TRANSFINITE ORDINALS
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1 Defining transfinite ordinal numbers

Natural numbers can be represented by sets. Each number is represented by the set of all numbers smaller than it.

- $0 = \{\}$ (the empty set)
- $1 = \{0\} = \{\{\}\}$
- $2 = \{0, 1\} = \{\{\}, \{\{\}\}\}$
- $3 = \{0, 1, 2\} = \{\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}\}$
- ...

The successor of a natural number can be defined by $suc(n) = n + 1 = n \cup \{n\}$.
We have $n \leq p$ if and only if $n \subseteq p$.
$\mathbb{N}$ is the set of all natural numbers : $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ The natural numbers can be generalized to what is called "transfinite ordinal numbers", or simply "ordinal numbers" or "ordinals", by considering that infinite sets represent ordinal numbers. $\mathbb{N}$ considered as an ordinal number is written ω. ω is the least ordinal which is greater than all the numbers 0, 1, 2, 3, ... We say that ω is a limit ordinal and 0, 1, 2, 3, ... is a fundamental sequence of ω. This is written : $\omega = sup\{0, 1, 2, 3, ...\}$ or $\omega = lim(n \mapsto n)$ because the n-th element (starting with 0) of the sequence is n.
Any ordinal can be defined as the least ordinal strictly greater than all ordinals of a set : the empty set for 0, $\{\alpha\}$ for the successor of α, $\{\alpha_0, \alpha_1, \alpha_2, ...\}$ for an ordinal with fundamental sequence $\alpha_0, \alpha_1, \alpha_2, ...$
The successor can be generalized to transfinite ordinal numbers : $suc(\omega) = \omega + 1 = \omega \cup \{\omega\} = \{0, 1, 2, 3, \ldots, \omega\}; suc(suc(\omega)) = \omega + 2 = \{0, 1, 2, 3, \ldots, \omega, \omega + 1\}$ and so on.
Then we can consider the set $\{0, 1, 2, 3, \ldots, \omega, \omega + 1, omega + 2, omega + 3, \ldots\}$ which is a limit ordinal, and $\omega, \omega + 1, \omega + 2, \omega + 3, \ldots$ is a fundamental sequence of this ordinal. This ordinal is $\omega + \omega = \omega \cdot 2$ or $\omega \times 2$ or $\omega 2$.
Then we can go on building greater and greater ordinals : $\omega \cdot 3, \ldots, \omega \cdot \omega = \omega^2, \omega^3, \ldots, \omega^\omega, \omega^{\omega^\omega}, \ldots$.

The definitions of arithmetical operations can be generalized to ordinals :

- addition : $\alpha + 0 = \alpha; \alpha + suc(\beta) = suc(\alpha + \beta); \alpha + lim(f) = lim(n \mapsto \alpha + f(n))$
- multiplication : $\alpha \cdot 0 = 0; \alpha \cdot suc(\beta) = (\alpha \cdot \beta) + \alpha; \alpha \cdot lim(f) = lim(n \mapsto \alpha \cdot f(n))$
- exponentiation : $\alpha^0 = 1; \alpha^{suc(\beta)} = \alpha^\beta \cdot \alpha; \alpha^{lim(f)} = lim(n \mapsto \alpha^{f(n)})$

2 Veblen functions

The next step is the limit or least upper bound of $\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots$ which is called ϵ_0.
Then we can go on with $\epsilon_0 + 1, \epsilon_0 + 2, \ldots, \epsilon_0 + \epsilon_0 = \epsilon_0 \cdot 2, \ldots, \epsilon_0 \cdot \epsilon_0 = \epsilon_0^2, \epsilon_0^{\epsilon_0}, ...$
The limit of $\epsilon_0, \epsilon_0^{\epsilon_0}, \epsilon_0^{\epsilon_0^{\epsilon_0}}, \ldots$ is called ϵ_1. It can be shown that it is also the limit of $\epsilon_0 + 1, \omega^{\epsilon_0 + 1}, \omega^{\omega^{\epsilon_0 + 1}}, \ldots$.
In a similar way, the limit of $\epsilon_1, \epsilon_1^{\epsilon_1}, \epsilon_1^{\epsilon_1^{\epsilon_1}}, \ldots$ is called ϵ_2 and is also the limit of $\epsilon_1 + 1, \omega^{\epsilon_1 + 1}, \omega^{\omega^{\epsilon_1 + 1}}, \ldots$.
We can define the same way ϵ_n for any natural number n. Then ϵ_ω is defined as the limit of $\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3, \ldots$, and $\epsilon_{\omega+1}$ as the limit of $\epsilon_\omega, \epsilon_\omega^{\epsilon_\omega}, \epsilon_\omega^{\epsilon_\omega^{\epsilon_\omega}}, \ldots$ or $\epsilon_\omega + 1, \omega^{\epsilon^\omega + 1}, \omega^{\omega^{\epsilon_\omega + 1}}, \ldots$.
After comes ϵ_{ϵ_0}, and the limit of $\epsilon_0, \epsilon_{\epsilon_0}, \epsilon_{\epsilon_{\epsilon_0}}, \ldots$ which is called ζ_0.

