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Geometric Invariant Theory

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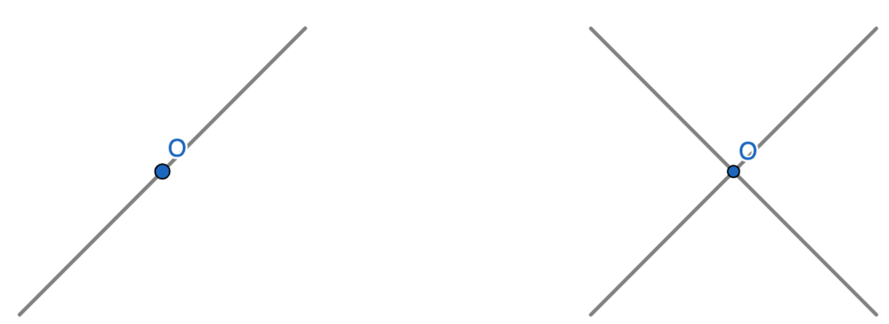
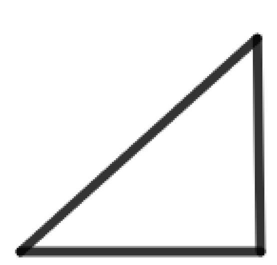
Introduction

- Invariant theory was first established by Arthur Cayley in 1845. David Hilbert brought a new approach to the subject in 1890, allowing it to now be realised as a common branch of representation theory, algebraic geometry, commutative algebra and algebraic combinatorics.
- Invariants can be thought of as properties which '*remain the same*' whenever a group acts upon an algebraic variety.
- A simple example is the well known discriminant of a quadratic binary form, which is given by $b^2 - ac$.



$\Delta = 0, \Delta \neq 0$

- Finding and classifying a complete set of invariants turns out to be *really* difficult.
- There are many mathematical methods to tackle this problem, but we want to make use of algebraic - geometric correspondences.

$a^2 + b^2 = c^2 \longleftrightarrow$ 

Aims

- Diagnose the invariance of a given algebraic object.
- Develop methods of constructing invariants, and analyse their usefulness.
- Develop methods to prove we have found them all, and to ensure we have no redundancy.
- Extract information via the algebraic - geometric correspondence about underlying properties shared by all objects of a given form.

Invariant Theory of Binary Forms

Definition

A *binary form of degree n* is a function of the form:

$$B_n(x, y) = a_0 x^n + \binom{n}{1} a_1 x^{n-1} y + \binom{n}{2} a_2 x^{n-2} y^2 + \cdots + a_n y^n,$$

where $(x, y, a_0, \ldots, a_n) \in \mathbb{C}^{n+3}$.

By setting $B_n(x, y)$ equal to zero, we obtain solutions we can understand geometrically. For example, we could get two intersecting lines. We want to understand the relation between these geometric conditions via a change of coordinates. The most general change of coordinates is given by

$$\begin{cases} x = \alpha X + \beta Y, \\ y = \gamma X + \delta Y, \end{cases}$$

which we may rewrite as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

where $(\alpha, \beta, \gamma, \delta) \in \mathbb{C}^4$.
We begin with an example to motivate the definition of an invariant.

Example

Consider the binary quadratic form

$$B_2(x, y) = a_0 x^2 + 2a_1 xy + a_2 y^2.$$

It is well known that the *discriminant* is given by $a_0 a_2 - a_1^2$, and that it is 'invariant' under a change of basis. Applying a general change of coordinates as above, we get

$$B_2(X, Y) = A_0 X^2 + 2A_1 XY + A_2 Y^2,$$

where

$$\begin{cases} A_0 = \alpha^2 a_0 + 2\alpha\gamma a_1 + \gamma^2 a_2, \\ A_1 = \alpha\beta a_0 + (\alpha\delta + \beta\gamma) a_1 + \gamma\delta a_2, \\ A_2 = \beta^2 a_0 + 2\beta\delta a_1 + \delta^2 a_2. \end{cases}$$

We discover that

$$A_0 A_2 - A_1^2 = \Delta^2 (a_0 a_2 - a_1^2),$$

where

$$\Delta = \det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

so the expression has only changed up to a factor of some power of Δ. For this reason it makes sense to assume that $\Delta \neq 0$, and we can therefore study $GL_2(\mathbb{C})$ transformations.

