

More information about this function can be found at :
http://googology.wikia.com/wiki/Veblen_function

Every non-zero ordinal $\alpha < \Gamma_0$, where Γ_0 is the smallest ordinal α such that $\varphi_\alpha(0) = \alpha$, can be uniquely written in normal form for the Veblen hierarchy:
 $\alpha = \varphi_{\beta_1}(\gamma_1) + \varphi_{\beta_2}(\gamma_2) + \dots + \varphi_{\beta_k}(\gamma_k)$,
 where
 $\varphi_{\beta_1}(\gamma_1) \geq \varphi_{\beta_2}(\gamma_2) \geq \dots \geq \varphi_{\beta_k}(\gamma_k)$ $\gamma_m < \varphi_{\beta_m}(\gamma_m)$ for $m \in \{1, \dots, k\}$

Now we will see how we can find the fundamental sequence of an ordinal written in this normal form.

From the rule defining addition of a limit ordinal :

$$\alpha + \lim(f) = \lim(n \mapsto \alpha + f(n))$$

we deduce the fundamental sequence :

$$(\alpha + \beta)[n] = \alpha + \beta[n]$$

if β is a limit ordinal.

In particular, we have :

$$(\varphi_{\beta_1}(\gamma_1) + \varphi_{\beta_2}(\gamma_2) + \dots + \varphi_{\beta_k}(\gamma_k))[n] = \varphi_{\beta_1}(\gamma_1) + \dots + \varphi_{\beta_{k-1}}(\gamma_{k-1}) + (\varphi_{\beta_k}(\gamma_k)[n]),$$

where $\varphi_{\beta_1}(\gamma_1) \geq \varphi_{\beta_2}(\gamma_2) \geq \dots \geq \varphi_{\beta_k}(\gamma_k)$ and $\gamma_m < \varphi_{\beta_m}(\gamma_m)$ for $m \in \{1, 2, \dots, k\}$,

Then, $\varphi_0(\gamma)$ is ω^γ .

For $\gamma = 0$ it is 1.

From the rule of multiplication by a limit ordinal :

$$\alpha \cdot \lim(f) = \lim(n \mapsto \alpha \cdot f(n))$$

we deduce the fundamental sequence :

$$(\alpha \cdot \beta)[n] = \alpha \cdot \beta[n] \text{ if } \beta \text{ is a limit ordinal.}$$

In particular, for ω :

$$(\alpha \cdot \omega)[n] = \alpha \cdot \omega[n] = \alpha \cdot n$$

Then we have :

$$\varphi_0(\gamma + 1) = \omega^{\gamma+1} = \omega^\gamma \cdot \omega = \varphi_0(\gamma) \cdot \omega$$

So the corresponding fundamental sequence is :

$$\varphi_0(\gamma + 1)[n] = (\varphi_0(\gamma) \cdot \omega)[n] = \varphi_0(\gamma) \cdot n$$

If γ is a limit ordinal, the fundamental sequence can be defined canonically:

$$\varphi_0(\gamma)[n] = \varphi_0(\gamma[n])$$

Then, $\varphi_{\beta+1}(\gamma)$ is the $1 + \gamma$ -th fixed point of the function $\xi \mapsto \varphi_\beta(\xi)$, or more simply the function φ_β .

In particular, $\varphi_{\beta+1}(0)$ is the least fixed point of this function, which is $\varphi_\beta^\omega(0)$. A fundamental sequence of this ordinal is $\varphi_{\beta+1}(0)[n] = \varphi_\beta^n(0)$, which can also be written $\varphi_{\beta+1}(0)[0] = 0$ and $\varphi_{\beta+1}(0)[n+1] = \varphi_\beta(\varphi_{\beta+1}(0)[n])$.

$\varphi_{\beta+1}(\gamma+1)$ is the fixed point of φ_β that follows $\varphi_{\beta+1}(\gamma)$. It is $\varphi_\beta^\omega(\varphi_{\beta+1}(\gamma) + 1)$. This can also be written $\varphi_{\beta+1}(\gamma+1)[0] = \varphi_{\beta+1}(\gamma) + 1$ and $\varphi_{\beta+1}(\gamma+1)[n+1] = \varphi_\beta(\varphi_{\beta+1}(\gamma+1)[n])$.

If γ is a limit ordinal, the fundamental sequence can be defined canonically:

$$\varphi_{\beta+1}(\gamma)[n] = \varphi_{\beta+1}(\gamma[n]).$$

Finally, if β is a limit ordinal, we can define canonically:

$$\varphi_\beta(0)[n] = \varphi_{\beta[n]}(0) \text{ if } \beta < \varphi_\beta(0)$$

and

$$\varphi_\beta(\gamma + 1)[n] = \varphi_{\beta[n]}(\varphi_\beta(\gamma) + 1)$$

Note that if we remove the condition $\beta < \varphi_\beta(0)$ there is a problem. For example, if we take $\beta = \Gamma_0$ the least fixed point of the function $\xi \mapsto \varphi_\xi(0)$, then we have $\varphi_{\Gamma_0}(0) = \Gamma_0$. A fundamental sequence of Γ_0 is $\Gamma_0[0] = 0, \Gamma_0[1] = \varphi_0(0) = \omega^0 = 1, \Gamma_0[2] = \varphi_1(0) = \varepsilon_0, \dots$. Then we have $\varphi_{\Gamma_0}(0)[0] = \Gamma_0[0] = 0$, but $\varphi_{\Gamma_0[0]}(0) = \varphi_0(0) = \omega^0 = 1$.

