Analytic Solution of Free Space Optical Beam Steering Using Risley Prisms

Yaguang Yang

***Abstract*—This paper provides some mathematical formulae for Risley beam steer control system that is used for free space optical interconnection system. Given the prisms' orientations, the analytic formulae can be used to calculate the pointing position. For the inverse problem, i.e., given the desired pointing position, the required prisms' orientations can be obtained efficiently by applying any numerical method for systems of nonlinear equations related to the formulae. Unlike some existing methods, these formulae are not first-order approximation but an accurate closed analytical form. It does not need a good initial guess on the motor positions required by some other methods. It also does not need a feedback loop to get the correct pointing position but it can be used in feedback implementation. This feature is especially important for systems that need to slew the beam to the desired position quickly and accurately.**

***Index Terms*—Beam steering, optical interconnection, Risley prisms.**

I. Introduction

RISLEY prisms have many applications. Recently, Risley beam steerers have been proved [1]–[5] to be a simple and cheap way of aligning free space interconnects. In this type of application, one of the most important steps is to find the relation between the pointing position and the corresponding prisms' orientations (or motor positions). There are two basic problems associated with this relationship. First, given two prisms' orientations, what is the pointing position of the system? This has turned out to be an easier problem. A more difficult problem is: given the pointing position, what are the required orientations of the two prisms that will guide the light beam to the given pointing position? The second problem is actually more important to the beam steering system and is directly related to the first problem. If there is an analytic solution to the first problem, theoretically, one can use any method for systems of nonlinear equations to solve the second problem iteratively. However, in practice, any method may fail to find some real solution from some initial guess point for some systems of nonlinear equations. Therefore, finding an effective (avoid possible failure of finding the solution) and efficient (real time control requirement) method suitable for this problem is a very important task.

Boisset *et al.* [1] investigated a first-order paraxial method. They used approximated solution for the first problem and provided an iterative method to solve the second problem (to find motor positions for a desired pointing position based on the measurement information in a feedback loop). This method needs a good initial guess. It cannot be used when the measurement does not exist. In many applications, the optical receiver field of view window for the measurement is small, the measurement information initially may not be available. Therefore, an extensive "hunting" process was suggested in [1].

Degnan [4] described a different way to use Risley prism in NASA's SLR2000 photo-counting satellite laser ranging system. Though in the point ahead application, this method (based on paraxial ray matrices developed in [6]) is also a first-order approximation method, it does not need an iterative algorithm to find prisms' orientations for a given pointing position, some explicit formulae are used to calculate prisms' orientations (therefore, this method does not need an initial guess).

However, both methods developed in [1] and [4] use the first-order approximation to describe the relation between pointing position and prisms' orientations, and they explicitly or implicitly assumed [4] that the beam deflection is in the direction of the thickest part of the prism and has a constant magnitude $\delta = (n-1)\theta$ (where δ is the deflection angle, n is the index of the prism, and θ is the opening angle of the prism). These assumptions are inappropriate for tilted and decentered systems [7, p. 52] like prisms; their methods work *only* when the approximation is good enough.

Lacoursiere *et al.* [3] also realized the challenging when the Risley pointing system is designed for large deviations (from 0 to about 25 degrees). But they did not investigate the pointing formula in their paper. As a matter of truth, for a pair of silicon prisms, the pointing system can achieve much larger angular deviations (depending on the opening angles and refraction indices of the prisms). For example, for a pair of silicon prisms with 9 degrees opening angle, the deviation can be as large as about 60 degrees.

Some commercial optical design software can be used to solve the first problem, for example, ZEMAX [7]. However, this kind of software cannot delt with the second problem because they use numerical iterative method for the first problem (there is no analytic relation between pointing position and the orientations of the Risley prisms).

Therefore, there is a need to investigate some analytic formulae on the relation between the pointing position and the orientations of the Risley prisms.

In this paper, we will present an analytical relation between the pointing position and the orientations of the prisms, i.e., we will provide analytical pointing position functions in terms of orientations of the prisms. We will also propose methods for the inverse problem, i.e., for any given pointing position, the

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The author is with Orbital Sciences Corporation, Dulles, VA 21090 USA (e-mail: yaguang.yang@att.net).

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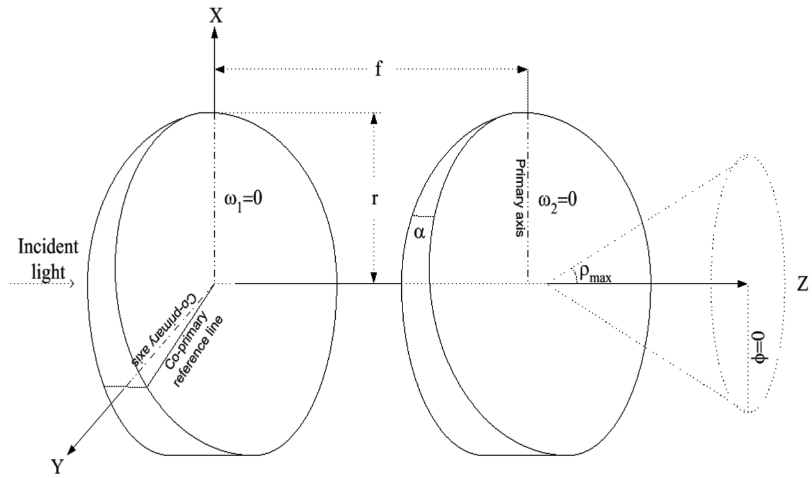

Fig. 1. Coordinator system used in this paper.

methods calculate the desired prisms' orientations (motor positions). These methods need neither a good initial guess, nor the measurement information for dynamic corrections. But it can be used when measurement information is available.

II. Relation Between Pointing Position and Prisms' Orientations

In this section, we will present the analytic formulae to describe the relation between any given pointing position and the corresponding prisms' orientations.

Fig. 1 is the diagram of the Cartesian coordinate system we will use for the Risley prism optical beam steering system. The Cartesian coordinate system is selected as follows. The geometry center of the first prism is at the origin (0, 0, 0) of the Cartesian coordinate system. The circular plane of the first prism with the center at the geometrical center is on the $x-y$ plane. The second prism is parallel to the first prism with its geometrical center on z axis. For the sake of easy reference, the zero orientation of both prisms is as described with the opening angle α pointing up. We refer the line passing through the center and connecting the edges on the opening angle to the point on the other side of circumference as to the primary axis. We refer the corresponding lines on the prism surfaces as to the primary reference lines. If the first prism is at zero orientation, its primary axis is vertical and coincides with the x axis. We refer the line on the circular plane as to co-primary axis if it is orthogonal to the primary axis and passes through the geometrical center. Finally, we refer the lines on the prism surfaces corresponding to the co-primary axis as to the co-primary reference lines.

Let f be the distance between the geometrical centers of the two prisms, then the geometrical center of the second prism is at (0, 0, f). We assume that the input optical signal always comes along z axis with the direction from negative side to positive side. Both prisms can be rotated independently around z axis. The positive rotational angle will follows the right hand rule around z axis.

