**TRANSFINITE ORDINALS**

by Jacques Bailhache, January-february 2018

# 1 Defining transfinite ordinal numbers

Natural numbers can be represented by sets. Each number is represented by the set of all numbers smaller than it.

- $0 = \{\}$ (the empty set)
- $1 = \{0\} = \{\{\}\}$
- $2 = \{0,1\} = \{\{\}, \{\{\}\}\}$
- $3 = \{0,1,2\} = \{\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}\}$
- ...

The successor of a natural number can be defined by $suc(n) = n + 1 = n \cup \{n\}$.

We have $n \leq p$ if and only if $n \subseteq p$.

$\mathbb{N}$ is the set of all natural numbers : $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ The natural numbers can be generalized to what is called "transfinite ordinal numbers", or simply "ordinal numbers" or "ordinals", by considering that infinite sets represent ordinal numbers. $\mathbb{N}$ considered as an ordinal number is written $\omega$. $\omega$ is the least ordinal which is greater than all the numbers 0, 1, 2, 3, ... We say that $\omega$ is a limit ordinal and 0, 1, 2, 3, ... is a fundamental sequence of $\omega$. This is written : $\omega = sup\{0, 1, 2, 3, \ldots\}$ or $\omega = lim(n \mapsto n)$ because the n-th element (starting with 0) of the sequence is n. An ordinal does not have a unique fundamental sequence, for example 1, 2, 3, 4, ... is also a fundamental sequence of $\omega$, because the least ordinal that is greater than all ordinals of this sequence is also $\omega$ (more generally the limit ordinal is the same if any number of the least items of a sequence are removed), and the same stands for the sequence 0, 2, 4, 6, ...

Any ordinal can be defined as the least ordinal strictly greater than all ordinals of a set : the empty set for 0, $\{\alpha\}$ for the successor of $\alpha$, $\{\alpha_0, \alpha_1, \alpha_2, \ldots\}$ for an ordinal with fundamental sequence $\alpha_0, \alpha_1, \alpha_2, \ldots$

The successor can be generalized to transfinite ordinal numbers : $suc(\omega) = \omega + 1 = \omega \cup \{\omega\} = \{0, 1, 2, 3, \ldots, \omega\}$; $suc(suc(\omega)) = \omega + 2 = \{0, 1, 2, 3, \ldots, \omega, \omega + 1\}$ and so on.

Then we can consider the set $\{0, 1, 2, 3, \ldots, \omega, \omega + 1, omega + 2, omega + 3, \ldots\}$ which is a limit ordinal, and $\omega, \omega + 1, \omega + 2, \omega + 3, \ldots$ is a fundamental sequence of this ordinal. This ordinal is $\omega + \omega = \omega \cdot 2$ or $\omega \times 2$ or $\omega 2$.

Then we can go on building greater and greater ordinals : $\omega \cdot 3, \ldots, \omega \cdot \omega = \omega^2, \omega^3, \ldots, \omega^\omega, \omega^{\omega^\omega}, \ldots$.

The definitions of arithmetical operations can be generalized to ordinals :

- addition : $\alpha + 0 = \alpha; \alpha + suc(\beta) = suc(\alpha + \beta); \alpha + lim(f) = lim(n \mapsto \alpha + f(n))$
- multiplication : $\alpha \cdot 0 = 0; \alpha \cdot suc(\beta) = (\alpha \cdot \beta) + \alpha; \alpha \cdot lim(f) = lim(n \mapsto \alpha \cdot f(n))$
- exponentiation : $\alpha^0 = 1; \alpha^{suc(\beta)} = \alpha^\beta \cdot \alpha; \alpha^{lim(f)} = lim(n \mapsto \alpha^{f(n)})$

Remark that the addition and the multiplication are not commutative, for example $1 + \omega = \omega \neq \omega + 1$, because if we take 0, 1, 2, 3, ... as fundamental sequence of $\omega$, then a fundamental sequence of $1 + \omega$ is 1+0, 1+1, 1+2, 1+3, ... = 1, 2, 3, 4, ... and the least ordinal which is greater than all ordinals of this sequence is $\omega$.

# 2 Veblen functions

The next step is the limit or least upper bound of $\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots$ which is called $\varepsilon_0$. Note that we have $\omega^{\varepsilon_0} = \varepsilon_0$. We say that $\varepsilon_0$ is a fixed point (the least one) of the function $\alpha \mapsto \omega^\alpha$.

Then we can go on with $\varepsilon_0 + 1, \varepsilon_0 + 2, \ldots, \varepsilon_0 + \varepsilon_0 = \varepsilon_0 \cdot 2, \ldots, \varepsilon_0 \cdot \varepsilon_0 = \varepsilon_0{}^2, \varepsilon_0{}^{\varepsilon_0}, \ldots$

The limit of $\varepsilon_0, \varepsilon_0^{\varepsilon_0}, \varepsilon_0^{\varepsilon_0^{\varepsilon_0}}, \ldots$ is called $\varepsilon_1$. It can be shown that it is also the limit of $\varepsilon_0 + 1, \omega^{\varepsilon_0 + 1}, \omega^{\omega^{\varepsilon_0 + 1}}, \ldots$ (see proof below).

These two fundamental sequences are examples of two ways of ascending ordinals :

- Build greater ordinals by combining known ordinals with arithmetic operations.
- When we have found a function that, when applied to a given ordinal, gives a greater one (for example $\alpha \mapsto \omega^\alpha$), enumerate the fixed points of this function. A fixed point of a function f is a value (for example an ordinal) $\alpha$ with $f(\alpha) = \alpha$. Under some conditions (see below), the least fixed point of f is the limit of 0, f(0), f(f(0)), f(f(f(0))), ... If it is called $\alpha$, the next fixed point is the limit of $\alpha + 1, f(\alpha + 1), f(f(\alpha + 1)), f(f(f(\alpha + 1))), \ldots$. More generally, the least fixed point of f that is greater or equal to $\zeta$ is the limit of $\zeta, f(\zeta), f(f(\zeta)), \ldots$. The Veblen functions use this method.

The required conditions are described for example in http://www.cs.man.ac.uk/ hsimmons/ORDINAL-NOTATIONS/Fruitful.pdf page 8 lemma 3.9 :

For each fruitful function f and each ordinal $\zeta$, $f^\omega(\zeta + 1)$ is the least ordinal $\nu$ such that $\zeta < \nu = f(\nu)$, or the least fixed point of f that is strictly greater than $\zeta$ (or greater than or equal to $\zeta + 1$).

