TRANSFINITE ORDINALS by Jacques Bailhache, January-march 2018

Any ordinal can be defined as the least ordinal strictly greater than all ordinals of a set : the empty set for 0, $\{\alpha\}$ for the successor of α, $\{\alpha_0, \alpha_1, \alpha_2, ...\}$ for an ordinal with fundamental sequence $\alpha_0, \alpha_1, \alpha_2, ...$

1 Algebraic notation

We define the following operations on ordinals :

- addition : $\alpha + 0 = \alpha; \alpha + suc(\beta) = suc(\alpha+\beta); \alpha + lim(f) = lim(n \mapsto \alpha + f(n))$
- multiplication : $\alpha \times 0 = 0; \alpha \times suc(\beta) = (\alpha \times \beta) + \alpha; \alpha \times lim(f) = lim(n \mapsto \alpha \times f(n))$
- exponentiation : $\alpha^0 = 1; \alpha^{suc(\beta)} = \alpha^\beta \times \alpha; \alpha^{lim(f)} = lim(n \mapsto \alpha^{f(n)})$

2 Veblen functions

$\varepsilon_0 = lim\ \omega, \omega^\omega, \omega^{\omega^\omega}, \ldots; \varepsilon_1 = lim\ \varepsilon_0, {\varepsilon_0}^{\varepsilon_0}, {\varepsilon_0}^{{\varepsilon_0}^{\varepsilon_0}}, \ldots = lim\ \varepsilon_0 + 1, \omega^{\varepsilon_0+1}, \omega^{\omega^{\varepsilon_0+1}}, \ldots; \zeta_0 = lim\ 0, \varepsilon_0, \varepsilon_{\varepsilon_0}, \ldots$

$\omega^\alpha = \varphi(0,\alpha); \varepsilon_\alpha = \varphi(1,\alpha); \zeta_\alpha = \varphi(2,\alpha)$

$\varphi(\ldots,\beta,0,\ldots,0,\gamma)$ is the $(1+\gamma)^{th}$ common fixed point of the functions $\xi \mapsto \varphi(\ldots,\delta,\xi,0,\ldots,0)$ for all $\delta < \beta$.

$\varphi(\alpha_n,\ldots,\alpha_0,\beta)$ may also be written $\varphi_{\alpha_n,\ldots,\alpha_0}(\beta)$ or $\varphi_{\Omega^n \times \alpha_n + \ldots + \alpha_0}(\beta)$ or $\varphi(\Omega^n \times \alpha_n + \ldots + \alpha_0, \beta)$ or $\begin{pmatrix} \beta & \alpha_0 & \ldots & \alpha_n \\ 0 & 1 & \ldots & n+1 \end{pmatrix}$

3 Simmons notation

$Fixfz = f^w(z+1)$ = least fixed point of f strictly greater than z ; $Next = Fix(\alpha \mapsto \omega^\alpha)$

$[0]h = Fix(\alpha \mapsto h^\alpha \omega)$; $[1]hg = Fix(\alpha \mapsto h^\alpha g \omega)$; $[2]hgf = Fix(\alpha \mapsto h^\alpha gf\omega)$; etc...

Correspondence with Veblen's φ : $\varphi(1+\beta,\alpha) = ([0]^\beta Next)^{1+\alpha}\omega$

If $\gamma > 0, \varphi(\gamma,\beta,\alpha) = \varphi(\gamma \times \Omega + \beta, \alpha) = ([0]^{\gamma\times\Omega+\beta} Next)^{1+\alpha}\omega = ([0]^\beta(([0]^\Omega)^\gamma Next))^{1+\alpha}\omega = ([0]^\beta(([1][0])^\gamma Next))^{1+\alpha}\omega$

If $\delta > 0$ or $\gamma > 0, \varphi(\delta,\gamma,\beta,\alpha) = \varphi(\delta \times \Omega^2 + \gamma \times \Omega + \beta, \alpha) = ([0]^{\Omega^2\times\delta+\Omega\times\gamma+\beta} Next)^{1+\alpha}\omega = ([0]^\beta([0]^\Omega)^\gamma([0]^{\Omega^2})^\delta Next)))^{1+\alpha}\omega = ([0]^\beta(([1][0])^\gamma(([1]^2[0])^\delta Next)))^{1+\alpha}\omega$, with $[0]^{\Omega^n} = [1]^n[0]$.

Rationalization of φ : $\varphi(1+\beta,\alpha) = \varphi'(\beta, 1+\alpha) => \varphi'(\beta,\alpha) = ([0]^\beta Next)^\alpha \omega; \varphi(\gamma,\beta,\alpha) = \varphi'(\gamma,\beta,1+\alpha)$

4 RHS0 notation

We start from 0, if we don(t see any regularity we take the successor, if we see a regularity, if we have a notation for this regularity, we use it, else we invent it, then we jump to the limit.

$Hfx = lim\ x, fx, f(fx), \ldots; R_1fgx = lim\ gx, fgx, ffgx, \ldots; R_2fghx = lim\ hx, fghx, fgfghx, \ldots$

Correspondence with Simmons notation : $\ldots, [3] \to R_5, [2] \to R_4, [1] \to R_3, [0] \to R_2, Next \to R_1, \omega \to Hsuc\ 0$

5 Ordinal collapsing functions

These functions use uncountable ordinals to define countable ordinals.

