Cardinals, Veblen and Simmons

There is a correspondence between inaccessible cardinals and Veblen and Simmons hierarchies.

In both case, there is a function f that, given some ordinal α, produces a greater ordinal $f(\alpha)$. A way to get large ordinals is to enumerate the fixed points of this function. For Veblen and Simmons hierarchies, this function is $\xi \mapsto \omega^\xi$ or $[\omega^\bullet]$, and for inaccessible cardinals it is $\xi \mapsto \aleph_\xi$ or $[\aleph_\bullet]$.

The least fixed point of $\xi \mapsto \omega^\xi$ is $\varepsilon_0 = \varepsilon'_1 = \varphi(1,0) = \varphi'(0,1)$. It is the limit of $\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots$. In a similar way, we can define $E_0 = E'_1 = \Phi(1,0) = \Phi'(0,1)$ as the least fixed point of $\xi \mapsto \aleph_\xi$, which is the limit of $\aleph_0, \aleph_{\aleph_0}, \aleph_{\aleph_{\aleph_0}}, \ldots$.

Then, like we have defined $\varepsilon_1 = \varepsilon'_2 = \varphi(1,1) = \varphi'(0,2)$ as the second fixed point of $\xi \mapsto \omega^\xi$, we can define $E_1 = E'_2$ as the second fixed point of $\xi \mapsto \aleph_\xi$, which is the limit of $E_0+1, \aleph_{E_0+1}, \aleph_{\aleph_{E_0+1}}, \ldots$.

More generally, like we defined $\varepsilon_\alpha = \varepsilon'_{1+\alpha}$ as the $1+\alpha$-th fixed point of $\xi \mapsto \omega^\xi$, we can define $E_\alpha = E'_{1+\alpha}$ as the $1+\alpha$-th fixed point of $\xi \mapsto \aleph_\xi$.

Then, like we defined $\zeta_0 = \zeta'1 = \varphi(2,0) = \varphi'(1,1)$ as the least fixed point of $\xi \mapsto \varepsilon_\xi$, the limit of $\varepsilon_0, \varepsilon_{\varepsilon_0}, \ldots$, we can define $Z_0 = Z'_1$ as the least fixed point of $\xi \mapsto E_\xi$, the limit of $E_0, E_{E_0}, \ldots$. This is the least ordinal κ such that $\kappa = E_\kappa = \kappa$-th fixed point of $\xi \mapsto \aleph_\xi$ (the "1+" being absorbed). This is the least weakly inaccessible ordinal.

We can also use the Simmons notation to produce weakly inaccessible cardinals.

Remember this notation :

$Fixf\zeta = f^\omega(\zeta+1)$ is the least fixed point of f that is strictly greater than ζ.

$[0]h = Fix(\alpha \mapsto h^\alpha 0)$

$[1]hg = Fix(\alpha \mapsto h^\alpha g0$

Like we defined the function NEXT $= Fix(\xi \mapsto \omega^\xi$ which gives the next ε ordinal after a given ordinal, we can define the function NEXT $= Fix(\xi \mapsto \aleph_\xi$ which gives the next fixed point of $\xi \mapsto \aleph_\xi$ after a given ordinal or cardinal. For example, NEXT 0 is the least fixed point of $\xi \mapsto \aleph_\xi, \mathrm{NEXT}(\mathrm{NEXT}\,0) = \mathrm{NEXT}^2\,0$ is the second one, and more generally $\mathrm{NEXT}^\alpha\,0$ is the α-th fixed point.

[0] NEXT 0 = Fix ($\alpha \mapsto \mathrm{NEXT}^\alpha\,0)0$ is the least κ such that $\kappa = \mathrm{NEXT}^\kappa\,0 = \kappa$-th fixed point of $\xi \mapsto \aleph_\xi$, which is the least weakly inaccessible cardinal.

More generally, $([0]\,\mathrm{NEXT})^\alpha 0 = Z'_\alpha = \Phi'(1,\alpha)$ is the α-th weakly inaccessible cardinal.

