CP1 for NPRE 501

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1 Problem Definition

Table 1 lists the constants used in the problem.

Parameter	Value	[Unit]
Diamater	6	[cm]
Radius	**3**	[cm]
Geometry	Sphere	
k	15	$[\frac{W}{mK}]$
Density	8000	$[\frac{kg}{m^3}]$
Specific Heat	500	$[\frac{J}{kgK}]$
α	**3.75e-6**	$[\frac{m^2}{s}]$

Table 1: Problem Constants. Derived constants are in bold.

Differential Equation:

$$\frac{1}{\alpha} \cdot \frac{dT}{dt} = \frac{1}{r^2} \frac{d}{dr} r^2 \frac{dT}{dr}$$

Boundary Conditions:

$$T(0,t) = finite \quad OR \quad \frac{dT}{dr}(r=0,t) = 0$$

$$\frac{dT}{dr}(r=R) = 0$$

Initial Condition:

$$T(r,0) = \frac{T_0}{2}(1 - \cos(\frac{\pi \cdot r}{R}))$$

2 Analytical Term Study

For the term study, it is speculated that for higher time values, less terms (β_n) would be needed for reasonable convergence, since a higher time value would cause the exponential term, which has β in, less of a contribution.

$$\overline{T}(r,t) = \sum_{n=1}^{\infty} A_n e^{-\alpha\beta_n^2 t} sin(\beta_n r)$$

The study is conducted where different number of betas are used for each timestep. Six timesteps are used: $t = 0, 4, 17.7, 31.1, 62.2, 75$. A 'reasonable convergence' is met when

$$\sum_{r=0}^{R}(|r_r^n - r_r^{n+1}|) < 1e - 3$$

Where n is the number of terms used for the temperature profile.

Table 2 organizes the 'reasonable convergence' point of all timesteps.

Time [secs]	**Number of Terms for Convergence**
0	17
4	5
17.7	3
31.1	2
62.2	2
75	2

Table 2: Timestep and number of terms for convergence.

As expected, as time increases, the number of terms for convergence becomes minimal. Figure 1, 2, 3, 4, 5, 6 show the convergence study where the temperature profile is plotted with increasing number of terms. Note the visible difference is negligible. Also note the y axis range change for different times.

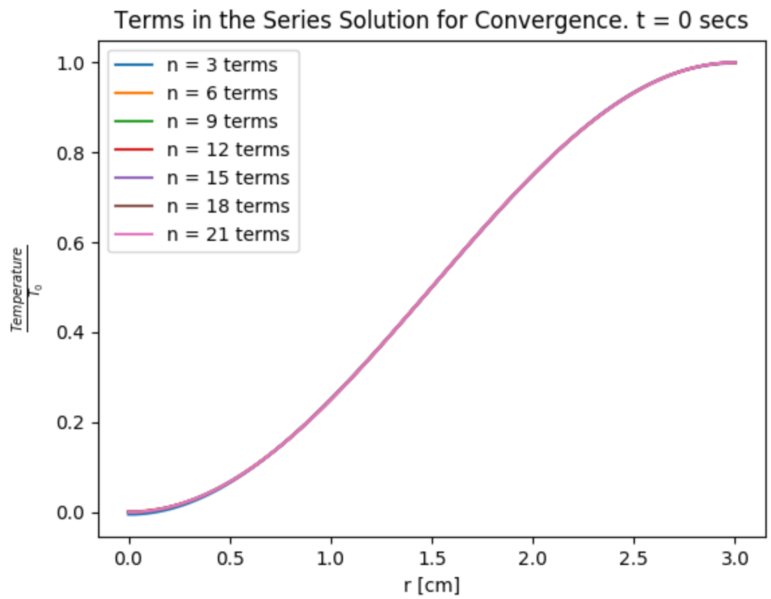

Figure 1: Number of terms convergence study for t = 0.