$f^\omega(\zeta + 1)$ is the limit of $\zeta + 1, f(\zeta + 1), f(f(\zeta + 1)), \ldots$.

A fruitful function is a function that is inflationary, monotone, big, and continuous.

A function f is inflationary if $\alpha \leq f(\alpha)$, monotone if $\alpha \leq \beta \Rightarrow f(\alpha) \leq f(\beta)$, big if $\omega^\alpha \leq f(\alpha)$ except possibly for $\alpha = 0$, continuous if f(VA) = Vf[A] where VA is the pointwise supremum of the set A.

We will now prove by induction the equivalence of the two fundamental sequences above.

We will use the notation $\alpha^{\cdot^{\cdot^{\cdot^{\alpha^\beta}}}}$ for an an "exponential tower" with $\alpha$ repeated n times.

Note that the ordinals respectively limits of the fondamental sequence whose n-th term is $\varepsilon_0^{\varepsilon_0^{\cdot^{\cdot^{\varepsilon_0^\omega}}}}$ and the one whose n-th term is $\varepsilon_0^{\varepsilon_0^{\cdot^{\cdot^{\varepsilon_0^{\varepsilon_0}}}}}$ is the same, the least fixed point of the function $\alpha \mapsto \varepsilon_0{}^\alpha$, which is greater than $\omega$ and also than $\varepsilon_0$.

So we have proved what we want if we prove that, for any n, we have $\omega^{\omega^{\cdot^{\cdot^{\omega^{\omega^{\varepsilon_0+1}}}}}} = \varepsilon_0^{\varepsilon_0^{\cdot^{\cdot^{\varepsilon_0^\omega}}}}$ .

For n = 0, we have $\omega^{\omega^{\varepsilon_0+1}} = \omega^{\omega^{\varepsilon_0}\cdot\omega} = \omega^{\varepsilon_0\cdot\omega} = (\omega^{\varepsilon_0})^\omega = \varepsilon_0{}^\omega$.

Now suppose we have $\omega^{\omega^{\cdot^{\cdot^{\omega^{\omega^{\varepsilon_0+1}}}}}} = \varepsilon_0^{\varepsilon_0^{\cdot^{\cdot^{\varepsilon_0^\omega}}}}$ .

We must prove the equality for n+1, which can be written $\omega^{\omega^{\omega^{\cdot^{\cdot^{\omega^{\omega^{\varepsilon_0+1}}}}}}} = \varepsilon_0^{\varepsilon_0^{\varepsilon_0^{\cdot^{\cdot^{\varepsilon_0^\omega}}}}}$ .

We have $\omega^{\omega^{\omega^{\cdot^{\cdot^{\omega^{\omega^{\varepsilon_0+1}}}}}}} = \omega^{\varepsilon_0^{\varepsilon_0^{\cdot^{\cdot^{\varepsilon_0^\omega}}}}}$ (by our hypothesis) $= \omega^{\varepsilon_0^{1+\varepsilon_0^{\cdot^{\cdot^{\varepsilon_0^\omega}}}}}$ (for the same reason than $1 + \omega = \omega$, see above) $= \omega^{\varepsilon_0\cdot\varepsilon_0^{\varepsilon_0^{\cdot^{\cdot^{\varepsilon_0^\omega}}}}} =$ $(\omega^{\varepsilon_0})^{\varepsilon_0^{\varepsilon_0^{\cdot^{\cdot^{\varepsilon_0^\omega}}}}} = \varepsilon_0^{\varepsilon_0^{\varepsilon_0^{\cdot^{\cdot^{\varepsilon_0^\omega}}}}}$ . QED.

In a similar way, the limit of $\varepsilon_1, \varepsilon_1^{\varepsilon_1}, \varepsilon_1^{\varepsilon_1^{\varepsilon_1}}, \ldots$ is called $\varepsilon_2$ and is also the limit of $\varepsilon_1 + 1, \omega^{\varepsilon_1+1}, \omega^{\omega^{\varepsilon_1+1}}, \ldots$.

We can define the same way $\varepsilon_n$ for any natural number n. Then $\varepsilon_\omega$ is defined as the limit of $\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$, and $\varepsilon_{\omega+1}$ as the limit of $\varepsilon_\omega, \varepsilon_\omega^{\varepsilon_\omega}, \varepsilon_\omega^{\varepsilon_\omega^{\varepsilon_\omega}}, \ldots$ or $\varepsilon_\omega + 1, \omega^{\varepsilon^\omega+1}, \omega^{\omega^{\varepsilon_\omega+1}}, \ldots$.

After comes $\varepsilon_{\varepsilon_0}$, and the limit of $\varepsilon_0, \varepsilon_{\varepsilon_0}, \varepsilon_{\varepsilon_{\varepsilon_0}}, \ldots$ which is called $\zeta_0$. This is the least fixed point of $\alpha \mapsto \varepsilon_\alpha$. The next one is $\zeta_1$ which is the limit of $\zeta_0 + 1, \varepsilon_{\zeta_0+1}, \varepsilon_{\varepsilon_{\zeta_0+1}}, \ldots$. Then we get $\zeta_2, \zeta_3, \ldots, \zeta_\omega, \zeta_{\omega+1}, \ldots, \zeta_{\varepsilon_0}, \ldots, \zeta_{\zeta_0}, \ldots, \zeta_{\zeta_{\zeta_0}}, \ldots$. The limit of $0, \zeta_0, \zeta_{\zeta_0}, \zeta_{\zeta_{\zeta_0}}, \ldots$ is called $\eta_0$.

We could go on using successively different greek letters, or define a function $\varphi$ with two variables by :

- $\varphi(0, \alpha) = \omega^\alpha$
- $\varphi(1, \alpha) = \varepsilon_\alpha$
- $\varphi(2, \alpha) = \zeta_\alpha$
- $\varphi(3, \alpha) = \eta_\alpha$
- $\varphi(\alpha + 1, \beta)$ is the $(1 + \beta)$-th fixed point of $\xi \mapsto \varphi(\alpha, \xi)$ .

Then we can enumerate the fixed points of the function $\alpha \mapsto \varphi(\alpha, 0)$ and define $\Gamma_\alpha$ as the $(1 + \alpha)$-th fixed point of this function, or add another variable to the $\varphi$ function and define $\varphi(1, 0, \alpha)$ as the $(1 + \alpha)$-th fixed point of this function. So we have $\Gamma_\alpha = \varphi(1, 0, \alpha)$. More generally, we can define $\varphi(\alpha_n, \alpha_{n-1}, \ldots, \alpha_1, \alpha_0)$.

