A Tutorial Overview of Ordinal Notations

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1 Ordinal collapsing functions

1.1 Deedlit's extension of hierarchy of ϑ-functions with φ and Ω_α

1.1.1 Definition

- $C_0(\nu, \alpha, \beta) = \beta \cup \Omega_\nu \cup \{0\}$
- $C_{n+1}(\nu, \alpha, \beta) = \{\gamma + \delta, \varphi(\gamma, \delta), \Omega_\gamma, \vartheta_\gamma(\eta) : \gamma, \delta, \eta \in C_n(\nu, \alpha, \beta); \eta < \alpha\}$
- $C(\nu, \alpha, \beta) = \cup_{n<\omega} C_n(\nu, \alpha, \beta)$
- $\vartheta_\nu(\alpha) = \min(\{\beta < \Omega_{\nu+1} : C(\nu, \alpha, \beta) \cap \Omega_{\nu+1} \subseteq \beta \wedge \alpha \in C(\nu, \alpha, \beta)\} \cup \{\Omega_{\nu+1}\})$

1.1.2 Standard form

- If $\alpha = 0$, then the standard form for α is 0.
- If α is not additively principal, then the standard form for α is $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, where the α_i are principal ordinals with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$, and the α_i are expressed in standard form.
- If α is an additively principal ordinal but not a strongly critical ordinal, then the standard form for α is $\alpha = \varphi(\beta, \gamma)$ where $\gamma < \alpha$ where β and γ are expressed in standard form.
- If α is of the form Ω_β, then Ω_β is the standard form for α.
- If α is a strongly critical ordinal but not of the form Ω_β, then α is expressible in the form $\vartheta_\nu(\gamma)$. Then the standard form for α is $\alpha = \vartheta_\nu(\gamma)$ where γ and ν are expressed in standard form.

1.2 Fundamental sequences

For ordinals $\alpha < \vartheta(\Omega_{\Omega_{\Omega_{\dots}}})$, written in normal form, fundamental sequences are defined as follows:

- If $\alpha = 0$, then $\text{cof}(\alpha) = 0$ and α has fundamental sequence the empty set.
- If $\alpha = \varphi(0, 0) = 1$ then $\text{cof}(\alpha) = 1$ and $\alpha[0] = 0$
- If $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, then $\text{cof}(\alpha) = \text{cof}(\alpha_n)$ and $\alpha[\eta] = \alpha_1 + \alpha_2 + \cdots + (\alpha_n[\eta])$
- If $\alpha = \varphi(\beta, \gamma)$ where γ is a limit ordinal then $\text{cof}(\alpha) = \text{cof}(\gamma)$ and $\alpha[\eta] = \varphi(\beta, \gamma[\eta])$
- If $\alpha = \varphi(0, \gamma + 1)$ then $\text{cof}(\alpha) = \omega$ and $\alpha[\eta] = \varphi(0, \gamma) \cdot \eta$
- If $\alpha = \varphi(\beta + 1, 0)$ then $\text{cof}(\alpha) = \omega$ and $\alpha[0] = 0$ and $\alpha[\eta + 1] = \varphi(\beta, \alpha[\eta])$
- If $\alpha = \varphi(\beta + 1, \gamma + 1)$ then $\text{cof}(\alpha) = \omega$ and $\alpha[0] = \varphi(\beta + 1, \gamma) + 1$ and $\alpha[\eta + 1] = \varphi(\beta, \alpha[\eta])$
- If $\alpha = \varphi(\beta, 0)$ where β is a limit ordinal then $\text{cof}(\alpha) = \text{cof}(\beta)$ and $\alpha[\eta] = \varphi(\beta[\eta], 0)$
- If $\alpha = \varphi(\beta, \gamma + 1)$ where β is a limit ordinal then $\text{cof}(\alpha) = \text{cof}(\beta)$ and $\alpha[\eta] = \varphi(\beta[\eta], \varphi(\beta, \gamma) + 1)$
- If $\alpha = \Omega_{\beta+1}$ then $\text{cof}(\alpha) = \Omega_{\beta+1}$ and $\alpha[\eta] = \eta$
- If $\alpha = \Omega_\beta$ where β is a limit ordinal then $\text{cof}(\alpha) = \text{cof}(\beta)$ and $\alpha[\eta] = \Omega_{\beta[\eta]}$
- If $\alpha = \vartheta_\nu(\beta + 1)$ then $\text{cof}(\alpha) = \omega$ and $\alpha[0] = \vartheta_\nu(\beta) + 1$ and $\alpha[\eta + 1] = \varphi(\alpha[\eta], 0)$
- If $\alpha = \vartheta_\nu(\beta)$ where $\omega \leq \text{cof}(\beta) \leq \Omega_\nu$ then $\text{cof}(\alpha) = \text{cof}(\beta)$ and $\alpha[\eta] = \vartheta_\nu(\beta[\eta])$
- If $\alpha = \vartheta_\nu(\beta)$ where $\omega \leq \text{cof}(\beta) = \Omega_{\mu+1} > \Omega_\nu$ then $\text{cof}(\alpha) = \omega$ and $\alpha[\eta] = \vartheta_\nu(\beta[\gamma[\eta]])$ with $\gamma[0] = \Omega_\mu$ and $\gamma[\eta + 1] = \vartheta_\mu(\beta[\gamma[\eta]])$

