Geometric Invariant Theory

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Introduction to Invariant Theory

- Invariant theory was first established by Cayley in 1845. Hilbert brought a new approach to the subject in 1890, allowing it to now be realised as a common branch of representation theory, algebraic geometry, commutative algebra and algebraic combinatorics.
- Invariants can be thought of as properties which '*remain the same*' whenever a group acts upon an algebraic variety.
- Example - the well known discriminant of a quadratic binary form is given by $b^2 - ac$.

Figure 1: $\Delta = 0, \Delta \neq 0$

- Finding and classifying a complete set of invariants is *really* difficult.
- Many mathematical methods to tackle this problem.
- Want to make use of algebraic - geometric correspondences.

$$a^2 + b^2 = c^2 \longleftrightarrow$$

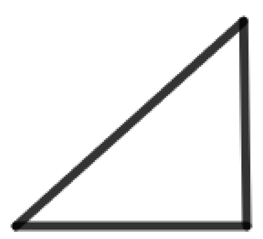

Figure 2: A simple algebraic-geometric correspondence

Main Aims of My Research

- Diagnose the invariance of a given algebraic object.
- Develop methods of constructing invariants, and analyse their usefulness.
- Develop methods to prove we have found them all, and to ensure we have no redundancy.
- Extract information via the algebraic - geometric correspondence about underlying properties shared by all objects of a given form.

Binary Forms- a Classical Approach

Setup

A *binary form of degree* n is a function of the form:

$$B_n(x,y) = a_0x^n + \binom{n}{1}a_1x^{n-1}y + \binom{n}{2}a_2x^{n-2}y^2 + \cdots + a_ny^n,$$

where $(x, y, a_0, \dots, a_n) \in \mathbb{C}^{n+3}$.

$GL_2(\mathbb{C})$ acts naturally as a linear transformation on (x, y):

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x + \beta y \\ \gamma x + \delta y \end{pmatrix}$$

The transformation maps degree n binary forms to degree n binary forms with coefficients $a_i \mapsto A_i$.

Diagnosis and Criteria

$\mathcal{I}$ is an invariant $\iff \mathcal{I}(A_0, \dots, A_n) = \Delta^p \mathcal{I}(a_0, \dots, a_n)$.

We find that an invariant $\mathcal{I}$ must satisfy:

$$ng = 2p,$$

and

$$a_0\frac{\partial \mathcal{I}}{\partial a_1} + 2a_1\frac{\partial \mathcal{I}}{\partial a_2} + 3a_2\frac{\partial \mathcal{I}}{\partial a_3} + \cdots + na_{n-1}\frac{\partial \mathcal{I}}{\partial a_n} = 0,$$

where g is the degree of the term as a polynomial in $\mathbb{C}[a_0, \dots, a_n]$.

The number of invariants of degree g and weight p is given by:

$$T_n(g,p) = \left[\frac{(1-x^{n+1})(1-x^{n+2})\cdots(1-x^{n+g})}{(1-x^2)(1-x^3)\cdots(1-x^g)}\right]_{x^p},$$

the coefficient of x^p when the fraction is expanded in terms of powers of x.

Example - for the binary quartic, we have $p = 2g$. Then

g	1	2	3	4	5	6	7	8	9
$T_n(g,2g)$		${\color{red}\bullet}$	${\color{blue}\bullet}$	${\color{red}\bullet\bullet}$	${\color{red}\bullet}{\color{blue}\bullet}$	${\color{red}\bullet\bullet\bullet}$, ${\color{blue}\bullet\bullet}$	${\color{red}\bullet\bullet}{\color{blue}\bullet}$	${\color{red}\bullet\bullet\bullet\bullet}$, ${\color{red}\bullet}{\color{blue}\bullet\bullet}$	${\color{red}\bullet\bullet\bullet}{\color{blue}\bullet}$, ${\color{blue}\bullet\bullet\bullet}$

The ring of invariants is generated by the invariants of degree 2 and degree 3.