See for example https://en.wikipedia.org/wiki/Veblen_function :

"Let $z$ be an empty string or a string consisting of one or more comma-separated zeros $0, 0, \ldots, 0$ and $s$ be an empty string or a string consisting of one or more comma-separated ordinals $\alpha_1, \alpha_2, \ldots, \alpha_n$ with $\alpha_1 > 0$. The binary function $\varphi(\beta, \gamma)$ can be written as $\varphi(s, \beta, z, \gamma)$ where both $s$ and $z$ are empty strings.

The finitary Veblen functions are defined as follows:

- $\varphi(\gamma) = \omega^\gamma$
- $\varphi(z, s, \gamma) = \varphi(s, \gamma)$
- if $\beta > 0$, then $\varphi(s, \beta, z, \gamma)$ denotes the $(1 + \gamma)$-th common fixed point of the functions $\xi \mapsto \varphi(s, \delta, \xi, z)$ for each $\delta < \beta$

(...)
The limit of the $\varphi(1, 0, \ldots, 0)$ where the number of zeroes ranges over $\omega$, is sometimes known as the "small" Veblen ordinal.
Every non-zero ordinal $\alpha$ less than the small Veblen ordinal (SVO) can be uniquely written in normal form for the finitary Veblen function:
$\alpha = \varphi(s_1) + \varphi(s_2) + \cdots + \varphi(s_k)$
where

- $k$ is a positive integer
- $\varphi(s_1) \geq \varphi(s_2) \geq \cdots \geq \varphi(s_k)$
- $s_m$ is a string consisting of one or more comma-separated ordinals $\alpha_{m,1}, \alpha_{m,2}, \ldots, \alpha_{m,n_m}$ where $\alpha_{m,1} > 0$ and each $\alpha_{m,i} < \varphi(s_m)$

For limit ordinals $\alpha < SVO$, written in normal form for the finitary Veblen function:

- $(\varphi(s_1) + \varphi(s_2) + \cdots + \varphi(s_k))[n] = \varphi(s_1) + \varphi(s_2) + \cdots + \varphi(s_k)[n]$,
- $\varphi(\gamma)[n] =$
    - n if $\gamma = 1$

- $\varphi(\gamma - 1) \cdot n$ if $\gamma$ is a successor ordinal
- $\varphi(\gamma[n])$ if $\gamma$ is a limit ordinal

- $\varphi(s, \beta, z, \gamma)[0] = 0$ and $\varphi(s, \beta, z, \gamma)[n+1] = \varphi(s, \beta - 1, \varphi(s, \beta, z, \gamma)[n], z)$ if $\gamma = 0$ and $\beta$ is a successor ordinal,
- $\varphi(s, \beta, z, \gamma)[0] = \varphi(s, \beta, z, \gamma - 1) + 1$ and $\varphi(s, \beta, z, \gamma)[n+1] = \varphi(s, \beta - 1, \varphi(s, \beta, z, \gamma)[n], z)$ if $\gamma$ and $\beta$ are successor ordinals,
- $\varphi(s, \beta, z, \gamma)[n] = \varphi(s, \beta, z, \gamma[n])$ if $\gamma$ is a limit ordinal,
- $\varphi(s, \beta, z, \gamma)[n] = \varphi(s, \beta[n], z, \gamma)$ if $\gamma = 0$ and $\beta$ is a limit ordinal,
- $\varphi(s, \beta, z, \gamma)[n] = \varphi(s, \beta[n], \varphi(s, \beta, z, \gamma - 1) + 1, z)$ if $\gamma$ is a successor ordinal and $\beta$ is a limit ordinal. "

The Veblen function can be generalized to transfinitely many variables with a finite number different from 0. Instead of writing the list of all the variable of the Veblen function, we can write only the non zero variables with position as indice, for example $\varphi(\alpha, 0, \beta, \gamma) = \varphi(\alpha_3, \beta_1, \gamma_0)$. We can then generalize the Veblen function by allowing any ordinal as indices, writing for example $SVO = \varphi(1_\omega)$. The limit of the ordinals that can be written with this notation is called the large Veblen ordinal (LVO).

According to Wikipedia, "The definition can be given as follows: let $\alpha$ be a transfinite sequence of ordinals (i.e., an ordinal function with finite support) which ends in zero (i.e., such that $\alpha_0=0$), and let $\alpha[0 \mapsto \gamma]$ denote the same function where the final 0 has been replaced by $\gamma$. Then $\gamma \mapsto \varphi(\alpha[0 \mapsto \gamma])$ is defined as the function enumerating the common fixed points of all functions $\xi \mapsto \varphi(\beta)$ where $\beta$ ranges over all sequences which are obtained by decreasing the smallest-indexed nonzero value of $\alpha$ and replacing some smaller-indexed value with the indeterminate $\xi$ (i.e., $\beta = \alpha[\iota_0 \mapsto \zeta, \iota \mapsto \xi]$ meaning that for the smallest index $\iota_0$ such that $\alpha_{\iota_0}$ is nonzero the latter has been replaced by some value $\zeta < \alpha_{\iota_0}$ and that for some smaller index $\iota < \iota_0$, the value $\alpha_\iota = 0$ has been replaced with $\xi$ )."

Schütte brackets or Klammersymbols are another way to write Veblen fuctions with transfinitely many variables. A Schütte bracket consists in a matrix with two lines, with the positions of the variables in the second line in increasing order, and the corresponding values in the first line. This matrix is preceded by the function $\xi \mapsto \varphi(\xi)$. If we take $\xi \mapsto \omega^\xi$, we get the equivalent of the Veblen function. With this notation, the previous example is written :

$$(\xi \mapsto \omega^\xi) \begin{pmatrix} \gamma & \beta & \alpha \\ 0 & 1 & 3 \end{pmatrix}$$

When the function at the left of the matrix is $\xi \mapsto \omega^\xi$, it is sometimes omitted. Example :

$$\begin{pmatrix} \gamma & \beta & \alpha \\ 0 & 1 & 3 \end{pmatrix}$$

The corresponding fundamental sequences can be found in https://sites.google.com/site/travelingtotheinfinity/fundamental-sequences-for-extended-veblen-function .