1.3 Deedlit's extension of hierarchy of ϑ-functions without φ and Ω_α

1.3.1 Definition

- $C_0(\alpha, \beta) = \beta$
- $C_{n+1}(\alpha, \beta) = \{\gamma + \delta, \vartheta_\gamma(\eta) : \gamma, \delta, \eta \in C_n(\alpha, \beta); \eta < \alpha\}$
- $C(\alpha, \beta) = \cup_{n<\omega} C_n(\alpha, \beta)$
- $\vartheta_\nu(\alpha) = \min\{\beta : |\omega\beta| = \Omega_\nu; C(\alpha, \beta) \cap \Omega_{\nu+1} \subseteq \beta; \alpha \in C(\alpha, \beta)\}$

1.3.2 Standard form

- If $\alpha = 0$, then the standard form for α is 0.
- If α is not additively principal, then the standard form for α is $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, where the α_i are principal ordinals with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$, and the α_i are expressed in standard form.
- If α is additively principal, then α is expressible in the form $\vartheta_\nu(\gamma)$. Then the standard form for α is $\alpha = \vartheta_\nu(\gamma)$ where γ and ν are expressed in standard form.

1.3.3 Fundamental sequences

For ordinals $\alpha < \vartheta(\Omega_{\Omega_{\Omega_{\dots}}})$, written in normal form, fundamental sequences are defined as follows:

- If $\alpha = 0$, then $\mathrm{cof}(\alpha) = 0$ and α has fundamental sequence the empty set.
- If $\alpha = \vartheta_0(0) = 1$ then $\mathrm{cof}(\alpha) = 1$ and $\alpha[0] = 0$
- If $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, then $\mathrm{cof}(\alpha) = \mathrm{cof}(\alpha_n)$ and $\alpha[\eta] = \alpha_1 + \alpha_2 + \cdots + (\alpha_n[\eta])$
- If $\alpha = \vartheta_{\beta+1}(0)$ then $\mathrm{cof}(\alpha) = \Omega_{\beta+1}$ and $\alpha[\eta] = \eta$
- If $\alpha = \vartheta_\beta(0)$ where β is a limit ordinal then $\mathrm{cof}(\alpha) = \mathrm{cof}(\beta)$ and $\alpha[\eta] = \vartheta_{\beta[\eta]}(0)$
- If $\alpha = \vartheta_\nu(\beta + 1)$ then $\mathrm{cof}(\alpha) = \omega$ and $\alpha[\eta] = \vartheta_\nu(\beta)\eta$
- If $\alpha = \vartheta_\nu(\beta)$ where $\omega \leq \mathrm{cof}(\beta) \leq \Omega_\nu$ then $\mathrm{cof}(\alpha) = \mathrm{cof}(\beta)$ and $\alpha[\eta] = \vartheta_\nu(\beta[\eta])$
- If $\alpha = \vartheta_\nu(\beta)$ where $\mathrm{cof}(\beta) = \Omega_{\mu+1} > \Omega_\nu$ then $\mathrm{cof}(\alpha) = \omega$ and $\alpha[\eta] = \vartheta_\nu(\beta[\gamma[\eta]])$ with $\gamma[0] = \Omega_\mu$ and $\gamma[\eta+1] = \vartheta_\mu(\beta[\gamma[\eta]])$

These fundamental sequences can be reformulated :

- $(0 = 0)$
- $\vartheta_0(0) = 1$
- (standard definition of addition of a limit ordinal)
- $\vartheta_{\beta+1}(0) = \Omega_{\beta+1}$
- $\vartheta_{Lim_\mu f}(0) = Lim_\mu(\xi \mapsto \vartheta_{f(\xi)}(0))$
- $\vartheta_\nu(\beta + 1) = \vartheta_\nu(\beta) \cdot \omega$
- $\vartheta_\nu(Lim_\mu f) = Lim_\mu(\vartheta_\nu \circ f)$ if $\mu \leq \nu$
- $\vartheta_\nu(Lim_{\mu+1} f) = lim(\xi \mapsto \vartheta_\nu(f((\vartheta_\mu \circ f)^\xi(\Omega_\mu))$ if $\mu + 1 > \nu$