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Level-4 Honours project

Serre's Global Dimension Theorem

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Abstract

A proof of Serre's global dimension theorem is presented using homological techniques. The proof avoids the classical construction of a Koszul complex and instead uses methods developed by Auslander and Buchsbaum. The eponymous theorem characterises regular rings as precisely those of finite global dimension, which allows for the application of homological techniques in commutative algebra.

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1 Introduction

In commutative algebra, the global dimension of a commutative ring is an important invariant that measures the complexity of the ring. The global dimension of a ring is defined as the supremum of the projective dimensions of all modules over the ring. In 1955, the mathematician Jean-Pierre Serre proved a deep and powerful theorem that relates the global dimension of a ring, a homological condition, to its property of being regular, an ideal theoretic condition. This result has been used to study the singularities of algebraic varieties and to prove important results classifying regular rings.

In this paper we will provide a detailed introduction to Serre's global dimension theorem and its connections to commutative algebra. We will begin in section 2 by reviewing some basic ring and module theory and will look at some examples. It is here where we will introduce the notion of a regular local ring which is the type of ring Serre's theorem sets out to classify. We then introduce the concept of localisation of a ring which allows us to simplify computations and study their behaviour by looking at a simplified version of the ring which admits desirable properties.

In section 3 we provide a brief introduction to category theory, with the goal of motivating the definition of an abelian category by realising all of the nice properties of the category of modules over a commutative ring and abstracting them. We end this section with an overview of the Freyd-Mitchell embedding theorem and an example of its usefulness in proving results for abelian categories by simplifying the problem to one involving the category of modules over a ring.

With the definition of an abelian category in hand, we move on to section 4 where we introduce some basic homological algebra tools such as exact sequences and the Ext functor. These allow us to understand the definition of the global dimension of a ring, which describes the complexity of a ring as a module over itself.

Finally in section 5 we relate regular local rings with their global dimension by proving Serre's global dimension theorem, bringing together all of the work in the paper. We conclude by sketching a proof of the fact that regular local rings are stable under localisation and use this fact to define regular rings, which are no longer dependent on being local.

2 Rings and modules

Throughout this paper all rings are tacitly assumed to be commutative unital rings unless stated otherwise. We begin by defining some special types of rings and modules and will proceed to study the idea of localisation which is a process by which we can simplify problems involving complicated rings to simpler ones by creating new local ones by using their prime ideals.

2.1 Rings

We assume that the reader has taken an introductory course in ring and ideal theory. In particular we assume the definition of a ring, ideal, ring homomorphism, maximal and prime ideals and the spectrum of a ring, all of which can be found in ([2], chapter 1).

Definition 2.1. (Local Ring) A ring R is said to be **local** if it has precisely one maximal ideal $\mathfrak{m}$, and we denote this as $(R, \mathfrak{m})$.

Local rings are ubiquitous in commutative algebra and arise naturally from a process known as localisation which we will see shortly.

Definition 2.2 (Noetherian Ring)**.** Let R be a ring. We say that R is **Noetherian** if it satisfies the ascending chain condition. This means that for every increasing sequence of ideals $I_1 \subseteq I_2 \subseteq \cdots$ in R there exists some $N \in \mathbb{N}$ such that for every $n \geq N$, $I_n = I_N$.

This definition makes apparent that Noetherian rings are interesting as they admit a finiteness property. However, this definition is evidently difficult to work with, and so we may reformulate the definition as follows.

Proposition 2.1. A ring R is Noetherian if and only if every ideal of R is finitely generated.

Proof. First suppose that R is a Noetherian ring. Let I be an ideal of R, and define:

$$\mathcal{S} = \{K : K \trianglelefteq R, K \subseteq I, K \text{ is finitely generated}\}.$$

The trivial ideal is certainly in $\mathcal{S}$, and so $\mathcal{S}$ is non-empty. It follows that $\mathcal{S}$ must contain a maximal element M. Since M is finitely generated, it has a basis $\mathcal{B} = \{r_1, \ldots, r_n\}$ with $M = \langle \mathcal{B} \rangle$. Now let $x \in I$ be arbitrary and let $J = \langle r_1, \ldots, r_n, x \rangle$. Clearly $J \trianglelefteq R, J \subseteq I$ and J is finitely generated, so that $J \in \mathcal{S}$. This then means that since $M \subseteq J$ and since M is maximal, we must have that $M = J$. Since $x \in I$ was arbitrary, we have that $I \subseteq M$. Since $M \subseteq I$ it follows that $I = M$ and therefore I is finitely generated.

Conversely suppose that $I_1 \subseteq I_2 \subseteq \cdots$ is an increasing sequence of ideals of R and define $I = \bigcup_{n=1}^{\infty} I_n$. Since countable unions of ideals are ideals, we have that I is an ideal and in particular it is finitely generated with some basis $\mathcal{B} = \{r_1, \ldots, r_m\}$. For each $i \in \{1, \ldots, m\}$ we have that $r_i \in I$, which means that there exists some $n_i \in \mathbb{N}$ such that each $r_i \in I_{n_i}$. Define $N = \max\{n_1, \ldots, n_m\}$. Then each $n_i \leq N$ so for each $i \in \{1, \ldots, m\}$ we have that $r_i \in I_{n_i} \subseteq I_N$, so that $I \subseteq I_N$. We have that if $j \geq N$ then $I_j \subseteq I \subseteq I_N \subseteq I_j$ so that $I_j = I_N$. Therefore the ascending chain condition is satisfied and so R is a Noetherian ring. $\square$

Examples. We provide some examples of Noetherian rings.

- Any field is a Noetherian ring because fields have only two ideals, namely itself and (0).
- Principal ideal rings are Noetherian because every ideal is generated by a single element.
- The Hilbert basis theorem (see [7], page 77) states that for any field k, the polynomial ring $k[x_1, \ldots, x_n]$ is Noetherian, since every ideal is finitely generated.

Definition 2.3. (Krull Dimension) Let R be a ring. Given $\mathfrak{p} \in \operatorname{Spec}(R)$, we define its **height** to be:

$$\operatorname{ht}(\mathfrak{p}) = \sup\{n : \text{there is an increasing sequence of distinct prime ideals } \mathfrak{p}_0 \subseteq \mathfrak{p}_1 \subseteq \cdots \subseteq \mathfrak{p}_n = \mathfrak{p}\}.$$

We then define the **Krull dimension** $\dim(R)$ of R to be:

$$\dim(R) = \sup\{\operatorname{ht}(\mathfrak{p}) : \mathfrak{p} \in \operatorname{Spec}(R)\}.$$

Examples. We provide some examples of rings and their Krull dimensions.

- Any field k has only one prime ideal, namely the trivial one. Therefore all fields have Krull dimension 0. If we consider the polynomial ring $k[x]$ we find that it has Krull dimension 1 since in that case every ideal is principal.
- In the ring $\mathbb{Z}$ the ideals are the trivial one and those of the form $p\mathbb{Z}$ for prime numbers p. It follows that every ideal of the form $p\mathbb{Z}$ is maximal, and therefore $\dim(\mathbb{Z}) = 1$.
- Every Noetherian local ring has finite Krull dimension but there do exist Noetherian rings of infinite Krull dimension, as demonstrated by Nagata (see [17], page 203).

Definition 2.4. (Embedding Dimension) Let $(R, \mathfrak{m})$ be a local ring. We define the **embedding dimension** $emb.\dim(R)$ to be the dimension of $\mathfrak{m}/\mathfrak{m}^2$ as a $R/\mathfrak{m}$-vector space.

We will take for granted that if $(R, \mathfrak{m})$ is a Noetherian local ring, then we have the inequality $\dim(R) \leq emb.\dim(R)$. This result is explained in ([15], page 78). With this theory in hand, we are now able to introduce the main character in this paper.

Definition 2.5. (Regular Local Ring) Let $(R, \mathfrak{m})$ be a Noetherian local ring. We say that R is a **regular local ring** if $emb.\dim(R) = \dim(R)$.

2.2 Modules

We move on to defining modules which are one of the central focuses of commutative algebra. Modules are generalisations of vector spaces, in the sense that the scalars now come from a ring instead of a field. Let us briefly recall the definition of a module.

Definition 2.6. (Module) Let R be a ring. An abelian group $(M, +)$ is said to be a left R-**module** if there exists a mapping $\cdot : R \times M \to M$ for which the following hold for all $a, b \in R$ and all $x, y \in M$:

1. $(a + b) \cdot x = a \cdot x + b \cdot x$,
2. $a \cdot (x + y) = a \cdot x + a \cdot y$,
3. $(ab) \cdot x = a \cdot (b \cdot x)$,
4. $1 \cdot x = x$.

A right R-module is defined in a similar fashion and these definitions coincide since R is commutative.

Moreover a R-**submodule** M' is a subgroup $(M', +)$ of $(M, +)$ which is closed under such a mapping. In other words for all $r \in R$ and $m \in M'$, we have that $r \cdot m \in M'$.

Examples. We look at some examples of modules.

- If k is a field, then k-vector spaces are k-modules.
- If $R = \mathbb{Z}$ then every abelian group G is a $\mathbb{Z}$-module. In particular if a group G is a $\mathbb{Z}$-module, then any subgroup $H \leq G$ is a submodule.
- If R any ring, then the set of all ideals in R forms a R-module with the operations being addition and multiplication of ideals.

We now want to define module homomorphisms. Just as modules are generalisations of vector spaces, module homomorphisms arise from generalising the notion of a linear map.

Definition 2.7. (Module Homomorphism) Let R be a ring and let M, N be R-modules. We say that $f : M \to N$ is a R-**module homomorphism** if for all $x, y \in M$ and all $r \in R$ we have $f(x + y) = f(x) + f(y)$ and $f(rx) = rf(x)$.

R-module homomorphisms are incredibly useful as they allow us to relate the structure of modules to each other and give rise to some powerful results, which are analogous to the isomorphism theorems of group theory. These will not be discussed here but the interested reader is referred to [2]. We will also use R-module homomorphisms when constructing the category R-**Mod** later on.

We now proceed to discuss an interesting type of module with some special properties.

Definition 2.8. (Free Modules) Let R be a ring and let M be a R-module. We say that M is **free** as a R-module if it has a basis. In that case it is isomorphic to:

$$\bigoplus_{i \in \mathcal{I}} M_i$$

for some family $\{M_i : i \in \mathcal{I}\}$ of R-modules, where each $M_i \cong R$.

Examples. We give some examples of free modules.

- If k is a field then every k-vector space has a basis and is therefore a free k-module.
- Let R be a ring. Then R is a free R-module with basis $\{1\}$. More generally if $n \in \mathbb{N}$ then $\bigoplus_{i \in \{1,\ldots,n\}} R = R^n$ is a free R-module.
- $\mathbb{Z}/2\mathbb{Z}$ is not a free $\mathbb{Z}$-module. The only plausible basis would be $\{1 + 2\mathbb{Z}\}$, however this is not linearly independent as $2(1 + 2\mathbb{Z}) = 0 + 2\mathbb{Z}$.

2.3 Localisation

Localisation is a central tool of commutative algebra which can be intuitively understood as a method of honing in on a special part of a ring in order to give it some properties the original ring previously lacked. It allows us to generalise the idea of embedding $\mathbb{Z}$ into $\mathbb{Q}$ by constructing fractions to general rings. We will introduce the idea of local properties which are properties of a ring which are stable under localisation.

Definition 2.9. (Multiplicatively Closed Subset) Let R be a ring. We define a **multiplicatively closed subset** of R to be a subset $S \subseteq R$ such that $1 \in S$ and S is closed under multiplication. More concisely, S is a submonoid of the multiplicative monoid of R.

We want to generalise the idea given in the exposition by defining an equivalence relation as follows.

