CP1 for NPRE 501

Jin Whan Bae

2017-11-01

---

1 Problem Definition

Table 1 lists the constants used in the problem.

Parameter	Value	[Unit]
Diamater	6	[cm]
Radius	**3**	[cm]
Geometry	Sphere	
k	15	$[\frac{W}{mK}]$
Density	8000	$[\frac{kg}{m^3}]$
Specific Heat	500	$[\frac{J}{kgK}]$
α	**3.75e-6**	$[\frac{m^2}{s}]$

Table 1: Problem Constants. Derived constants are in bold.

Differential Equation:

$$\frac{1}{\alpha} \cdot \frac{dT}{dt} = \frac{1}{r^2} \frac{d}{dr} r^2 \frac{dT}{dr}$$

Boundary Conditions:

$$T(0,t) = finite \quad OR \quad \frac{dT}{dr}(r=0,t) = 0$$

$$\frac{dT}{dr}(r=R) = 0$$

Initial Condition:

$$T(r,0) = \frac{T_0}{2}(1 - \cos{(\frac{\pi \cdot r}{R})})$$

1.1 Finding Steady State

Since there is no heat generation or leakage (r=R is insulated), the steady state temperature is a constant throughout the sphere.

Expressing mathematically,

T_{ss} is when $\frac{dT}{dt} = 0$:

$$0 = \frac{1}{r^2}\frac{d}{dr}r^2(\frac{dT_{ss}}{dr})$$

$$\frac{C_1}{r^2} = \frac{dT_{ss}}{dr}$$

$$-\frac{C_1}{r} + C_2 = T_{ss}$$

applying BC:

at r = 0, finite, makes $C_1 = 0$

$$T_{ss} = C_2$$

the numerical value of this constant can be found doing an energy balance of the system.

$$\int_0^R \frac{T_0}{2}(1-\cos(\frac{\pi r}{R}))r^2 dr = \int_0^R Cr^2 dr$$

$$\frac{T_0}{2}\int_0^R (r^2 - r^2\cos(\frac{\pi r}{R}))dr = C\frac{R^3}{3}$$

using separation of variables:

$$\frac{T_0}{2}(\frac{R^3}{3} - (\frac{r^2 R}{\pi}\sin(\frac{\pi r}{R})\Big|_0^R - \frac{2R}{\pi}\int_0^R r\sin(\frac{\pi r}{R})))dr = C\frac{R^3}{3}$$

Solving this:

$$\frac{T_0}{2}(\frac{R^3}{3} + \frac{2R^3}{\pi^2}) = C\frac{R^3}{3}$$

$$C = T_{ss} = \frac{3T_0}{2}(\frac{\pi^2+6}{3\pi^2}) \approx 0.804T_0$$

2 Numerical Solution

$$\frac{1}{\alpha} \cdot \frac{dT}{dt} = \frac{2}{r}\frac{dT}{dr} + \frac{d^2T}{dr^2}$$

Applying the finite difference method, central for r and explicit for t:

$$\frac{1}{\alpha} \cdot \frac{T_k^u - T_k^{u-1}}{\Delta t} = \frac{2}{r_k}\left(\frac{T_{k+1}^{u-1} - T_{k-1}^{u-1}}{2\Delta r}\right) + \frac{T_{k-1}^{u-1} - 2T_k^{u-1} + T_{k+1}^{u-1}}{\Delta r^2}$$

where u is the temporal step and k is the spacial step.

Solving for T_k^{u+1}:

$$T_k^{u+1} = T_k^u + \alpha\frac{\Delta t}{\Delta r \cdot r}(T_{k+1}^u + T_{k-1}^u) + \alpha * \frac{\Delta t}{(\Delta r)^2}(T_{k+1}^u - 2T_k^u + T_{k-1}^u)$$

