

More information about this function can be found at :  
[http://googology.wikia.com/wiki/Veblen\\_function](http://googology.wikia.com/wiki/Veblen_function)

Every non-zero ordinal  $\alpha < \Gamma_0$ , where  $\Gamma_0$  is the smallest ordinal  $\alpha$  such that  $\varphi_\alpha(0) = \alpha$ , can be uniquely written in normal form for the Veblen hierarchy:  
 $\alpha = \varphi_{\beta_1}(\gamma_1) + \varphi_{\beta_2}(\gamma_2) + \cdots + \varphi_{\beta_k}(\gamma_k)$ ,  
 where  
 $\varphi_{\beta_1}(\gamma_1) \geq \varphi_{\beta_2}(\gamma_2) \geq \cdots \geq \varphi_{\beta_k}(\gamma_k)$   $\gamma_m < \varphi_{\beta_m}(\gamma_m)$  for  $m \in \{1, \dots, k\}$

Now we will see how we can find the fundamental sequence of an ordinal written in this normal form.

From the rule defining addition of a limit ordinal :

$$\alpha + \lim(f) = \lim(n \mapsto \alpha + f(n))$$

we deduce the fundamental sequence :

$$(\alpha + \beta)[n] = \alpha + \beta[n]$$

if  $\beta$  is a limit ordinal.

In particular, we have :

$(\varphi_{\beta_1}(\gamma_1) + \varphi_{\beta_2}(\gamma_2) + \cdots + \varphi_{\beta_k}(\gamma_k))[n] = \varphi_{\beta_1}(\gamma_1) + \cdots + \varphi_{\beta_{k-1}}(\gamma_{k-1}) + (\varphi_{\beta_k}(\gamma_k)[n])$ , where  $\varphi_{\beta_1}(\gamma_1) \geq \varphi_{\beta_2}(\gamma_2) \geq \cdots \geq \varphi_{\beta_k}(\gamma_k)$  and  $\gamma_m < \varphi_{\beta_m}(\gamma_m)$  for  $m \in \{1, 2, \dots, k\}$ ,

Then,  $\varphi_0(\gamma)$  is  $\omega^\gamma$ .

For  $\gamma = 0$  it is 1.

From the rule of multiplication by a limit ordinal :

$$\alpha \cdot \lim(f) = \lim(n \mapsto \alpha \cdot f(n))$$

we deduce the fundamental sequence :

$$(\alpha \cdot \beta)[n] = \alpha \cdot \beta[n] \text{ if } \beta \text{ is a limit ordinal.}$$

In particular, for  $\omega$  :

$$(\alpha \cdot \omega)[n] = \alpha \cdot \omega[n] = \alpha \cdot n$$

Then we have :

$$\varphi_0(\gamma + 1) = \omega^{\gamma+1} = \omega^\gamma \cdot \omega = \varphi_0(\gamma) \cdot \omega$$

So the corresponding fundamental sequence is :

$$\varphi_0(\gamma + 1)[n] = (\varphi_0(\gamma) \cdot \omega)[n] = \varphi_0(\gamma) \cdot n$$

If  $\gamma$  is a limit ordinal, the fundamental sequence is defined canonically :

$$\varphi_0(\gamma)[n] = \varphi_0(\gamma[n])$$

This function can be defined with fundamental sequences.

The fundamental sequences for the Veblen functions  $\varphi_\beta(\gamma) = \varphi(\beta, \gamma)$  are :

1.  $(\varphi_{\beta_1}(\gamma_1) + \varphi_{\beta_2}(\gamma_2) + \cdots + \varphi_{\beta_k}(\gamma_k))[n] = \varphi_{\beta_1}(\gamma_1) + \cdots + \varphi_{\beta_{k-1}}(\gamma_{k-1}) + (\varphi_{\beta_k}(\gamma_k)[n])$ , where  $\varphi_{\beta_1}(\gamma_1) \geq \varphi_{\beta_2}(\gamma_2) \geq \cdots \geq \varphi_{\beta_k}(\gamma_k)$  and  $\gamma_m < \varphi_{\beta_m}(\gamma_m)$  for  $m \in \{1, 2, \dots, k\}$ ,
2.  $\varphi_0(0) = 1$ ,
3.  $\varphi_0(\gamma + 1)[n] = \varphi_0(\gamma)n$

4.  $\varphi_{\beta+1}(0)[0] = 0$  and  $\varphi_{\beta+1}(0)[n+1] = \varphi_{\beta}(\varphi_{\beta+1}(0)[n])$ ,
5.  $\varphi_{\beta+1}(\gamma+1)[0] = \varphi_{\beta+1}(\gamma)+1$  and  $\varphi_{\beta+1}(\gamma+1)[n+1] = \varphi_{\beta}(\varphi_{\beta+1}(\gamma+1)[n])$ ,
6.  $\varphi_{\beta}(\gamma)[n] = \varphi_{\beta}(\gamma[n])$  for a limit ordinal  $\gamma < \varphi_{\beta}(\gamma)$ ,
7.  $\varphi_{\beta}(0)[n] = \varphi_{\beta[n]}(0)$  for a limit ordinal  $\beta < \varphi_{\beta}(0)$ ,
8.  $\varphi_{\beta}(\gamma+1)[n] = \varphi_{\beta[n]}(\varphi_{\beta}(\gamma)+1)$  for a limit ordinal  $\beta$ .

These fundamental sequences can be reformulated to get a definition of the function  $\varphi$ .

1. This does not concern the definition of the  $\varphi$  function but the definition of addition
2. and
3. are equivalent to  $\varphi_0(\gamma) = \omega^{\gamma}$ .
4.  $\varphi_{\beta+1}(0) = \lim(n \mapsto \varphi_{\beta}^n(0)) = \varphi_{\beta}^{\omega}(0)$  which is the least fixed point of  $\varphi_{\beta}$ .
5.  $\varphi_{\beta+1}(\gamma+1) = \lim(n \mapsto \varphi_{\beta}^n(\varphi_{\beta+1}(\gamma)+1))$ , which is the least fixed point of  $\varphi_{\beta}$  strictly greater than  $\varphi_{\beta+1}(\gamma)$ , so  $\varphi_{\beta+1}(\gamma)$  is the  $1+\gamma$ -th fixed point of  $\varphi_{\beta}$ .