The least 1-weakly inaccessible cardinal is the least κ such that κ is the κ-th weakly inaccessible cardinal, which can be written $\kappa = ([0]\,\mathrm{NEXT})^\kappa 0$. This κ is $[0]([0]\,\mathrm{NEXT})0 = [0]^2\,\mathrm{NEXT}\,0 = \Phi(3,0) = \Phi'(2,1)$.

The α-th 1-weakly inaccessible cardinal is $[0]^2\,\mathrm{NEXT})^\alpha 0$.

The least 2-weakly inaccessible cardinal is the least κ such that κ is the κ-th 1-weakly inaccessible cardinal, which can be written $\kappa = ([0]^2\,\mathrm{NEXT})^\kappa 0$. This κ is $[0]([0]^2\,\mathrm{NEXT})0 = [0]^3\,\mathrm{NEXT}\,0 = \Phi(4,0) = \Phi'(3,1)$.

More generally, the least α-weakly inaccessible cardinal is $[0]^{1+\alpha}\,\mathrm{NEXT}\,0 = \Phi(2+a,0) = \Phi'(1+a,1)$ and the β-th α-weakly inaccessible cardinal is $([0]^{1+\alpha}\,\mathrm{NEXT})^\beta 0 = \Phi'(1+\alpha,\beta)$.

The least hyper-weakly inaccessible cardinal is the least κ such that κ is κ-inaccessible, which can be written $\kappa = [0]^\kappa\,\mathrm{NEXT}\,0$. This κ is $[1][0]\,\mathrm{NEXT}\,0 = \Phi(1,0,0)$.

The second one is $([1][0]\,\mathrm{NEXT})^2 0$, and more generally the α-th one is $([1][0]\,\mathrm{NEXT})^\alpha 0$.

Then, κ is 1-hyper-weakly inaccessible if κ is the κ-th hyper-weakly inaccessible cardinal, which can be written $\kappa = ([1][0]\,\mathrm{NEXT})^\kappa 0$. This is [0] ([1] [0] NEXT) 0. The second one is $([0]([1][0]\,\mathrm{NEXT}))^2 0$, and the α-th one is $([0]([1][0]\,\mathrm{NEXT}))^\alpha 0$.

Similarily, the least 2-hyper-weakly inaccessible cardinal is $[0]^2([1][0]\,\mathrm{NEXT})0$ and the α-th one is $([0]^2([1][0]\,\mathrm{NEXT}))^\alpha 0$.

More generally, the α-th β-hyper-weakly inaccessible cardinal is $([0]^\beta([1][0]\,\mathrm{NEXT}))^\alpha 0 = \Phi'(1,\beta,\alpha)$.

The least hyper-hyper-weakly inaccessible cardinal, or hyper^2 weakly inaccessible cardinal is the least κ such that κ is κ-hyper-weakly inaccessible, or $\kappa = [0]^\kappa([1][0]\,\mathrm{NEXT})0$, which is $[1][0]([1][0]\,\mathrm{NEXT})0 = ([1][0])^2\,\mathrm{NEXT}\,0$.

More generally, the least hyper$^\gamma$-weakly inaccessible cardinal is $([1][0])^\gamma\,\mathrm{NEXT}\,0$, and the α-th one is $(([1][0])^\gamma\,\mathrm{NEXT})^\alpha 0$.

The least 1-hyper$^\gamma$-weakly inaccessible cardinal is the least κ such that κ is the κ-th hyper$^\gamma$-weakly inaccessible cardinal, or $\kappa = (([1][0])^\gamma\,\mathrm{NEXT})^\kappa 0$. This κ is $[0](([1][0])^\gamma\,\mathrm{NEXT})0$.

More generally, the least β-hyper$^\gamma$-weakly inaccessible cardinal is $[0]^\beta(([1][0])^\gamma\,\mathrm{NEXT})0$.

Finally, the α-th β-hyper$^\gamma$-weakly inaccessible cardinal is $([0]^\beta(([1][0])^\gamma\,\mathrm{NEXT}))^\alpha 0 = \Phi'(\gamma,\beta,\alpha)$.