Applying neuman boundary condition at r=0:

$$\frac{dT(0,t)}{dr} = 0$$

$$\frac{T_1^u - T_{-1}^u}{2\Delta r} = 0$$

$$T_1^u = T_{-1}^u$$

This gives:

$$T_0^{u+1} = T_0^u + \alpha * \frac{\Delta t}{(\Delta r)^2}(2T_1^u - 2T_0^u)$$

Applying neuman boundary condition at r=R (k=K at r=R):

$$-k\frac{dT(R,t)}{dr} = 0$$

$$\frac{T_{K+1}^u - T_{K-1}^u}{2\Delta r} = 0$$

$$T_{K+1}^u = T_{K-1}^u$$

This gives:

$$T_K^{u+1} = T_K^u + \alpha * \frac{\Delta t}{(\Delta r)^2}(2T_{K-1}^u - 2T_K^u)$$

3 Appendix A

3.1 a. Final Temperature Distribution in the Sphere

The final temperature distribution will be a cosine curve with the highest point at r = 0, gradually going down to the minimum value at r = R.

4 Appendix B

4.1 Analytical solution for solving T(r,t) directly

$$\frac{1}{\alpha} \cdot \frac{dT}{dt} = \frac{1}{r^2}\frac{d}{dr}r^2\frac{dT}{dr}$$

Boundary Conditions:

$$T(0,t) = finite$$

$$\frac{dT}{dr}(r = R) = 0$$

Initial Condition:

$$T(r,0) = \frac{T_0}{2}(1 - \cos(\frac{\pi \cdot r}{R}))$$

set:

$$T(r,t) = \frac{\overline{T}(r,t)}{r} + T_s s$$

$$\overline{T}(r,t) = T(r,t)r + T_s sr$$

plugging this back into the differential equation:

$$\frac{1}{\alpha} \cdot \frac{d\overline{T}}{dt}\frac{1}{r} = \frac{1}{r^2}\frac{d}{dr}r^2(\frac{d\overline{T}}{dr}\frac{1}{r} - \frac{1}{r^2}\overline{T})$$

$$\frac{1}{\alpha} \cdot \frac{d\overline{T}}{dt} = \frac{1}{r}\frac{d}{dr}(\frac{d\overline{T}}{dr}r - \overline{T})$$

$$\frac{1}{\alpha} \cdot \frac{d\overline{T}}{dt} = \frac{1}{r}(\frac{d^2\overline{T}}{dr^2}r + \frac{d\overline{T}}{dr} - \frac{d\overline{T}}{dr})$$

$$\frac{1}{\alpha} \cdot \frac{d\overline{T}}{dt} = \frac{d^2\overline{T}}{dr^2}$$

where

$$\overline{T}(0,t) = 0$$

$$\frac{d\overline{T}}{dr} = \frac{\overline{T}}{R}$$

and initial condition

$$\overline{T}(r,0) = r\frac{T_0}{2}(1 - cos(\frac{\pi r}{R})) - T_{ss}r$$

This turns into a cartesian problem.
Applying Separation of Variables:

$$\overline{T}(r,t) = \Gamma(t)\Psi(r)$$

Applying the new variables, dividing both sides by $\Gamma(t)\Psi(r)$, and setting it to a new variable $-\beta^2$:

$$\frac{1}{\alpha\Gamma} \cdot \frac{d\Gamma}{dt} = \frac{d^2\Psi}{dr^2}\frac{1}{\Psi} = -\beta^2$$

Solving for Ψ first:

$$\frac{d^2\Psi}{dr^2}\frac{1}{\Psi} = -\beta^2$$

$$\frac{d^2\Psi}{dr^2} + \beta^2\Psi = 0$$

$$\Psi(r) = C_1 sin(\beta r) + C_2 cos(\beta r)$$

Boundary Conditions:

Since:

$$\frac{dT}{dr} = -\frac{\overline{T}}{r^2} + \frac{1}{r}\frac{d\overline{T}}{dr}$$

The boundary conditions become:

$$-\Psi(r = 0) = 0$$

$$-k(\frac{\Psi}{R^2} + \frac{d\Psi}{dr}\frac{1}{R}) = 0$$

Applying the first boundary condition, interpreting as $\Psi(r = 0) = 0, C_2 = 0$