Proposition 2.2. Let $\sim$ be a relation on $R \times S$ defined as follows:

$$(a, s) \sim (b, t) \iff \exists\, u \in S \text{ such that } (at - bs)u = 0.$$

Then $\sim$ defines an equivalence relation on $R \times S$.

Proof. We will show each of the properties of an equivalence relation in turn.

1. Reflexivity: Let $(a, s) \in R \times S$. Then $(as - as) \cdot 1 = 0 \cdot 1 = 0$ and so $(a, s) \sim (a, s)$.

2. Symmetry: Suppose that $(a, s), (b, t) \in R \times S$ such that $(a, s) \sim (b, t)$. Then there exists a $u \in S$ such that $(at-bs)u = 0$. It follows that $-(at-bs)u = (bs-at)u = 0$ and so $(b, t) \sim (a, s)$.

3. Transitivity: Suppose that $(a, s), (b, t), (c, w) \in R \times S$ such that $(a, s) \sim (b, t)$ and $(b, t) \sim (c, w)$. This means that there exist $u, v \in S$ such that:

   $$\begin{cases} (at - bs)u = 0, \\ (bw - ct)v = 0. \end{cases}$$

   It follows that $atu = bsu$ and $bvw = ctv$. Therefore $atuvw = bsuvw$ and $bsuvw = cstuv$, from which it follows that $atuvw = cstuv$ and so $(aw - cs)tuv = 0$.

   Since S is a multiplicatively closed subset we have that $tuv \in S$ and therefore $(a, s) \sim (c, w)$.

$\sim$ is reflexive, symmetric and transitive and therefore defines an equivalence relation on $R \times S$. $\square$

In the context of the above proposition, we let a/s denote the equivalence class $(a, s)_\sim$ and write $S^{-1}R$ for the set of all such equivalence classes.

We can now rigorously define our notion of fractions as follows.

Definition 2.10. (Ring Of Fractions) Let R be a ring and let S be a multiplicatively closed subset of R. We can endow $S^{-1}R$ with a ring structure by defining addition and multiplication as follows:

$$a/s + b/t = (at + bs)/st,$$

and

$$(a/s)(b/t) = ab/st.$$

We take for granted that these operations are well defined, that $0/1$ is the additive identity and that $1/1$ is multiplicative unity. In this case we call $S^{-1}R$ together with these operations the **ring of fractions** of R with respect to S.

This gives rise to a canonical ring homomorphism $f : R \to S^{-1}R$ defined by $f(r) = r/1$. The construction of a ring of fractions admits a universal property which can be characterized as follows.

Proposition 2.3. Let A, B be rings and let S be a multiplicatively closed subset of A. Let f be as above and let $g : A \to B$ be a ring homomorphism such that $g(s)$ is a unit in B for every $s \in S$. Then there exists a unique ring homomorphism $h : S^{-1}A \to B$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[rr, "g"] \arrow[dr, "f"'] & & B \\ & S^{-1}A \arrow[ur, dashed, "\exists!\, h"'] & \end{tikzcd}$$

Proof. Assuming the existence of such a h, we must have that for all $a \in A$:

$$h(a/1) = (h \circ f)(a) = g(a).$$

Moreover if $s \in S$ then:

$$h(1/s) = h((s/1)^{-1}) = h(s/1)^{-1} = g(s)^{-1}.$$

Putting these results together we have that if $a/s \in S^{-1}A$ then:

$$h(a/s) = h(a/1)h(1/s) = g(a)g(s)^{-1},$$

and therefore g uniquely determines h. It is clear that h respects addition and multiplication on $S^{-1}A$ so it will suffice to show that h is well defined. Assume that $a/s = a'/s'$ so that there is some $u \in S$ with $(as' - a's)u = 0$. It follows that:

$$(g(a)g(s') - g(a')g(s))\, g(u) = 0.$$

Since $g(u)$ is a unit in B, it follows that $g(a)g(s)^{-1} = g(a')g(s')^{-1}$ which proves that h is well defined and therefore that it is indeed a ring homomorphism. □

Proposition 2.4. Let R be a ring and let $\mathfrak{p}$ be an ideal in R. Then $S = R\backslash\mathfrak{p}$ is a multiplicatively closed subset of R if and only if $\mathfrak{p} \in \mathrm{Spec}(R)$.

Proof. Let $\mathfrak{p} \in \mathrm{Spec}(R)$. Since $1 \notin \mathfrak{p}$ we have that $1 \in R\backslash\mathfrak{p}$. We can then take the contrapositive of the definition of $\mathfrak{p}$ being a prime ideal to get:

$$x \in R\backslash\mathfrak{p} \text{ and } y \in R\backslash\mathfrak{p} \implies xy \in R\backslash\mathfrak{p},$$

which is precisely the definition of $R\backslash\mathfrak{p}$ being a multiplicatively closed subset of R.

Conversely suppose that $\mathfrak{p}$ is an ideal in R and that $R\backslash\mathfrak{p}$ is a multiplicatively closed subset of R. Since $1 \in R\backslash\mathfrak{p}$ we have that $1 \notin \mathfrak{p}$ and so $\mathfrak{p}$ is a proper ideal. We can then take the contrapositive of the definition of $R\backslash\mathfrak{p}$ being a multiplicatively closed subset of R to get:

$$xy \in \mathfrak{p} \implies x \in \mathfrak{p} \text{ or } y \in \mathfrak{p},$$

which means that $\mathfrak{p} \in \mathrm{Spec}(R)$. □

Notation. We write $R_\mathfrak{p}$ for $S^{-1}R$ in the case where $S = R\backslash\mathfrak{p}$ for some $\mathfrak{p} \in \mathrm{Spec}(R)$. This particular type of ring of fractions admits lots of nice properties which we will now explore.

Proposition 2.5. Let R be a ring and let $\mathfrak{p} \in \mathrm{Spec}(R)$. Then the set:

$$\mathfrak{m} = \{a/s : a \in \mathfrak{p} \text{ and } s \in R\backslash\mathfrak{p}\}$$

is the unique maximal ideal in $R_\mathfrak{p}$.

Proof. It is straightforward to verify that $\mathfrak{m}$ forms an ideal in $R_\mathfrak{p}$. Now suppose that $\mathfrak{q}$ is an ideal in $R_\mathfrak{p}$ such that $\mathfrak{q} \not\subseteq \mathfrak{m}$. Then $\mathfrak{q}$ contains an element a/s such that $a \in R\backslash\mathfrak{p}$. We then have that $s/a \in R_\mathfrak{p}$ and so a/s is a unit, which forces $\mathfrak{q} = R_\mathfrak{p}$. □

It follows from this proposition that $R_\mathfrak{p}$ is in fact a local ring.

Definition 2.11. (Localisation) Let R be a ring and let $\mathfrak{p} \in \mathrm{Spec}(R)$. We call the local ring $R_\mathfrak{p}$ the **localisation** of R at $\mathfrak{p}$.

2.4 Examples and applications of localisation

Examples. We give some examples of the localisation of rings.

- The localisation of $\mathbb{Z}$ at the trivial prime ideal (0) is $\mathbb{Q}$. In general if a ring R is an integral domain, the localisation of R at the trivial ideal is called the **field of fractions** of R.
- Let p be a prime number. We have that $\mathbb{Z}_{(p)} = \{a/s : a, s \in \mathbb{Z} \text{ and } p \nmid s\}$.
- Let R be a ring and consider the polynomial ring over R in one variable, $R[x]$. The localisation of $R[x]$ at $\mathfrak{p} = \{x^n : n \in \mathbb{N}_0\} \in \operatorname{Spec}(R[x])$ yields the ring of Laurent polynomials over x, denoted by $R[x, x^{-1}]$. These differ from the polynomials we are used to since we now have terms of negative degree.

Localisation is an incredibly useful tool as it allows for the construction of rings with desirable properties. It is therefore natural to study which properties of a ring are preserved under localisation.

Definition 2.12. (Local Property) Let R be a ring. We say that a property P of R is a **local property** if for every $\mathfrak{p} \in \operatorname{Spec}(R)$ we have:

$$R \text{ has } P \iff R_{\mathfrak{p}} \text{ has } P.$$

A trivial example of a local property is that of being the trivial ring. An important result we can now state is that every localisation of a Noetherian ring R is also Noetherian. To prove this we will need the following proposition.

Proposition 2.6. Let R be a ring and let $\mathfrak{p} \in \operatorname{Spec}(R)$. Then every ideal of $R_{\mathfrak{p}}$ is of the form $f(I)R_{\mathfrak{p}}$, where I is an ideal in R and $f : R \to R_{\mathfrak{p}}$ is the canonical ring homomorphism.

Proof. First let J be an ideal in $R_{\mathfrak{p}}$ so that $I = f^{-1}(J)$ is an ideal in R. Let $a/s \in J$ and observe that:

$$(a/s)f(s) = f(a) \in J \implies a \in f^{-1}(J) = I.$$

We then have that:

$$a/s = f(a)(1/s) \in f(I)R_{\mathfrak{p}},$$

and so $J \subseteq f(I)R_{\mathfrak{p}}$. Conversely since J is an ideal it follows that $f(I)R_{\mathfrak{p}} \subseteq JR_{\mathfrak{p}} = J$ and therefore we have established equality. □

Proposition 2.7. Suppose that R is a Noetherian ring and that $\mathfrak{p} \in \operatorname{Spec}(R)$. Then $R_{\mathfrak{p}}$ is also a Noetherian ring.

Proof. Let $J_1 \subseteq J_2 \subseteq \cdots$ be an increasing sequence of ideals in $R_{\mathfrak{p}}$. Then by proposition 2.6, any J_i is of the form $f(I_i)R_{\mathfrak{p}}$ for an ideal I_i of R. Then we may rewrite the increasing sequence as:

$$f(I_1)R_{\mathfrak{p}} \subseteq f(I_2)R_{\mathfrak{p}} \subseteq \cdots$$

Now since R is Noetherian there exists a $N \in \mathbb{N}$ such that for every $n \geq N, I_n = I_N$ and therefore $f(I_n) = f(I_N)$. Therefore $R_{\mathfrak{p}}$ satisfies the ascending chain condition and so is a Noetherian ring. □

Remark. Being Noetherian is not a local property as the converse is not true. Indeed, consider:

$$R = \bigoplus_{i=1}^{\infty} \mathbb{Z}/2\mathbb{Z}.$$

Then R is not a Noetherian ring since it is not finitely generated. However we find that its localisation at any prime ideal is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, which is Noetherian.

3 Categories

We will now move on to provide a general overview of category theory. Categories are a collection of objects with a shared property together with morphisms between them, which allow us to make universal statements about many mathematical fields all at once. Our goal will be to introduce the notion of an abelian category which is the general setting of homological algebra, of which our main example will be R-**Mod**. We will also see the Freyd-Mitchell embedding theorem which allows us to develop abstract theory by focusing on the category of R-modules.

3.1 Introduction to category theory

Our fundamental definitions in category theory will closely follow those presented in [14].

Definition 3.1. (Category) A **category** $\mathcal{C}$ is a class of objects $\mathrm{ob}(\mathcal{C})$ together with a class of morphisms $\hom(\mathcal{C})$ such that for every $f \in \hom(\mathcal{C})$, f has a unique domain $X \in \mathrm{ob}(\mathcal{C})$ and a unique codomain $Y \in \mathrm{ob}(\mathcal{C})$.

