# Jäger Collapsing Function

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# 1 Functions collapsing large cardinals

## 1.1 Jäger's collapsing functions

Jäger's collapsing functions are a hierarchy of single-argument ordinal functions $\psi_\pi$ introduced by German mathematician Gerhard Jäger in 1984. This is an extension of Buchholz's notation.

### 1.1.1 Basic Notions

$M_0$ is the least Mahlo cardinal, small Greek letters denote ordinals less than $M_0$. Each ordinal $\alpha$ is identified with the set of its predecessors $\alpha = \{\beta | \beta < \alpha\}$.
$L$ denotes the set of all limit ordinals less than $M_0$.
An ordinal $\alpha$ is an additive principal number if $\alpha > 0$ and $\xi + \eta < \alpha$ for all $\xi, \eta < \alpha$. Let $P$ denote the set of all additive principal numbers less than $M_0$.
$\alpha =_{NF} \alpha_1 + \cdots + \alpha_n :\Leftrightarrow \alpha = \alpha_1 + \cdots + \alpha_n \wedge \alpha_1 \geq \cdots \geq \alpha_n \wedge \alpha_1, ..., \alpha_n \in P$
Cofinality $\mathrm{cof}(\alpha)$ of an ordinal $\alpha$ is the least $\beta$ such that there exists a function $f : \beta \to \alpha$ with $\sup\{f(\xi) | \xi < \beta\} = \alpha$. An ordinal $\alpha$ is regular, if $\alpha$ is a limit ordinal and $\mathrm{cof}(\alpha) = \alpha$. Let $R$ denote the set of all regular ordinals $\in (\omega, M_0)$.
An ordinal $\alpha$ is (weakly) inaccessible if $\alpha$ is a regular limit cardinal larger than $\omega$.
Enumeration function $F$ of class of ordinals $X$ is the unique increasing function such that $X = \{F(\alpha) | \alpha \in \mathrm{dom}(F)\}$ where domain of $F$, $\mathrm{dom}(F)$ is an ordinal number. We use $\mathrm{Enum}(X)$ to donate $F$.

### 1.1.2 Veblen function

$\varphi_\alpha = \mathrm{Enum}(\{\beta \in P | \forall \gamma < \alpha(\varphi_\gamma(\beta) = \beta)\})$
Normal form
$\alpha =_{NF} \varphi_\beta(\gamma) :\Leftrightarrow \alpha = \varphi_\beta(\gamma) \wedge \beta, \gamma < \alpha$
An ordinal $\alpha$ is a strongly critical if $\varphi(\alpha, 0) = \alpha$. Let $S$ denote the set of all strongly critical ordinals less than $M_0$.
Definition of $S(\gamma)$ for arbitrary $\gamma$.
$S(\gamma) = \{\gamma\}$ if $\gamma \in S \cup \{0\}$
$S(\gamma) = \{\alpha_1, ..., \alpha_n\}$ if $\gamma =_{NF} \alpha_1 + \cdots + \alpha_n \notin P$
$S(\gamma) = \{\alpha, \beta\}$ if $\gamma =_{NF} \varphi_\alpha(\beta) \notin S$

### 1.1.3 $\rho$-Inaccessible Ordinals

An ordinal is $\rho$-inaccessible if it is a regular cardinal and limit of $\alpha$-inaccessible ordinals for all $\alpha < \rho$. So the 0-inaccessible ordinals are exactly the regular cardinals $> \omega$, the 1-inaccessible ordinals are the inaccessible ordinals. Functions $I_\rho : M_0 \to M_0$ enumerate the $\rho$-inaccessible ordinals less than $M_0$ and their limits.
$I_\alpha = \mathrm{Enum}(\{\beta \in R | \forall \gamma < \alpha(I_\gamma(\beta) = \beta)\})$
Normal form
$\alpha =_{NF} I_\beta(\gamma) :\Leftrightarrow \alpha = I_\beta(\gamma) \wedge \gamma \notin L$
Definition of $\gamma^-$ for $\gamma \in R$.
$\gamma^- = 0$ if $\gamma =_{NF} I_\alpha(0)$
$\gamma^- = I_\alpha(\beta)$ if $\gamma =_{NF} I_\alpha(\beta + 1)$
"'Properties"'

