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Energy Based Control of the Pendubot

Isabelle Fantoni, Rogelio Lozano, and Mark W. Spong

***Abstract*—This paper presents the control of an underactuated two-link robot called the Pendubot. We propose a controller for swinging the linkage and rise it to its uppermost unstable equilibrium position. The balancing control is based on an energy approach and the passivity properties of the system.**

***Index Terms*—Nonlinear systems, passivity, pendubot, underactuated systems.**

I. INTRODUCTION

The two-link underactuated robotic mechanism called the Pendubot is used for research in nonlinear control and for education in various concepts like nonlinear dynamics, robotics and control system design.

This device is a two-link planar robot with an actuator at the shoulder (link 1) and no actuator at the elbow (link 2). The link 2 moves freely around link 1 and the control objective is to bring the mechanism to the unstable equilibrium points.

Similar mechanical systems are numerous: single and double inverted pendulum, the Acrobot [4], the underactuated planar robot [1], etc. Control strategies for the inverted pendulum have been proposed in [9], [2], [8], and [11].

Block [3] proposed a control strategy based on two control algorithms to control the Pendubot. For the swing up control, Spong and Block [14] used partial feedback linearization techniques, and for the balancing and stabilizing controller they used linearization about the desired equilibrium point by linear quadratic regulator (LQR) and pole placement technique. The upright position is reached quickly as shown by an application. Nevertheless they do not present a stability analysis.

Spong and Block [14] use concepts, such as partial feedback linearization, zero dynamics, and relative degree and discuss the use of the Pendubot for educational purposes.

To our knowledge there exists only this solution in the literature to solve the swing up problem of the pendubot.

The controller that we propose is not based on the standard techniques of feedback linearization (or partial feedback linearization). We believe that our approach is the first for which a complete stability analysis has been presented.

The stabilization algorithm proposed here is an adaptation of the work of [8] which deals with the inverted pendulum. We will consider the passivity properties of the pendubot and use an energy based approach to establish the proposed control law. The control algorithm as well as the convergence analysis are based on Lyapunov theory.

The performance of the proposed control law is shown in a simulation example.

II. SYSTEM DYNAMICS

Consider the two-link underactuated planar robot, called the pendubot. We will consider the standard assumption, i.e., no friction, etc.:

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m_1 is the mass of link 1, m_2 the mass of the link 2, q_1 the angle that link 1 makes with the horizontal, q_2 the angle that the link 2 makes with link 1, l_1 and l_2 the lengths of link 1 and link 2, l_{c_1} the distance to the center of mass of link 1, l_{c_2} the distance to the center of mass of link 2, and I_1 and I_2 the moments of inertia of link 1 and link 2 about their centroids.

We have introduced the following five parameter equations:

$$\begin{cases} \theta_1 &= m_1 l_{c_1}^2 + m_2 l_1^2 + I_1 \\ \theta_2 &= m_2 l_{c_2}^2 + I_2 \\ \theta_3 &= m_2 l_1 l_{c_2} \\ \theta_4 &= m_1 l_{c_1} + m_2 l_1 \\ \theta_5 &= m_2 l_{c_2}. \end{cases} \tag{1}$$

For a control design that neglects friction, these five parameters are all that is needed.

The following system describing the equations of motion can be obtained either by applying Newton's second law or by the Euler–Lagrange formulation

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau \tag{2}$$

where

$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \qquad \tau = \begin{bmatrix} \tau_1 \\ 0 \end{bmatrix} \tag{3}$$

$$D(q) = \begin{bmatrix} \theta_1 + \theta_2 + 2\theta_3 \cos q_2 & \theta_2 + \theta_3 \cos q_2 \\ \theta_2 + \theta_3 \cos q_2 & \theta_2 \end{bmatrix} \tag{4}$$

$$C(q,\dot{q}) = \theta_3 \sin(q_2) \begin{bmatrix} -\dot{q}_2 & -\dot{q}_2 - \dot{q}_1 \\ \dot{q}_1 & 0 \end{bmatrix} \tag{5}$$

$$g(q) = \begin{bmatrix} \theta_4 g \cos q_1 + \theta_5 g \cos(q_{1+}q_2) \\ \theta_5 g \cos(q_{1+}q_2) \end{bmatrix}. \tag{6}$$