The structure on $\hom(\mathcal{C})$ is provided by the obvious binary operation of composition, denoted $\circ$. For a pair $f, g \in \hom(\mathcal{C})$ for which the codomain of f is the domain of g, we define $\circ(f, g) = g \circ f$. Moreover we demand the following two axioms of composition:

1. Composition must be associative.
2. Every object must have an identity morphism. In other words, for every $X \in \mathrm{ob}(\mathcal{C})$ there must exist a morphism $1_X : X \to X \in \hom(\mathcal{C})$ such that for all morphisms $f : A \to X$ and $g : X \to B$ we have that $1_X \circ f = f$ and $g \circ 1_X = g$.

$$
\begin{tikzcd}
A \arrow[loop above, "1_A"] \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[loop above, "1_B"] \arrow[d, "g"] \\
 & C \arrow[loop below, "1_C"]
\end{tikzcd}
$$

Remark. Notice that in the above definition we referred to $\mathrm{ob}(\mathcal{C})$ and $\hom(\mathcal{C})$ as classes, not sets. A **class** is a collection of objects which can be characterized by a shared property. These are carefully defined as such so as to avoid running into problems such as Russell's paradox when constructing some categories. A **proper class** is a class which is not a set, such as the class of all sets.

Examples. We provide some examples of categories.

1. **Set** is the category in which the objects are sets and the morphisms are functions between them.
2. **Grp** is the category in which the objects are groups and the morphisms are group homomorphisms.
3. $\mathbf{Top}_*$ is the category in which the objects are based topological spaces and the morphisms are basepoint preserving continuous maps.
4. If k is any field, then $\mathbf{Vect}_k$ is the category in which the objects are k-vector spaces and the morphisms are k-linear maps.

The most important category for our purposes will be R-**Mod**, in which the objects are R-modules and the morphisms are module homomorphisms.

Proposition 3.1. *R-**Mod** forms a category for any ring R.*

Proof. Let ob(R-**Mod**) be the class of all R-modules, and let hom(R-**Mod**) be the class of all R-module homomorphisms. Since all R-module homomorphisms are functions and function composition is associative, it follows that composition of R-module homomorphisms is associative.

Moreover for every $X \in \text{ob}(R\text{-}\mathbf{Mod})$ we can define $1_X : X \to X$ by $1_X(x) = x$. Then if $f : A \to X$ and $g : X \to B \in \text{hom}(R\text{-}\mathbf{Mod})$, we have that for every $x \in X$:

$$(1_X \circ f)(x) = 1_X(f(x)) = f(x),$$

and

$$(g \circ 1_X)(x) = g(1_X(x)) = g(x).$$

Therefore $1_X \circ f = f$ and $g \circ 1_X = g$, so R-**Mod** forms a category. □

There is a simple notion of duality here which allows us to construct a potentially new category.

Definition 3.2. (Opposite Category) Let $\mathcal{C}$ be a category. We define the **opposite category** $\mathcal{C}^{\text{op}}$ to be the category with $\text{ob}(\mathcal{C}') = \text{ob}(\mathcal{C})$, but we reverse the direction of each morphism.

It is natural to look for some sub-structure in categories and we can do so with subcategories in the natural way.

Definition 3.3. (Subcategory) Let $\mathcal{C}$ be a category. A **subcategory** $\mathcal{S}$ of $\mathcal{C}$ is a class $\text{ob}(\mathcal{S})$ contained in $\text{ob}(\mathcal{C})$ together with a class $\text{hom}(\mathcal{S})$ contained in $\text{hom}(\mathcal{C})$ such that the following conditions hold:

1. If $X \in \text{ob}(\mathcal{S})$ then $1_X \in \text{hom}(\mathcal{S})$.
2. If $f : X \to Y \in \text{hom}(\mathcal{S})$, then $X, Y \in \text{ob}(\mathcal{S})$.
3. If $f : X \to Y$ and $g : Y \to Z$ are both in $\text{hom}(\mathcal{S})$, then $g \circ f \in \text{hom}(\mathcal{S})$.

The next natural question is to ask how we can study relations between categories. In order to do so we will introduce functors, which are maps we will define on the objects and morphisms of a category. We will require them to preserve all of the axioms of a category; that is, they must take identity morphisms to identity morphisms and must also preserve composition of morphisms.

It is therefore natural to assume that functors must therefore preserve the orientation of all morphisms. This type of functor is called a covariant functor. Interestingly, we actually discover that this is not a requirement. The so called contravariant functor reverses the orientation of every morphism and also preserves the axioms of a category. We will now formally define both of these functors.

Definition 3.4. (Covariant Functor) Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A **covariant functor** $F : \mathcal{C} \to \mathcal{D}$ is a map which maps objects $X \in \text{ob}(\mathcal{C})$ to objects $F(X) \in \text{ob}(\mathcal{D})$ and morphisms $f : X \to Y \in \text{hom}(\mathcal{C})$ to morphisms $F(f) : F(X) \to F(Y) \in \text{hom}(\mathcal{D})$. Moreover, the following conditions are satisfied for all $X \in \text{ob}(\mathcal{C})$ and all $f, g \in \text{hom}(\mathcal{C})$ for which $g \circ f$ is well-defined:

1. $F(1_X) = 1_{F(X)}$.
2. $F(g \circ f) = F(g) \circ F(f)$.

The definition of a contravariant functor is completely analogous but instead reverses the orientations of each morphism.

Definition 3.5. (Contravariant Functor) Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A **contravariant functor** $F : \mathcal{C} \to \mathcal{D}$ is a map which maps objects $X \in \mathrm{ob}(\mathcal{C})$ to objects $F(X) \in \mathrm{ob}(\mathcal{D})$ and morphisms $f : X \to Y \in \hom(\mathcal{C})$ to morphisms $F(f) : F(Y) \to F(X) \in \hom(\mathcal{D})$. Moreover, the following conditions are satisfied for all $X \in \mathrm{ob}(\mathcal{C})$ and all $f, g \in \hom(\mathcal{C})$ for which $g \circ f$ is well-defined:

1. $F(1_X) = 1_{F(X)}$.
2. $F(g \circ f) = F(f) \circ F(g)$.

Remark. A contravariant functor $F : \mathcal{C} \to \mathcal{D}$ can be viewed as a covariant functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

We now want to introduce some special types of categories, in particular the abelian category, which has many uses in commutative and homological algebra. First we introduce the idea of a Hom-set.

Definition 3.6. (Hom-set) Let $\mathcal{C}$ be a category and let $X, Y \in \mathrm{ob}(\mathcal{C})$. We define the **Hom-set** of X and Y to be the collection of all morphisms in $\hom(\mathcal{C})$ with domain X and codomain Y, which we denote by $\mathrm{Hom}_{\mathcal{C}}(X, Y)$.

An important observation we can make about R-**Mod** is that we can endow the Hom-sets with the structure of a R-module. Indeed if $M, N \in \mathrm{ob}(R\text{-}\mathbf{Mod})$ then for all $f, g \in \mathrm{Hom}_{R\text{-}\mathbf{Mod}}(M, N)$ and all $r \in R$ we can define:

$$(f + g)(m) = f(m) + g(m),$$

and

$$(rf)(m) = rf(m).$$

In particular we can notice that with these definitions we have given $\mathrm{Hom}_{R\text{-}\mathbf{Mod}}(M, N)$ the structure of an abelian group, which leads us on to the following definition.

Definition 3.7. (Preadditive Category) Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is a **preadditive category** if for every $X, Y \in \mathrm{ob}(\mathcal{C})$, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ has the structure of an abelian group and compositions of morphisms are bilinear. In particular, if we have a sequence of morphisms in $\mathcal{C}$ of the form:

$$\begin{tikzcd} W \arrow[r, "f"] & X \arrow[r, shift left, "g'"] \arrow[r, shift right, "g"'] & Y \arrow[r, "h"] & Z \end{tikzcd}$$

we have that:

$$h \circ (g + g') \circ f = (h \circ g \circ f) + (h \circ g' \circ f)$$

in $\mathrm{Hom}_{\mathcal{C}}(W, Z)$.

Examples. We give some examples of preadditive categories.

- The simplest example of a preadditive category is **Ab**, the category of abelian groups. For all abelian groups A, B we have that $\mathrm{Hom}_{\mathbf{Ab}}(A, B)$ contains the trivial homomorphism, addition of group homomorphisms is associative and each group homomorphism f has inverse $-f$, where for each $a \in A$ we have that $(-f)(a) = -f(a)$. Moreover composition of group homomorphisms is bilinear.
- If k is any field, then the category $\mathbf{Vect}_k$ is a preadditive category for similar reasons.
- The structure given above for R-**Mod** is also bilinear, and so R-**Mod** is a preadditive category.

We now introduce zero objects, the definition of which will be exactly what one would expect.

Definition 3.8. (Zero Object) Let $\mathcal{C}$ be a category and let $I \in \text{ob}(\mathcal{C})$. We say that I is an **initial object** if for all X in $\text{ob}(\mathcal{C})$, there exists exactly one morphism $I \to X$. We say that I is a **terminal object** if for all $X \in \text{ob}(\mathcal{C})$, there exists exactly one morphism $X \to I$. If I is both an initial object and a terminal object, we say that it is a **zero object**.

Examples. We give some examples of categories which have initial, terminal and zero objects.

- In **Set**, the empty set is the unique initial object and the only terminal objects are the singletons. It follows that **Set** has no zero objects.
- In **Grp**, the trivial group is both an initial object and a terminal object, so is therefore a zero object. The same is true for the trivial module in R-**Mod**.
- The category of fields **Field** has no initial or terminal objects and hence no zero objects. This follows directly from the standard fact that there are no field homomorphisms between fields of different characteristics.

In a similar vein we want to define zero morphisms, which have relatively 'boring' properties.

Definition 3.9. (Zero Morphism) Let $\mathcal{C}$ be a category and let $f : X \to Y \in \text{hom}(\mathcal{C})$. We say that f is a **left zero morphism** if for any $W \in \text{ob}(\mathcal{C})$ and any $g, h : W \to X \in \text{hom}(\mathcal{C})$, we have that $f \circ g = f \circ h$. We say that f is a **right zero morphism** if for any $Z \in \text{ob}(\mathcal{C})$ and any $g, h : Y \to Z \in \text{hom}(\mathcal{C})$, we have that $g \circ f = h \circ f$. We say that f is a **zero morphism** if it is both a left zero morphism and a right zero morphism. If every Hom-set in $\mathcal{C}$ has a zero morphism, we say that $\mathcal{C}$ **has zero morphisms**.

Definition 3.10. (Equaliser, Coequaliser) Let $\mathcal{C}$ be a category, let $X, Y \in \text{ob}(\mathcal{C})$ and $f, g : X \to Y$. We define the **equaliser** of f and g to be a pair consisting of an object $E \in \text{ob}(\mathcal{C})$ and a morphism $e : E \to X \in \text{hom}(\mathcal{C})$ such that the following conditions hold:

1. $f \circ e = g \circ e$.
2. If another pair $(E', e' : E' \to X)$ satisfies condition 1, then there exists a unique morphism $\eta : E' \to E$ such that $e' = e \circ \eta$. In other words, the following diagram commutes:

$$\begin{tikzcd}
E \arrow[r, "e"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\
E' \arrow[u, dashed, "\eta"] \arrow[ur, "e'"'] & &
\end{tikzcd}$$

Similarly the **coequaliser** of f and g is a pair consisting of an object $Q \in \text{ob}(\mathcal{C})$ and a morphism $q : Y \to Q \in \text{hom}(\mathcal{C})$ such that the following conditions hold:

1. $q \circ f = q \circ g$.
2. If another pair $(Q', q' : Q' \to X)$ satisfies condition 1, then there exists a unique morphism $\zeta : Q \to Q'$ such that $q' = \zeta \circ q$. In other words, the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "q"] \arrow[dr, "q'"'] & Q \arrow[d, dashed, "\zeta"] \\
& & Q'
\end{tikzcd}$$

Now we continue to abstract some more familiar concepts into the language of category theory. Another important notion we are familiar with in most settings from working with morphisms is that of kernels and cokernels. The following definitions are quite different than those we are used to seeing in concrete settings, but boil down to the same thing.

