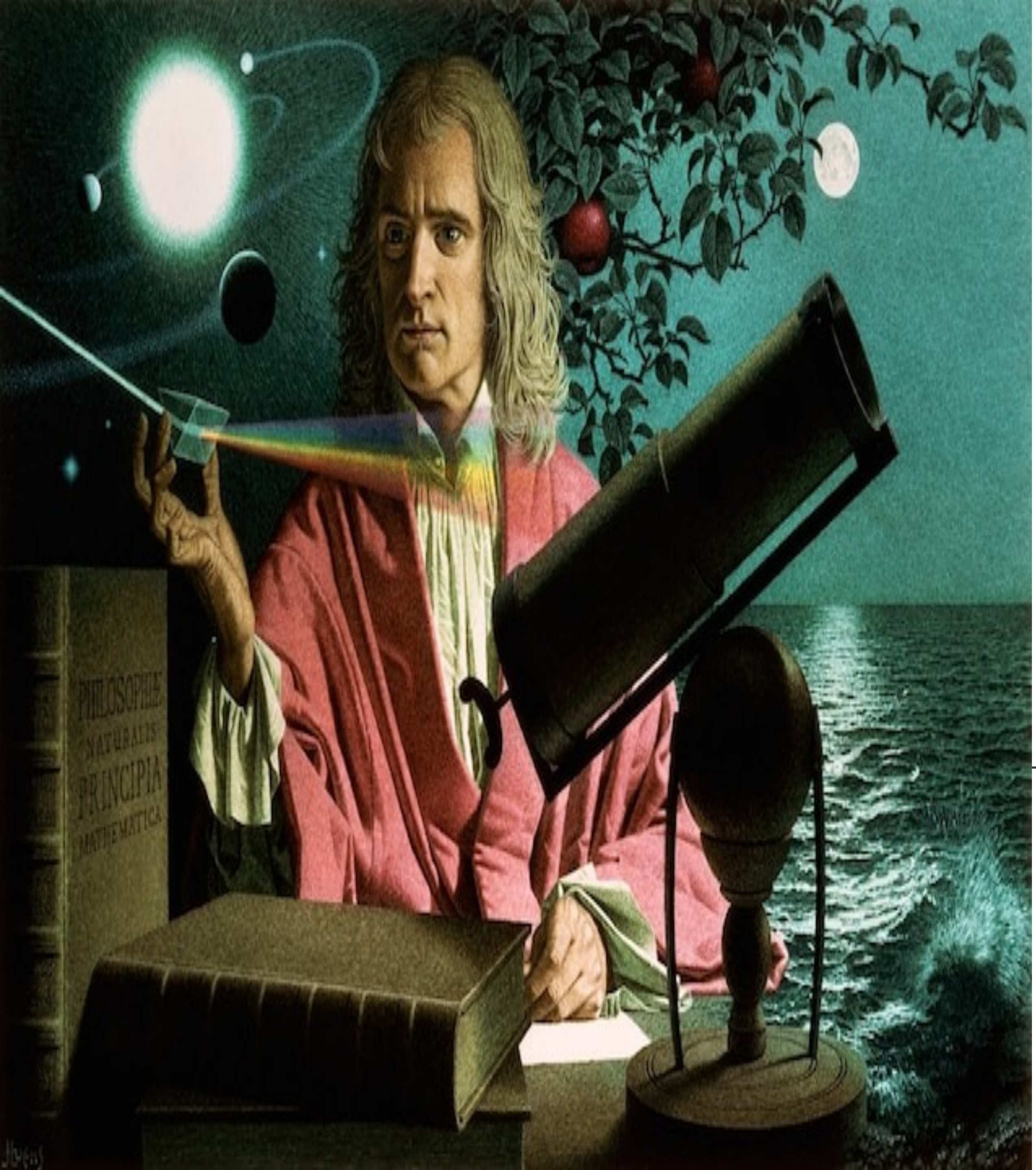


The Three Body Problem

PHYS4150

JORGE BERMEO

"I can calculate the motion of heavenly bodies, but not the madness of people."

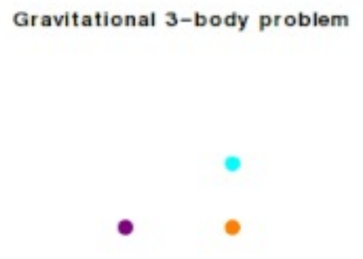
- ISAAC NEWTON

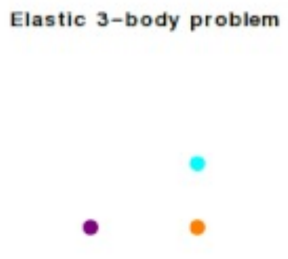
What is it?

In physics and classical mechanics, the three-body problem is the problem of taking the initial positions and velocities (or momenta) of three point masses and solving for their subsequent motion according to Newton's laws of motion and Newton's law of universal gravitation[1].

$$F = \frac{Gm_1m_2}{r^2}$$

Historically, the first specific three-body problem to receive extended study was the one involving the Moon, the Earth, and the Sun[2]. In an extended modern sense, a three-body problem is any problem in classical mechanics or quantum mechanics that models the motion of three particles.