Here is an Agda implementation of this notation :

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   A definition of the large Veblen ordinal in Agda
   by Jacques Bailhache, March 2016

   See https://en.wikipedia.org/wiki/Veblen_function

   (1) phi(a)=w**a for a single variable,

   (2) phi(0,an-1,...,a0)=phi(an-1,...,a0), and

   (3) for a>0, c->phi(an,...,ai+1,a,0,...,0,c) is the function enumerating the common fixed points of the
       functions x->phi(an,...,ai+1,b,x,0,...,0) for all b<a.

   (4) Let a be a transfinite sequence of ordinals (i.e., an ordinal function with finite support) which ends in
       zero  (i.e., such that a0=0), and let a[0->c] denote the same function where the final 0 has been replaced
       by c.
       Then c->phi(a[0->c]) is defined as the function enumerating the common fixed points of all functions
       x->phi(b) where b ranges over all sequences which are obtained by decreasing the smallest-indexed nonzero
       value of a  and replacing some smaller-indexed value with the indeterminate x (i.e., b=a[i0->z,i->x]
       meaning that for the smallest index i0 such that ai0 is nonzero the latter has been replaced by some value
       z<ai0 and that for some smaller index i<i0, the value ai=0 has been replaced with x).

-}


module LargeVeblen where
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```
data Nat : Set where
 O : Nat
 1+ : Nat -> Nat

data Ord : Set where
 Zero : Ord
 Suc : Ord -> Ord
 Lim : (Nat -> Ord) -> Ord

-- rpt n f x = f^n(x)
rpt : {t : Set} -> Nat -> (t -> t) -> t -> t
rpt O f x = x
rpt (1+ n) f x = rpt n f (f x)

-- smallest fixed point of f greater than x, limit of x, f x, f (f x), ...
fix : (Ord -> Ord) -> Ord -> Ord
fix f x = Lim (\n -> rpt n f x)

w = fix Suc Zero -- not a fixed point in this case !

-- cantor a b = b + w^a
cantor : Ord -> Ord -> Ord
cantor Zero a = Suc a
cantor (Suc b) a = fix (cantor b) a
cantor (Lim f) a = Lim (\n -> cantor (f n) a)

-- phi0 a = w^a
phi0 : Ord -> Ord
phi0 a = cantor a Zero

-- Another possibility is to use phi'0 instead of phi0 in the definition of phi,
-- this gives a phi function which grows slower
phi'0 : Ord -> Ord
phi'0 Zero = Suc Zero
phi'0 (Suc a) = Suc (phi'0 a)
phi'0 (Lim f) = Lim (\n -> phi'0 (f n))

-- Associative list of ordinals
infixr 40 _=>_&_
data OrdAList : Set where
 Zeros : OrdAList
 _=>_&_ : Ord -> Ord -> OrdAList -> OrdAList

-- Usage : phi al, where al is the associative list of couples index => value ordered by increasing values,
-- absent indexes corresponding to Zero values

phi : OrdAList -> Ord
phi                 Zeros  = phi0 Zero -- (1) phi(0) = w**0 = 1
phi (Zero => a & Zeros) = phi0 a     -- (1) phi(a) = w**a
phi (           k => Zero & al) = phi al -- eliminate unnecessary Zero value
phi (Zero => a & k => Zero & al) = phi (Zero => a & al) -- idem
phi (Zero => a & Zero => b & al) = phi (Zero => a & al) -- should not appear but necessary for completeness
phi (Zero => Lim f & al) = Lim (\n -> phi (Zero => f n & al)) -- canonical treatment of limit
phi (           Suc k => Suc b & al) = fix (\x -> phi (k => x & Suc k => b & al)) Zero
  -- (3) least fixed point
phi (Zero => Suc a & Suc k => Suc b & al) = fix (\x -> phi (k => x & Suc k => b & al)) (Suc (phi (Zero => a &
Suc k => Suc b & al))) -- (3) following fixed points
 phi (           Suc k => Lim f & al) = Lim (\n -> phi (Suc k => f n & al)) -- idem
 phi (Zero => Suc a & Suc k => Lim f & al) = Lim (\n -> phi (k => Suc (phi (Zero => a & Suc k => Lim f & al)) &
Suc k => f n & al)) -- idem
 phi (           Lim f => Suc b & al) = Lim (\n -> phi (f n => (Suc Zero) & Lim f => b & al))
 phi (Zero => Suc a & Lim f => Suc b & al) = Lim (\n -> phi (f n => phi (Zero => a & Lim f => Suc b & al) & Lim
f => b & al))
 phi (           Lim f => Lim g & al) = Lim (\n -> phi (Lim f => g n & al))
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```
phi (Zero => Suc a & Lim f => Lim g & al) = Lim (\n -> phi (f n => phi (Zero => a & Lim f => Lim g & al) & Lim
f => g n & al))

 SmallVeblen = phi (w => Suc Zero & Zeros)

 LargeVeblen = fix (\x -> phi (x => Suc Zero & Zeros)) (Suc Zero)

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Normally it should terminate because the parameter of phi lexicographically decreases, but Agda is not clever enoug
so it must be called with no termination check option :