Definition 3.11. (Kernel, Cokernel) Let $\mathcal{C}$ be a category with zero morphisms. Let $X, Y \in \text{ob}(\mathcal{C})$ and $f : X \to Y \in \text{hom}(\mathcal{C})$. We define a **kernel** of f to be an equaliser of f and the zero morphism from X to Y. Similarly we define a **cokernel** of f to be a coequaliser of f and the zero morphism from X to Y.

R-**Mod** has both kernels and cokernels. To see that this is the case, let $M, N \in \text{ob}(R\text{-}\mathbf{Mod})$ and let $\varphi : M \to N \in \text{hom}(R\text{-}\mathbf{Mod})$. Then the kernel of φ is given by $(\ker\varphi, i : \ker\varphi \to M)$, where i is inclusion. Similarly the cokernel of φ is given by $\left(\frac{N}{\text{im}\varphi}, \pi : N \to \frac{N}{\text{im}\varphi}\right)$, where π is projection.

We now want to define a way of taking the product of two objects in a category. In doing so we want to emulate the process involved in constructing the product of two concrete things we already understand, such as the direct product of two groups or the cartesian product of two sets.

Definition 3.12. (Product) Let $\mathcal{C}$ be a category and let $X, Y \in \text{ob}(\mathcal{C})$. A **product** of X and Y is an object in $\text{ob}(\mathcal{C})$, denoted $X \times Y$, together with a pair of morphisms $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y \in \text{hom}(\mathcal{C})$ called projection morphisms such that for every $Z \in \text{ob}(\mathcal{C})$ and every pair of morphisms $f_1 : Z \to X$ and $f_2 : Z \to Y$, there exists a unique morphism $f : Z \to X \times Y$ such that the following diagram commutes:

$$\begin{tikzcd}
 & Z \arrow[dl, "f_1"'] \arrow[d, dashed, "f"] \arrow[dr, "f_2"] & \\
X & X \times Y \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & Y
\end{tikzcd}$$

Remark. Note that products don't necessarily need to exist. This may depend on the category or even just on our choice of objects.

Examples. We give some examples of categories and their products.

- In **Grp**, the product of two groups is their direct product together with the two natural projection homomorphisms.
- In R-**Mod**, the product of two R-modules is given by their cartesian product with addition defined componentwise and distributive multiplication, together with the two natural projection module homomorphisms.
- In the category of topological spaces **Top**, the product of two topological spaces is the space whose underlying set is their cartesian product which carries the product topology, together with the two natural projection maps.
- In **Field** the product of two fields cannot be defined. Indeed if K_1 and K_2 are fields, we find that the element $(1, 0) \in K_1 \times K_2$ is a zero divisor, which would mean the product is not again a field.

We can sum up the theory we have developed so far in the following definition of a desirable category.

Definition 3.13. (Additive Category) Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is an **additive category** if it has a zero object and a product $X \times Y$ for every $X, Y \in \text{ob}(\mathcal{C})$.

Examples. We give some examples of additive categories.

- R-**Mod** is an additive category as we have seen that it has a zero object and products.
- **Ab** is also an additive category, as we already have demonstrated.
- **Set** is not an additive category, as it has no zero objects. Neither is **Field**.

Definition 3.14. (Monomorphism, Epimorphism) Let $\mathcal{C}$ be a category and let $A, B, C, D \in \text{ob}(\mathcal{C})$. We say that a morphism $f : B \to C \in \hom(\mathcal{C})$ is a **monomorphism** if for any two morphisms $e_1, e_2 : A \to B \in \hom(\mathcal{C})$ we have that:

$$f \circ e_1 = f \circ e_2 \implies e_1 = e_2.$$

We say that f is an **epimorphism** if for any two morphisms $g_1, g_2 : C \to D \in \hom(\mathcal{C})$ we have that:

$$g_1 \circ f = g_2 \circ f \implies g_1 = g_2.$$

It is possible for these definitions to coincide. Indeed if a morphism is both a monomorphism and an epimorphism, we say that it is an **isomorphism**.

It would be desirable if these were the abstractions of injective and surjective functions. It turns out that these notions aren't always the same, but they are in R-**Mod**.

Proposition 3.2. Let $M, N \in R$-**Mod** and let $f \in \text{Hom}_{R\text{-}\mathbf{Mod}}(M, N)$. Then:

- f is a monomorphism $\iff$ f is injective.
- f is an epimorphism $\iff$ f is surjective.

Proof. We will only prove the forward directions of both statements here – the reverse directions follow from the fact that in **Set**, injective functions are monomorphisms and surjective functions are epimorphisms. The reader is referred to ([14], page 123) for full details.

First assume that f is a monomorphism. Let $i : \ker f \to M$ be inclusion and let $j : \ker f \to M$ be the map which sends everything to 0_M. It follows that $f \circ i = f \circ j$. By the definition of a monomorphism, this means that $i = j$. Therefore $\text{im}(i) = \text{im}(j)$, which implies that $\ker f = 0_M$. This means that f is injective.

Now assume that f is an epimorphism. Let $\pi : N \to \frac{N}{\text{im} f}$ be projection and let $j : N \to \frac{N}{\text{im} f}$ be the map which sends everything to $0_{\frac{N}{\text{im} f}}$. It follows that $\pi \circ f = j \circ f$. By the definition of an epimorphism, this means that $\pi = j$. Therefore $\text{im}(\pi) = \text{im}(j)$, which implies that $\frac{N}{\text{im} f} = 0_{\frac{N}{\text{im} f}}$ and so $N = \text{im} f$. This means that f is surjective. □

The definition of an abelian category is motivated by an observation about R-**Mod**, which we summarise in the following proposition.

Proposition 3.3. In R-**Mod** we have the following:

- Every monomorphism in $\hom(R\text{-}\mathbf{Mod})$ is the kernel of its cokernel.
- Every epimorphism in $\hom(R\text{-}\mathbf{Mod})$ is the cokernel of its kernel.

Proof. Suppose that $\varphi : M \to N \in \hom(R\text{-}\mathbf{Mod})$ is a monomorphism. By proposition 3.2, it is in particular injective and so we have that $M \cong \text{im}\varphi$. Now the cokernel of φ is $\left(\frac{N}{\text{im}\varphi}, \pi : N \to \frac{N}{\text{im}\varphi}\right)$, where π is projection. Moreover since $\ker \pi = \text{im}\varphi \cong M$, the kernel of π is $(M, i : \ker \pi \to N)$, where i is inclusion. Therefore there must exist an isomorphism $\phi : \ker \pi \to M$.

Suppose that $M' \in \text{ob}(R\text{-}\mathbf{Mod})$ such that there exists some $\alpha : M' \to N \in \hom(R\text{-}\mathbf{Mod})$ with the property that $\pi \circ \alpha = 0$. By the property of equalisers there exists a unique map $\beta : M' \to \ker \pi$ such that $\alpha = i \circ \beta$. Therefore $\phi \circ \beta$ is a R-module homomorphism from M' to M, and so we can form the following commutative diagram.

$$
\begin{tikzcd}
\ker\pi \arrow[r, "i"] \arrow[dr, bend left, "\phi"] & N \arrow[r, "\pi"] & \frac{N}{\mathrm{im}\varphi} \\
M' \arrow[u, dashed, "\exists!\beta"] \arrow[ur, bend right, "\alpha"'] \arrow[r, dashed, "\exists!\phi\circ\beta"'] & M \arrow[u, "\varphi"']
\end{tikzcd}
$$

We can diagram chase to discover that:

$$\varphi \circ (\phi \circ \beta) = (\varphi \circ \phi) \circ \beta = i \circ \beta = \alpha.$$

Suppose that there is a R-module homomorphism $\gamma : M' \to M \in \hom(R\text{-}\mathbf{Mod})$ which also has the property that $\varphi \circ \gamma = \alpha$. Then observe that:

$$i \circ (\phi^{-1} \circ \gamma) = (i \circ \phi^{-1}) \circ \gamma = \varphi \circ \gamma = \alpha.$$

Since β is the unique map with this property, it follows that $\beta = \phi^{-1} \circ \gamma$. Therefore $\gamma = \phi \circ \beta$, which proves that $\phi \circ \beta$ is the unique map with this property.

It follows that (M, φ) is the kernel of π, which is the cokernel of φ. Therefore every monomorphism in $\hom(R\text{-}\mathbf{Mod})$ is the kernel of its cokernel, as claimed. The proof for the other case is analogous. $\square$

These properties of R-**Mod** motivate the following definition of some particularly nice categories over which we can perform homological algebra.

Definition 3.15. (Abelian Category) Let $\mathcal{A}$ be an additive category. We say that $\mathcal{A}$ is an **abelian category** if the following conditions hold:

1. Every morphism in $\hom(\mathcal{A})$ has a kernel and a cokernel.
2. Every monomorphism in $\hom(\mathcal{A})$ is the kernel of its cokernel.
3. Every epimorphism in $\hom(\mathcal{A})$ is the cokernel of its kernel.

A useful result we shall assume that can be proven from this definition is that in abelian categories injections and monomorphisms coincide, as do surjections and epimorphisms, as we saw was the case for R-**Mod** in proposition 3.2. The interested reader is referred to [9] for a more comprehensive treatment of abelian categories.

Examples. We give some examples of abelian categories.

- R-**Mod** is an abelian category.
- **Ab** is an abelian category.
- Let k be a field and let A be a k-algebra. Then the category $\mathbf{Rep}_k(A)$ of representations of A is an abelian category.
- Consider the category of torsion-free abelian groups **TorsFreeAb**. It is straightforward to verify that this is an additive category. However, consider the monomorphism $\varphi : \mathbb{Z} \to \mathbb{Z}$ given by $\varphi(x) = 2x$. If there was a torsion-free abelian group A such that φ was the kernel of a non-zero homomorphism $f : \mathbb{Z} \to A$, we would have that $f(2x) = 0$ for all $x \in \mathbb{Z}$. This forces $f(1) = 0$ which contradicts f being non-zero and therefore A cannot be torsion free, so **TorsFreeAb** not an abelian category.

3.2 Applications

We finish this chapter by stating a powerful theorem. Roughly speaking, it states that we can prove results about small abelian categories by proving them to be true in the case of the category of R-modules, where the ring R may not necessarily be commutative.

Definition 3.16. (Small Category) A category $\mathcal{C}$ is said to be **small** if its objects and morphisms are sets as opposed to proper classes.

Definition 3.17. (Fully Faithful Functor) Let $\mathcal{C}$ and $\mathcal{D}$ be small categories and let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then for every pair $X, Y \in \text{ob}(\mathcal{C})$ we have that F induces a functor:

$$F_{X,Y} : \text{Hom}_{\mathcal{C}}(X, Y) \to \text{Hom}_{\mathcal{D}}(F(X), F(Y)).$$

We say that F is **fully faithful** if $F_{X,Y}$ is bijective for every pair $X, Y \in \text{ob}(\mathcal{C})$.

Theorem 3.1. (Freyd-Mitchell Embedding Theorem) Let $\mathcal{A}$ be an abelian category. For every small abelian subcategory $\overline{\mathcal{A}} \subset \mathcal{A}$ there exists a ring R (not necessarily commutative, but it is unitary) and a fully faithful, exact functor $F : \mathcal{A} \to R$-**Mod** which embeds $\overline{\mathcal{A}}$ as a full subcategory in the sense that $\text{Hom}_{\bar{\mathcal{A}}}(M, N) \cong \text{Hom}_{R\text{-}\mathbf{Mod}}(F \circ M, F \circ N)$.