We define sets of ordinals that can be built from given ordinals and operations, then we take the least ordinal which is not in this set, or the least ordinal which is greater than all contable ordinals of this set.

These functions are extensions of functions on countable ordinals, whose fixed points can be reached by applying them to an uncountable ordinal, for example :

- Madore's ψ : $\psi(\alpha) = \varepsilon_\alpha$ if $\alpha < \zeta_0; \psi(\Omega) = \zeta_0$ which is the least fixed point of $\alpha \mapsto \varepsilon_\alpha$.
- Feferman's θ : $\theta(\alpha,\beta) = \varphi(\alpha,\beta)$ if $\alpha < \Gamma_0$ and $\beta < \Gamma_0; \theta(\Omega,0) = \Gamma_0$ which is the least fixed point of $\alpha \mapsto \varphi(\alpha,0)$.
- Taranovsky's C : $C(\alpha,\beta) = \beta + \omega^\alpha$ if α is countable; $C(\Omega_1,0) = \varepsilon_0$ which is the least fixed point of $\alpha \mapsto \omega^\alpha$.

Name	Symbol	Algebraic	Veblen	Simmons	RHS0	Madore	Taranovsky
Zero	0	0			0		0
One	1	1	$\varphi(0,0)$		suc 0		C(0,0)
Two	2	2			suc (suc 0)		C(0,C(0,0))
Omega	ω	ω	$\varphi(0,1)$	ω	H suc 0		C(1,0)
		$\omega+1$			suc (H suc 0)		C(0,C(1,0))
		$\omega\times 2$			H suc (H suc 0)		C(1,C(1,0))
		ω^2	$\varphi(0,2)$		H (H suc) 0		C(C(0,C(0,0)),0)
		ω^ω	$\varphi(0,\omega)$		H H suc 0		C(C(1,0),0)
Epsilon zero	ε_0	ε_0	$\varphi(1,0)$	$Next\ \omega$	$R_1Hsuc\ 0$	$\psi(0)$	$C(\Omega_1,0)$
		ε_1	$\varphi(1,1)$	$Next^2\omega$	$R_1(R_1H)suc\ 0$	$\psi(1)$	$C(\Omega_1,C(\Omega_1,0))$
		ε_ω	$\varphi(1,\omega)$	$Next^\omega\omega$	$HR_1Hsuc\ 0$	$\psi(\omega)$	$C(C(0,\Omega_1),0)$
		$\varepsilon_{\varepsilon_0}$	$\varphi(1,\varphi(1,0))$	$Next^{Next\omega}\omega$	$R_1HR_1Hsuc\ 0$	$\psi(\psi(0))$	$C(C(C(\Omega_1,0),\Omega_1),0)$
Zeta zero	ζ_0	ζ_0	$\varphi(2,0)$	$[0]Next\ \omega$	$R_2R_1Hsuc\ 0$	$\psi(\Omega)$	$C(C(\Omega_1,\Omega_1),0)$
Eta zero	η_0	η_0	$\varphi(3,0)$	$[0]^2Next\ \omega$	$R_2(R_2R_1)Hsuc\ 0$		$C(C(\Omega,C(\Omega,\Omega)),0)$
			$\varphi(\omega,0)$	$[0]^\omega Next\ \omega$	$HR_2R_1Hsuc\ 0$		$C(C(C(0,\Omega_1),\Omega_1),0)$
Feferman -Schütte	Γ_0	Γ_0	$\varphi(1,0,0)$ $=\varphi(2\mapsto 1)$	$[1][0]Next\ \omega$	$R_3R_2R_1Hsuc\ 0$ $= R_{3\ldots1}Hsuc\ 0$	$\psi(\Omega^\Omega)$	$C(C(C(\Omega_1,\Omega_1),$ $\Omega_1),0)$
Ackermann			$\varphi(1,0,0,0)$ $=\varphi(3\mapsto 1)$	$[1]^2[0]Next\ \omega$	$R_3(R_3R_2)R_1Hsuc\ 0$	$\psi(\Omega^{\Omega^2})$	
Small Veblen ordinal			$\varphi(\omega\mapsto 1)$	$[1]^\omega[0]Next\ \omega$	$HR_3R_2R_1Hsuc\ 0$	$\psi(\Omega^{\Omega^\omega})$	$C(\Omega_1^\omega,0)$ $=C(C(C(C(0,\Omega_1),$ $\Omega_1),\Omega_1),0)$
Large Veblen ordinal			least ord. not rep.	$[2][1][0]Next\ \omega$	$R_4R_3R_2R_1Hsuc\ 0$ $=R_{4\ldots1}Hsuc\ 0$	$\psi(\Omega^{\Omega^\Omega})$	$C(\Omega_1^{\Omega_1},0)$ $=C(C(C(C(\Omega_1,\Omega_1),$ $\Omega_1),\Omega_1),0)$
Bachmann-Howard				least ord. not rep.	$R_{\omega\ldots1}Hsuc\ 0$	$\psi(\varepsilon_{\Omega+1})$	$C(C(\Omega_2,\Omega_1),0)$