TRANSFINITE ORDINALS
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1 Defining transfinite ordinal numbers

Natural numbers can be represented by sets. Each number is represented by the set of all numbers smaller than it.

- $0 = \{\}$ (the empty set)
- $1 = \{0\} = \{\{\}\}$
- $2 = \{0, 1\} = \{\{\}, \{\{\}\}\}$
- $3 = \{0, 1, 2\} = \{\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}\}$
- ...

The successor of a natural number can be defined by $suc(n) = n + 1 = n \cup \{n\}$.

We have $n \leq p$ if and only if $n \subseteq p$.

$\mathbb{N}$ is the set of all natural numbers : $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ The natural numbers can be generalized to what is called "transfinite ordinal numbers", or simply "ordinal numbers" or "ordinals", by considering that infinite sets represent ordinal numbers. $\mathbb{N}$ considered as an ordinal number is written ω. ω is the least ordinal which is greater than all the numbers 0, 1, 2, 3, ... We say that ω is a limit ordinal and 0, 1, 2, 3, ... is a fundamental sequence of ω. This is written : $\omega = sup\{0, 1, 2, 3, \ldots\}$ or $\omega = lim(n \mapsto n)$ because the n-th element (starting with 0) of the sequence is n. An ordinal does not have a unique fundamental sequence, for example 1, 2, 3, 4, ... is also a fundamental sequence of ω, because the least ordinal that is greater than all ordinals of this sequence is also ω (more generally the limit ordinal is the same if any number of the least items of a sequence are removed), and the same stands for the sequence 0, 2, 4, 6, ...

Any ordinal can be defined as the least ordinal strictly greater than all ordinals of a set : the empty set for 0, $\{\alpha\}$ for the successor of α, $\{\alpha_0, \alpha_1, \alpha_2, \ldots\}$ for an ordinal with fundamental sequence $\alpha_0, \alpha_1, \alpha_2, \ldots$

The successor can be generalized to transfinite ordinal numbers : $suc(\omega) = \omega + 1 = \omega \cup \{\omega\} = \{0, 1, 2, 3, \ldots, \omega\}$; $suc(suc(\omega)) = \omega + 2 = \{0, 1, 2, 3, \ldots, \omega, \omega + 1\}$ and so on.

Then we can consider the set $\{0, 1, 2, 3, \ldots, \omega, \omega + 1, \omega + 2, \omega + 3, \ldots\}$ which is a limit ordinal, and $\omega, \omega + 1, \omega + 2, \omega + 3, \ldots$ is a fundamental sequence of this ordinal. This ordinal is $\omega + \omega = \omega \cdot 2$ or $\omega \times 2$ or $\omega 2$.

Then we can go on building greater and greater ordinals : $\omega \cdot 3, \ldots, \omega \cdot \omega = \omega^2, \omega^3, \ldots, \omega^\omega, \omega^{\omega^\omega}, \ldots$

For natural numbers, arithmetical operations are defined as follows :

- addition : $a + 0 = a; a + suc(b) = suc(a + b)$
- multiplication : $a \cdot 0 = 0; a \cdot suc(b) = (a \cdot b) + a$
- exponentiation : $a^0 = 1; a^{suc(b)} = a^b \cdot a$

The definitions of arithmetical operations can be generalized to ordinals by adding canonical rules for limit ordinals :

- addition : $\alpha + 0 = \alpha; \alpha + suc(\beta) = suc(\alpha + \beta); \alpha + lim(f) = lim(n \mapsto \alpha + f(n))$
- multiplication : $\alpha \cdot 0 = 0; \alpha \cdot suc(\beta) = (\alpha \cdot \beta) + \alpha; \alpha \cdot lim(f) = lim(n \mapsto \alpha \cdot f(n))$
- exponentiation : $\alpha^0 = 1; \alpha^{suc(\beta)} = \alpha^\beta \cdot \alpha; \alpha^{lim(f)} = lim(n \mapsto \alpha^{f(n)})$

Note that addition and multiplication are not commutative, for example $1 + \omega = \omega \neq \omega + 1$, because if we take 0, 1, 2, 3, ... as fundamental sequence of ω, then a fundamental sequence of $1 + \omega$ is 1+0, 1+1, 1+2, 1+3, ... = 1, 2, 3, 4, ... and the least ordinal which is greater than all ordinals of this sequence is ω. We will say that "1+" is "absorbed" by ω. More generally, we have $1 + \alpha = \alpha$ for any ordinal $\alpha \geq \omega$.

2 Veblen functions

The next step is the limit or least upper bound of $\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots$ which is called ε_0. Note that we have $\omega^{\varepsilon_0} = \varepsilon_0$. We say that ε_0 is a fixed point (the least one) of the function $\alpha \mapsto \omega^\alpha$.

Then we can go on with $\varepsilon_0 + 1, \varepsilon_0 + 2, \ldots, \varepsilon_0 + \varepsilon_0 = \varepsilon_0 \cdot 2, \ldots, \varepsilon_0 \cdot \varepsilon_0 = \varepsilon_0^2, \varepsilon_0^{\varepsilon_0}, \ldots$

The limit of $\varepsilon_0, \varepsilon_0^{\varepsilon_0}, \varepsilon_0^{\varepsilon_0^{\varepsilon_0}}, \ldots$ is called ε_1. It can be shown that it is also the limit of $\varepsilon_0 + 1, \omega^{\varepsilon_0 + 1}, \omega^{\omega^{\varepsilon_0 + 1}}, \ldots$ (see proof below). These two fundamental sequences are examples of two ways of ascending ordinals :

- Build greater ordinals from known ones by increasing them using operations like successor, addition, multiplication, exponentiation, ... This method is used by the RSH0 notation which we will study later.
- When we have found a function that, when applied to a given ordinal, gives a greater one (for example $\alpha \mapsto \omega^\alpha$), use it ad infinitum starting for example with 0, and then to go further use it ad infinitum starting with the successor of the previous result, and so on. This is called an enumeration of the fixed points of this function. A fixed point of a function f is a value (for example an ordinal) α with $f(\alpha) = \alpha$. Under some conditions (see below), the least fixed point of f is the limit of 0, f(0), f(f(0)), f(f(f(0))), ... If it is called α, the next fixed point is the limit of $\alpha+1, f(\alpha+1), f(f(\alpha+1)), f(f(f(\alpha+1))), \ldots$. More generally, the least fixed point of f that is greater or equal to ζ is the limit of $\zeta, f(\zeta), f(f(\zeta)), \ldots$. The Veblen functions use this method.

The required conditions are described for example in http://www.cs.man.ac.uk/ hsimmons/ORDINAL-NOTATIONS/Fruitful.pdf page 8 lemma 3.9 :
For each fruitful function f and each ordinal ζ, $f^\omega(\zeta+1)$ is the least ordinal ν such that $\zeta < \nu = f(\nu)$, or the least fixed point of f that is strictly greater than ζ (or greater than or equal to $\zeta + 1$).
$f^\omega(\zeta+1)$ is the limit of $\zeta+1, f(\zeta+1), f(f(\zeta+1)), \ldots$.
A fruitful function is a function that is inflationary, monotone, big, and continuous.
A function f is inflationary if $\alpha \le f(\alpha)$, monotone if $\alpha \le \beta \Rightarrow f(\alpha) \le f(\beta)$, big if $\omega^\alpha \le f(\alpha)$ except possibly for $\alpha = 0$, continuous if f(VA) = Vf[A] where VA is the pointwise supremum of the set A.

We will now prove by induction the equivalence of the two fundamental sequences above.

We will use the notation $\alpha^{\cdot^{\cdot^{\alpha^\beta}}}$ for an an "exponential tower" with α repeated n times.

Note that the ordinals respectively limits of the fondamental sequence whose n-th term is $\varepsilon_0^{\cdot^{\cdot^{\varepsilon_0^{\varepsilon_0}}}}$ and the one whose n-th term is $\varepsilon_0^{\cdot^{\cdot^{\varepsilon_0^{\varepsilon_0^0}}}}$ is the same, the least fixed point of the function $\alpha \mapsto \varepsilon_0{}^\alpha$, which is greater than ω and also than ε_0.

So we have proved what we want if we prove that, for any n, we have $\omega^{\omega^{\cdot^{\cdot^{\omega^{\omega^{\varepsilon_0+1}}}}}} = \varepsilon_0^{\cdot^{\cdot^{\varepsilon_0^{\varepsilon_0^\omega}}}}$.
For n = 0, we have $\omega^{\omega^{\varepsilon_0+1}} = \omega^{\omega^{\varepsilon_0}\cdot\omega} = \omega^{\varepsilon_0\cdot\omega} = (\omega^{\varepsilon_0})^\omega = \varepsilon_0{}^\omega$.

Now suppose we have $\omega^{\omega^{\cdot^{\cdot^{\omega^{\omega^{\varepsilon_0+1}}}}}} = \varepsilon_0^{\cdot^{\cdot^{\varepsilon_0^{\varepsilon_0^\omega}}}}$.

We must prove the equality for n+1, which can be written $\omega^{\omega^{\omega^{\cdot^{\cdot^{\omega^{\omega^{\varepsilon_0+1}}}}}}} = \varepsilon_0^{\cdot^{\cdot^{\varepsilon_0^{\varepsilon_0^{\varepsilon_0^\omega}}}}}$.

We have $\omega^{\omega^{\cdot^{\cdot^{\omega^{\omega^{\varepsilon_0+1}}}}}} = \omega^{\varepsilon_0^{\cdot^{\cdot^{\varepsilon_0^{\varepsilon_0^\omega}}}}}$ (by our hypothesis) $= \omega^{\varepsilon_0^{1+\varepsilon_0^{\cdot^{\cdot^{\varepsilon_0^\omega}}}}}$ (for the same reason than $1 + \omega = \omega$, see above) $= \omega^{\varepsilon_0\cdot\varepsilon_0^{\cdot^{\cdot^{\varepsilon_0^\omega}}}} =$
$(\omega^{\varepsilon_0})^{\varepsilon_0^{\cdot^{\cdot^{\varepsilon_0^\omega}}}} = \varepsilon_0^{\cdot^{\cdot^{\varepsilon_0^{\varepsilon_0^\omega}}}}$. QED.
In a similar way, the limit of $\varepsilon_1, \varepsilon_1^{\varepsilon_1}, \varepsilon_1^{\varepsilon_1^{\varepsilon_1}}, \ldots$ is called ε_2 and is also the limit of $\varepsilon_1 + 1, \omega^{\varepsilon_1+1}, \omega^{\omega^{\varepsilon_1+1}}, \ldots$.
We can define the same way ε_n for any natural number n. Then ε_ω is defined as the limit of $\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$, and $\varepsilon_{\omega+1}$ as the limit of $\varepsilon_\omega, \varepsilon_\omega^{\varepsilon_\omega}, \varepsilon_\omega^{\varepsilon_\omega^{\varepsilon_\omega}}, \ldots$ or $\varepsilon_\omega + 1, \omega^{\varepsilon_\omega+1}, \omega^{\omega^{\varepsilon_\omega+1}}, \ldots$.
After comes $\varepsilon_{\varepsilon_0}$, and the limit of $\varepsilon_0, \varepsilon_{\varepsilon_0}, \varepsilon_{\varepsilon_{\varepsilon_0}}, \ldots$ which is called ζ_0. This is the least fixed point of $\alpha \mapsto \varepsilon_\alpha$. The next one is ζ_1 which is the limit of $\zeta_0 + 1, \varepsilon_{\zeta_0+1}, \varepsilon_{\varepsilon_{\zeta_0+1}}, \ldots$. Then we get $\zeta_2, \zeta_3, \ldots, \zeta_\omega, \zeta_{\omega+1}, \ldots, \zeta_{\varepsilon_0}, \ldots, \zeta_{\zeta_0}, \ldots, \zeta_{\zeta_{\zeta_0}}, \ldots$. The limit of $0, \zeta_0, \zeta_{\zeta_0}, \zeta_{\zeta_{\zeta_0}}, \ldots$ is called η_0.
We could go on using successively different greek letters, or define a function φ with two variables by :

- $\varphi(0, \alpha) = \varphi(\alpha) = \omega^\alpha$
- $\varphi(1, \alpha) = \varepsilon_\alpha$
- $\varphi(2, \alpha) = \zeta_\alpha$
- $\varphi(3, \alpha) = \eta_\alpha$
- $\varphi(\alpha + 1, \beta)$ is the $(1 + \beta)$-th fixed point of $\xi \mapsto \varphi(\alpha, \xi)$.

Then we can enumerate the fixed points of the function $\alpha \mapsto \varphi(\alpha, 0)$ and define Γ_α as the $(1 + \alpha)$-th fixed point of this function, or add another variable to the φ function and define $\varphi(1, 0, \alpha)$ as the $(1 + \alpha)$-th fixed point of this function. So we have $\Gamma_\alpha = \varphi(1, 0, \alpha)$.

More generally, we can define $\varphi(\alpha_n, \alpha_{n-1}, \ldots, \alpha_1, \alpha_0)$.

It is sometimes written $\varphi_{\alpha_n, \alpha_{n-1}, \ldots, \alpha_1}(\alpha_0)$ since α_0 plays a different role than the other variables.