Applying the second boundary condition,

$$-\frac{C_1 \sin(\beta R)}{R^2} + C_1 \cos(\beta r)\beta = 0$$

$$-\sin(\beta R) + R\beta \cos \beta R = 0$$

$$\tan(\beta R) = R\beta$$

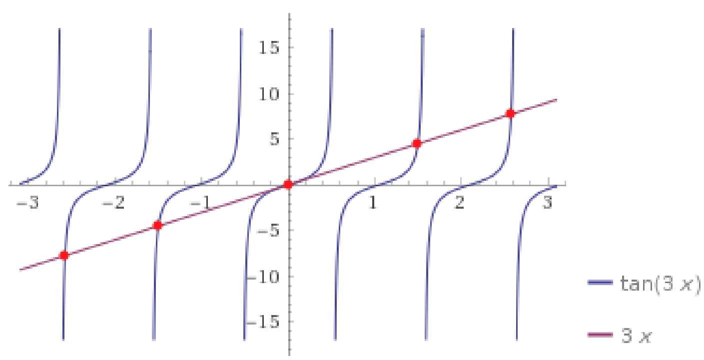

Figure 1: Plot of tan(3x) = 3x.

β is the zeros of that equation (illustrated in fig. 1)

Solving for $\Gamma(t)$:

$$\frac{1}{\alpha\Gamma} \cdot \frac{d\Gamma}{dt} = -\beta_n^2$$

$$\frac{1}{\alpha\Gamma} \cdot \frac{d\Gamma}{dt} = -\beta_n^2\alpha\Gamma$$

$$\Gamma(t) = A_1 e^{-\alpha\beta_n^2 t}$$

This makes $\overline{T}(r,t)$:

$$\overline{T}(r,t) = \sum_{n=1}^{\infty} A_n e^{-\alpha\beta_n^2 t} sin(\beta_n r)$$

Applying initial condition and orthogonality:

$$T\sum_{n=1}^{\infty} A_n \sin(\beta_n r) = r\frac{T_0}{2}(1-\cos(\frac{\pi r}{R})) - T_{ss}r$$

$$A_n = \frac{\int_0^R (r\frac{T_0}{2}(1-\cos(\frac{\pi r}{R})) - T_{ss}r)\sin(\beta_n r)dr}{\int_0^R (\sin(\beta_n r))^2 dr}$$

Plugging all this into $T(r,t)$:

$$T(r,t) = \sum_{n=1}^{\infty} A_n \frac{\sin(\beta_n r)}{r} e^{-\beta_n^2 \alpha t} + T_{ss}$$

4.2 Analytical solution by defining new temperature

$$T'(r,t) = T(r,t) - T_{ss}$$

$$T'(r,t) + T_{ss} = T(r,t)$$

where T_{ss} is:

$$0 = \frac{1}{r^2}\frac{d}{dr}r^2(\frac{dT_{ss}}{dr})$$

$$\frac{C_1}{r^2} = \frac{dT_{ss}}{dr}$$

$$-\frac{C_1}{r} + C_2 = T_{ss}$$

applying BC:
at r = 0, finite, makes $C_1 = 0$

$$T_{ss} = C_2$$

Plugging the new definition into the original differential equation:
differential equation becomes:

$$\frac{1}{\alpha}(\frac{dT'}{dt} + \frac{dT_{ss}}{dt}) = \frac{1}{r^2}\frac{d}{dr}r^2(\frac{dT'}{dr} + \frac{dT_{ss}}{dr})$$

Considering $\frac{dT_{ss}}{dt} = \frac{dT_{ss}}{dr} = 0$,

$$\frac{1}{\alpha} \cdot \frac{dT'}{dt} = \frac{1}{r^2}\frac{d}{dr}r^2 \cdot \frac{dT'}{dr}$$

Boundary Conditions:

$$T'(0,t) = finite$$

$$\frac{dT'}{dr}(r = R) = 0$$

Initial Condition:

$$T'(r,0) = \frac{T_0}{2}(1 - \cos{(\frac{\pi \cdot r}{R})}) - C_2$$

set:

$$T'(r,t) = \frac{\overline{T'}}{r}$$

$$\frac{1}{\alpha} \cdot \frac{d\overline{T'}}{dt}\frac{1}{r} = \frac{1}{r^2}\frac{d}{dr}r^2(\frac{d\overline{T'}}{dr}\frac{1}{r} - \frac{1}{r^2}\overline{T'})$$

$$\frac{1}{\alpha}\cdot\frac{d\overline{T'}}{dt}=\frac{1}{r}\frac{d}{dr}(\frac{d\overline{T'}}{dr}r-\overline{T'})$$

$$\frac{1}{\alpha}\cdot\frac{d\overline{T'}}{dt}=\frac{1}{r}(\frac{d^2\overline{T'}}{dr^2}r+\frac{d\overline{T'}}{dr}-\frac{d\overline{T'}}{dr})$$

$$\frac{1}{\alpha}\cdot\frac{d\overline{T'}}{dt}=\frac{d^2\overline{T'}}{dr^2}$$

turns into a cartesian problem.