| Veblen function | $\rho$-Inaccessible Ordinals |
|---|---|
| $\varphi_\alpha(\beta) \in P$ | $I_\alpha(0), I_\alpha(\beta+1) \in R$ |
| $\gamma < \alpha \Rightarrow \varphi_\gamma(\varphi_\alpha(\beta)) = \varphi_\alpha(\beta)$ | $-\gamma < \alpha \Rightarrow I_\gamma(I_\alpha(\beta)) = I_\alpha(\beta)$ |
| $\beta < \gamma \Rightarrow \varphi_\alpha(\beta) < \varphi_\alpha(\gamma)$ | $\beta < \gamma \Rightarrow I_\alpha(\beta) < I_\alpha(\gamma)$ |
| $\alpha < \beta \Rightarrow \varphi_\alpha(0) < \varphi_\beta(0)$ | $\alpha < \beta \Rightarrow I_\alpha(0) < I_\beta(0)$ |

### 1.1.4 The Ordinal Functions $\psi_\kappa$

Every $\psi_\kappa$ is a function from $M_0$ to $\kappa$ which "collapses" the elements of $M_0$ below $\kappa$. By the Greek letters $\kappa$ and $\pi$ we shall denote uncountable regular cardinals less than $M_0$.
"'Inductive Definition"' of $C_\kappa(\alpha)$ and $\psi_\kappa(\alpha)$.
$\{\kappa^-\} \cup \kappa^- \subset C^n_\kappa(\alpha)$
$S(\gamma) \subset C^n_\kappa(\alpha) \Rightarrow \gamma \in C^{n+1}_\kappa(\alpha)$
$\beta, \gamma \in C^n_\kappa(\alpha) \Rightarrow I_\beta(\gamma) \in C^{n+1}_\kappa(\alpha)$
$\gamma < \pi < \kappa \wedge \pi \in C^n_\kappa(\alpha) \Rightarrow \gamma \in C^{n+1}_\kappa(\alpha)$
$\gamma < \alpha \wedge \gamma, \pi \in C^n_\kappa(\alpha) \wedge \gamma \in C_\pi(\gamma) \Rightarrow \psi_\pi(\gamma) \in C^{n+1}_\kappa(\alpha)$
$C_\kappa(\alpha) = \cup\{C^n_\kappa(\alpha) | n < \omega\}$
$\psi_\kappa(\alpha) = \min\{\xi | \xi \notin C_\kappa(\alpha)\}$
Normal form
$\alpha =_{NF} \psi_\kappa(\beta) :\Leftrightarrow \alpha = \psi_\kappa(\beta) \wedge \beta \in C_\kappa(\beta)$

### 1.1.5 Fundamental sequences

The fundamental sequence for an ordinal number $\alpha$ with cofinality $\mathrm{cof}(\alpha) = \beta$ is a strictly increasing sequence $(\alpha[\eta])_{\eta<\beta}$ with length $\beta$ and with limit $\alpha$, where $\alpha[\eta]$ is the $\eta$-th element of this sequence.
"'Inductive Definition"' of $T$.

- $0 \in T$
- $\alpha =_{NF} \alpha_1 + \cdots + \alpha_n \wedge \alpha_1, ..., \alpha_n \in T \Rightarrow \alpha \in T$
- $\alpha =_{NF} \varphi_\beta(\gamma) \wedge \beta, \gamma \in T \Rightarrow \alpha \in T$
- $\alpha =_{NF} I_\beta(\gamma) \wedge \beta, \gamma \in T \Rightarrow \alpha \in T$
- $\alpha =_{NF} \psi_\kappa(\beta) \wedge \kappa, \beta \in T \Rightarrow \alpha \in T$

Below we write $I(\alpha, \beta)$ for $I_\alpha(\beta)$ and $\varphi(\alpha, \beta)$ for $\varphi_\alpha(\beta)$
For non-zero ordinals $\alpha \in T$ we define the fundamental sequences as follows:

- If $\alpha = \varphi(0, \beta+1)$ then $\mathrm{cof}(\alpha) = \omega$ and $\alpha[\eta] = \varphi(0, \beta) \times \eta$
- If $\alpha = \varphi(\beta+1, 0)$ then $\mathrm{cof}(\alpha) = \omega$ and $\alpha[0] = 0$ and $\alpha[\eta+1] = \varphi(\beta, \alpha[\eta])$
- If $\alpha = \varphi(\beta+1, \gamma+1)$ then $\mathrm{cof}(\alpha) = \omega$ and $\alpha[0] = \varphi(\beta+1, \gamma) + 1$ and $\alpha[\eta+1] = \varphi(\beta, \alpha[\eta])$
- If $\alpha = \varphi(\beta, 0)$ and $\beta \in L$ then $\mathrm{cof}(\alpha) = \mathrm{cof}(\beta)$ and $\alpha[\eta] = \varphi(\beta[\eta], 0)$
- If $\alpha = \varphi(\beta, \gamma+1)$ and $\beta \in L$ then $\mathrm{cof}(\alpha) = \mathrm{cof}(\beta)$ and $\alpha[\eta] = \varphi(\beta[\eta], \varphi(\beta, \gamma) + 1)$
- If $\alpha = \varphi(\beta, \gamma)$ and $\gamma \in L$ then $\mathrm{cof}(\alpha) = \mathrm{cof}(\gamma)$ and $\alpha[\eta] = \varphi(\beta, \gamma[\eta])$