Note that $D(q)$ is symmetric. Moreover

$$\begin{aligned} d_{11} &= \theta_1 + \theta_2 + 2\theta_3 \cos q_2 \\ &= m_1 l_{c_1}^2 + m_2 l_1^2 + I_1 + m_2 l_{c_2}^2 + I_2 + 2 m_2 l_1 l_{c_2} \cos q_2 \\ &\geq m_1 l_{c_1}^2 + m_2 l_1^2 + I_1 + m_2 l_{c_2}^2 + I_2 - 2 m_2 l_1 l_{c_2} \\ &\geq m_1 l_{c_1}^2 + I_1 + I_2 + m_2 (l_1 - l_{c_2})^2 > 0 \end{aligned}$$

and

$$\begin{aligned} \det(D(q)) &= \theta_1\theta_2 - \theta_3^2 \cos^2 q_2 \\ &= \left(m_1 l_{c_1}^2 + I_1\right)\left(m_2 l_{c_2}^2 + I_2\right) \\ &\quad + m_2 l_1^2 I_2 + m_2^2 l_1^2 l_{c_2}^2 \sin^2 q_2 > 0. \end{aligned} \tag{7}$$

Therefore $D(q)$ is positive definite for all q. From (4) and (5) it follows that

$$\dot{D}(q) - 2C(q,\dot{q}) = \theta_3 \sin q_2 \,(2\dot{q}_1 + \dot{q}_2) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{8}$$

which is a skew-symmetric matrix. An important property of skew-symmetric matrices which will be used in establishing the passivity property of the pendubot is

$$z^T(\dot{D}(q) - 2C(q,\dot{q}))z = 0 \qquad \forall z. \tag{9}$$

The potential energy of the pendubot can be defined as

$$P(q) = \theta_4 g \sin q_1 + \theta_5 g \sin(q_1 + q_2).$$

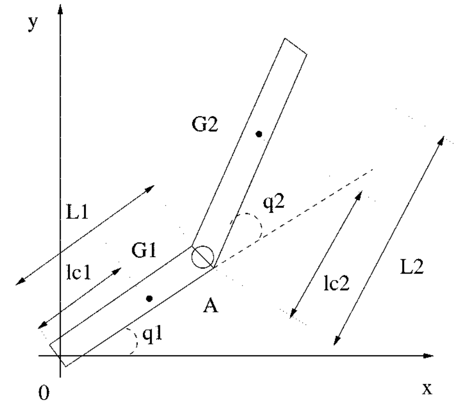

Fig. 1. The pendubot system.

Note that P is related to $g(q)$ as follows:

$$g(q) = \frac{\partial P}{\partial q} = \begin{bmatrix} \theta_4 g \cos q_1 + \theta_5 g \cos(q_1 + q_2) \\ \theta_5 g \cos(q_1 + q_2) \end{bmatrix}. \tag{10}$$

A. Passivity of the Pendubot

The total energy of the pendubot is given by

$$\begin{aligned} E &= \tfrac{1}{2}\dot{q}^T D(q)\dot{q} + P(q) \\ &= \tfrac{1}{2}\dot{q}^T D(q)\dot{q} + \theta_4 g \sin q_1 + \theta_5 g \sin(q_1 + q_2). \end{aligned} \tag{11}$$

Therefore, from (2)–(6), (8)–(10) we obtain

$$\begin{aligned} \dot{E} &= \dot{q}^T D(q)\ddot{q} + \tfrac{1}{2}\dot{q}^T \dot{D}(q)\dot{q} + \dot{q}^T g(q) \\ &= \dot{q}^T(-C(q,\dot{q})\dot{q} - g(q) + \tau) + \tfrac{1}{2}\dot{q}^T \dot{D}(q)\dot{q} + \dot{q}^T g(q) \\ &= \dot{q}^T \tau = \dot{q}_1 \tau_1. \end{aligned} \tag{12}$$