Mathematical description

Using Newtonian equations of motion for vectors with positions $r_i=(x_i, y_i, z_i)$ of three gravitationally interacting bodies with masses m_i

$$\ddot{\mathbf{r}}_1 = -Gm_2\frac{\mathbf{r}_1-\mathbf{r}_2}{|\mathbf{r}_1-\mathbf{r}_2|^3} - Gm_3\frac{\mathbf{r}_1-\mathbf{r}_3}{|\mathbf{r}_1-\mathbf{r}_3|^3},$$
$$\ddot{\mathbf{r}}_2 = -Gm_3\frac{\mathbf{r}_2-\mathbf{r}_3}{|\mathbf{r}_2-\mathbf{r}_3|^3} - Gm_1\frac{\mathbf{r}_2-\mathbf{r}_1}{|\mathbf{r}_2-\mathbf{r}_1|^3},$$
$$\ddot{\mathbf{r}}_3 = -Gm_1\frac{\mathbf{r}_3-\mathbf{r}_1}{|\mathbf{r}_3-\mathbf{r}_1|^3} - Gm_2\frac{\mathbf{r}_3-\mathbf{r}_2}{|\mathbf{r}_3-\mathbf{r}_2|^3}.$$

Restricted three body problem

In the restricted three-body problem, a body of negligible mass (the "planetoid") moves under the influence of two massive bodies.
Having negligible mass, the force that the planetoid exerts on the two massive bodies may be neglected, and the system can be analyzed and can therefore be described in terms of a two-body motion.
This is of practical interest as well since it accurately describes many real-world problems, the most important example being the Earth–Moon–Sun system. For these reasons, it has occupied an important role in the historical development of the three-body problem[3].

Restricted three body problem

Combining the Gravitational Force equation with Newton's Second law, the three equations of motion can be derived

$$m_1\ddot{\vec{r}}_1 = \frac{Gm_1m_2}{r_{12}^3}\vec{r}_{12} + \frac{Gm_1m_3}{r_{13}^3}\vec{r}_{13}$$

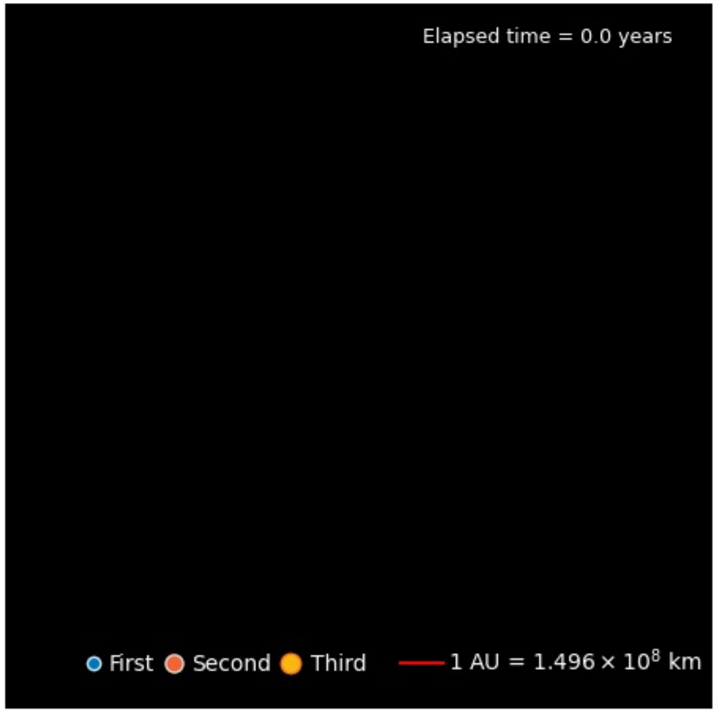
$$m_2\ddot{\vec{r}}_2 = -\frac{Gm_1m_2}{r_{12}^3}\vec{r}_{12} + \frac{Gm_2m_3}{r_{23}^3}\vec{r}_{23}$$

$$m_3\ddot{\vec{r}}_3 = -\frac{Gm_1m_3}{r_{13}^3}\vec{r}_{13} - \frac{Gm_2m_3}{r_{23}^3}\vec{r}_{23}$$

Where $r_{12} = \sqrt{(x_2 - x_1)^2 - (y_2 - y_1)^2}$, with x and y being the coordinates of A and B*

Kepler-16

This is a simulation of a binary star system that also has a planet with a mass of 1/3 of Jupiter's. Both binary stars are smaller than the Sun. The orbital plane of the planet is located edge-on to us. Consequently, when the planet moves in front of the stars, it blocks some of their light. This is precisely how this planet was discovered[4]. The simulation tells us that the system appears to be stable, at least in the short term.