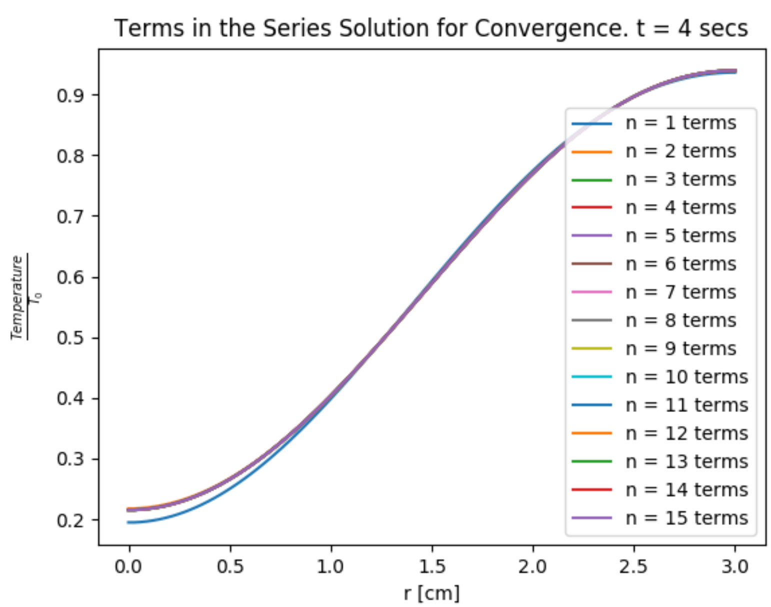

Figure 2: Number of terms convergence study for t = 4.

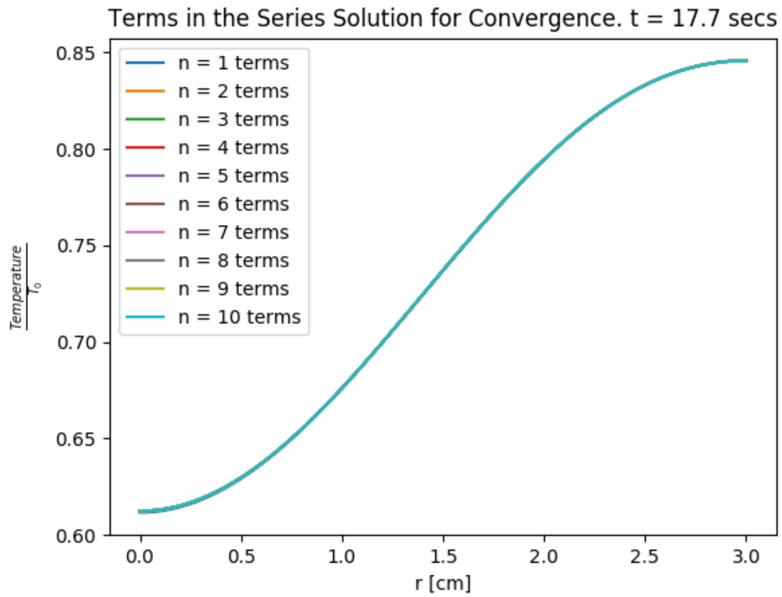

Figure 3: Number of terms convergence study for t = 17.7.

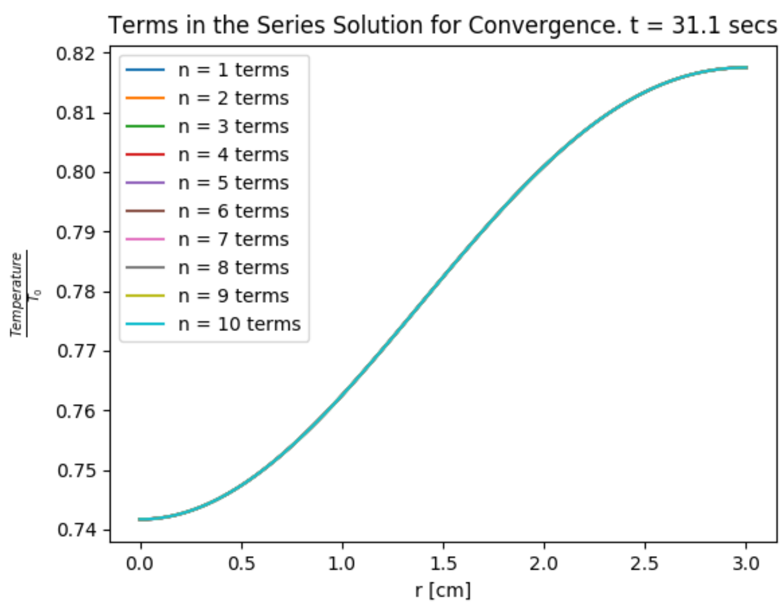

Figure 4: Number of terms convergence study for t = 31.1.

