SCATTERING CONVOLUTIONAL HIDDEN MARKOV TREE

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ABSTRACT

A Scattering convolutional hidden Markov tree proposes a new inference mechanism for high-dimensional signals by combining the interesting signal representation created by the scattering transform to a powerful probabilistic graphical model. A wavelet scattering network computes a signal translation invariant and stable to deformations representation that still preserves the informative content of the signal. Such properties are acquired by cascading wavelet transform convolutions with non-linear modulus and averaging operators. The network's structure and its distributions are described using a Hidden Markov Tree. This yield a generative model for high-dimensional inference. It offers a mean for performing several inference tasks among which are predictions. The scattering convolutional hidden Markov tree displays promising results on classification tasks of complex images in the difficult case were the number of training examples is extremely limited.

***Index Terms*—** Scattering network, Hidden Markov Model, Classification, Deep network

1 Introduction

The standard approach to classify high dimensional signals can be expressed as a two steps procedure. First the data are projected in a feature space where the task at hand is simplified. Then prediction is done using a simple predictor in this new representational space. The mapping can either be hand-built —e.g. Fourier transform, wavelet transform— or learned. In the last decade methods for learning the projection have drastically improved under the impulsion of the so called deep learning. Deep neural networks (sometime enriched by convolutional architecture) have been able to learn very effective representations for a given dataset and a given task [1] [2] [3]. Such method have achieved state of the art on many standard problems [4] [5] as well as real world applications [6].

However deep learning methods are only efficient when having access to a vast quantity of training examples [7]. But in some cases such as —TODO : examples — training examples are expensive to collect and learning has to be performed on smaller datasets. In that case using a fix hand crafted set of filters seems to be one of the best solution [8]. Recently Mallat described the scattering transform [9] a fixed bank of wavelet filters used to generate data representation in a convolutional neural networks like architecture. This representational method associated to a support vector machine classifier achieved close to state of the art performances of some standard datasets [10]. Furthermore this method can be accurately applied to relatively smaller datasets [11].

When an extremely low number of training examples are available —one-shot learning [12]— generative classification methods have the upper hand [13]. Modelling a a signal representation using the generative probabilistic graphical models have been successfully done for various wavelet transform [14] [15]. In those work Hidden Markov trees are used to model the wavelet coefficients distribution.

We proposes a method combining the recently proposed deterministic analytically tractable transformation inspired by deep convolutional to a probabilistic graphical model in order to create a powerful probabilistic tool to handle high dimensional prediction problems. In a similar fashion to the work done by Crouse on wavelet trees, we propose to describe Mallat's scattering convolutional scattering transform using a hidden Markov tree. Doing so we develop a new framework to model high-dimensional inputs which, in contrast to passing the raw scattering coefficients into a classifier, captures dependencies between different layers in a generative probabilistic model and, unlike Crouse's HMT, our proposed framework accommodates invariances to deformations. Moreover as opposed to the commonly used simple classification method, once trained our model can tackle prediction problems but also other inference tasks —e.g. generation, sensitivity analysis... — and can also outperform SVMs when only a very low number of training examples are available.

The remainder of this paper introduces the scattering convolutional hidden Markov tree and is organised as follow. In section 2 we review the scattering transform and some of its properties. Section 3 introduces the proposed Scattering Hidden Markov Tree (SCHMT). Section 4 we perform classification on a selection of standard datasets restricted to only a few training samples. We draw conclusions in Section **??**

J.-B. Regli is funded by a Dstl/UCL Impact studentship. J. D. B. Nelson is partially supported by grants from the Dstl and Innovate UK/EPSRC.

2 Scattering networks

Scattering convolutional networks (SCNs) [16] are Convolutional Neural Networks (CNNs) [3] using a fixed filter bank of wavelets. Those filters can be hand-crafted to yield descriptors with the desired invariances [9]. For image classification tasks, one is interested in descriptors that are —at least— stable to deformations and invariant to translations. Note that SCNs producing more complexes set of invariances exist but on the remainder of this paper we consider only on descriptors with the previously mentioned properties.

2.1 Scattering transform

Wavelets are localized functions stable to deformations. They are thus well adapted to construct descriptor that would also be translation invariant. A two-dimensional spatial wavelet transform W is obtained by scaling by 2^j and rotating by r_θ a mother wavelet ψ,

$$\psi_\lambda(u) = \psi_{j,\theta}(u) = 2^{-2j}\psi(2^{-j}r_\theta p) \tag{1}$$

In the remainder of this paper we restrict to Morlet wavelet transforms defined on $\Lambda = G \times [\![0, J]\!]$ where G is a finite group of rotations of cardinal L and where the wavelet is taken at scale J,

$$W_J\mathbf{x} = \{\mathbf{x} * \phi_J(u); \mathbf{x} * \psi_\lambda(u)\}_{p\in\mathbb{R}^2,\lambda\in\Lambda} \tag{2}$$

While the averaging part ϕ_J of the wavelet transform is invariant to translations, the high frequency part ψ_λ is covariant to them [9]. Invariance within a limited range inferior 2^J can be achieved by averaging the positive envelope with a smooth window,

$$S_J[\lambda]\mathbf{x}(u) = |\mathbf{x} * \psi_\lambda| * \phi_J(u) \tag{3}$$

Such non-linearised averaged wavelet coefficients are used in various form in computer vision (SIFT [17], DAISY [18]), but the scattering transform proposes a new non-linearity as well as a layer based architecture.

