An Intuitive Explanation of Bayes' Theorem

Bayes' Theorem
for the curious and bewildered;
an excruciatingly gentle introduction.

Your friends and colleagues are talking about something called "Bayes' Theorem" or "Bayes' Rule", or something called Bayesian reasoning. They sound really enthusiastic about it, too, so you google and find a webpage about Bayes' Theorem and...

It's this equation. That's all. Just one equation. The page you found gives a definition of it, but it doesn't say what it is, or why it's useful, or why your friends would be interested in it. It looks like this random statistics thing.

So you came here. Maybe you don't understand what the equation says. Maybe you understand it in theory, but every time you try to apply it in practice you get mixed up trying to remember the difference between p(a|x) and p(x|a), and whether p(a)*p(x|a) belongs in the numerator or the denominator. Maybe you see the theorem, and you understand the theorem, and you can use the theorem, but you can't understand why your friends and/or research colleagues seem to think it's the secret of the universe. Maybe your friends are all wearing Bayes' Theorem T-shirts, and you're feeling left out. Maybe you're a girl looking for a boyfriend, but the boy you're interested in refuses to date anyone who "isn't Bayesian". What matters is that Bayes is cool, and if you don't know Bayes, you aren't cool.

Why does a mathematical concept generate this strange enthusiasm in its students? What is the so-called Bayesian Revolution now sweeping through the sciences, which claims to subsume even the experimental method itself as a special case? What is the secret that the adherents of Bayes know? What is the light that they have seen?

Soon you will know. Soon you will be one of us.

While there are a few existing online explanations of Bayes' Theorem, my experience with trying to introduce people to Bayesian reasoning is that the existing online explanations are too abstract. Bayesian reasoning is very counterintuitive. People do not employ Bayesian reasoning intuitively, find it very difficult to learn Bayesian reasoning when tutored, and rapidly forget Bayesian methods once the tutoring is over. This holds equally true for novice students and highly trained professionals in a field. Bayesian reasoning is apparently one of those things which, like quantum mechanics or the Wason Selection Test, is inherently difficult for humans to grasp with our built-in mental faculties.

Or so they claim. Here you will find an attempt to offer an intuitive explanation of Bayesian reasoning - an excruciatingly gentle introduction that invokes all the human ways of grasping numbers, from natural frequencies to spatial visualization. The intent is to convey, not abstract rules for manipulating numbers, but what the numbers mean, and why the rules are what they are (and cannot possibly be anything else). When you are finished reading this page, you will see Bayesian problems in your dreams.

And let's begin.

Here's a story problem about a situation that doctors often encounter:
1% of women at age forty who participate in routine screening have breast cancer. 80% of women with breast cancer will get positive mammographies. 9.6% of women without breast cancer will also get positive mammographies. A woman in this age group had a positive mammography in a routine screening. What is the probability that she actually has breast cancer?

What do you think the answer is? If you haven't encountered this kind of problem before, please take a moment to come up with your own answer before continuing.

Next, suppose I told you that most doctors get the same wrong answer on this problem - usually, only around 15% of doctors get it right. ("Really? 15%? Is that a real number, or an urban legend based on an Internet poll?" It's a real number. See Casscells, Schoenberger, and Grayboys 1978; Eddy 1982; Gigerenzer and Hoffrage 1995; and many other studies. It's a surprising result which is easy to replicate, so it's been extensively replicated.)

Do you want to think about your answer again? Here's a Javascript calculator if you need one. This calculator has the usual precedence rules; multiplication before addition and so on. If you're not sure, I suggest using parentheses.
Calculator: Result:

On the story problem above, most doctors estimate the probability to be between 70% and 80%, which is wildly incorrect.

Here's an alternate version of the problem on which doctors fare somewhat better:
10 out of 1000 women at age forty who participate in routine screening have breast cancer. 800 out of 1000 women with breast cancer will get positive mammographies. 96 out of 1000 women without breast cancer will also get positive mammographies. If 1000 women in this age group undergo a routine screening, about what fraction of women with positive mammographies will actually have breast cancer? Calculator: Result:

And finally, here's the problem on which doctors fare best of all, with 46% - nearly half - arriving at the correct answer:

100 out of 10,000 women at age forty who participate in routine screening have breast cancer. 80 of every 100 women with breast cancer will get a positive mammography. 950 out of 9,900 women without breast cancer will also get a positive mammography. If 10,000 women in this age group undergo a routine screening, about what fraction of women with positive mammographies will actually have breast cancer? Calculator: Result:

The correct answer is 7.8%, obtained as follows: Out of 10,000 women, 100 have breast cancer; 80 of those 100 have positive mammographies. From the same 10,000 women, 9,900 will not have breast cancer and of those 9,900 women, 950 will also get positive mammographies. This makes the total number of women with positive mammographies 950+80 or 1,030. Of those 1,030 women with positive mammographies, 80 will have cancer. Expressed as a proportion, this is 80/1,030 or 0.07767 or 7.8%.

To put it another way, before the mammography screening, the 10,000 women can be divided into two groups: - Group 1: 100 women with breast cancer. - Group 2: 9,900 women without breast cancer.

Summing these two groups gives a total of 10,000 patients, confirming that none have been lost in the math. After the mammography, the women can be divided into four groups: - Group A: 80 women with breast cancer, and a positive mammography. - Group B: 20 women with breast cancer, and a negative mammography. - Group C: 950 women without breast cancer, and a positive mammography. - Group D: 8,950 women without breast cancer, and a negative mammography.

Calculator: Result: As you can check, the sum of all four groups is still 10,000. The sum of groups A and B, the groups with breast cancer, corresponds to group 1; and the sum of groups C and D, the groups without breast cancer, corresponds to group 2; so administering a mammography does not actually change the number of women with breast cancer. The proportion of the cancer patients (A + B) within the complete set of patients (A + B + C + D) is the same as the 1% prior chance that a woman has cancer: (80 + 20) / (80 + 20 + 950 + 8950) = 100 / 10000 = 1%.

The proportion of cancer patients with positive results, within the group of all patients with positive results, is the proportion of (A) within (A + C): 80 / (80 + 950) = 80 / 1030 = 7.8%. If you administer a mammography to 10,000 patients, then out of the 1030 with positive mammographies, 80 of those positive-mammography patients will have cancer. This is the correct answer, the answer a doctor should give a positive-mammography patient if she asks about the chance she has breast cancer; if thirteen patients ask this question, roughly 1 out of those 13 will have cancer.

The most common mistake is to ignore the original fraction of women with breast cancer, and the fraction of women without breast cancer who receive false positives, and focus only on the fraction of women with breast cancer who get positive results. For example, the vast majority of doctors in these studies seem to have thought that if around 80% of women with breast cancer have positive mammographies, then the probability of a women with a positive mammography having breast cancer must be around 80%.

Figuring out the final answer always requires all three pieces of information - the percentage of women with breast cancer, the percentage of women without breast cancer who receive false positives, and the percentage of women with breast cancer who receive (correct) positives.

To see that the final answer always depends on the original fraction of women with breast cancer, consider an alternate universe in which only one woman out of a million has breast cancer. Even if mammography in this world detects breast cancer in 8 out of 10 cases, while returning a false positive on a woman without breast cancer in only 1 out of 10 cases, there will still be a hundred thousand false positives for every real case of cancer detected. The original probability that a woman has cancer is so extremely low that, although a positive result on the mammography does increase the estimated probability, the probability isn't increased to certainty or even "a noticeable chance"; the probability goes from 1:1,000,000 to 1:100,000.

Similarly, in an alternate universe where only one out of a million women does not have breast cancer, a positive result on the patient's mammography obviously doesn't mean that she has an 80% chance of having breast cancer! If this were the case her estimated probability of having cancer would have been revised drastically downward after she got a positive result on her mammography - an 80% chance of having cancer is a lot less than 99.9999%! If you administer mammographies to ten million women in this world, around eight million women with breast cancer will get correct positive results, while one woman without breast cancer will get false positive results. Thus, if you got a positive mammography in this alternate universe, your chance of having cancer would go from 99.9999% up to 99.999987%. That is, your chance of being healthy would go from 1:1,000,000 down to 1:8,000,000.

These two extreme examples help demonstrate that the mammography result doesn't replace your old information about the patient's chance of having cancer; the mammography slides the estimated probability in the direction of the result. A positive result slides the original probability upward; a negative result slides the probability downward. For example, in the original problem where 1% of the women have cancer, 80% of women with cancer get positive mammographies, and 9.6% of women without cancer get positive mammographies, a positive result on the mammography slides the 1% chance upward to 7.8%.

Most people encountering problems of this type for the first time carry out the mental operation of replacing the original 1% probability with the 80% probability that a woman with cancer gets a positive mammography. It may seem like a good idea, but it just doesn't work. "The probability that a woman with a positive mammography has breast cancer" is not at all the same thing as "the probability that a woman with breast cancer has a positive mammography"; they are as unlike as apples and cheese. Finding the final answer, "the probability that a woman with a positive mammography has breast cancer", uses all three pieces of problem information - "the prior probability that a woman has breast cancer", "the probability that a woman with breast cancer gets a positive mammography", and "the probability that a woman without breast cancer gets a positive mammography".

Fun
Fact! Q. What is the Bayesian Conspiracy?
A. The Bayesian Conspiracy is a multinational, interdisciplinary, and shadowy group of scientists that controls publication, grants, tenure, and the illicit traffic in grad students. The best way to be accepted into the Bayesian Conspiracy is to join the Campus Crusade for Bayes in high school or college, and gradually work your way up to the inner circles. It is rumored that at the upper levels of the Bayesian Conspiracy exist nine silent figures known only as the Bayes Council.

To see that the final answer always depends on the chance that a woman without breast cancer gets a positive mammography, consider an alternate test, mammography+. Like the original test, mammography+ returns positive for 80% of women with breast cancer. However, mammography+ returns a positive result for only one out of a million women without breast cancer - mammography+ has the same rate of false negatives, but a vastly lower rate of false positives. Suppose a patient receives a positive mammography+. What is the chance that this patient has breast cancer? Under the new test, it is a virtual certainty - 99.988%, i.e., a 1 in 8082 chance of being healthy.
Calculator: Result: Remember, at this point, that neither mammography nor mammography+ actually change the number of women who have breast cancer. It may seem like "There is a virtual certainty you have breast cancer" is a

terrible thing to say, causing much distress and despair; that the more hopeful verdict of the previous mammography test - a 7.8% chance of having breast cancer - was much to be preferred. This comes under the heading of “Don’t shoot the messenger”. The number of women who really do have cancer stays exactly the same between the two cases. Only the accuracy with which we detect cancer changes. Under the previous mammography test, 80 women with cancer (who already had cancer, before the mammography) are first told that they have a 7.8% chance of having cancer, creating X amount of uncertainty and fear, after which more detailed tests will inform them that they definitely do have breast cancer. The old mammography test also involves informing 950 women without breast cancer that they have a 7.8% chance of having cancer, thus creating twelve times as much additional fear and uncertainty. The new test, mammography+, does not give 950 women false positives, and the 80 women with cancer are told the same facts they would have learned eventually, only earlier and without an intervening period of uncertainty. Mammography+ is thus a better test in terms of its total emotional impact on patients, as well as being more accurate. Regardless of its emotional impact, it remains a fact that a patient with positive mammography+ has a 99.988% chance of having breast cancer.

Of course, that mammography+ does not give 950 healthy women false positives means that all 80 of the patients with positive mammography+ will be patients with breast cancer. Thus, if you have a positive mammography+, your chance of having cancer is a virtual certainty. It is because mammography+ does not generate as many false positives (and needless emotional stress), that the (much smaller) group of patients who do get positive results will be composed almost entirely of genuine cancer patients (who have bad news coming to them regardless of when it arrives).