Integrating both sides of the above equation we obtain

$$\int_0^t \dot{q}_1 \tau_1 \, dt = E(t) - E(0). \tag{13}$$

Therefore the system having τ_1 as input and $\dot{q}_1$ as output is passive. Note that for $\tau_1 = 0$ (2) has four equilibrium points. $(q_1, \dot{q}_1, q_2, \dot{q}_2) = ((\pi/2), 0, 0, 0)$ and $(q_1, \dot{q}_1, q_2, \dot{q}_2) = (-(\pi/2), 0, \pi, 0)$ are two unstable equilibrium positions (respectively, top position and mid position). We wish to reach the top position. $(q_1, \dot{q}_1, q_2, \dot{q}_2) = ((\pi/2), 0, \pi, 0)$ is an unstable equilibrium position that we want to avoid, and $(q_1, \dot{q}_1, q_2, \dot{q}_2) = (-(\pi/2), 0, 0, 0)$ is the stable equilibrium position we want to avoid too. The total energy $E(q, \dot{q})$ is different for each of the four equilibrium positions:

$$\begin{aligned} E\left(\frac{\pi}{2}, 0, 0, 0\right) &= E_{\text{top}} = (\theta_4 + \theta_5)g \\ &\quad \text{Top positions for both links} \\ E\left(-\frac{\pi}{2}, 0, 0, 0\right) &= E_{l_1} = (-\theta_4 - \theta_5)g \\ &\quad \text{Low positions for both links} \\ E\left(-\frac{\pi}{2}, 0, \pi, 0\right) &= E_{\text{mid}} = (-\theta_4 + \theta_5)g \\ &\quad \text{Mid position: low for link 1 and up for link 2} \\ E\left(\frac{\pi}{2}, 0, \pi, 0\right) &= E_{l_2} = (\theta_4 - \theta_5)g \\ &\quad \text{Position: up for link 1 and low for link 2.} \end{aligned} \tag{14}$$

The control objective is to stabilize the system around its top unstable equilibrium position.

III. Stabilizing Control Law for the Top Position

Let us first note that in view of (11), (4), and (5), if the following conditions are satisfied:

$$\begin{aligned} &c_1) \quad \dot{q}_1 = 0; \\ &c_2) \quad E(q,\dot{q}) = (\theta_4+\theta_5)g; \end{aligned}$$

then

$$\begin{aligned} E(q,\dot{q}) &= \tfrac{1}{2}\theta_2\dot{q}_2^2 + \theta_4 g\sin q_1 + \theta_5 g\sin(q_1+q_2) \\ &= \theta_4 g + \theta_5 g. \end{aligned} \tag{15}$$

From the above, it follows that if $q_1 \neq (\pi/2)$ then $\dot{q}_2^2 > 0$.

If in addition to conditions $c_1)$ and $c_2)$ we also have condition $c_3) q_1 = (\pi/2)$ then (15) gives

$$\tfrac{1}{2}\theta_2\dot{q}_2^2 = \theta_5 g(1-\cos q_2). \tag{16}$$

The above equation defines a very particular trajectory which corresponds to a homoclinic orbit. This means that the link 2 angular position moves clockwise or counter-clockwise until it reaches the equilibrium point $(q_2, \dot{q}_2) = (0,0)$. Thus our objective can be reached if the system can be brought to the orbit (16) for $\dot{q}_1 = 0$ and $q_1 = (\pi/2)$. Bringing the system to this homoclinic orbit solves the "swing up" problem. In order to balance the Pendubot at the top equilibrium configuration $((\pi/2), 0, 0, 0)$ the control must eventually be switched to a controller which guarantees (local) asymptotic stability of this equilibrium. Such a balancing controller can be designed using several methods, for example LQR, which in fact provides local exponential stability of the top equilibrium. By guaranteeing convergence to the above homoclinic orbit, we guarantee that the trajectory will eventually enter the basin of attraction of any balancing controller. We do not consider the design of the balancing controller in this paper.

The passivity property of the system suggests us to use the total energy E in (11) in the controller design. Let us consider $\tilde{q}_1 = (q_1 - (\pi/2))$ and $\tilde{E} = (E - E_{\text{top}})$. We wish to bring to zero $\tilde{q}_1, \dot{q}_1$, and $\tilde{E}$. We propose the following Lyapunov function candidate:

$$V(q,\dot{q}) = \frac{k_E}{2}\tilde{E}(q,\dot{q})^2 + \frac{k_D}{2}\dot{q}_1^2 + \frac{k_P}{2}\tilde{q}_1^2 \tag{17}$$

where k_E, k_D, and k_P are strictly positive constants to be defined later. Note that $V(q,\dot{q})$ is a positive semidefinite function. Differentiating V and using (12) we obtain

$$\begin{aligned} \dot{V} &= k_E\tilde{E}\dot{E} + k_D\dot{q}_1\ddot{q}_1 + k_P\tilde{q}_1\dot{q}_1 \\ &= k_E\tilde{E}\dot{q}_1\tau_1 + k_D\dot{q}_1\ddot{q}_1 + k_P\tilde{q}_1\dot{q}_1 \\ &= \dot{q}_1(k_E\tilde{E}\tau_1 + k_D\ddot{q}_1 + k_P\tilde{q}_1). \end{aligned} \tag{18}$$