Proof. The proof of this result can be found in ([9], page 101) and will not be reproduced here. ☐

Exact functors mentioned in the theorem above will be defined in definition 4.2. We will conclude this section by proving the so-called snake lemma for abelian categories. By the Freyd-Mitchell embedding theorem it will suffice to prove the lemma in the category of modules over unital rings.

Theorem 3.2. (Snake Lemma) Let $\mathcal{A}$ be an abelian category. Let $A, A', B, B', C, C' \in \text{ob}(\mathcal{A})$ and let $f, g, h \in \text{hom}(\mathcal{A})$ be as shown in the commutative diagram below:

$$\begin{tikzcd}
 & \ker f \arrow[r] \arrow[d] & \ker g \arrow[r] \arrow[d] & \ker h \arrow[d] \arrow[dddll, rounded corners, "\partial"] & \\
 & A' \arrow[r] \arrow[d, "f"] & B' \arrow[r] \arrow[d, "g"] & C' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] & \\
 & \operatorname{coker} f \arrow[r] & \operatorname{coker} g \arrow[r] & \operatorname{coker} h &
\end{tikzcd}$$

Suppose that the diagram in black commutes with exact rows. Then the sequence given by:

$$\ker f \to \ker g \to \ker h \xrightarrow{\partial} \operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h$$

exists, and in particular is exact.

Proof. (Sketch) Take a small abelian subcategory $\bar{\mathcal{A}} \subset \mathcal{A}$ containing the commutative diagram above. By the Freyd-Mitchell embedding theorem, there exists a unitary ring R for which we may embed $\bar{\mathcal{A}}$ into R-**Mod**. Since this embedding is exact, it follows that the image of the diagram will also be commutative with exact rows. It therefore follows that if we can establish the sequence in R-**Mod**, then it must necessarily also hold in $\mathcal{A}$. It therefore suffices to prove the result in R-**Mod**, which is a standard result covered in many homological algebra texts such as in ([19], page 131). ☐

4 Homological algebra

In this section we will introduce some basic concepts from homological algebra. Homological algebra is performed over abelian categories and this permits us to study the structure and properties of commutative rings and their ideals, as well as the structure of modules over these rings. The most important result from this section will be introducing the Ext functor and understanding the definition of the global dimension of a ring, which is a tool allowing us to measure how 'far away' a R-module is from being projective. These notions will then allow us to state and prove Serre's global dimension theorem in section 5.

4.1 Introduction to homological algebra

We begin by defining some basic concepts in homological algebra. Where possible, definitions will be given in terms of a general abelian category before we hone in on the category R-**Mod**.

Definition 4.1. (Exact Sequence) Let $\mathcal{A}$ be an abelian category. Let $X, Y, Z \in \text{ob}(\mathcal{A})$ and let $f : X \to Y, g : Y \to Z \in \text{hom}(\mathcal{A})$. The sequence of morphisms:

$$X \xrightarrow{\quad f \quad} Y \xrightarrow{\quad g \quad} Z$$

is said to be **exact** if $\ker(g) = \text{coker}(\ker(f))$. This implies that $g \circ f$ is the zero morphism from X to Z. More generally, the sequence:

$$\cdots \xrightarrow{\quad f_{n+1} \quad} M_n \xrightarrow{\quad f_n \quad} M_{n-1} \xrightarrow{\quad f_{n-1} \quad} \cdots$$

is said to be an **exact sequence** if for each $n \in \mathbb{N}$, the sequence:

$$M_{n+1} \xrightarrow{\quad f_{n+1} \quad} M_n \xrightarrow{\quad f_n \quad} M_{n-1}$$

is exact. Finally, an exact sequence of the form:

$$0 \xrightarrow{\qquad} X \xrightarrow{\quad f \quad} Y \xrightarrow{\quad g \quad} Z \xrightarrow{\qquad} 0$$

is said to be **short exact**, where 0 denotes the zero object in $\mathcal{A}$. In this case, f is necessarily a monomorphism and g is necessarily an epimorphism.

It is then natural to ask what functors preserve exactness, and to do so we make the following definition.

Definition 4.2. (Exact Functor) Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and let $F : \mathcal{A} \to \mathcal{B}$ be a covariant functor. Furthermore let $X, Y, Z \in \text{ob}(\mathcal{A})$ be any triple of objects such that $0 \to X \to Y \to Z \to 0$ is a short exact sequence. We say that F is a **left exact functor** if $0 \to F(X) \to F(Y) \to F(Z)$ is an exact sequence. We say that F is a **right exact functor** if $F(X) \to F(Y) \to F(Z) \to 0$ is an exact sequence. We say that F is an **exact functor** if $0 \to F(X) \to F(Y) \to F(Z) \to 0$ is an exact sequence. The definition for a contravariant exact functor is analogous.

We can use the concept of Hom-sets to create a covariant functor from any abelian category $\mathcal{A}$ to **Ab**. Indeed for any $A \in \text{ob}(\mathcal{A})$ we can define the covariant functor $\text{Hom}_{\mathcal{A}}(A, -) : \mathcal{A} \to \textbf{Ab}$ by $\text{Hom}_{\mathcal{A}}(A, -)(X) = \text{Hom}_{\mathcal{A}}(A, X)$. Moreover if $f : X \to Y \in \text{hom}(\mathcal{A})$ we get a naturally induced map $f_* : \text{Hom}_{\mathcal{A}}(A, X) \to \text{Hom}_{\mathcal{A}}(A, Y)$ which is defined by $f_*(\varphi) = f \circ \varphi$. We can create a contravariant functor in which we take homomorphisms onto A in a completely analogous fashion.

It turns out that neither the covariant or contravariant Hom-functors are in general exact. It is however true that they are both always left exact, as demonstrated below in the covariant case.

Proposition 4.1. Let $\mathcal{A}$ be an abelian category and let $A \in \mathrm{ob}(\mathcal{A})$. Then $\mathrm{Hom}_{\mathcal{A}}(A, -)$ is a left exact functor.

Proof. Suppose that $X, Y, Z \in \mathrm{ob}(\mathcal{A})$ and $f : X \to Y, g : Y \to Z \in \mathrm{hom}(\mathcal{A})$ such that:

$$0 \longrightarrow X \xrightarrow{\quad f \quad} Y \xrightarrow{\quad g \quad} Z \longrightarrow 0$$

is a short exact sequence in $\mathcal{A}$. We claim that:

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(A, X) \xrightarrow{\quad f_* \quad} \mathrm{Hom}_{\mathcal{A}}(A, Y) \xrightarrow{\quad g_* \quad} \mathrm{Hom}_{\mathcal{A}}(A, Z)$$

is then an exact sequence in **Ab**. To prove it is exact at $\mathrm{Hom}_{\mathcal{A}}(A, X)$ it suffices to prove that f_* is injective. In other words we need that for all $h, h' \in \mathrm{Hom}_{\mathcal{A}}(A, X), f \circ h = f \circ h' \implies h = h'$. This is trivially true since f is a monomorphism.

Now we want to show that the sequence is exact at $\mathrm{Hom}_{\mathcal{A}}(A, Y)$. Let $p \in \mathrm{im}(f_*)$ so that there exists some $\bar{p} \in \mathrm{Hom}_{\mathcal{A}}(A, X)$ such that $p = f \circ \bar{p}$. In that case:

$$g \circ p = g \circ (f \circ \bar{p}) = (g \circ f) \circ \bar{p} = 0 \circ \bar{p} = 0,$$

and so $p \in \ker(g_*)$. This proves that $\mathrm{im}(f_*) \subseteq \ker(g_*)$.

Now let $p \in \ker(g_*)$ so that $g \circ p = 0$. Since (X, f) is the kernel of g, we may form the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \\
& A \arrow[ul, dashed, "\exists! \bar{p}"] \arrow[u, "p"] \arrow[ur, "0"'] &
\end{tikzcd}$$

Therefore there exists a unique morphism $\bar{p} \in \mathrm{Hom}_{\mathcal{A}}(A, X)$ such that $p = f \circ \bar{p}$, and so $p \in \mathrm{im}(f_*)$ as desired. This proves that $\ker(g_*) \subseteq \mathrm{im}(f_*)$ and so we conclude that $\ker(g_*) = \mathrm{im}(f_*)$. Therefore this is an exact sequence in **Ab** as claimed, and so $\mathrm{Hom}_{\mathcal{A}}(A, -)$ is a left exact functor. The proof for the contravariant Hom-functor is analogous. $\square$

Despite this nice result, it turns out that Hom-functors are not right exact in general.

Example. Consider the following short exact sequence in **Ab**:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\quad f \quad} \mathbb{Z} \xrightarrow{\quad \pi \quad} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

where $f(x) = 2x$ is multiplication by 2, and $\pi(x) = x \pmod 2$ is projection. We can apply the covariant Hom-functor $\mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}/2\mathbb{Z}, -)$ to get:

$$0 \longrightarrow \mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \xrightarrow{f_*} \mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \xrightarrow{\pi_*} \mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \longrightarrow 0$$

Notice that there are no non-trivial group homomorphisms from $\mathbb{Z}/2\mathbb{Z}$ to $\mathbb{Z}$, so we have that $\mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \cong \{0\}$. Also $\mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, so this sequence simplifies to:

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

which is clearly not exact at $\mathbb{Z}/2\mathbb{Z}$. We could have also applied the contravariant Hom-functor $\mathrm{Hom}_{\mathcal{A}}(-, \mathbb{Z})$ to get the same result in that case.

The obstacle to $\mathrm{Hom}_{\mathcal{A}}(A, -)$ being an exact functor above was that g_* was not an epimorphism. In view of the previous exploration it is then natural to ask the question: given any morphism $h : A \to Z$ and any epimorphism $g : Y \to Z$, when is h of the form $h = g \circ i$ for some morphism $i : A \to B$? This motivates the following definition.

Definition 4.3. (Projective Object) Let $\mathcal{A}$ be an abelian category. We say that $P \in \mathrm{ob}(\mathcal{A})$ is a **projective object** if given an epimorphism $g : Y \to Z$ and a morphism $h : P \to Z$, there exists a morphism $p : P \to Y$ such that the following diagram commutes:

$$\begin{tikzcd}
 & P \arrow[dl, dashed, "\exists p"'] \arrow[d, "h"] \\
Y \arrow[r, two heads, "g"'] & Z
\end{tikzcd}$$

The dual notion for the contravariant Hom-functor follows from similar motivation and is as follows.

Definition 4.4. (Injective Object) Let $\mathcal{A}$ be an abelian category. We say that $Q \in \mathrm{ob}(\mathcal{A})$ is an **injective object** if given a monomorphism $f : X \to Y$ and a morphism $g : X \to Q$, there exists a morphism $q : Y \to Q$ such that the following diagram commutes:

$$\begin{tikzcd}
 & Q \\
X \arrow[ur, "g"] \arrow[r, hook, "f"'] & Y \arrow[u, dashed, "\exists q"']
\end{tikzcd}$$

Since we have defined our way out of this problem, it should be of no surprise that this property of being a projective object is exactly what we require of an object A in an abelian category $\mathcal{A}$ in order for $\mathrm{Hom}_{\mathcal{A}}(A, -)$ to be an exact functor. We formalise this in the following proposition.

Proposition 4.2. Let $\mathcal{A}$ be an abelian category and let $P \in \mathrm{ob}(\mathcal{A})$. Then $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is an exact functor if and only if P is a projective object.

Proof. Suppose that $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is an exact functor. In that case there is an epimorphism $g : Y \to Z$ which induces an epimorphism $g_* : \mathrm{Hom}_{\mathcal{A}}(P, Y) \to \mathrm{Hom}_{\mathcal{A}}(P, Z)$. Consider a morphism $h : P \to Z$. Then since g_* is an epimorphism there is a morphism $p \in \mathrm{Hom}_{\mathcal{A}}(P, Y)$ such that $h = g \circ p = g_*(p)$, which means that P is a projective object.

