PS 1 Final Exam Fall 2017

1.

a) $\ln(10w + 3) = 8$

$10w + 3 = e^8$

$w = \dfrac{e^8 - 3}{10}$

b) $4\tan\left(\frac{2\pi}{3}x\right) = 5$ $\qquad -2 \le x \le 1$

$\tan\left(\frac{2\pi}{3}x\right) = \frac{5}{4}$ $\qquad P = \frac{\pi}{B} \Rightarrow P = \pi \cdot \frac{3}{2\pi} = \frac{3}{2}$

Calculator is going to give you the closest solution to the origin

Let $w = \frac{2\pi}{3}x \Rightarrow \tan(w) = \frac{5}{4}$

$w_1 = \tan^{-1}\left(\frac{5}{4}\right) \approx .896$

$w_2 = \pi + \tan^{-1}\left(\frac{5}{4}\right) \approx 4.037$

$w_1 = \frac{2\pi}{3}x_1 \Rightarrow \boxed{x_1 = \tan^{-1}\left(\frac{5}{4}\right) \cdot \frac{3}{2\pi} \approx 0.428}$

$w_2 = \frac{2\pi}{3}x_2 \Rightarrow x_2 = \pi + \tan^{-1}\left(\frac{5}{4}\right) \cdot \frac{3}{2\pi} \approx 1.927$ (outside of range)

Subtract Period from w_1

$\boxed{x_3 = \tan^{-1}\left(\frac{5}{4}\right) \cdot \frac{3}{2\pi} - \frac{3}{2} \approx -1.07}$

$x_4 = \tan^{-1}\left(\frac{5}{4}\right) \cdot \frac{3}{2\pi} - \frac{3}{2} - \frac{3}{2} \approx -2.57$ x

2.

a) $f(x)$ = periodic, period = 5, amp = 2, mid = $y = -1$

$g(x) = -7 f(2(x-3)) + 1$

horizontal compression by 2 ⇒ divide period by 2

multiply the period by $|-7| = 7$ ⇒ amp = 14

midline is stretched, then shifted up (order of transformations)

$(y = -1) \cdot -7 + 1 \Rightarrow (y = 7) + 1 \Rightarrow y = 8$

Outside changes affect the outputs values (range)

b) $a, b, c > 0$

$$\lim_{x \to \infty} \frac{(\sqrt{x}+2)^6}{ax^3 + bx + c} \approx \frac{(x^{1/2})^6}{ax^3} \approx \lim_{x \to \infty} \frac{x^3}{ax^3} = \frac{1}{a}$$

c) odd (goes to different ∞'s as $x \to \infty$, $x \to -\infty$)

leading coefficient: negative

smallest possible degree: 5 zeros with odd degree

d) quadratic ✓

$$\frac{M(3) - M(0)}{3-0} = \frac{900 - 625}{3} = 91.6$$

$$\frac{M(5) - M(3)}{5-3} = \frac{1296 - 900}{2} = 198$$

⇒ not linear could be quadratic

- $M(t)$ proportional to t^{2}?
  no, need $M(0) = 0$
  $M(t) \sim kt^2$

A quantity is proportional to x^n if
$y = kx^n$
where k is the proportionality constant
If n is positive, then x and y increase together

A quantity is inversely proportional to x^n if
$y = \frac{k}{x^n}$

$y = \frac{2}{x^2}$ $\Rightarrow$ y is inversely proportional to x^2

e.) $H(t) = 3e^{1.2t-3}$ grams of mealworm (grams), t days after it hatches

①. Find the weight when it hatches
②. Find the daily (non-continuous): rate
③. Amount of time it takes to triple in weight

① $H(t) = 3e^{1.2t-3} = \frac{3e^{1.2t}}{e^3}$ $\Rightarrow$ $H(0) = \frac{3}{e^3}$ grams

② $e^k = b$ $\Rightarrow$ $b = 1+r$ $\Rightarrow$ $r = b-1 = e^{1.2} - 1$

③ $3H(0) = 3 \cdot 3e^{-3} = 3e^{1.2t} \cdot e^{-3}$
$3 = e^{1.2t}$ $\rightarrow$ $\ln(3) = 1.2t$ $\rightarrow$ $t = \frac{\ln(3)}{1.2}$ days