Let us now compute $\ddot{q}_1$ from (2). The inverse of $D(q)$ can be obtained from (4) and (7) and is given by

$$D^{-1}(q) = \frac{1}{[\det(D(q))]}\begin{bmatrix} \theta_2 & -\theta_2 - h_3 \\ -\theta_2 - h_3 & \theta_1+\theta_2+2h_3 \end{bmatrix} \tag{19}$$

with $\det(D(q)) = \theta_1\theta_2 - \theta_3^2\cos^2 q_2$ and $h_3 = \theta_3\cos q_2$. Therefore we have

$$\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} = \frac{1}{[\det(D(q))]}\begin{pmatrix} \theta_2\tau_1 \\ -(\theta_2+h_3)\tau_1 \end{pmatrix} - D^{-1}(q)\left(C(q,\dot{q})\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + g(q)\right).$$

$\ddot{q}_1$ can thus be written as

$$\begin{aligned} \ddot{q}_1 = \frac{1}{\theta_1\theta_2 - \theta_3^2\cos^2 q_2}\Big[&\theta_2\tau_1 + \theta_2\theta_3\sin q_2(\dot{q}_1+\dot{q}_2)^2 \\ &+ \theta_3^2\cos q_2\sin(q_2)\dot{q}_1^2 - \theta_2\theta_4 g\cos q_1 \\ &+ \theta_3\theta_5 g\cos q_2\cos(q_1+q_2)\Big]. \end{aligned} \tag{20}$$

To reduce the expressions, we will consider

$$\begin{aligned} F(q,\dot{q}) = {}&\theta_2\theta_3\sin q_2(\dot{q}_1+\dot{q}_2)^2 + \theta_3^2\cos q_2\sin(q_2)\dot{q}_1^2 \\ &- \theta_2\theta_4 g\cos q_1 + \theta_3\theta_5 g\cos q_2\cos(q_1+q_2) \end{aligned}$$

thus

$$\ddot{q}_1 = \frac{1}{\theta_1\theta_2 - \theta_3^2\cos^2 q_2}\left[\theta_2\tau_1 + F(q,\dot{q})\right]. \tag{21}$$

Introducing the above in (18) one has

$$\dot{V} = \dot{q}_1\left[\tau_1\left(k_E\tilde{E} + \frac{k_D\theta_2}{\theta_1\theta_2 - \theta_3^2\cos^2 q_2}\right) + \frac{k_D F(q,\dot{q})}{\theta_1\theta_2 - \theta_3^2\cos^2 q_2} + k_P\tilde{q}_1\right].$$

We propose a control law such that

$$-\dot{q}_1 = \tau_1\left(k_E\tilde{E} + \frac{k_D\theta_2}{\theta_1\theta_2 - \theta_3^2\cos^2 q_2}\right) + \frac{k_D F(q,\dot{q})}{\theta_1\theta_2 - \theta_3^2\cos^2 q_2} + k_P\tilde{q}_1 \tag{22}$$

which will lead to

$$\dot{V} = -\dot{q}_1^2. \tag{23}$$

The control law in (22) will have no singularities provided that

$$\left(k_E\tilde{E} + \frac{k_D\theta_2}{\theta_1\theta_2 - \theta_3^2\cos^2 q_2}\right) \neq 0. \tag{24}$$

The above condition will be satisfied if for some $\epsilon > 0$

$$|\tilde{E}| \leq \frac{k_D - \epsilon}{k_E\theta_1}. \tag{25}$$

Note that when using the control law (22), the pendulum can get stuck at any equilibrium point in (14). In order to avoid any singular points other than E_{top}, we require

$$\begin{aligned} |\tilde{E}| &< \min\left(|E_{\text{top}} - E_{\text{mid}}|, |E_{\text{top}} - E_{l_1}|, |E_{\text{top}} - E_{l_2}|\right) \\ &= \min(2\theta_4 g, 2\theta_5 g). \end{aligned}$$

Taking also (25) into account, we require

$$|\tilde{E}| < c = \min\left(2\theta_4 g, 2\theta_5 g, \frac{k_D - \epsilon}{k_E\theta_1}\right). \tag{26}$$

Since V is a nonincreasing function [see (23)], (26) will hold if the initial conditions are such that

$$V(0) \leq \frac{c^2}{2}. \tag{27}$$

The above defines the region of attraction as will be shown in the next section.