The final pointing position will be defined by two angles (ϕ, ρ) in a cone with the vertex on the second surface of the second prism. $\phi \in [0, 2\pi)$ is the angle between the $-x$ axis and the projection of the deflection ray of the second prism on $x-y$ plane. $\phi = 0$ if the projection is on the $-x$ axis. The positive angle follows the right hand rule around z axis. $\rho \le \rho_{max}$ is the angle between the z axis and the deflection ray of the second prism. Our goal is to find a system of nonlinear equations $\phi = g_1(\omega_1, \omega_2)$ and $\rho = g_2(\omega_1, \omega_2)$ that depend on control variables ω_1, ω_2 (the prisms' rotational angles), and design parameters n and α.

To calculate the pointing position (ϕ, ρ), we will use some known result [8] to simplify the derivation. Given the wedge angle α, the refraction index of the air n_a, and the refraction index of the prism n, after optical signal passing through the first prism, it will generate an angle deviation (deflection) δ from the original path (along z axis). Using the fact that the incident light beam in the first prism is along the z axis with the incident angle $\alpha/2$, this angle deviation can be calculated in a similar way described in [8].

Lemma 2.1: The deflection angle δ between the light ray coming out of the first prism and the original path (along z axis) is given by

$$\delta = -\frac{\alpha}{2} + \sin^{-1}\left(\sin(\alpha)\sqrt{\left(\frac{n}{n_a}\right)^2 - \sin^2\left(\frac{\alpha}{2}\right)} + \sin\left(\frac{\alpha}{2}\right)\cos(\alpha)\right). \quad (1)$$

It is worthwhile to note that in this case, δ is a constant because α, n, and n_a are constants after the prism system is designed. In the rest of the paper, all vectors are presented in the coordinate frame depicted in Fig. 1. Now we will present our main result of this paper.

Theorem 2.1: Let ω_1 and ω_2 be the rotational angles of prism 1 and prism 2, respectively. The final pointing position (ϕ, ρ) can be calculated by tracing the light ray using the following steps.

1) The normal vectors of the input/output surfaces of the second prism are given by

$$\boldsymbol{\nu_3} = \left(\cos(\omega_2)\sin\left(\frac{\alpha}{2}\right), \sin(\omega_2)\sin\left(\frac{\alpha}{2}\right), -\cos\left(\frac{\alpha}{2}\right)\right)^{\mathrm{T}}$$
$$\boldsymbol{\nu_4} = \left(\cos(\omega_2)\sin\left(\frac{\alpha}{2}\right), \sin(\omega_2)\sin\left(\frac{\alpha}{2}\right), \cos\left(\frac{\alpha}{2}\right)\right)^{\mathrm{T}}. \quad (2)$$

2) The deflection ray coming out of the first prism with deflection angle δ can be represented as

$$\mathbf{R_1} = (-\cos(\omega_1)\tan(\delta), -\sin(\omega_1)\tan(\delta), 1). \tag{3}$$

3) The projection of $\mathbf{R_1}$ onto the input surface of the second prism can be calculated by (4), shown at the bottom of the page.
4) The incident angle of $\mathbf{R_1}$ into the second prism is given by (5), shown at the bottom of the page.
5) The refracted angle of the first surface of the second prism is given by

$$\lambda_2 = \sin^{-1}\left(\frac{\sin(\lambda_1)}{n/n_a}\right). \tag{6}$$

6) The incident ray of the second surface of the second prism $\mathbf{R_2}$ is given by

$$\mathbf{R_2} = \frac{\cos(\lambda_2)\cos(\delta)}{\cos(\lambda_1)}\mathbf{R_1} + \frac{\cos(\lambda_1-\lambda_2) - \cos(\lambda_2)/\cos(\lambda_1)}{\cos(\delta)\langle \mathbf{R_1}, \mathbf{P_1}\rangle}\mathbf{P_1}. \tag{7}$$

7) The projection of $\mathbf{R_2}$ on the second surface of the second prism is given by

$$\mathbf{P_2} = \left(I - \boldsymbol{\nu}_4\boldsymbol{\nu}_4^{\mathrm{T}}\right)\left(\frac{\cos(\lambda_2)\cos(\delta)}{\cos(\lambda_1)}\mathbf{R_1} + \frac{\cos(\lambda_1-\lambda_2) - \cos(\lambda_2)/\cos(\lambda_1)}{\cos(\delta)\langle \mathbf{R_1}, \mathbf{P_1}\rangle}\mathbf{P_1}\right). \tag{8}$$

8) The incident angle of $\mathbf{R_2}$ onto the second surface of the second prism is given by

$$\lambda_3 = \frac{\pi}{2} - \cos^{-1}\left(\frac{\langle \mathbf{P_2}, \mathbf{R_2}\rangle}{||\mathbf{P_2}||}\right). \tag{9}$$

9) The refraction angle from the second surface of the second prism is given by

$$\lambda_4 = \sin^{-1}\left(\frac{n}{n_a}\sin(\lambda_3)\right). \tag{10}$$

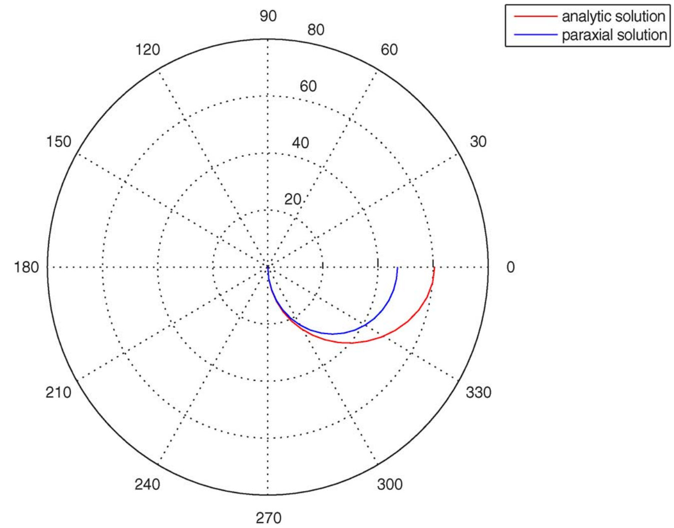

Fig. 2. Comparison of calculated pointing positions using zemax and first-order paraxial method.

10) The light path coming out of the second prism is given by

$$\mathbf{R_3} = \frac{\cos(\lambda_4)}{\cos(\lambda_3)}\mathbf{R_2} + \frac{(\cos(\lambda_4-\lambda_3) - \cos(\lambda_4)/\cos(\lambda_3))}{\langle \mathbf{R_2}, \mathbf{P_2}\rangle}\mathbf{P_2}. \tag{11}$$

11) Finally, let I be a three by three identity matrix, $\mathbf{e_1}$ and $\mathbf{e_3}$ be its first column and the third column. Let

$$\mathbf{R_z} = \left(I - \mathbf{e_3}\mathbf{e_3^{\mathrm{T}}}\right)\mathbf{R_3}. \tag{12}$$