Applying Separation of Variables:

$$\overline{T'}(r,t)=\Gamma(t)\Psi(r)$$

Boundary Conditions:

$$\Psi(r=0)=finite$$

$$\frac{d\Psi}{dr}(r=R)=0$$

Applying the new variables, dividing both sides by $\Gamma(t)\Psi(r)$, and setting it to a new variable $-\beta^2$:

$$\frac{1}{\alpha\Gamma}\cdot\frac{d\Gamma}{dt}=\frac{d^2\Psi}{dr^2}\frac{1}{\Psi}=-\beta^2$$

Solving for Ψ first:

$$\frac{d^2\Psi}{dr^2}\frac{1}{\Psi}=-\beta^2$$

$$\frac{d^2\Psi}{dr^2}+\beta^2\Psi=0$$

$$\Psi(r)=C_1sin(\beta r)+C_2cos(\beta r)$$

Applying the first boundary condition, interpreting as $\Psi(r=0)=0, C_2=0$
Applying the second boundary condition,

$$-\frac{C_1\sin(\beta R)}{R^2}+C_1\cos(\beta r)\beta=0$$

$$-\sin(\beta R)+R\beta\cos\beta R=0$$

$$\tan(\beta R)=R\beta$$

β is the zeros of that equation.
Solving for $\Gamma(t)$:

$$\frac{1}{\alpha\Gamma}\cdot\frac{d\Gamma}{dt}=-\beta_n^2$$

$$\frac{1}{\alpha\Gamma}\cdot\frac{d\Gamma}{dt}=-\beta_n^2\alpha\Gamma$$

$$\Gamma(t)=A_1e^{-\alpha\beta_n^2t}$$

This makes $\overline{T'}(r,t)$:

$$\overline{T'}(r,t)=A_ne^{-\alpha\beta_n^2t}sin(\beta_nr)$$

Solving back for $T'(r,t)$:

$$T'(r,t)=\frac{A_n}{r}e^{-\alpha\beta_n^2t}sin(\beta_nr)$$

Applying initial condition and orthogonality:

$$T'(r,0)=\frac{A_n}{r}sin(\beta_nr)=\frac{T_0}{2}(1-\cos(\frac{\pi\cdot r}{R}))+C_2$$

$$A_n=\frac{\int_0^R r^3sin(\beta_nr)(\frac{T_0}{2}\cdot(1-\cos\frac{\pi r}{R})+C_2)dr}{\int_0^R r^2\sin^2\beta_nrdr}$$

The analytical solution is:

$$T'(r,t) = \frac{\int_0^R r^3 sin(\beta_n r)(\frac{T_0}{2} \cdot (1 - \cos\frac{\pi r}{R}) + T_{ss})dr}{\int_0^R r^2 \sin^2 \beta_n r dr} e^{-\alpha \beta_n^2 t} sin(\beta_n r)$$

$$T = \frac{\int_0^R r^3 sin(\beta_n r)(\frac{T_0}{2} \cdot (1 - \cos\frac{\pi r}{R}) + T_{ss})dr}{\int_0^R r^2 \sin^2 \beta_n r dr} e^{-\alpha \beta_n^2 t} sin(\beta_n r) + T_{ss}$$

Since the solution from section 4.1 with $n = 0$ covers the steady state case, this problem is redundant and provides no new insight into the problem from section 4.1.

4.3 Code1

```
<institution>
      <name>NO_ddd</name>
       <config><NO_ddd/></config>
      <reactor_list>
        <val>lwr</val>
        <val>sfr</val>
        <val>mox_lwr</val>
      </reactor_list>
```