See for example https://en.wikipedia.org/wiki/Veblen_function :

"Let z be an empty string or a string consisting of one or more comma-separated zeros $0, 0, \ldots, 0$ and s be an empty string or a string consisting of one or more comma-separated ordinals $\alpha_1, \alpha_2, \ldots, \alpha_n$ with $\alpha_1 > 0$. The binary function $\varphi(\beta, \gamma)$ can be written as $\varphi(s, \beta, z, \gamma)$ where both s and z are empty strings.

The finitary Veblen functions are defined as follows:

- $\varphi(\gamma) = \omega^\gamma$
- $\varphi(z, s, \gamma) = \varphi(s, \gamma)$
- if $\beta > 0$, then $\varphi(s, \beta, z, \gamma)$ denotes the $(1 + \gamma)$-th common fixed point of the functions $\xi \mapsto \varphi(s, \delta, \xi, z)$ for each $\delta < \beta$

(...)

The limit of the $\varphi(1, 0, \ldots, 0)$ where the number of zeroes ranges over ω, is sometimes known as the "small" Veblen ordinal.

Every non-zero ordinal α less than the small Veblen ordinal (SVO) can be uniquely written in normal form for the finitary Veblen function:

$\alpha = \varphi(s_1) + \varphi(s_2) + \cdots + \varphi(s_k)$

where

- k is a positive integer
- $\varphi(s_1) \geq \varphi(s_2) \geq \cdots \geq \varphi(s_k)$
- s_m is a string consisting of one or more comma-separated ordinals $\alpha_{m,1}, \alpha_{m,2}, \ldots, \alpha_{m,n_m}$ where $\alpha_{m,1} > 0$ and each $\alpha_{m,i} < \varphi(s_m)$

For limit ordinals $\alpha < SVO$, written in normal form for the finitary Veblen function:

- $(\varphi(s_1) + \varphi(s_2) + \cdots + \varphi(s_k))[n] = \varphi(s_1) + \varphi(s_2) + \cdots + \varphi(s_k)[n]$,
- $\varphi(\gamma)[n] =$
    - n if $\gamma = 1$
    - $\varphi(\gamma - 1) \cdot n$ if γ is a successor ordinal
    - $\varphi(\gamma[n])$ if γ is a limit ordinal
- $\varphi(s, \beta, z, \gamma)[0] = 0$ and $\varphi(s, \beta, z, \gamma)[n+1] = \varphi(s, \beta - 1, \varphi(s, \beta, z, \gamma)[n], z)$ if $\gamma = 0$ and β is a successor ordinal,
- $\varphi(s, \beta, z, \gamma)[0] = \varphi(s, \beta, z, \gamma - 1) + 1$ and $\varphi(s, \beta, z, \gamma)[n+1] = \varphi(s, \beta - 1, \varphi(s, \beta, z, \gamma)[n], z)$ if γ and β are successor ordinals,
- $\varphi(s, \beta, z, \gamma)[n] = \varphi(s, \beta, z, \gamma[n])$ if γ is a limit ordinal,
- $\varphi(s, \beta, z, \gamma)[n] = \varphi(s, \beta[n], z, \gamma)$ if $\gamma = 0$ and β is a limit ordinal,
- $\varphi(s, \beta, z, \gamma)[n] = \varphi(s, \beta[n], \varphi(s, \beta, z, \gamma - 1) + 1, z)$ if γ is a successor ordinal and β is a limit ordinal. "

The Veblen function can be generalized to transfinitely many variables with a finite number different from 0. Instead of writing the list of all the variable of the Veblen function, we can write only the non zero variables with position as indice, for example $\varphi(\alpha, 0, \beta, \gamma) = \varphi(\alpha_3, \beta_1, \gamma_0)$. We can then generalize the Veblen function by allowing any ordinal as indices, writing for example $SVO = \varphi(1_\omega)$. The limit of the ordinals that can be written with this notation is called the large Veblen ordinal (LVO).

According to Wikipedia, "The definition can be given as follows: let α be a transfinite sequence of ordinals (i.e., an ordinal function with finite support) which ends in zero (i.e., such that $\alpha_0 = 0$), and let $\alpha[0 \mapsto \gamma]$ denote the same function where the final 0 has been replaced by γ. Then $\gamma \mapsto \varphi(\alpha[0 \mapsto \gamma])$ is defined as the function enumerating the common fixed points of all functions $\xi \mapsto \varphi(\beta)$ where β ranges over all sequences which are obtained by decreasing the smallest-indexed nonzero value of α and replacing some smaller-indexed value with the indeterminate ξ (i.e., $\beta = \alpha[\iota_0 \mapsto \zeta, \iota \mapsto \xi]$ meaning that for the smallest index ι_0 such that α_{ι_0} is nonzero the latter has been replaced by some value $\zeta < \alpha_{\iota_0}$ and that for some smaller index $\iota < \iota_0$, the value $\alpha_\iota = 0$ has been replaced with ξ)."

Schütte brackets or Klammersymbols are another way to write Veblen fuctions with transfinitely many variables. A Schütte bracket consists in a matrix with two lines, with the positions of the variables in the second line in increasing order, and the corresponding values in the first line. This matrix is preceded by the function $\xi \mapsto \varphi(\xi)$. If we take $\xi \mapsto \omega^\xi$, we get the equivalent of the Veblen function. With this notation, the previous example is written :

$$(\xi \mapsto \omega^\xi) \begin{pmatrix} \gamma & \beta & \alpha \\ 0 & 1 & 3 \end{pmatrix}$$

In some of his papers, Harold Simmons puts the function after the matrix, which is more logical, the matrix being considered as a function which, when applied to a function, gives an ordinal :

$$\begin{pmatrix} \gamma & \beta & \alpha \\ 0 & 1 & 3 \end{pmatrix} (\xi \mapsto \omega^\xi)$$

When the function at the left of the matrix is $\xi \mapsto \omega^\xi$, it is sometimes omitted. Example :

$$\begin{pmatrix} \gamma & \beta & \alpha \\ 0 & 1 & 3 \end{pmatrix}$$

The corresponding fundamental sequences can be found in https://sites.google.com/site/travelingtotheinfinity/fundamental-sequences-for-extended-veblen-function .

Another possible notation is to represent the parameters of the φ function by a polynom of variable Ω where the exponent corresponds to the position of the variable, for example $\varphi(\alpha, 0, \beta, \gamma) = \varphi(\gamma_0, \beta_1, \alpha_3) = (\xi \mapsto \omega^\xi) \begin{pmatrix} \gamma & \beta & \alpha \\ 0 & 1 & 3 \end{pmatrix} = \varphi(\Omega^3 \cdot \alpha + \Omega \cdot \beta + \gamma)$.

For Ω, we can choose an ordinal which is greater than all the ordinals we want to produce. Since they all are countable, we can take for example $\Omega = \omega_1$ which is the least uncountable ordinal. The method consisting in using uncountable ordinals to define countable ordinals is called "collapsing". We will see later other examples of notations using this method.
Note that $\varphi(1, 0) = \varphi(\Omega)$ is the least α such that $\alpha = \varphi(\alpha) = \omega^\alpha$ (the least fixed point of $\alpha \mapsto \omega^\alpha$); $\varphi(1, 0, 0) = \varphi(\Omega^2) = \varphi(\Omega \cdot \Omega)$ is the least α such that $\alpha = \varphi(\alpha, 0) = \varphi(\Omega \cdot \alpha)$. Generally speaking, we can see that $f(\Omega)$ is the least fixed point of f. We shall see other examples of this equality later concerning ordinal collapsing functions. Note also that "Ω" can be replaced by "1,0" in the formulas.

Here is an Agda implementation of the Veblen function with transfinitely many variables :

```
{-
    A definition of the large Veblen ordinal in Agda
    by Jacques Bailhache, March 2016

    See https://en.wikipedia.org/wiki/Veblen_function

    (1) phi(a)=w**a for a single variable,

    (2) phi(0,an-1,...,a0)=phi(an-1,...,a0), and

    (3) for a>0, c->phi(an,...,ai+1,a,0,...,0,c) is the function enumerating the common fixed points of the
        functions x->phi(an,...,ai+1,b,x,0,...,0) for all b<a.

    (4) Let a be a transfinite sequence of ordinals (i.e., an ordinal function with finite support) which ends i
        zero  (i.e., such that a0=0), and let a[0->c] denote the same function where the final 0 has been replac
        by c.
        Then c->phi(a[0->c]) is defined as the function enumerating the common fixed points of all functions
        x->phi(b) where b ranges over all sequences which are obtained by decreasing the smallest-indexed nonzer
        value of a  and replacing some smaller-indexed value with the indeterminate x (i.e., b=a[i0->z,i->x]
        meaning that for the smallest index i0 such that ai0 is nonzero the latter has been replaced by some val
        z<ai0 and that for some smaller index i<i0, the value ai=0 has been replaced with x).

-}


module LargeVeblen where
```
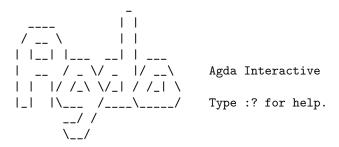
```
data Nat : Set where
 O : Nat
 1+ : Nat -> Nat

data Ord : Set where
 Zero : Ord
 Suc : Ord -> Ord
 Lim : (Nat -> Ord) -> Ord

-- rpt n f x = f^n(x)
rpt : {t : Set} -> Nat -> (t -> t) -> t -> t
rpt O f x = x
rpt (1+ n) f x = rpt n f (f x)

-- smallest fixed point of f greater than x, limit of x, f x, f (f x), ...
fix : (Ord -> Ord) -> Ord -> Ord
fix f x = Lim (\n -> rpt n f x)

w = fix Suc Zero -- not a fixed point in this case !

-- cantor a b = b + w^a
cantor : Ord -> Ord -> Ord
cantor Zero a = Suc a
cantor (Suc b) a = fix (cantor b) a
cantor (Lim f) a = Lim (\n -> cantor (f n) a)

-- phi0 a = w^a
phi0 : Ord -> Ord
phi0 a = cantor a Zero

-- Another possibility is to use phi'0 instead of phi0 in the definition of phi,
-- this gives a phi function which grows slower
phi'0 : Ord -> Ord
phi'0 Zero = Suc Zero
phi'0 (Suc a) = Suc (phi'0 a)
phi'0 (Lim f) = Lim (\n -> phi'0 (f n))

-- Associative list of ordinals
infixr 40 _=>_&_
data OrdAList : Set where
 Zeros : OrdAList
 _=>_&_ : Ord -> Ord -> OrdAList -> OrdAList

-- Usage : phi al, where al is the associative list of couples index => value ordered by increasing values,
-- absent indexes corresponding to Zero values

phi : OrdAList -> Ord
phi                Zeros  = phi0 Zero -- (1) phi(0) = w**0 = 1
phi (Zero => a & Zeros) = phi0 a     -- (1) phi(a) = w**a
phi (           k => Zero & al) = phi al -- eliminate unnecessary Zero value
phi (Zero => a & k => Zero & al) = phi (Zero => a & al) -- idem
phi (Zero => a & Zero => b & al) = phi (Zero => a & al) -- should not appear but necessary for completeness
phi (Zero => Lim f & al) = Lim (\n -> phi (Zero => f n & al)) -- canonical treatment of limit
phi (            Suc k => Suc b & al) = fix (\x -> phi (k => x & Suc k => b & al)) Zero
 -- (3) least fixed point
```

```
 phi (Zero => Suc a & Suc k => Suc b & al) = fix (\x -> phi (k => x & Suc k => b & al)) (Suc (phi (Zero => a &
Suc k => Suc b & al))) -- (3) following fixed points
 phi (                Suc k => Lim f & al) = Lim (\n -> phi (Suc k => f n & al)) -- idem
 phi (Zero => Suc a & Suc k => Lim f & al) = Lim (\n -> phi (k => Suc (phi (Zero => a & Suc k => Lim f & al)) &
Suc k => f n & al)) -- idem
 phi (                Lim f => Suc b & al) = Lim (\n -> phi (f n => (Suc Zero) & Lim f => b & al))
 phi (Zero => Suc a & Lim f => Suc b & al) = Lim (\n -> phi (f n => phi (Zero => a & Lim f => Suc b & al) & Lim
f => b & al))
 phi (                Lim f => Lim g & al) = Lim (\n -> phi (Lim f => g n & al))
 phi (Zero => Suc a & Lim f => Lim g & al) = Lim (\n -> phi (f n => phi (Zero => a & Lim f => Lim g & al) & Lim
f => g n & al))

 SmallVeblen = phi (w => Suc Zero & Zeros)

 LargeVeblen = fix (\x -> phi (x => Suc Zero & Zeros)) (Suc Zero)

{-
Normally it should terminate because the parameter of phi lexicographically decreases, but Agda is not clever en
so it must be called with no termination check option :

$ agda -I --no-termination-check LargeVeblen.agda

                  _
   ____         | |
  / __ \        | |
 | |__| |___  __| | ___
 |  __  / _ \/ _  |/ __\      Agda Interactive
 | |  |/ /_\ \/_| / /_| \
 |_|  |\___  /_____/     Type :? for help.
       __/ /
       \__/

The interactive mode is no longer supported. Don't complain if it doesn't work.
Checking LargeVeblen (/perso/ord/LargeVeblen.agda).
Finished LargeVeblen.
Main> phi Zeros
Suc Zero
```

3 Going beyond Veblen function with transfinitely many variables

We start with the large Veblen ordinal which is the least fixed point of the function $\alpha \mapsto \varphi(1_\alpha)$. Then we consider a function F which enumerates the fixed points of $\alpha \mapsto \varphi(1_\alpha)$. So we have LVO = F(0). The next fixed point F(1) is the limit of $LVO + 1, \varphi(1_{LVO+1}), \varphi(1_{\varphi(1_{LVO+1})}), ...$

Then we can consider the fixed points of the function F and define a function G which enumerates these fixed points, then a function H which enumerates the fixed points of G, and so on.