We have

$$\phi = \begin{cases} \cos^{-1}\left(\frac{\mathbf{e_1^{\mathrm{T}}R_3} - \mathbf{e_1^{\mathrm{T}}e_3e_3^{\mathrm{T}}R_3}}{||\mathbf{R_z}||}\right), & \text{for } \mathbf{R_z}(2) \geq 0 \\ 2\pi - \cos^{-1}\left(\frac{\mathbf{e_1^{\mathrm{T}}R_3} - \mathbf{e_1^{\mathrm{T}}e_3e_3^{\mathrm{T}}R_3}}{||\mathbf{R_z}||}\right), & \text{for } \mathbf{R_z}(2) < 0 \end{cases} \tag{13}$$

and

$$\rho = \cos^{-1}\left(\langle \mathbf{e_3}, \mathbf{R_3}\rangle\right). \tag{14}$$

$$\begin{aligned} \mathbf{P_1} &= \mathbf{R_1} - \boldsymbol{\nu}_3\left(-\cos\left(\frac{\alpha}{2}\right) - \tan(\delta)\sin\left(\frac{\alpha}{2}\right)\cos(\omega_1-\omega_2)\right) \\ &= \begin{pmatrix} \tan(\delta)\left(-\cos(\omega_1) + \cos(\omega_2)\sin^2\left(\frac{\alpha}{2}\right)\cos(\omega_1-\omega_2)\right) + \cos(\omega_2)\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right) \\ \tan(\delta)\left(-\sin(\omega_1) + \sin(\omega_2)\sin^2\left(\frac{\alpha}{2}\right)\cos(\omega_1-\omega_2)\right) + \sin(\omega_2)\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right) \\ \sin^2\left(\frac{\alpha}{2}\right) - \tan(\delta)\cos(\omega_1-\omega_2)\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right) \end{pmatrix} \end{aligned} \tag{4}$$

$$\lambda_1 = \frac{\pi}{2} - \cos^{-1}\left(\frac{\left(\sin^2\left(\frac{\alpha}{2}\right) - 2\tan(\delta)\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)\cos(\omega_1-\omega_2) + \tan^2(\delta)\left(-1 + \sin^2\left(\frac{\alpha}{2}\right)\cos^2(\omega_1-\omega_2)\right)\right)\cos(\delta)}{\sin^2\left(\frac{\alpha}{2}\right) + \tan^2(\delta)\left(1 - \sin^2\left(\frac{\alpha}{2}\right)\cos^2(\omega_1-\omega_2)\right) - 2\tan(\delta)\cos(\omega_1-\omega_2)\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)}\right) \tag{5}$$

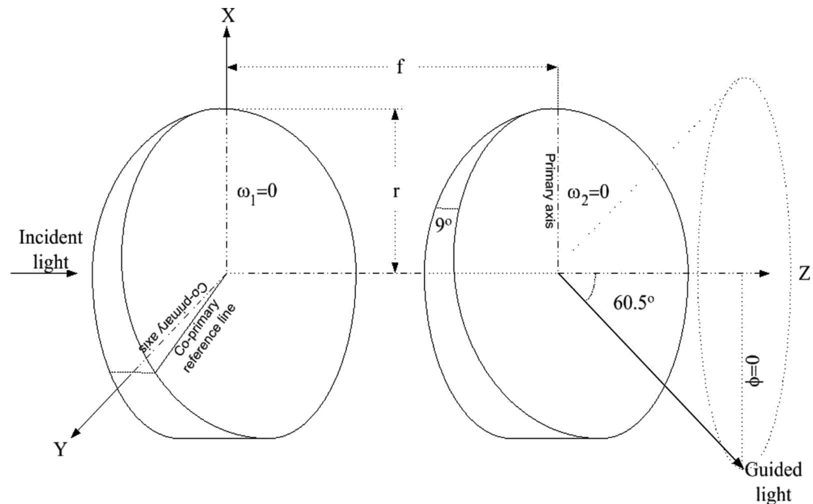

Fig. 3. Guided light for $\omega_1 = \omega_2 = 0$ degree.

Simulation of the new formula has been conducted and the result is compared with the result obtained by the first-order paraxial approximation (cf. Fig. 2). The result is shown in Fig. 2, where the polar coordinate is used. $\phi \in [0, 360]$ is the pointing angle that rotates on $x-y$ plane. ρ is the angle between the input ray (z axis) and the output ray. In the simulation, $\omega_1 = 0$ and ω_2 varies from 0 to 180 degrees with increment of 10 degrees. For each combination of the orientations, there is a point (ϕ, ρ) (pointing position) in the polar coordinate system. Connecting all this position points generates a curve that starts at $(\phi, \rho) = (0, 60.5)$ with $\omega_1 = \omega_2 = 0$ degree, and ends at $(\phi, \rho) = (0, 0)$. For a special case of $\omega_1 = \omega_2 = 0$ degree, the guided light is shown in Fig. 3. If the second prism rotates an entire circle from 0 to 360, the pointing position also rotates an entire loop clock-wise on the polar coordinate system around the center point $(\phi, \rho) = (0, 30.25)$. It starts at $(\phi, \rho) = (0, 60.5)$, passes $(\phi, \rho) = (0, 0)$, and then comes back at $(\phi, \rho) = (0, 60.5)$. For each combination of the prisms, fixing the relative positions of the two prisms but rotating the two prisms an entire circle altogether, the pointing position will rotate an entire circle around the center point. Therefore, for any given pointing position, there is at least one combination of the orientations of the prisms that will steer beam to the given pointing position. For the given position $(\phi, \rho) = (0, 0)$, there are infinite many solutions as long as $\omega_1 = \omega_2 + 180$. For most of other positions, there are two solutions, one for $\omega_1 > \omega_2$ and one for $\omega_1 < \omega_2$. For the pointing positions with ρ equal to the maximum, there is only one solution.

The results obtained by first-order paraxial approximation and our method are close for small ρ but are significantly different for large ρ. In Fig. 3, the upper part of the curve is calculated by the first-order paraxial approximation, and the low part of the curve is obtained by our method. Clearly, paraxial approximation can only be used when the required ρ is small.

III. Solution for the Inverse Problem

The inverse problem can be solved by many numerical iteration methods, for example, Gauss-Newton algorithm [9], or trust region algorithms [10], or Levenberg–Marquardt algorithm [11], [12], or simply the original Newton method. Though numerical Jocobian matrix is often a choice, it is more accurate and normally more efficient to have the explicit Jocobian matrix formulae for the systems of nonlinear equations [9, pp. 2–7]. The main result of this section is to give the analytic formula of the Jocobian

$$J = -\begin{bmatrix} \frac{d\rho}{d\omega_1} & \frac{d\rho}{d\omega_2} \\ \frac{d\phi}{d\omega_1} & \frac{d\phi}{d\omega_2} \end{bmatrix}. \tag{15}$$

Theorem 3.1: The elements of the Jocobian matrix of (15) is as follows [see (16)–(19), shown at the bottom of the next page], and [see (20)–(36), shown at the top of the following pages].

Proof: The proof is straightforward but needs some detailed calculations. (16) and (17) are derivatives of (13). (18) and (19) are derivatives of (14). (20) and (21) are derivatives of (11). (22) and (23) are derivatives of (12). (24) and (25) are derivatives of (8). (26) is the derivative of (9). (27) is the derivative of (10). (28) and (29) are derivatives of (7). (30) is the derivative of (2). (31) is the derivative of (3). (32), (33), and (34) are derivatives of (4). (35) is the derivative of (5). Finally, (36) is the derivative of (6).

The Jocobian matrix has been used with trust region algorithm and the original Newton algorithm implemented in MATLAB. We choose trust region method because it normally converges for a broader class of problems than other methods but normally slower than Newton method. We choose Newton method because it normally converges faster than other methods but may fail for some problems. For this particular problem with special structure, Newton method never fails in the simulation and is faster than Trust region method as expected. Therefore, Newton method is a better choice for this particular problem. The following table presents 10 randomly generated desired pointing positions and the solutions obtained by using trust region algorithm and Newton algorithm.