Equations of motion

In order to use the equations in our program we need to write each equation in terms of x and y coordinates. This gives six equations:

$$\ddot{x}_1 = \frac{Gm_2}{r_{12}^3}(x_2 - x_1) + \frac{Gm_3}{r_{13}^3}(x_3 - x_1)$$

$$\ddot{y}_1 = \frac{Gm_2}{r_{12}^3}(y_2 - y_1) + \frac{Gm_3}{r_{13}^3}(y_3 - y_1)$$

$$\ddot{x}_2 = -\frac{Gm_1}{r_{12}^3}(x_2 - x_1) + \frac{Gm_3}{r_{23}^3}(x_3 - x_2)$$

$$\ddot{y}_2 = -\frac{Gm_1}{r_{12}^3}(y_2 - y_1) + \frac{Gm_3}{r_{23}^3}(y_3 - y_2)$$

$$\ddot{x}_3 = -\frac{Gm_1}{r_{13}^3}(x_3 - x_1) - \frac{Gm_2}{r_{23}^3}(x_3 - x_2)$$

$$\ddot{y}_3 = -\frac{Gm_1}{r_{13}^3}(y_3 - y_1) - \frac{Gm_2}{r_{23}^3}(y_3 - y_2)$$

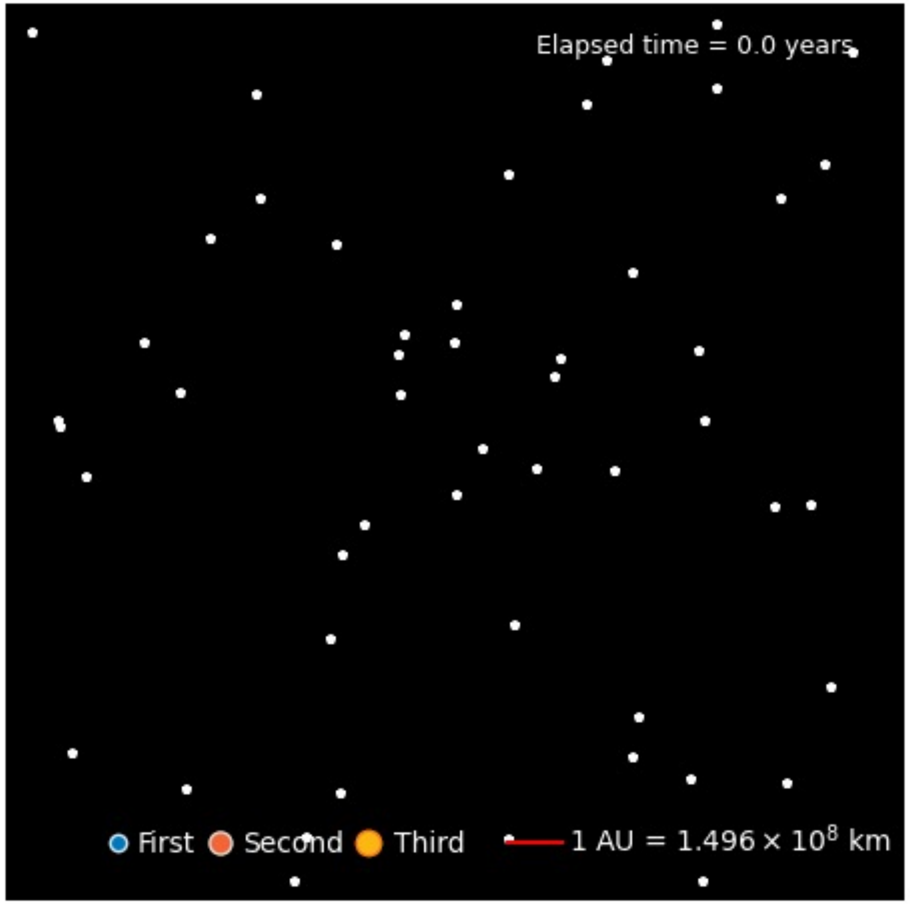
Programming

The equations of motion are solved via the Runge-Kutta method, and a loop that updates the locations and velocities of the respective bodies
Before the first run, however, we need to choose some initial positions and velocities of the three bodies and store them in an array, which are used for the animation later.

```
for i in range(0, N - 1):
    [earth_position[i + 1, :], earth_velocity[i + 1, :]] = Runge_Kutta(t[i], earth_position[i, :],
                                                                        earth_velocity[i, :], h, 'earth',
                                                                        planet_position[i, :], planet_velocity[i, :])
    [planet_position[i + 1, :], planet_velocity[i + 1, :]] = Runge_Kutta(t[i], planet_position[i, :],
                                                                          planet_velocity[i, :], h, 'Planet',
                                                                          earth_position[i, :], earth_velocity[i, :])
    [sun_position[i + 1, :], sun_velocity[i + 1, :]] = Runge_Kutta(t[i], sun_position[i, :],
                                                                    sun_velocity[i, :], h, 'Planet',
                                                                    earth_position[i, :], earth_velocity[i, :])
```