This construction is similar to ε which enumerates the fixed points of $\alpha \mapsto \omega^\alpha$, ζ which enumerates the fixed points of ε, η which enumerates the fixed points of ζ.

Like we have defined :
- $\varphi_0(\alpha) = \omega^\alpha$
- $\varphi_1(\alpha) = \varepsilon(\alpha)$
- $\varphi_2(\alpha) = \zeta(\alpha)$
...
we can define :
- $\varphi_0^+(\alpha) = F(\alpha)$
- $\varphi_1^+(\alpha) = G(\alpha)$
- $\varphi_2^+[\alpha] = H(\alpha)$

...

With this notation we can write $LVO = \varphi_0^+(0)$.

Then $\varphi_\alpha^+(\beta)$ can be written as a binary function $\varphi^+(\alpha, \beta)$ which can be generalized to finitely many variables like $\varphi^+(\alpha, \beta, \gamma)$ and transfinitely many variables like $\varphi^+(1_\omega)$.

Then we can consider the fixed points of the function $\alpha \mapsto \varphi^+(1_\alpha)$ and define a function φ_0^{++} which enumerates these fixed points.

The same way we can define φ^{+++}, φ^{++++}, ...

We can then define a new notation :

- $\Phi_0 = \varphi$
- $\Phi_1 = \varphi^+$
- $\Phi_2 = \varphi^{++}$

...

There is another way to express this construction.

There are different conventions for $\varphi_0(x)$, like ω^x or ε_x. We can write explicitely the convention chosen for φ_0 by writing "$\varphi_f(\alpha, \beta)$" for "$\varphi_\alpha(\beta)$ with function f used for φ_0". With this notation we have:

- $\varphi_f(0, \beta) = f(\beta)$
- $\varphi_f(\alpha + 1, \beta) = (1 + \beta)$th fixed point of the function $\beta \mapsto \varphi_f(\alpha, \beta)$
- $\varphi_f(\lambda, \beta) = (1 + \beta)$th common fixed point of the function $\beta \mapsto \varphi_f(\alpha, \beta)$ for all $\alpha < \lambda$, if λ is a limit ordinal.

(See http://www.cs.man.ac.uk/ hsimmons/TEMP/OrdNotes.pdf)

Then we generalize the binary function $\varphi_f(\alpha, \beta)$ to finitely many variables: for example $\varphi_f(1, 0, \alpha) = (1 + \alpha)$th common fixed point of the function $\xi \mapsto \varphi(\xi, 0)$ (see https://en.wikipedia.org/wiki/Veblen_function) and to infinitely many variables with a finite number of them different from 0, for example $\varphi_f(1_\omega)$.

Then we can define new φ functions by taking for φ_0 the function $\xi \mapsto \varphi_f(1_\xi)$ and define functions $\varphi_{\xi \mapsto \varphi_f(1_\xi)}$ with 2 variables, with finitely many variables and with transfinitely many variables.

To make a correspondence with my previous construction, if f is the function $\xi \mapsto \omega^\xi$, then $\varphi_f(\alpha, \beta)$ corresponds to what I wrote $\varphi_\alpha(\beta)$, and $\varphi_{\xi \mapsto \varphi_f(1_\xi)}(\alpha, \beta)$ to $\varphi_\alpha^+(\beta)$.

If we define the function S by $S(f)(\xi) = \varphi_f(1_\xi)$, then $\varphi_{\xi \mapsto \varphi_f(1_\xi)}$ can be written $\varphi_{S(f)}$. We can then consider $\varphi_{S(S(f))}$ and so on. Given an ordinal α, we can iterate transfinitely "α times" the application of S to an initial function f_0, for example $f_0(\xi) = \omega^\xi$, to obtain a function which I will write $S^\alpha(f_0)$. We can use this function to define a function $\varphi_{S^\alpha(f_0)}$ which permits to construct big ordinals.

4   Simmons notation

Harold Simmons defined a notation (see http://www.cs.man.ac.uk/ hsimmons/ORDINAL-NOTATIONS/ordinal-notations.html) based on fixed points enumeration which "contains" Veblen functions and permits to go further.

He uses the lambda calculus formalism, in which f x represents the application of function f to x, and f x y = (f x) y the application of function f to x which gives another function which is applied to y giving the final result. He uses tho notation $x \mapsto y$ to represent the function which, when applied to x, gives y (instead of the traditional lambda calculus notation $\lambda x.y$).

He also uses the notation $\omega^\bullet$ for $\alpha \mapsto \omega^\alpha$.

$f \circ g$ represents the composition of functions f and g : $(f \circ g)\alpha = f(g\alpha)$.

f^α is a canonical generalization of of exponentiation of a function to an ordinal power : f^n represents $f \circ f \circ \ldots \circ f$ with f repeated n times, $f^\omega \zeta$ is the limit of $\zeta, f \zeta, f(f \zeta), \ldots, f^{\omega+1}\zeta = f(f^\omega \zeta)$ and so on.

More precisely, Simmons gives the following definitions in http://www.cs.man.ac.uk/ hsimmons/TEMP/OrdNotes.pdf page 11 :

- $g^0 \zeta = \zeta$
- $g^{\alpha+1} \zeta = g(g^\alpha \zeta)$
- $g^\lambda \zeta = V\{g^\alpha \zeta | \alpha < \lambda\}$ (if λ is a limit ordinal, where V denotes the poinwise supremum)

and the following equivalent definitions in http://www.cs.man.ac.uk/ hsimmons/ORDINAL-NOTATIONS/Fruitful.pdf page 4 :

- $g^0 = id$
- $g^{\alpha+1} = g \circ g^\alpha$
- $g^\lambda = V\{g^\alpha | \alpha < \lambda\}$

and he generalizes these definitions to higher order functions.

Then Simmons defines the following functions :
$Fix\ f\zeta = f^\omega(\zeta+1) =$ limit of $\zeta+1, f(\zeta+1), f(f(\zeta+1)), \ldots$ is the least fixed point of the function f which is strictly greater than ζ, which means the least ordinal ν satisfying $f\ \nu = \nu$ and $\nu > \zeta$.
$Next = Fix\ \omega^\bullet = Fix(\alpha \mapsto \omega^\alpha)$; $Next\ \zeta$ is the next ε_α after ζ.
$[0]h = Fix(\alpha \mapsto h^\alpha 0)$
$[1]hg = Fix(\alpha \mapsto h^\alpha g0)$
$[2]hgf = Fix(\alpha \mapsto h^\alpha gf0)$
... and so on ...

In http://www.cs.man.ac.uk/ hsimmons/ORDINAL-NOTATIONS/OrdSlides.pdf Simmons gives another equivalent definition :
$[0]h = Fix(\alpha \mapsto h^\alpha \omega)$
$[1]hg = Fix(\alpha \mapsto h^\alpha g\omega)$
$[2]hgf = Fix(\alpha \mapsto h^\alpha gf\omega)$

Simmons also defines :
$Veb\ f\ \zeta = (Fix\ f)^{1+\zeta}0$ is the $(1+\zeta)$-th fixed point of f
$Enm\ h\ \alpha = h^{1+\alpha}0$
$Veb = Enm \circ Fix$
$[0] = Fix \circ Enm$
$Fix \circ Veb = Fix \circ Enm \circ Fix = [0] \circ Fix$
$Fix \circ Veb^\alpha = [0]^\alpha \circ Fix$
$\Delta[0] = \omega$
$\Delta[1] = Next\ \omega = \varepsilon_0$
$\Delta[2] = [0]Next\ \omega =$ least ν with $\nu = Next^\nu\omega = \zeta_0$
$\Delta[3] = [1][0]Next\ \omega =$ least ν with $\nu = [0]^\nu Next\ \omega = \Gamma_0$
$\Delta[4] = [2][1][0]Next\ \omega =$ least ν with $\nu = [1]^\nu[0]Next\ \omega = LVO$ (large Veblen ordinal)
... and so on ...

4.1 Implementation

Here is an implementation of the Simmons hierarchy in Haskell :

```
module Simmons where

 -- Natural numbers
 data Nat
  = ZeroN
  | SucN Nat

 -- Ordinals
 data Ord
  = Zero
  | Suc Ord
  | Lim (Nat -> Ord)

 -- Ordinal corresponding to a given natural
 ordOfNat ZeroN = Zero
 ordOfNat (SucN n) = Suc (ordOfNat n)

 -- omega
 w = Lim ordOfNat

 lim0 s = Lim s
 lim1 f x = lim0 (\n -> f n x)
```

```
lim2 f x = lim1 (\n -> f n x)

-- this does not work :
-- lim ZeroN s = Lim s
-- lim (SucN p) f = \x -> lim p (\n -> f n x)

-- f^a(x)
fpower0 f Zero x = x
fpower0 f (Suc a) x = f (fpower0 f a x)
fpower0 f (Lim s) x = Lim (\n -> fpower0 f (s n) x)

fpower l f Zero x = x
fpower l f (Suc a) x = f (fpower l f a x)
fpower l f (Lim s) x = l (\n -> fpower l f (s n) x)

-- fix f z = least fixed point of f which is > z
fix f z = fpower lim0 f w (Suc z) -- Lim (\n -> fpower0 f (ordOfNat n) (Suc z))

-- cantor b a = a + w^b
cantor Zero a = Suc a
cantor (Suc b) a = fix (cantor b) a
cantor (Lim s) a = Lim (\n -> cantor (s n) a)

-- expw a = w^a
expw a = cantor a Zero

-- next a = least epsilon_b > a
next = fix expw

-- [0]
simmons0 h = fix (\a -> fpower lim0 h a Zero)

-- [1]
simmons1 h1 h0 = fix (\a -> fpower lim1 h1 a h0 Zero)

-- [2]
simmons2 h2 h1 h0 = fix (\a -> fpower lim2 h2 a h1 h0 Zero)

-- Large Veblen ordinal
 lvo = simmons2 simmons1 simmons0 next w
```

```
$ hugs
--   -- -- -- ----  ---      ----------------------------------------
||   || || || || || ||__     Hugs 98: Based on the Haskell 98 standard
||___|| ||__|| ||__||  __||  Copyright (c) 1994-2005
||---||         ___||        World Wide Web: http://haskell.org/hugs
||   ||                      Bugs: http://hackage.haskell.org/trac/hugs
||   || Version: September 2006 ----------------------------------------

Haskell 98 mode: Restart with command line option -98 to enable extensions

Type :? for help
Hugs> :load simmons
Simmons> lvo
```

```
ERROR - Cannot find "show" function for:
*** Expression : lvo
*** Of type    : Ord

Simmons>
```

4.2 Correspondence with Veblen functions

ε_0 is the next ε_α after 0 (or after ω, or after any ordinal less than ε_0, so we have $\varepsilon_0 = Next\ 0 = Next\ \omega$.
ε_1 is the next ε_α after ε_0, so we have $\varepsilon_1 = Next\ \varepsilon_0 = Next\ (Next\ 0) = Next^2 0 = Next\ (Next\ \omega) = Next^2 \omega$.
ε_2 is the next ε_α after ε_1, so we have $\varepsilon_2 = Next\ \varepsilon_1 = Next\ (Next\ (Next\ 0)) = Next^3 0 = Next\ (Next\ (Next\ \omega)) = Next^3 \omega$.
...
ε_ω is the limit of $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots$. It is the limit of $Next^1 0, Next^2 0, Next^3 0, \ldots$ which is $Next^\omega 0$.
More generally, we have $\varepsilon_\alpha = \varphi(1, \alpha) = Next^{1+\alpha} 0 = Next^{1+\alpha} \omega$.

$\zeta_0 = \varphi(2, 0)$ is the least fixed point of $\alpha \mapsto \varepsilon_\alpha$ (greater than 0), so $\zeta_0 = Fix(\alpha \mapsto \varepsilon_\alpha)0 = Fix(\alpha \mapsto Next^{1+\alpha} 0)0 = Fix(\alpha \mapsto Next^\alpha 0)0$ (because the "1+" is "absorbed" after a few iterations) $= [0]Next\ 0$. Since ζ_0 is also greater than ω, it is also $[0]Next\ \omega$ according to a similar computation.
$\zeta_1 = \varphi(2, 1)$ is the next fixed point of $\alpha \mapsto \varepsilon_\alpha$, the least one which is strictly greater than ζ_0, so $\zeta_1 = Fix(\alpha \mapsto \varepsilon_\alpha)\zeta_0 = Fix(\alpha \mapsto Next^\alpha 0)\zeta_0 = [0]Next\ \zeta_0 = [0]Next([0]Next\ 0) = ([0]Next)^2 0$.
More generally, $\zeta_\alpha = ([0]Next)^{1+\alpha} 0$.
Similar computations give $\eta_0 = \varphi(3, 0) = [0]^2 Next\ 0$ and $\eta_\alpha = ([0]^2 Next)^{1+\alpha} 0$.
More generally, $\varphi(1 + \beta, \alpha) = ([0]^\beta Next)^{1+\alpha} 0$ or $([0]^\beta Next)^{1+\alpha} \omega$.