IV. Conclusions

We have developed a complete mathematical solution for the Risley prism optical beam steering system. Given prisms' orientations, we have obtained a set of analytic formulae to calculate the beam pointing position. For the inverse problem,

TABLE I
NUMERICAL EXAMPLES

ρ	ϕ	Trust region			Newton		
		iter	ω_1	ω_2	iter	ω_1	ω_2
26	335	7	214.2321	97.1268	3	214.2321	97.1268
27	150	9	28.0155	-86.6184	3	28.0155	-86.6184
50	189	7	-18.4200	34.8814	3	-18.4200	34.8814
12	241	8	-14.9326	136.2453	3	-14.9326	136.2453
50	7	10	-200.4200	-147.1186	4	-200.4200	-147.1186
40	136	8	-2.3796	-83.9231	2	-2.3796	-83.9231
49	81	7	-27.9682	38.3900	4	-27.9682	38.3900
42	154	9	12.9495	-63.2485	3	12.9495	-63.2485
18	68	9	-180.8355	-44.1657	3	-180.8355	-44.1657
11	245	8	-12.1096	141.4774	3	-12.1096	141.4774

given any desired beam pointing position, we provided the analytic formulae to calculate the Jocobian matrix. By applying this Jocobian matrix, we can use many different numerical algorithms for systems of nonlinear equations to find the required prisms' orientations that will steer the beam to the desired pointing position. Among these methods, trust region method and Newton method are implemented in MATLAB and tested. It turns out that both are very effective and efficient methods to solve this inverse problem. In most cases, the algorithms converge to the solution at a rate equal to or faster than the quadratic rate.

APPENDIX I
PROOF OF THEOREM 1

We start with the normals of surfaces of the Risley prisms. It is easy to see that the intersections of the four surfaces of the prisms and z axis are $(0, 0, -r\tan(\alpha/2))$, $(0, 0, r\tan(\alpha/2))$, $(0, 0, f - r\tan(\alpha/2))$, $(0, 0, f + r\tan(\alpha/2))$. The normal vectors of the four surfaces at these four intersections with both prisms at zero orientation are

$$\boldsymbol{\mu_1} = \left(\tan\left(\frac{\alpha}{2}\right), 0, -1\right)^{\mathrm{T}} / N$$
$$\boldsymbol{\mu_2} = \left(\tan\left(\frac{\alpha}{2}\right), 0, 1\right)^{\mathrm{T}} / N$$
$$\boldsymbol{\mu_3} = \left(\tan\left(\frac{\alpha}{2}\right), 0, -1\right)^{\mathrm{T}} / N$$
$$\boldsymbol{\mu_4} = \left(\tan\left(\frac{\alpha}{2}\right), 0, 1\right)^{\mathrm{T}} / N$$

where

$$N = \sqrt{1 + \tan^2\left(\frac{\alpha}{2}\right)} = \frac{1}{\cos\left(\frac{\alpha}{2}\right)}.$$

If the first prism rotates ω_1 degrees, and the second prism rotates ω_2 degrees, these four normal vectors are

$$\boldsymbol{\nu_1} = \left(\cos(\omega_1)\sin\left(\frac{\alpha}{2}\right), \sin(\omega_1)\sin\left(\frac{\alpha}{2}\right), -\cos\left(\frac{\alpha}{2}\right)\right)^{\mathrm{T}}$$
$$\boldsymbol{\nu_2} = \left(\cos(\omega_1)\sin\left(\frac{\alpha}{2}\right), \sin(\omega_1)\sin\left(\frac{\alpha}{2}\right), \cos\left(\frac{\alpha}{2}\right)\right)^{\mathrm{T}}$$
$$\boldsymbol{\nu_3} = \left(\cos(\omega_2)\sin\left(\frac{\alpha}{2}\right), \sin(\omega_2)\sin\left(\frac{\alpha}{2}\right), -\cos\left(\frac{\alpha}{2}\right)\right)^{\mathrm{T}}$$
$$\boldsymbol{\nu_4} = \left(\cos(\omega_2)\sin\left(\frac{\alpha}{2}\right), \sin(\omega_2)\sin\left(\frac{\alpha}{2}\right), \cos\left(\frac{\alpha}{2}\right)\right)^{\mathrm{T}}.$$

$$\frac{d\phi}{d\omega_1} = \begin{cases} -\dfrac{\left(\mathbf{e}_1^{\mathrm{T}} - \mathbf{e}_1^{\mathrm{T}}\mathbf{e_3}\mathbf{e}_3^{\mathrm{T}}\right)\frac{d\mathbf{R_3}}{d\omega_1}\|\mathbf{R_z}\| - \frac{\mathbf{R}_\mathbf{z}^{\mathrm{T}}\frac{d\mathbf{R_z}}{d\omega_1}\left(\mathbf{e}_1^{\mathrm{T}} - \mathbf{e}_1^{\mathrm{T}}\mathbf{e_3}\mathbf{e}_3^{\mathrm{T}}\right)\mathbf{R_3}}{\|\mathbf{R_z}\|}}{\sqrt{1 - \left(\frac{\left(\mathbf{e}_1^{\mathrm{T}} - \mathbf{e}_1^{\mathrm{T}}\mathbf{e_3}\mathbf{e}_3^{\mathrm{T}}\right)\mathbf{R_3}}{\|\mathbf{R_z}\|}\right)^2}\left(\mathbf{R}_\mathbf{z}^{\mathrm{T}}\mathbf{R_z}\right)}, & \text{for } \mathbf{R_z}(2) \geq 0 \\ \dfrac{\left(\mathbf{e}_1^{\mathrm{T}} - \mathbf{e}_1^{\mathrm{T}}\mathbf{e_3}\mathbf{e}_3^{\mathrm{T}}\right)\frac{d\mathbf{R_3}}{d\omega_1}\|\mathbf{R_z}\| - \frac{\mathbf{R}_\mathbf{z}^{\mathrm{T}}\frac{d\mathbf{R_z}}{d\omega_1}\left(\mathbf{e}_1^{\mathrm{T}} - \mathbf{e}_1^{\mathrm{T}}\mathbf{e_3}\mathbf{e}_3^{\mathrm{T}}\right)\mathbf{R_3}}{\|\mathbf{R_z}\|}}{\sqrt{1 - \left(\frac{\left(\mathbf{e}_1^{\mathrm{T}} - \mathbf{e}_1^{\mathrm{T}}\mathbf{e_3}\mathbf{e}_3^{\mathrm{T}}\right)\mathbf{R_3}}{\|\mathbf{R_z}\|}\right)^2}\left(\mathbf{R}_\mathbf{z}^{\mathrm{T}}\mathbf{R_z}\right)}, & \text{for } \mathbf{R_z}(2) < 0 \end{cases} \tag{16}$$