Diagnosis and Criteria

Definition

An *invariant* of a binary form of degree n is a polynomial function of the coefficients $a_0, \ldots, a_n$ which changes only by a factor equal to a power of the transformation determinant Δ if one replaces the coefficients $a_0, \ldots, a_n$ of the given binary form by the corresponding coefficients $A_0, \ldots, A_n$ of the linearly transformed binary form. In other words, $\mathcal{I}$ is an invariant if and only if there exists a non-negative integer p such that

$$\mathcal{I}(A_0, \ldots, A_n) = \Delta^p \mathcal{I}(a_0, \ldots, a_n).$$

Necessary Conditions

We find that an invariant $\mathcal{I}$ must satisfy:

$$ng = 2p,$$

and

$$a_0 \frac{\partial \mathcal{I}}{\partial a_1} + 2a_1 \frac{\partial \mathcal{I}}{\partial a_2} + 3a_2 \frac{\partial \mathcal{I}}{\partial a_3} + \cdots + n a_{n-1} \frac{\partial \mathcal{I}}{\partial a_n} = 0,$$

where g is the degree of the term as a polynomial in $\mathbb{C}[a_0, \ldots, a_n]$.

The number of invariants of degree g and weight p is given by:

$$T_n(g, p) = \left[\frac{(1 - x^{n+1})(1 - x^{n+2}) \cdots (1 - x^{n+g})}{(1 - x^2)(1 - x^3) \cdots (1 - x^g)} \right]_{x^p},$$

the coefficient of x^p when the fraction is expanded in terms of powers of x.

Example

There is only one invariant of degree 2 and only one invariant of degree 3 for the binary quartic. These are given by

$$I_2 = a_0 a_4 - 4a_1 a_3 + 3a_2^2,$$

and

$$I_3 = a_0 a_2 a_4 - a_0 a_3^2 - a_1^2 a_4 + 2a_1 a_2 a_3 - a_2^3.$$

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Representation theory of $\mathfrak{sl}_2(\mathbb{C})$

Consider the Lie Algebra $\mathfrak{sl}_2(\mathbb{C}) = \mathrm{span}_{\mathbb{C}}\langle \mathbf{e}, \mathbf{f}, \mathbf{h}\rangle$, where

$$\mathbf{e} = y\partial_x, \quad \mathbf{f} = x\partial_y, \quad \mathbf{h} = [\mathbf{e}, \mathbf{f}] = y\partial_y - x\partial_x.$$

By treating the a_i's as variables, we can find the action of these operators on the a_i's by noting that they must "kill" the binary forms:

$$\begin{cases} \mathbf{e}(a_i) = -ia_{i-1}, \\ \mathbf{f}(a_i) = -(n-i)a_{i+1}, \\ \mathbf{h}(a_i) = (n-2i)a_i. \end{cases}$$

We can consider the $\mathbf{h}$-eigenspace:

If $\mathbf{h}(v) = \lambda v$ then $\mathbf{e}(v) = (\lambda{+}2)v$ and $\mathbf{f}(v) = (\lambda{-}2)v$.

Since $\mathbf{h} = [\mathbf{e}, \mathbf{f}]$, it follows that invariants must have weight 0. Putting this together, we get another definition of an invariant.

Definition

An *invariant* of a binary form of degree n is a polynomial function $\mathcal{I}$ of the coefficients $a_0, \ldots, a_n$ of weight 0 which satisfies $\mathbf{e}(\mathcal{I}) = 0$ and $\mathbf{f}(\mathcal{I}) = 0$.

If we let $V = \langle a_0, \ldots, a_n\rangle$ then we can tensor up to get

$$V \otimes V = \bigoplus_{i=1}^{n} \mathbf{2i - 1},$$

where the right hand side is the direct sum of odd dimensional representations. This can be proven using partition theory arguments on the weights. The 1-dimensional representation is invariant under the actions of both $\mathbf{e}$ and $\mathbf{f}$, and so by definition must contain an invariant. In particular, it must be a linear combination of the weight 0 elements in $V \otimes V$. We can consider the g-fold tensor product of V with itself to find more invariants. The g-fold symmetric tensor product $\odot^g V$ is naturally isomorphic to degree g polynomials in $\mathbb{C}[a_0, \ldots, a_n]$,which allows us to relate our invariants back to polynomials. Higher dimensional representations are also useful for finding invariants, as we see in this example.