$\Gamma_0 = \varphi(1, 0, 0)$ is the least fixed point (greater than 0) of the function $\alpha \mapsto \varphi(\alpha, 0)$ or $\alpha \mapsto \varphi(1 + \alpha, 0)$ (for the same reason of "absorbsion" of "1+" than previously), so $\Gamma_0 = Fix(\alpha \mapsto \varphi(1 + \alpha, 0))0 = Fix(\alpha \mapsto ([0]^\alpha Next)^{(1 + 0)}0)0 = Fix(\alpha \mapsto [0]^\alpha Next\ 0)0 = [1][0]Next\ 0$.
$\Gamma_1 = \varphi(1, 0, 1)$ is the next fixed point : $\Gamma_1 = Fix(\alpha \mapsto [0]^\alpha Next\ 0)\Gamma_0 = [1][0]Next\ \Gamma_0 = [1][0]Next\ ([1][0]Next\ 0) = ([1][0]Next)^2 0$.
More generally, we have $\varphi(1, 0, \alpha) = ([1][0]Next)^{1+\alpha} 0$.
$\varphi(1, 1, 0)$ is the least fixed point (greater than 0) of the function $\alpha \mapsto \varphi(1, 0, \alpha)$, so it is $Fix(\alpha \mapsto \varphi(1, 0, \alpha))0 = Fix(\alpha \mapsto ([1][0]Next)^{1+\alpha} 0)0 = Fix(\alpha \mapsto ([1][0]Next)^\alpha 0)0$ (absorbsion of 1+) $= [0]([1][0]Next)0$.
$\varphi(1, 1, 1)$ is the next fixed point $Fix(\alpha \mapsto ([1][0]Next)^\alpha 0)\varphi(1, 1, 0) = ([0]([1][0]Next)([0]([1][0]Next)0) = ([0]([1][0]Next))^2 0$.
More generally, $\varphi(1, 1, \alpha) = ([0]([1][0]Next))^{1+\alpha} 0$.
$\varphi(1, 2, 0)$ is the least fixed point (greater than 0) of the function $\alpha \mapsto \varphi(1, 1, \alpha), Fix(\alpha \mapsto \varphi(1, 1, \alpha))0 = Fix([0]([1][0]Next)^{1+\alpha} 0)0 = Fix(\alpha \mapsto ([0]([1][0]Next))^\alpha 0)0 = [0]([0]([1][0]Next))0 = [0]^2([1][0]Next)0$.
Like previously, $\varphi(1, 2, \alpha)$ is the $(1 + \alpha)$-th fixed point of the previous function, which is $([0]^2([1][0]Next))^{1+\alpha} 0$.
More generally, $\varphi(1, \beta, \alpha) = ([0]^\beta([1][0]Next))^{1+\alpha} 0$.
$\varphi(2, 0, 0)$ is the least fixed point (greater than 0) of the function $\beta \mapsto \varphi(1, \beta, 0)$, which is $Fix(\alpha \mapsto \varphi(1, \beta, 0))0 = Fix(\beta \mapsto ([0]^\beta([1][0]Next))^{1+0}0)0 = Fix(\beta \mapsto [0]^\beta([1][0]Next)0)0 = [1][0]([1][0]Next)0 = ([1][0])^2 Next\ 0$.
The $(1 + \alpha)$-th fixed point of the previous function is $\varphi(2, 0, \alpha) = (([1][0])^2 Next)^{1+\alpha} 0$.
The least fixed point of the function $\alpha \mapsto \varphi(2, 0, \alpha)$ is $\varphi(2, 1, 0) = Fix(\alpha \mapsto \varphi(2, 0, \alpha))0 = Fix(\alpha \mapsto (([1][0])^2 Next)^{(1 + \alpha)}0)0 = Fix(\alpha \mapsto (([1][0])^2 Next)^\alpha 0) = [0]((([1][0])^2 Next)0$ and its $(1 + \alpha)$-th fixed point is $\varphi(2, 1, \alpha) = ([0]((([1][0])^2 Next))^{1+\alpha} 0$.
More generally, we have $\varphi(2, \beta, \alpha) = ([0]^\beta(([1][0])^2 Next))^{1+\alpha} 0$.

The general formula with three variables (with $\gamma \neq 0$) is $\varphi(\gamma, \beta, \alpha) = ([0]^\beta(([1][0])^\gamma Next))^{1+\alpha} 0$.
In particular, we have $\varphi(\gamma, 0, 0) = ([1][0])^\gamma Next 0$.

$\varphi(1, 0, 0, 0)$ is the least fixed point of the function $\gamma \mapsto \varphi(\gamma, 0, 0), Fix(\gamma \mapsto \varphi(\gamma, 0, 0))0 = Fix(\gamma \mapsto ([1][0])^\gamma Next\ 0)0 = [1]([1][0])Next\ 0 = [1]^2[0]Next\ 0$.
All of these computations could be done with ω instead of 0 at the end of the formulas so we also have $\varphi(\gamma, \beta, \alpha) = ([0]^\beta(([1][0])^\gamma Next))^{1+\alpha}$
In a similar way, we can obtain the formula with 4 variables :
$\varphi(1, 0, 0, \alpha) = ([1]^2[0]Next)^{1+\alpha} 0$
$\varphi(1, 0, 1, 0) = Fix(\alpha \mapsto ([1]^2[0]Next)^\alpha 0)0 = [0]([1]^2[0])0$

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$\varphi(1,0,1,\alpha) = ([0]([1]^2[0]Next))^{1+\alpha}0$

$\varphi(1,0,\beta,\alpha) = ([0]^\beta([1]^2[0]Next))^{1+\alpha}0$

$\varphi(1,1,0,0) = Fix(\alpha \mapsto \varphi(1,0,\alpha,0)]0 = Fix(\alpha \mapsto [0]^\alpha([1]^2[0]Next)0]0 = [1][0]([1]^2[0]Next)0$

$\varphi(1,1,0,\alpha) = ([1][0]([1]^2[0]Next))^{1+\alpha}0$

$\varphi(1,1,1,0) = Fix(\alpha \mapsto \varphi(1,1,0,\alpha))0 = Fix(\alpha \mapsto ([1][0]([1]^2[0]Next))^\alpha0)0 = [0]([1][0]([1]^2[0]Next))0$

$\varphi(1,1,1,\alpha) = ([0]([1][0]([1]^2[0]next)))^{1+\alpha}0$

$\varphi(1,1,\beta,\alpha) = ([0]^\beta([1][0]([1]^2[0]Next)))^{1+\alpha}0$

$\varphi(1,2,0,0) = Fix(\alpha \mapsto \varphi(1,1,\alpha,0))0 = Fix(\alpha \mapsto [0]^\alpha([1][0]([1]^2[0]next))0)0 = [1][0]([1][0]([1]^2[0]Next))0 = ([1][0])^2([1]^2[0]Next)0$

$\varphi(1,0,0,0) = [1]^2[0]Next0$

$\varphi(1,1,0,0) = [1][0]([1]^2[0]Next)0$

$\varphi(1,2,0,0) = ([1][0])^2([1]^2[0]Next)0$

$\varphi(1,\gamma,0,0) = ([1][0])^\gamma([1]^2[0]Next)0$

$\varphi(1,\gamma,\beta,\alpha) = ([0]^\beta(([1][0])^\gamma([1]^2[0]Next)))^{1+\alpha}0$

$\varphi(2,0,0,0) = Fix(\alpha \mapsto \varphi(1,\alpha,0,0)]0 = Fix(\alpha \mapsto ([1][0])^\alpha([1]^2[0]Next)0]0 = [1](([1][0])([1]^2[0]Next)0 = [1]^2[0]([1]^2[0]Next)0 = ([1]^2[0])^2Next0$

$\varphi(\delta,0,0,0) = ([1]^2[0])^\delta Next\ 0$

The general formula with four variables is :

$\varphi(\delta,\gamma,\beta,\alpha) = ([0]^\beta(([1][0])^\gamma(([1]^2[0])^\delta Next)))^{1+\alpha}0 = ([0]^\beta(([1][0])^\gamma(([1]^2[0])^\delta Next)))^{1+\alpha}\omega$

and so on.

The small Veblen ordinal is the limit of :

$\varphi(1) = \omega, \varphi(1,0) = Next\ \omega, \varphi(1,0,0) = [1][0]Next\ \omega, \varphi(1,0,0,0) = [1]^2[0]Next\ \omega, \varphi(1,0,0,0,0) = [1]^3[0]Next\ \omega, \ldots$.

This limit is $[1]^\omega[0]Next\ \omega = [1]^\omega[0]Next\ 0$.

Allowing variables at any finite or transfinite positions (which is equivalent to Schütte brackets or Klammersymbols) gives ordinals smaller than the large Veblen ordinal which is the least fixed point of the function $\alpha \mapsto \varphi(1_\alpha)$. It is $Fix(\alpha \mapsto \varphi(1_\alpha))0 = Fix(\alpha \mapsto [1]^\alpha[0]Next\ 0)0 = [2][1][0]Next\ 0$

$[2][1][0]Next\ 0$ or $[2][1][0]Next\ \omega$.

The conversion rule from Schütte Klammersymbol to Simmons notation are described by Simmons in his paper : http://www.cs.man.ac.u NOTATIONS/FromBelow.pdf (Simmons also wrote other papers but it seems to me that they contain inaccuracies and maybe even errors).

In summary :

$Fix\ f\zeta = f^\omega(\zeta + 1)$

$Enm\ h\ \alpha = h^{1+\alpha}0$

$Next = Fix(\alpha \mapsto \omega^\alpha)$

$[0]h = Fix(\alpha \mapsto h^\alpha0)$

$[1]hg = Fix(\alpha \mapsto h^\alpha g0)$

$\nabla \begin{bmatrix} \alpha+1 \\ i+1 \end{bmatrix} = ([1]^i[0])^{1+\alpha}$ if $i \neq 0$; $[0]^\alpha$ if $i = 0$

$\nabla \begin{bmatrix} \alpha_1+1 & \ldots & \alpha_s+1 \\ i_1+1 & \ldots & 1_s+1 \end{bmatrix} = \nabla \begin{bmatrix} \alpha_1+1 \\ i_1+1 \end{bmatrix} \circ [0] \circ \ldots \circ [0] \circ \nabla \begin{bmatrix} \alpha_s+1 \\ i_s+1 \end{bmatrix}$

where $f \circ g$ is the composition of functions f and g : $(f \circ g)x = f\ (g\ x)$

$Sch \begin{bmatrix} 1+\alpha_1 & \ldots & 1+\alpha_s \\ 1+i_1 & \ldots & 1+i_s \end{bmatrix} = Enm\ \circ \nabla \begin{bmatrix} 1+\alpha_1 & \ldots & 1+\alpha_s \\ 1+i_1 & \ldots & 1+i_s \end{bmatrix} o\ Fix$

f may be any function but it is usually $\alpha \mapsto \omega^\alpha$.

$f \begin{pmatrix} \zeta & 1+\alpha_1 & \ldots & 1+\alpha_s \\ 0 & 1+i_1 & \ldots & 1+i_s \end{pmatrix}$

$= Sch \begin{bmatrix} 1+\alpha_1 & \ldots & 1+\alpha_s \\ 1+i_1 & \ldots & 1+i_s \end{bmatrix} f\zeta$

$= (Enm \circ \nabla \begin{bmatrix} 1+\alpha_1 & \ldots & 1+\alpha_s \\ 1+i_1 & \ldots & 1+i_s \end{bmatrix} \circ Fix)f\zeta$

$= (Enm \circ \nabla \begin{bmatrix} \alpha_1+1 \\ i_1+1 \end{bmatrix} \circ [0] \circ \ldots \circ [0] \circ \nabla \begin{bmatrix} \alpha_s+1 \\ i_s+1 \end{bmatrix} \circ Fix)f\zeta$

$= Enm((\nabla \begin{bmatrix} \alpha_1+1 \\ i_1+1 \end{bmatrix} \circ [0] \circ \ldots \circ [0] \circ \nabla \begin{bmatrix} \alpha_s+1 \\ i_s+1 \end{bmatrix})(Fixf))\zeta$

$$= (\nabla \begin{bmatrix} \alpha_1 + 1 \\ i_1 + 1 \end{bmatrix} \circ [0] \circ \ldots \circ [0] \circ \nabla \begin{bmatrix} \alpha_s + 1 \\ i_s + 1 \end{bmatrix})(Fix f))^{1+\zeta} 0$$

If $f = \alpha \mapsto \omega^\alpha$, then Fix f = Next and

$$f \begin{pmatrix} \zeta & 1 + \alpha_1 & \ldots & 1 + \alpha_s \\ 0 & 1 + i_1 & \ldots & 1 + i_s \end{pmatrix} = (\nabla \begin{bmatrix} \alpha_1 + 1 \\ i_1 + 1 \end{bmatrix} \circ [0] \circ \ldots \circ [0] \circ \nabla \begin{bmatrix} \alpha_s + 1 \\ i_s + 1 \end{bmatrix}) Next)^{1+\zeta} 0$$

