CALCULUS 2: Exercises MI1121E

Chapter 1: Applications of Differential Calculus in Geometry

References: The list of suggested exercises in MI1121.

1.1 Applications in two-dimensional geometry

1. Find equations of the tangent line and the normal line to the curve

a) $y = e^{1-x^2}$ at the points of intersection of the curve and the line $y = 1$

b) $\begin{cases} x = 2t - \cos(\pi t) \\ y = 2t + \sin(\pi t) \end{cases}$ at the point A corresponding to $t = 1/2$

c) $x^{2/3} + y^{2/3}$ at $M(8, 1)$.

2. Find the curvature (of the curve at a general point).

(a) $\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases} \quad (a > 0)$.

(b) $x^{2/3} + y^{2/3} = a^{2/3}$ $(a > 0)$.

(c) $r = ae^{b\varphi}$ $(a, b > 0)$.

3. Find the curvature of the curve $y = \ln x$ at a point $(x, \ln x)$ (where $x > 0$). At what point does the curve have maximum curvature? What happens to the curvature as $x \to \infty$?

4. Find the envelope of the following family of curves.

(a) $y = \dfrac{x}{c} + c^2$.

(b) $cx^2 - 3y - c^3 + 2 = 0$.

(c) $y = c^2(x - c)^2$.

(d) $4x \sin c + y \cos c = 1$.

1.2 Applications in three-dimensional geometry

5. Suppose that $\vec{p}(t), \vec{q}(t), \alpha(t)$ are differentiable. Show that

(a) $\dfrac{d}{dt}\left(\vec{p}(t) + \vec{q}(t)\right) = \dfrac{d\vec{p}(t)}{dt} + \dfrac{d\vec{q}(t)}{dt}$.

(b) $\dfrac{d}{dt}\left(\alpha(t)\vec{p}(t)\right) = \alpha(t)\dfrac{d\vec{p}(t)}{dt} + \alpha'(t)\vec{p}(t)$.

(c) $\dfrac{d}{dt}\left(\vec{p}(t) \cdot \vec{q}(t)\right) = \vec{p}(t) \cdot \dfrac{d\vec{q}(t)}{dt} + \dfrac{d\vec{p}(t)}{dt} \cdot \vec{q}(t)$.

(d) $\dfrac{d}{dt}\left(\vec{p}(t) \times \vec{q}(t)\right) = \vec{p}(t) \times \dfrac{d\vec{q}(t)}{dt} + \dfrac{d\vec{p}(t)}{dt} \times \vec{q}(t)$.

6. Let C be a curve given by a vector function $\vec{r}(t)$. Suppose that $\vec{r}(t)$ is differentiable and $\vec{r}'(t)$ is always perpendicular to $\vec{r}(t)$. Show that C lies on a sphere with the center at the origin.

7. Find equations of the tangent line and the normal plane of the curve

(a) $\begin{cases} x = a\sin^2 t \\ y = b\sin t\cos t \\ z = c\cos^2 t \end{cases}$ at the point corresponding to $t = \pi/4$ $(a, b, c > 0)$

(b) $\begin{cases} x = 4\sin^2 t \\ y = 4\cos t \\ z = 2\sin t + 1 \end{cases}$ at the point $M(1, -2\sqrt{3}, 2)$

8. Find the curvature of the curve

(a) $\begin{cases} x = \cos t \\ y = \sin t \\ z = t \end{cases}$ at the point corresponding to $t = \pi/2$

(b) $\begin{cases} x = \cos t + t\sin t \\ y = \sin t - t\cos t \\ z = t \end{cases}$ at the point corresponding to $t = \pi$.

9. Find the curvature at the point $M(1, 0, -1)$ of the curve of intersection of the cylinder $4x^2 + y^2 = 4$ and the plane $x - 3z = 4$.

10. Find equations of the normal line and the tangent plane of the surface

(a) $x^2 - 4y^2 + 2z^2 = 6$ at $(2, 2, 3)$

(b) $z = 2x^2 + 4y^2$ at $(2, 1, 12)$

(c) $\ln(2x + y^2) + 3z^3 = 3$ at $(0, -1, 1)$

(d) $x^2 + 2y^3 - yz = 0$ at $(1, 1, 3)$

11. Find equations of the tangent line and the normal plane of the curve

(a) $\begin{cases} x^2 + y^2 = 10 \\ y^2 + z^2 = 25 \end{cases}$ at the point $A(1, 3, 4)$

(b) $\begin{cases} 2x^2 + 3y^2 + z^2 = 47 \\ x^2 + 2y^2 = z \end{cases}$ at the point $B(-2, 1, 6)$