Similarly, let’s suppose that we have a less discriminating test, mammography*, that still has a 20% rate of false negatives, as in the original case. However, mammography* has an 80% rate of false positives. In other words, a patient without breast cancer has an 80% chance of getting a false positive result on her mammography* test. If we suppose the same 1% prior probability that a patient presenting herself for screening has breast cancer, what is the chance that a patient with positive mammography* has cancer? - Group 1: 100 patients with breast cancer. - Group 2: 9,900 patients without breast cancer.

After mammography* screening: - Group A: 80 patients with breast cancer and a “positive” mammography*. - Group B: 20 patients with breast cancer and a “negative” mammography*. - Group C: 7920 patients without breast cancer and a “positive” mammography*. - Group D: 1980 patients without breast cancer and a “negative” mammography*.

Calculator: Result: The result works out to 80 / 8,000, or 0.01. This is exactly the same as the 1% prior probability that a patient has breast cancer! A "positive" result on mammography* doesn't change the probability that a woman has breast cancer at all. You can similarly verify that a "negative" mammography* also counts for nothing. And in fact it must be this way, because if mammography* has an 80% hit rate for patients with breast cancer, and also an 80% rate of false positives for patients without breast cancer, then mammography* is completely uncorrelated with breast cancer. There's no reason to call one result "positive" and one result "negative"; in fact, there's no reason to call the test a "mammography". You can throw away your expensive mammography* equipment and replace it with a random number generator that outputs a red light 80% of the time and a green light 20% of the time; the results will be the same. Furthermore, there's no reason to call the red light a "positive" result or the green light a "negative" result. You could have a green light 80% of the time and a red light 20% of the time, or a blue light 80% of the time and a purple light 20% of the time, and it would all have the same bearing on whether the patient has breast cancer: i.e., no bearing whatsoever.

We can show algebraically that this must hold for any case where the chance of a true positive and the chance of a false positive are the same, i.e: - Group 1: 100 patients with breast cancer. - Group 2: 9,900 patients without breast cancer.

Now consider a test where the probability of a true positive and the probability of a false positive are the same number M (in the example above, M=80% or M = 0.8): - Group A: 100*M patients with breast cancer and a "positive" result. - Group B: 100*(1 - M) patients with breast cancer and a "negative" result. - Group C: 9,900*M patients without breast cancer and a "positive" result. - Group D: 9,900*(1 - M) patients without breast cancer and a "negative" result.

The proportion of patients with breast cancer, within the group of patients with a "positive" result, then equals 100*M / (100*M + 9900*M) = 100 / (100 + 9900) = 1%. This holds true regardless of whether M is 80%, 30%, 50%, or 100%. If we have a mammography* test that returns "positive" results for 90% of patients with breast cancer and returns "positive" results for 90% of patients without breast cancer, the proportion of "positive"-testing patients who have breast cancer will still equal the original proportion of patients with breast cancer, i.e., 1%.

You can run through the same algebra, replacing the prior proportion of patients with breast cancer with an arbitrary percentage P: - Group 1: Within some number of patients, a fraction P have breast cancer. - Group 2: Within some number of patients, a fraction (1 - P) do not have breast cancer.

After a "cancer test" that returns "positive" for a fraction M of patients with breast cancer, and also returns "positive" for the same fraction M of patients without cancer: - Group A: P*M patients have breast cancer and a "positive" result. - Group B: P*(1 - M) patients have breast cancer and a "negative" result. - Group C: (1 - P)*M patients have no breast cancer and a "positive" result. - Group D: (1 - P)*(1 - M) patients have no breast cancer and a "negative" result.

The chance that a patient with a "positive" result has breast cancer is then the proportion of group A within the combined group A + C, or P*M / [P*M + (1 - P)*M], which, cancelling the common factor M from the numerator and denominator, is P / [P + (1 - P)] or P / 1 or just P. If the rate of false positives is the same as the rate of true positives, you always have the same probability after the test as when you started.

Which is common sense. Take, for example, the "test" of flipping a coin; if the coin comes up heads, does it tell you anything about whether a patient has breast cancer? No; the coin has a 50% chance of coming up heads if the patient has breast cancer, and also a 50% chance of coming up heads if the patient does not have breast cancer. Therefore there is no reason to call either heads or tails a "positive" result. It's not the probability being "50/50" that makes the coin a bad test; it's that the two probabilities, for "cancer patient turns up heads" and "healthy patient turns up heads", are the same. If the coin was slightly biased, so that it had a 60% chance of coming up heads, it still wouldn't be a cancer test - what makes a coin a poor test is not that it has a 50/50 chance of coming up heads if the patient has cancer, but that it also has a 50/50 chance of coming up heads if the patient does not have cancer. You can even use a test that comes up "positive" for cancer patients 100% of the time, and still not learn anything. An example of such a test is "Add 2 + 2 and see if the answer is 4." This test returns positive 100% of the time for patients with breast cancer. It also returns positive 100% of the time for patients without breast cancer. So you learn nothing.

The original proportion of patients with breast cancer is known as the prior probability. The chance that a patient with breast cancer gets a positive mammography, and the chance that a patient without breast cancer gets a positive mammography, are known as the two conditional probabilities. Collectively, this initial information is known as the priors. The final answer - the estimated probability that a patient has breast cancer, given that we know she has a positive result on her mammography - is known as the revised probability or the posterior probability. What we've just shown is that if the two conditional probabilities are equal, the posterior probability equals the prior probability.

Fun
Fact! Q. How can I find the priors for a problem?
A. Many commonly used priors are listed in the Handbook of Chemistry and Physics.

Q. Where do priors originally come from?
A. Never ask that question.

Q. Uh huh. Then where do scientists get their priors?
A. Priors for scientific problems are established by annual vote of the AAAS. In recent years the vote has become fractious and controversial, with widespread acrimony, factional polarization, and several outright assassinations. This may be a front for infighting within the Bayes Council, or it may be that the disputants have too much spare time. No one is really sure.

Q. I see. And where does everyone else get their priors?
A. They download their priors from Kazaa.

Q. What if the priors I want aren't available on Kazaa?
A. There's a small, cluttered antique shop in a back alley of San Francisco's Chinatown. Don't ask about the bronze rat.

Actually, priors are true or false just like the final answer - they reflect reality and can be judged by comparing them against reality. For example, if you think that 920 out of 10,000 women in a sample have breast cancer, and the actual number is 100 out of 10,000, then your priors are wrong. For our particular problem, the priors might have been established by three studies - a study on the case histories of women with breast cancer to see how many of them tested positive on a mammography, a study on women without breast cancer to see how many of them test positive on a mammography, and an epidemiological study on the prevalence of breast cancer in some specific demographic.

Suppose that a barrel contains many small plastic eggs. Some eggs are painted red and some are painted blue. 40% of the eggs in the bin contain pearls, and 60% contain nothing. 30% of eggs containing pearls are painted blue, and 10% of eggs containing nothing are painted blue. What is the probability that a blue egg contains a pearl? For this example the arithmetic is simple enough that you may be able to do it in your head, and I would suggest trying to do so. But just in case... Result: A more compact way of specifying the problem: - p(pearl) = 40% - p(blue|pearl) = 30% - p(blue|$\sim$pearl) = 10% - p(pearl|blue) = ?

"$\sim$" is shorthand for "not", so $\sim$pearl reads "not pearl".

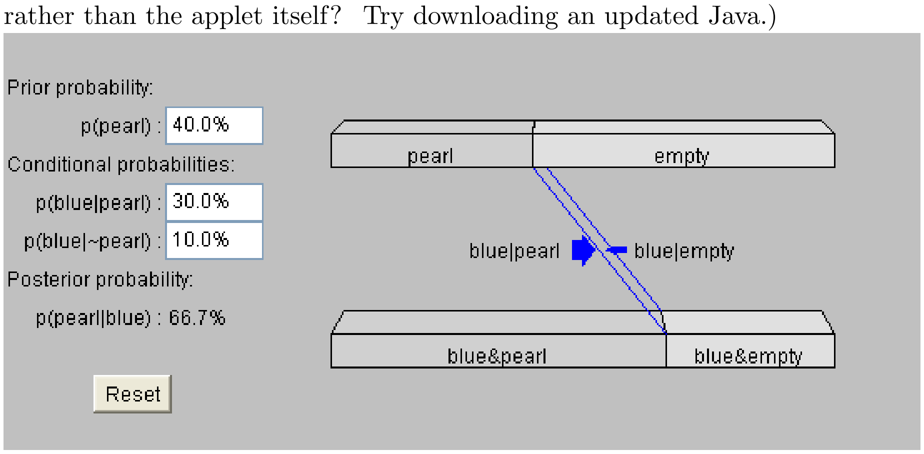
blue|pearl is shorthand for "blue given pearl" or "the probability that an egg is painted blue, given that the egg contains a pearl". One thing that's confusing about this notation is that the order of implication is read right-to-left, as in Hebrew or Arabic. blue|pearl means "blue<-pearl", the degree to which pearl-ness implies blue-ness, not the degree to which blue-ness implies pearl-ness. This is confusing, but it's unfortunately the standard notation in probability theory.

Readers familiar with quantum mechanics will have already encountered this peculiarity; in quantum mechanics, for example, <d|c><c|b><b|a> reads as "the probability that a particle at A goes to B, then to C, ending up at D". To follow the particle, you move your eyes from right to left. Reading from left to right, "|" means "given"; reading from right to left, "|" means "implies" or "leads to". Thus, moving your eyes from left to right, blue|pearl reads "blue given pearl" or "the probability that an egg is painted blue, given that the egg contains a pearl". Moving your eyes from right to left, blue|pearl reads "pearl implies blue" or "the probability that an egg containing a pearl is painted blue".

The item on the right side is what you already know or the premise, and the item on the left side is the implication or conclusion. If we have p(blue|pearl) = 30%, and we already know that some egg contains a pearl, then we can conclude there is a 30% chance that the egg is painted blue. Thus, the final fact we're looking for - "the chance that a blue egg contains a pearl" or "the probability that an egg contains a pearl, if we know the egg is painted blue" - reads p(pearl|blue).

Let's return to the problem. We have that 40% of the eggs contain pearls, and 60% of the eggs contain nothing. 30% of the eggs containing pearls are painted blue, so 12% of the eggs altogether contain pearls and are painted blue. 10% of the eggs containing nothing are painted blue, so altogether 6% of the eggs contain nothing and are painted blue. A total of 18% of the eggs are painted blue, and a total of 12% of the eggs are painted blue and contain pearls, so the chance a blue egg contains a pearl is 12/18 or 2/3 or around 67%.

The applet below, courtesy of Christian Rovner, shows a graphic representation of this problem:
(Are you having trouble seeing this applet? Do you see an image of the applet rather than the applet itself? Try downloading an updated Java.)



Looking at this applet, it's easier to see why the final answer depends on all three probabilities; it's the differential pressure between the two conditional probabilities, p(blue|pearl) and p(blue|∼pearl), that slides the prior probability p(pearl) to the posterior probability p(pearl|blue).

As before, we can see the necessity of all three pieces of information by considering extreme cases (feel free to type them into the applet). In a (large) barrel in which only one egg out of a thousand contains a pearl, knowing that an egg is painted blue slides the probability from 0.1% to 0.3% (instead of sliding the probability from 40% to 67%). Similarly, if 999 out of 1000 eggs contain pearls, knowing that an egg is blue slides the probability from 99.9% to 99.966%; the probability that the egg does not contain a pearl goes from 1/1000 to around 1/3000. Even when the prior probability changes, the differential pressure of the two conditional probabilities always slides the probability in the same direction. If you learn the egg is painted blue, the probability the egg contains a pearl always goes up - but it goes up from the prior probability, so you need to know the prior probability in order to calculate the final answer. 0.1% goes up to 0.3%, 10% goes up to 25%, 40% goes up to 67%, 80% goes up to 92%, and 99.9% goes up to 99.966%. If you're interested in knowing how any other probabilities slide, you can type your own prior probability into the Java applet. You can also click and drag the dividing line between pearl and ~pearl in the upper bar, and watch the posterior probability change in the bottom bar.