Examples :

$$\varphi(1 + \beta, \alpha)$$
$$= (\xi \mapsto \omega^\xi) \begin{pmatrix} \alpha & 1 + \beta \\ 0 & 1 \end{pmatrix}$$
$$= ((\nabla \begin{bmatrix} \beta + 1 \\ 1 \end{bmatrix})(Fix(\xi \mapsto \omega^\xi)))^{1+\alpha} 0$$
$$= ((\nabla \begin{bmatrix} \beta + 1 \\ 1 \end{bmatrix}) Next)^{1+\alpha} 0$$
$$= ([0]^\beta Next)^{1+\alpha} 0$$

$$\varphi(1 + \gamma, 1 + \beta, \alpha)$$
$$= (\xi \mapsto \omega^\xi) \begin{pmatrix} \alpha & 1 + \beta & 1 + \gamma \\ 0 & 1 & 2 \end{pmatrix}$$
$$= ((\nabla \begin{bmatrix} \beta + 1 \\ 1 \end{bmatrix} \circ [0] \circ \nabla \begin{bmatrix} \gamma + 1 \\ 2 \end{bmatrix})(Fix(\xi \mapsto \omega^\xi)))^{1+\alpha} 0$$
$$= ((\nabla \begin{bmatrix} \beta + 1 \\ 1 \end{bmatrix} \circ [0] \circ \nabla \begin{bmatrix} \gamma + 1 \\ 2 \end{bmatrix}) Next)^{1+\alpha} 0$$
$$= (([0]^\beta \circ [0] \circ ([1][0])^{1+\gamma}) Next)^{1+\alpha} 0$$
$$= ([0]^{1+\beta} (([1][0])^{1+\gamma} Next))^{1+\alpha} 0$$
Compare with the previously found formula :
if $\gamma > 0$, $\varphi(\gamma, \beta, \alpha) = ([0]^\beta (([1][0])^\gamma Next))^{1+\alpha} 0$
and note the "round trip" $1 + \gamma \to \gamma + 1 \to 1 + \gamma$.

$$\varphi(1 + \delta, 1 + \gamma, 1 + \beta, \alpha)$$
$$= (\xi \mapsto \omega^\xi) \begin{pmatrix} \alpha & 1 + \beta & 1 + \gamma & 1 + \delta \\ 0 & 1 & 2 & 3 \end{pmatrix}$$
$$= ((\nabla \begin{bmatrix} \beta + 1 \\ 1 \end{bmatrix} \circ [0] \circ \nabla \begin{bmatrix} \gamma + 1 \\ 2 \end{bmatrix} \circ [0] \circ \nabla \begin{bmatrix} \delta + 1 \\ 3 \end{bmatrix})(Fix(\xi \mapsto \omega^\xi)))^{1+\alpha} 0$$
$$= ((\nabla \begin{bmatrix} \beta + 1 \\ 1 \end{bmatrix} \circ [0] \circ \nabla \begin{bmatrix} \gamma + 1 \\ 2 \end{bmatrix} \circ [0] \circ \nabla \begin{bmatrix} \delta + 1 \\ 3 \end{bmatrix}) Next)^{1+\alpha} 0$$
$$= (([0]^\beta \circ [0] \circ ([1][0])^{1+\gamma} \circ [0] \circ ([1]^2[0])^{1+\delta}) Next)^{1+\alpha} 0$$
$$= ([0]^{1+\beta} (([1][0])^{1+\gamma} ([0](([1]^2[0])^{1+\delta} Next))))^{1+\alpha} 0$$
$$= ([0]^{1+\beta} (([1][0])^{1+\gamma} (([1]^2[0])^{1+\delta} Next)))^{1+\alpha} 0$$
because [0] is absorbed by the following operator (see http://www.cs.man.ac.uk/ hsimmons/ORDINAL-NOTATIONS/FromBelow.pdf p 33, 6.7)
Compare with the previously mentioned formula :
$$\varphi(\delta, \gamma, \beta, \alpha) = ([0]^\beta (([1][0])^\gamma (([1]^2[0])^\delta Next)))^{1+\alpha} 0$$

The equality
$$(\xi \mapsto \omega^\xi) \begin{pmatrix} \zeta & 1 + \alpha_1 & \ldots & 1 + \alpha_s \\ 0 & 1 + i_1 & \ldots & 1 + i_s \end{pmatrix} = (\nabla \begin{bmatrix} \alpha_1 + 1 \\ i_1 + 1 \end{bmatrix} \circ [0] \circ \ldots \circ [0] \circ \nabla \begin{bmatrix} \alpha_s + 1 \\ i_s + 1 \end{bmatrix}) Next)^{1+\zeta} 0$$
can be reformulated, distinguishing four cases :

- $(\xi \mapsto \omega^\xi) \begin{pmatrix} \zeta \\ 0 \end{pmatrix} = \varphi(0, \zeta) = \omega^\zeta$

- $(\xi \mapsto \omega^\xi) \begin{pmatrix} \zeta & 1 + \alpha \\ 0 & 1 \end{pmatrix} = \varphi(1 + \alpha, \zeta) = (\nabla \begin{bmatrix} \alpha + 1 \\ 1 \end{bmatrix} Next)^{1+\zeta} 0 = ([0]^\alpha Next)^{1+\zeta} 0$

- $(\xi \mapsto \omega^\xi) \begin{pmatrix} \zeta & 1+\alpha_1 & 1+\alpha_2 & \dots & 1+\alpha_s \\ 0 & 1 & 1+i_2 & \dots & 1+i_s \end{pmatrix}$

$= ((\nabla \begin{bmatrix} \alpha_1+1 \\ 1 \end{bmatrix} \circ [0] \circ \nabla \begin{pmatrix} \alpha_2+1 \\ i_2+1 \end{pmatrix} \circ [0] \circ \dots \circ [0] \circ \nabla \begin{bmatrix} \alpha_s+1 \\ i_s+1 \end{bmatrix}) Next)^{1+\zeta} 0$

$= (([0]^{\alpha_1} \circ [0] \circ ([1]^{i_2}[0])^{1+\alpha_2} \circ [0] \circ \dots \circ [0] \circ ([1]^{i_s}[0])^{1+\alpha_s}) Next)^{1+\zeta} 0$

$= (([0]^{1+\alpha_1} \circ ([1]^{i_2}[0])^{1+\alpha_2} \circ [0] \circ \dots \circ [0] \circ ([1]^{i_s}[0])^{1+\alpha_s}) Next)^{1+\zeta} 0$

$= (([0]^{1+\alpha_1} \circ ([1]^{i_2}[0])^{1+\alpha_2} \circ \dots \circ ([1]^{i_s}[0])^{1+\alpha_s}) Next)^{1+\zeta} 0$

The first separating $[0]$ is combined with $[0]^{\alpha_1}$ giving $[0]^{1+\alpha_1}$ and the other are absorbed.

- $(\xi \mapsto \omega^\xi) \begin{pmatrix} \zeta & 1+\alpha_1 & \dots & 1+\alpha_s \\ 0 & 1+i_1 & \dots & 1+i_s \end{pmatrix}$ with $i_1 \neq 0$

$= ((\nabla \begin{bmatrix} \alpha_1+1 \\ i_1+1 \end{bmatrix} \circ [0] \circ \dots \circ [0] \circ \nabla \begin{bmatrix} \alpha_s+1 \\ i_s+1 \end{bmatrix}) Next)^{1+\zeta} 0$

$= ((([1]^{i_1}[0])^{1+\alpha_1} \circ [0] \circ \dots \circ [0] \circ ([1]^{i_s}[0])^{1+\alpha_s}) Next)^{1+\zeta} 0$

$= ((([1]^{i_1}[0])^{1+\alpha_1} \circ \dots \circ ([1]^{i_s}[0])^{1+\alpha_s}) Next)^{1+\zeta} 0$

The separating $[0]$ are absorbed.

We can see that the third case is contained in the fourth one if we remove the restriction $i_1 \neq 0$ because if $i_1 = 0$ we have $([1]^{i_1}[0])^{1+\alpha_1} = [0]^{1+\alpha_1}$ like in the third case.

For more information concerning the correspondence between Simmons notation and Schütte Klammersymbols, see : http://www.cs.man.ac.uk/ hsimmons/ORDINAL-NOTATIONS/FromBelow.pdf pages 28 - 34.

The Simmons notation can also be used to represent the notation going beyond Veblen functions that we saw previously.

As we saw previously, the large Veblen ordinal is the least fixed point of the function $\alpha \mapsto \varphi(1_\alpha)$ or $\alpha \mapsto (\xi \mapsto \omega^\xi) \begin{pmatrix} 1 \\ \alpha \end{pmatrix}$. It is $Fix(\alpha \mapsto \varphi(1_\alpha))0 = Fix(\alpha \mapsto [1]^\alpha[0]Next\,0)0 = [2][1][0]Next\,0$. The fixed points of this function are enumerated by the function F, so we have LVO = F(0). More generally, the $(1+\alpha$-th fixed point of $\alpha \mapsto \varphi(1_\alpha)$ is $F(\alpha) = \varphi_1^+(\alpha) = ([2][1][0]Next)^{1+\alpha}0$.
Then the fixed points of $F = \varphi_1^+$ are enumerated by $G = \varphi_2^+$. The least fixed point of F is $G(0) = \varphi_2^+(0) = Fix(\alpha \mapsto ([2][1][0]Next)^{1+\alpha}0)0 = [0]([2][1][0]Next)0$ (because of the absorbsion of "1+") and its $(1+\alpha)$-th fixed point is $G(\alpha) = \varphi_2^+(\alpha) = ([0]([2][1][0]Next))^{1+\alpha}0$.
Then the fixed points of $G = \varphi_2^+$ are enumerated by $H = \varphi_3^+$. The least fixed point of H is $H(0) = \varphi_3^+(0) = Fix(\alpha \mapsto ([0]([2][1][0]Next)^{1+\alpha}0)0 = [0]([0]([2][1][0]Next))0 = [0]^2([2][1][0]Next)0$ and its $(1+\alpha)$-th fixed point is $H(\alpha) = \varphi_3^+(\alpha) = ([0]^2([2][1][0]Next))^{1+\alpha}0$.
More generally, we have $\varphi_{1+\alpha}^+(0) = [0]^\alpha([2][1][0]Next)0$ and $\varphi_{1+\alpha}^+(\beta) = ([0]^\alpha([2][1][0]Next))^{1+\beta}0$.
Then we generalize the function φ^+ to any number of variables :
$\varphi^+(\alpha, \beta) = \varphi_\alpha^+(\beta)$
$\varphi^+(1, 0, 0)$ is the least fixed point of the function $\alpha \mapsto \varphi^+(\alpha, 0) = \alpha \mapsto [0]^\alpha([2][1][0]Next)0$. It is $Fix(\alpha \mapsto [0]^\alpha([2][1][0]Next)0)0 = [1][0]([2][1][0]Next)0$.
Compare with $\varphi(1, 0, 0) = [1][0]Next0$.
More generally, like we found $\varphi(\gamma, \beta, \alpha) = ([0]^\beta(([1][0])^\gamma Next))^{1+\alpha}0$, we have $\varphi^+(\gamma, \beta, \alpha) = ([0]^\beta(([1][0])^\gamma([2][1][0]Next)))^{1+\alpha}0$.
Like we generalized the φ function to transfinitely many variables reaching all ordinals less than $LVO = [2][1][0]Next\,0$, we can generalize the φ^+ function to transfinitely many variables and reach all ordinals less than a new limit which we will call $LVO^+ = [2][1][0]([2][1][0]Next)0$ which is the least fixed point of $\alpha \mapsto [1]^\alpha[0]([2][1][0]Next)0$.
Then we can do the same with $\varphi^{++} = \Phi_2$ and we shall get similar results with $([2][1][0])^2 Next$, and generally with Φ_α, getting formulas with $([2][1][0])^\alpha Next$.

The limit of Next 0, [0] Next 0, [1] [0] Next 0, [2] [1] [0] Next 0, [3] [2] [1] [0] Next 0, ...
or $Next\,\omega, [0]Next\,\omega, [1][0]Next\,\omega, [2][1][0]Next\,\omega, [3][2][1][0]Next\,\omega, \dots$
is called the Bachmann-Howard ordinal (BHO).
It could be written $[\omega \dots 0]Next\,0$ or $[\omega \dots 0]Next\,\omega$.

5 Rationalization of the Veblen functions

When we have defined the different notations, we have arbitrarily chosen some conventions, for example the limit of $\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots$ have been called ε_0. We could have called it ε_1. In this case, ε_α would have been the α-th fixed point of $\xi \mapsto \omega^\xi$ instead of the the $(1+\alpha)$-th one. Also we chose to define $\varphi(0,\alpha) = \omega^\alpha$. We could have chosen to define $\varphi(0,\alpha) = \varepsilon_\alpha$. The "1+" which appear in the correspondence between Simmons and Veblen notations may be due to the fact that the choices that have been made are not the most logical.

