Binary Veblen function

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Every non-zero ordinal $\alpha < \Gamma_0$, where Γ_0 is the smallest ordinal α such that $\varphi_\alpha(0) = \alpha$, can be uniquely written in normal form for the Veblen hierarchy:
$\alpha = \varphi_{\beta_1}(\gamma_1) + \varphi_{\beta_2}(\gamma_2) + \cdots + \varphi_{\beta_k}(\gamma_k)$,
where
$\varphi_{\beta_1}(\gamma_1) \geq \varphi_{\beta_2}(\gamma_2) \geq \cdots \geq \varphi_{\beta_k}(\gamma_k)$ $\gamma_m < \varphi_{\beta_m}(\gamma_m)$ for $m \in \{1, ..., k\}$

Now we will see how we can find the fundamental sequence of an ordinal written in this notmal form.
From the rule defining addition of a limit ordinal :
$\alpha + lim(f) = lim(n \mapsto \alpha + f(n))$
we deduce the fundamental sequence :
$(\alpha + \beta)[n] = \alpha + \beta[n]$
if β is a limit ordinal.
In particular, we have :
$(\varphi_{\beta_1}(\gamma_1) + \varphi_{\beta_2}(\gamma_2) + \cdots + \varphi_{\beta_k}(\gamma_k))[n] = \varphi_{\beta_1}(\gamma_1) + \cdots + \varphi_{\beta_{k-1}}(\gamma_{k-1}) + (\varphi_{\beta_k}(\gamma_k)[n])$, where $\varphi_{\beta_1}(\gamma_1) \geq \varphi_{\beta_2}(\gamma_2) \geq \cdots \geq \varphi_{\beta_k}(\gamma_k)$ and $\gamma_m < \varphi_{\beta_m}(\gamma_m)$ for $m \in \{1, 2, ..., k\}$,
Then, $\varphi_0(\gamma)$ is ω^γ.
For $\gamma = 0$ it is 1.
From the rule of multiplication by a limit ordinal :
$\alpha \cdot lim(f) = lim(n \mapsto \alpha \cdot f(n))$
we deduce the fundamental sequence :
$(\alpha \cdot \beta)[n] = \alpha \cdot \beta[n]$ if β is a limit ordinal.
In particular, for ω :
$(\alpha \cdot \omega)[n] = \alpha \cdot \omega[n] = \alpha \cdot n$
Then we have :
$\varphi_0(\gamma + 1) = \omega^{\gamma+1} = \omega^\gamma \cdot \omega = \varphi_0(\gamma) \cdot \omega$
So the corresponding fundamental sequence is :
$\varphi_0(\gamma + 1)[n] = (\varphi_0(\gamma) \cdot \omega)[n] = \varphi_0(\gamma) \cdot n$
If γ is a limit ordinal and $\gamma < \varphi_0(\gamma)$, the fundamental sequence can be defined canonically:
$\varphi_0(\gamma)[n] = \varphi_0(\gamma[n])$
Note that if we remove the condition $\gamma < \varphi_0(\gamma)$ there is a problem. For example, for $\gamma = \varepsilon_0$, we have $\gamma = \varphi_0(\gamma) = \omega^\gamma$. Then, if we take as fundamental sequence of ε_0 the sequence $\varepsilon_0[0] = 0$ and $\varepsilon_0[n+1] = \omega^{\varepsilon_0[n]}$, then $\varphi_0(\gamma)[0] = \omega^{\varepsilon_0}[0] = \varepsilon_0[0] = 0$, but $\varphi_0(\gamma[0]) = \omega^{\varepsilon_0[0]} = \omega^0 = 1$.
Then, $\varphi_{\beta+1}(\gamma)$ is the $1 + \gamma$-th fixed point of the function $\xi \mapsto \varphi_\beta(\xi)$, or more simply the function φ_β.
In particular, $\varphi_{\beta+1}(0)$ is the least fixed point of this function, which is ${\varphi_\beta}^\omega(0)$. A fundamental sequence of this ordinal is $\varphi_{\beta+1}(0)[n] = {\varphi_\beta}^n(0)$, which can also be written $\varphi_{\beta+1}(0)[0] = 0$ and $\varphi_{beta+1}(0)[n+1] = \varphi_\beta(\varphi_{\beta+1}(0)[n])$.
$\varphi_{\beta+1}(\gamma + 1)$ is the fixed point of φ_β that follows $\varphi_{\beta+1}(\gamma)$. It is ${\varphi_\beta}^\omega(\varphi_{\beta+1}(\gamma) + 1)$. This can also be written $\varphi_{\beta+1}(\gamma + 1)[0] = \varphi_{\beta+1}(\gamma) + 1$ and $\varphi_{\beta+1}(\gamma + 1)[n + 1] = \varphi_\beta(\varphi_{\beta+1}(\gamma + 1)[n])$.
If γ is a limit ordinal, the fundamental sequence can be defined canonically:
$\varphi_{\beta+1}(\gamma)[n] = \varphi_{\beta+1}(\gamma[n])$ if $\gamma < \varphi_\beta(\gamma)$.
Finally, if β is a limit ordinal, we define the following fundamental sequences:
$\varphi_\beta(0)[n] = \varphi_{\beta[n]}(0)$ if $\beta < \varphi_\beta(0)$
$\varphi_\beta(\gamma + 1)[n] = \varphi_{\beta[n]}(\varphi_\beta(\gamma) + 1)$
$\varphi_\beta(\gamma)[n] = \varphi_\beta(\gamma[n])$ for a limit ordinal $\gamma < \varphi_\beta(\gamma)$.