Studies of clinical reasoning show that most doctors carry out the mental operation of replacing the original 1% probability with the 80% probability that a woman with cancer would get a positive mammography. Similarly, on the pearl-egg problem, most respondents unfamiliar with Bayesian reasoning would probably respond that the probability a blue egg contains a pearl is 30%, or perhaps 20% (the 30% chance of a true positive minus the 10% chance of a false positive). Even if this mental operation seems like a good idea at the time, it makes no sense in terms of the question asked. It's like the experiment in which you ask a second-grader: "If eighteen people get on a bus, and then seven more people get on the bus, how old is the bus driver?" Many second-graders will respond: "Twenty-five." They understand when they're being prompted to carry out a particular mental procedure, but they haven't quite connected the procedure to reality. Similarly, to find the probability that a woman with a positive mammography has breast cancer, it makes no sense whatsoever to replace the original probability that the woman has cancer with the probability that a woman with breast cancer gets a positive mammography. Neither can you subtract the probability of a false positive from the probability of the true positive. These operations are as wildly irrelevant as adding the number of people on the bus to find the age of the bus driver.

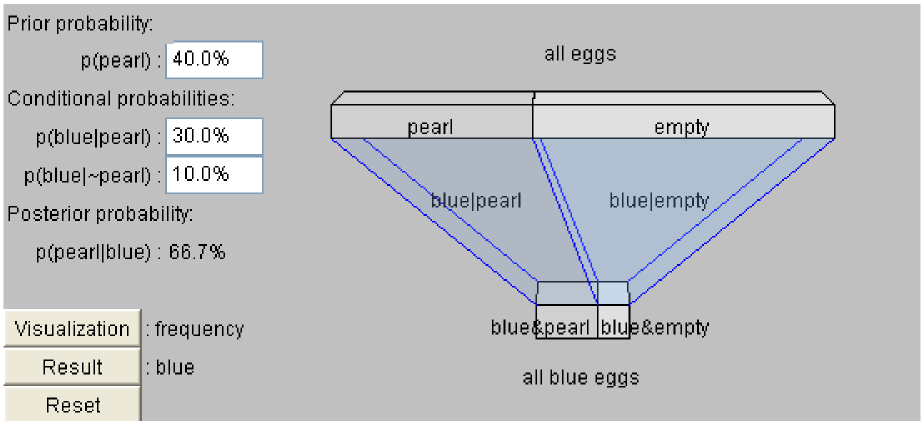
I keep emphasizing the idea that evidence slides probability because of research that shows people tend to use spatial intutions to grasp numbers. In particular, there's interesting evidence that we have an innate sense of quantity that's localized to left inferior parietal cortex - patients with damage to this area can selectively lose their sense of whether 5 is less than 8, while retaining their ability to read, write, and so on. (Yes, really!) The parietal cortex processes our sense of where things are in space (roughly speaking), so an innate "number line", or rather "quantity line", may be responsible for the human sense of numbers. This is why I suggest visualizing Bayesian evidence as sliding the probability along the number line; my hope is that this will translate Bayesian reasoning into something that makes sense to innate human brainware. (That, really, is what an "intuitive explanation" is.) For more information, see Stanislas Dehaene's The Number Sense.

A study by Gigerenzer and Hoffrage in 1995 showed that some ways of phrasing story problems are much more evocative of correct Bayesian reasoning. The least evocative phrasing used probabilities. A slightly more evocative phrasing used frequencies instead of probabilities; the problem remained the same, but instead of saying that 1% of women had breast cancer, one would say that 1 out of 100 women had breast cancer, that 80 out of 100 women with breast cancer would get a positive mammography, and so on. Why did a higher proportion of subjects display Bayesian reasoning on this problem? Probably because saying "1 out of 100 women" encourages you to concretely visualize X women with cancer, leading you to visualize X women with cancer and a positive mammography, etc.

The most effective presentation found so far is what's known as natural frequencies - saying that 40 out of 100 eggs contain pearls, 12 out of 40 eggs containing pearls are painted blue, and 6 out of 60 eggs containing nothing are painted blue. A natural frequencies presentation is one in which the information about the prior probability is included in presenting the conditional probabilities. If you were just learning about the eggs' conditional probabilities through natural experimentation, you would - in the course of cracking open a hundred eggs - crack open around 40 eggs containing pearls, of which 12 eggs would be painted blue, while cracking open 60 eggs containing nothing, of which about 6 would be painted blue. In the course of learning the conditional probabilities, you'd see examples of blue eggs containing pearls about twice as often as you saw examples of blue eggs containing nothing.

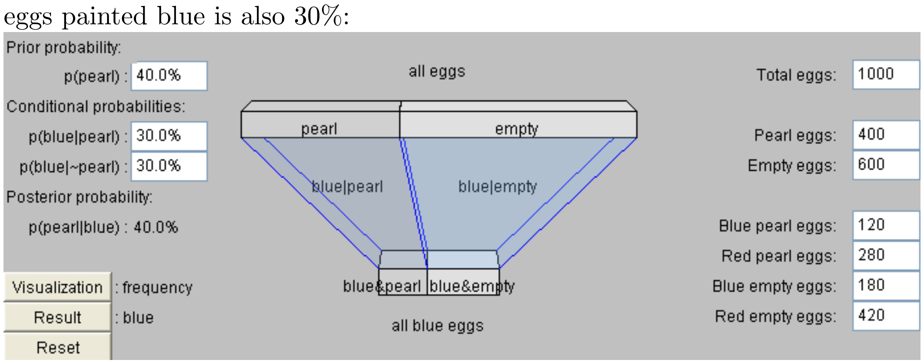
It may seem like presenting the problem in this way is "cheating", and indeed if it were a story problem in a math book, it probably would be cheating. However, if you're talking about real doctors, you want to cheat; you want the doctors to draw the right conclusions as easily as possible. The obvious next move would be to present all medical statistics in terms of natural frequencies. Unfortunately, while natural frequencies are a step in the right direction, it probably won't be enough. When problems are presented in natural frequences, the proportion of people using Bayesian reasoning rises to around half. A big improvement, but not big enough when you're talking about real doctors and real patients.

A presentation of the problem in natural frequencies might be visualized like this:



In the frequency visualization, the selective attrition of the two conditional probabilities changes the proportion of eggs that contain pearls. The bottom bar is shorter than the top bar, just as the number of eggs painted blue is less than the total number of eggs. The probability graph shown earlier is really just the frequency graph with the bottom bar "renormalized", stretched out to the same length as the top bar. In the frequency applet you can change the conditional probabilities by clicking and dragging the left and right edges of the graph. (For example, to change the conditional probability blue|pearl, click and drag the line on the left that stretches from the left edge of the top bar to the left edge of the bottom bar.)

In the probability applet, you can see that when the conditional probabilities are equal, there's no differential pressure - the arrows are the same size - so the prior probability doesn't slide between the top bar and the bottom bar. But the bottom bar in the probability applet is just a renormalized (stretched out) version of the bottom bar in the frequency applet, and the frequency applet shows why the probability doesn't slide if the two conditional probabilities are equal. Here's a case where the prior proportion of pearls remains 40%, and the proportion of pearl eggs painted blue remains 30%, but the number of empty eggs painted blue is also 30%:

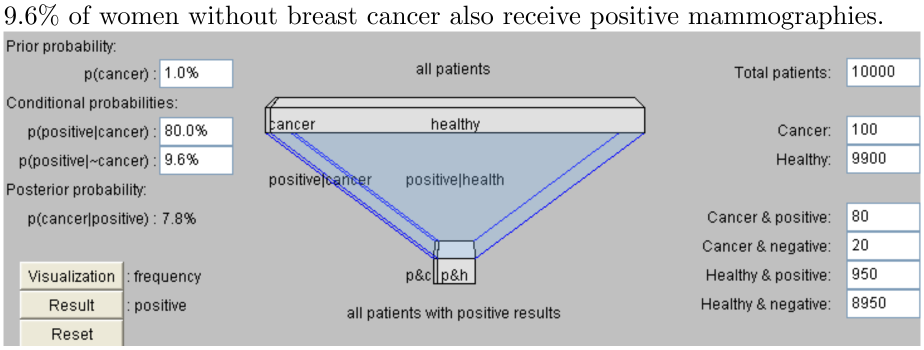


If you diminish two shapes by the same factor, their relative proportion will be the same as before. If you diminish the left section of the top bar by the same factor as the right section, then the bottom bar will have the same proportions as the top bar - it'll just be smaller. If the two conditional probabilities are equal, learning that the egg is blue doesn't change the probability that the egg contains a pearl - for the same reason that similar triangles have identical angles; geometric figures don't change shape when you shrink them by a constant factor.

In this case, you might as well just say that 30% of eggs are painted blue, since the probability of an egg being painted blue is independent of whether the egg contains a pearl. Applying a "test" that is statistically independent of its condition just shrinks the sample size. In this case, requiring that the egg be painted blue doesn't shrink the group of eggs with pearls any more or less than it shrinks the group of eggs without pearls. It just shrinks the total number of eggs in the sample.

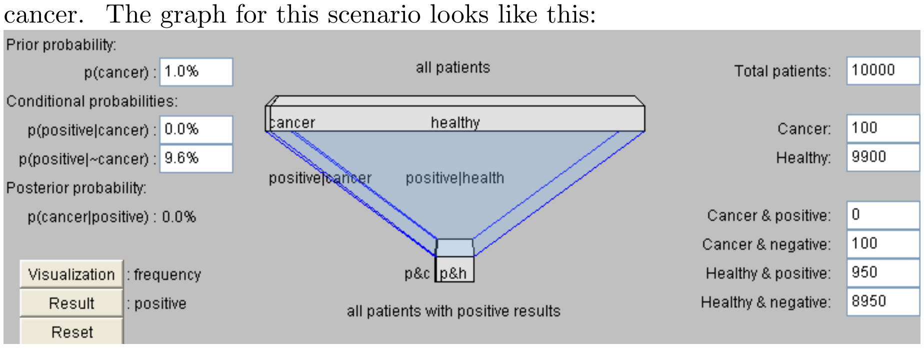
Fun
Fact! Q. Why did the Bayesian reasoner cross the road?
A. You need more information to answer this question.

Here's what the original medical problem looks like when graphed. 1% of women have breast cancer, 80% of those women test positive on a mammography, and 9.6% of women without breast cancer also receive positive mammographies.



As is now clearly visible, the mammography doesn't increase the probability a positive-testing woman has breast cancer by increasing the number of women with breast cancer - of course not; if mammography increased the number of women with breast cancer, no one would ever take the test! However, requiring a positive mammography is a membership test that eliminates many more women without breast cancer than women with cancer. The number of women without breast cancer diminishes by a factor of more than ten, from 9,900 to 950, while the number of women with breast cancer is diminished only from 100 to 80. Thus, the proportion of 80 within 1,030 is much larger than the proportion of 100 within 10,000. In the graph, the left sector (representing women with breast cancer) is small, but the mammography test projects almost all of this sector into the bottom bar. The right sector (representing women without breast cancer) is large, but the mammography test projects a much smaller fraction of this sector into the bottom bar. There are, indeed, fewer women with breast cancer and positive mammographies than there are women with breast cancer - obeying the law of probabilities which requires that p(A) >= p(A&B). But even though the left sector in the bottom bar is actually slightly smaller, the proportion of the left sector within the bottom bar is greater - though still not very great. If the bottom bar were renormalized to the same length as the top bar, it would look like the left sector had expanded. This is why the proportion of "women with breast cancer" in the group "women with positive mammographies" is higher than the proportion of "women with breast cancer" in the general population - although the proportion is still not very high. The evidence of the positive mammography slides the prior probability of 1% to the posterior probability of 7.8%.