We will define a rationalized variant of the Veblen notations which simplifies the correspondence with the Simmons notation :

- $\varepsilon_\alpha = \varphi(1,\alpha) = \varepsilon'_{1+\alpha} = \varphi'(0, 1+\alpha)$

- $\zeta_\alpha = \varphi(2,\alpha) = \zeta'_{1+\alpha} = \varphi'(1, 1+\alpha)$

- $\eta_\alpha = \varphi(3,\alpha) = \eta'_{1+\alpha} = \varphi'(2, 1+\alpha)$

- Generally, $\varphi(1+\beta, \alpha) = \varphi'(\beta, 1+\alpha)$

- $\Gamma_0 = \varphi(1,0,0) = \varphi'(1,0,1)$

- Generally, if $\gamma \neq 0, \varphi(\gamma, \beta, \alpha) = \varphi'(\gamma, \beta, 1+\alpha)$

- In a similar way, if $\gamma \neq 0$ or $\delta \neq 0, \varphi(\delta, \gamma, \beta, \alpha) = \varphi'(\delta, \gamma, \beta, 1+\alpha)$ and so on.

With this notation, the correspondence with Simmons notation becomes simpler, for example we have :

- $\varepsilon'_\alpha = Next^\alpha 0$ instead of $\varepsilon_\alpha = Next^{1+\alpha} 0$

- $\varphi'(\beta, \alpha) = ([0]^\beta Next)^\alpha 0$ instead of $\varphi(1+\beta, \alpha) = ([0]^\beta Next)^{1+\alpha} 0$

- $\varphi'(\gamma, \beta, \alpha) = ([0]^\beta (([1][0])^\gamma Next))^\alpha 0$ instead of $\varphi(\gamma, \beta, \alpha) = ([0]^\beta (([1][0])^\gamma Next))^{1+\alpha} 0$

- $\varphi'(\delta, \gamma, \beta, \alpha) = ([0]^\beta (([1][0])^\gamma (([1]^2[0])^\delta Next)))^{1+\alpha} 0$ instead of $\varphi(\delta, \gamma, \beta, \alpha) = ([0]^\beta (([1][0])^\gamma (([1]^2[0])^\delta Next)))^{1+\alpha} 0$

6 RHS0 notation

```
This method is based on combinatory logic and lambda calculus formalism.
Intuitively, the method consists in :

    Start from 0
    If we don't see any regularity, take the successor (add 1)
    If we see a regularity and we don't have a notation for it, invent it and jump to the limit
    If we see a regularity and we already have a notation for it, use it and jump to the limit.
```

The difficulty, which requires intelligence, is to see the regularities. It gives the following sequence :

```
    0 : no regularity, take the successor
    suc 0 : no regularity, take the successor
    suc (suc 0) : regularity : suc repeatedly applied to 0. No notation, invent it :
     H f x = limit of x, f x, f (f x), ...
    H suc 0 : no regularity, take the successor
    suc (H suc 0) : no regularity, take the successor
    suc (suc (H suc 0)) : regularity : suc repeatedly applied to H suc 0, notation exists
    H suc (H suc 0) : regularity : H suc repeatedly applied to 0, notation exists
    H (H suc) 0 : regularity : H repeatedly applied to suc, notation exists
    H H suc 0 : regularity (suc 0, ..., H suc 0, ... H H suc 0, ... H H H suc 0, ...),
     invent notation R1 H suc 0 for the limit of this sequence
    R1 H suc 0 : no regularity, take the successor
    suc (R1 H suc 0)
    suc (suc (R1 H suc 0))
```

```
    H suc (R1 H suc 0)
    suc (H suc (R1 H suc 0))
    suc (suc (H suc (R1 H suc 0)))
    H suc (H suc (R1 H suc 0))
    H (H suc) (R1 H suc 0)
    H H suc (R1 H suc 0)
    R1 H suc (R1 H suc 0)
    H (R1 H suc) 0
    suc (H (R1 H suc) 0)
    suc (suc (H (R1 H suc) 0))
    H suc (H (R1 H suc) 0)
    suc (suc (H suc (H (R1 H suc) 0)))
    H suc (H suc (H (R1 H suc) 0)))
    H (H suc) (H (R1 H suc) 0)
    H H suc (H (R1 H suc) 0)
    R1 H suc (H (R1 H suc) 0)
    suc (R1 H suc (H (R1 H suc) 0))
    suc (suc (R1 H suc (H (R1 H suc) 0)))
    H suc (R1 H suc (H (R1 H suc) 0))
    suc (H suc (R1 H suc (H (R1 H suc) 0)))
    suc (suc (H suc (R1 H suc (H (R1 H suc) 0))))
    H suc (H suc (R1 H suc (H (R1 H suc) 0)))
    H (H suc) (R1 H suc (H (R1 H suc) 0))
    H H suc (R1 H suc (H (R1 H suc) 0))
    R1 H suc (R1 H suc (H (R1 H suc) 0))
    H (R1 H suc) (H (R1 H suc) 0)
    H (H (R1 H suc)) 0
    H H (R1 H suc) 0
    R1 H (R1 H suc) 0
    H (R1 H) suc 0
    ...
    R1 H (R1 H) suc 0
    R1 (R1 H) suc 0
    H R1 H suc 0
    ...
    R1 H R1 H suc 0 : invent notation R2 R1 H suc 0 = limit of suc 0, R1 H suc 0, R1 H R1 H suc 0, ...
    ...
    R3 R2 R1 H suc 0 : invent notation R3...1 H suc 0 and jump to limit
    Rw...1 H suc 0
    ...
    R2 Rw...1 H suc 0 : invent notation Rw+1...1 H suc 0
    ...


To progress faster, we can use the following rule :
If we have found an ordinal a, and later another ordinal b of the form f (s (s z)),
we may produce an ordinal c = f ([suc->s,0->z] a)
where [suc->f,0->x] a means the ordinal obtained by replacing suc by z and 0 by z in a.
For example :

a = R1 H suc 0
b = R1 H (R1 H suc) 0
s = R1 H
z = suc
f x = x 0
[suc->R1 H,0->suc] a = R1 H (R1 H) suc
```

```
c = f ([suc->R1 H,0->suc) a) = R1 H (R1 H) suc 0
```

With the following rules :

```
0 : -> 0
suc : x -> suc x
H : f (f x) -> H f x
R1 : f f -> R1 f
R2 : f g f g -> R2 f g
R3 : f g h f g h -> R3 f g h
...
Repl : a, f (s (s z)) -> f([suc->s,0->z] a)
```

we can produce the following sequence of ordinals :

```
0 : 0 : 0
1 : suc 0 : suc 0
2 : suc 1 : suc (suc 0)
3 : H 2 : H suc 0
4 : suc 3 : suc (H suc 0)
5 : suc 4 : suc (suc (H suc 0))
6 : H 5 : H suc (H suc 0)
7 : H 6 : H (H suc) 0
8 : H 7 : H H suc 0
9 : R1 8 : R1 H suc 0
10 : suc 9 : suc (R1 H suc 0)
11 : suc 10 : suc (suc (R1 H suc 0))
12 : Repl 9 11 [suc->suc,0->R1 H suc 0] : R1 H suc (R1 H suc 0)
13 : Repl 9 12 [suc->R1 H suc,0->0] : R1 H (R1 H suc) 0
14 : Repl 9 13 [suc->R1 H,0->suc] : R1 H (R1 H) suc 0
15 : R1 14 : R1 (R1 H) suc 0
16 : Repl 9 15 [suc->R1,0->H] : R1 H R1 H suc 0
17 : R2 16 : R2 R1 H suc 0
```

The rules R1, R2, R3, ... may be replaced by H or Repl if f1 ... fn ... f1 ... fn
is reformulated in <f1,...,fn> (... (<f1,...,fn> I)...) with <f1,...,fn> g = g f1 ... fn :

```
0 : 0 : 0
1 : suc 0 : suc 0
2 : suc 1 : suc (suc 0)
3 : H 2 : H suc 0
4 : suc 3 : suc (H suc 0)
5 : suc 4 : suc (suc (H suc 0))
6 : Repl 3 5 [suc->suc,0->H suc 0] : H suc (H suc 0)
7 : Repl 3 6 [suc->H suc,0->0] : H (H suc) 0
8 : Repl 3 7 [suc->H,0->suc] : H H suc 0 = <H> (<H> I) suc 0
9 : Repl 3 8 [suc-><H>,0->I] : H <H> I suc 0
10 : suc 9 : suc (H <H> I suc 0)
11 : suc 10 : suc (suc (H <H> I suc 0))
12 : Repl 9 10 [suc->suc,0->H <H> I suc 0] : H <H> I suc (H <H> I suc 0)
13 : Repl 9 12 [suc->H <H>: I suc,0->0] : H <H> I (H <H> I suc) 0
14 : Repl 9 13 [suc->H <H> I,0->suc] : H <H> I (H <H> I) suc 0 = <H <H> I> (<H <H> I> I) suc 0
15 : Repl 3 14 [suc-><H <H> I>,0->I] : H <H <H> I> I suc 0 = [H <*> I] ([H <*> I] H) suc 0
16 : Repl 9 15 [suc->[H <*> I],0->H] : H <H> I [H <*> I] H suc 0
 = [H <*> I] H [H <*> I] H suc 0 = <[H <*> I],H> (<[H <*> I], H> I) suc 0
17 : Repl 3 16 [suc-><[H <*> I],H>,0->I] : H <[H <*> I], H> I suc 0
```

```
More formally, this notation uses combinatory logic with De Bruijn indexes.
lambda.x is written [ x ] and variables are written *, **, ***...,
for example [ ... * ... ] = \x ( ... x ... )


CI = C I is defined by CI x f = f x.
CI x = <x>
<x1,...,xn> f = f x1 ... xn
tuple n f x1 ... xn = f <x1,...,xn>
tuple 0 = <I>
tuple (n+1) f x0 = tuple n [ f (insert x0 *) ]
with insert x0 a f  = a (f x0)


r 0 f x = x
r (n+1) f x = f (r n f x)
r (lim g) f x = lim [r * f x]


H f x represents the limit of x, f x, f (f x), ...
H f x = r w f x


R1 = [H <*> I] = tuple 1 [H * I]
R2 = [[H <**,*> I]] = tuple 2 [H * I]
R3 = [[[H <***,**,*> I]]] = tuple 3 [H * I]
Rn = tuple n [H * I]
R(n...1) = Rn ... R1
S(n...1) = [S(*...1)] n = <Rn,...,R1>
R(n...1) = S(n...1) I
[S(*...1)] 0 = I
[S(*...1)] (n+1) = insert (tuple (n+1) [H * I]) ([S(*...1)] n)


L f = lim f 0, f 1, ...
L f x = L [f * x]
H = [[L [r * *** **]]]
or
L0 = lim f 0, f 1, ...
L n f = tuple n [ L0 [ ** (f *) ]]
L n = [ tuple n [ L0 [ ** (*** *) ]]]
L = [[ tuple ** [ L0 [ ** (*** *) ]]]
 = \n \f (tuple n \a (L0 \i (a (f i)) ) )


To represent the replacement [suc->s,0->z] we can represent ordinals by ordinal functions which, when
applied to suc and 0, give the considered ordinal. For example, R1 H suc 0 is represented by the ordinal
function s -> z -> R1 H s z, R1 H (R1 H suc) 0 by s -> z -> R1 H (R1 H s) z. From these ordinals, with
the replacement [suc->R1 H,0->suc] we can produce a new ordinal represented by
s -> z -> ((s -> z -> R1 H s z) (R1 H) s z) = s -> z -> R1 H (R1 H) s z)
which, when applied to suc and 0, gives R1 H (R1 H) suc 0.


Operations can be represented with replacements :

    a+b = [0->a] b
    a.b = [suc->[*+a]] b = [suc->[[0->**] a]] b
    a^b = [suc->[*.a], 0->1] b = [suc->[[suc->[[0->**] ***]] a], 0->suc 0] b
    w^a = [suc->[suc->H suc], 0->suc 0] b = [suc->H,0->suc] b 0
    epsilon_0^a = [suc->R1 H,0->suc] a 0
```

```
    epsilon_a = [suc->R1, 0->H] (1+a) suc 0
    1+a = [0->suc 0] a


The ordinal ascension can be continued after Ra...1 H suc 0 by taking the fixed point :
H [ R(*->1) H suc 0 ] 0
= R(1;1) H suc 0
with R(1;1) = [[[ H [ R(*...1) **** *** ** ] 0 ]]]
= tuple 3 [ H [ ** R(*...1) ] 0]
with tuple n f x1 ... xn = f <x1, ... , xn>
and <x1, ... , xn> f = f x1 ... xn
and then we can go on with :

    H [R*...1 H suc 0] 0 = R(1;1) H suc 0
    R(1;1) (R(1;1) H) suc 0
    H R(1;1) H suc 0
    R(1;1) H R(1;1) H suc 0
    R2 R(1;1) H suc 0
    R3 R2 R(1;1) H suc 0 = R(3...2) R(1;1) H suc 0
    H [ R(*...2) R(1;1) H suc 0 ] 0 = R(1;2) R(1;1) H suc 0 with R(1;2)
     = [[[[H[R(*...2) ***** **** *** **]0]]]] = R(1;2...1) H suc 0
    R(1;3) R(1;2) R(1;1) H suc 0 = R(1;3...1) H suc 0
    H [ R(1;*...1) H suc 0 ] 0 = R(2;1) H suc 0 with R(2;1) = [[[ H [ R(1;*...1) **** *** ** ] 0 ]]]
    R(3;1) H suc 0
    R(w;1) H suc 0
    H [ R(*;1) H suc 0] 0 = R(W;1) H suc 0 or R(1,0;1) H suc 0 with R(W;1)
     = R(1,0;1) = [[[ H [ R(*;1) **** *** ** ] 0 ]]] or R(1:1;1) H suc 0
    R(1,1;1) H suc 0 = R(1:1,0:1;1)
    R(1,2;1) H suc 0 = R(1:1,0:2;1) H suc 0
    R(2,0;1) H suc 0 = R(1:2;1) H suc 0
    R(1,0,0;1) H suc 0 = R(2:1;1) H suc 0
    R(1,0,...,0;1) H suc 0 = R(H suc 0:1;1) H suc 0
    H [ R(*:1;1) H suc 0 ] 0 = [[[ H [ R(*:1;1) **** *** ** ] 0 ]]] H suc 0
     = R(1,0:1;1) H suc 0 = R(1:1:1;1) H suc 0
    R(1,0,0:1;1) H suc 0 = R(2:1:1;1) H suc 0
    R(w:1:1;1) H suc 0 (?) with a:b:c = (a:b):c
    R(1:1:1:1;1) H suc 0 = R(1::3::1;1) H suc 0 = R(r 3 [*:1] 1;1) H suc 0
    R(1:1:1:1:1;1) H suc 0 = R(1::4::1;1) H suc 0 = R(r 4 [*:1] 1;1) H suc 0
    H [R(1::*::1;1) H suc 0] 0 = H [R(r * [*:1] 1;1) H suc 0] 0
    ...


Correspondence with Simmons notation :
The RHS0 notation can be obtained from the Simmons notation (ending with w) by replacing
w by H, Next by R1, [0] by R2, [1] by R3, [2] by R4, and so on, and putting suc 0 at the end.
For example :

    w = H suc 0
    epsilon_0 = Next w = R1 H suc 0
    epsilon_1 = Next^2 w = Next (Next w) = R1 (R1 H) suc 0
    zeta_0 = [0] Next w = R2 R1 H suc 0
    Gamma_0 = [1] [0] Next w = R3 R2 R1 H suc 0
    LVO = [2] [1] [0] Next w = R4 R3 R2 R1 H suc 0
```