2.2 Scattering convolutional network

While providing local translation invariance, the averaging convolution introduced in 3 also removes the spatial variability of the wavelet transform. SCNs cascade this wavelet modulus operator to recover the lost information and compute progressively more invariant descriptors. Let combine the wavelet transform and modulus operations into a single wavelet modulus operator,

$$\mathcal{U}_J\mathbf{x} = \{S[\emptyset]\mathbf{x}; U[\lambda]\mathbf{x}\}_{\lambda\in\Lambda_J} = \{\mathbf{x}*\phi_J; |\mathbf{x} * \psi_\lambda|\}_{\lambda\in\Lambda_J}, \tag{4}$$

A scattering transform can be interpreted as a CNN [19] illustrated in Figure 1 which propagates a signal $\mathbf{x}$ across multiple layers of the network and which outputs at each layer m the scattering invariant coefficients $S[p_m]\mathbf{x}$ where $p_m = (\lambda_1 \dots \lambda_m)$ is a path of m orientations and scales.

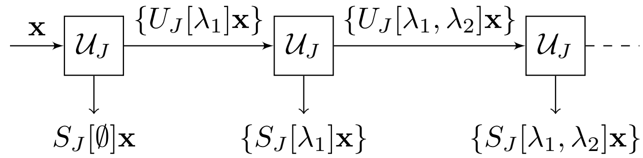

Fig. 1. Scattering networks can be seen as neural networks iterating over wavelet modulus operators $\mathcal{U}_j$. Each layer m outputs the averaged invariants $S[p_m]\mathbf{x}$ and covariant coefficients $U[p_{m+1}]\mathbf{x}$.

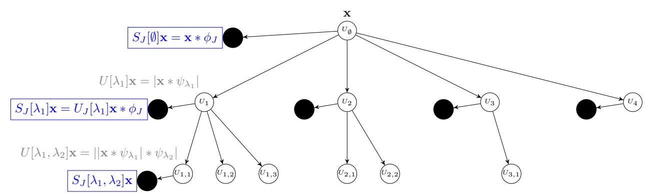

Fig. 2. Frequency decreasing scattering convolution network with $J = 4$, $L = 1$ and $M = 2$. A node i at scale j_i generates $(j_i - 1) \times L$ nodes.

The scattering energy is mainly concentrated along frequency decreasing paths, i.e. for which $|\lambda_{k+1}| \leq |\lambda_k|$ [9]. The energy contained in the other paths is negligible and thus for applications only frequency decreasing paths are considered. Moreover there exist a path length $M > 0$ after which all longer paths can be neglected. For signal processing applications, this decay appears to be exponential. And for classification applications, paths of length $M = 3$, i.e. two convolutions, provides the most interesting results [20], [10].

This restrictions yield an easier parametrization of a scattering network. Indeed its now completely defined by the mother wavelet ϕ, the maximum path length considered M, the finest scale level considered J and the number of orientation considered L.

Hence for a given set of parameter (ψ, M, J, L), let $ST_{(\psi,M,J,L)}(\mathbf{x})$ denotes the unique frequency decreasing windowed scattering convolutional network with those parameters evaluated for signal $\mathbf{x}$. Each node i of this network generates a -possibly empty- set of of nodes of size $(j_i - 1) \times L$ where j_i is the scale of node i and L is the number of orientations considered and it has the architecture displayed by Figure 2.

2.3 Scattering convolutional classifier

In the original framework [21], the scattering network $ST_{(\psi,M,J,L)}(.)$ is used for classification task using a SVM classifier on the outputs of the network. Performance can be slightly improved by adding a feature selection step performing PCA on the scattering coefficients and keeping only the most informative ones. This classification framework provides results comparable with the state of the art on several datasets [10].

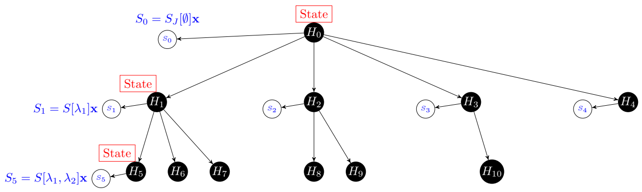

Fig. 3. Scattering convolutional hidden Markov tree.