$ agda -I --no-termination-check LargeVeblen.agda
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         ____         | |
        / __ \        | |
       | |__| |___  __| | ___
       |  __ / _ \/ _ |/ __\      Agda Interactive
       | | |/ /_\ \/_| / /_| \
       |_|  |\___  /_____/     Type :? for help.
              __/ /
              \__/
```

The interactive mode is no longer supported. Don't complain if it doesn't work.
Checking LargeVeblen (/perso/ord/LargeVeblen.agda).
Finished LargeVeblen.
Main> phi Zeros
Suc Zero

# 3  Simmons notation

Harold Simmons defined a notation ( see http://www.cs.man.ac.uk/ hsimmons/ORDINAL-NOTATIONS/ordinal-notations.html )
based on fixed points enumeration which "contains" Veblen functions and permits to go further.
He uses the lambda calculus formalism, in which f x represents the application of function f to x, and f x y = (f x) y the application of
function f to x which gives another function which is applied to y giving the final result. He uses tho notation $x \mapsto y$ to represent the
function which, when applied to x, gives y (instead of the traditional lambda calculus notation $\lambda x.y$ ).

$Fix\ f\zeta = f^\omega(\zeta + 1) = $ limit of $\zeta + 1, f(\zeta + 1), f(f(\zeta + 1)), \dots$ is the least fixed point of the function f which is strictly greater than $\zeta$,
which means the least ordinal $\nu$ satisfying $f\ \nu = \nu$ and $\nu > \zeta$.
$Next = Fix(\alpha \mapsto \omega^\alpha)$ ; $Next\ \zeta$ is the next $\varepsilon_\alpha$ after $\zeta$.
$[0]h = Fix(\alpha \mapsto h^\alpha 0)$
$[1]hg = Fix(\alpha \mapsto h^\alpha g0)$
$[2]hgf = Fix(\alpha \mapsto h^\alpha gf0)$
... and so on ...

## 3.1  Correspondence with Veblen functions

$\varepsilon_0$ is the next $\varepsilon_\alpha$ after 0 (or after $\omega$, or after any ordinal less than $\varepsilon_0$, so we have $\varepsilon_0 = Next\ 0 = Next\ \omega$.
$\varepsilon_1$ is the next $\varepsilon_\alpha$ after $\varepsilon_0$, so we have $\varepsilon_1 = Next\ \varepsilon_0 = Next\ (Next\ 0) = Next^2 0 = Next\ (Next\ \omega) = Next^2 \omega$.
$\varepsilon_2$ is the next $\varepsilon_\alpha$ after $\varepsilon_1$, so we have $\varepsilon_2 = Next\ \varepsilon_1 = Next\ (Next\ (Next\ 0)) = Next^3 0 = Next\ (Next\ (Next\ \omega)) = Next^3 \omega$.
...
$\varepsilon_\omega$ is the limit of $\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots$. It is the limit of $Next^1 0, Next^2 0, Next^3 0, \dots$ which is $Next^\omega 0$.
More generally, we have $\varepsilon_\alpha = \varphi(1, \alpha) = Next^{1+\alpha} 0 = Next^{1+\alpha} \omega$.
$\zeta_0 = \varphi(2, 0)$ is the least fixed point of $\alpha \mapsto \epsilon_\alpha$ (greater than 0), so $\zeta_0 = Fix(\alpha \mapsto \epsilon_\alpha)0 = Fix(\alpha \mapsto Next^{1+\alpha} 0)0 = Fix(\alpha \mapsto Next^\alpha 0)0$
(because the "1+" is "absorbed" after a few iterations) $= [0]Next\ 0$. Since $\zeta_0$ is also greater than $\omega$, it is also $[0]Next\ \omega$ according to
a similar computation.
$\zeta_1 = \varphi(2, 1)$ is the next fixed point of $\alpha \mapsto \epsilon_\alpha$, the least one which is strictly greater than $\zeta_0$, so $\zeta_1 = Fix(\alpha \mapsto \epsilon_\alpha)\zeta_0 = Fix(\alpha \mapsto Next^\alpha 0)\zeta_0 = [0]Next\ \zeta_0 = [0]Next([0]Next\ 0) = ([0]Next)^2 0$.
More generally, $\zeta_\alpha = ([0]Next)^{1+\alpha} 0$.
Similar computations give $\eta_0 = \varphi(3, 0) = [0]^2 Next\ 0$ and $\eta_\alpha = ([0]^2 Next)^{1+\alpha} 0$.
More generally, $\varphi(1 + \beta, \alpha) = ([0]^\beta Next)^{1+\alpha} 0$ or $([0]^\beta Next)^{1+\alpha} \omega$.
$\Gamma_0 = \varphi(1, 0, 0)$ is the least fixed point (greater than 0) of the function $\alpha \mapsto \varphi(\alpha, 0)$ or $\alpha \mapsto \varphi(1+\alpha, 0)$ (for the same reason of "absorbsion"
of "1+" than previously), so $\Gamma_0 = Fix(\alpha \mapsto \varphi(1 + \alpha, 0)0 = Fix(\alpha \mapsto ([0]^\alpha Next)^{(1 + 0)}0)0 = Fix(\alpha \mapsto [0]^\alpha Next\ 0)0 = [1][0]Next\ 0$.

$\Gamma_1 = \varphi(1,0,1)$ is the next fixed point : $\Gamma_1 = Fix(\alpha \mapsto [0]^\alpha Next\ 0)\Gamma_0 = [1][0]Next\ \Gamma_0 = [1][0]Next\ ([1][0]Next\ 0) = ([1][0]Next)^2 0.$
More generally, we have $\varphi(1,0,\alpha) = ([1][0]Next)^{1+\alpha}0.$
With similar computations it can be shown that $\varphi(\gamma,\beta,\alpha) = ([0]^\beta(([1][0])^\gamma Next))^{1+\alpha}0 = ([0]^\beta(([1][0])^\gamma Next))^{1+\alpha}\omega, \varphi(\delta,\gamma,\beta,\alpha) = ([0]^\beta(([1][0])^\gamma(([1]^2[0])^\delta Next)))^{1+\alpha}0$ and so on, the limit being the small Veblen ordinal which is $[1]^\omega[0]Next\ 0$ or $[1]^\omega[0]Next\ \omega.$
Allowing variables at any finite or transfinite positions gives ordinals smaller than the large Veblen ordinal which is $[2][1][0]Next\ 0$ or $[2][1][0]Next\ \omega.$
The limit of Next 0, [0] Next 0, [1] [0] Next 0, [2] [1] [0] Next 0, [3] [2] [1] [0] Next 0, ... is called the Bachmann-Howard ordinal (BHO).


```
epsilon_a = phi(1,a) = Next^(1+a) 0 = Next^(1+a) w
zeta_0 = phi(2,0) = least fixed point of (x -> epsilon_x) = Fix (x -> epsilon_x) 0
 = Fix (x -> Next^(1+x) 0] 0 = Fix (x -> Next^x 0) 0 = [0] Next 0
Next fixed point : zeta_1 = phi(2,1) = Fix (x -> Next^x 0) ([0] Next 0) = [0] Next ([0] Next 0) = ([0] Next)^2 0
zeta_a = phi(2,a) = ([0] Next)^(1+a) 0
eta_0 = phi(3,0) = Fix (x -> zeta_x) 0 = Fix (x -> ([0] Next)^(1+x) 0) 0 = Fix (x -> ([0] Next)^x 0) 0
 = [0] ([0] Next) 0 = [0]^2 Next 0
eta_a = phi(3,a) = ([0]^2 Next)^(1+a) 0
phi(1+b,a) = ([0]^b Next)^(1+a) 0
Gamma_0 = phi(1,0,0) = least fixed point of x -> phi(x,0) = Fix (x -> phi(x,0)) 0 = Fix (x -> [0]^x Next 0) 0
 = [1] [0] Next 0
phi(c,b,a) = ([0]^b (([1] [0])^c Next))^(1+a) 0
phi(d,c,b,a) = ([0]^b (([1] [0])^c (([1]^2 [0])^d Next)))^(1+a) 0
Small Veblen ordinal = [1]^w [0] Next 0
Large Veblen ordinal = [2] [1] [0] Next 0 = [2] [1] [0] Next w