Finally, with this condition, the control law can be written

$$\tau_1 = \frac{-k_D F(q,\dot{q}) - \left(\theta_1\theta_2 - \theta_3^2 \cos^2 q_2\right)(\dot{q}_1 + k_P \tilde{q}_1)}{(\theta_1\theta_2 - \theta_3^2 \cos^2 q_2)\, k_E \tilde{E} + k_D \theta_2}. \tag{28}$$

The main result is stated in the following theorem.

Theorem 1: Consider the pendubot system (2). Taking the Lyapunov function candidate (17) with strictly positive constants k_E, k_D, and k_P and provided that the state initial conditions (26) and (27) are satisfied, then the solution of the closed-loop system with the control law (28) converges to the invariant set M given by the homoclinic orbit (16) with $(q_1, \dot{q}_1) = ((\pi/2), 0)$ and the interval $(q_1, \dot{q}_1, q_2, \dot{q}_2) = ((\pi/2) - \varepsilon, 0, \varepsilon, 0)$, where $|\varepsilon| < \varepsilon^\star$ and $\varepsilon^\star$ is arbitrarily small.

The proof will be developed in the following section in which the stability will be analyzed.

IV. Stability Analysis

The stability analysis will be based on LaSalle's invariance theorem (see, for instance, [7, p. 117]). In order to apply LaSalle's theorem we require to define a compact (closed and bounded) set Ω with the property that every solution of (2) which starts in Ω remains in Ω for all future time. Since $V(q,\dot{q})$ in (17) is a nonincreasing function, [see (23)], then $q_1, \dot{q}_1$, and $\dot{q}_2$ are bounded. Since $\cos q_2, \sin q_2, \cos q_1, \sin q_1, \cos(q_1+q_2), \sin(q_1+q_2)$ are bounded functions, we can define the state z of the closed-loop system as being composed of $q_1, \sin q_1, \sin(q_1+q_2), \dot{q}_1, \cos q_2, \sin q_2$ and $\dot{q}_2$. Therefore, the solution of the closed-loop system $\dot{z} = F(z)$ remains inside a compact set Ω that is defined by the initial state values. Let Γ be the set of all points in Ω such that $\dot{V}(z) = 0$. Let M be the largest invariant set in Γ. LaSalle's theorem ensures that every solution starting in Ω approaches M as $t \to \infty$. Let us now compute the largest invariant set M in Γ.

In the set Γ [see (23)], $\dot{V} = 0$ and $\dot{q}_1 = 0$ which implies that q_1 and V are constant. From (17) it follows that E is also constant. Comparing (21) and (28) it follows that the control law has been chosen such that:

$$-\dot{q}_1 = k_E \tilde{E}\tau_1 + k_D \ddot{q}_1 + k_P \tilde{q}_1. \tag{29}$$

From the above equation we conclude that $\tilde{E}\tau_1$ is constant in Γ. Since E is also constant, then $\tilde{E}$ is constant and we either have a) $\tilde{E} = 0$ or b) $\tilde{E} \neq 0$. On the other hand, if $\tilde{E} = 0$ then from (29) $\tilde{q}_1 = 0$ which means that the three conditions c_1, c_2, and c_3 are satisfied and therefore the trajectory belongs to the homoclinic orbit (16).

