TRANSFINITE ORDINALS
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1 Defining transfinite ordinal numbers

Natural numbers can be represented by sets. Each number is represented by the set of all numbers smaller than it.

- $0 = \{\}$ (the empty set)
- $1 = \{0\} = \{\{\}\}$
- $2 = \{0, 1\} = \{\{\}, \{\{\}\}\}$
- $3 = \{0, 1, 2\} = \{\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}\}$
- ...

The successor of a natural number can be defined by $suc(n) = n + 1 = n \cup \{n\}$.
We have $n \leq p$ if and only if $n \subseteq p$.
$\mathbb{N}$ is the set of all natural numbers : $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ The natural numbers can be generalized to what is called "transfinite ordinal numbers", or simply "ordinal numbers" or "ordinals", by considering that infinite sets represent ordinal numbers. $\mathbb{N}$ considered as an ordinal number is written ω. ω is the least ordinal which is greater than all the numbers 0, 1, 2, 3, ... We say that ω is a limit ordinal and 0, 1, 2, 3, ... is a fundamental sequence of ω. This is written : $\omega = sup\{0, 1, 2, 3, ...\}$ or $\omega = lim(n \mapsto n)$ because the n-th element (starting with 0) of the sequence is n. An ordinal does not have a unique fundamental sequence, for example 1, 2, 3, 4, ... is also a fundamental sequence of ω, because the least ordinal that is greater than all ordinals of this sequence is also ω (more generally the limit ordinal is the same if any number of the least items of a sequence are removed), and the same stands for the sequence 0, 2, 4, 6, ...
Any ordinal can be defined as the least ordinal strictly greater than all ordinals of a set : the empty set for 0, $\{\alpha\}$ for the successor of α, $\{\alpha_0, \alpha_1, \alpha_2, ...\}$ for an ordinal with fundamental sequence $\alpha_0, \alpha_1, \alpha_2, ...$
The successor can be generalized to transfinite ordinal numbers : $suc(\omega) = \omega + 1 = \omega \cup \{\omega\} = \{0, 1, 2, 3, \ldots, \omega\}; suc(suc(\omega)) = \omega + 2 = \{0, 1, 2, 3, \ldots, \omega, \omega + 1\}$ and so on.
Then we can consider the set $\{0, 1, 2, 3, \ldots, \omega, \omega + 1, omega + 2, omega + 3, \ldots\}$ which is a limit ordinal, and $\omega, \omega + 1, \omega + 2, \omega + 3, \ldots$ is a fundamental sequence of this ordinal. This ordinal is $\omega + \omega = \omega \cdot 2$ or $\omega \times 2$ or $\omega 2$.
Then we can go on building greater and greater ordinals : $\omega \cdot 3, \ldots, \omega \cdot \omega = \omega^2, \omega^3, \ldots, \omega^\omega, \omega^{\omega^\omega}, \ldots$.

The definitions of arithmetical operations can be generalized to ordinals :

- addition : $\alpha + 0 = \alpha; \alpha + suc(\beta) = suc(\alpha + \beta); \alpha + lim(f) = lim(n \mapsto \alpha + f(n))$
- multiplication : $\alpha \cdot 0 = 0; \alpha \cdot suc(\beta) = (\alpha \cdot \beta) + \alpha; \alpha \cdot lim(f) = lim(n \mapsto \alpha \cdot f(n))$
- exponentiation : $\alpha^0 = 1; \alpha^{suc(\beta)} = \alpha^\beta \cdot \alpha; \alpha^{lim(f)} = lim(n \mapsto \alpha^{f(n)})$

Remark that the addition and the multiplication are not commutative, for example $1 + \omega = \omega \neq \omega + 1$, because if we take 0, 1, 2, 3, ... as fundamental sequence of ω, then a fundamental sequence of $1 + \omega$ is 1+0, 1+1, 1+2, 1+3, ... = 1, 2, 3, 4, ... and the least ordinal which is greater than all ordinals of this sequence is ω.