- If $\alpha = \psi_{I(0,0)}(0)$ then $\mathrm{cof}(\alpha) = \omega$ and $\alpha[0] = 0$ and $\alpha[\eta+1] = \varphi(\alpha[\eta], 0)$
- If $\alpha = \psi_{I(0,\beta+1)}(0)$ then $\mathrm{cof}(\alpha) = \omega$ and $\alpha[0] = I(0, \beta) + 1$ and $\alpha[\eta+1] = \varphi(\alpha[\eta], 0)$
- If $\alpha = \psi_{I(0,\beta)}(\gamma+1)$ then $\mathrm{cof}(\alpha) = \omega$ and $\alpha[0] = \psi_{I(0,\beta)}(\gamma) + 1$ and $\alpha[\eta+1] = \varphi(\alpha[\eta], 0)$

- If $\alpha = \psi_{I(\beta+1,0)}(0)$ then $\mathrm{cof}(\alpha) = \omega$ and $\alpha[0] = 0$ and $\alpha[\eta+1] = I(\beta, \alpha[\eta])$
- If $\alpha = \psi_{I(\beta+1,\gamma+1)}(0)$ then $\mathrm{cof}(\alpha) = \omega$ and $\alpha[0] = I(\beta+1, \gamma) + 1$ and $\alpha[\eta+1] = I(\beta, \alpha[\eta])$
- If $\alpha = \psi_{I(\beta+1,\gamma)}(\delta+1)$ then $\mathrm{cof}(\alpha) = \omega$ and $\alpha[0] = \psi_{I(\beta+1,\gamma)}(\delta) + 1$ and $\alpha[\eta+1] = I(\beta, \alpha[\eta])$

- If $\alpha = \psi_{I(\beta,0)}(0)$ and $\beta \in L$ then $\mathrm{cof}(\alpha) = \mathrm{cof}(\beta)$ and $\alpha[\eta] = I(\beta[\eta], 0)$
- If $\alpha = \psi_{I(\beta,\gamma+1)}(0)$ and $\beta \in L$ then $\mathrm{cof}(\alpha) = \mathrm{cof}(\beta)$ and $\alpha[\eta] = I(\beta[\eta], I(\beta, \gamma) + 1)$
- If $\alpha = \psi_{I(\beta,\gamma)}(\delta+1)$ and $\beta \in L$ then $\mathrm{cof}(\alpha) = \mathrm{cof}(\beta)$ and $\alpha[\eta] = I(\beta[\eta], \psi_{I(\beta,\gamma)}(\delta) + 1)$

- If $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ with $n \geq 2$ then $\mathrm{cof}(\alpha) = \mathrm{cof}(\alpha_n)$ and $\alpha[\eta] = \alpha_1 + \alpha_2 + \cdots + (\alpha_n[\eta])$
- If $\alpha = \varphi(0, 0)$ then $\mathrm{cof}(\alpha) = \alpha = 1$ and $\alpha[0] = 0$

- If $\alpha = I(\beta, 0)$ or $\alpha = I(\beta, \gamma + 1)$ then $\mathrm{cof}(\alpha) = \alpha$ and $\alpha[\eta] = \eta$
- If $\alpha = I(\beta, \gamma)$ and $\gamma \in L$ then $\mathrm{cof}(\alpha) = \mathrm{cof}(\gamma)$ and $\alpha[\eta] = I(\beta, \gamma[\eta])$
- If $\alpha = \psi_\pi(\beta)$ and $\omega \leq \mathrm{cof}(\beta) < \pi$ then $\mathrm{cof}(\alpha) = \mathrm{cof}(\beta)$ and $\alpha[\eta] = \psi_\pi(\beta[\eta])$
- If $\alpha = \psi_\pi(\beta)$ and $\mathrm{cof}(\beta) = \rho \geq \pi$ then $\mathrm{cof}(\alpha) = \omega$ and $\alpha[\eta] = \psi_\pi(\beta[\gamma[\eta]])$ with $\gamma[0] = 1$ and $\gamma[\eta + 1] = \psi_\rho(\beta[\gamma[\eta]])$