Suppose there's yet another variant of the mammography test, mammography@, which behaves as follows. 1% of women in a certain demographic have breast cancer. Like ordinary mammography, mammography@ returns positive 9.6% of the time for women without breast cancer. However, mammography@ returns positive 0% of the time (say, once in a billion) for women with breast cancer. The graph for this scenario looks like this:



What is it that this test actually does? If a patient comes to you with a positive result on her mammography@, what do you say?

"Congratulations, you're among the rare 9.5% of the population whose health is definitely established by this test."

Mammography@ isn't a cancer test; it's a health test! Few women without breast cancer get positive results on mammography@, but only women without breast cancer ever get positive results at all. Not much of the right sector of the top bar projects into the bottom bar, but none of the left sector projects into the bottom bar. So a positive result on mammography@ means you definitely don't have breast cancer.

What makes ordinary mammography a positive indicator for breast cancer is not that someone named the result "positive", but rather that the test result stands in a specific Bayesian relation to the condition of breast cancer. You could call the same result "positive" or "negative" or "blue" or "red" or "James Rutherford", or give it no name at all, and the test result would still slide the probability in exactly the same way. To minimize confusion, a test result which slides the probability of breast cancer upward should be called "positive". A test result which slides the probability of breast cancer downward should be called "negative". If the test result is statistically unrelated to the presence or absence of breast cancer - if the two conditional probabilities are equal - then we shouldn't call the procedure a "cancer test"! The meaning of the test is determined by the two conditional probabilities; any names attached to the results are simply convenient labels.

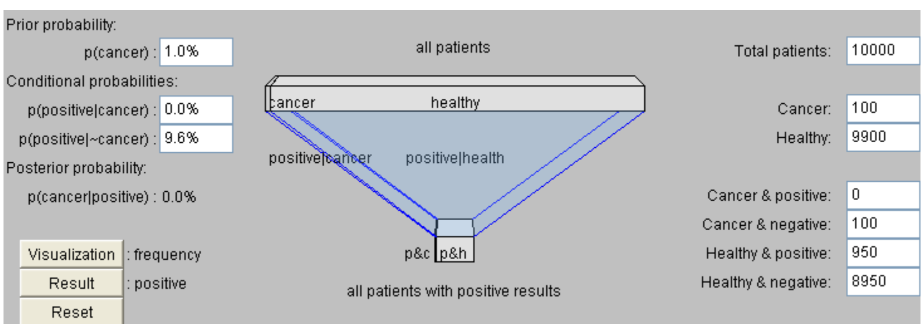


Figure 1: image

The bottom bar for the graph of mammography@ is small; mammography@ is a test that's only rarely useful. Or rather, the test only rarely gives strong evidence, and most of the time gives weak evidence. A negative result on mammography@ does slide probability - it just doesn't slide it very far. Click the "Result" switch at the bottom left corner of the applet to see what a negative result on mammography@ would imply. You might intuit that since the test could have returned positive for health, but didn't, then the failure of the test to return positive must mean that the woman has a higher chance of having breast cancer - that her probability of having breast cancer must be slid upward by the negative result on her health test.

This intuition is correct! The sum of the groups with negative results and positive results must always equal the group of all women. If the positive-testing group has "more than its fair share" of women without breast cancer, there must be an at least slightly higher proportion of women with cancer in the negative-testing group. A positive result is rare but very strong evidence in one direction, while a negative result is common but very weak evidence in the opposite direction. You might call this the Law of Conservation of Probability - not a standard term, but the conservation rule is exact. If you take the revised probability of breast cancer after a positive result, times the probability of a positive result, and add that to the revised probability of breast cancer after a negative result, times the probability of a negative result, then you must always arrive at the prior probability. If you don't yet know what the test result is, the expected revised probability after the test result arrives - taking both possible results into account - should always equal the prior probability.

On ordinary mammography, the test is expected to return "positive" 10.3% of the time - 80 positive women with cancer plus 950 positive women without cancer equals 1030 women with positive results. Conversely, the mammography should return negative 89.7% of the time: 100% - 10.3% = 89.7%. A positive result slides the revised probability from 1% to 7.8%, while a negative result slides the revised probability from 1% to 0.22%. So p(cancer|positive)*p(positive) + p(cancer|negative)*p(negative) = 7.8%*10.3% + 0.22%*89.7% = 1% = p(cancer), as expected.
Calculator: Result:

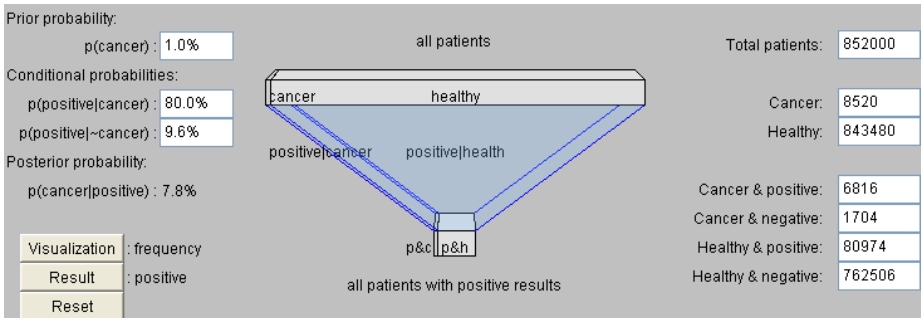
Why "as expected"? Let's take a look at the quantities involved:
p(cancer): 0.01 Group 1: 100 women with breast cancer p(~cancer): 0.99 Group 2: 9900 women without breast cancer p(positive|cancer): 80.0% 80% of women with breast cancer have positive mammographies p(~positive|cancer): 20.0% 20% of women with breast cancer have negative mammographies p(positive|~cancer): 9.6% 9.6% of women without breast cancer have positive mammographies p(~positive|~cancer): 90.4% 90.4% of women without breast cancer have negative mammographies p(cancer&positive): 0.008 Group A: 80 women with breast cancer and positive mammographies p(cancer&~positive): 0.002 Group B: 20 women with breast cancer and negative mammographies p(~cancer&positive): 0.095 Group C: 950 women without breast cancer and positive mammographies p(~cancer&~positive): 0.895 Group D: 8950 women without breast cancer and negative mammographies p(positive): 0.103 1030 women with positive results p(~positive): 0.897 8970 women with negative results p(cancer|positive): 7.80% Chance you have breast cancer if mammography is positive: 7.8% p(~cancer|positive): 92.20% Chance you are healthy if mammography is positive: 92.2% p(cancer|~positive): 0.22% Chance you have breast cancer if mammography is negative: 0.22% p(~cancer|~positive): 99.78% Chance you are healthy if mammography is negative: 99.78%

One of the common confusions in using Bayesian reasoning is to mix up some or all of these quantities - which, as you can see, are all numerically different and have different meanings. p(A&B) is the same as p(B&A), but p(A|B) is not the same thing as p(B|A), and p(A&B) is completely different from p(A|B). (I don't know who chose the symmetrical "|" symbol to mean "implies", and then made the direction of implication right-to-left, but it was probably a bad idea.)

To get acquainted with all these quantities and the relationships between them, we'll play "follow the degrees of freedom". For example, the two quantities p(cancer) and p(∼cancer) have 1 degree of freedom between them, because of the general law p(A) + p(∼A) = 1. If you know that p(∼cancer) = .99, you can obtain p(cancer) = 1 - p(∼cancer) = .01. There's no room to say that p(∼cancer) = .99 and then also specify p(cancer) = .25; it would violate the rule p(A) + p(∼A) = 1.

p(positive|cancer) and p(∼positive|cancer) also have only one degree of freedom between them; either a woman with breast cancer gets a positive mammography or she doesn't. On the other hand, p(positive|cancer) and p(positive|∼cancer) have two degrees of freedom. You can have a mammography test that returns positive for 80% of cancerous patients and 9.6% of healthy patients, or that returns positive for 70% of cancerous patients and 2% of healthy patients, or even a health test that returns "positive" for 30% of cancerous patients and 92% of healthy patients. The two quantities, the output of the mammography test for cancerous patients and the output of the mammography test for healthy patients, are in mathematical terms independent; one cannot be obtained from the other in any way, and so they have two degrees of freedom between them.

What about p(positive&cancer), p(positive|cancer), and p(cancer)? Here we have three quantities; how many degrees of freedom are there? In this case the equation that must hold is p(positive&cancer) = p(positive|cancer) * p(cancer). This equality reduces the degrees of freedom by one. If we know the fraction of patients with cancer, and chance that a cancerous patient has a positive mammography, we can deduce the fraction of patients who have breast cancer and a positive mammography by multiplying. You should recognize this operation from the graph; it's the projection of the top bar into the bottom bar. p(cancer) is the left sector of the top bar, and p(positive|cancer) determines how much of that sector projects into the bottom bar, and the left sector of the bottom bar is p(positive&cancer).



Similarly, if we know the number of patients with breast cancer and positive mammographies, and also the number of patients with breast cancer, we can estimate the chance that a woman with breast cancer gets a positive mammography by dividing: p(positive|cancer) = p(positive&cancer) / p(cancer). In fact, this is exactly how such medical diagnostic tests are calibrated; you do a study on 8,520 women with breast cancer and see that there are 6,816 (or thereabouts) women with breast cancer andpositive mammographies, then divide 6,816 by 8520 to find that 80% of women with breast cancer had positive mammographies. (Incidentally, if you accidentally divide 8520 by 6,816 instead of the other way around, your calculations will start doing strange things, such as insisting that 125% of women with breast cancer and positive mammographies have breast cancer. This is a common mistake in carrying out Bayesian arithmetic, in my experience.) And finally, if you know p(positive&cancer) and p(positive|cancer), you can deduce how many cancer patients there must have been originally. There are two degrees of freedom shared out among the three quantities; if we know any two, we can deduce the third.

How about p(positive), p(positive&cancer), and p(positive&∼cancer)? Again there are only two degrees of freedom among these three variables. The equation occupying the extra degree of freedom is p(positive) = p(positive&cancer) + p(positive&∼cancer). This is how p(positive) is computed to begin with; we figure out the number of women with breast cancer who have positive mammographies, and the number of women without breast cancer who have positive mammographies, then add them together to get the total number of women with positive mammographies. It would be very strange to go out and conduct a study to determine the number of women with positive mammographies - just that one number and nothing else - but in theory you could do so. And if you then conducted another study and found the number of those women who had positive mammographies and breast cancer, you would also know the number of women with positive mammographies and no breast cancer - either a woman with a positive mammography has breast cancer or she doesn't. In general, p(A&B) + p(A&∼B) = p(A). Symmetrically, p(A&B) + p(∼A&B) = p(B).

What about p(positive&cancer), p(positive&∼cancer), p(∼positive&cancer), and p(∼positive&∼cancer)? You might at first be tempted to think that there are only two degrees of freedom for these four quantities - that you can, for example, get p(positive&~cancer) by multiplying p(positive) * p(~cancer), and thus that all four quantities can be found given only the two quantities p(positive) and p(cancer). This is not the case! p(positive&~cancer) = p(positive) * p(~cancer) only if the two probabilities are statistically independent - if the chance that a woman has breast cancer has no bearing on whether she has a positive mammography. As you'll recall, this amounts to requiring that the two conditional probabilities be equal to each other - a requirement which would eliminate one degree of freedom. If you remember that these four quantities are the groups A, B, C, and D, you can look over those four groups and realize that, in theory, you can put any number of people into the four groups. If you start with a group of 80 women with breast cancer and positive mammographies, there's no reason why you can't add another group of 500 women with breast cancer and negative mammographies, followed by a group of 3 women without breast cancer and negative mammographies, and so on. So now it seems like the four quantities have four degrees of freedom. And they would, except that in expressing them as probabilities, we need to normalize them to fractions of the complete group, which adds the constraint that p(positive&cancer) + p(positive&~cancer) + p(~positive&cancer) + p(~positive&~cancer) = 1. This equation takes up one degree of freedom, leaving three degrees of freedom among the four quantities. If you specify the fractions of women in groups A, B, and D, you can deduce the fraction of women in group C.