3 Summary

Any ordinal can be defined as the least ordinal strictly greater than all ordinals of a set : the empty set for 0, $\{\alpha\}$ for the successor of α, $\{\alpha_0, \alpha_1, \alpha_2, ...\}$ for an ordinal with fundamental sequence $\alpha_0, \alpha_1, \alpha_2, ...$

4 Algebraic notation

We define the following operations on ordinals :

- addition : $\alpha + 0 = \alpha; \alpha + suc(\beta) = suc(\alpha + \beta); \alpha + lim(f) = lim(n \mapsto \alpha + f(n))$
- multiplication : $\alpha \times 0 = 0; \alpha \times suc(\beta) = (\alpha \times \beta) + \alpha; \alpha \times lim(f) = lim(n \mapsto \alpha \times f(n))$
- exponentiation : $\alpha^0 = 1; \alpha^{suc(\beta)} = \alpha^\beta \times \alpha; \alpha^{lim(f)} = lim(n \mapsto \alpha^{f(n)})$

5 Veblen functions

These functions use fixed points enumaration : $\varphi(\ldots, \beta, 0, \ldots, 0, \gamma)$ represents the $(1 + \gamma)^{th}$ common fixed point of the functions $\xi \mapsto \varphi(\ldots, \delta, \xi, 0, \ldots, 0)$ for all $\delta < \beta$.

6 Simmons notation

$Fixfz = f^w(z + 1)$ = least fixed point of f strictly greater than z.
$Next = Fix(\alpha \mapsto \omega^\alpha)$
$[0]h = Fix(\alpha \mapsto h^\alpha \omega)$; $[1]hg = Fix(\alpha \mapsto h^\alpha g\omega)$; $[2]hgf = Fix(\alpha \mapsto h^\alpha gf\omega)$; etc...
Correspondence with Veblen's ϕ : $\phi(1 + \alpha, \beta) = ([0]^\alpha Next)^{1+\beta}\omega; \phi(\alpha, \beta, \gamma) = ([0]^\beta(([1][0])^\alpha Next))^{1+\gamma}\omega$

7 RHS0 notation

We start from 0, if we don(t see any regularity we take the successor, if we see a regularity, if we have a notation for this regularity, we use it, else we invent it, then we jump to the limit.

$H f x = lim\ x, fx, f(fx), \ldots ; R_1 fgx = lim\ gx, fgx, ffgx, \ldots ; R_2 fghx = lim\ hx, fghx, fgfghx, \ldots$

Correspondence with Simmons notation : $\ldots, [3] \to R5, [2] \to R4, [1] \to R3, [0] \to R2, Next \to R1, \omega \to Hsuc\ 0$

8 Ordinal collapsing functions

These functions use uncountable ordinals to define countable ordinals.

We define sets of ordinals that can be built from given ordinals and operations, then we take the least ordinal which is not in this set, or the least ordinal which is greater than all contable ordinals of this set.

These functions are extensions of functions on countable ordinals, whose fixed points can be reached by applying them to an uncountable ordinal.

Examples :

- Madore's ψ : $\psi(\alpha) = \varepsilon_\alpha$ if $\alpha < \zeta_0; \psi(\Omega) = \zeta_0$ which is the least fixed point of $\alpha \mapsto \varepsilon_\alpha$.
- Feferman's θ : $\theta(\alpha, \beta) = \varphi(\alpha, \beta)$ if $\alpha < \Gamma_0$ and $\beta < \Gamma_0; \theta(\Omega, 0) = \Gamma_0$ which is the least fixed point of $\alpha \mapsto \varphi(\alpha, 0)$.
- Taranovsky's C : $C(\alpha, \beta) = \beta + \omega^\alpha$ if α is countable; $C(\Omega_1, 0) = \varepsilon_0$ which is the least fixed point of $\alpha \mapsto \omega^\alpha$.