3.

a) 24 hr cycle → period = 24   min sleep = 2.7 snores at 9:30 pm, max sleep = 8.9 snores at 9.30 am

- $W(t)$ = sleepiness t hours after 5pm on Friday
- This means that this function isn't starting at a min or max ⇒ need to shift

$y = A\cos(Bt) + k$

Let's find this function first

$B = \frac{2\pi}{P} = \frac{2\pi}{24} = \frac{\pi}{12}$,   mid. $y = \frac{\text{max}+\text{mid}}{2} = \frac{8.9+2.7}{2} = \frac{11.6}{2} = 5.8$

amp : max - mid = $8.9 - 5.8 = 3.1$

$y = 3.1\cos\left(\frac{\pi}{12}t\right) + 5.8$

- We know the max happens at 9:30 am
- For this function, the max happens at 5:00 pm.

We need to shift this over to the right by 16.5 (16 hours and 30 mins)

$y = 3.1\cos\left(\frac{\pi}{12}(t - 16.5)\right) + 5.8$

b) $T(t) = -2\sin\left(\frac{\pi}{8}(t+27)\right) + 7$ *HARD*

How to find when sleepiness rose above 5 for the first time

Pg 5

- Set $I(t) = 8$

$$-2\sin\left(\frac{\pi}{8}(t+2)\right)+7 = 8$$

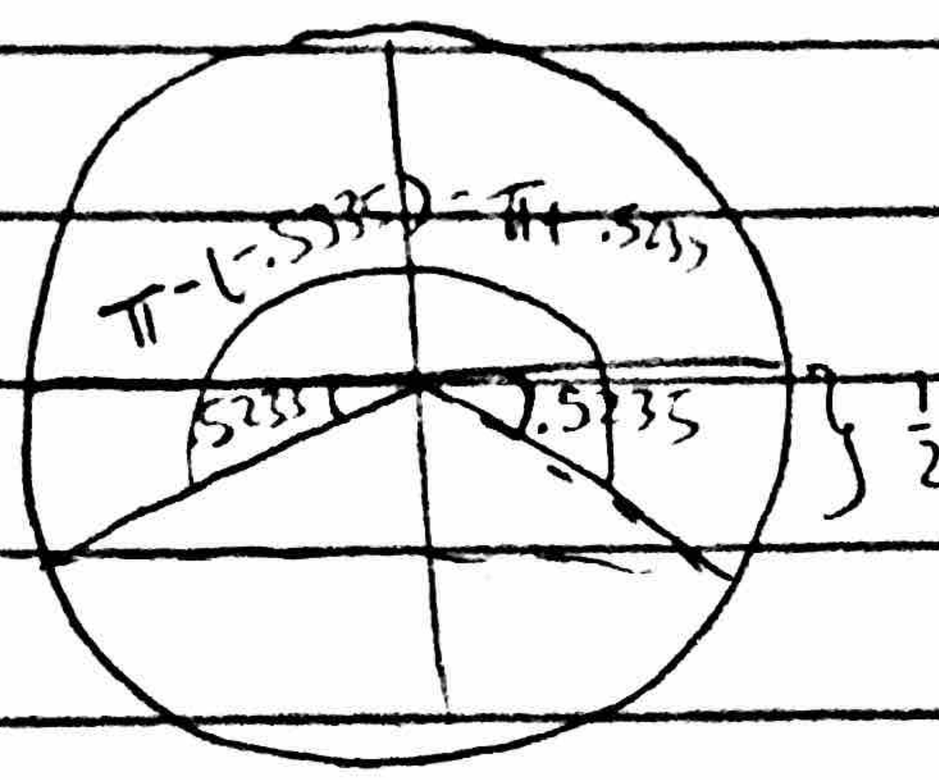
$$-2\sin\left(\frac{\pi}{8}(t+2)\right) = 1$$