Let us recap now the results we obtained.

The fundamental sequences for the Veblen functions $\varphi_\beta(\gamma) = \varphi(\beta, \gamma)$ are :

- (1) $(\varphi_{\beta_1}(\gamma_1) + \varphi_{\beta_2}(\gamma_2) + \dots + \varphi_{\beta_k}(\gamma_k))[n] = \varphi_{\beta_1}(\gamma_1) + \dots + \varphi_{\beta_{k-1}}(\gamma_{k-1}) + (\varphi_{\beta_k}(\gamma_k)[n])$, where $\varphi_{\beta_1}(\gamma_1) \geq \varphi_{\beta_2}(\gamma_2) \geq \dots \geq \varphi_{\beta_k}(\gamma_k)$ and $\gamma_m < \varphi_{\beta_m}(\gamma_m)$ for $m \in \{1, 2, \dots, k\}$,
- (2) $\varphi_0(0) = 1$,
- (3) $\varphi_0(\gamma + 1)[n] = \varphi_0(\gamma)n$
- (4) $\varphi_{\beta+1}(0)[0] = 0$ and $\varphi_{\beta+1}(0)[n+1] = \varphi_\beta(\varphi_{\beta+1}(0)[n])$,
- (5) $\varphi_{\beta+1}(\gamma+1)[0] = \varphi_{\beta+1}(\gamma) + 1$ and $\varphi_{\beta+1}(\gamma+1)[n+1] = \varphi_\beta(\varphi_{\beta+1}(\gamma+1)[n])$,
- (6) $\varphi_\beta(\gamma)[n] = \varphi_\beta(\gamma[n])$ for a limit ordinal $\gamma < \varphi_\beta(\gamma)$,
- (7) $\varphi_\beta(0)[n] = \varphi_{\beta[n]}(0)$ for a limit ordinal $\beta < \varphi_\beta(0)$,
- (8) $\varphi_\beta(\gamma + 1)[n] = \varphi_{\beta[n]}(\varphi_\beta(\gamma) + 1)$ for a limit ordinal β .

From these fundamental sequences, we can retrieve the initial definition of the function φ .

(1) This does not concern the definition of the φ function but the definition of addition

(2) and (3) and (6) for $\beta = 0$ are equivalent to $\varphi_0(\gamma) = \omega^\gamma$.

(4) $\varphi_{\beta+1}(0) = \lim(n \mapsto \varphi_\beta^n(0)) = \varphi_\beta^\omega(0)$ which is the least fixed point of φ_β .

(5) $\varphi_{\beta+1}(\gamma+1) = \lim(n \mapsto \varphi_\beta^n(\varphi_{\beta+1}(\gamma) + 1))$, which is the least fixed point of φ_β strictly greater than $\varphi_{\beta+1}(\gamma)$, so with (6) it gives $\varphi_{\beta+1}(\gamma)$ is the $1 + \gamma$ -th fixed point of φ_β .

(7), (8) and (6) for β limit ordinal complete the definition of $\varphi_\beta(\gamma)$ for β limit ordinal.

Here is an Haskell implementation of the φ function :

module Phi where

```
data Nat
  = ZeroN
  | SucN Nat
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data Ord
```

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= Zero
| Suc Ord
| Lim (Nat -> Ord)

iter f ZeroN x = x
iter f (SucN n) x = f (iter f n x)

opLim f a = Lim (\n -> f n a)

opItw f = opLim (iter f)

cantor a Zero = Suc a
cantor a (Suc b) = opItw (\x -> cantor x b) a
cantor a (Lim f) = Lim (\n -> cantor a (f n))

nabla f Zero = f Zero
nabla f (Suc a) = f (Suc (nabla f a))
nabla f (Lim h) = Lim (\n -> nabla f (h n))

deriv f = nabla (opItw f)

phi Zero = cantor Zero
phi (Suc a) = deriv (phi a)
phi (Lim f) = nabla (opLim (\n -> phi (f n)))

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$\text{iter } f \ n \ x$ is $f^n(x)$.
 $\text{opLim } f \ a$ is the limit of $f \ 0 \ a, f \ 1 \ a, f \ 2 \ a, \dots$
 $\text{opItw } f$ is f^ω
 $\text{cantor } a \ b$ is $a + \omega^b$.
 $\text{deriv } f \ a$ is the $(1+a)$ -th fixed point of f .