Given the four groups A, B, C, and D, it is very straightforward to compute everything else: p(cancer) = A + B, p(~positive|cancer) = B / (A + B), and so on. Since ABCD contains three degrees of freedom, it follows that the entire set of 16 probabilities contains only three degrees of freedom. Remember that in our problems we always needed three pieces of information - the prior probability and the two conditional probabilities - which, indeed, have three degrees of freedom among them. Actually, for Bayesian problems, any three quantities with three degrees of freedom between them should logically specify the entire problem. For example, let's take a barrel of eggs with p(blue) = 0.40, p(blue|pearl) = 5/13, and p(~blue&~pearl) = 0.20. Given this information, you can compute p(pearl|blue).

As a story problem:
Suppose you have a large barrel containing a number of plastic eggs. Some eggs contain pearls, the rest contain nothing. Some eggs are painted blue, the rest are painted red. Suppose that 40% of the eggs are painted blue, 5/13 of the eggs containing pearls are painted blue, and 20% of the eggs are both empty and painted red. What is the probability that an egg painted blue contains a pearl?

Try it - I assure you it is possible.
Calculator: Result: You probably shouldn't try to solve this with just a Javascript

calculator, though. I used a Python console. (In theory, pencil and paper should also work, but I don't know anyone who owns a pencil so I couldn't try it personally.)

As a check on your calculations, does the (meaningless) quantity p(~pearl|~blue)/p(pearl) roughly equal .51? (In story problem terms: The likelihood that a red egg is empty, divided by the likelihood that an egg contains a pearl, equals approximately .51.) Of course, using this information in the problem would be cheating.

If you can solve that problem, then when we revisit Conservation of Probability, it seems perfectly straightforward. Of course the mean revised probability, after administering the test, must be the same as the prior probability. Of course strong but rare evidence in one direction must be counterbalanced by common but weak evidence in the other direction.

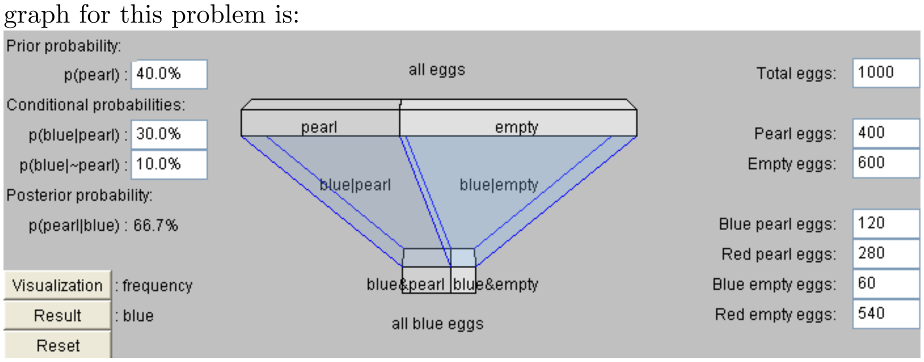
Because:

$$
\begin{aligned}
&\ p(\text{cancer}|\text{positive})*p(\text{positive}) \\
&+ p(\text{cancer}|{\sim}\text{positive})*p({\sim}\text{positive}) \\
&= p(\text{cancer})
\end{aligned}
$$

In terms of the four groups:

$$
\begin{aligned}
&p(\text{cancer}|\text{positive}) &&= A\ /\ (A + C) \\
&p(\text{positive}) &&= A + C \\
&p(\text{cancer}\&\text{positive}) &&= A \\
&p(\text{cancer}|{\sim}\text{positive}) &&= B\ /\ (B + D) \\
&p({\sim}\text{positive}) &&= B + D \\
&p(\text{cancer}\&{\sim}\text{positive}) &&= B \\
&p(\text{cancer}) &&= A + B
\end{aligned}
$$

Let's return to the original barrel of eggs - 40% of the eggs containing pearls, 30% of the pearl eggs painted blue, 10% of the empty eggs painted blue. The graph for this problem is:



What happens to the revised probability, p(pearl|blue), if the proportion of eggs containing pearls is kept constant, but 60% of the eggs with pearls are painted blue (instead of 30%), and 20% of the empty eggs are painted blue (instead of 10%)? You could type 60% and 20% into the inputs for the two conditional probabilities, and see how the graph changes - but can you figure out in advance what the change will look like?

If you guessed that the revised probability remains the same, because the bottom bar grows by a factor of 2 but retains the same proportions, congratulations! Take a moment to think about how far you've come. Looking at a problem like
1% of women have breast cancer. 80% of women with breast cancer get positive mammographies. 9.6% of women without breast cancer get positive mammographies. If a woman has a positive mammography, what is the probability she has breast cancer?

the vast majority of respondents intuit that around 70–80% of women with positive mammographies have breast cancer. Now, looking at a problem like
Suppose there are two barrels containing many small plastic eggs. In both barrels, some eggs are painted blue and the rest are painted red. In both barrels, 40% of the eggs contain pearls and the rest are empty. In the first barrel, 30% of the pearl eggs are painted blue, and 10% of the empty eggs are painted blue. In the second barrel, 60% of the pearl eggs are painted blue, and 20% of the empty eggs are painted blue. Would you rather have a blue egg from the first or second barrel?

you can see it's intuitively obvious that the probability of a blue egg containing a pearl is the same for either barrel. Imagine how hard it would be to see that using the old way of thinking!

It's intuitively obvious, but how to prove it? Suppose that we call P the prior probability that an egg contains a pearl, that we call M the first conditional probability (that a pearl egg is painted blue), and N the second conditional probability (that an empty egg is painted blue). Suppose that M and N are both increased or diminished by an arbitrary factor X - for example, in the problem above, they are both increased by a factor of 2. Does the revised probability that an egg contains a pearl, given that we know the egg is blue, stay the same? - p(pearl) = P - p(blue|pearl) = M*X - p(blue|~pearl) = N*X - p(pearl|blue) = ?

From these quantities, we get the four groups: - Group A: p(pearl&blue) = P*M*X - Group B: p(pearl&~blue) = P*(1 - (M*X)) - Group C: p(~pearl&blue) = (1 - P)*N*X - Group D: p(~pearl&~blue) = (1 - P)*(1 - (N*X))

The proportion of eggs that contain pearls and are blue, within the group of all blue eggs, is then the proportion of group (A) within the group (A + C), equalling P*M*X / (P*M*X + (1 - P)*N*X). The factor X in the numerator and denominator cancels out, so increasing or diminishing both conditional probabilities by a constant factor doesn't change the revised probability.

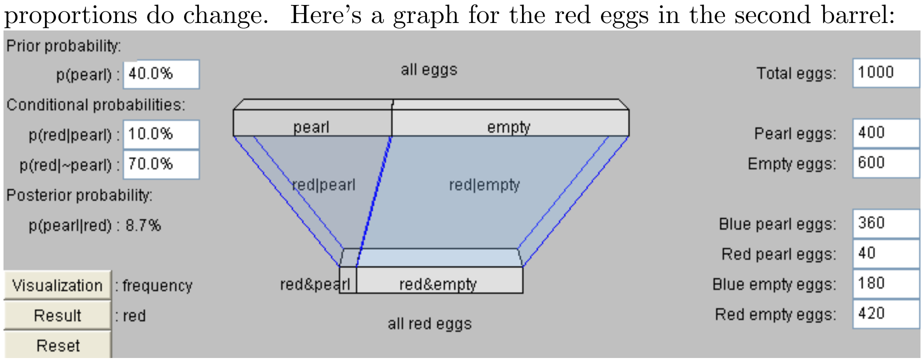
Fun Fact! Q. Suppose that there are two barrels, each containing a number of plastic eggs. In both barrels, some eggs are painted blue and the rest are painted red. In the first barrel, 90% of the eggs contain pearls and 20% of the pearl eggs are painted blue. In the second barrel, 45% of the eggs contain pearls and 60% of the empty eggs are painted red. Would you rather have a blue pearl egg from the first or second barrel?
A. Actually, it doesn't matter which barrel you choose! Can you see why?

The probability that a test gives a true positive divided by the probability that a test gives a false positive is known as the likelihood ratio of that test. Does the likelihood ratio of a medical test sum up everything there is to know about the usefulness of the test?

No, it does not! The likelihood ratio sums up everything there is to know about the meaning of a positive result on the medical test, but the meaning of a negative result on the test is not specified, nor is the frequency with which the test is useful. If we examine the algebra above, while p(pearl|blue) remains constant, p(pearl|~blue) may change - the X does not cancel out. As a story problem, this strange fact would look something like this:
Suppose that there are two barrels, each containing a number of plastic eggs. In both barrels, 40% of the eggs contain pearls and the rest contain nothing. In both barrels, some eggs are painted blue and the rest are painted red. In the first barrel, 30% of the eggs with pearls are painted blue, and 10% of the empty eggs are painted blue. In the second barrel, 90% of the eggs with pearls are painted blue, and 30% of the empty eggs are painted blue. Would you rather have a blue egg from the first or second barrel? Would you rather have a red egg from the first or second barrel?

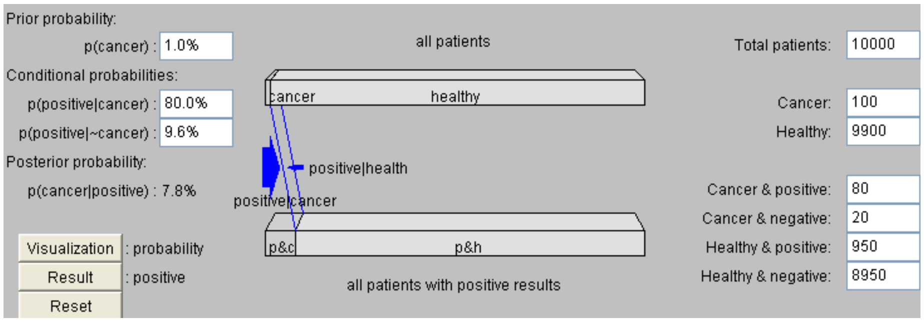
For the first question, the answer is that we don't care whether we get the blue egg from the first or second barrel. For the second question, however, the probabilities do change - in the first barrel, 34% of the red eggs contain pearls, while in the second barrel 8.7% of the red eggs contain pearls! Thus, we should prefer to get a red egg from the first barrel. In the first barrel, 70% of the pearl eggs are painted red, and 90% of the empty eggs are painted red. In the second barrel, 10% of the pearl eggs are painted red, and 70% of the empty eggs are painted red.

Calculator: Result: What goes on here? We start out by noting that, counter to intuition, p(pearl|blue) and p(pearl|∼blue) have two degrees of freedom among them even when p(pearl) is fixed - so there's no reason why one quantity shouldn't change while the other remains constant. But we didn't we just get through establishing a law for "Conservation of Probability", which says that p(pearl|blue)*p(blue) + p(pearl|∼blue)*p(∼blue) = p(pearl)? Doesn't this equation take up one degree of freedom? No, because p(blue) isn't fixed between the two problems. In the second barrel, the proportion of blue eggs containing pearls is the same as in the first barrel, but a much larger fraction of eggs are painted blue! This alters the set of red eggs in such a way that the proportions do change. Here's a graph for the red eggs in the second barrel:



Let's return to the example of a medical test. The likelihood ratio of a medical test - the number of true positives divided by the number of false positives - tells us everything there is to know about the meaning of a positive result. But it doesn't tell us the meaning of a negative result, and it doesn't tell us how often the test is useful. For example, a mammography with a hit rate of 80% for patients with breast cancer and a false positive rate of 9.6% for healthy patients has the same likelihood ratio as a test with an 8% hit rate and a false positive rate of 0.96%. Although these two tests have the same likelihood ratio, the first test is more useful in every way - it detects disease more often, and a negative result is stronger evidence of health.