3 The Scattering hidden Markov tree

State of the art performance can be achieved using SCNs associated to SVMs. However this approach is not adapted to very small training sets or to deliver a probabilistic output. To overcome those limitations we propose an adaptation of Crouse [14] and Durand [22] wavelet hidden Markov trees to the non-regular non-homogeneous tree structure of SCNs.

3.1 Hidden Markov tree model

The HMT models the marginal distribution of each real ST coefficient S_i as a Gaussian mixture. To each S_i, we associate a discrete hidden state H_i that takes on values in $[\![1, K]\!]$ with probability mass function (pmf) $P(H_i)$. Conditioned on $H_i = k$, S_i is Gaussian with mean $\mu_{i,k}$ and variance $\sigma_{i,k}$. Thus, its overall marginal PDF is given by $P(w_i) = \sum_{k=1}^{K} P(H_i = m)P(S_i|H_i = k)$ with $P(S_i|H_i = k) \sim \mathcal{N}(\mu_{i,k}, \sigma_{i,k})$. While each scattering coefficient S_i is conditionally gaussian given its state variable H_i, overall it has a non-Gaussian density. Finally the probability for the hidden node H_i to be in a state k given its father's state g is characterized by a transition probability such that $\epsilon_i^{(gk)} = P(H_i = k|H_{\rho(i)} = g)$. This yields $P(H_i = k) = \sum_{g=1}^{K} \epsilon_i^{(gk)} P(H_{\rho(i)} = g)$.

Such a model is pictured in Figure 3 and for a given scattering architecture —i.e. fixed M, J and L— the SCHMT model is fully parametrized by,

$$\Theta = \big(\pi_0, \{\epsilon_i, \{\theta_{k,i}\}_{k\in[\![1,K]\!]}\}_{i\in\mathcal{T}}\big). \tag{5}$$

This model implies two assumptions on the scattering transform. First — K-populations— that a signal's scattering coefficients can be described by K clusters. This is a common assumptions for standard wavelets [15] and hence it can be extended to the scattering transform. The SCHMT also assumed —persistence— that the informative character of a coefficients is propagated across layers. This assumption is sound since scattering coefficients are highly correlated [19].

3.2 Learning the tree parameters

The SCHMT is trained using the smoothed version of the Expectation-Maximization (EM) algorithm [?] for hidden Markov trees proposed by [22] and adapted to non-homogeneous and non-binary trees.

Let $\bar{\mathcal{S}}_i = \bar{s}_i$ be the observed sub-tree rooted at node i. By convention $\bar{\mathcal{S}}_0$ denotes the entire observed tree. The smoothed version of the E-step requires the computation of the conditional probability distributions $\xi_i(k) = P(H_i = k|\bar{\mathcal{S}}_i = \bar{s}_i)$ (smoothed probability) and $P(H_i = k, H_{\rho(i)} = g|\bar{\mathcal{S}}_i = \bar{s}_i)$ for each node $i \in \mathcal{T}$ and states k and g. This can be achieved through an upward-downward recursion displayed in Algorithm 1 and 2.

// Initialization :
for *All the nodes i of the tree $\mathcal{T}$* **do**
  $P_{\theta_{k,i}}(s_i) = \mathcal{N}(s_i|\mu_{k,i}, \sigma_{k,i})$
end
for *All the leaves i of the tree $\mathcal{T}$* **do**
  $\beta_i(k) = \frac{P_{\theta_{k,i}}(s_i)P(H_i=k)}{\sum_{g=1}^{K} P_{\theta_{g,i}}(s_i)P(H_i=g)}$
  $\beta_{i,\rho(i)}(k) = \sum_{g=1}^{K} \frac{\beta_i(g)\epsilon_i^{(kg)}}{P(H_i=g)}.P(H_{\rho(i)} = k)$
  $l_i = 0$
end
// Induction :
for *All non-leaf nodes i of the tree $\mathcal{T}$ (Bottom-up)* **do**
  $M_i = \sum_{k=1}^{K} P_{\theta_{k,i}}(s_i) \prod_{j\in c(i)} \frac{\beta_{j,i}(k)}{P(H_i=k)^{n_i-1}}$
  $l_i = \log(M_i) + \sum_{j\in c(i)} l_j$
  $\beta_i(k) = \frac{P_{\theta_{k,i}(s_i)} \prod_{j\in c(i)}(\beta_{j,i}(k))}{P(H_i=k)^{n_i-1}M_i}$
  for *All the children nodes j of node i* **do**
    $\beta_{i\backslash c(i)}(k) = \frac{\beta_i(k)}{\beta_{i,j}(k)}$
  end
  $\beta_{i,\rho(i)}(k) = \sum_{g=1}^{K} \frac{\beta_i(g)\epsilon_i^{(kg)}}{P(H_i=g)}.P(H_{\rho(i)} = k)$
end

Algorithm 1: Smoothed upward algorithm.