Fix f z = f^w (z+1) = H f (suc z)
Next = Fix [w^*]
epsilon_a = phi(1,a) = Next^(1+a) 0
[0] h = Fix [h^* 0]
zeta0 = phi(2,0) = least fixed point of [epsilon_*] = Fix [epsilon_*] 0 = Fix [Next^(1+*) 0] 0 = Fix [Next^* 0] 0 =
Fix [Next^* 0] 0 = lim 1, Next 0 = epsilon0, Next^epsilon0 0 =Next^(1+epsilon0) 0 = epsilon_epsilon0, ... = epsilo
Fix [Next^(1+*) 0] 0 = lim 1, Next^2 0 = Next (Next 0) = epsilon1, ... epsilon_..._epsilon1 = zeta0
zeta1 = phi(2,1) = second fixed point of [epsilon_*] = Fix [epsilon_*] (zeta0 + 1) = Fix [Next^(1+*) 0] ([0] Next (
 = [0] Next ([0] Next 0)
zeta_a = phi(2,a) = ([0] Next)^(1+a) 0
eta0 = phi(3,0) = least fixed point of [zeta_*] = Fix [zeta_*] 0 = Fix [([0] Next)^(1+*) 0] 0 = Fix [([0] Next)^* (
eta1 = phi(3,1) = next fixed point = Fix [zeta_*] ([0]^2 Next 0) = Fix [([0] Next)^(1+*) 0] ([0]^2 Next 0) = Fix [
 = [0] ([0] Next) ([0]^2 Next 0) = [0]^2 Next ([0]^2 Next 0) = ([0]^2 Next)^2 0
eta_a = phi(3,a) = ([0]^2 Next)^(1+a) 0

phi(1,a) = Next^(1+a) 0
phi(2,a) = ([0] Next)^(1+a) 0
phi(3,a) = ([0]^2 Next)^(1+a) 0
phi(1+b,a) = ([0]^b Next)^(1+a) 0

Gamma0 = phi(1,0,0) = least fixed point of [phi(*,0)] = least fixed point of [phi(1+*,0)]
 = Fix [phi(1+*,0)] 0
 = Fix [[0]^* Next 0] 0
 = [1] [0] Next 0
with [1] h g = Fix [h^* g 0]
phi(1,0,1) = next fixed point = Fix [[0]^* Next 0] ([1] [0] Next 0) = [1] [0] Next ([1] [0] Next 0) = ([1] [0] Next
phi(1,0,a) = ([1] [0] Next)^(1+a) 0
phi(1,1,0) = least fixed point of [phi(1,0,*)] = Fix [phi(1,0,*)] 0 = Fix [([1] [0] Next)^(1+*) 0] 0 = Fix [([1] [0
phi(1,1,1) = next fixed point = Fix [([1] [0] Next)^* 0] ([0] ([1] [0] Next) 0) = [0] ([1] [0] Next) ([0] ([1] [0]
phi(1,1,a) = ([0] ([1] [0] Next))^(1+a) 0
phi(1,2,0) = least fixed point of [phi(1,1,*)] = Fix [([0] ([1] [0] Next))^(1+*) 0] 0 = Fix [([0] ([1] [0] Next))^*
 = [0]^2 ([1] [0] Next) 0
phi(1,2,1) = next fixed point ...
phi(1,2,a) = ([0]^2 ([1] [0] Next))^(1+a) 0

phi(1,0,a) = ([1] [0] Next)^(1+a) 0
phi(1,1,a) = ([0] ([1] [0] Next))^(1+a) 0
phi(1,2,a) = ([0]^2 ([1] [0] Next))^(1+a) 0
```

```
phi(1,b,a) = ([0]^b ([1] [0] Next))^(1+a) 0

phi(2,0,0) = least fixed point of [phi(1,*,0)] = Fix [phi(1,*,0)] 0 = Fix [[0]^* ([1] [0] Next) 0] 0 = [1] [0] ([1]
phi(2,0,a) = (([1] [0])^2 Next)^(1+a) 0
phi(2,1,0) = least fixed point of [phi(2,1,*)] = Fix [phi(2,1,*)] 0 = Fix [(([1] [0])^2 Next)^* 0] 0 = [0] (([1] [0]
phi(2,1,a) = ([0] (([1] [0])^2 Next))^(1+a) 0
phi(2,b,a) = ([0]^b (([1] [0])^2 Next))^(1+a) 0
phi(c,b,a) = ([0]^b (([1] [0])^c Next))^(1+a) 0

phi(1,0,0,0) = least fixed point of [phi(*,0,0)] = Fix [phi(*,0,0)] 0 = Fix [([1] [0])^* Next 0] 0 = [1] ([1] [0])
phi(1,0,0,a) = ([1]^2 [0] Next)^(1+a) 0
phi(1,0,1,0) = Fix [([1]^2 [0] Next)^* 0] 0 = [0] ([1]^2 [0]) 0
phi (1,0,1,a) = ([0] ([1]^2 [0] Next))^(1+a) 0
phi(1,0,b,a) = ([0]^b ([1]^2 [0] Next))^(1+a) 0
phi(1,1,0,0) = Fix [phi(1,0,*,0)] 0 = Fix [[0]^* ([1]^2 [0] Next) 0] 0 = [1] [0] ([1]^2 [0] Next) 0
phi(1,1,0,a) = ([1] [0] ([1]^2 [0] Next))^(1+a) 0
phi(1,1,1,0) = Fix [phi(1,1,0,*)] 0 = Fix [([1] [0] ([1]^2 [0] Next))^* 0] 0 = [0] ([1] [0] ([1]^2 [0] Next)) 0
phi(1,1,1,a) = ([0] ([1] [0] ([1]^2 [0] next)))^(1+a) 0
phi(1,1,b,a) = ([0]^b ([1] [0] ([1]^2 [0] Next)))^(1+a) 0
phi(1,2,0,0) = Fix [phi(1,1,*,0)] 0 = Fix [[0]^* ([1] [0] ([1]^2 [0] next)) 0] 0 = [1] [0] ([1] [0] ([1]^2 [0] Next