Now suppose that P is a projective object. Suppose that $g : Y \to Z$ and choose a morphism $h \in \mathrm{Hom}_{\mathcal{A}}(P, Z)$. Since P is a projective object there exists a morphism $f \in \mathrm{Hom}_{\mathcal{A}}(P, Y)$ such that $h = g \circ f = g_*(f)$. Since h was arbitrary we must have that g_* is surjective which proves that g_* is an epimorphism. Therefore $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is a right exact functor and is therefore an exact functor. □

The corresponding result for the contravariant Hom-functor is that $\mathrm{Hom}_{\mathcal{A}}(-, A)$ is exact if and only if A is an injective object.

We can now introduce the notion of a projective resolution which is a useful tool for studying the structure and properties of objects in an abelian category.

Definition 4.5. (Projective Resolution, Injective Resolution) Let $\mathcal{A}$ be an abelian category and let $A \in \mathrm{ob}(\mathcal{A})$. A **projective resolution** of A is an exact sequence:

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

such that each P_i is a projective object in $\mathcal{A}$. An injective resolution is defined analogously.

The category R-**Mod** has a special property in that every R-module admits a projective resolution, which follows directly from the next proposition.

Proposition 4.3. Every free module is projective.

Proof. Let $F, M, N \in \mathrm{ob}(R\text{-}\mathbf{Mod})$ such that F is a free module with basis $\mathcal{B} = \{x_i : i \in \mathcal{I}\}$. Suppose that $\psi \in \mathrm{Hom}_{R\text{-}\mathbf{Mod}}(M, N)$ is an epimorphism and that $f \in \mathrm{Hom}_{R\text{-}\mathbf{Mod}}(F, N)$. Then for each $i \in \mathcal{I}$, there is a $y_i \in M$ such that $\psi(y_i) = f(x_i)$. Therefore we can define $g : F \to M$ by $g(x_i) = y_i$ for each $i \in \mathcal{I}$. Then $(\psi \circ g)(x_i) = f(x_i)$ for every $i \in \mathcal{I}$ and thus F is a projective module. $\square$

Let $M \in \mathrm{ob}(R\text{-}\mathbf{Mod})$. In order to construct a projective resolution of M we can take P_0 to be any free R-module on a set of generators for M. We can then define a surjection $\epsilon : P_0 \to M$ which begins the resolution. Next, take P_1 to be any free module mapping into the submodule $\ker \epsilon$ of P_0. We can carry on with this process and we will obtain a resolution of M by free (and therefore projective) modules.

It is then natural to ask what conditions a general abelian category must have in order for every object to have a projective resolution.

Definition 4.6. (Enough Projectives) Let $\mathcal{A}$ be an abelian category. We say that $\mathcal{A}$ has **enough projectives** if every object in $\mathcal{A}$ can be covered by a projective object.

Proposition 4.4. Let $\mathcal{A}$ be an abelian category with enough projectives. Then every object in $\mathcal{A}$ admits a projective resolution.

Proof. Let $A \in \mathrm{ob}(\mathcal{A})$. We will construct a projective resolution of A inductively. Since $\mathcal{A}$ has enough projectives, there is some projective object P_0 such that there is an epimorphism from P_0 to A. Then trivially $P_0 \twoheadrightarrow A \to 0$ is exact at A. Now assume that we have constructed:

$$P_{n-1} \xrightarrow{d_{n-1}} P_{n-2} \to \cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A \to 0$$

and that we want to define P_n. An intuitively obvious candidate for P_n to make this sequence exact at P_{n-1} would be $\ker(d_{n-1})$, however this may not necessarily be a projective object. Instead, we cover this kernel by a projective object and take this cover to be our P_n.

$$\begin{tikzcd}
P_n \arrow[rr, "d_n"] \arrow[dr, "e_n"'] & & P_{n-1} \arrow[r, "d_{n-1}"] & P_{n-2} \arrow[r] & \cdots \\
& \ker(d_{n-1}) \arrow[ur, "i"'] & & &
\end{tikzcd}$$

In the diagram above i is the inclusion function. Since e_n is an epimorphism, we have that:

$$\mathrm{im}(d_n) = \mathrm{im}(i \circ e_n) = \mathrm{im}(i) = \ker(d_{n-1}).$$

Therefore the constructed sequence is exact at P_{n-1} as desired. $\square$

4.2 Chain complexes of R-Modules

Now we want to turn our attention to studying chain complexes, an algebraic structure which will allow us to make comparisons between sequences of R-modules. Chain complexes will be of particular use to us in section 4.3 as they will permit us to introduce the Ext functor and study its properties.

Definition 4.7. (Homology Module) Let X, Y, Z be R-modules and let $f : X \to Y, g : Y \to Z$ be R-module homomorphisms such that $g \circ f = 0$. Then we call $H = \ker(g)/\mathrm{im}(f)$ a **homology module**.

Definition 4.8. (Chain Complex) Let R be a ring. A chain complex $\mathfrak{C}$ of R-modules is a collection of R-modules $\{C_n\}_{n\in\mathbb{Z}}$, together with R-module homomorphisms $d = \{d_n : C_n \to C_{n-1}\}_{n\in\mathbb{Z}}$ such that for each $n \in \mathbb{Z}$, $d_{n-1} \circ d_n = 0$.

$$\cdots \xrightarrow{d_{n+2}} C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots$$

We call the maps d_n the **differentials** of $\mathfrak{C}$ and denote the n^{th} homology module by:

$$H_n(\mathfrak{C}) = \ker(d_n)/\mathrm{im}(d_{n+1}).$$

So far this is quite abstract, so we look at a concrete example.

Example. Let $C_n = \mathbb{Z}/8\mathbb{Z}$ for each $n \geq 0$ and $C_n = 0$ otherwise. For each $n \in \mathbb{N}$ we define:

$$d_n : \mathbb{Z}/8\mathbb{Z} \to \mathbb{Z}/8\mathbb{Z}, \qquad d_n(x) = 4x.$$

Then this forms a chain complex of R-modules: for $n \geq 2$ we have that $(d_{n-1} \circ d_n)(x) = 16x = 0$ and for $n \leq 1$ we have that $(d_{n-1} \circ d_n)(x) = 0$. Moreover, we can compute the homology modules for the chain complex. We find that:

$$H_n(\mathfrak{C}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & n \geq 1, \\ \mathbb{Z}/4\mathbb{Z} & n = 0, \\ 0 & n \leq -1. \end{cases}$$

It turns out that we can create an abelian category where the objects are chain complexes. In this case for two chain complexes $\mathfrak{C}$ and $\mathfrak{D}$, the morphisms are chain complex maps or in other words a family of R-module homomorphisms $u = \{u_n : C_n \to D_n\}_{n\in\mathbb{Z}}$ such that the following diagram commutes:

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1} \arrow[r, "c_{n+1}"] \arrow[d, "u_{n+1}"] & C_n \arrow[r, "c_n"] \arrow[d, "u_n"] & C_{n-1} \arrow[r] \arrow[d, "u_{n-1}"] & \cdots \\ \cdots \arrow[r] & D_{n+1} \arrow[r, "d_{n+1}"'] & D_n \arrow[r, "d_n"'] & D_{n-1} \arrow[r] & \cdots \end{tikzcd}$$

We denote the category of chain complexes of R-modules by $\mathbf{Ch}(R\text{-}\mathbf{Mod})$. The homology modules of chain complexes give a measure of how far a sequence is from being exact. It is therefore of importance to study the images and kernels of differentials under chain complex maps.

Definition 4.9. (Cycles, Boundaries) Let $\mathfrak{C}$ be a chain complex with differentials $(d_n)_{n\in\mathbb{Z}}$. We define the **cycles** of $\mathfrak{C}$ to be the kernels of the differentials and for each $n \in \mathbb{Z}$ write $Z_n(\mathfrak{C}) = \ker d_n$. We define the **boundaries** of $\mathfrak{C}$ to be the images of the differentials and for each $n \in \mathbb{Z}$ write $B_n(\mathfrak{C}) = \mathrm{im} d_{n+1}$.

Proposition 4.5. Let $u : \mathfrak{C} \to \mathfrak{D}$ be a morphism of chain complexes. Then u maps boundaries to boundaries and cycles to cycles, and hence homology modules to homology modules. In particular, each H_n is a covariant functor from $\mathbf{Ch}(R\text{-}\mathbf{Mod})$ to R-$\mathbf{Mod}$.

Proof. Fix $n \in \mathbb{Z}$ and let $x \in Z_n(\mathfrak{C})$. Then $c_n(x) = 0$ and since u_{n-1} is a homomorphism, $u_{n-1}(c_n(x)) = 0$. Since the diagram commutes we therefore have that $d_n(u_n(x)) = 0$ and so $u_n(x) \in Z_n(\mathfrak{D})$. Therefore u maps cycles to cycles.

Now let $x \in B_n(\mathfrak{C})$. Then there exists a $y \in C_{n+1}$ such that $c_{n+1}(y) = x$. It follows that $u_n(c_{n+1}(y)) = u_n(x)$ and since the diagram commutes we therefore have that $d_{n+1}(u_{n+1}(y)) = u_n(x)$. Therefore $u_n(x) \in B_n(\mathfrak{D})$ and so u maps boundaries to boundaries.

Consider $x \in H_n(\mathfrak{C})$. We may write $x = z + B_n(\mathfrak{C})$ for some $z \in Z_n(\mathfrak{C})$ and it follows that we have $u_n(x) = u_n(z) + B_n(\mathfrak{D}) \in H_n(\mathfrak{D})$. Therefore u maps homology modules to homology modules. It follows that u induces a well defined homomorphism $u_* : H_n(\mathfrak{C}) \to H_n(\mathfrak{D})$ given by $u_*(z + B_n(\mathfrak{C})) = u_n(z) + B_n(\mathfrak{D})$. Now if $v : \mathfrak{B} \to \mathfrak{C}$ is a chain complex map it is straightforward to see that $(u \circ v)_* = u_* \circ v_*$ and that for the identity map $\mathrm{id}_* = \mathrm{id}$. Therefore we conclude that H_n is a covariant functor from **Ch**(R-**Mod**) to R-**Mod**. □

Our next goal is to develop a method of making comparisons between chain complex maps, which we do with a chain homotopy.

Definition 4.10. (Chain Homotopy) Let $\mathfrak{C}$ and $\mathfrak{D}$ be chain complexes. Let $f : \mathfrak{C} \to \mathfrak{D}$ and $g : \mathfrak{C} \to \mathfrak{D}$ be chain complex maps. We say that there is a **chain homotopy** between f and g if there exists a collection of R-module homomorphisms $\{s_n : C_n \to D_{n+1}\}_{n \in \mathbb{Z}}$ such that for every $n \in \mathbb{Z}$ we have that $d_{n+1} \circ s_n + s_{n-1} \circ c_n = f_n - g_n$.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "c_{n+1}"] \arrow[d] & C_n \arrow[r, "c_n"] \arrow[d] \arrow[dl, "s_n"'] & C_{n-1} \arrow[r] \arrow[d] \arrow[dl, "s_{n-1}"'] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "d_{n+1}"'] & D_n \arrow[r, "d_n"'] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

We will take for granted that this consitutes an equivalence relation and that such maps induce the same homomorphisms on homology groups.

Definition 4.11. Let $\mathcal{A}$ be an abelian category and let $A, B \in \mathrm{ob}(\mathcal{A})$. Let $\mathcal{P}$ be a projective resolution of A:

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

We define $\mathrm{Hom}_{\mathcal{A}}(\mathcal{P}, B)$ to be the canonically induced sequence:

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(P_0, B) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(P_1, B) \longrightarrow \cdots$$

obtained by applying the Hom-functor and deleting $\mathrm{Hom}_{\mathcal{A}}(A, B)$.