$$\frac{d\phi}{d\omega_2} = \begin{cases} -\dfrac{\left(\mathbf{e}_1^{\mathrm{T}} - \mathbf{e}_1^{\mathrm{T}}\mathbf{e_3}\mathbf{e}_3^{\mathrm{T}}\right)\frac{d\mathbf{R_3}}{d\omega_2}\|\mathbf{R_z}\| - \frac{\mathbf{R}_\mathbf{z}^{\mathrm{T}}\frac{d\mathbf{R_z}}{d\omega_2}\left(\mathbf{e}_1^{\mathrm{T}} - \mathbf{e}_1^{\mathrm{T}}\mathbf{e_3}\mathbf{e}_3^{\mathrm{T}}\right)\mathbf{R_3}}{\|\mathbf{R_z}\|}}{\sqrt{1 - \left(\frac{\left(\mathbf{e}_1^{\mathrm{T}} - \mathbf{e}_1^{\mathrm{T}}\mathbf{e_3}\mathbf{e}_3^{\mathrm{T}}\right)\mathbf{R_3}}{\|\mathbf{R_z}\|}\right)^2}\left(\mathbf{R}_\mathbf{z}^{\mathrm{T}}\mathbf{R_z}\right)}, & \text{for } \mathbf{R_z}(2) \geq 0 \\ \dfrac{\left(\mathbf{e}_1^{\mathrm{T}} - \mathbf{e}_1^{\mathrm{T}}\mathbf{e_3}\mathbf{e}_3^{\mathrm{T}}\right)\frac{d\mathbf{R_3}}{d\omega_2}\|\mathbf{R_z}\| - \frac{\mathbf{R}_\mathbf{z}^{\mathrm{T}}\frac{d\mathbf{R_z}}{d\omega_2}\left(\mathbf{e}_1^{\mathrm{T}} - \mathbf{e}_1^{\mathrm{T}}\mathbf{e_3}\mathbf{e}_3^{\mathrm{T}}\right)\mathbf{R_3}}{\|\mathbf{R_z}\|}}{\sqrt{1 - \left(\frac{\left(\mathbf{e}_1^{\mathrm{T}} - \mathbf{e}_1^{\mathrm{T}}\mathbf{e_3}\mathbf{e}_3^{\mathrm{T}}\right)\mathbf{R_3}}{\|\mathbf{R_z}\|}\right)^2}\left(\mathbf{R}_\mathbf{z}^{\mathrm{T}}\mathbf{R_z}\right)}, & \text{for } \mathbf{R_z}(2) < 0 \end{cases} \tag{17}$$

$$\frac{d\rho}{d\omega_1} = -\frac{\mathbf{e}_3^{\mathrm{T}}\frac{d\mathbf{R_3}}{d\omega_1}}{\sqrt{1 - \left(\mathbf{e}_3^{\mathrm{T}}\mathbf{R_3}\right)^2}} \tag{18}$$

$$\frac{d\rho}{d\omega_2} = -\frac{\mathbf{e}_3^{\mathrm{T}}\frac{d\mathbf{R_3}}{d\omega_2}}{\sqrt{1 - \left(\mathbf{e}_3^{\mathrm{T}}\mathbf{R_3}\right)^2}} \tag{19}$$

$$
\begin{aligned}
\frac{d\mathbf{R_3}}{d\omega_1} =& \frac{-\sin(\lambda_4)\frac{d\lambda_4}{d\omega_1}\cos(\lambda_3)+\cos(\lambda_4)\sin(\lambda_3)\frac{d\lambda_3}{d\omega_1}}{\cos(\lambda_3)^2}\mathbf{R_2} \\
&+\frac{\cos(\lambda_4)}{\cos(\lambda_3)}\frac{d\mathbf{R_2}}{d\omega_1}+\frac{\left(-\sin(\lambda_4-\lambda_3)\left(\frac{d\lambda_4}{d\omega_1}-\frac{d\lambda_3}{d\omega_1}\right)-\frac{-\sin(\lambda_4)\cos(\lambda_3)\frac{d\lambda_4}{d\omega_1}+\cos(\lambda_4)\sin(\lambda_3)\frac{d\lambda_3}{d\omega_1}}{\cos(\lambda_3)^2}\right)}{\left(\mathbf{R_2^T P_2}\right)}\mathbf{P_2} \\
&-\frac{\left(\cos(\lambda_4-\lambda_3)-\frac{\cos(\lambda_4)}{\cos(\lambda_3)}\right)\left(\mathbf{P_2^T}\frac{d\mathbf{R_2}}{d\omega_1}+\mathbf{R_2^T}\frac{d\mathbf{P_2}}{d\omega_1}\right)}{\left(\mathbf{R_2^T P_2}\right)^2}\mathbf{P_2}+\frac{\left(\cos(\lambda_4-\lambda_3)-\frac{\cos(\lambda_4)}{\cos(\lambda_3)}\right)}{\mathbf{R_2^T P_2}}\frac{d\mathbf{P_2}}{d\omega_1}
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
\frac{d\mathbf{R_3}}{d\omega_2} =& \frac{-\sin(\lambda_4)\frac{d\lambda_4}{d\omega_2}\cos(\lambda_3)+\cos(\lambda_4)\sin(\lambda_3)\frac{d\lambda_3}{d\omega_2}}{\cos(\lambda_3)^2}\mathbf{R_2}+\frac{\cos(\lambda_4)}{\cos(\lambda_3)}\frac{d\mathbf{R_2}}{d\omega_2} \\
&+\frac{\left(-\sin(\lambda_4-\lambda_3)\left(\frac{d\lambda_4}{d\omega_2}-\frac{d\lambda_3}{d\omega_2}\right)-\frac{-\sin(\lambda_4)\cos(\lambda_3)\frac{d\lambda_4}{d\omega_2}+\cos(\lambda_4)\sin(\lambda_3)\frac{d\lambda_3}{d\omega_2}}{\cos(\lambda_3)^2}\right)}{\left(\mathbf{R_2^T P_2}\right)}\mathbf{P_2} \\
&-\frac{\left(\cos(\lambda_4-\lambda_3)-\frac{\cos(\lambda_4)}{\cos(\lambda_3)}\right)\left(\mathbf{P_2^T}\frac{d\mathbf{R_2}}{d\omega_2}+\mathbf{R_2^T}\frac{d\mathbf{P_2}}{d\omega_2}\right)}{\left(\mathbf{R_2^T P_2}\right)^2}\mathbf{P_2}+\frac{\left(\cos(\lambda_4-\lambda_3)-\frac{\cos(\lambda_4)}{\cos(\lambda_3)}\right)}{\mathbf{R_2^T P_2}}\frac{d\mathbf{P_2}}{d\omega_2}
\end{aligned} \tag{21}
$$

$$\frac{d\mathbf{R_z}}{d\omega_1} = \left(I-\mathbf{e_3 e_3^T}\right)\frac{d\mathbf{R_3}}{d\omega_1} \tag{22}$$

$$\frac{d\mathbf{R_z}}{d\omega_2} = \left(I-\mathbf{e_3 e_3^T}\right)\frac{d\mathbf{R_3}}{d\omega_2} \tag{23}$$

$$\frac{d\mathbf{P_2}}{d\omega} = \left[\left(I-\boldsymbol{\nu_4}\boldsymbol{\nu_4^T}\right)\frac{d\mathbf{R_2}}{d\omega_1},\ \left(I-\boldsymbol{\nu_4}\boldsymbol{\nu_4^T}\right)\frac{d\mathbf{R_2}}{d\omega_2}-\boldsymbol{\nu_4}\frac{d\boldsymbol{\nu_4}}{d\omega_2}^{\mathrm{T}}\mathbf{R_2}-\frac{d\boldsymbol{\nu_4}}{d\omega_2}\boldsymbol{\nu_4^T}\mathbf{R_2}\right] \tag{24}$$