The likelihood ratio for a positive result summarizes the differential pressure of the two conditional probabilities for a positive result, and thus summarizes how much a positive result will slide the prior probability. Take a probability graph, like this one:



The likelihood ratio of the mammography is what determines the slant of the line. If the prior probability is 1%, then knowing only the likelihood ratio is enough to determine the posterior probability after a positive result.

But, as you can see from the frequency graph, the likelihood ratio doesn't tell the whole story - in the frequency graph, the proportions of the bottom bar can stay fixed while the size of the bottom bar changes. p(blue) increases but p(pearl|blue) doesn't change, because p(pearl&blue) and p(∼pearl&blue) increase by the same factor. But when you flip the graph to look at p(∼blue), the proportions of p(pearl&∼blue) and p(∼pearl&∼blue) do not remain constant.

Of course the likelihood ratio can't tell the whole story; the likelihood ratio and the prior probability together are only two numbers, while the problem has three degrees of freedom.

Suppose that you apply two tests for breast cancer in succession - say, a standard mammography and also some other test which is independent of mammography. Since I don't know of any such test which is independent of mammography, I'll invent one for the purpose of this problem, and call it the Tams-Braylor Division Test, which checks to see if any cells are dividing more rapidly than other cells. We'll suppose that the Tams-Braylor gives a true positive for 90% of patients with breast cancer, and gives a false positive for 5% of patients without cancer. Let's say the prior prevalence of breast cancer is 1%. If a patient gets a positive result on her mammography and her Tams-Braylor, what is the revised probability she has breast cancer?

One way to solve this problem would be to take the revised probability for a positive mammography, which we already calculated as 7.8%, and plug that into the Tams-Braylor test as the new prior probability. If we do this, we find that the result comes out to 60%.
Calculator: Result: But this assumes that first we see the positive mammography result, and then the positive result on the Tams-Braylor. What if first the woman gets a positive result on the Tams-Braylor, followed by a positive result

on her mammography. Intuitively, it seems like it shouldn't matter. Does the math check out?

First we'll administer the Tams-Braylor to a woman with a 1% prior probability of breast cancer.
Calculator: Result: Then we administer a mammography, which gives 80% true positives and 9.6% false positives, and it also comes out positive.
Calculator: Result: Lo and behold, the answer is again 60%. (If it's not exactly the same, it's due to rounding error - you can get a more precise calculator, or work out the fractions by hand, and the numbers will be exactly equal.)

An algebraic proof that both strategies are equivalent is left to the reader. To visualize, imagine that the lower bar of the frequency applet for mammography projects an even lower bar using the probabilities of the Tams-Braylor Test, and that the final lowest bar is the same regardless of the order in which the conditional probabilities are projected.

We might also reason that since the two tests are independent, the probability a woman with breast cancer gets a positive mammography and a positive Tams-Braylor is 90% * 80% = 72%. And the probability that a woman without breast cancer gets false positives on mammography and Tams-Braylor is 5% * 9.6% = 0.48%. So if we wrap it all up as a single test with a likelihood ratio of 72%/0.48%, and apply it to a woman with a 1% prior probability of breast cancer:
Calculator: Result: …we find once again that the answer is 60%.

Suppose that the prior prevalence of breast cancer in a demographic is 1%. Suppose that we, as doctors, have a repertoire of three independent tests for breast cancer. Our first test, test A, a mammography, has a likelihood ratio of 80%/9.6% = 8.33. The second test, test B, has a likelihood ratio of 18.0 (for example, from 90% versus 5%); and the third test, test C, has a likelihood ratio of 3.5 (which could be from 70% versus 20%, or from 35% versus 10%; it makes no difference). Suppose a patient gets a positive result on all three tests. What is the probability the patient has breast cancer?

Here's a fun trick for simplifying the bookkeeping. If the prior prevalence of breast cancer in a demographic is 1%, then 1 out of 100 women have breast cancer, and 99 out of 100 women do not have breast cancer. So if we rewrite the probability of 1% as an odds ratio, the odds are:

1:99

And the likelihood ratios of the three tests A, B, and C are:

8.33:1 = 25:3
18.0:1 = 18:1
3.5:1 = 7:2

The odds for women with breast cancer who score positive on all three tests, versus women without breast cancer who score positive on all three tests, will equal:

1*25*18*7:99*3*1*2 =
3,150:594

To recover the probability from the odds, we just write:
3,150 / (3,150 + 594) = 84%

This always works regardless of how the odds ratios are written; i.e., 8.33:1 is just the same as 25:3 or 75:9. It doesn't matter in what order the tests are administered, or in what order the results are computed. The proof is left as an exercise for the reader.

E. T. Jaynes, in "Probability Theory With Applications in Science and Engineering", suggests that credibility and evidence should be measured in decibels.

Decibels?

Decibels are used for measuring exponential differences of intensity. For example, if the sound from an automobile horn carries 10,000 times as much energy (per square meter per second) as the sound from an alarm clock, the automobile horn would be 40 decibels louder. The sound of a bird singing might carry 1,000 times less energy than an alarm clock, and hence would be 30 decibels softer. To get the number of decibels, you take the logarithm base 10 and multiply by 10.

decibels = $10 \log_{10}$ (intensity)
or
intensity = $10^{(\text{decibels}/10)}$

Suppose we start with a prior probability of 1% that a woman has breast cancer, corresponding to an odds ratio of 1:99. And then we administer three tests of likelihood ratios 25:3, 18:1, and 7:2. You could multiply those numbers... or you could just add their logarithms:

$10 \log_{10} (1/99) = -20$
$10 \log_{10} (25/3) = 9$
$10 \log_{10} (18/1) = 13$
$10 \log_{10} (7/2) = 5$

It starts out as fairly unlikely that a woman has breast cancer - our credibility level is at –20 decibels. Then three test results come in, corresponding to 9, 13, and 5 decibels of evidence. This raises the credibility level by a total of 27 decibels, meaning that the prior credibility of –20 decibels goes to a posterior credibility of 7 decibels. So the odds go from 1:99 to 5:1, and the probability goes from 1% to around 83%.

In front of you is a bookbag containing 1,000 poker chips. I started out with two such bookbags, one containing 700 red and 300 blue chips, the other containing 300 red and 700 blue. I flipped a fair coin to determine which bookbag to use, so your prior probability that the bookbag in front of you is the red bookbag is 50%. Now, you sample randomly, with replacement after each chip. In 12 samples, you get 8 reds and 4 blues. What is the probability that this is the predominantly red bag?

Just for fun, try and work this one out in your head. You don't need to be exact - a rough estimate is good enough. When you're ready, continue onward.

According to a study performed by Lawrence Phillips and Ward Edwards in 1966, most people, faced with this problem, give an answer in the range 70% to 80%. Did you give a substantially higher probability than that? If you did, congratulations - Ward Edwards wrote that very seldom does a person answer this question properly, even if the person is relatively familiar with Bayesian reasoning. The correct answer is 97%.

The likelihood ratio for the test result "red chip" is 7/3, while the likelihood ratio for the test result "blue chip" is 3/7. Therefore a blue chip is exactly the same amount of evidence as a red chip, just in the other direction - a red chip is 3.6 decibels of evidence for the red bag, and a blue chip is –3.6 decibels of evidence. If you draw one blue chip and one red chip, they cancel out. So the ratio of red chips to blue chips does not matter; only the excess of red chips over blue chips matters. There were eight red chips and four blue chips in twelve samples; therefore, four more red chips than blue chips. Thus the posterior odds will be:

7^4:3^4 = 2401:81
which is around 30:1, i.e., around 97%.

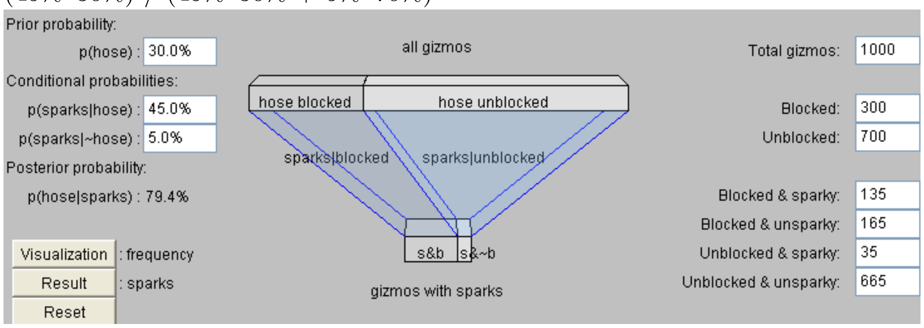
The prior credibility starts at 0 decibels and there's a total of around 14 decibels of evidence, and indeed this corresponds to odds of around 25:1 or around 96%. Again, there's some rounding error, but if you performed the operations using exact arithmetic, the results would be identical.

We can now see intuitively that the bookbag problem would have exactly the same answer, obtained in just the same way, if sixteen chips were sampled and we found ten red chips and six blue chips.

You are a mechanic for gizmos. When a gizmo stops working, it is due to a blocked hose 30% of the time. If a gizmo's hose is blocked, there is a 45%

probability that prodding the gizmo will produce sparks. If a gizmo's hose is unblocked, there is only a 5% chance that prodding the gizmo will produce sparks. A customer brings you a malfunctioning gizmo. You prod the gizmo and find that it produces sparks. What is the probability that a spark-producing gizmo has a blocked hose? Calculator: Result: What is the sequence of arithmetical operations that you performed to solve this problem?

(45%*30%) / (45%*30% + 5%*70%)



Similarly, to find the chance that a woman with positive mammography has breast cancer, we computed:

$$\frac{p(\text{positive}|\text{cancer})*p(\text{cancer})}{p(\text{positive}|\text{cancer})*p(\text{cancer}) + p(\text{positive}|{\sim}\text{cancer})*p({\sim}\text{cancer})}$$

which is

p(positive&cancer) / [p(positive&cancer) + p(positive&~cancer)]

which is

p(positive&cancer) / p(positive)

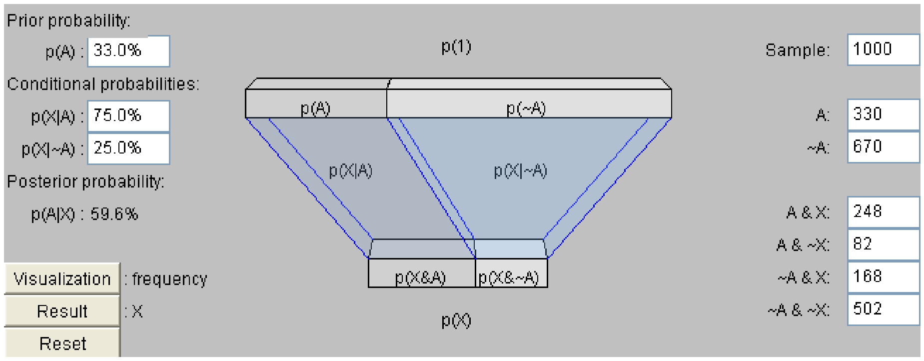
which is

p(cancer|positive)

The fully general form of this calculation is known as Bayes' Theorem or Bayes' Rule:

$$p(A|X) = \frac{p(X|A)*p(A)}{p(X|A)*p(A) + p(X|{\sim}A)*p({\sim}A)}$$

Given some phenomenon A that we want to investigate, and an observation X that is evidence about A - for example, in the previous example, A is breast cancer and X is a positive mammography - Bayes' Theorem tells us how we should update our probability of A, given the new evidence X.



By this point, Bayes' Theorem may seem blatantly obvious or even tautological, rather than exciting and new. If so, this introduction has entirely succeeded in its purpose.

Fun Fact! Q. Who originally discovered Bayes' Theorem?