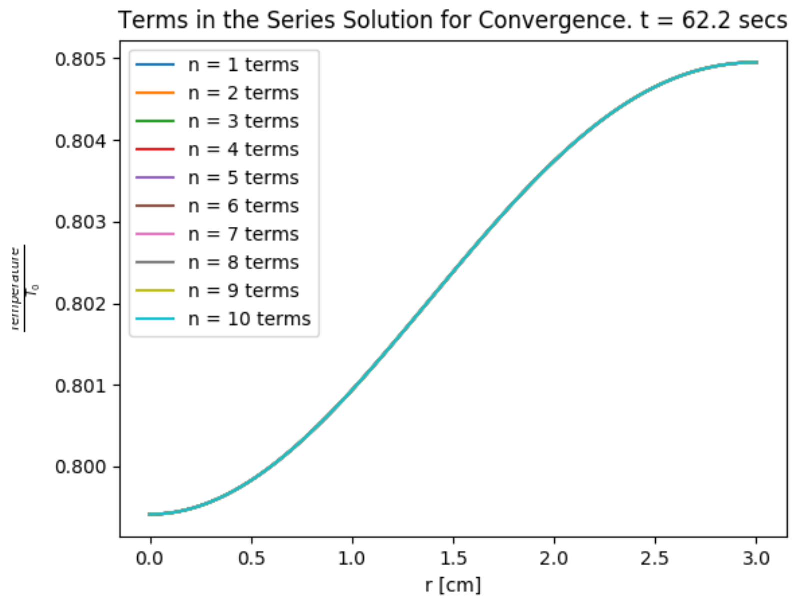

Figure 5: Number of terms convergence study for t = 62.2.

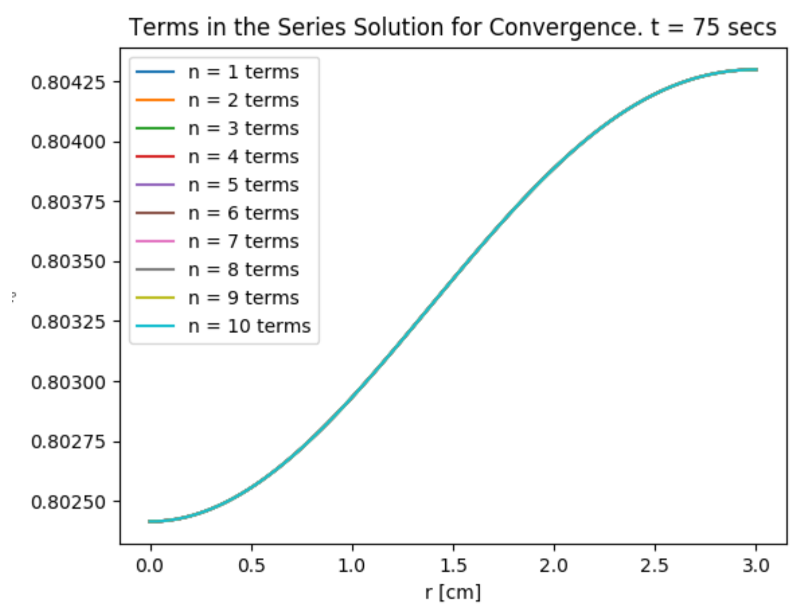

Figure 6: Number of terms convergence study for t = 75.

3 Appendix A

3.1 Simple Energy Balance

Since there is no heat generation or leakage (r=R is insulated), the steady state temperature is a constant throughout the sphere.

Doing a simple energy balance,

$$\frac{dE}{dt} = E_{gain} - E_{loss}$$

$$\frac{dE}{dt} = E_{gen} + E_{in} - E_{out}$$

Since there is no heat generation, flux in, or flux out (insulated BC), the steady state is a constant temperature in time.