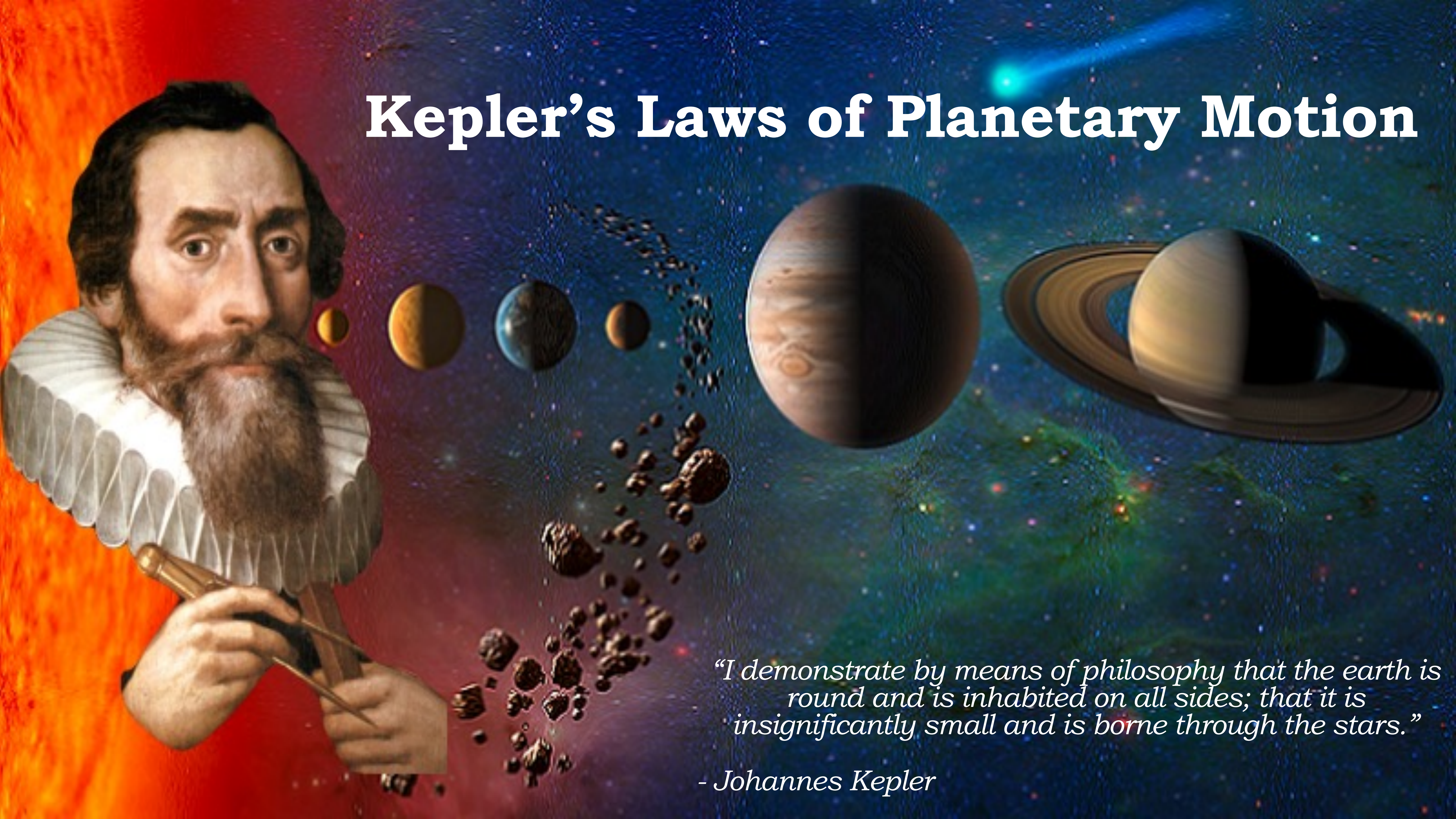
Results

Adding randomly placed stars make the animation look a little better:



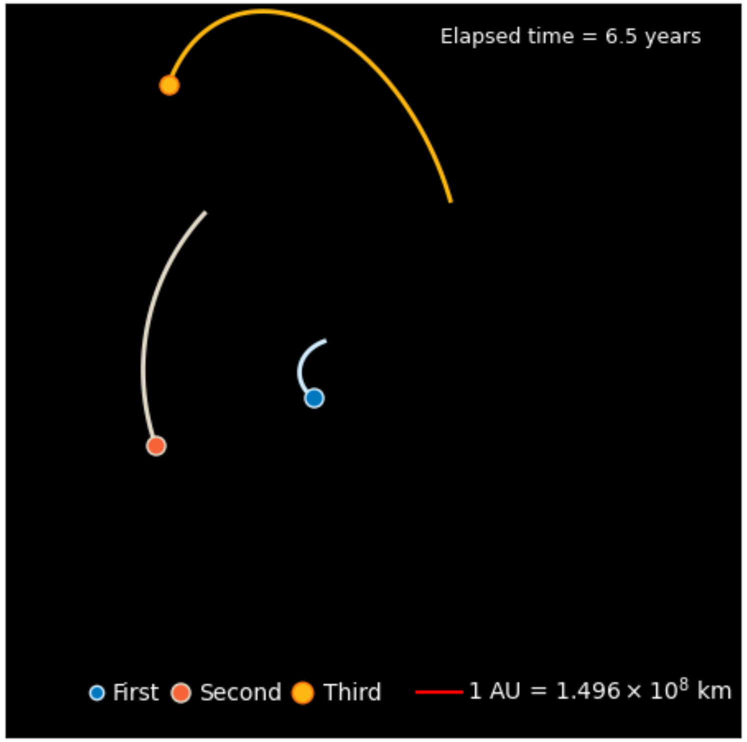
But Wait...