A. The Reverend Thomas Bayes, by far the most enigmatic figure in mathematical history. Almost nothing is known of Bayes's life, and very few of his manuscripts survived. Thomas Bayes was born in 1701 or 1702 to Joshua Bayes and Ann Carpenter, and his date of death is listed as 1761. The exact date of Thomas Bayes's birth is not known for certain because Joshua Bayes, though a surprisingly wealthy man, was a member of an unusual, esoteric, and even heretical religious sect, the "Nonconformists". The Nonconformists kept their birth registers secret, supposedly from fear of religious discrimination; whatever the reason, no true record exists of Thomas Bayes's birth. Thomas Bayes was raised a Nonconformist and was soon promoted into the higher ranks of the Nonconformist theosophers, whence comes the "Reverend" in his name.

In 1742 Bayes was elected a Fellow of the Royal Society of London, the most prestigious scientific body of its day, despite Bayes having published no scientific or mathematical works at that time. Bayes's nomination certificate was signed by sponsors including the President and the Secretary of the Society, making his election almost certain. Even today, however, it remains a mystery why such weighty names sponsored an unknown into the Royal Society.

Bayes's sole publication during his known lifetime was allegedly a mystical book entitled Divine Benevolence, laying forth the original causation and ultimate purpose of the universe. The book is commonly attributed to Bayes, though it is said that no author appeared on the title page, and the entire work is sometimes considered to be of dubious provenance.

Most mysterious of all, Bayes' Theorem itself appears in a Bayes manuscript presented to the Royal Society of London in 1764, three years after Bayes's supposed death in 1761!

Despite the shocking circumstances of its presentation, Bayes' Theorem was soon forgotten, and was popularized within the scientific community only by the later efforts of the great mathematician Pierre-Simon Laplace. Laplace himself is almost as enigmatic as Bayes; we don't even know whether it was "Pierre" or "Simon" that was his actual first name. Laplace's papers are said to have contained a design for an AI capable of predicting all future events, the so-called "Laplacian superintelligence". While it is generally believed that Laplace never tried to implement his design, there remains the fact that Laplace presciently fled the guillotine that claimed many of his colleagues during the Reign of Terror. Even today, physicists sometimes attribute unusual effects to a "Laplacian Operator" intervening in their experiments.

In summary, we do not know the real circumstances of Bayes's birth, the ultimate origins of Bayes' Theorem, Bayes's actual year of death, or even whether Bayes ever really died. Nonetheless "Reverend Thomas Bayes", whatever his true identity, has the greatest fondness and gratitude of Earth's scientific community.

So why is it that some people are so excited about Bayes' Theorem?

"Do you believe that a nuclear war will occur in the next 20 years? If no, why not?" Since I wanted to use some common answers to this question to make a point about rationality, I went ahead and asked the above question in an IRC channel, #philosophy on EFNet.

One EFNetter who answered replied "No" to the above question, but added that he believed biological warfare would wipe out "99.4%" of humanity within the next ten years. I then asked whether he believed 100% was a possibility. "No," he said. "Why not?", I asked. "Because I'm an optimist," he said. (Roanoke of #philosophy on EFNet wishes to be credited with this statement, even having been warned that it will not be cast in a complimentary light. Good for him!) Another person who answered the above question said that he didn't expect a nuclear war for 100 years, because "All of the players involved in decisions regarding nuclear war are not interested right now." "But why extend that out for 100 years?", I asked. "Pure hope," was his reply.

What is it exactly that makes these thoughts "irrational" - a poor way of arriving at truth? There are a number of intuitive replies that can be given to this; for example: "It is not rational to believe things only because they are comforting." Of course it is equally irrational to believe things only because they are discomforting; the second error is less common, but equally irrational. Other intuitive arguments include the idea that "Whether or not you happen to be an optimist has nothing to do with whether biological warfare wipes out the human species", or "Pure hope is not evidence about nuclear war because it is not an observation about nuclear war."

There is also a mathematical reply that is precise, exact, and contains all the intuitions as special cases. This mathematical reply is known as Bayes' Theorem.

For example, the reply "Whether or not you happen to be an optimist has nothing to do with whether biological warfare wipes out the human species" can be translated into the statement:

p(you are currently an optimist | biological war occurs within ten years and wipes out humanity) =
p(you are currently an optimist | biological war occurs within ten years and does not wipe out humanity)

Since the two probabilities for $p(X|A)$ and $p(X|{\sim}A)$ are equal, Bayes' Theorem says that $p(A|X) = p(A)$; as we have earlier seen, when the two conditional probabilities are equal, the revised probability equals the prior probability. If X and A are unconnected - statistically independent - then finding that X is true cannot be evidence that A is true; observing X does not update our probability for A; saying "X" is not an argument for A.

But suppose you are arguing with someone who is verbally clever and who says something like, "Ah, but since I'm an optimist, I'll have renewed hope for tomorrow, work a little harder at my dead-end job, pump up the global economy a little, eventually, through the trickle-down effect, sending a few dollars into the pocket of the researcher who ultimately finds a way to stop biological warfare - so you see, the two events are related after all, and I can use one as valid evidence about the other." In one sense, this is correct - any correlation, no matter how weak, is fair prey for Bayes' Theorem; but Bayes' Theorem distinguishes between weak and strong evidence. That is, Bayes' Theorem not only tells us what is and isn't evidence, it also describes the strength of evidence. Bayes' Theorem not only tells us when to revise our probabilities, but how much to revise our probabilities. A correlation between hope and biological warfare may exist, but it's a lot weaker than the speaker wants it to be; he is revising his probabilities much too far.

Let's say you're a woman who's just undergone a mammography. Previously, you figured that you had a very small chance of having breast cancer; we'll suppose that you read the statistics somewhere and so you know the chance is 1%. When the positive mammography comes in, your estimated chance should now shift to 7.8%. There is no room to say something like, "Oh, well, a positive mammography isn't definite evidence, some healthy women get positive mammographies too. I don't want to despair too early, and I'm not going to revise my probability until more evidence comes in. Why? Because I'm a optimist." And there is similarly no room for saying, "Well, a positive mammography may not be definite evidence, but I'm going to assume the worst until I find otherwise. Why? Because I'm a pessimist." Your revised probability should go to 7.8%, no more, no less.

Bayes' Theorem describes what makes something "evidence" and how much evidence it is. Statistical models are judged by comparison to the Bayesian method because, in statistics, the Bayesian method is as good as it gets - the Bayesian method defines the maximum amount of mileage you can get out of a given piece of evidence, in the same way that thermodynamics defines the maximum amount of work you can get out of a temperature differential. This is why you hear cognitive scientists talking about Bayesian reasoners. In cognitive science, Bayesian reasoner is the technically precise codeword that we use to mean rational mind.

There are also a number of general heuristics about human reasoning that you can learn from looking at Bayes' Theorem.

For example, in many discussions of Bayes' Theorem, you may hear cognitive psychologists saying that people do not take prior frequencies sufficiently into account, meaning that when people approach a problem where there's some evidence X indicating that condition A might hold true, they tend to judge A's likelihood solely by how well the evidence X seems to match A, without taking into account the prior frequency of A. If you think, for example, that under the mammography example, the woman's chance of having breast cancer is in the range of 70%–80%, then this kind of reasoning is insensitive to the prior frequency given in the problem; it doesn't notice whether 1% of women or 10% of women start out having breast cancer. "Pay more attention to the prior frequency!" is one of the many things that humans need to bear in mind to partially compensate for our built-in inadequacies.

A related error is to pay too much attention to p(X|A) and not enough to p(X|~A) when determining how much evidence X is for A. The degree to which a result X is *evidence for A* depends, not only on the strength of the statement *we'd expect to see result X if A were true,* but also on the strength of the statement *we wouldn't expect to see result X if A weren't true.* For example, if it is raining, this very strongly implies the grass is wet - p(wetgrass|rain) ~ 1 - but seeing that the grass is wet doesn't necessarily mean that it has just rained; perhaps the sprinkler was turned on, or you're looking at the early morning dew. Since p(wetgrass|~rain) is substantially greater than zero, p(rain|wetgrass) is substantially less than one. On the other hand, if the grass was never wet when it wasn't raining, then knowing that the grass was wet would always show that it was raining, p(rain|wetgrass) ~ 1, even if p(wetgrass|rain) = 50%; that is, even if the grass only got wet 50% of the times it rained. Evidence is always the result of the differential between the two conditional probabilities. Strong evidence is not the product of a very high probability that A leads to X, but the product of a very low probability that not-A could have led to X.

The Bayesian revolution in the sciences is fueled, not only by more and more cognitive scientists suddenly noticing that mental phenomena have Bayesian structure in them; not only by scientists in every field learning to judge their statistical methods by comparison with the Bayesian method; but also by the idea that science itself is a special case of Bayes' Theorem; experimental evidence is Bayesian evidence. The Bayesian revolutionaries hold that when you perform an experiment and get evidence that "confirms" or "disconfirms" your theory, this confirmation and disconfirmation is governed by the Bayesian rules. For example, you have to take into account, not only whether your theory predicts the phenomenon, but whether other possible explanations also predict the phenomenon. Previously, the most popular philosophy of science was probably Karl Popper's falsificationism - this is the old philosophy that the Bayesian revolution is currently dethroning. Karl Popper's idea that theories can be definitely falsified, but never definitely confirmed, is yet another special case of the Bayesian rules; if $p(X|A) \sim 1$ - if the theory makes a definite prediction - then observing $\sim$X very strongly falsifies A. On the other hand, if $p(X|A) \sim 1$, and we observe X, this doesn't definitely confirm the theory; there might be some other condition B such that $p(X|B) \sim 1$, in which case observing X doesn't favor A over B. For observing X to definitely confirm A, we would have to know, not that $p(X|A) \sim 1$, but that $p(X|{\sim}A) \sim 0$, which is something that we can't know because we can't range over all possible alternative explanations. For example, when Einstein's theory of General Relativity toppled Newton's incredibly well-confirmed theory of gravity, it turned out that all of Newton's predictions were just a special case of Einstein's predictions.

You can even formalize Popper's philosophy mathematically. The likelihood ratio for X, $p(X|A)/p(X|{\sim}A)$, determines how much observing X slides the probability for A; the likelihood ratio is what says how strong X is as evidence. Well, in your theory A, you can predict X with probability 1, if you like; but you can't control the denominator of the likelihood ratio, $p(X|{\sim}A)$ - there will always be some alternative theories that also predict X, and while we go with the simplest theory that fits the current evidence, you may someday encounter some evidence that an alternative theory predicts but your theory does not. That's the hidden gotcha that toppled Newton's theory of gravity. So there's a limit on how much mileage you can get from successful predictions; there's a limit on how high the likelihood ratio goes for confirmatory evidence.

On the other hand, if you encounter some piece of evidence Y that is definitely not predicted by your theory, this is enormously strong evidence against your theory. If $p(Y|A)$ is infinitesimal, then the likelihood ratio will also be infinitesimal. For example, if $p(Y|A)$ is 0.0001%, and $p(Y|{\sim}A)$ is 1%, then the likelihood ratio $p(Y|A)/p(Y|{\sim}A)$ will be 1:10000. –40 decibels of evidence! Or flipping the likelihood ratio, if $p(Y|A)$ is very small, then $p(Y|{\sim}A)/p(Y|A)$ will be very large, meaning that observing Y greatly favors $\sim$A over A. Falsification is much stronger than confirmation. This is a consequence of the earlier point that very strong evidence is not the product of a very high probability that A leads to X, but the product of a very low probability that not-A could have led to X. This is the precise Bayesian rule that underlies the heuristic value of Popper's falsificationism.

Similarly, Popper's dictum that an idea must be falsifiable can be interpreted as a manifestation of the Bayesian conservation-of-probability rule; if a result X is positive evidence for the theory, then the result ~X would have disconfirmed the theory to some extent. If you try to interpret both X and ~X as "confirming" the theory, the Bayesian rules say this is impossible! To increase the probability of a theory you must expose it to tests that can potentially decrease its probability; this is not just a rule for detecting would-be cheaters in the social process of science, but a consequence of Bayesian probability theory. On the other hand, Popper's idea that there is only falsification and no such thing as confirmation turns out to be incorrect. Bayes' Theorem shows that falsification is very strong evidence compared to confirmation, but falsification is still probabilistic in nature; it is not governed by fundamentally different rules from confirmation, as Popper argued.