Limit of this notation $\lambda$. If $\alpha = \lambda$ then $\mathrm{cof}(\alpha) = \omega$ and $\alpha[0] = 0$ and $\alpha[\eta + 1] = I(\alpha[\eta], 0)$
These fundamental sequences can be reformulated to produce recursive definitions :

- $\varphi(0, \beta) = \omega^\beta$
- $\varphi(\beta + 1, 0) = [\varphi(\beta, \bullet]^\omega 0 = H[\varphi(\beta, \bullet)]0$
- $\varphi(\beta + 1, \gamma + 1) = [\varphi(\beta, \bullet)]^\omega(\varphi(\beta + 1, \gamma) + 1)$
- $\varphi(Lim_\nu f, 0) = Lim_\nu[\varphi(f\bullet, 0)]$
- $\varphi(Lim_\nu f, \gamma + 1) = Lim_\nu[\varphi(f\bullet, \varphi(Lim_\nu f, \gamma) + 1)]$
- $\varphi(\beta, Lim_\nu g) = Lim_\nu[\varphi(\beta, g\bullet)]$

- $\psi_{I(0,0)}(0) = [\varphi(\bullet, 0)]^\omega 0 = \Gamma_0$
- $\psi_{I(0,\beta+1)}(0) = [\varphi(\bullet, 0)]^\omega(I(0, \beta) + 1)$
- $\psi_{I(0,\beta)}(\gamma + 1) = [\varphi(\bullet, 0)]^\omega(\psi_{I(0,\beta)}(\gamma) + 1)$

- $\psi_{I(\beta+1,0)}(0) = [I(\beta, \bullet)]^\omega 0$
- $\psi_{I(\beta+1,\gamma+1)}(0) = [I(\beta, \bullet)]^\omega(I(\beta + 1, \gamma) + 1)$
- $\psi_{I(\beta+1,\gamma)}(\delta + 1) = [I(\beta, \bullet)]^\omega(\psi_{I(\beta+1,\gamma)}(\delta) + 1)$

- $\psi_{I(Lim_\nu f,0)}(0) = Lim_\nu[I(f\bullet, 0)]$
- $\psi_{I(Lim_\nu f,\gamma+1)}(0) = Lim_\nu[I(f\bullet, I(Lim_\nu f, \gamma) + 1)]$
- $\psi_{I(Lim_\nu f,\gamma)}(\delta + 1) = Lim_\nu[I(f\bullet, \psi_{I(Lim_\nu f,\gamma)}(\delta) + 1)]$

- $\beta + Lim_\nu g = Lim_\nu[\beta + g\bullet]$
- $\varphi(0, 0) = 1$

- $I(\beta, 0) = I(\beta, \gamma + 1) = Lim_{cof(I(\beta,0)}[\bullet]$ where $[\bullet]$ is the identity function
- $I(\beta, Lim_n ug) = Lim_\nu[I(\beta, g\bullet]$
- $\psi_\pi(Lim_\nu f) = Lim_\nu[\psi_\pi(f\bullet)]$ if $\omega \leq \nu \leq \pi$
- $\psi_\pi(Lim_\nu f) = lim[\psi_\pi(f(g\bullet)]$ with $g(0) = 1$ and $g(n + 1) = \psi_\nu(f(g(n)))$ if $\nu \geq \pi$

### 1.1.6 See also

Other ordinal collapsing functions:
[[Madore's $\psi$ function]]
[[Buchholz's $\psi$ functions]]
[[User blog:Denis Maksudov/Ordinal functions collapsing the least weakly Mahlo cardinal; a system of fundamental sequences—collapsing functions based on a weakly Mahlo cardinal]]

### 1.1.7 References

1. W.Buchholz. A New System of Proof-Theoretic Ordinal Functions. Annals of Pure and Applied Logic (1986),32
2. M.Jäger. $\rho$-inaccessible ordinals, collapsing functions and a recursive notation system. Arch. Math. Logik Grundlagenforsch (1984),24
3. http://cantorsattic.info/J%C3%A4ger%27s_collapsing_functions_and_%CF%81-inaccessible_ordinals