2 Veblen functions

The next step is the limit or least upper bound of $\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots$ which is called ε_0. Note that we have $\omega^{\varepsilon_0} = \varepsilon_0$. We say that ε_0 is a fixed point (the least one) of the function $\alpha \mapsto \omega^\alpha$.
Then we can go on with $\varepsilon_0 + 1, \varepsilon_0 + 2, \ldots, \varepsilon_0 + \varepsilon_0 = \varepsilon_0 \cdot 2, \ldots, \varepsilon_0 \cdot \varepsilon_0 = \varepsilon_0^2, \varepsilon_0^{\varepsilon_0}, ...$
The limit of $\varepsilon_0, \varepsilon_0^{\varepsilon_0}, \varepsilon_0^{\varepsilon_0^{\varepsilon_0}}, \ldots$ is called ε_1. It can be shown that it is also the limit of $\varepsilon_0 + 1, \omega^{\varepsilon_0+1}, \omega^{\omega^{\varepsilon_0+1}}, \ldots$ (see proof below).
These two fundamental sequences are examples of two ways of ascending ordinals :

- Build greater ordinals by combining known ordinals with arithmetic operations.
- When we have found a function that, when applied to a given ordinal, gives a greater one (for example $\alpha \mapsto \omega^\alpha$), enumerate the fixed points of this function. A fixed point of a function f is a value (for example an ordinal) α with $f(\alpha) = \alpha$. Under some conditions (see below), the least fixed point of f is the limit of 0, f(0), f(f(0)), f(f(f(0))), ... If it is called α, the next fixed point is the limit of $\alpha + 1, f(\alpha + 1), f(f(\alpha + 1)), f(f(f(\alpha + 1))), \ldots$. More generally, the least fixed point of f that is greater or equal to ζ is the limit of $\zeta, f(\zeta), f(f(\zeta)), \ldots$. The Veblen functions use this method.

The required conditions are described for example in http://www.cs.man.ac.uk/ hsimmons/ORDINAL-NOTATIONS/Fruitful.pdf page 8 lemma 3.9 :
For each fruitful function f and each ordinal ζ, $f^\omega(\zeta + 1)$ is the least ordinal ν such that $\zeta < \nu = f(\nu)$, or the least fixed point of f that is strictly greater than ζ (or greater than or equal to $\zeta + 1$).
$f^\omega(\zeta + 1)$ is the limit of $\zeta + 1, f(\zeta + 1), f(f(\zeta + 1)), \ldots$.
A fruitful function is a function that is inflationary, monotone, big, and continuous.
A function f is inflationary if $\alpha \leq f(\alpha)$, monotone if $\alpha \leq \beta \Rightarrow f(\alpha) \leq f(\beta)$, big if $\omega^\alpha \leq f(\alpha)$ except possibly for $\alpha = 0$, continuous if f(VA) = Vf[A] where VA is the pointwise supremum of the set A.

We will now prove by induction the equivalence of the two fundamental sequences above.

We will use the notation $\alpha^{\cdot^{\cdot^{\cdot^{\alpha^{\beta}}}}}$ for an an "exponential tower" with α repeated n times.

Note that the ordinals respectively limits of the fondamental sequence whose n-th term is $\varepsilon_0^{\varepsilon_0^{\cdot^{\cdot^{\cdot^{\varepsilon_0^{\omega}}}}}}$ and the one whose n-th term is $\varepsilon_0^{\varepsilon_0^{\cdot^{\cdot^{\cdot^{\varepsilon_0^{\varepsilon_0}}}}}}$ is the same, the least fixed point of the function $\alpha \mapsto {\varepsilon_0}^{\alpha}$, which is greater than ω and also than ε_0.

So we have proved what we want if we prove that, for any n, we have $\omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega^{\omega^{\varepsilon_0+1}}}}}}} = \varepsilon_0^{\varepsilon_0^{\cdot^{\cdot^{\cdot^{\varepsilon_0^{\omega}}}}}}$.
For n = 0, we have $\omega^{\omega^{\varepsilon_0+1}} = \omega^{\omega^{\varepsilon_0}\cdot\omega} = \omega^{\varepsilon_0\cdot\omega} = (\omega^{\varepsilon_0})^{\omega} = {\varepsilon_0}^{\omega}$.