Concerning $\varphi_\beta(0)[n]$, note that if we remove the condition $\beta < \varphi_\beta(0)$ there is a problem. For example, if we take $\beta = \Gamma_0$ the least fixed point of the function $\xi \mapsto \varphi_\xi(0)$, then we have $\varphi_{\Gamma_0}(0) = \Gamma_0$. A fundamental sequence of Γ_0 is $\Gamma_0[0] = 0, \Gamma_0[1] = \varphi_0(0) = \omega^0 = 1, \Gamma_0[2] = \varphi_1(0) = \varepsilon_0, \ldots$. Then we have $\varphi_{\Gamma_0}(0)[0] = \Gamma_0[0] = 0$, but $\varphi_{\Gamma_0[0]}(0) = \varphi_0(0) = \omega^0 = 1$.
For more explanations about the fundaamental sequence $\varphi_\beta(\gamma+1)[n] = \varphi_{\beta[n]}(\varphi_\beta(\gamma)+1)$ see :
https://www.physicsforums.com/threads/fundamental-sequences-for-the-veblen-hierarchy-of-ordinals.933538/

Let us recap now the results we obtained.
The fundamental sequences for the Veblen functions $\varphi_\beta(\gamma) = \varphi(\beta, \gamma)$ are :

(1) $(\varphi_{\beta_1}(\gamma_1) + \varphi_{\beta_2}(\gamma_2) + \cdots + \varphi_{\beta_k}(\gamma_k))[n] = \varphi_{\beta_1}(\gamma_1) + \cdots + \varphi_{\beta_{k-1}}(\gamma_{k-1}) + (\varphi_{\beta_k}(\gamma_k)[n])$, where $\varphi_{\beta_1}(\gamma_1) \geq \varphi_{\beta_2}(\gamma_2) \geq \cdots \geq \varphi_{\beta_k}(\gamma_k)$ and $\gamma_m < \varphi_{\beta_m}(\gamma_m)$ for $m \in \{1, 2, ..., k\}$,
(2) $\varphi_0(0) = 1$,
(3) $\varphi_0(\gamma+1)[n] = \varphi_0(\gamma)n$
(4) $\varphi_{\beta+1}(0)[0] = 0$ and $\varphi_{\beta+1}(0)[n+1] = \varphi_\beta(\varphi_{\beta+1}(0)[n])$,
(5) $\varphi_{\beta+1}(\gamma+1)[0] = \varphi_{\beta+1}(\gamma)+1$ and $\varphi_{\beta+1}(\gamma+1)[n+1] = \varphi_\beta(\varphi_{\beta+1}(\gamma+1)[n])$,
(6) $\varphi_\beta(\gamma)[n] = \varphi_\beta(\gamma[n])$ for a limit ordinal $\gamma < \varphi_\beta(\gamma)$,
(7) $\varphi_\beta(0)[n] = \varphi_{\beta[n]}(0)$ for a limit ordinal $\beta < \varphi_\beta(0)$,
(8) $\varphi_\beta(\gamma+1)[n] = \varphi_{\beta[n]}(\varphi_\beta(\gamma)+1)$ for a limit ordinal β.

From these fundamental sequences, we can retrieve the initial definition of the function φ.

(1) This does not concern the definition of the φ function but the definition of addition
(2) and (3) and (6) for $\beta = 0$ are equivalent to $\varphi_0(\gamma) = \omega^\gamma$.
(4) $\varphi_{\beta+1}(0) = lim(n \mapsto {\varphi_\beta}^n(0)) = {\varphi_\beta}^\omega(0)$ which is the least fixed point of φ_β.
(5) $\varphi_{\beta+1}(\gamma+1) = lim(n \mapsto {\varphi_\beta}^n(\varphi_{\beta+1}(\gamma)+1))$, which is the least fixed point of φ_β strictly greater than $\varphi_{\beta+1}(\gamma)$, so with (6) it gives $\varphi_{\beta+1}(\gamma)$ is the $1+\gamma$-th fixed point of φ_β.
(7), (8) and (6) for β limit ordinal complete the definition of $\varphi_\beta(\gamma)$ for β limit ordinal.

Here is an Haskell implementation of the φ function :

```
module Phi where

 data Nat
  = ZeroN
  | SucN Nat

 data Ord
  = Zero
  | Suc Ord
  | Lim (Nat -> Ord)

 iter f ZeroN x = x
 iter f (SucN n) x = f (iter f n x)

 opLim f a = Lim (\n -> f n a)

 opItw f = opLim (iter f)

 cantor a Zero = Suc a
 cantor a (Suc b) = opItw (\x -> cantor x b) a
 cantor a (Lim f) = Lim (\n -> cantor a (f n))

 nabla f Zero = f Zero
 nabla f (Suc a) = f (Suc (nabla f a))
 nabla f (Lim h) = Lim (\n -> nabla f (h n))
```

```
deriv f = nabla (opItw f)

phi Zero = cantor Zero
phi (Suc a) = deriv (phi a)
phi (Lim f) = nabla (opLim (\n -> phi (f n)))
```

iter f n x = $f^n(x)$.
opLim f a = limit of f 0 a, f 1 a, f 2 a, ...
opItw f = f^ω.
cantor a b = $a + \omega^b$.
deriv f a = the (1+a)-th fixed point of f.
phi a b = $\varphi_a(b)$.