Expressing mathematically for when time goes to infinity:

T_{ss} is when $\frac{dT}{dt} = 0$:

$$0 = \frac{1}{r^2}\frac{d}{dr}r^2(\frac{dT_{ss}}{dr})$$

$$\frac{C_1}{r^2} = \frac{dT_{ss}}{dr}$$

$$-\frac{C_1}{r} + C_2 = T_{ss}$$

applying BC:

at r = 0, finite, makes $C_1 = 0$

$$T_{ss} = C_2$$

the numerical value of this constant can be found doing an energy balance of the system with the initial condition:

$$\int_0^R \frac{T_0}{2}(1 - \cos(\frac{\pi r}{R}))r^2 dr = \int_0^R Cr^2 dr$$

$$\frac{T_0}{2}\int_0^R (r^2 - r^2\cos(\frac{\pi r}{R}))dr = C\frac{R^3}{3}$$

using separation of variables:

$$\frac{T_0}{2}(\frac{R^3}{3}-(\frac{r^2R}{\pi}\sin(\frac{\pi r}{R})\Big|_0^R-\frac{2R}{\pi}\int_0^R r\sin(\frac{\pi r}{R})))dr = C\frac{R^3}{3}$$

Solving this:

$$\frac{T_0}{2}(\frac{R^3}{3}+\frac{2R^3}{\pi^2}) = C\frac{R^3}{3}$$

$$C = T_{ss} = \frac{3T_0}{2}(\frac{\pi^2+6}{3\pi^2}) \approx 0.804T_0$$

The steady state is plotted in figure fig. 7.

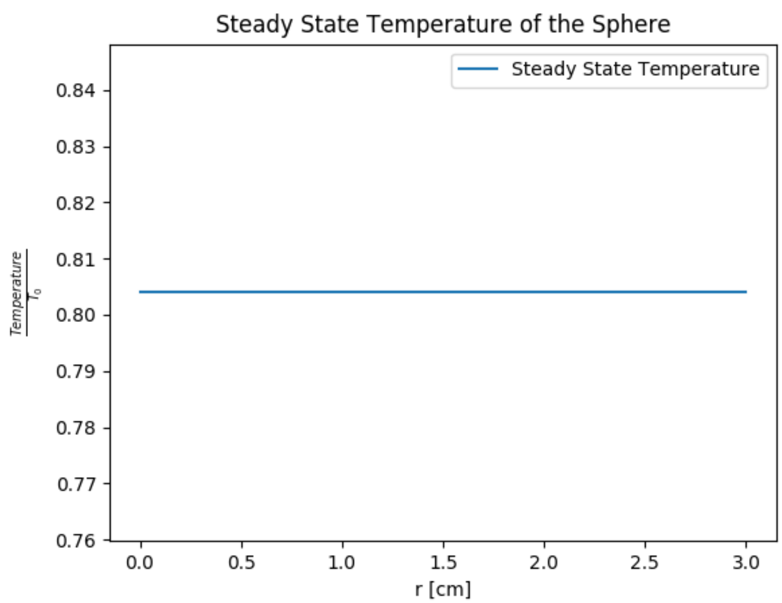

Figure 7: Plot of Steady State Solution for the Sphere.

4 Sketch of Different Temperature Profiles in Time

The time evolution of the temperature profile starts from t=0, with the two boundary points, $T(0,0) = 0$ and $T(R,0) = T_0$. Through time, the temperature profile flattens out, since the only thing affecting the profile is heat diffusion. The steady state (as time goes to infinity) becomes a constant throughout space. A sketch is made in Figure 8.