Now suppose we have $\omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega^{\omega^{\varepsilon_0+1}}}}}}} = \varepsilon_0^{\varepsilon_0^{\cdot^{\cdot^{\cdot^{\varepsilon_0^{\omega}}}}}}$.

We must prove the equality for n+1, which can be written $\omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega^{\omega^{\varepsilon_0+1}}}}}}}} = \varepsilon_0^{\varepsilon_0^{\varepsilon_0^{\cdot^{\cdot^{\cdot^{\varepsilon_0^{\omega}}}}}}}$.

We have $\omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega^{\omega^{\varepsilon_0+1}}}}}}}} = \omega^{\varepsilon_0^{\varepsilon_0^{\cdot^{\cdot^{\cdot^{\varepsilon_0^{\omega}}}}}}}$ (by our hypothesis) $= \omega^{\varepsilon_0^{1+\varepsilon_0^{\cdot^{\cdot^{\cdot^{\varepsilon_0^{\omega}}}}}}}$ (for the same reason than $1 + \omega = \omega$, see above) $= \omega^{\varepsilon_0 \cdot \varepsilon_0^{\varepsilon_0^{\cdot^{\cdot^{\cdot^{\varepsilon_0^{\omega}}}}}}} = (\omega^{\varepsilon_0})^{\varepsilon_0^{\varepsilon_0^{\cdot^{\cdot^{\cdot^{\varepsilon_0^{\omega}}}}}}} = \varepsilon_0^{\varepsilon_0^{\varepsilon_0^{\cdot^{\cdot^{\cdot^{\varepsilon_0^{\omega}}}}}}}$. QED.
In a similar way, the limit of $\varepsilon_1, \varepsilon_1^{\varepsilon_1}, \varepsilon_1^{\varepsilon_1^{\varepsilon_1}}, \ldots$ is called ε_2 and is also the limit of $\varepsilon_1 + 1, \omega^{\varepsilon_1+1}, \omega^{\omega^{\varepsilon_1+1}}, \ldots$.
We can define the same way ε_n for any natural number n. Then ε_ω is defined as the limit of $\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$, and $\varepsilon_{\omega+1}$ as the limit of $\varepsilon_\omega, \varepsilon_\omega^{\varepsilon_\omega}, \varepsilon_\omega^{\varepsilon_\omega^{\varepsilon_\omega}}, \ldots$ or $\varepsilon_\omega + 1, \omega^{\varepsilon^\omega+1}, \omega^{\omega^{\varepsilon_\omega+1}}, \ldots$.
After comes $\varepsilon_{\varepsilon_0}$, and the limit of $\varepsilon_0, \varepsilon_{\varepsilon_0}, \varepsilon_{\varepsilon_{\varepsilon_0}}, \ldots$ which is called ζ_0. This is the least fixed point of $\alpha \mapsto \varepsilon_\alpha$. The next one is ζ_1 which is the limit of $\zeta_0 + 1, \varepsilon_{\zeta_0+1}, \varepsilon_{\varepsilon_{\zeta_0+1}}, \ldots$. Then we get $\zeta_2, \zeta_3, \ldots, \zeta_\omega, \zeta_{\omega+1}, \ldots, \zeta_{\varepsilon_0}, \ldots, \zeta_{\zeta_0}, \ldots, \zeta_{\zeta_{\zeta_0}}, \ldots$. The limit of $0, \zeta_0, \zeta_{\zeta_0}, \zeta_{\zeta_{\zeta_0}}, \ldots$ is called η_0.
We could go on using successively different greek letters, or define a function φ with two variables by :
$\varphi(0, \alpha) = \omega^\alpha$
$\varphi(1, \alpha) = \varepsilon_\alpha$
$\varphi(2, \alpha) = \zeta_\alpha$
$\varphi(3, \alpha) = \eta_\alpha$
$\varphi(\alpha + 1, \beta)$ is the $(1 + \beta)$-th fixed point of $\xi \mapsto \varphi(\alpha, \xi)$.