4.3 The Ext functor

We now introduce the Ext functor which, formally, is the right derived functor of the contravariant Hom-functor. Together with the Tor functor, which won't be discussed here, Ext is one of the central functors of interest in homological algebra. For a comprehensive discussion of the Tor functor and of derived functors in general, see ([23], chapter 2).

Definition 4.12. (Ext Functor) Let X, Y be R-modules and let $\mathcal{P}$ be a projective resolution of X. We define the **Ext functor** as:

$$\mathrm{Ext}^n_R(X, Y) = H_n\left(\mathrm{Hom}_{R\text{-}\mathbf{Mod}}(\mathcal{P}, Y)\right),$$

where each $n \in \mathbb{N}_0$. In order for the Ext functor to be well defined, it should not depend on our choice of projective resolution for X. In order to verify this we will need the following propositions.

Proposition 4.6. Let $f : X \to Y$ be a R-module homomorphism and let $\mathcal{P}$ and $\mathcal{Q}$ be projective resolutions of X and Y respectively. Then for every $n \in \mathbb{N}_0$ there exists a lift f_n of f such that the following diagram commutes:

$$
\begin{tikzcd}
\cdots \arrow[r, "p_2"] & P_1 \arrow[r, "p_1"] \arrow[d, "f_1"'] & P_0 \arrow[r, "p_0"] \arrow[d, "f_0"'] & X \arrow[r] \arrow[d, "f"'] & 0 \\
\cdots \arrow[r, "q_2"'] & Q_1 \arrow[r, "q_1"'] & Q_0 \arrow[r, "q_0"'] & Y \arrow[r] & 0
\end{tikzcd}
$$

Proof. We proceed by strong induction on n starting at $n = 0$. Since projective resolutions are exact sequences, we have that q_0 is an epimorphism. Since P_0 is projective it follows that $f \circ p_0 : P_0 \to Y$ lifts to a map $f_0 : P_0 \to Q_0$ such that $q_0 \circ f_0 = f \circ p_0$. Now assume that we have constructed $f_0, \dots, f_{n-1}$ such that the following diagram commutes:

$$
\begin{tikzcd}
P_n \arrow[r, "p_n"] & P_{n-1} \arrow[r, "p_{n-1}"] \arrow[d, "f_{n-1}"'] & P_{n-2} \arrow[r, "p_{n-2}"] \arrow[d, "f_{n-2}"'] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d, "f_0"'] & X \arrow[r] \arrow[d, "f"'] & 0 \\
Q_n \arrow[r, "q_n"'] & Q_{n-1} \arrow[r, "q_{n-1}"'] & Q_{n-2} \arrow[r, "q_{n-2}"'] & \cdots \arrow[r] & Q_0 \arrow[r] & Y \arrow[r] & 0
\end{tikzcd}
$$

Now since the diagram commutes we have that:

$$q_{n-1} \circ (f_{n-1} \circ p_n) = f_{n-2} \circ (p_{n-1} \circ p_n) = f_{n-2} \circ 0 = 0.$$

Therefore we have that $\mathrm{im}(f_{n-1} \circ p_n) \subseteq \ker(q_{n-1}) = \mathrm{im}(q_n)$ by exactness. Since P_n is projective, we may form the following commutative diagram and lift $f_{n-1} \circ p_n$ to some f_n:

$$
\begin{tikzcd}
 & P_n \arrow[dl, dashed, "\exists\, f_n"'] \arrow[d, "f_{n-1} \circ p_n"] \\
Q_n \arrow[r, two heads, "q_n"'] & B_{n-1}(\mathcal{Q})
\end{tikzcd}
$$

We have constructed an appropriate lift f_n and so the proposition is true by strong induction. $\square$

Proposition 4.7. Let $f : X \to Y$ be a R-module homomorphism and let $\mathcal{P}$ and $\mathcal{Q}$ be projective resolutions of X and Y respectively. Then for each $n \in \mathbb{N}_0$ we have an induced R-module homomorphism $\varphi_n : \mathrm{Ext}^n_R(X, Z) \to \mathrm{Ext}^n_R(Y, Z)$ on the homology modules. Moreover these maps depend only on f and not on the choice of lifts.

Proof. The induced R-module homomorphism is the canonical one which is well defined since chain complex maps take homology modules to homology modules. Now to prove uniqueness we let a_n and b_n be two lifts of f. It will suffice to show that a is chain homotopic to b, or equivalently that $c = a - b$ is chain homotopic to the zero chain map. Observe that $c_{-1} = f - f = 0$ so that $c_0 = 0$ as well since the diagram must commute. Since P_0 is projective, we may form the following commutative diagram and lift the zero map to a map $s_0 : P_0 \to Q_1$:

$$
\begin{tikzcd}
 & P_0 \arrow[r, "p_0"] \arrow[dl, dashed, "\exists\, s_0"'] \arrow[d, "0"] & X \arrow[d, "0"] \\
Q_1 \arrow[r, two heads, "q_1"'] & B_0(\mathcal{Q}) \arrow[r, "q_0"'] & Y
\end{tikzcd}
$$

This map allows us to begin induction. Suppose that we have constructed $s_i : P_i \to Q_{i+1}$ such that $c_i = s_{i-1} \circ p_i + q_{i+1} \circ s_i$ or equivalently that $q_{i+1} \circ s_i = c_i - s_{i-1} \circ p_i$. Now it follows that:

$$
\begin{aligned}
q_{i+1} \circ (c_{i+1} - s_i \circ p_{i+1}) &= q_{i+1} \circ c_{i+1} - c_i \circ p_{i+1} + s_{i-1} \circ (p_i \circ p_{i+1}) \\
&= q_{i+1} \circ c_{i+1} - c_i \circ p_{i+1} \\
&= 0,
\end{aligned}
$$

by exactness and commutativity and so $c_{i+1} - s_i \circ p_{i+1}$ maps P_{i+1} into cycles $Z_{i+1}(\mathcal{Q})$. Therefore we have the following commutative diagram:

$$
\begin{tikzcd}
 & P_{i+1} \arrow[dl, "s_{i+1}"'] \arrow[d, "c_{i+1}-s_i\circ p_{i+1}"] \\
Q_{i+2} \arrow[r, two heads, "q_{i+2}"'] & Z_{i+1}(\mathcal{Q})
\end{tikzcd}
$$

Since P_{i+1} is projective we have constructed $s_{i+1} : P_{i+1} \to Q_{i+2}$ such that $c_{i+1} = s_i \circ p_{i+1} + q_{i+2} \circ s_{i+1}$ as required. Hence there is a chain homotopy between our two lifts and therefore they induce the same homomorphisms on homology groups. $\square$

We are now ready to prove that the Ext functor is well defined in the sense that it is independent of our choice of projective resolution.

Proposition 4.8. The modules $\mathrm{Ext}^n_R(X, Y)$ depend only on X and Y. In other words, they are independent of the projective resolution of X we choose.

Proof. Let $X = X'$ and Y be R-modules and let $\mathcal{P}$ and $\mathcal{Q}$ be projective resolutions of X. Let f_n and g_n be lifts of the identity map and form the following commutative diagram:

$$
\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "p_2"] \arrow[d, "f_2"] & P_1 \arrow[r, "p_1"] \arrow[d, "f_1"] & P_0 \arrow[r, "p_0"] \arrow[d, "f_0"] & X \arrow[d, "f=\mathrm{id}"] \\
\cdots \arrow[r] & Q_2 \arrow[r, "q_2"] \arrow[d, "g_2"] & Q_1 \arrow[r, "q_1"] \arrow[d, "g_1"] & Q_0 \arrow[r, "q_0"] \arrow[d, "g_0"] & X' \arrow[d, "g=\mathrm{id}"] \\
\cdots \arrow[r] & P_2 \arrow[r, "p_2"'] & P_1 \arrow[r, "p_1"'] & P_0 \arrow[r, "p_0"'] & X
\end{tikzcd}
$$

The f_n result in the homology module homomorphisms $\varphi_n : \mathrm{Ext}^n_R(X, Y) \to \mathrm{Ext}^n_R(X', Y)$ and the g_n in $\psi_n : \mathrm{Ext}^n_R(X', Y) \to \mathrm{Ext}^n_R(X, Y)$. The maps $g_n \circ f_n$ are now lifts of the identity map $g \circ f$ which induce the homomorphisms $\varphi_n \circ \psi_n$ on the homology modules.

However, since the first and last rows are the same, taking the identity map from P_n to itself is in particular a lift of $g \circ f$, which induces the identity homomorphism on the homology modules. By our previous proposition this implies that $\varphi_n \circ \psi_n = \mathrm{id}_{\mathrm{Ext}^n_R(X,Y)}$. By a completely analagous argument we see that $\psi_n \circ \varphi_n = \mathrm{id}_{\mathrm{Ext}^n_R(X',Y)}$ which proves that φ_n and ψ_n are isomorphisms as desired. $\square$

Now we can begin to study some properties of the Ext functor.

Proposition 4.9. Let A, B be R-modules and let $n \in \mathbb{N}$. Then:

1. $\mathrm{Ext}^0_R(A, B) \cong \mathrm{Hom}_{R\text{-}\mathbf{Mod}}(A, B)$.

2. $\mathrm{Ext}^n_R(A, B) = 0$ if A is projective or B is injective.

Proof. 1. Let $\cdots P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} A \to 0$ be a projective resolution of A. Since the Hom-functor is left exact, it follows that $0 \to \mathrm{Hom}_{R\text{-}\mathbf{Mod}}(A, B) \xrightarrow{p_{0*}} \mathrm{Hom}_{R\text{-}\mathbf{Mod}}(P_0, B) \xrightarrow{p_{1*}} \mathrm{Hom}_{R\text{-}\mathbf{Mod}}(P_1, B)$ is also exact. Hence $\mathrm{Ext}^0_R(A, B) \cong \ker p_{1*} = \mathrm{im} p_{0*} \cong \mathrm{Hom}_{R\text{-}\mathbf{Mod}}(A, B)$, since p_{0*} is injective.

2. Suppose A is projective. Then $\cdots \to 0 \to A \xrightarrow{\mathrm{id}} A \to 0$ is a projective resolution of A. The induced complex is then $0 \to \mathrm{Hom}_{R\text{-}\mathbf{Mod}}(A, B) \to 0 \to 0 \to \cdots$ from which the result follows. The case where B is injective follows trivially since in that case Hom will be an exact functor.

□

4.4 Homological dimension

We are now ready to introduce and study the concept of global dimension. This allows us to measure the complexity of a ring as a module over itself and gives rise to some interesting classifications.

Definition 4.13. (Projective and Injective Dimension) Let R be a ring and let M be a R-module.

1. The **projective dimension** of M, $\mathrm{pd}(M)$ is the smallest non-negative integer n such that there is a projective resolution of M by R-modules:

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

2. The **injective dimension** of M, $\mathrm{id}(M)$ is the smallest non-negative integer n such that there is an injective resolution of M by R-modules:

$$0 \longrightarrow M \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots \longrightarrow E^n \longrightarrow 0$$

In either case if no such n exists, we say that the projective or injective dimension is infinite.

Examples. We give some examples of projective and injective dimensions of R-modules.

- Let R be any ring. Then R is free as a R-module with basis $\{1\}$, so is in particular projective. Therefore $0 \to R \xrightarrow{\mathrm{id}} R \to 0$ is a projective resolution of R, and so $\mathrm{pd}(R) = 0$. Using this idea it is straightforward to show that in general, $\mathrm{pd}(M) = 0$ if and only if M is projective.
- Let $R = \mathbb{Z}$. Then the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ is an injective resolution of $\mathbb{Z}$, so that $\mathrm{id}(\mathbb{Z}) \leq 1$. Techniques used to prove equality are developed in ([13], chapter 5).