Concrete Classifications

Example

We will use the theory developed thus far to derive the invariants of the binary cubic.
Let $V = \langle a_0, a_1, a_2, a_3\rangle$. We can tensor up to get

$$V \otimes V = \mathbf{7} \oplus \mathbf{5} \oplus \mathbf{3} \oplus \mathbf{1}.$$

Explicitly, we can write

$$\mathbf{3} = \langle a_1^2 - a_0a_2, a_0a_3 - a_1a_2, a_2^2 - a_1a_3\rangle,$$

which we can visualise as follows:

$$a_2^2 - a_1a_3 \xrightarrow{\mathbf{e}} a_0a_3 - a_1a_2 \xrightarrow{\mathbf{e}/2} a_1^2 - a_0a_2 \xrightarrow{\mathbf{e}/3} 0$$

$$0 \xleftarrow{\mathbf{f}/3} a_2^2 - a_1a_3 \xleftarrow{\mathbf{f}/2} a_0a_3 - a_1a_2 \xleftarrow{\mathbf{f}} a_1^2 - a_0a_2$$

From our analysis of the 3-dimensional case we know that $A_1^2 - 4A_0A_2$ is an invariant. So

$$(a_0a_3 - a_1a_2)^2 - 4(a_1^2 - a_0a_2)(a_2^2 - a_1a_3)$$

is an invariant, namely it is I_4, the discriminant of the cubic.
For an invariant, we require $ng = 2p$. So for the binary cubic, we have that $p = \frac{3g}{2}$. Using our counting formula we find:

$$T_3\left(g, \frac{3g}{2}\right) = \left[\frac{1}{1-x^4}\right]_{x^g},$$

and so

$$\begin{cases} T_3\left(g, \frac{3g}{2}\right) = 1 & \text{if } g \equiv 0 \mod 4, \\ T_3\left(g, \frac{3g}{2}\right) = 0 & \text{otherwise.} \end{cases}$$

This means that the ring of invariants of the binary cubic is given by $\langle I_4, I_4^2, I_4^3, \ldots\rangle$. Clearly these are linearly independent, and so we have a complete classification for the binary cubic.

We have already seen the invariants of degree 2 and 3 of the binary quartic. Using our counting formula, we find that there are as many linearly independent invariants of degree g as there are solutions to the equation

$$2m + 3n = g,$$

where both m and n are non-negative integers. But then $I_2^m I_3^n$ is an invariant, and it is possible to prove that these are all indeed linearly independent. So we have a complete classification for the binary quartic.

Visualisation

g	1	2	3	4	5	6	7	8	9	10
$T_n(g, 2g)$		$\color{red}{\bullet}$	$\color{blue}{\bullet}$	$\color{red}{\bullet\bullet}$	$\color{red}{\bullet}\color{blue}{\bullet}$	$\color{red}{\bullet\bullet\bullet}$, $\color{blue}{\bullet\bullet}$	$\color{red}{\bullet\bullet}\color{blue}{\bullet}$	$\color{red}{\bullet\bullet\bullet\bullet}$, $\color{red}{\bullet}\color{blue}{\bullet\bullet}$	$\color{red}{\bullet\bullet\bullet}\color{blue}{\bullet}$, $\color{blue}{\bullet\bullet\bullet}$	$\color{red}{\bullet\bullet\bullet\bullet\bullet}$, $\color{red}{\bullet\bullet}\color{blue}{\bullet\bullet}$

Resultants and Joint Invariants

Definition

Let a and b be binary forms of degree m and n, respectively, where we drop the binomial coefficients for brevity. Then their *resultant* is given by the determinant of the $(n+m) \times (n+m)$ matrix:

$$R_{a,b} = \det \begin{pmatrix} a_0 & a_1 & \cdots & a_m & 0 & \cdots & 0 \\ 0 & a_0 & a_1 & \cdots & a_m & \cdots & 0 \\ & & & \ddots & & & \\ 0 & \cdots & 0 & a_0 & a_1 & \cdots & a_m \\ b_0 & b_1 & \cdots & b_n & 0 & \cdots & 0 \\ 0 & b_0 & b_1 & \cdots & b_n & \cdots & 0 \\ & & & \ddots & & & \\ 0 & \cdots & 0 & b_0 & b_1 & \cdots & b_n \end{pmatrix}.$$

For a degree m polynomial to have a repeated root, we require that the root is also a root of the derivative. In other words, $R_{a,a'} = 0$. For the quadratic case where $a = a_0x^2 + 2a_1x + a_2$, we find that $a_0(a_0a_2 - a_1^2) = 0$. Since a_0 is arbitrary, we get that $a_0a_2 - a_1^2 = 0$. This method allows us to construct discriminants of polynomials of any degree.

Example

Consider two binary quadratics

$a = a_0x^2 + 2a_1xy + a_2y^2, b = b_0x^2 + 2b_1xy + b_2y^2$.