phi(1,0,0,0) = [1]^2 [0] Next 0
phi(1,1,0,0) = [1] [0] ([1]^2 [0] Next) 0
phi(1,2,0,0) = ([1] [0])^2 ([1]^2 [0] Next) 0
phi(1,c,0,0) = ([1] [0])^c ([1]^2 [0] Next) 0
phi(1,c,b,a) = ([0]^b (([1] [0])^c ([1]^2 [0] Next)))^(1+a) 0
phi(2,0,0,0) = Fix [phi(1,*,0,0)] 0 = Fix [([1] [0])^* ([1]^2 [0] Next) 0] 0 = [1] ([1] [0]) ([1]^2 [0] Next) 0 =
phi(d,0,0,0) = ([1]^2 [0])^d Next 0
phi(d,c,b,a) = ([0]^b (([1] [0])^c (([1]^2 [0])^d Next)))^(1+a) 0 = ((([1]^0 [0])^b (([1]^1 [0])^c (([1]^2 [0])^d Ne

phi(1,0) = Next 0
phi(1,0,0) = [1] [0] Next 0
phi(1,0,0,0) = [1]^2 [0] Next 0
...
SVO = [1]^w [0] Next 0
LVO = [2] [1] [0] Next 0 = [2...0] Next 0
BHO = [w...0] Next 0
[1] [w..0] Next 0 = [w+1...0] Next 0
Fix [[*...0] Next 0] 0
```

## 3.2 Implementation

Here is an implementation of the Simmons hierarchy in Haskell :

```
module Simmons where

 -- Natural numbers
 data Nat
  = ZeroN
  | SucN Nat

 -- Ordinals
 data Ord
  = Zero
  | Suc Ord
  | Lim (Nat -> Ord)

 -- Ordinal corresponding to a given natural
 ordOfNat ZeroN = Zero
 ordOfNat (SucN n) = Suc (ordOfNat n)

 -- omega
```

```
w = Lim ordOfNat

lim0 s = Lim s
lim1 f x = lim0 (\n -> f n x)
lim2 f x = lim1 (\n -> f n x)

-- this does not work :
-- lim ZeroN s = Lim s
-- lim (SucN p) f = \x -> lim p (\n -> f n x)

-- f^a(x)
fpower0 f Zero x = x
fpower0 f (Suc a) x = f (fpower0 f a x)
fpower0 f (Lim s) x = Lim (\n -> fpower0 f (s n) x)

fpower l f Zero x = x
fpower l f (Suc a) x = f (fpower l f a x)
fpower l f (Lim s) x = l (\n -> fpower l f (s n) x)

-- fix f z = least fixed point of f which is > z
fix f z = fpower lim0 f w (Suc z) -- Lim (\n -> fpower0 f (ordOfNat n) (Suc z))

-- cantor b a = a + w^b
cantor Zero a = Suc a
cantor (Suc b) a = fix (cantor b) a
cantor (Lim s) a = Lim (\n -> cantor (s n) a)

-- expw a = w^a
expw a = cantor a Zero

-- next a = least epsilon_b > a
next = fix expw

-- [0]
simmons0 h = fix (\a -> fpower lim0 h a Zero)

-- [1]
simmons1 h1 h0 = fix (\a -> fpower lim1 h1 a h0 Zero)

-- [2]
simmons2 h2 h1 h0 = fix (\a -> fpower lim2 h2 a h1 h0 Zero)

-- Large Veblen ordinal
  lvo = simmons2 simmons1 simmons0 next w



$ hugs

--   -- -- -- ----  ---      ----------------------------------------
||   || || || || || ||__       Hugs 98: Based on the Haskell 98 standard
||___|| ||__|| ||__|| __||      Copyright (c) 1994-2005
||---||        ___||            World Wide Web: http://haskell.org/hugs
||   ||                         Bugs: http://hackage.haskell.org/trac/hugs
||   || Version: September 2006 ----------------------------------------