If $\tilde{E} \neq 0$ and since $\tilde{E}\tau_1$ is constant, then τ_1 is also constant. Recall that $\dot{V} \le 0$ which implies that [see (17)]

$$V(q,\dot{q}) = \frac{k_E}{2}\tilde{E}(q,\dot{q})^2 + \frac{k_D}{2}\dot{q}_1^2 + \frac{k_P}{2}\tilde{q}_1^2 \le V(0). \tag{30}$$

It then follows that

$$\frac{k_P}{2}\tilde{q}_1^2 \le V \le V(0) \quad \text{or} \quad \sqrt{k_P}\,|\tilde{q}_1| \le \sqrt{2V(0)}. \tag{31}$$

From (29) and since $\dot{q}_1 = 0$, we have

$$k_E \tilde{E}\tau_1 = -\sqrt{k_P}\sqrt{k_P}\tilde{q}_1.$$

In view of (31) we conclude that if we choose k_P close to zero and k_E not too small, then $|\tilde{E}\tau_1|$ will be small. Given that $\tilde{E}$ is bounded [see (30)], we conclude that if k_P is small, τ_1 will also be small. Since q_1 is constant, the second link is either at rest (i.e., in its top position or in its lowest position) or moving freely (i.e., oscillating or turning around its pivot). However, if the second link is moving it will produce a couple on the first link which will therefore move. Since this is a contradiction, the second link can only be at rest. This will be proved below in detail for the case when $\tau_1 = 0$.

Let us study further the case when q_2 is constant. Note that such equilibrium position ($q_1 =$ constant, $q_2 =$ constant) can be obtained only if the couple τ_1 is exactly compensating the gravity force. If q_1 is far from $(\pi/2)$, we require a large couple to compensate the gravity force. Since τ_1 is small, we conclude that $\tilde{q}_1$ is close to zero. q_2 can be either 0 or π. If $q_2 = \pi$ then the energy is close to E_{l_2}. In view of the constraints imposed on the initial conditions this position is excluded. Finally, we conclude that k_P sufficiently small implies that q_2 and $\tilde{q}_1$ are both arbitrarily close to zero.

Let us go back to the case when $\tau_1 = 0, \dot{q}_1 = 0, \ddot{q}_1 = 0$ and $\tilde{q}_1 = 0$. We will now present a detailed proof that q_2 should be identically zero.

With $\tau_1 = 0, \dot{q}_1 = 0, \ddot{q}_1 = 0$ and $\tilde{q}_1 = 0$, (2) becomes

$$(\theta_2 + \theta_3 \cos q_2)\ddot{q}_2 - \theta_3 \sin(q_2)\dot{q}_2^2 = \theta_5 g \sin(q_2) \tag{32}$$

$$\theta_2 \ddot{q}_2 = \theta_5 g \sin(q_2). \tag{33}$$

Introducing (33) into (32), we obtain

$$\frac{\theta_5 g}{\theta_2}\cos(q_2)\sin(q_2) - \sin(q_2)\dot{q}_2^2 = 0. \tag{34}$$

Thus we have either

$$\dot{q}_2^2 = \frac{\theta_5 g}{\theta_2}\cos q_2 \tag{35}$$

or

$$\sin q_2 = 0. \tag{36}$$

Let us study each case separately. If $\sin q_2 = 0$, then q_2 is either equal to 0 or π. $q_2 = \pi$ has been excluded by imposing conditions (26) and (27). Therefore (36) implies $q_2 = 0$.

Let us now study (35). Differentiating (35) we obtain

$$2\dot{q}_2\ddot{q}_2 = -\frac{\theta_5 g}{\theta_2}\dot{q}_2 \sin q_2. \tag{37}$$

We will next study (37) in two different cases.

- *Case 1:* If $\dot{q}_2 \neq 0$, (37) becomes $2\ddot{q}_2 = -(\theta_5 g/\theta_2)\sin q_2$. Combining this equation with (33) we conclude that $\sin q_2 = 0$ which implies that $q_2 = 0$ as proved above.
- *Case 2:* If $\dot{q}_2 = 0$ then $\ddot{q}_2 = 0$ which together with (33) implies that $\sin q_2 = 0$ which implies that $q_2 = 0$ as proved above.

Finally, the largest invariant set M is given by the homoclinic orbit (16) with $(q_1, \dot{q}_1) = ((\pi/2), 0)$ and the interval $(q_1, \dot{q}_1, q_2, \dot{q}_2) = ((\pi/2) - \varepsilon, 0, \varepsilon, 0)$, where $|\varepsilon| < \varepsilon^\star$ and $\varepsilon^\star$ is arbitrarily small. Provided that the state initial conditions satisfy (26) and (27), and $k_P > 0$ is sufficiently small then all the solutions converge to the invariant set M. This ends the proof of Theorem 1.