So we find that many phenomena in the cognitive sciences, plus the statistical methods used by scientists, plus the scientific method itself, are all turning out to be special cases of Bayes' Theorem. Hence the Bayesian revolution.

Fun Fact! Q. Are there any limits to the power of Bayes' Theorem?
A. According to legend, one who fully grasped Bayes' Theorem would gain the ability to create and physically enter an alternate universe using only off-the-shelf equipment and a short computer program. One who fully grasps Bayes' Theorem, yet remains in our universe to aid others, is known as a Bayesattva.

Bayes' Theorem:

$$p(A|X) = \frac{p(X|A)*p(A)}{p(X|A)*p(A) + p(X|\sim A)*p(\sim A)}$$

Why wait so long to introduce Bayes' Theorem, instead of just showing it at the beginning? Well... because I've tried that before; and what happens, in my experience, is that people get all tangled up in trying to apply Bayes' Theorem as a set of poorly grounded mental rules; instead of the Theorem helping, it becomes one more thing to juggle mentally, so that in addition to trying to remember how many women with breast cancer have positive mammographies, the reader is also trying to remember whether it's p(X|A) in the numerator or p(A|X), and whether a positive mammography result corresponds to A or X, and which side of p(X|A) is the implication, and what the terms are in the denominator, and so on. In this excruciatingly gentle introduction, I tried to show all the workings of Bayesian reasoning without ever introducing the explicit Theorem as something extra to memorize, hopefully reducing the number of factors the reader needed to mentally juggle.

Even if you happen to be one of the fortunate people who can easily grasp and apply abstract theorems, the mental-juggling problem is still something to bear in mind if you ever need to explain Bayesian reasoning to someone else.

If you do find yourself losing track, my advice is to forget Bayes' Theorem as an equation and think about the graph. p(A) and p(∼A) are at the top. p(X|A) and p(X|∼A) are the projection factors. p(X&A) and p(X&∼A) are at the bottom. And p(A|X) equals the proportion of p(X&A) within p(X&A)+p(X&∼A). The graph isn't shown here - but can you see it in your mind?

And if thinking about the graph doesn't work, I suggest forgetting about Bayes' Theorem entirely - just try to work out the specific problem in gizmos, hoses, and sparks, or whatever it is.

Having introduced Bayes' Theorem explicitly, we can explicitly discuss its components.

$$p(A|X) = \frac{p(X|A)*p(A)}{p(X|A)*p(A) + p(X|\sim A)*p(\sim A)}$$

We'll start with p(A|X). If you ever find yourself getting confused about what's A and what's X in Bayes' Theorem, start with p(A|X) on the left side of the equation; that's the simplest part to interpret. A is the thing we want to know about. X is how we're observing it; X is the evidence we're using to make inferences about A. Remember that for every expression p(Q|P), we want to know about the probability for Q given P, the degree to which P implies Q - a more sensible notation, which it is now too late to adopt, would be p(Q<-P).

p(Q|P) is closely related to p(Q&P), but they are not identical. Expressed as a probability or a fraction, p(Q&P) is the proportion of things that have property Q and property P within all things; i.e., the proportion of "women with breast cancer and a positive mammography" within the group of all women. If the total number of women is 10,000, and 80 women have breast cancer and a positive mammography, then p(Q&P) is 80/10,000 = 0.8%. You might say that the absolute quantity, 80, is being normalized to a probability relative to the group of all women. Or to make it clearer, suppose that there's a group of 641 women with breast cancer and a positive mammography within a total sample group of 89,031 women. 641 is the absolute quantity. If you pick out a random woman from the entire sample, then the probability you'll pick a woman with breast cancer and a positive mammography is p(Q&P), or 0.72% (in this example).

On the other hand, p(Q|P) is the proportion of things that have property Q and property P within all things that have P; i.e., the proportion of women with breast cancer and a positive mammography within the group of all women with positive mammographies. If there are 641 women with breast cancer and positive mammographies, 7915 women with positive mammographies, and 89,031 women, then p(Q&P) is the probability of getting one of those 641 women if you're picking at random from the entire group of 89,031, while p(Q|P) is the probability of getting one of those 641 women if you're picking at random from the smaller group of 7915.

In a sense, p(Q|P)really means p(Q&P|P), but specifying the extra P all the time would be redundant. You already know it has property P, so the property you're investigating is Q - even though you're looking at the size of group Q&P within group P, not the size of group Q within group P (which would be nonsense). This is what it means to take the property on the right-hand side as given; it means you know you're working only within the group of things that have property P. When you constrict your focus of attention to see only this smaller group, many other probabilities change. If you're taking P as given, then p(Q&P) equals just p(Q) - at least, relative to the group P. The old p(Q), the frequency of "things that have property Q within the entire sample", is revised to the new frequency of "things that have property Q within the subsample of things that have property P". If P is given, if P is our entire world, then looking for Q&P is the same as looking for just Q.

If you constrict your focus of attention to only the population of eggs that are painted blue, then suddenly "the probability that an egg contains a pearl" becomes a different number; this proportion is different for the population of blue eggs than the population of all eggs. The given, the property that constricts our focus of attention, is always on the right side of p(Q|P); the P becomes our world, the entire thing we see, and on the other side of the "given" P always has probability 1 - that is what it means to take P as given. So p(Q|P) means "If P has probability 1, what is the probability of Q?" or "If we constrict our attention to only things or events where P is true, what is the probability of Q?" Q, on the other side of the given, is not certain - its probability may be 10% or 90% or any other number. So when you use Bayes' Theorem, and you write the part on the left side as p(A|X) - how to update the probability of A after seeing X, the new probability of A given that we know X, the degree to which X implies A - you can tell that X is always the observation or the evidence, and A is the property being investigated, the thing you want to know about.

The right side of Bayes' Theorem is derived from the left side through these steps:

p(A|X) = p(A|X) p(A|X) = p(X&A)
p(X) p(A|X) = p(X&A)
p(X&A) + p(X&∼A) p(A|X) = p(X|A)*p(A)
p(X|A)*p(A) + p(X|∼A)*p(∼A)

The first step, p(A|X) to p(X&A)/p(X), may look like a tautology. The actual math performed is different, though. p(A|X) is a single number, the normalized probability or frequency of A within the subgroup X. p(X&A)/p(X) are usually the percentage frequencies of X&A and X within the entire sample, but the calculation also works if X&A and X are absolute numbers of people, events, or things. p(cancer|positive) is a single percentage/frequency/probability, always between 0 and 1. (positive&cancer)/(positive) can be measured either in probabilities, such as 0.008/0.103, or it might be expressed in groups of women, for example 194/2494. As long as both the numerator and denominator are measured in the same units, it should make no difference.

Going from p(X) in the denominator to p(X&A)+p(X&∼A) is a very straightforward step whose main purpose is as a stepping stone to the last equation. However, one common arithmetical mistake in Bayesian calculations is to divide p(X&A) by p(X&∼A), instead of dividing p(X&A) by [p(X&A) + p(X&∼A)]. For example, someone doing the breast cancer calculation tries to get the posterior probability by performing the math operation 80 / 950, instead of 80 / (80 + 950). I like to think of this as a rose-flowers error. Sometimes if you show young children a picture with eight roses and two tulips, they'll say that the picture contains more roses than flowers. (Technically, this would be called a class inclusion error.) You have to add the roses and the tulips to get the number of flowers, which you need to find the proportion of roses within the flowers. You can't find the proportion of roses in the tulips, or the proportion of tulips in the roses. When you look at the graph, the bottom bar consists of all the patients with positive results. That's what the doctor sees - a patient with a positive result. The question then becomes whether this is a healthy patient with a positive result, or a cancerous patient with a positive result. To figure the odds of that, you have to look at the proportion of cancerous patients with positive results within all patients who have positive results, because again, "a patient with a positive result" is what you actually see. You can't divide 80 by 950 because that would mean you were trying to find the proportion of cancerous patients with positive results within the group of healthy patients with positive results; it's like asking how many of the tulips are roses, instead of asking how many of the flowers are roses. Imagine using the same method to find the proportion of healthy patients. You would divide 950 by 80 and find that 1,187% of the patients were healthy. Or to be exact, you would find that 1,187% of cancerous patients with positive results were healthy patients with positive results.

The last step in deriving Bayes' Theorem is going from p(X&A) to p(X|A)*p(A), in both the numerator and the denominator, and from p(X&∼A) to p(X|∼A)*p(∼A), in the denominator.

Why? Well, one answer is because $p(X|A)$, $p(X|\sim A)$, and $p(A)$ correspond to the initial information given in all the story problems. But why were the story problems written that way?

Because in many cases, $p(X|A)$, $p(X|\sim A)$, and $p(A)$ are what we actually know; and this in turn happens because $p(X|A)$ and $p(X|\sim A)$ are often the quantities that directly describe causal relations, with the other quantities derived from them and $p(A)$ as statistical relations. For example, $p(X|A)$, the implication from A to X, where A is what we want to know and X is our way of observing it, corresponds to the implication from a woman having breast cancer to a positive mammography. This is not just a statistical implication but a direct causal relation; a woman gets a positive mammography because she has breast cancer. The mammography is designed to detect breast cancer, and it is a fact about the physical process of the mammography exam that it has an 80% probability of detecting breast cancer. As long as the design of the mammography machine stays constant, $p(X|A)$ will stay at 80%, even if $p(A)$ changes - for example, if we screen a group of woman with other risk factors, so that the prior frequency of women with breast cancer is 10% instead of 1%. In this case, $p(X\&A)$ will change along with $p(A)$, and so will $p(X)$, $p(A|X)$, and so on; but $p(X|A)$ stays at 80%, because that's a fact about the mammography exam itself. (Though you do need to test this statement before relying on it; it's possible that the mammography exam might work better on some forms of breast cancer than others.) $p(X|A)$ is one of the simple facts from which complex facts like $p(X\&A)$ are constructed; $p(X|A)$ is an elementary causal relation within a complex system, and it has a direct physical interpretation. This is why Bayes' Theorem has the form it does; it's not for solving math brainteasers, but for reasoning about the physical universe.

Once the derivation is finished, all the implications on the right side of the equation are of the form $p(X|A)$ or $p(X|\sim A)$, while the implication on the left side is $p(A|X)$. As long as you remember this and you get the rest of the equation right, it shouldn't matter whether you happened to start out with $p(A|X)$ or $p(X|A)$ on the left side of the equation, as long as the rules are applied consistently - if you started out with the direction of implication $p(X|A)$ on the left side of the equation, you would need to end up with the direction $p(A|X)$ on the right side of the equation. This, of course, is just changing the variable labels; the point is to remember the symmetry, in order to remember the structure of Bayes' Theorem.

The symmetry arises because the elementary causal relations are generally implications from facts to observations, i.e., from breast cancer to positive mammography. The elementary steps in reasoning are generally implications from observations to facts, i.e., from a positive mammography to breast cancer. The left side of Bayes' Theorem is an elementary inferential step from the observation of positive mammography to the conclusion of an increased probability of breast cancer. Implication is written right-to-left, so we write $p(\text{cancer}|\text{positive})$ on the left side of the equation. The right side of Bayes' Theorem describes the elementary causal steps - for example, from breast cancer to a positive mammography - and so the implications on the right side of Bayes' Theorem take the form p(positive|cancer) or p(positive|~cancer).

And that's Bayes' Theorem. Rational inference on the left end, physical causality on the right end; an equation with mind on one side and reality on the other. Remember how the scientific method turned out to be a special case of Bayes' Theorem? If you wanted to put it poetically, you could say that Bayes' Theorem binds reasoning into the physical universe. Okay, we're done.