Haskell 98 mode: Restart with command line option -98 to enable extensions

Type :? for help
Hugs> :load simmons
Simmons> lvo
ERROR - Cannot find "show" function for:
*** Expression : lvo
*** Of type    : Ord
```

# 4   Summary

Any ordinal can be defined as the least ordinal strictly greater than all ordinals of a set : the empty set for 0, $\{\alpha\}$ for the successor of $\alpha$, $\{\alpha_0, \alpha_1, \alpha_2, ...\}$ for an ordinal with fundamental sequence $\alpha_0, \alpha_1, \alpha_2, ...$

# 5   Algebraic notation

We define the following operations on ordinals :

- addition : $\alpha + 0 = \alpha; \alpha + suc(\beta) = suc(\alpha + \beta); \alpha + lim(f) = lim(n \mapsto \alpha + f(n))$
- multiplication : $\alpha \times 0 = 0; \alpha \times suc(\beta) = (\alpha \times \beta) + \alpha; \alpha \times lim(f) = lim(n \mapsto \alpha \times f(n))$
- exponentiation : $\alpha^0 = 1; \alpha^{suc(\beta)} = \alpha^\beta \times \alpha; \alpha^{lim(f)} = lim(n \mapsto \alpha^{f(n)})$

# 6   Veblen functions

These functions use fixed points enumaration : $\varphi(\ldots, \beta, 0, \ldots, 0, \gamma)$ represents the $(1 + \gamma)^{th}$ common fixed point of the functions $\xi \mapsto \varphi(\ldots, \delta, \xi, 0, \ldots, 0)$ for all $\delta < \beta$.

# 7   Simmons notation

$Fix f z = f^w(z + 1) = $ least fixed point of f strictly greater than z.

$Next = Fix(\alpha \mapsto \omega^\alpha)$

$[0]h = Fix(\alpha \mapsto h^\alpha \omega)$ ; $[1]hg = Fix(\alpha \mapsto h^\alpha g\omega)$ ; $[2]hgf = Fix(\alpha \mapsto h^\alpha gf\omega)$ ; etc...

Correspondence with Veblen's $\phi$ : $\phi(1 + \alpha, \beta) = ([0]^\alpha Next)^{1+\beta}\omega; \phi(\alpha, \beta, \gamma) = ([0]^\beta(([1][0])^\alpha Next))^{1+\gamma}\omega$

# 8   RHS0 notation

We start from 0, if we don(t see any regularity we take the successor, if we see a regularity, if we have a notation for this regularity, we use it, else we invent it, then we jump to the limit.

$Hfx = lim\ x, fx, f(fx), \ldots; R_1 fgx = lim\ gx, fgx, ffgx, \ldots; R_2 fghx = lim\ hx, fghx, fgfghx, \ldots$

Correspondence with Simmons notation : $\ldots, [3] \to R5, [2] \to R4, [1] \to R3, [0] \to R2, Next \to R1, \omega \to Hsuc\ 0$

# 9   Ordinal collapsing functions

These functions use uncountable ordinals to define countable ordinals.

We define sets of ordinals that can be built from given ordinals and operations, then we take the least ordinal which is not in this set, or the least ordinal which is greater than all contable ordinals of this set.

These functions are extensions of functions on countable ordinals, whose fixed points can be reached by applying them to an uncountable ordinal.

Examples :

- Madore's $\psi$ : $\psi(\alpha) = \varepsilon_\alpha$ if $\alpha < \zeta_0; \psi(\Omega) = \zeta_0$ which is the least fixed point of $\alpha \mapsto \varepsilon_\alpha$.
- Feferman's $\theta$ : $\theta(\alpha, \beta) = \varphi(\alpha, \beta)$ if $\alpha < \Gamma_0$ and $\beta < \Gamma_0; \theta(\Omega, 0) = \Gamma_0$ which is the least fixed point of $\alpha \mapsto \varphi(\alpha, 0)$.
- Taranovsky's C : $C(\alpha, \beta) = \beta + \omega^\alpha$ if $\alpha$ is countable; $C(\Omega_1, 0) = \varepsilon_0$ which is the least fixed point of $\alpha \mapsto \omega^\alpha$.

| Nom | Symbole | Algebraic | Veblen | Simmons | RHS0 | Madore | Taranovsky |
|---|---|---|---|---|---|---|---|
| Zero | $0$ | $0$ | | | $0$ | | $0$ |
| One | $1$ | $1$ | $\varphi(0,0)$ | | suc 0 | | C(0,0) |
| Two | $2$ | $2$ | | | suc (suc 0) | | C(0,C(0,0)) |
| Omega | $\omega$ | $\omega$ | $\varphi(0,1)$ | $\omega$ | H suc 0 | | C(1,0) |
| | | $\omega+1$ | | | suc (H suc 0) | | C(0,C(1,0)) |
| | | $\omega \times 2$ | | | H suc (H suc 0) | | C(1,C(1,0)) |
| | | $\omega^2$ | $\varphi(0,2)$ | | H (H suc) 0 | | C(C(0,C(0,0)),0) |
| | | $\omega^\omega$ | $\varphi(0,\omega)$ | | H H suc 0 | | C(C(1,0),0) |
| | | $\omega^{\omega^\omega}$ | $\varphi(0,\omega^\omega)$ | | H H H suc 0 | | C(C(C(1,0),0),0) |
| Epsilon zero | $\varepsilon_0$ | $\varepsilon_0$ | $\varphi(1,0)$ | $Next\ \omega$ | $R_1 H suc\ 0$ | $\psi(0)$ | $C(\Omega_1,0)$ |
| | | $\varepsilon_1$ | $\varphi(1,1)$ | $Next^2\omega$ | $R_1(R_1 H)suc\ 0$ | $\psi(1)$ | $C(\Omega_1,C(\Omega_1,0))$ |
| | | $\varepsilon_\omega$ | $\varphi(1,\omega)$ | $Next^\omega\omega$ | $H R_1 H suc\ 0$ | $\psi(\omega)$ | $C(C(0,\Omega_1),0)$ |
| | | $\varepsilon_{\varepsilon_0}$ | $\varphi(1,\varphi(1,0))$ | $Next^{Next\omega}\omega$ | $R_1 H R_1 H suc\ 0$ | $\psi(\psi(0))$ | $C(C(C(\Omega_1,0),\Omega_1),0)$ |
| Zeta zero | $\zeta_0$ | $\zeta_0$ | $\varphi(2,0)$ | $[0]Next\ \omega$ | $R_2 R_1 H suc\ 0$ | $\psi(\Omega)$ | $C(C(\Omega_1,\Omega_1),0)$ |
| Eta zero | $\eta_0$ | $\eta_0$ | $\varphi(3,0)$ | $[0]^2 Next\ \omega$ | $R_2(R_2 R_1)H suc\ 0$ | | $C(C(\Omega,C(\Omega,\Omega)),0)$ |
| | | | $\varphi(\omega,0)$ | $[0]^\omega Next\ \omega$ | $H R_2 R_1 H suc\ 0$ | | $C(C(C(0,\Omega_1),\Omega_1),0)$ |
| Feferman -Schütte | $\Gamma_0$ | $\Gamma_0$ | $\varphi(1,0,0)$ $= \varphi(2 \mapsto 1)$ | $[1][0]Next\ \omega$ | $R_3 R_2 R_1 H suc\ 0$ $= R_{3\ldots1}H suc\ 0$ | $\psi(\Omega^\Omega)$ | $C(C(C(\Omega_1,\Omega_1),$ $\Omega_1),0)$ |
| Ackermann | | | $\varphi(1,0,0,0)$ $= \varphi(3 \mapsto 1)$ | $[1]^2[0]Next\ \omega$ | $R_3(R_3 R_2)R_1 H suc\ 0$ | $\psi(\Omega^{\Omega^2})$ | |
| Small Veblen ordinal | | | $\varphi(\omega \mapsto 1)$ | $[1]^\omega[0]Next\ \omega$ | $H R_3 R_2 R_1 H suc\ 0$ | $\psi(\Omega^{\Omega^\omega})$ | $C(\Omega_1^\omega,0)$ $= C(C(C(C(0,\Omega_1),$ $\Omega_1),\Omega_1),0)$ |
| Large Veblen ordinal | | | least ord. not rep. | $[2][1][0]Next\ \omega$ | $R_4 R_3 R_2 R_1 H suc\ 0$ $= R_{4\ldots1}H suc\ 0$ | $\psi(\Omega^{\Omega^\Omega})$ | $C(\Omega_1^{\Omega_1},0)$ $= C(C(C(C(\Omega_1,\Omega_1),$ $\Omega_1),\Omega_1),0)$ |
| Bachmann- Howard ordinal | | | | least ord. not rep. | $R_{\omega\ldots1}H suc\ 0$ | $\psi(\varepsilon_{\Omega+1})$ | $C(C(\Omega_2,\Omega_1),0)$ |