3 Summary

Any ordinal can be defined as the least ordinal strictly greater than all ordinals of a set : the empty set for 0, $\{\alpha\}$ for the successor of α, $\{\alpha_0, \alpha_1, \alpha_2, ...\}$ for an ordinal with fundamental sequence $\alpha_0, \alpha_1, \alpha_2, ...$

4 Algebraic notation

We define the following operations on ordinals :

- addition : $\alpha + 0 = \alpha; \alpha + suc(\beta) = suc(\alpha + \beta); \alpha + lim(f) = lim(n \mapsto \alpha + f(n))$
- multiplication : $\alpha \times 0 = 0; \alpha \times suc(\beta) = (\alpha \times \beta) + \alpha; \alpha \times lim(f) = lim(n \mapsto \alpha \times f(n))$
- exponentiation : $\alpha^0 = 1; \alpha^{suc(\beta)} = \alpha^\beta \times \alpha; \alpha^{lim(f)} = lim(n \mapsto \alpha^{f(n)})$

5 Veblen functions

These functions use fixed points enumaration : $\varphi(\ldots, \beta, 0, \ldots, 0, \gamma)$ represents the $(1 + \gamma)^{th}$ common fixed point of the functions $\xi \mapsto \varphi(\ldots, \delta, \xi, 0, \ldots, 0)$ for all $\delta < \beta$.

6 Simmons notation

$Fixfz = f^w(z + 1)$ = least fixed point of f strictly greater than z.
$Next = Fix(\alpha \mapsto \omega^\alpha)$
$[0]h = Fix(\alpha \mapsto h^\alpha\omega)$; $[1]hg = Fix(\alpha \mapsto h^\alpha g\omega)$; $[2]hgf = Fix(\alpha \mapsto h^\alpha gf\omega)$; etc...
Correspondence with Veblen's ϕ : $\phi(1 + \alpha, \beta) = ([0]^\alpha Next)^{1+\beta}\omega; \phi(\alpha, \beta, \gamma) = ([0]^\beta(([1][0])^\alpha Next))^{1+\gamma}\omega$

7 RHS0 notation

We start from 0, if we don(t see any regularity we take the successor, if we see a regularity, if we have a notation for this regularity, we use it, else we invent it, then we jump to the limit.
$Hfx = lim\ x, fx, f(fx), \ldots; R_1fgx = lim\ gx, fgx, ffgx, \ldots; R_2fghx = lim\ hx, fghx, fgfghx, \ldots$
Correspondence with Simmons notation : $\ldots, [3] \to R5, [2] \to R4, [1] \to R3, [0] \to R2, Next \to R1, \omega \to Hsuc\ 0$

8 Ordinal collapsing functions

These functions use uncountable ordinals to define countable ordinals.
We define sets of ordinals that can be built from given ordinals and operations, then we take the least ordinal which is not in this set, or the least ordinal which is greater than all contable ordinals of this set.
These functions are extensions of functions on countable ordinals, whose fixed points can be reached by applying them to an uncountable ordinal.
Examples :

- Madore's ψ : $\psi(\alpha) = \varepsilon_\alpha$ if $\alpha < \zeta_0; \psi(\Omega) = \zeta_0$ which is the least fixed point of $\alpha \mapsto \varepsilon_\alpha$.
- Feferman's θ : $\theta(\alpha,\beta) = \varphi(\alpha,\beta)$ if $\alpha < \Gamma_0$ and $\beta < \Gamma_0; \theta(\Omega,0) = \Gamma_0$ which is the least fixed point of $\alpha \mapsto \varphi(\alpha,0)$.
- Taranovsky's C : $C(\alpha,\beta) = \beta + \omega^\alpha$ if α is countable; $C(\Omega_1,0) = \varepsilon_0$ which is the least fixed point of $\alpha \mapsto \omega^\alpha$.

