pg 1 Quiz 4 Solutions

1.

a) Domain of $C(z)$? $C(z)$ is given by a graph. In order to find the domain, look at the graph and see where the function is defined (which inputs have an output).

$C(z)$ is a piecewise function. This means the domain is a combination of domains for each piece.

The first piece is defined for $-3 < z < -1$ (not $\leq$ because of the open circle)

The second piece is defined for $-1 < z \leq 0$.

The third piece is defined for $1 \leq z \leq 4$

To combine these domains and get the domain of $C(z)$, we take the union of these intervals.

Domain: $(-3,-1) \cup (-1,0] \cup [1,4]$ or $-3 < z < -1$, $-1 < z \leq 0$, $1 \leq z \leq 4$

b.) Range of $b(w)$?

$$b(w) = \begin{cases} 1.5w + 8 & -5 \leq w < -1 \\ -4 \cdot 2^{-w} & 1 \leq w \leq 5 \end{cases}$$

To find the range, plug in the endpoints for each piece

1st piece: $b(w) = 1.5w + 8$ $\quad -5 \le w < 1$

$b(-5) = 1.5(-5) + 8 = -7.5 + 8 = 0.5$ or $\frac{1}{2}$

$b(-1) = 1.5(-1) + 8 = 6.5$ or $\frac{13}{2}$

* Keep in mind that -1 is not a part of the domain for the 1st piece. We are plugging it so that we know where our open circle would (if we were graphing it)

Range: $[\frac{1}{2}, \frac{13}{2})$

2nd piece: $b(w) = -4 \cdot 2^{-w}$ $\quad 1 \le w < 5$

$b(1) = -4 \cdot 2^{-1} = -4 \cdot (\frac{1}{2}) = -2$

$b(5) = -4 \cdot 2^{-5} = -4 \cdot (\frac{1}{2^5}) = -4 \cdot (\frac{1}{32}) = -\frac{1}{8}$

Range: $[-2, -\frac{1}{8}]$

Combine these ranges to get the total range: $[-2, -\frac{1}{8}] \cup [\frac{1}{2}, \frac{13}{2}]$

c)

i) $(a(-1))^{-1} = \frac{1}{a(-1)}$; $a(-1) = 2$ from the table

$(a(-1))^{-1} = \frac{1}{2}$ $\quad$ * Remember $a^{-1}(-1) \neq (a(-1))^{-1}$ *

ii) $a(a(-10))$; This is a composition. Composing $a(y)$ with itself.

$a(-10) = 4 \Rightarrow a(a(-10)) = a(4) = 3$

iii) $c(b(-5) + 2.5)$

$b(-5) = 1.5(-5) + 8 = 0.5 \Rightarrow c(b(-5) + 2.5) = c(0.5 + 2.5) = c(3) = 2$

iv) $b^{-1}(2)$; This is an inverse. The way we want to think about this is as follows:

If our output of $b(w)$ is 2, what input got us there?

Just set our eqn equal to 2 and solve!

$1.5w + 8 = 2$ $\quad$ * We use this piece because 2 is in 1st range *

$$1.5w = -6$$
$$w = -6/1.5 = -4$$
$$\Rightarrow b^{-1}(2) = -4$$

d.) Find solution to these eqns.

i) $c(a(y)) = 2$

Let's see when $c(z) = 2$. By looking at the graph, we can see that $c(z) = 2$ when $z = 1$ and $z = 3$

Now let's look at $a(y)$. Does $a(y)$ have an output of 1 or 3? If so, what input(s) gives us these values?

$a(y) = 3$ when $\boxed{y = 4}$

ii) $b(w) = a(3)$

First, find out what $a(3)$ is. $a(3) = 0$

Now, we can rewrite this as:

$b(w) = 0$

Let's think back to what the range of $b(w)$ was:

Range: $[-2, -\frac{1}{8}] \cup [1/2, 13/2)$

Is 0 in any of these intervals? Nope! This means that there is $\boxed{\text{no solution}}$!

2.) $H(x) = -x^2 + \frac{\pi}{2}x + \frac{1}{2}$

a) put this into vertex form using "complete the square"

① Factor out the coefficient in front of the x^2 term from the non-constant terms (x^2, x)

$$H(x) = -[x^2 - \frac{\pi}{2}x] + \frac{1}{2}$$

② Take the new b-value and divide by 2, then square it.

$$"(-\tfrac{\pi}{2}/2) = -\frac{\pi}{4} \rightarrow (-\frac{\pi}{4})^2 = \frac{\pi}{16}$$

③ Add and subtract this inside the bracket and after the x-term

$$H(x) = -[x^2 - \frac{\pi}{2}x + \frac{\pi^2}{16} - \frac{\pi^2}{16}] + \frac{1}{2}$$

④ Group the first 3 terms. You should be able to turn this into a perfect square.

$$H(x) = -[(x^2 - \frac{\pi}{2}x + \frac{\pi^2}{16}) - \frac{\pi}{16}] + \frac{1}{2}$$

$$H(x) = -[(x - \frac{\pi}{4})^2 - \frac{\pi}{16}] + \frac{1}{2}$$

⑤ Distribute the coefficient in front of bracket

$$H(x) = -(x - \frac{\pi}{4})^2 + \frac{\pi}{16} + \frac{1}{2}$$

⑥ Combine the constant terms

$$H(x) = -(x - \frac{\pi}{4})^2 + \frac{\pi^2 + 8}{16} \quad ; \quad \text{Vertex} = \left(\frac{\pi}{4}, \frac{\pi^2 + 8}{16}\right)$$

Note: The coefficient in front of the perfect square is negative
$\Rightarrow$ The max is at the vertex

b) Max height of bottle? The max value happens at the vertex

$$H(\tfrac{\pi}{4}) = -(\frac{\pi}{4} - \frac{\pi}{4})^2 + \frac{\pi^2 + 8}{16} = \frac{\pi^2 + 8}{16}$$