$$\frac{d\|\mathbf{P_2}\|}{d\omega} = \begin{bmatrix}\mathbf{P_2^T}\frac{d\mathbf{P_2}}{d\omega_1}/\|\mathbf{P_2}\| \\ \mathbf{P_2^T}\frac{d\mathbf{P_2}}{d\omega_2}/\|\mathbf{P_2}\|\end{bmatrix} \tag{25}$$

$$\frac{d\lambda_3}{d\omega} = \begin{bmatrix}\frac{\|P2\|\left(\mathbf{P_2^T}\frac{d\mathbf{R_2}}{d\omega_1}+\mathbf{R_2^T}\frac{d\mathbf{P_2}}{d\omega_1}\right)-\mathbf{P_2^T R_2}\frac{d\|\mathbf{P_2}\|}{d\omega_1}}{\sin\left(\frac{\pi}{2}-\lambda_3\right)\left(\mathbf{P_2^T P_2}\right)} \\ \frac{\|P2\|\left(\mathbf{P_2^T}\frac{d\mathbf{R_2}}{d\omega_2}+\mathbf{R_2^T}\frac{d\mathbf{P_2}}{d\omega_2}\right)-\mathbf{P_2^T R_2}\frac{d\|\mathbf{P_2}\|}{d\omega_2}}{\sin\left(\frac{\pi}{2}-\lambda_3\right)\left(\mathbf{P_2^T P_2}\right)}\end{bmatrix} \tag{26}$$

$$\frac{d\lambda_4}{d\omega} = \begin{bmatrix}\frac{\frac{n}{n_a}\cos(\lambda_3)\frac{d\lambda_3}{d\omega_1}}{\cos(\lambda_4)} \\ \frac{\frac{n}{n_a}\cos(\lambda_3)\frac{d\lambda_3}{d\omega_2}}{\cos(\lambda_4)}\end{bmatrix} \tag{27}$$

$$
\begin{aligned}
\frac{d\mathbf{R_2}}{d\omega_1} =& \frac{-\sin(\lambda_2)\frac{d\lambda_2}{d\omega_1}\cos(\delta)\cos(\lambda_1)+\sin(\lambda_1)\frac{d\lambda_1}{d\omega_1}\cos(\lambda_2)\cos(\delta)}{\cos(\lambda_1)^2}\mathbf{R_1}+\frac{\cos(\lambda_2)\cos(\delta)}{\cos(\lambda_1)}\frac{d\mathbf{R_1}}{d\omega_1} \\
&+\frac{\mathbf{R_1^T P_1}\left(-\sin(\lambda_1-\lambda_2)\left(\frac{d\lambda_1}{d\omega_1}-\frac{d\lambda_2}{d\omega_1}\right)-\left(\frac{-\sin(\lambda_2)\frac{d\lambda_2}{d\omega_1}\cos(\lambda_1)+\cos(\lambda_2)\sin(\lambda_1)\frac{d\lambda_1}{d\omega_1}}{\cos(\lambda_1)^2}\right)\right)}{\cos(\delta)\left(\mathbf{R_1^T P_1}\right)^2}\mathbf{P_1} \\
&-\frac{\left(\cos(\lambda_1-\lambda_2)-\frac{\cos(\lambda_2)}{\cos(\lambda_1)}\right)\left(\mathbf{P_1^T}\frac{d\mathbf{R_1}}{d\omega_1}+\mathbf{R_1^T}\frac{d\mathbf{P_1}}{d\omega_1}\right)}{\cos(\delta)\left(\mathbf{R_1^T P_1}\right)^2}\mathbf{P_1}+\frac{\cos(\lambda_1-\lambda_2)-\frac{\cos(\lambda_2)}{\cos(\lambda_1)}}{\cos(\delta)\left(\mathbf{R_1^T P_1}\right)}\frac{d\mathbf{P_1}}{d\omega_1}
\end{aligned} \tag{28}
$$

If the first prism is at the zero orientation, the ray deflection of the first prism is along the path from $(0,0,r\tan(\alpha/2))$ to $(-\tan(\delta),0,1+r\tan(\alpha/2))$. When the first prism rotates ω_1 degrees, the ray deflection of the first prism along the path from $(0,0,r\tan(\alpha/2))$ to $(-\cos(\omega_1)\tan(\delta),-\sin(\omega_1)\tan(\delta),1+r\tan(\alpha/2))$. This ray can be represented by (3).

To obtain the light input axis, we project $\mathbf{R_1}$ on to the input surface of the second prism. Since $\boldsymbol{\nu_3^T}\mathbf{R_1} = -\cos(\alpha/2)-\tan(\delta)\sin(\alpha/2)\cos(\omega_1-\omega_2)$, this projection vector is given by (4) with

$$
\begin{aligned}
\|\mathbf{P_1}\|^2 =& \tan^2(\delta)\left(-\cos(\omega_1)+\cos(\omega_2)\sin^2\left(\frac{\alpha}{2}\right)\cos^2(\omega_1-\omega_2)\right)^2 \\
&+\cos^2(\omega_2)\sin^2\left(\frac{\alpha}{2}\right)\cos^2\left(\frac{\alpha}{2}\right) \\
&+2\tan(\delta)\cos(\omega_2)\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right) \\
&\times\left(-\cos(\omega_1)+\cos(\omega_2)\sin^2\left(\frac{\alpha}{2}\right)\cos(\omega_1-\omega_2)\right)
\end{aligned}
$$

$$\begin{aligned}\frac{d\mathbf{R_2}}{d\omega_2} &= \frac{\left(-\sin(\lambda_2)\frac{d\lambda_2}{d\omega_2}\cos(\delta)\cos(\lambda_1)+\sin(\lambda_1)\frac{d\lambda_1}{d\omega_2}\cos(\lambda_2)\cos(\delta)\right)}{\cos(\lambda_1)^2}\mathbf{R_1}+\frac{\cos(\lambda_2)\cos(\delta)\frac{d\mathbf{R_1}}{d\omega_2}}{\cos(\lambda_1)}\\ &+\frac{\mathbf{R_1^{\mathrm{T}}P_1}\left(-\sin(\lambda_1-\lambda_2)\left(\frac{d\lambda_1}{d\omega_2}-\frac{d\lambda_2}{d\omega_2}\right)-\left(-\sin(\lambda_2)\frac{d\lambda_2}{d\omega_2}\cos(\lambda_1)+\frac{\cos(\lambda_2)\sin(\lambda_1)\frac{d\lambda_1}{d\omega_2}}{\cos(\lambda_1)^2}\right)\right)}{\cos(\delta)(\mathbf{R_1^{\mathrm{T}}P_1})^2}\mathbf{P_1}\\ &-\frac{\left(\cos(\lambda_1-\lambda_2)-\frac{\cos(\lambda_2)}{\cos(\lambda_1)}\right)\left(\mathbf{P_1^{\mathrm{T}}}\frac{d\mathbf{R_1}}{d\omega_2}+\mathbf{R_1^{\mathrm{T}}}\frac{d\mathbf{P_1}}{d\omega_2}\right)}{\cos(\delta)\left(\mathbf{R_1^{\mathrm{T}}}P1\right)^2}\mathbf{P_1}+\frac{\left(\cos(\lambda_1-\lambda_2)-\frac{\cos(\lambda_2)}{\cos(\lambda_1)}\right)}{\cos(\delta)\mathbf{R_1^{\mathrm{T}}P_1}}\frac{d\mathbf{P_1}}{d\omega_2}\end{aligned} \tag{29}$$