Nom	Symbole	Algebraic	Veblen	Simmons	RHS0	Madore	Taranovsky
Zero	0	0			0		0
One	1	1	$\varphi(0,0)$		suc 0		C(0,0)
Two	2	2			suc (suc 0)		C(0,C(0,0))
Omega	ω	ω	$\varphi(0,1)$	ω	H suc 0		C(1,0)
		$\omega+1$			suc (H suc 0)		C(0,C(1,0))
		$\omega\times 2$			H suc (H suc 0)		C(1,C(1,0))
		ω^2	$\varphi(0,2)$		H (H suc) 0		C(C(0,C(0,0)),0)
		ω^ω	$\varphi(0,\omega)$		H H suc 0		C(C(1,0),0)
		ω^{ω^ω}	$\varphi(0,\omega^\omega)$		H H H suc 0		C(C(C(1,0),0),0)
Epsilon zero	ε_0	ε_0	$\varphi(1,0)$	$Next\ \omega$	$R_1 H suc\ 0$	$\psi(0)$	$C(\Omega_1,0)$
		ε_1	$\varphi(1,1)$	$Next^2\omega$	$R_1(R_1 H) suc\ 0$	$\psi(1)$	$C(\Omega_1, C(\Omega_1,0)$
		ε_ω	$\varphi(1,\omega)$	$Next^\omega\omega$	$HR_1 H suc\ 0$	$\psi(\omega)$	$C(C(0,\Omega_1),0)$
		$\varepsilon_{\varepsilon_0}$	$\varphi(1,\varphi(1,0))$	$Next^{Next\omega}\omega$	$R_1 H R_1 H suc\ 0$	$\psi(\psi(0))$	$C(C(C(\Omega_1,0),\Omega_1),0)$
Zeta zero	ζ_0	ζ_0	$\varphi(2,0)$	$[0]Next\ \omega$	$R_2 R_1 H suc\ 0$	$\psi(\Omega)$	$C(C(\Omega_1,\Omega_1),0)$
Eta zero	η_0	η_0	$\varphi(3,0)$	$[0]^2 Next\ \omega$	$R_2(R_2 R_1) H suc\ 0$		$C(C(\Omega, C(\Omega,\Omega)),0)$
			$\varphi(\omega,0)$	$[0]^\omega Next\ \omega$	$HR_2 R_1 H suc\ 0$		$C(C(C(0,\Omega_1),\Omega_1),0)$
Feferman -Schütte	Γ_0	Γ_0	$\varphi(1,0,0)$ $=\varphi(2\mapsto 1)$	$[1][0]Next\ \omega$	$R_3 R_2 R_1 H suc\ 0$ $= R_{3\ldots 1} H suc\ 0$	$\psi(\Omega^\Omega)$	$C(C(C(\Omega_1,\Omega_1),$ $\Omega_1),0)$
Ackermann			$\varphi(1,0,0,0)$ $=\varphi(3\mapsto 1)$	$[1]^2[0]Next\ \omega$	$R_3(R_3 R_2) R_1 H suc\ 0$	$\psi(\Omega^{\Omega^2})$	
Small Veblen ordinal			$\varphi(\omega\mapsto 1)$	$[1]^\omega[0]Next\ \omega$	$HR_3 R_2 R_1 H suc\ 0$	$\psi(\Omega^{\Omega^\omega})$	$C(\Omega_1^\omega,0)$ $=C(C(C(C(0,\Omega_1),$ $\Omega_1),\Omega_1),0)$
Large Veblen ordinal			least ord. not rep.	$[2][1][0]Next\ \omega$	$R_4 R_3 R_2 R_1 H suc\ 0$ $= R_{4\ldots 1} H suc\ 0$	$\psi(\Omega^{\Omega^\Omega})$	$C(\Omega_1^{\Omega_1},0)$ $=C(C(C(C(\Omega_1,\Omega_1),$ $\Omega_1),\Omega_1),0)$
Bachmann-Howard ordinal				least ord. not rep.	$R_{\omega\ldots 1} H suc\ 0$	$\psi(\varepsilon_{\Omega+1})$	$C(C(\Omega_2,\Omega_1),0)$