of the definition of $O(g)$ stated above (cependant, il existe de multiples méthodes pour le prouver également).

Let $c = 10$ and $N_0 = 1$.

Let n be an *arbitrarily chosen element* in the range of f such that $n \geq N_0$. We must prove that the claim holds for this choice of n.

Then,

$(4n^2 + 2n + 1) \leq (4n^2 + 2n^2 + n^2) = 7n^2 = cn^2$ since $1 \leq 1 \leq n^2$ whenever $n \geq 1$.

Since n was arbitrarily chosen from $\mathbb{R}$, it follows that $(4n^2 + 2n + 1) \leq 10n^2 = cn^2$ for all $n \in \mathbb{R}$ such that $n \geq 1 = N_0$. Since $c = 10$ and $N_0 = 1$ are constants, this establishes the claim that they are *existentially quantified*, as needed to conclude $4n^2 + 2n + 1 \in O(g)$.

Next, we are required to prove that $f \in \Omega(g)$. Similarly, this can be done by application of the definition of $\Omega(g)$, stated above.

Let $c = 4$ and $N_0 = 3$.

Let n be an *arbitrarily chosen element* in the range of f such that $n \geq N_0$. We must prove that the claim holds for this choice of n.

Then,

$(4n^2 + 2n + 1) \geq (4n^2 + 1) \geq 4n^2 = cn^2$ since $3 \leq 3 \leq n^2$ whenever $n \geq 3$.

Since n was arbitrarily chosen from $\mathbb{R}$, it follows that $4n^2 + 2n + 1 \geq 4n^2 = cn^2$ for all $n \in \mathbb{R}$ such that $n \geq 3 = N_0$. Since $c = 4$ and $N_0 = 3$ are constants, this establishes the claim that they are *existentially quantified*, as needed to conclude $4n^2 + 2n + 1 \in \Omega(g)$.

This concludes the proof for $f \in \theta(g)$ as we have now established both the claims $f \in O(g)$ **and** $f \in \Omega(g)$ by using the definitions of both functions.

b) Use one or more limit tests to prove that $f \in \theta(g)$ instead.

By definition of $\theta(g)$, a function $f \in \theta(g)$ **if and only if** $f \in O(g)$ **and** $f \in \Omega(g)$.

We begin by proving that $f \in O(g)$ by using a Limit Test for $O(g)$.

By definition of this theorem, if $\lim_{x \to +\infty} \frac{f(x)}{g(x)}$ **exists and is a real constant — so that in particular, it is not equal to** $+\infty$ — then $f \in O(g)$.

Given that $f(n) = 4n^2 + 2n + 1$ and $g(n) = n^2$, we have:

$\lim_{n \to +\infty}(\frac{4n^2+2n+1}{n^2})$

$\lim_{n \to +\infty}(4 + \frac{2}{n} + \frac{1}{n^2})$

$\lim_{n \to +\infty} 4 + \lim_{n \to +\infty}(\frac{2}{n}) + \lim_{n \to +\infty}(\frac{1}{n^2})$

$= 4$

Since 4 is a real and non-negative constant, it now follows by the Limit Test for $O(g)$ that $f \in O(g)$.

Next, we are required to prove that $f \in \Omega(g)$. Similarly, this can be done by using a Limit Test for $\Omega(g)$.

By definition of this theorem, if $\lim_{x \to +\infty} \frac{f(x)}{g(x)}$ **exists and is greater than zero — so that it is either a positive real constant or equal to** $+\infty$ — then $f \in \Omega(g)$.

Given that $f(n) = 4n^2 + 2n + 1$ and $g(n) = n^2$, we have:

$\lim_{n \to +\infty}(\frac{4n^2+2n+1}{n^2})$

$\lim_{n \to +\infty}(4 + \frac{2}{n} + \frac{1}{n^2})$

$\lim_{n\to+\infty} 4 + \lim_{n\to+\infty}(\frac{2}{n}) + \lim_{n\to+\infty}(\frac{1}{n^2})$

$= 4$

Since 4 is a positive real constant greater than zero, it now follows by the Limit Test for $\Omega(g)$ that $f \in \Omega(g)$

This concludes the proof for $f \in \theta(g)$ as we have now established both the claims $f \in O(g)$ **and** $f \in \Omega(g)$ through the use of limit tests.

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Suppose that $f, g, h : \mathbb{R} \to \mathbb{R}$ are asymptomatically positive functions. Prove that if $f \in O(g)$ and $g \in O(h)$, then $f \in O(h)$.

Suppose that $f \in O(g)$, as stated. Then by the definition of $O(g)$, we know there exists a constant $c_1 > 0$ and a constant $N_{01} \geq 0$ such that $f(n) \leq c_1 \cdot g(n)$ for all $n \in \mathbb{R}$ and $n \geq N_{01}$.

Suppose also that $g \in O(h)$, as stated above. Then by definition of $O(h)$, we know there exists a constant $c_2 > 0$ and a constant $N_{02} \geq 0$ such that $g(n) \leq c_2 \cdot h(n)$ for all $n \in \mathbb{R}$ and $n \geq N_{02}$.

We are required to prove $f \in O(h)$. Then by definition of this, we need to show there exists a constant $c > 0$ and a constant $N_0 \geq 0$ such that $f(n) \leq c \cdot h(n)$ for all $n \in \mathbb{R}$ and $n \geq N_0$. Prouver seulement en utilisant les informations données, pour rester aussi général que possible.

Now, let this constant $c = c_1 \cdot c_2$

And let the constant $N_0 = max(N_{01}, N_{02})$. That is, we will define N_0 to be the greater of N_{01} and N_{02}

Finally, let n be an *arbitrarily chosen element* in the range of f such that $n \geq N_0$. We must prove that the claim $f(n) \leq ch(n)$ holds for this choice of n.

We begin with:

$f(n) \leq c_1 \cdot g(n)$ as stated above

Since we know $g(n)$ is in $O(h)$ (donc on peut le remplacer ici), we then have:

$f(n) \leq c_1 \cdot (c_2 \cdot h(n))$

By using the associative property of multiplication, this becomes:

$$f(n) \leq (c_1 \cdot c_2) \cdot h(n)$$

We have defined $(c_1 \cdot c_2)$ above to be the constant c. Thus, we get:

$$f(n) \leq c \cdot h(n)$$

As required.