There's MORE!


Kepler's Laws of Planetary Motion
"I demonstrate by means of philosophy that the earth is round and is inhabited on all sides; that it is insignificantly small and is borne through the stars."
- Johannes Kepler

First Law

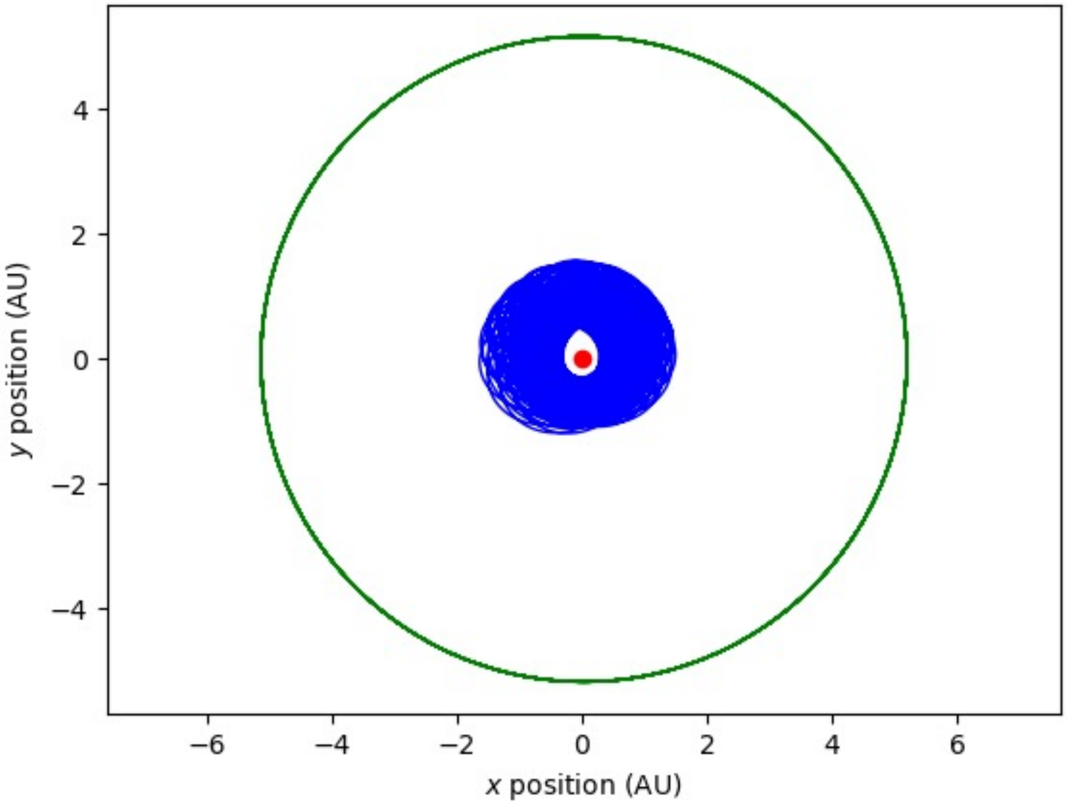
Each planet moves in an elliptical orbit with its star at one focus.