7 Ordinal collapsing functions

An ordinal collapsing function is a function which, when applied to an uncountable ordinal, gives a countable ordinal.

The general idea is to define a set of ordinals C(a) or C(a,b) where a and b are ordinals, which contains all ordinals that can be built using an initial set of ordinals and some operations or functions, and then define $\psi(a)$ or $\psi(a,b)$ as the smallest ordinal which is not in C(a) or C(a,b), or the least ordinal which is greater than than all countable ordinals of C(a) or C(a,b).

Different ordinal collapsing functions are described in http://googology.wikia.com/wiki/Ordinal_notation .

These functions are extensions of functions on countable ordinals, whose fixed points can be reached by applying them to an uncountable ordinal.

Here is a correspondence between basic notation systems and their collapsing extensions based on formula : least fixed point of $f = f(\Omega)$:

Basic notation	Formula	Limit	Extension	Correspondence	Crossing
Cantor	$cantor(\alpha, \beta)$ $= \beta + \omega^\alpha$	least $\alpha = cantor(\alpha, 0)$ $= \omega^\alpha = \varepsilon_0$	Taranovsky's C	$C(\alpha, \beta) = \beta + \omega^\alpha$ iff $C(\alpha, \beta) \geq \alpha$	$C(\Omega, 0) = \varepsilon_0$
Epsilon	ε_α	least $\alpha = \varepsilon_\alpha$ $= \zeta_0$	Madore's ψ	$\psi(\alpha) = \varepsilon_\alpha$ for all $\alpha < \zeta_0$	$\psi(\Omega) = \zeta_0$
Binary Veblen	$\varphi_\alpha(\beta)$ or $\varphi(\alpha, \beta)$	least $\alpha = \varphi(\alpha, 0)$ $= \Gamma_0$	θ	$\theta(\alpha, \beta) = \varphi(\alpha, \beta)$ below Γ_0	$\theta(\Omega, 0) = \Gamma_0$

7.1 Madore's ψ

This ordinal collapsing function is described in https://en.wikipedia.org/wiki/Ordinal_collapsing_function and http://quibb.blogspot.fr/? impredicative-ordinals.html .

The definition of this function uses the ordinal Ω which is the least uncountable ordinal.

$C(\alpha)$ is the set of all ordinals constructible using only $0, 1, \omega, \Omega$ and addition, multiplication, exponentiation, and the function ψ (which will be defined later) restricted to ordinals smaller than α.

$\psi(\alpha)$ is the smallest ordinal not in $C(\alpha)$.

The smallest ordinal not in C(0) is the limit of $\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots$ which is ε_0, so $\psi(0) = \varepsilon_0$. More generally, $\psi(\alpha) = \varepsilon_\alpha$ for all $\alpha < \zeta_0, \psi(\alpha) = \zeta_0$ for $\zeta_0 \leq a \leq \Omega$, and $\psi(\Omega + \alpha) = \varepsilon_(\zeta_0 + \alpha)$ for $\alpha <= \zeta_1$.

Note that $\psi(\Omega) = \zeta_0$ is the least fixed point of $\alpha \mapsto \varepsilon_\alpha$; we already saw such an equality when we introduced collapsing in the Veblen function.

The limit $\psi(\epsilon_(\Omega + 1))$ of $\psi(\Omega), \psi(\Omega^\Omega), \psi(\Omega^{\Omega^\Omega}), \ldots$ is the Bachmann-Howard ordinal.

Some examples of fundamental sequences (FS) are :

A FS of ω is 0, 1, 2, 3, ...

A FS of $\psi(0)$ is $\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots$.

A FS of $\psi(\alpha + 1)$ is $\psi(\alpha), \psi(\alpha)^{\psi(\alpha)}, \psi(\alpha)^{\psi(\alpha)^{\psi(\alpha)}}, \ldots$.

A FS of $\psi(f(\Omega))$ is $\psi(0), \psi(f(\psi(0))), \psi(f(\psi(f(\psi(0))))), \ldots$.

For example :

A FS of $\psi(\Omega)$ is $\psi(0), \psi(\psi(0)), \psi(\psi(\psi(0))), \ldots$.

A FS of $\psi(\Omega \cdot 2)$ is $\psi(0), \psi(\Omega + \psi(0)), \psi(\Omega + \psi(\Omega + \psi(0))), \ldots$.

A FS of $\psi(\Omega^\Omega \cdot 3)$ is $\psi(0), \psi(\Omega^\Omega \cdot 2 + \Omega^{\psi(0)}), \psi(\Omega^\Omega \cdot 2 + \Omega^{\Omega^\Omega \cdot 2 + \Omega^{\psi(0)}}), \cdots$.

7.2 Correspondence between Veblen functions and Madore's ψ

To distinguish between the different Veblen functions, let us call φ_F the Veblen function with finitely many variables, and φ_T the Veblen function with transfinitely many variables.

φ_F is a function which, when applied to a list of countable ordinals, gives a countable ordinal. A list of countable ordinals can be seen as a function which, when applied to a natural number, gives a countable ordinal, with the restriction that the result differs from 0 for finitely many integers. If we denote ω the set of natural numbers and Ω the set of countable ordinals, then this can be written : $\varphi_F : (\omega \to \Omega) \to \Omega$. If we replace $\alpha \to \beta$ by β^α, we get Ω^{Ω^ω}, and if we apply ψ to it, we get $\psi(\Omega^{\Omega^\omega})$, which is the small Veblen ordinal, the least ordinal that cannot be reached using φ_F.

For φ_T, the position of a variable is represented by a countable ordinal instead of a natural number, also with the restriction that finitely many variables differ from 0, so we have $\varphi_T : (\Omega \to \Omega) \to \Omega$. If we replace $\alpha \to \beta$ by β^α, we get Ω^{Ω^Ω}, and if we apply ψ to it, we get $\psi(\Omega^{\Omega^\Omega})$, which is the large Veblen ordinal, the least ordinal that cannot be reached using φ_T.

7.3 θ function

θ function is a binary function. It's defined as follows:

- $C_0(\alpha, \beta) = \{\gamma | \gamma < \beta\} \cup \{0\}$.

- $C_{n+1}(\alpha, \beta) = \{\gamma + \delta | \gamma, \delta \in C_n(\alpha, \beta)\} \cup \{\theta(\gamma, \delta) | \gamma < \alpha \& \gamma, \delta \in C_n(\alpha, \beta)\} \cup \{\Omega_c | c \in C_n(\alpha, \beta)\}$.

- $C(\alpha, \beta) = \cup_{n<\omega} C_n(\alpha, \beta)$

- $\theta(\alpha, \beta) = min\{c | \ (c \in C(\alpha, \gamma) \& (\forall \delta < \beta : \gamma > \theta(\alpha, \delta))\}$

  where $\Omega_0 = 0$ and Ω_a represents the a-th uncountable ordinal.

It means that $\theta(\alpha, \beta)$ is the $(1+\beta)$-th ordinal such that it cannot be built from ordinals less than it by addition, applying $\theta(\delta, \dots)$ where $\delta < \alpha$ and getting an uncountable cardinal.
It seems that $\theta(\alpha, \beta) = \varphi(\alpha, \beta)$ below Γ_0, making θ function an extension of φ function. Even $\theta(\Gamma_0, \beta) = \varphi(\Gamma_0, \beta)$ is true.

Other important values are :

- $\theta(\Omega, \alpha) = \Gamma_\alpha$

- $\theta(\Omega^\omega, 0) = $ small Veblen ordinal

- $\theta(\Omega^\Omega, 0) = $ large Veblen ordinal

- $\theta(\varepsilon_{\Omega+1}, 0) = $ Bachmann Howard ordinal

See https://stepstowardinfinity.wordpress.com/2015/05/04/ordinal2/ for more information.

7.4 Taranovsky's C

C(a,b) is the least element above b that has degree a.
Definition: A degree for a well-ordered set S is a binary relation on S such that :

- Every element $c \in S$ has degree 0_S (the least element of S). 0_S only has degree 0_S.

- For a limit a, c has degree a iff it has every degree less than a.

- For a successor a'=a+1, either of the following holds:

  - An element has degree a' iff it is a limit of elements of degree a.
  - There is a limit element d ¡= a such that for every c in S, c has degree a' iff it has degree a and either c ¡= d or c is a limit of elements of degree a (or both).

Note: The third condition can be equivalently written as $\forall a(C_{a+1} = lim(C_a) \lor \exists d \in lim(S) \cap (a+1) C_{a+1} = lim(C_a) \cup (C_a \cap (d+1)))$, where S is identified with an ordinal (so a+1 consists of ordinals $\leq a$), C_a is the set of elements that have degree a, and lim is limit points.
In other terms : Let η be an ordinal, and let 0_S and let Ld(a,b) be the statement that a is a limit of ordinals c such that $(c, b) \in D$. Let D be the following binary relation over η :

- $\forall a < \eta : (a, 0) \in D$

- $\forall a < \eta : a \neq 0 \Rightarrow (0, a) \notin D$

- $\forall b \in Lim \cup \eta : (a, b) \in D \Leftrightarrow \forall c < b : (a, c) \in D$

- $\forall b : (a, b) \in D \Leftrightarrow Ld(a, b+1) \forall b : (a, b) \in D \Leftrightarrow Ld(a, b+1)$

- $\forall b : \exists d \in Lim \cup \eta : d <= b \Rightarrow \forall c : (c, a+1) \in D \Leftrightarrow (c <= d \lor Ld(c, b))$

Then $C(a, b) = min c : c \in \eta \wedge c > b \wedge (c, a) \in D$.

$C(a, b) = b + \omega^a$ iff $C(a, b) \geq a$.

For ordinals in the standard representation written in the postfix form, the comparison is done in the lexicographical order where $'C' <' 0' <' \Omega'$: For example, $C(C(0,0),0) < C(\Omega, 0)$ because $000CC < 0\omega C$. (This does not hold for non-standard representations of ordinals.)

The fundamental sequences of Taranovsky's notation can be easily defined.

Let $L(\alpha)$ be the amount of C's in standard representation of α, then $\alpha[n] = max\{\beta | \beta < \alpha \wedge L(\beta) \leq L(\alpha) + n\}$.