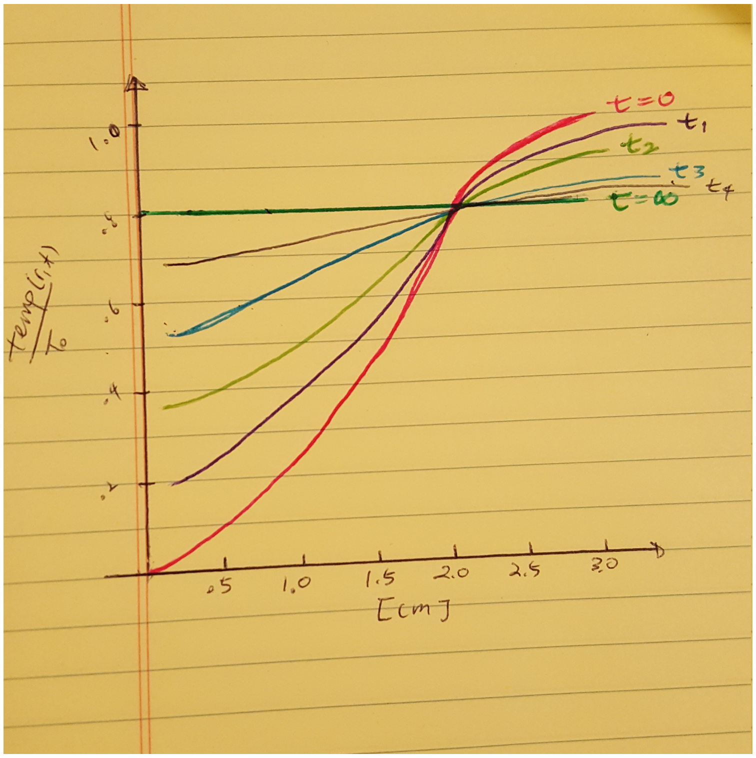

Figure 8: Sketch of Temperature profile over time.

5 Solution of $T(r,t)$ for $t > 0$

Since the numerical value of T_{ss} has been found through the energy balance, that can be added to the solution for $T(r,t)$ for $t > 0$.

The application and solution for $T(r,t)$ is shown in Appendix B.

6 Appendix B

6.1 Analytical solution for solving T(r,t) directly

$$\frac{1}{\alpha} \cdot \frac{dT}{dt} = \frac{1}{r^2}\frac{d}{dr}r^2\frac{dT}{dr}$$

Boundary Conditions:

$$T(0,t) = finite$$

$$\frac{dT}{dr}(r = R) = 0$$

Initial Condition:

$$T(r,0) = \frac{T_0}{2}(1 - \cos{(\frac{\pi \cdot r}{R}}))$$

set:

$$T(r,t) = \frac{\overline{T}(r,t)}{r} + T_s s$$

$$\overline{T}(r,t) = T(r,t)r + T_s sr$$

plugging this back into the differential equation:

$$\frac{1}{\alpha} \cdot \frac{d\overline{T}}{dt}\frac{1}{r} = \frac{1}{r^2}\frac{d}{dr}r^2(\frac{d\overline{T}}{dr}\frac{1}{r} - \frac{1}{r^2}\overline{T})$$

$$\frac{1}{\alpha} \cdot \frac{d\overline{T}}{dt} = \frac{1}{r}\frac{d}{dr}(\frac{d\overline{T}}{dr}r - \overline{T})$$

$$\frac{1}{\alpha} \cdot \frac{d\overline{T}}{dt} = \frac{1}{r}(\frac{d^2\overline{T}}{dr^2}r + \frac{d\overline{T}}{dr} - \frac{d\overline{T}}{dr})$$

$$\frac{1}{\alpha} \cdot \frac{d\overline{T}}{dt} = \frac{d^2\overline{T}}{dr^2}$$

where

$$\overline{T}(0,t) = 0$$

$$\frac{d\overline{T}}{dr} = \frac{\overline{T}}{R}$$

and initial condition

$$\overline{T}(r,0) = r\frac{T_0}{2}(1 - cos(\frac{\pi r}{R})) - T_{ss}r$$

This turns into a cartesian problem.
Applying Separation of Variables:

$$\overline{T}(r,t) = \Gamma(t)\Psi(r)$$

Applying the new variables, dividing both sides by $\Gamma(t)\Psi(r)$, and setting it to a new variable $-\beta^2$:

$$\frac{1}{\alpha\Gamma} \cdot \frac{d\Gamma}{dt} = \frac{d^2\Psi}{dr^2}\frac{1}{\Psi} = -\beta^2$$

Solving for Ψ first:

$$\frac{d^2\Psi}{dr^2}\frac{1}{\Psi} = -\beta^2$$

$$\frac{d^2\Psi}{dr^2} + \beta^2\Psi = 0$$

$$\Psi(r) = C_1 sin(\beta r) + C_2 cos(\beta r)$$

Boundary Conditions:

Since:

$$\frac{dT}{dr} = -\frac{\overline{T}}{r^2} + \frac{1}{r}\frac{d\overline{T}}{dr}$$