Nom	Symbole	Algebraic	Veblen	Simmons	RHS0	Madore	Taranovsky
Zero	0	0			0		0
One	1	1	$\varphi(0,0)$		suc 0		C(0,0)
Two	2	2			suc (suc 0)		C(0,C(0,0))
Omega	ω	ω	$\varphi(0,1)$	ω	H suc 0		C(1,0)
		$\omega+1$			suc (H suc 0)		C(0,C(1,0))
		$\omega \times 2$			H suc (H suc 0)		C(1,C(1,0))
		ω^2	$\varphi(0,2)$		H (H suc) 0		C(C(0,C(0,0)),0)
		ω^ω	$\varphi(0,\omega)$		H H suc 0		C(C(1,0),0)
		ω^{ω^ω}	$\varphi(0,\omega^\omega)$		H H H suc 0		C(C(C(1,0),0),0)
Epsilon zero	ε_0	ε_0	$\varphi(1,0)$	$Next\ \omega$	$R_1 Hsuc\ 0$	$\psi(0)$	$C(\Omega_1, 0)$
		ε_1	$\varphi(1,1)$	$Next^2\omega$	$R_1(R_1 H)suc\ 0$	$\psi(1)$	$C(\Omega_1, C(\Omega_1, 0)$
		ε_ω	$\varphi(1,\omega)$	$Next^\omega\omega$	$HR_1 Hsuc\ 0$	$\psi(\omega)$	$C(C(0,\Omega_1),0)$
		$\varepsilon_{\varepsilon_0}$	$\varphi(1,\varphi(1,0))$	$Next^{Next\omega}\omega$	$R_1 HR_1 Hsuc\ 0$	$\psi(\psi(0))$	$C(C(C(\Omega_1,0),\Omega_1),0)$
Zeta zero	ζ_0	ζ_0	$\varphi(2,0)$	$[0]Next\ \omega$	$R_2 R_1 Hsuc\ 0$	$\psi(\Omega)$	$C(C(\Omega_1,\Omega_1),0)$
Eta zero	η_0	η_0	$\varphi(3,0)$	$[0]^2 Next\ \omega$	$R_2(R_2 R_1)Hsuc\ 0$		$C(C(\Omega, C(\Omega,\Omega)),0)$
			$\varphi(\omega,0)$	$[0]^\omega Next\ \omega$	$HR_2 R_1 Hsuc\ 0$		$C(C(C(0,\Omega_1),\Omega_1),0)$
Feferman -Schütte	Γ_0	Γ_0	$\varphi(1,0,0)$ $=\varphi(2 \mapsto 1)$	$[1][0]Next\ \omega$	$R_3 R_2 R_1 Hsuc\ 0$ $= R_{3\ldots1} Hsuc\ 0$	$\psi(\Omega^\Omega)$	$C(C(C(\Omega_1,\Omega_1),$ $\Omega_1),0)$
Ackermann			$\varphi(1,0,0,0)$ $=\varphi(3 \mapsto 1)$	$[1]^2[0]Next\ \omega$	$R_3(R_3 R_2)R_1 Hsuc\ 0$	$\psi(\Omega^{\Omega^2})$	
Small Veblen ordinal			$\varphi(\omega \mapsto 1)$	$[1]^\omega[0]Next\ \omega$	$HR_3 R_2 R_1 Hsuc\ 0$	$\psi(\Omega^{\Omega^\omega})$	$C(\Omega_1^\omega, 0)$ $= C(C(C(C(0,\Omega_1),$ $\Omega_1),\Omega_1),0)$
Large Veblen ordinal			least ord. not rep.	$[2][1][0]Next\ \omega$	$R_4 R_3 R_2 R_1 Hsuc\ 0$ $= R_{4\ldots1} Hsuc\ 0$	$\psi(\Omega^{\Omega^\Omega})$	$C(\Omega_1^{\Omega_1}, 0)$ $= C(C(C(C(\Omega_1,\Omega_1),$ $\Omega_1),\Omega_1),0)$
Bachmann-Howard ordinal				least ord. not rep.	$R_{\omega\ldots1} Hsuc\ 0$	$\psi(\varepsilon_{\Omega+1})$	$C(C(\Omega_2,\Omega_1),0)$