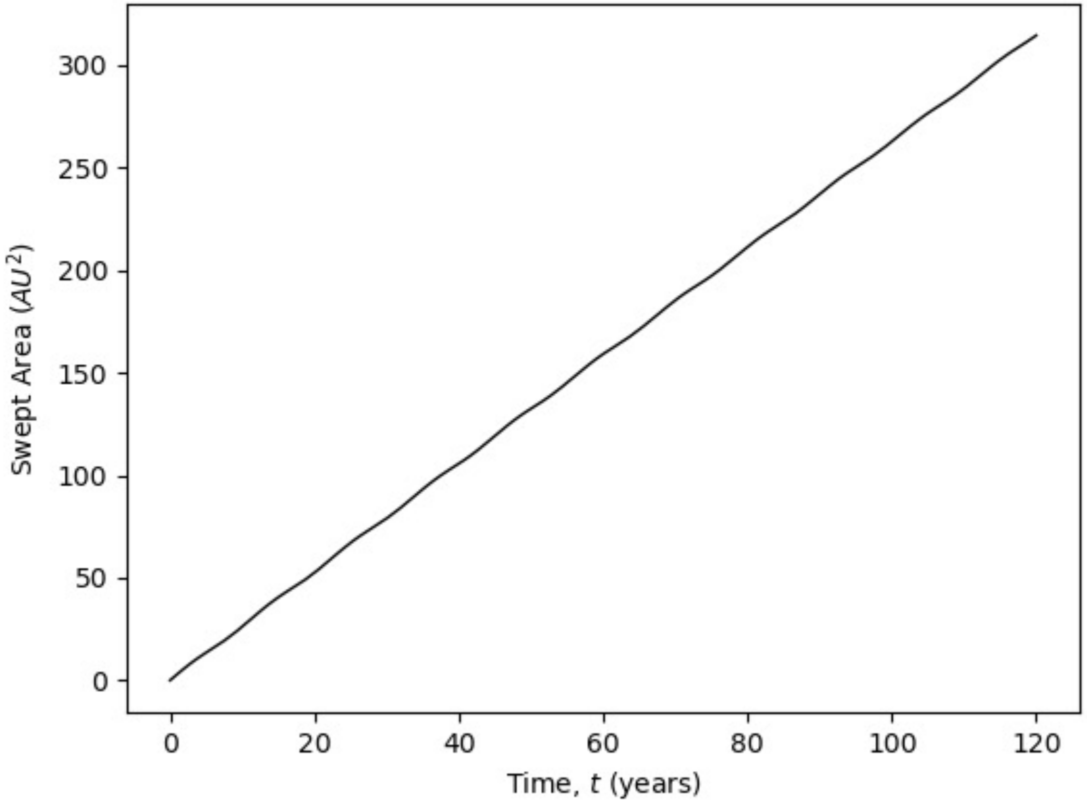


Second Law

An orbiting object will take the same amount of time to travel between points A and B as it takes to travel between points C and D.

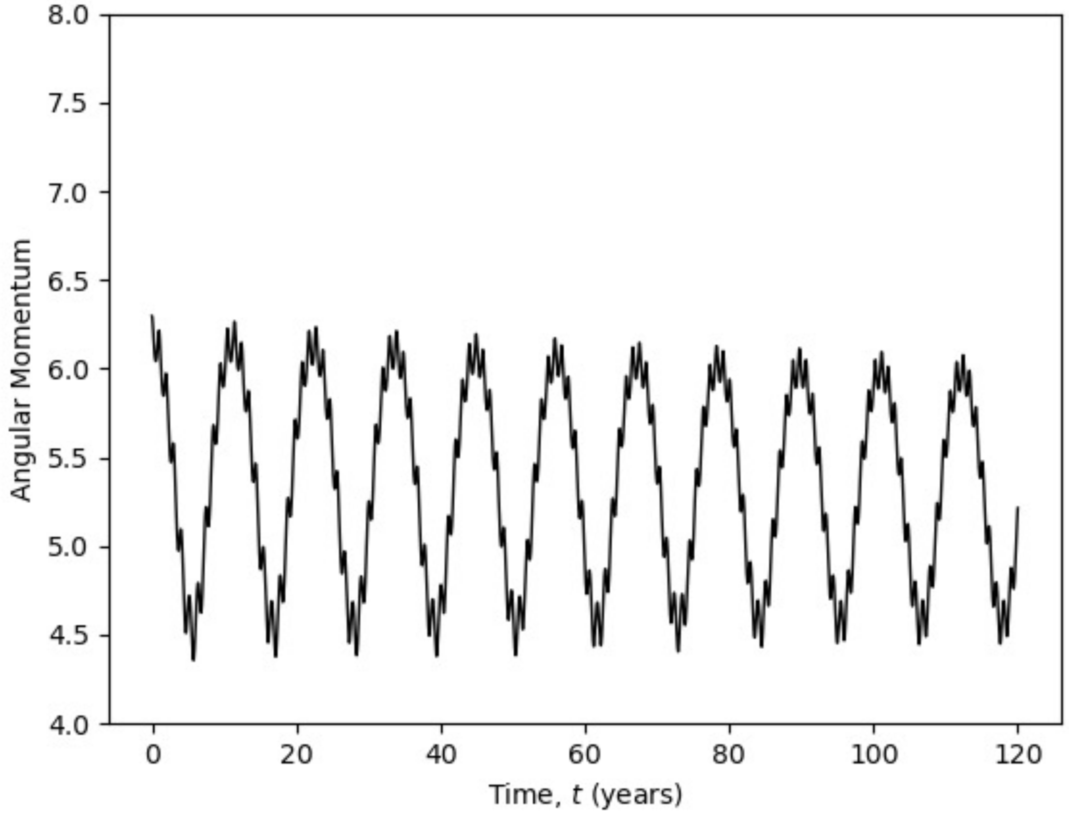
A planet travels faster when closer to the Sun, then slower when farther from the Sun. Kepler's second law states that the blue sector has constant area.