The boundary conditions become:

$$-\Psi(r=0) = 0$$

$$-k(\frac{\Psi}{R^2} + \frac{d\Psi}{dr}\frac{1}{R}) = 0$$

Applying the first boundary condition, interpreting as $\Psi(r=0) = 0, C_2 = 0$
Applying the second boundary condition,

$$-\frac{C_1 \sin(\beta R)}{R^2} + C_1 \cos(\beta r)\beta = 0$$

$$-\sin(\beta R) + R\beta \cos \beta R = 0$$

$$\tan(\beta R) = R\beta$$

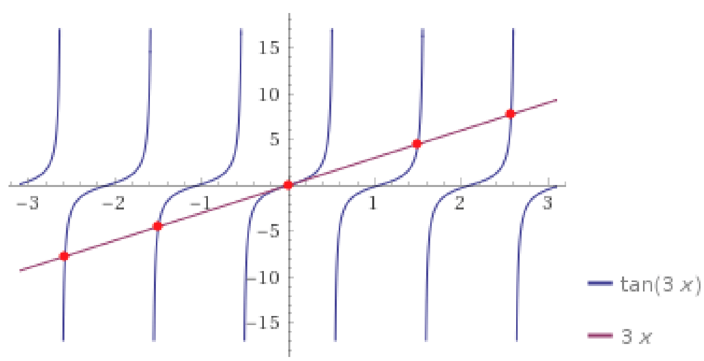

Figure 9: Plot of tan(3x) = 3x.

β is the zeros of that equation (illustrated in fig. 9)
Solving for $\Gamma(t)$:

$$\frac{1}{\alpha\Gamma} \cdot \frac{d\Gamma}{dt} = -\beta_n^2$$

$$\frac{1}{\alpha\Gamma} \cdot \frac{d\Gamma}{dt} = -\beta_n^2 \alpha\Gamma$$

$$\Gamma(t) = A_1 e^{-\alpha\beta_n^2 t}$$

This makes $\overline{T}(r,t)$:

$$\overline{T}(r,t) = \sum_{n=1}^{\infty} A_n e^{-\alpha\beta_n^2 t} sin(\beta_n r)$$

Applying initial condition and orthogonality:

$$T\sum_{n=1}^{\infty} A_n \sin(\beta_n r) = r\frac{T_0}{2}(1-\cos(\frac{\pi r}{R})) - T_{ss} r$$

$$A_n = \frac{\int_0^R (r\frac{T_0}{2}(1-\cos(\frac{\pi r}{R})) - T_{ss} r)\sin(\beta_n r)dr}{\int_0^R (\sin(\beta_n r))^2 dr}$$

Plugging all this into $T(r,t)$:

$$T(r,t) = \sum_{n=1}^{\infty} A_n \frac{\sin(\beta_n r)}{r} e^{-\beta_n^2 \alpha t} + T_{ss}$$

6.2 Analytical solution by defining new temperature

$$T'(r,t) = T(r,t) - T_{ss}$$

$$T'(r,t) + T_{ss} = T(r,t)$$

where T_{ss} is:

$$0 = \frac{1}{r^2}\frac{d}{dr}r^2(\frac{dT_{ss}}{dr})$$

$$\frac{C_1}{r^2} = \frac{dT_{ss}}{dr}$$

$$-\frac{C_1}{r} + C_2 = T_{ss}$$

applying BC:
at r = 0, finite, makes $C_1 = 0$

$$T_{ss} = C_2$$

Plugging the new definition into the original differential equation:
differential equation becomes:

$$\frac{1}{\alpha}(\frac{dT'}{dt} + \frac{dT_{ss}}{dt}) = \frac{1}{r^2}\frac{d}{dr}r^2(\frac{dT'}{dr} + \frac{dT_{ss}}{dr})$$

Considering $\frac{dT_{ss}}{dt} = \frac{dT_{ss}}{dr} = 0$,

$$\frac{1}{\alpha} \cdot \frac{dT'}{dt} = \frac{1}{r^2}\frac{d}{dr}r^2 \cdot \frac{dT'}{dr}$$