$$\frac{d\boldsymbol{\nu_4}}{d\omega} = \begin{bmatrix} 0 & -\sin\left(\frac{\alpha}{2}\right)\sin(\omega_2)\\ 0 & \sin\left(\frac{\alpha}{2}\right)\cos(\omega_2)\\ 0 & 0\end{bmatrix} \tag{30}$$

$$\frac{d\mathbf{R_1}}{d\omega} = \begin{bmatrix} \sin(\omega_1)\tan(\delta) & 0\\ -\cos(\omega_1)tan(\delta) & 0\\ 0 & 0\end{bmatrix} \tag{31}$$

$$\frac{d\mathbf{P_1}}{d\omega_1} = \begin{bmatrix} \tan(\delta)\left(\sin(\omega_1)-\cos(\omega_2)\sin^2\left(\frac{\alpha}{2}\right)\sin(\omega_1-\omega_2)\right)\\ \tan(\delta)(-\cos(\omega_1)-\sin(\omega_2)\sin^2\left(\frac{\alpha}{2}\right)\sin(\omega_1-\omega_2)\\ \tan(\delta)\sin(\omega_1-\omega_2)\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)\end{bmatrix} \tag{32}$$

$$\frac{d\mathbf{P_1}}{d\omega_2} = \begin{bmatrix} \tan(\delta)\left(-\sin(\omega_2)\sin^2\left(\frac{\alpha}{2}\right)\cos(\omega_1-\omega_2)+\cos(\omega_2)\sin^2\left(\frac{\alpha}{2}\right)\sin(\omega_1-\omega_2)\right)+\sin(\omega_2)\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)\\ \tan(\delta)\left(\cos(\omega_2)\sin^2\left(\frac{\alpha}{2}\right)\cos(\omega_1-\omega_2)+\sin(\omega_2)\sin^2\left(\frac{\alpha}{2}\right)\sin(\omega_1-\omega_2)\right)+\cos(\omega_2)\sin\left(\frac{\alpha}{2}\right)\cos(\left(\frac{\alpha}{2}\right)\\ -\tan(\delta)\sin(\omega_1-\omega_2)\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)\end{bmatrix} \tag{33}$$

$$\frac{d\|\mathbf{P_1}\|}{d\omega} = \begin{bmatrix} \frac{\tan^2(\delta)\sin^2\left(\frac{\alpha}{2}\right)\cos(\omega_1-\omega_2)\sin(\omega_1-\omega_2)+\tan(\delta)\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)\sin(\omega_1-\omega_2)}{\|\mathbf{P_1}\|}\\ -\frac{\tan^2(\delta)\sin^2\left(\frac{\alpha}{2}\right)\cos(\omega_1-\omega_2)\sin(\omega_1-\omega_2)+\tan(\delta)\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)\sin(\omega_1-\omega_2)}{\|\mathbf{P_1}\|}\end{bmatrix} \tag{34}$$

$$\frac{d\lambda_1}{d\omega} = \begin{bmatrix} \frac{\left(\mathbf{R_1^{\mathrm{T}}}\frac{d\mathbf{P_1}}{d\omega_1}+\mathbf{P_1^{\mathrm{T}}}\frac{d\mathbf{R_1}}{d\omega_1}\right)\|\mathbf{P_1}\|-\mathbf{P_1^{\mathrm{T}}R_1}\frac{d\|\mathbf{P_1}\|}{d\omega_1}}{\sin\left(\frac{\pi}{2}-\lambda_1\right)\cos(\delta)\left(\mathbf{P_1^{\mathrm{T}}P_1}\right)\left(\mathbf{R_1^{\mathrm{T}}R_1}\right)}\\ \frac{\left(\mathbf{R_1^{\mathrm{T}}}\frac{d\mathbf{P_1}}{d\omega_2}+\mathbf{P_1^{\mathrm{T}}}\frac{d\mathbf{R_1}}{d\omega_2}\right)\|\mathbf{P_1}\|-\mathbf{P_1^{\mathrm{T}}R_1}\frac{d\|\mathbf{P_1}\|}{d\omega_2}}{\sin\left(\frac{\pi}{2}-\lambda_1\right)\cos(\delta)\left(\mathbf{P_1^{\mathrm{T}}P_1}\right)\left(\mathbf{R_1^{\mathrm{T}}R_1}\right)}\end{bmatrix} \tag{35}$$

$$\frac{d\lambda_2}{d\omega} = \begin{bmatrix} \frac{n_a\cos(\lambda_1)\frac{d\lambda_1}{d\omega_1}}{n\cos(\lambda_2)}\\ \frac{n_a\cos(\lambda_1)\frac{d\lambda_1}{d\omega_2}}{n\cos(\lambda_2)}\end{bmatrix} \tag{36}$$

$$\begin{aligned}&+\tan^2(\delta)\left(-\sin(\omega_1)+\sin(\omega_2)\sin^2\left(\frac{\alpha}{2}\right)\cos(\omega_1-\omega_2)\right)\\ &+\sin^2(\omega_2)\sin^2\left(\frac{\alpha}{2}\right)\cos^2\left(\frac{\alpha}{2}\right)\\ &+2\tan(\delta)\sin(\omega_2)\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)\\ &\times\left(\sin(\omega_1)+\sin(\omega_2)\sin^2\left(\frac{\alpha}{2}\right)\cos(\omega_1-\omega_2)\right)\\ &+\left(\sin^2\left(\frac{\alpha}{2}\right)-\tan(\delta)\cos(\omega_1-\omega_2)\ \sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)\right)^2.\end{aligned} \tag{37}$$

The first line plus the fifth line in (37) yields

$$\tan^2(\delta)\left(1+\sin^4\left(\frac{\alpha}{2}\right)\cos^2(\omega_1-\omega_2)-2\sin^2\left(\frac{\alpha}{2}\right)\cos^2(\omega_1-\omega_2)\right).$$

The second line plus the sixth line in (37) yields

$$\sin^2\left(\frac{\alpha}{2}\right)\cos^2\left(\frac{\alpha}{2}\right).$$

The third and fourth lines plus the seventh and eighth lines in (37) yields

$$-2\tan(\delta)\sin\left(\frac{\alpha}{2}\right)\cos^3\left(\frac{\alpha}{2}\right)\cos(\omega_1-\omega_2).$$

Rewriting the last line in (37) yields

$$\sin^4\left(\frac{\alpha}{2}\right)+\tan^2(\delta)\cos^2(\omega_1-\omega_2)\sin^2\left(\frac{\alpha}{2}\right)\cos^2\left(\frac{\alpha}{2}\right)-2\tan(\delta)\cos(\omega_1-\omega_2)\sin^3\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right).$$

Adding them all together and simplifying the expression gives

$$\|\mathbf{P_1}\|^2=\sin^2\left(\frac{\alpha}{2}\right)+\tan^2(\delta)\left(1-\sin^2\left(\frac{\alpha}{2}\right)\cos^2(\omega_1-\omega_2)\right)-2\tan(\delta)\cos(\omega_1-\omega_2)\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right). \tag{38}$$

We also need to know the incident angle of the second prism, which is given by

$$\lambda_1=\frac{\pi}{2}-\cos^{-1}\left(\frac{\langle\mathbf{P_1},\mathbf{R_1}\rangle}{\|\mathbf{P_1}\|\cdot\|\mathbf{R_1}\|}\right). \tag{39}$$