In order for them to have a common factor, we demand $R_{a,b}{=}0$. But

$$R_{a,b} = (a_0b_2 - 2a_1b_1 + a_2b_0)^2 - 4I_aI_b,$$

where $I_a = a_0a_2 - a_1^2$ and $I_b = b_0b_2 - b_1^2$ are invariants. This implies that

$$I_{a,b} = a_0b_2 - 2a_1b_1 + a_2b_0$$

is also an invariant, which we then call a *joint invariant*. Note that we could also get this from considering

$$\langle a_0, a_1, a_2\rangle \otimes \langle b_0, b_1, b_2\rangle.$$

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Free Resolutions

Definition

A short exact sequence is a sequence of morphisms between objects of the form

$$0 \to A \overset{f}{\hookrightarrow} B \overset{g}{\twoheadrightarrow} C \to 0,$$

where f is a monomorphism and g is an epimorphism with the property that $\operatorname{Im} f = \ker g$.

Example

Consider the following sequence of abelian groups:

$$0 \to \mathbb{Z} \overset{\times 2}{\hookrightarrow} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z} \to 0.$$

Then this sequence is exact because the image of f is $2\mathbb{Z}$, which is precisely the kernel of g.

Definition

A *free resolution* of a module M over a ring R is an exact sequence of free R-modules

$$\cdots \xrightarrow{d_{n+1}} E_n \xrightarrow{d_n} \cdots \xrightarrow{d_1} E_0 \xrightarrow{\varepsilon} M \longrightarrow 0.$$

Combined with a grading on the modules, this allows for a clean description of the dimensions of the graded elements, allowing one to solve combinatorial problems, identify invariants and describe a simple set of generators for relations.

Lemma

If the sequence

$$0 \to V_1 \to V_2 \to \cdots \to V_n \to 0$$

is exact, then

$$\sum_{i=1}^{n} (-1)^i \dim V_i = 0.$$

Non-Intersection Varieties

Lemma

$$\omega_1 \wedge \cdots \wedge \omega_p = 0 \iff \exists\, c_1, \ldots, c_p \in R \text{ such that } \sum_i c_i \omega_i = 0.$$

Suppose that we have two binary forms a and b of degree n and m, respectively, with common factor h of degree p. In that case

$$\begin{cases} a = h\alpha, \\ b = h\beta, \end{cases}$$

for some degree $n-p$ and degree $m-p$ binary forms α and β. These can be related by the simple equation $\alpha b - \beta a = 0$, which gives an equation like

$$\beta_0 x^{m-p} a + \beta_1 x^{m-p-1} y a + \cdots + \beta_{m-p} y^{m-p} a - \alpha_0 x^{n-p} b - \cdots - \alpha_{n-p} y^{n-p} b = 0.$$

So we have a linear relation between $\langle x^{m-p}a, x^{m-p-1}ya, \ldots, y^{n-p}b\rangle$, which means they are linearly dependent. This is a linear relation between $m+n-2p+2$ elements, each of which lies in an $m+n-p+1$ dimensional space. Now we notice that

$$x^{m-p}a \wedge \cdots \wedge y^{m-p}a \wedge x^{n-p}b \wedge \cdots \wedge y^{n-p}b = 0 \in \bigwedge^{m+n-2p+2},$$

which has dimension

$$\binom{m+n-p+1}{m+n-2p+2}.$$

This can be interpreted as the number of conditions this relation is imposing on the relation between the original forms. In the case $p = 1$, we find that it imposes one condition on the original forms, which is equivalent to the resultant $R_{a,b} = 0$. In the case $p = 2$, we find that it imposes $m+n-1$ conditions on the original forms. How can we view these?

Example

Consider the case where a and b are both binary cubics, where $p = 2$. Then we have that

$$xa \wedge ya \wedge xb \wedge yb = 0.$$

We can then wedge this with an arbitrary 1-form c to get the determinant

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 & 0 \\ 0 & b_0 & b_1 & b_2 & b_3 \\ c_0 & c_1 & c_2 & c_3 & c_4 \end{vmatrix}.$$

Since c was arbitrary, we can expand along the bottom row and get that each of the five 4×4 minors involved must be 0. This gives us the 5 desired conditions.

Exterior Algebra

Definition

The exterior algebra of a vector space V over a field F is defined as the quotient algebra of the tensor algebra $T(V)$ by the two-sided ideal I generated by all elements of the form $x \otimes x$ for $x \in V$.

The k^{th} *exterior power* of V is the vector subspace of the exterior algebra of V, given by

$$\bigwedge^k(V) = \langle x_1 \wedge x_2 \wedge \cdots \wedge x_k, \quad x_i \in V\rangle.$$

The wedge product is the 'morally correct' product to use when computing volume elements. Sometimes this vanishes, but other times we are left with an invariant.

Conclusions

- It is possible to relate coordinate independent geometrical statements to coordinate dependent invariant algebraic statements using the methods presented here.
- We can understand the geometric meaning of invariants by using projective varieties.
- Further research -we could find a systematic way to identify relations between invariants, and a generalised method of discovering the geometrical significance of invariants of binary forms.

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