Boundary Conditions

$$T'(0,t) = finite$$

$$\frac{dT'}{dr}(r = R) = 0$$

Initial Condition:

$$T'(r,0) = \frac{T_0}{2}(1 - \cos(\frac{\pi \cdot r}{R})) - C_2$$

set:

$$T'(r,t) = \frac{\overline{T'}}{r}$$

$$\frac{1}{\alpha} \cdot \frac{d\overline{T'}}{dt}\frac{1}{r} = \frac{1}{r^2}\frac{d}{dr}r^2(\frac{d\overline{T'}}{dr}\frac{1}{r} - \frac{1}{r^2}\overline{T'})$$

$$\frac{1}{\alpha} \cdot \frac{d\overline{T'}}{dt} = \frac{1}{r}\frac{d}{dr}(\frac{d\overline{T'}}{dr}r - \overline{T'})$$

$$\frac{1}{\alpha} \cdot \frac{d\overline{T'}}{dt} = \frac{1}{r}(\frac{d^2\overline{T'}}{dr^2}r + \frac{d\overline{T'}}{dr} - \frac{d\overline{T'}}{dr})$$

$$\frac{1}{\alpha} \cdot \frac{d\overline{T'}}{dt} = \frac{d^2\overline{T'}}{dr^2}$$

turns into a cartesian problem.
Applying Separation of Variables:

$$\overline{T'}(r,t) = \Gamma(t)\Psi(r)$$

Boundary Conditions:

$$\Psi(r=0) = finite$$

$$\frac{d\Psi}{dr}(r=R) = 0$$

Applying the new variables, dividing both sides by $\Gamma(t)\Psi(r)$, and setting it to a new variable $-\beta^2$:

$$\frac{1}{\alpha\Gamma} \cdot \frac{d\Gamma}{dt} = \frac{d^2\Psi}{dr^2}\frac{1}{\Psi} = -\beta^2$$

Solving for Ψ first:

$$\frac{d^2\Psi}{dr^2}\frac{1}{\Psi} = -\beta^2$$

$$\frac{d^2\Psi}{dr^2} + \beta^2\Psi = 0$$

$$\Psi(r) = C_1 sin(\beta r) + C_2 cos(\beta r)$$

Applying the first boundary condition, interpreting as $\Psi(r = 0) = 0, C_2 = 0$
Applying the second boundary condition,

$$-\frac{C_1 \sin(\beta R)}{R^2} + C_1 \cos(\beta r)\beta = 0$$

$$-\sin(\beta R) + R\beta \cos \beta R = 0$$

$$\tan(\beta R) = R\beta$$

β is the zeros of that equation.
Solving for $\Gamma(t)$:

$$\frac{1}{\alpha\Gamma} \cdot \frac{d\Gamma}{dt} = -\beta_n^2$$

$$\frac{1}{\alpha\Gamma} \cdot \frac{d\Gamma}{dt} = -\beta_n^2 \alpha\Gamma$$

$$\Gamma(t) = A_1 e^{-\alpha\beta_n^2 t}$$

This makes $\overline{T'}(r,t)$:

$$\overline{T'}(r,t) = A_n e^{-\alpha\beta_n^2 t} sin(\beta_n r)$$

Solving back for $T'(r,t)$:

$$T'(r,t) = \frac{A_n}{r} e^{-\alpha\beta_n^2 t} sin(\beta_n r)$$

Applying initial condition and orthogonality:

$$T'(r,0) = \frac{A_n}{r} sin(\beta_n r) = \frac{T_0}{2}(1 - \cos(\frac{\pi \cdot r}{R})) + C_2$$

$$A_n = \frac{\int_0^R r^3 sin(\beta_n r)(\frac{T_0}{2} \cdot (1 - \cos\frac{\pi r}{R}) + C_2)dr}{\int_0^R r^2 \sin^2 \beta_n r dr}$$

The analytical solution is:

$$T'(r,t) = \frac{\int_0^R r^3 sin(\beta_n r)(\frac{T_0}{2}\cdot(1-\cos\frac{\pi r}{R}) + T_{ss})dr}{\int_0^R r^2 \sin^2\beta_n r dr} e^{-\alpha\beta_n^2 t} sin(\beta_n r)$$

$$T = \frac{\int_0^R r^3 sin(\beta_n r)(\frac{T_0}{2}\cdot(1-\cos\frac{\pi r}{R}) + T_{ss})dr}{\int_0^R r^2 \sin^2\beta_n r dr} e^{-\alpha\beta_n^2 t} sin(\beta_n r) + T_{ss}$$

Since the solution from section 4.1 with $n = 0$ covers the steady state case, this problem is redundant and provides no new insight into the problem from section 4.1.