It is straightforward to see

$$\|\mathbf{R_1}\|=\frac{1}{\cos(\delta)}. \tag{40}$$

The inner product of $\mathbf{R_1}$ and $\mathbf{P_1}$ is obtained as follows:

$$\begin{aligned}\langle \mathbf{R_1},\mathbf{P_1}\rangle =& \tan^2(\delta)\cos(\omega_1)\left(-\cos(\omega_1)+\cos(\omega_2)\sin^2\left(\frac{\alpha}{2}\right)\cos(\omega_1-\omega_2)\right)\\ &-\tan(\delta)\cos(\omega_1)\cos(\omega_2)\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)\\ &-\tan^2(\delta)\sin(\omega_1)\left(-\sin(\omega_1)+\sin(\omega_2)\sin^2\left(\frac{\alpha}{2}\right)\cos(\omega_1-\omega_2)\right)\\ &-\tan(\delta)\sin(\omega_1)\sin(\omega_2)\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)\\ &+\sin^2\left(\frac{\alpha}{2}\right)-\tan(\delta)\cos(\omega_1-\omega_2)\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right). \end{aligned}\tag{41}$$

Combining the first and the third lines of (41) yields

$$-\tan^2(\delta)\left(-1+\sin^2\left(\frac{\alpha}{2}\right)\cos^2(\omega_1-\omega_2)\right).$$

Combining the second and the fourth lines of (41) yields

$$-\tan(\delta)\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)\cos(\omega_1-\omega_2).$$

Adding all items in (41) yields

$$\begin{aligned}\langle \mathbf{R_1},\mathbf{P_1}\rangle =& \sin^2\left(\frac{\alpha}{2}\right)-2\tan(\delta)\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)\cos(\omega_1-\omega_2)\\ &+\tan^2(\delta)\left(-1+\sin^2\left(\frac{\alpha}{2}\right)\cos^2(\omega_1-\omega_2)\right).\end{aligned}\tag{42}$$

Substituting (38), (40), and (42) into (39) gives the simplified expression of λ_1 as (5).

It is then straightforward to get the refracted angle of the first surface of the second prism as in (6).

However, the formulae will quickly get too complicated to be manageable if we try to express all of the rest formulae in terms of parameters n, n_a, α, and δ, and control variables ω_1 and ω_2. Therefore, we will express the rest functions in an iterative way, i.e., every function depends explicitly on the previous functions that are functions of ω_1 and ω_2. This will also simplify the task for us to get the analytical derivatives of $d\phi/d\omega$ and $d\rho/d\omega$ that will be used to solve the inverse problem.

We proceed the derivation by getting the equation to describe the light ray path $\mathbf{R_2}$ inside the second prism. Since $\mathbf{R_2} = a\mathbf{R_1} + b\mathbf{P_1}$, we need to determine a and b. Let a and b be selected such that $\|\mathbf{R_2}\| = 1$, we have

$$\begin{aligned}&\langle \mathbf{R_2}, -\boldsymbol{\nu_3}\rangle = \cos(\lambda_2)\\ \Longleftrightarrow\quad & -a\langle \mathbf{R_1}, \boldsymbol{\nu_3}\rangle = \cos(\lambda_2)\\ \Longleftrightarrow\quad & a = \frac{-\cos(\lambda_2)}{\langle \mathbf{R_1}, \boldsymbol{\nu_3}\rangle} = \frac{\cos(\lambda_2)\cos(\delta)}{\cos(\lambda_1)}\end{aligned}\tag{43}$$

and

$$\begin{aligned}&\langle \mathbf{R_1}, \mathbf{R_2}\rangle = \|\mathbf{R_1}\|\cos(\lambda_1-\lambda_2)\\ \Longleftrightarrow\quad & \langle a\mathbf{R_1}+b\mathbf{P_1}, \mathbf{R_1}\rangle = \|\mathbf{R_1}\|\cos(\lambda_1-\lambda_2)\\ \Longleftrightarrow\quad & b = \frac{\|\mathbf{R_1}\|\cos(\lambda_1-\lambda_2)-a\|\mathbf{R_1}\|^2}{\langle \mathbf{R_1}, \mathbf{P_1}\rangle}\\ \Longleftrightarrow\quad & b = \frac{\cos(\lambda_1-\lambda_2)-\cos(\lambda_2)/\cos(\lambda_1)}{\cos(\delta)\langle \mathbf{R_1}, \mathbf{P_1}\rangle}.\end{aligned}\tag{44}$$

Therefore, $\mathbf{R_2}$ is given by (7).

The incident angle of $\mathbf{R_2}$ on the second surface of the second prism can be obtained by the projection of $\mathbf{R_2}$ on to the second surface of the second prism. This projection $\mathbf{P_2}$ is given by (8). Hence, the incident angle of $\mathbf{R_2}$ on to the second surface of the second prism is (9). The refraction angle from the second surface of the second prism is (10).

We can calculate the light path $\mathbf{R_3}$ coming out the second prism similarly. Since $\mathbf{R_3} = c\mathbf{R_2} + d\mathbf{P_2}$, we need to determine c and d. Let c and d be selected such that $\|\mathbf{R_3}\| = 1$, we have

$$\begin{aligned}&\langle \mathbf{R_3}, -\boldsymbol{\nu_4}\rangle = \cos(\lambda_4)\\ \Longleftrightarrow\quad & -c\langle \mathbf{R_2}, \boldsymbol{\nu_4}\rangle = \cos(\lambda_4)\\ \Longleftrightarrow\quad & c = \frac{-\cos(\lambda_4)}{\langle \mathbf{R_2}, \boldsymbol{\nu_4}\rangle} = \frac{\cos(\lambda_4)}{\cos(\lambda_3)}\end{aligned}\tag{45}$$

and

$$\begin{aligned}&\langle \mathbf{R_2}, \mathbf{R_3}\rangle = \cos(\lambda_4-\lambda_3)\\ \Longleftrightarrow\quad & \langle c\mathbf{R_2}+d\mathbf{P_2}, \mathbf{R_2}\rangle = \cos(\lambda_4-\lambda_3)\\ \Longleftrightarrow\quad & d = \frac{\cos(\lambda_4-\lambda_3)-c}{\langle \mathbf{R_2}, \mathbf{P_2}\rangle}\\ \Longleftrightarrow\quad & d = \frac{(\cos(\lambda_4-\lambda_3)-\cos(\lambda_4)/\cos(\lambda_3))}{\langle \mathbf{R_2}, \mathbf{P_2}\rangle}.\end{aligned}\tag{46}$$

Therefore, $\mathbf{R_3}$ is given by (11).

Finally, (ϕ, ρ) is easily verified to be given as (13) and (14).

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Yaguang Yang received the B.S.E.E. and M.S.E.E. degrees from the Control Engineering Department, Huazhong University of Sciences and Technology, in 1982 and 1985, respectively, and the Ph.D. degree from the Electrical Engineering Department, University of Maryland, College Park, in 1996.

He is a senior principal GNC (guidance, navigation, and control) engineer at Orbital Sciences Corporation. He was a Lecturer at Zhejiang University, China, from 1985 to 1991. He was a Research Assistant at West Virginia University, Morgantown, from 1991 to 1992. He was a Research Assistant in the Electrical Engineering Department, University of Maryland, from 1992 to 1996. Since then, he has been working in different industries in the U.S. He has published more than 20 journal papers. His interests are in optical communication system control, aerospace system control, optimization techniques, and real-time computer control system.