Here is a summary of the system by Taranovsky (see https://cs.nyu.edu/pipermail/fom/2012-March/016349.html) :

```
I discovered a conjectured ordinal notation system that I conjecture
reaches full second order arithmetic.  I implemented the system in a
python module/program:
http://web.mit.edu/dmytro/www/other/OrdinalArithmetic.py
along with ordinal arithmetic operations (addition, multiplication,
exponentiation, etc.) and other functions.  The ordinal arithmetic
functionality is useful even if you are only interested in ordinals
below epsilon_0.

The notation system is simple enough to be defined in full here.

Definition: An ordinal a is 0-built from below from b iff a<=b
a is n+1-built from below from b iff the standard representation of a
does not use ordinals above a except in the scope of an ordinal n-built
from below from b.

(Note: "in the scope of" means "as a subterm of".)

The nth (n is a positive integer) ordinal notation system is defined as
follows.

Syntax: Two constants (0, W_n) and a binary function C.
Comparison: For ordinals in the standard representation written in the
postfix form, the comparison is done in the lexicographical order where
'C' < '0' < 'W_n': For example, C(C(0,0),0) < C(W_n, 0) because 0 0 0 C
C < 0 W_n C.

Standard Form:
0, W_n are standard
"C(a, b)" is standard iff
1. "a" and "b" are standard,
2. b is 0 or W_n or "C(c, d)" with a<=c, and
3. a in n-built from below from b.

I conjecture that the strength of the nth ordinal notation system is
between Pi^1_{n-1}-CA and Pi^1_n-CA_0, and thus the sum of the order
types of these ordinal notation systems is the proof-theoretical ordinal
of second order arithmetic.

The full notation system is obtained by combining these notation systems
as follows:
Constants 0 and W_i (for every positive integer i), and a binary function C.
W_i = C(W_{i+1}, 0) and the standard form always uses W_i instead of
C(W_{i+1}, 0).
To check for standard form and compare ordinals use W_i = C(W_{i+1}, 0)
to convert each W to W_n for a single positive integer n (it does not
matter which n) and then use the nth ordinal notation system.
```

To make C a total function for a and b in the notation system (this is
not required for standard forms), let C(a, b) be the least ordinal (in
the notation system) of degree >=a above b, where the degree of W_i is
W_{i+1} and the degree of C(c,d) is c if "C(c,d)" is the standard form.
A polynomial time computation of C(a, b) (that I believe is correct) is
included in the program.

To complete ordinal analysis of second order arithmetic, one would need:
* A canonical assignment of notations to formulas that provably in
second order arithmetic denote an ordinal, and such that for every two
ordinals/formulas, comparison is provable in second order arithmetic.
The idea is that the notation system captures not only provably
recursive ordinals of second order arithmetic but all ordinals that have
a provable canonical definition in second order arithmetic. For
example, W_1 is best assigned to the least admissible ordinal above
omega. A partial assignment is in my paper. (It is because of such
assignment that I believe that the system reaches full second order
arithmetic.)
* Proof that the system is well-founded and that it has the right
strength, etc. (If you do not fully understand the notation system, or
if you think that it is not well-founded, let me know.)

Historical Note: In 2005, I discovered the right general form of C,
defined a notation system at the level of alpha-recursively inaccessible
ordinals (FOM postings in August 2005), and had an idea for reaching
second order arithmetic. In January 2006 (or possibly late 2005), I
defined the notation system with W_2 and in 2009 (June 29, 2009 FOM
posting) implemented it is a computer program. This year I defined the
key concept -- n-built from below -- that allowed me to complete the
full notation system.

Details about the ordinal notation system and its initial segments are
in my paper:
http://web.mit.edu/dmytro/www/other/OrdinalNotation.htm

Sincerely,
Dmytro Taranovsky

Here are some examples of representations of some ordinals :

- $0 = 0$

- $1 = 0 + \omega^0 = C(0,0)$

- $2 = 1 + \omega^0 = C(0,1) = C(0, C(0,0))$

- $\omega = 0 + \omega^1 = C(1,0)$

- $\omega + 1 = \omega + \omega^0 = C(0,\omega) = C(0, C(1,0))$

- $\omega \cdot 2 = \omega + \omega^1 = C(1,\omega) = C(1, C(1,0))$

- $\omega^2 = 0 + \omega^2 = C(2,0)$

- $\omega^\omega = 0 + \omega^\omega = C(\omega, 0) = C(C(1,0), 0)$

- $\omega^{\omega^\omega} = 0 + \omega^{\omega^\omega} = C(\omega^\omega, 0) = C(C(C(1,0),0),0)$

- $\varepsilon_0 = \varphi(1,0) = \varphi'(0,1) = C(\Omega_1, 0)$

- $\varepsilon_1 = \varphi(1,1) = \varphi'(0,2) = C(W, C(W,0))$ (note that the correspondence with φ' is simpler than with φ)

- $\zeta_0 = \varphi(2,0) = \varphi'(1,1) = C(C(\Omega_1, \Omega_1), 0) = C(\Omega_1 \cdot 2, 0)$ with $\Omega_1 \cdot 2 = C(\Omega_1, \Omega_1)$

- $\zeta_1 = \varphi(2,1) = \varphi'(1,2) = C(\Omega_1 \cdot 2, C(\Omega_1 \cdot 2, 0))$

- $\eta_0 = \varphi(3,0) = \varphi'(2,1) = C(\Omega_1 \cdot 3, 0)$ with $\Omega_1 \cdot 3 = C(\Omega_1, C(\Omega_1, \Omega))$

- $\Gamma_0 = \varphi(1,0,0) = \varphi'(1,0,1) = C(C(\Omega_1 \cdot 2, \Omega_1), 0) = C(\Omega_1{}^2, 0)$ with $\Omega_1{}^2 = C(\Omega_1 \cdot 2, \Omega_1)$

- $\Gamma_1 = C(\Omega_1{}^2, C(\Omega_1{}^2, 0))$

- $\Gamma_\omega = C(\Omega_1{}^2 + 1, 0)$

- Small Veblen ordinal $= C(\Omega_1{}^\omega, 0)$

- Large Veblen ordinal $= C(\Omega_1{}^{\Omega_1}, 0)$

- Bachmann Howard ordinal $= C(C(\Omega_2, \Omega_1), 0)$

See http://web.mit.edu/dmytro/www/other/OrdinalNotation.htm and
https://stepstowardinfinity.wordpress.com/2015/06/22/ordinal3/ for more information.

8 Summary

Any ordinal can be defined as the least ordinal strictly greater than all ordinals of a set : the empty set for 0, $\{\alpha\}$ for the successor of α, $\{\alpha_0, \alpha_1, \alpha_2, ...\}$ for an ordinal with fundamental sequence $\alpha_0, \alpha_1, \alpha_2, ...$

9 Algebraic notation

We define the following operations on ordinals :
- addition : $\alpha + 0 = \alpha; \alpha + suc(\beta) = suc(\alpha + \beta); \alpha + lim(f) = lim(n \mapsto \alpha + f(n))$
- multiplication : $\alpha \times 0 = 0; \alpha \times suc(\beta) = (\alpha \times \beta) + \alpha; \alpha \times lim(f) = lim(n \mapsto \alpha \times f(n))$
- exponentiation : $\alpha^0 = 1; \alpha^{suc(\beta)} = \alpha^\beta \times \alpha; \alpha^{lim(f)} = lim(n \mapsto \alpha^{f(n)})$

10 Veblen functions

These functions use fixed points enumaration : $\varphi(\ldots, \beta, 0, \ldots, 0, \gamma)$ represents the $(1 + \gamma)^{th}$ common fixed point of the functions $\xi \mapsto \varphi(\ldots, \delta, \xi, 0, \ldots, 0)$ for all $\delta < \beta$.

11 Simmons notation

$Fix f z = f^w(z + 1) =$ least fixed point of f strictly greater than z.
$Next = Fix(\alpha \mapsto \omega^\alpha)$
$[0]h = Fix(\alpha \mapsto h^\alpha \omega)$; $[1]hg = Fix(\alpha \mapsto h^\alpha g \omega)$; $[2]hgf = Fix(\alpha \mapsto h^\alpha g f \omega)$; etc...
Correspondence with Veblen's ϕ : $\phi(1 + \alpha, \beta) = ([0]^\alpha Next)^{1+\beta} \omega; \phi(\alpha, \beta, \gamma) = ([0]^\beta (([1][0])^\alpha Next))^{1+\gamma} \omega$

12 RHS0 notation

We start from 0, if we don(t see any regularity we take the successor, if we see a regularity, if we have a notation for this regularity, we use it, else we invent it, then we jump to the limit.
$H f x = lim\ x, fx, f(fx), \ldots; R_1 fgx = lim\ gx, fgx, ffgx, \ldots; R_2 fghx = lim\ hx, fghx, fgfghx, \ldots$
Correspondence with Simmons notation : $\ldots, [3] \to R5, [2] \to R4, [1] \to R3, [0] \to R2, Next \to R1, \omega \to Hsuc\ 0$

13 Ordinal collapsing functions

These functions use uncountable ordinals to define countable ordinals.
We define sets of ordinals that can be built from given ordinals and operations, then we take the least ordinal which is not in this set, or the least ordinal which is greater than all contable ordinals of this set.
These functions are extensions of functions on countable ordinals, whose fixed points can be reached by applying them to an uncountable ordinal.
Examples :
- Madore's ψ : $\psi(\alpha) = \varepsilon_\alpha$ if $\alpha < \zeta_0; \psi(\Omega) = \zeta_0$ which is the least fixed point of $\alpha \mapsto \varepsilon_\alpha$.
- Feferman's θ : $\theta(\alpha, \beta) = \varphi(\alpha, \beta)$ if $\alpha < \Gamma_0$ and $\beta < \Gamma_0; \theta(\Omega, 0) = \Gamma_0$ which is the least fixed point of $\alpha \mapsto \varphi(\alpha, 0)$.
- Taranovsky's C : $C(\alpha, \beta) = \beta + \omega^\alpha$ if α is countable; $C(\Omega_1, 0) = \varepsilon_0$ which is the least fixed point of $\alpha \mapsto \omega^\alpha$.

Nom	Symbole	Algebraic	Veblen	Simmons	RHS0	Madore	Taranovsky
Zero	0	0			0		0
One	1	1	$\varphi(0,0)$		suc 0		C(0,0)
Two	2	2			suc (suc 0)		C(0,C(0,0))
Omega	ω	ω	$\varphi(0,1)$	ω	H suc 0		C(1,0)
		$\omega+1$			suc (H suc 0)		C(0,C(1,0))
		$\omega\times 2$			H suc (H suc 0)		C(1,C(1,0))
		ω^2	$\varphi(0,2)$		H (H suc) 0		C(C(0,C(0,0)),0)
		ω^ω	$\varphi(0,\omega)$		H H suc 0		C(C(1,0),0)
		ω^{ω^ω}	$\varphi(0,\omega^\omega)$		H H H suc 0		C(C(C(1,0),0),0)
Epsilon zero	ε_0	ε_0	$\varphi(1,0)$	$Next\ \omega$	$R_1 H suc\ 0$	$\psi(0)$	$C(\Omega_1,0)$
		ε_1	$\varphi(1,1)$	$Next^2\omega$	$R_1(R_1 H) suc\ 0$	$\psi(1)$	$C(\Omega_1,C(\Omega_1,0)$
		ε_ω	$\varphi(1,\omega)$	$Next^\omega\omega$	$H R_1 H suc\ 0$	$\psi(\omega)$	$C(C(0,\Omega_1),0)$
		$\varepsilon_{\varepsilon_0}$	$\varphi(1,\varphi(1,0))$	$Next^{Next\omega}\omega$	$R_1 H R_1 H suc\ 0$	$\psi(\psi(0))$	$C(C(C(\Omega_1,0),\Omega_1),0)$
Zeta zero	ζ_0	ζ_0	$\varphi(2,0)$	$[0]Next\ \omega$	$R_2 R_1 H suc\ 0$	$\psi(\Omega)$	$C(C(\Omega_1,\Omega_1),0)$
Eta zero	η_0	η_0	$\varphi(3,0)$	$[0]^2 Next\ \omega$	$R_2(R_2 R_1)H suc\ 0$		$C(C(\Omega,C(\Omega,\Omega)),0)$
			$\varphi(\omega,0)$	$[0]^\omega Next\ \omega$	$H R_2 R_1 H suc\ 0$		$C(C(C(0,\Omega_1),\Omega_1),0)$
Feferman -Schütte	Γ_0	Γ_0	$\varphi(1,0,0)$ $=\varphi(2\mapsto 1)$	$[1][0]Next\ \omega$	$R_3 R_2 R_1 H suc\ 0$ $=R_{3\ldots 1}H suc\ 0$	$\psi(\Omega^\Omega)$	$C(C(C(\Omega_1,\Omega_1),$ $\Omega_1),0)$
Ackermann			$\varphi(1,0,0,0)$ $=\varphi(3\mapsto 1)$	$[1]^2[0]Next\ \omega$	$R_3(R_3 R_2)R_1 H suc\ 0$	$\psi(\Omega^{\Omega^2})$	
Small Veblen ordinal			$\varphi(\omega\mapsto 1)$	$[1]^\omega[0]Next\ \omega$	$H R_3 R_2 R_1 H suc\ 0$	$\psi(\Omega^{\Omega^\omega})$	$C(\Omega_1^\omega,0)$ $=C(C(C(C(0,\Omega_1),$ $\Omega_1),\Omega_1),0)$
Large Veblen ordinal			least ord. not rep.	$[2][1][0]Next\ \omega$	$R_4 R_3 R_2 R_1 H suc\ 0$ $=R_{4\ldots 1}H suc\ 0$	$\psi(\Omega^{\Omega^\Omega})$	$C(\Omega_1^{\Omega_1},0)$ $=C(C(C(C(\Omega_1,\Omega_1),$ $\Omega_1),\Omega_1),0)$
Bachmann- Howard ordinal				least ord. not rep.	$R_{\omega\ldots 1}H suc\ 0$	$\psi(\varepsilon_{\Omega+1})$	$C(C(\Omega_2,\Omega_1),0)$