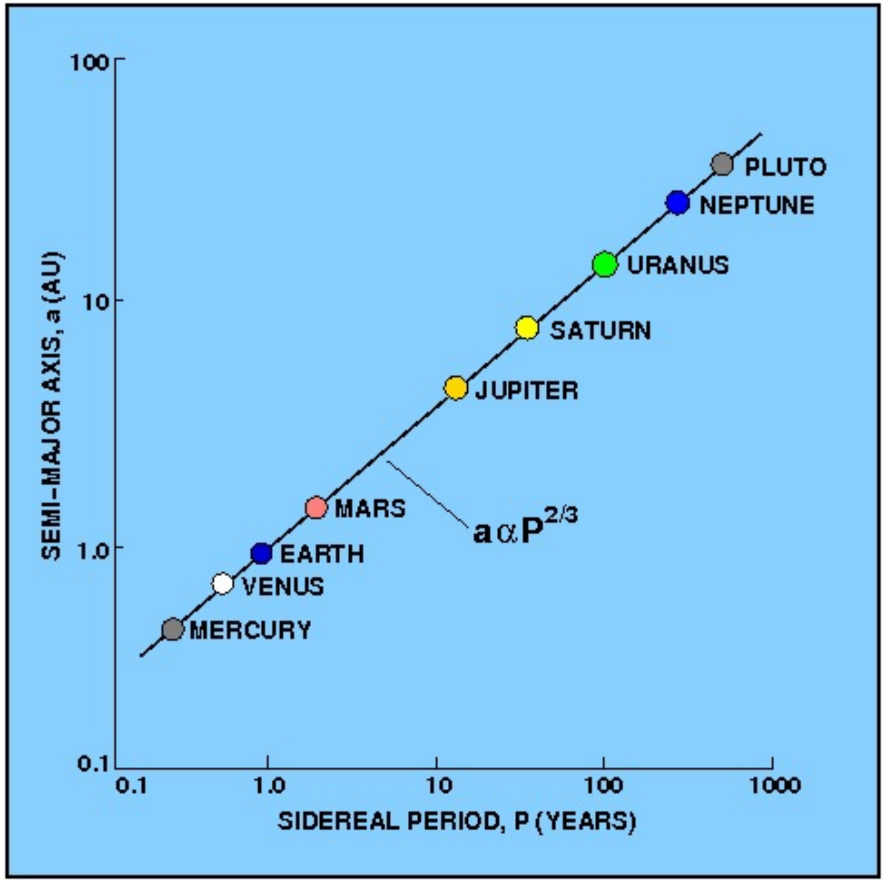
Conservation of Angular momentum

Second law continued:



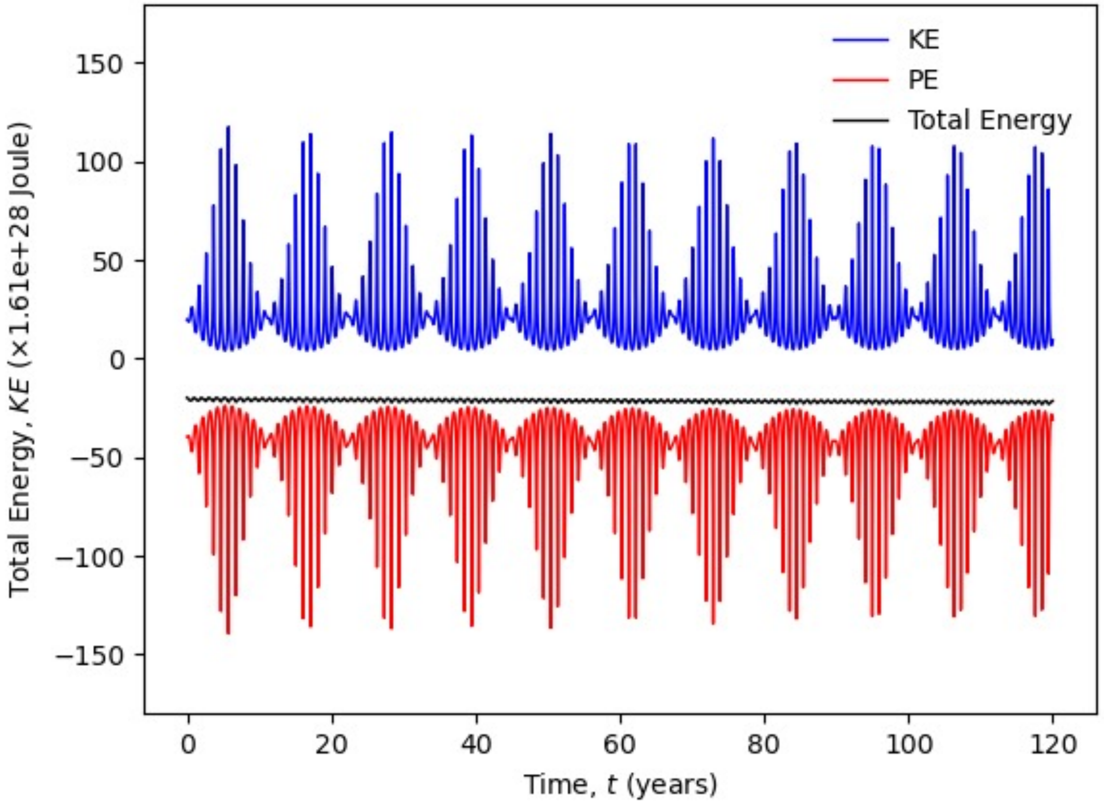
Third Law

The squares of the orbital periods of the planets are directly proportional to the cubes of the semi major axes of their orbits. Kepler's Third Law implies that the period for a planet to orbit the Sun increases rapidly with the radius of its orbit.