V. Simulation Results

In order to observe the performance of the proposed control law based on passivity we have performed simulations on MATLAB, using SIMULINK.

We have considered the system taking the parameters $\theta_{i,1\le i\le 5}$ as defined in [3]. We have some freedom in the choice of the coefficient k_D.

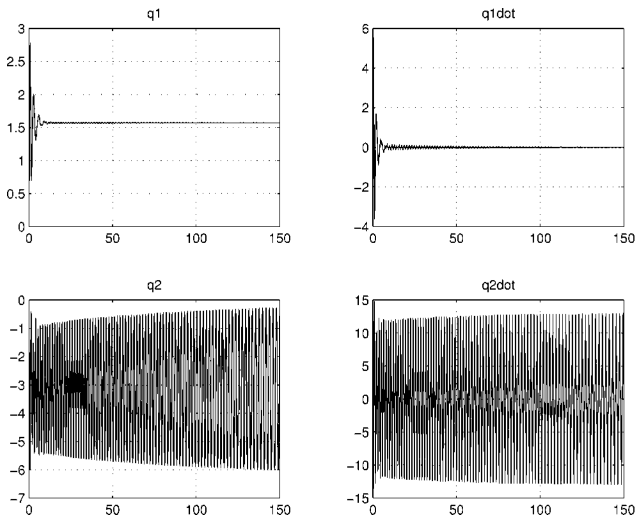

Fig. 2. States of the system.

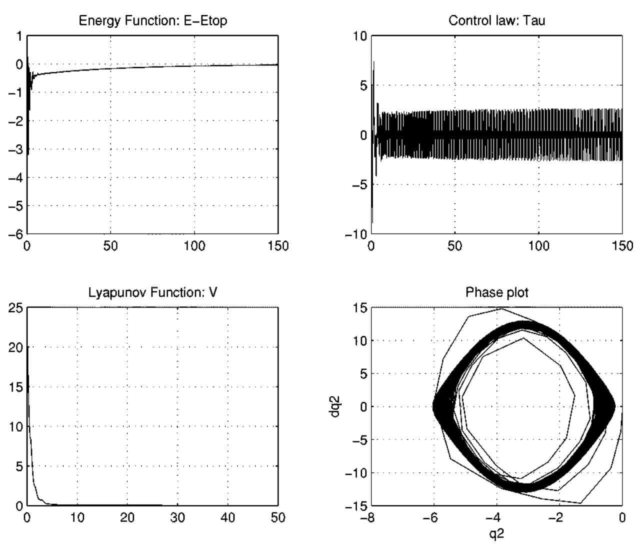

Fig. 3. Performance of the system and convergence to a homoclinic orbit.

On the other hand, we have to choose k_P sufficiently small, in order to reach the invariant set. Our algorithm allows us to bring the pendubot close to the top position, but the second link will remain swinging while getting closer and closer to the top position. Once the system is close enough to the top position, we could switch to a linear control law as has being proposed in previous papers on the subject (see [3]). Figs. 2 and 3 show the results for $k_E = 1.5, k_P = 1, k_D = 1$ and for an initial position

$$\begin{aligned} q_1 &= 0, \qquad q_2 = 0 \\ \dot{q}_1 &= 0, \qquad \dot{q}_2 = 0. \end{aligned}$$

Simulations showed that our control law brings the state of the system to the homoclinic orbit. Note that $\tilde{E}$ goes to zero, i.e. that the energy E goes to the energy at the top position: E_{top}. The Lyapunov function V is always decreasing and converges to zero.

VI. Conclusions

We have presented a control strategy for the pendubot that brings the state either arbitrarily close to the top position or to a homoclinic orbit that will eventually enter the basin of attraction of any locally convergent controller. The control strategy is based on an energy approach and the passivity properties of the pendubot. A Lyapunov function is obtained using the total energy of the system. The analysis is carried out using the LaSalle's theorem.

It has been proved that the first link converges to the upright position while the second oscillates and converges to the homoclinic orbit. This has also been observed in simulation.

In order to compare our controller with the one proposed by [14], we can remark that in our approach the control input amplitude does not need to be very large since at every cycle (of the second link) we only require to slightly increase the energy. In other words we do not need high gain controllers.

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