7 Appendix C - Numerical Solution

$$\frac{1}{\alpha}\cdot\frac{dT}{dt} = \frac{2}{r}\frac{dT}{dr} + \frac{d^2T}{dr^2}$$

Applying the finite difference method, central for r and explicit for t:

$$\frac{1}{\alpha}\cdot\frac{T_k^u - T_k^{u-1}}{\Delta t} = \frac{2}{r_k}(\frac{T_{k+1}^{u-1} - T_{k-1}^{u-1}}{2\Delta r}) + \frac{T_{k-1}^{u-1} - 2T_k^{u-1} + T_{k+1}^{u-1}}{\Delta r^2}$$

where u is the temporal step and k is the spacial step.

Solving for T_k^{u+1}:

$$T_k^{u+1} = T_k^u + \alpha\frac{\Delta t}{\Delta r\cdot r}(T_{k+1}^u + T_{k-1}^u) + \alpha * \frac{\Delta t}{(\Delta r)^2}(T_{k+1}^u - 2T_k^u + T_{k-1}^u)$$

Applying neuman boundary condition at r=0:

$$\frac{dT(0,t)}{dr} = 0$$

$$\frac{T_1^u - T_{-1}^u}{2\Delta r} = 0$$

$$T_1^u = T_{-1}^u$$

This gives:

$$T_0^{u+1} = T_0^u + \alpha * \frac{\Delta t}{(\Delta r)^2}(2T_1^u - 2T_0^u)$$

Applying neuman boundary condition at r=R (k=K at r=R):

$$-k\frac{dT(R,t)}{dr} = 0$$

$$\frac{T^u_{K+1} - T^u_{K-1}}{2\Delta r} = 0$$

$$T^u_{K+1} = T^u_{K-1}$$

This gives:

$$T^{u+1}_K = T^u_K + \alpha * \frac{\Delta t}{(\Delta r)^2}(2T^u_{K-1} - 2T^u_K)$$

7.1 Numerical Solution Code

```
def numerical (r_grid, t_max):
    # GRID AND STUFF
    R = 3
    k = 0.15
    rho = 8000. / 1000000.
    cp = 500
    alpha = k / (rho*cp)
    T_0 = 1

    t_grid = 1000

    r_list = np.linspace(0, R, r_grid)
    dr = r_list[1] - r_list[0]
    t_list = np.linspace(0, t_max, t_grid)
    dt = t_list[1] - t_list[0]

    if dt/dr > 1e8:
        raise ValueError('Too high of dt/dr dude')

    t = np.zeros((len(r_list), len(t_list)), dtype=float)

    # Apply Initial Condition
    t[:,0] = (T_0 / 2) * (1 - np.cos(np.pi * r_list / R))
    print(t[:,0])
```

```
    x = dt/dr

    plt.plot(r_list, t[:,0], label='t = 0 secs')

    for timestep in range(1, len(t_list)):
        t[0,timestep] = t[0,timestep-1] + (alpha * (x/dr) * (2*t[1,timestep
        t[-1,timestep] = t[-1, timestep-1] + alpha * (x/dr) * (2*t[-2,times
        for space in range(1, len(r_list)-1):
            first_term = alpha * (x/r_list[space]) * (t[space+1,timestep-1]
            second_term = alpha * (x/dr) * (t[space+1, timestep-1] - (2*t[sp
            t[space, timestep] = t[space, timestep-1] + first_term + second_

        if timestep%100 ==0:
            plt.plot(r_list, t[:,timestep], label='t = %s secs' %str(t_list

    print(t)
    plt.ylabel(r'$\frac{Temperature}{T_0}$')
    plt.xlabel('r [cm]')
    plt.legend()
    plt.title('Numerical Solution of Time-evolution of Heat Profile')
    plt.savefig('Numerical.png', format='png')
    plt.show()
```

7.2 Code1

```
<institution>
      <name>NO_ddd</name>
       <config><NO_ddd/></config>
      <reactor_list>
        <val>lwr</val>
        <val>sfr</val>
        <val>mox_lwr</val>
      </reactor_list>
```