// Initialization :
$\alpha_0(k) = 1$
// Induction :
for *All nodes i of the tree $\mathcal{T}\backslash\{0\}$ (Top-Down)* **do**
  $\alpha_i(k) = \frac{1}{P(H_i=k)} \sum_{g=1}^{K} \alpha_{\rho(i)}(g)\epsilon_i^{(gk)}\beta_{\rho(i)\backslash i}(g)P(H_{\rho(i)} = g)$
end

Algorithm 2: Smoothed downward algorithm.

// Initialization :
$\pi_0(k) = \frac{1}{N}\sum_{n=1}^{N} P(H_0^n = m|s_0^n, \Theta^l)$
// Induction :
for *All nodes i of the tree $\mathcal{T}\backslash\{0\}$* **do**
  $P(H_i = k) = \frac{1}{N}\sum_{n=1}^{N} P(H_i^n = k|\bar{s}_0^n, \Theta^l)$,
  $\epsilon_i^{gk} = \frac{\sum_{n=1}^{N} P(H_i^n=k, H_{\rho(i)}^n=g|\bar{s}_0^n, \Theta^l)}{NP(H_{\rho(i)}=k)}$,
  $\mu_{k,i} = \frac{\sum_{n=1}^{N} s_i^n P(H_i^n=k|\bar{s}_0^n, \Theta^l)}{NP(H_i=k)}$,
  $\sigma_{k,i}^2 = \frac{\sum_{n=1}^{N}(s_i^n-\mu_{k,i})^2 P(H_i^n=k|\bar{s}_0^n, \Theta^l)}{NP(H_i=k)}$.
end

Algorithm 3: M-step of the EM algorithm.

3.3 MAP classification

Let Θ_c now be a set of parameters for an SCHMT $\mathcal{T}$ learned on a training set $\{\bar{S}_{0,c}^n\}_{n\in[\![1,N]\!]} = \{ST_{(\psi,J,M,L)}(\mathbf{x}_c^n)\}_{n\in[\![1,N]\!]}$ composed of the scattering representations of N realizations of a signal of class c . Let also $\mathbf{x}^{new}$ be another realization of

Fig. 4. Average and best classification scores on MNIST trained on $N = 2, 5, 10, 20$ training examples per class.

this signal, not used for training and $\mathcal{T}^{new}$ be the instance of the SCHMT generated by this realization.

In this context the MAP algorithm [23] aims at finding the optimal hidden tree $\hat{\bar{h}}_0^{new} = (\hat{h}_0^{new} \dots \hat{h}_{I-1}^{new})$ maximizing the probability of this sequence given the model's parameters $P(\bar{\mathcal{H}}_0 = \hat{\bar{h}}_0^{new} | \mathcal{T}^{new}, \Theta_c)$. The MAP framework also provides $\hat{P}$ the value of this maximum.

The MAP algorithm can be used in a multi-class classification problem by training an SCHMT model per class and then when presented with a new realization $\mathbf{x}^{new}$ comparing the probability of the MAP hidden tree provided by each model

4 Classification results

We compare the performance of SCHMT to those of a SCN combined to an SVM (SCN+SVM) on restrictions to small number of training examples of two standard datasets.

4.1 MNIST

We first test SCHMTs on the digit classification dataset MNIST [24]. SCHMT and SCN+SVM are both trained on a limited number of training examples per class of MNIST and tested on 200 test samples (20 per class). The two methods use a scattering transform with $M = 3$ orders, $J = 5$ scales, $L = 3$ orientations and a Morlet wavelet. The hidden Markov tree has $K = 2$ states and is using a mixture of Gaussian to describe the relationship between the scattering coefficients and the hidden states. The SVM parameters are selected by cross-validation. This experiment is run 100 times per number of training samples and results are displayed in Figure 4

When the number of training example is really limited —i.e. 2 or 5— are slightly higher for SCHMT compare to SCN+SVM. However the EM algorithm performances are undermined by convergence to local minima issues [25]. When convergence occurs correctly SCHMTs reach much better performances than the best SVMs — 80% compared to 54% for 5 training examples. Finally, as expected, when the number of training samples grow large enough —i.e. 10 and onward— for the SVMs, SCN+SVMs reach both better maximum and average score.

To assess the generalisation quality even further, we test the best SCHMT and SCN+SVM models trained on 5 images per class on the full 10000 test samples.

This experiment confirm the superiority of generative model for limited number of training points. It also highly some weakness of the SCHMTs as sometime convergence is problematic and the problem seems to get stronger when the number of training samples increases. However when convergence occurs correctly SCHMT provides great classification score for low number of training examples.

4.2 KTH Texture

Further experiments are run on the KTH-TIPS texture dataset [26]. This time

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5 Conclusion

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