Representation theory of $\mathfrak{sl}_2(\mathbb{C})$

Setup

Consider the Lie Algebra $\mathfrak{sl}_2(\mathbb{C}) = \text{span}_{\mathbb{C}}\langle \mathbf{e}, \mathbf{f}, \mathbf{h} \rangle$, where

$$\mathbf{e} = y\partial_x, \quad \mathbf{f} = x\partial_y, \quad \mathbf{h} = y\partial_y - x\partial_x.$$

The action of these operators on the a_i's are as follows:

$$\mathbf{e}(a_i) = -ia_{i-1}, \quad \mathbf{f}(a_i) = -(n-i)a_{i+1}, \quad \mathbf{h}(a_i) = (n-2i)a_i.$$

We can consider the $\mathbf{h}$- eigenspace:

If $\mathbf{h}(v) = \lambda v$ then $\mathbf{e}(v) = (\lambda + 2)v$ and $\mathbf{f}(v) = (\lambda - 2)v$

Diagnosis and Criteria

Our diagnosis of an invariant is slightly different:

$$\mathcal{I} \text{ is an invariant } \iff \mathbf{e}(\mathcal{I}) = \mathbf{f}(\mathcal{I}) = 0.$$

Since $\mathbf{h} = [\mathbf{e}, \mathbf{f}]$ it also follows that invariants have weight 0. If we let $V = \langle a_0, \dots, a_n \rangle$ then we can tensor up to get

$$V \otimes V = \bigoplus_{i=1}^{n} \mathbf{2i-1},$$

the direct sum of $(2i-1)$-dimensional representations. The 1-dimensional representation is invariant under the actions of both $\mathbf{e}$ and $\mathbf{f}$, and so must contain an invariant - it must be a linear combination of the weight 0 elements in $V \otimes V$.

$$a_2^2 - a_1a_3 \xrightarrow{\mathbf{e}} a_0a_3 - a_1a_2 \xrightarrow{\mathbf{e}/2} a_1^2 - a_0a_2 \xrightarrow{\mathbf{e}/3} 0$$

$$0 \xleftarrow{\mathbf{f}/3} a_2^2 - a_1a_3 \xleftarrow{\mathbf{f}/2} a_0a_3 - a_1a_2 \xleftarrow{\mathbf{f}} a_1^2 - a_0a_2$$

Figure 3: An irreducible $\mathbf{3}$ dimensional representation in $\langle a_0, \dots, a_3 \rangle \otimes \langle a_0, \dots, a_3 \rangle$.

The symmetric tensor product $\bigodot^g V$ is isomorphic to degree g polynomials in $\mathbb{C}[a_0, \dots, a_n]$ allowing us to link this back to polynomials. Higher dimensional representations are also useful for finding invariants.

Analysis of these methods

Both aforementioned methods give us a systematic way to construct invariants. The generating function allows us to know when to stop looking for invariants of a given degree, which is powerful. However, neither give us an immediate way to spot relations (or redundancy) between invariants.

Other methods

Exterior Algebra

The k^{th} exterior power of a vector space V is given by

$$\bigwedge^k(V) = \{x_1 \wedge x_2 \wedge \cdots \wedge x_k \mid x_i \in V, i = 1, 2, \dots, k.\}$$

- Intuitive notion of determinant.
- Top exterior power is one dimensional.

Free Resolutions

$$0 \to \ker A \xhookrightarrow{\iota} V \xrightarrow{A} \text{im}\, A \to 0,$$

- $\dim(\text{im}\, A) - \dim(V) + \dim(\ker A) = 0.$
- Can construct Hilbert–Poincaré series.

Non-Intersection Varieties

Two binary forms with common factor h.

$$\begin{cases} a = \alpha h, \\ b = \beta h. \end{cases}$$

Their resultant $R_{a,b} = 0$, but there are often more relations between the coefficients which must be satisfied.

Conclusions

- Geometrical statements$\leftrightarrow$Invariant algebraic statements.
- Can understand invariants using projective varieties.
- Directions for further research could be finding a systematic way to identify relations between invariants, and a generalised method of discovering the geometrical significance of invariants of binary forms, as this gets quite complicated for large n.

References

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