Proposition 4.10. Let R be a ring and let $n \in \mathbb{N}_0$. Then the following conditions are equivalent:

1. $\mathrm{pd}(M) \leq n$ for every R-module M.
2. $\mathrm{pd}(M) \leq n$ for every finite R-module M.
3. $\mathrm{id}(M) \leq n$ for every R-module M.
4. $\mathrm{Ext}^{n+1}_R(M, N) = 0$ for all R-modules M and N.

Proof. The proof of this result can be found in ([16], page 155) and will not be reproduced here. □

Definition 4.14. (Global Dimension) Let R be a ring. We define the **global dimension** of R to be:

$$\mathrm{gl\,dim}(R) = \sup\{\mathrm{pd}(M) : M \text{ is a } R\text{-module}\}.$$

Examples. We give some examples of rings and their global dimension.

- It can be shown that $\mathbb{Z}/n\mathbb{Z}$ has infinite global dimension if n is a square number, and 0 otherwise. The reader is referred to ([13], page 171) for a proof.
- Let k be a field. Hilbert's syzygy theorem (see [8], page 474) states that the ring of polynomials $k[x_1, \ldots, x_n]$ has global dimension n.

5 Classification of regular local rings

In this section we bring together the theory discussed so far in this paper in order to characterise regular local rings as precisely those of finite global dimension. This was first done by Serre in [20], who proved the converse of Auslander and Buchsbaum's hypothesis as presented in [4].

5.1 Preliminaries

We will begin by defining the notion of depth, which contrary to the name doesn't admit any particularly nice geometric properties but is nonetheless an incredibly important homological invariant.

Definition 5.1. (Depth) Let R be a ring and I an ideal in R. Let M be a finitely generated R-module such that $IM \subsetneq M$. Then the **depth** of M with respect to I is defined as:

$$\operatorname{depth}_I(M) = \inf\{i \in \mathbb{N}_0 : \operatorname{Ext}^i_R(R/I, M) \neq 0\}.$$

A key ingredient in our proof will be a result due to Auslander and Buchsbaum presented in [5], which says that we can think of projective dimension and depth of a module over a Noetherian local ring as being complementary to one another, and allows for a simpler method of calculating depth.

Theorem 5.1. (Auslander-Buchsbaum formula) Let $(R, \mathfrak{m})$ be a Noetherian local ring and suppose that M is a non-zero finitely generated R-module such that $\operatorname{pd}(M)$ is finite. Then we have that:

$$\operatorname{pd}_R(M) + \operatorname{depth}_{\mathfrak{m}}(M) = \operatorname{depth}_{\mathfrak{m}}(R).$$

We will also use the following proposition which allows us to relate depth and various dimensions.

Proposition 5.1. Let $(R, \mathfrak{m})$ be a Noetherian local ring. Then:

1. $\operatorname{depth}_{\mathfrak{m}}(R) \leq \dim(R)$.
2. $emb. \dim(R) \leq \operatorname{gl}\dim(R) = \operatorname{pd}(R/\mathfrak{m})$.

Proof. Proofs of these results can be found in ([15], chapter 7) and will not be reproduced here. □

Recalling that for a Noetherian local ring $(R, \mathfrak{m})$ we have $\dim(R) \leq emb. \dim(R)$, the main takeaway is that:

$$\dim(R) \leq emb. \dim(R) \leq \operatorname{gl}\dim(R) = \operatorname{pd}(R/\mathfrak{m}).$$

The above results are enough to prove one direction of our result. For the other direction we will require the following proposition.

Proposition 5.2. Let R be a Noetherian local ring and let $x \in R$ be a non-zero-divisor. Suppose that M is a non-zero finitely generated R-module such that $\operatorname{pd}(M)$ is finite. Then we have that:

$$\operatorname{pd}(M/xM) = \operatorname{pd}(M) + 1.$$

Proof. We refer the reader to ([23], page 104) for a comprehensive proof. □

An important fact we will need is that regular systems of parameters are examples of so-called R-regular sequences which we define below. For a proof of this fact, see ([15], page 108).

Definition 5.2. (R-Regular Sequence) Let R be a ring, let M be an R-module and let $x_1, \ldots, x_r \in R$. We say that $\{x_1, \ldots, x_r\}$ is a R-**regular sequence** if the following conditions are satisfied:

1. For each $1 \leq i \leq r, x_i$ is a non-zero-divisor on $M/\langle x_1, \ldots, x_{i-1}\rangle M$.
2. $M \neq \sum_{i=1}^r x_i M$.

5.2 Proof of Serre's global dimension theorem

Theorem 5.2. [20] Let R be a Noetherian local ring. Then:

$$R \text{ is regular local} \iff \operatorname{gl}\dim(R) = \dim(R) \iff \operatorname{gl}\dim(R) < \infty$$

Proof. Suppose that $(R, \mathfrak{m})$ is a n-dimensional Noetherian local ring with $\operatorname{gl}\dim R = n$. By the Auslander-Buchsbaum formula with $M = R/\mathfrak{m}$, we have:

$$\operatorname{pd}(R/\mathfrak{m}) + \operatorname{depth}_{\mathfrak{m}}(R/\mathfrak{m}) = \operatorname{depth}_{\mathfrak{m}}(R).$$

Now we claim that $\operatorname{depth}_{\mathfrak{m}}(R/\mathfrak{m}) = 0$. Since $R/\mathfrak{m}$ is cyclic as a R-module, proposition 4.9 states:

$$\operatorname{Ext}^0_R(R/\mathfrak{m}, R/\mathfrak{m}) = \operatorname{Hom}_{R-\mathbf{Mod}}(R/\mathfrak{m}, R/\mathfrak{m}) \cong R/\mathfrak{m} \neq 0.$$

It follows that $\operatorname{pd}(R/\mathfrak{m}) = \operatorname{depth}_{\mathfrak{m}}(R)$. Combining this with proposition 5.1 we can form the following chain of inequalities:

$$\dim(R) \leq emb.\dim(R) \leq \operatorname{gl}\dim(R) = \operatorname{pd}(R/\mathfrak{m}) = \operatorname{depth}_{\mathfrak{m}}(R) \leq \dim(R).$$

Therefore we have that $\dim(R) = emb.\dim(R)$ and therefore R is a regular local ring as desired.

Conversely, suppose that $(R, \mathfrak{m})$ is a regular local ring such that $\dim(R) = n$. Since R is free, it is in particular projective as a module over itself. Therefore as we have already seen, $\operatorname{pd}(R) = 0$. Since R is a regular local ring there is a regular system of parameters $\{x_1, \ldots, x_n\}$ generating $\mathfrak{m}$, which form a R-regular sequence. Now by repeated application of proposition 5.2 we find that:

$$\begin{aligned} \operatorname{pd}(R/x_1R) &= \operatorname{pd}(R) + 1 = 1, \\ &\vdots \\ \operatorname{pd}(R/\langle x_1, \ldots, x_n\rangle R) &= \operatorname{pd}(R/\langle x_1, \ldots, x_{n-1}\rangle R) + 1 = n. \end{aligned}$$

Since $R/\langle x_1, \ldots, x_n\rangle R = R/\mathfrak{m}$, we have that $\operatorname{pd}(R/\mathfrak{m}) = \operatorname{gl}\dim(R) = n$ as desired.

□

5.3 Applications

We defined the process of localisation of rings in section 2. Localisation of modules is done in a similar fashion with the only difference being that the numerators come from the chosen module. This process can be used to prove the following corollary of our characterisation of regular local rings.

Proposition 5.3. Let R be a regular local ring and let $\mathfrak{p} \in \operatorname{Spec}(R)$. Then $R_{\mathfrak{p}}$ is a regular local ring.

Proof. (Sketch) Since R is a regular local ring, we know by proposition 2.6 that $R_{\mathfrak{p}}$ is also a Noetherian local ring and by Serre's global dimension theorem that $\operatorname{gl}\dim(R)$ is finite. We may therefore take an arbitrary projective resolution of any R-module M of finite length. Localising each R-module in this resolution at $\mathfrak{p}$ (which preserves exactness and projectivity as $R_{\mathfrak{p}}$-modules, see [2], page 39) we get a projective resolution of the $R_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ which can clearly have length at most that of the projective resolution of M. Since the projective resolution of M chosen was arbitrary, we may conclude that $\operatorname{gl}\dim(R_{\mathfrak{p}}) \leq \operatorname{gl}\dim(R) < \infty$ and so by Serre's global dimension theorem, it follows that $R_{\mathfrak{p}}$ is a regular local ring. □

This motivates the following definition which removes the local condition and hints at the geometric interpretation of such rings as regular schemes, which were introduced by Grothendieck in [10].

Definition 5.3. (Regular ring) Let R be a Noetherian ring. We say that R is a **regular ring** if $R_{\mathfrak{p}}$ is a regular local ring for every $\mathfrak{p} \in \operatorname{Spec}(R)$.

References

[1] E. F. Assmus Jr, *On the homology of local rings*, Illinois Journal of Mathematics, 3 (1959).

[2] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, 1969.

[3] M. Auslander, *On the dimension of modules and algebras. III. Global dimension*, Nagoya Mathematical Journal, 9 (1955).

[4] M. Auslander and D. A. Buchsbaum, *Homological dimension in noetherian rings*, Proceedings of the National Academy of Sciences, 42 (1956).

[5] M. Auslander and D. A. Buchsbaum, *Homological dimension in local rings*, Transactions of the American Mathematical Society, 85 (1957).

[6] W. Bruns and H. J. Herzog, *Cohen-macaulay rings*, no. 39, Cambridge University Press, 1998.

[7] D. Cox, J. Little, and D. OShea, *Ideals, varieties, and algorithms: an introduction to computational algebraic geometry and commutative algebra*, Springer Science & Business Media, 2013.

[8] D. Eisenbud, *Commutative algebra: with a view toward algebraic geometry*, vol. 150, Springer Science & Business Media, 2013.

[9] P. J. Freyd, *Abelian categories*, Harper & Row New York, 1964.

[10] A. Grothendieck, *Éléments de géométrie algébrique: I. le langage des schémas*, Publications Mathématiques de l'I.H.É.S, 4 (1960).

[11] D. Hilbert, *Ueber die theorie der algebraischen formen*, in Algebra· Invariantentheorie Geometrie, Springer, 1970.

[12] I. Kaplansky, *Fields and rings*, University of Chicago Press, 1972.

[13] T.-Y. Lam, *Lectures on modules and rings*, vol. 189, Springer Science & Business Media, 2012.

[14] T. Leinster, *Basic category theory*, vol. 143, Cambridge University Press, 2014.

[15] H. Matsumura, *Commutative algebra*, vol. 120, WA Benjamin New York, 1970.

[16] H. Matsumura, *Commutative ring theory*, no. 8, Cambridge University Press, 1989.

[17] M. Nagata et al., *Local rings*, vol. 13, Interscience publishers New York, 1962.

[18] M. S. Osborne, *Basic homological algebra*, vol. 196, Springer Science & Business Media, 2012.

[19] J. J. Rotman, *An introduction to homological algebra*, vol. 2, Springer, 2009.

[20] J. P. Serre, *Sur la dimension homologique des anneaux et des modules noethériens*, in Proceedings of the International Symposium on Algebraic Number Theory, Tokyo & Nikko, 1955.

[21] J. P. Serre, *Algèbre locale. Multiplicités: cours au Collège de France, 1957-1958, rédigé par Pierre Gabriel*, Springer, 1965.

[22] J. Von Neumann, *On regular rings*, Proceedings of the National Academy of Sciences, 22 (1936).

[23] C. A. Weibel, *An introduction to homological algebra*, no. 38, Cambridge University Press, 1995.