Bonus: Conservation of Energy

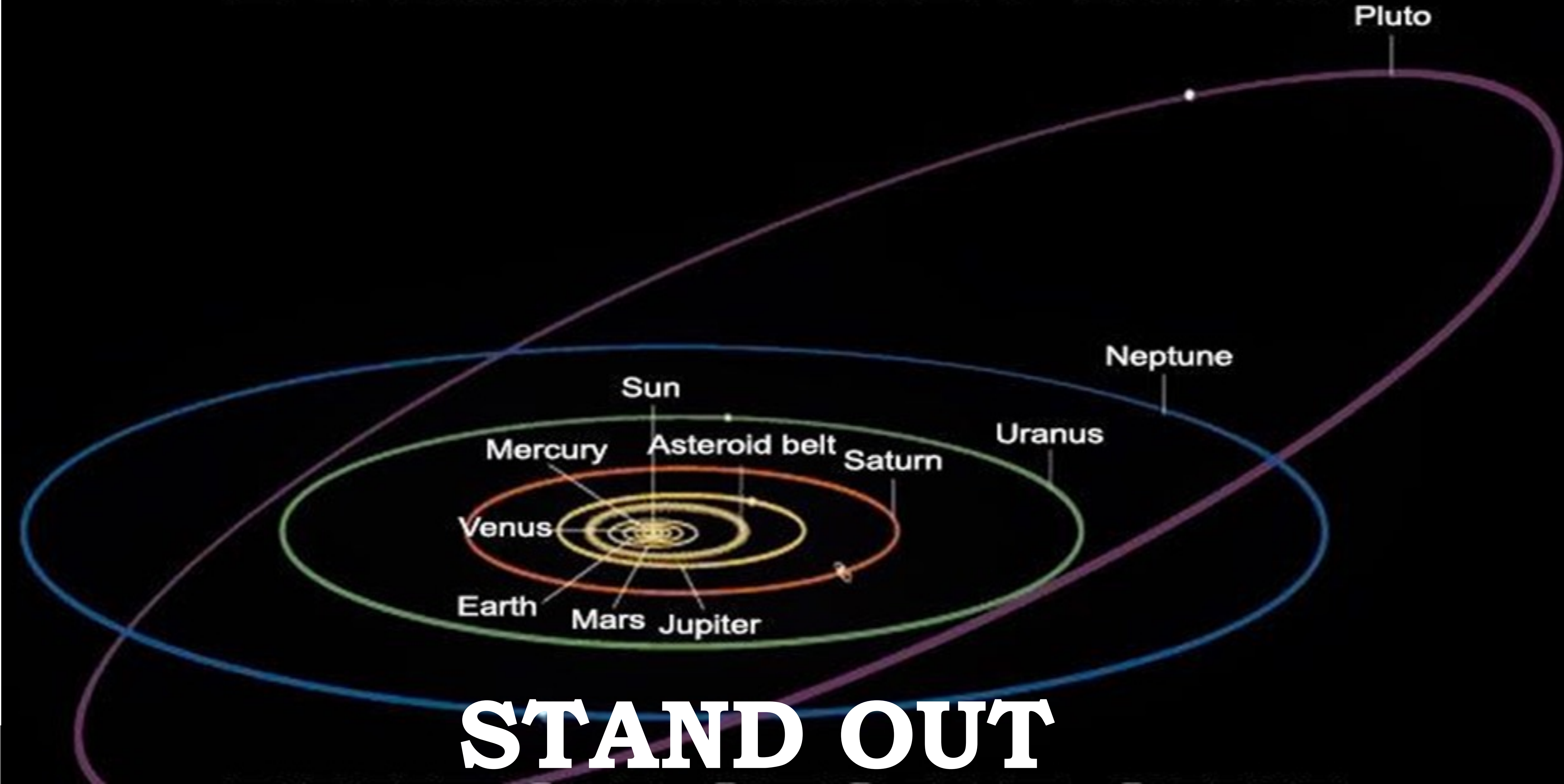
The Kinetic and Potential Energy are conserved at all time



Observations

- The initial approach was to build this problem basing if off the Earth-Moon-Sun orbits, but then it developed to a slightly more complex version of the problem
- There are many known solutions to the Three Body Problem such as: Figure Eight, Lagrange point L5, and a chaotic orbits. This version resembles that of Kepler-16 since is the most stable
- The laws of planetary motion for the earth and the sun are based on graphs that consider the third planet's mass neglectable, but further work into the code could help visualize how the laws hold up with more parameters into consideration
- While understanding the math portion of it could be complicated (and thus converting it into code), thus problem can be solved much faster when done by a computer.

BE MORE LIKE PLUTO



STAND OUT

Thank you

References

1) Newton, Principia, Corollary III to the laws of motion
2) "Historical Notes: Three-Body Problem". Retrieved 19 July 2017.
3) Barrow-Green, June (1997). Poincaré and the Three Body Problem. American Mathematical Soc. pp. 8–12. Bibcode:1997ptbp.book.....B. ISBN 978-0-8218-0367-7.
4) Kepler-16 system: Doyle, L. R., Carter, J. A., Fabrycky, D. C., et al. 2011, Science, 333, 1602.
5) Google Images
6) Wikipedia