$$\sin\left(\frac{\pi}{8}(t+2)\right) = -\frac{1}{2}$$

Let $w = \frac{\pi}{8}(t+2)$

$$\sin(w) = -\frac{1}{2}$$

$$w_1 = \sin^{-1}\left(-\frac{1}{2}\right) = -.5235$$



$$w_2 = \pi - \sin^{-1}\left(-\frac{1}{2}\right)$$

$$w_1 = \sin^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{8}(t+2) \Rightarrow$$

$$\frac{8}{\pi}\sin^{-1}\left(-\frac{1}{2}\right) = t_1+2 \Rightarrow t_1 = \frac{8}{\pi}\sin^{-1}\left(-\frac{1}{2}\right)-2 \approx -3.33$$

$$w_2 = \pi - \sin^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{8}(t+2)$$

$$\frac{8}{\pi}\left(\pi - \sin^{-1}\left(-\frac{1}{2}\right)\right) = t_2+2 \Rightarrow t_2 = \frac{8}{\pi}\left(\pi - \sin^{-1}\left(-\frac{1}{2}\right)\right)-2$$

So we have $t_1 = \frac{8}{\pi}\sin^{-1}\left(-\frac{1}{2}\right)-2 \approx -3.33$

$$t_2 = \frac{8}{\pi}\left(\pi - \sin^{-1}\left(-\frac{1}{2}\right)\right)-2 \approx 7.33$$

t_1 is not the answer because it's a negative time; Is t_2 the first time his sleepiness reaches 8?

Look at the graph of $T(t)$ and plot $y=8$ on the same graph on your calculators

Our first solution is the one closest to the origin and happens to be our first negative solution

Our second solution is the first time he reaches 8 snores! (you can check your answer by using the intersect function on your calculator

$$t = \frac{8}{\pi}\left(\pi - \sin^{-1}\left(\tfrac{1}{2}\right)\right) - 2 \approx 7.33 \text{ mins} \approx \frac{22}{3} \text{ mins}$$

4.

a) $A(x)$ has zeros at $x = 0, 2, 3$

b) $A(x) = k(x-0)^2(x-2)^2(x-3)$

$(x-0)$ and $(x-2)$ are squared becaus the graph bounces off the zero rather than crossing across at the zero (like in $(x-3)$)

Use other point to solve for k: $(1, 0.5)$

$A(x) = k\,x^2(x-2)^2(x-3)$

$A(1) = 0.5 = k(1)^2(1-2)^2(1-3)$

$0.5 = k(-2) \Rightarrow k = -0.25$

$\Rightarrow A(x) = -\frac{1}{4}x^2(x-2)^2(x-3)$

c) graph of $B(x)$ has V.A at $x=-1, x=1$ and a H.A at $y=0.8$

$B(x) = \frac{p(x)}{q(x)}$, where $p(x), q(x)$ are polynomials

$\Rightarrow B(x)$ is a rational func

$p(x)$ has zeros at $-2, 0, 2$
$q(x)$ has zeros at $x=-1, x=1$
} based off graph of $B(x)$

d) $B(x) = \frac{k(x+2)(x)(x-2)}{(x+1)^2(x-1)}$ ← (single zeros because graph crosses at zeros)

↑ squared because the graph blows up in the same direction on both sides of V.A at $x=-1$

READ

Remark: when we have a finite (constant # that isn't ∞ or $-\infty$) and non-zero H.A, that means the polynomials in the numerator and denominator have the same degree

when the H.A is zero, that means the degree of denominator > degree of numerator

Otherwise (no. H.A), then the degree of numerator > degree of denominator

Remember, we can think of the following limit:

$\lim_{x\to\infty} B(x)$ as the H.A

$\Rightarrow \lim_{x\to\infty} B(x) = 0.8$

$\lim_{x\to\infty} \frac{k(x+2)(x)(x-2)}{(x+1)^2(x-1)} = 0.8$ * This should look familiar; long run behavior of rational functions *

$\lim_{x\to\infty} \frac{k(x+2)(x)(x-2)}{(x+1)^2(x-1)} \approx \lim_{x\to\infty} \frac{(x)(x)(x)\cdot k}{(x)^2(x)} = \lim_{x\to\infty} \frac{kx^3}{x^3} = \lim_{x\to\infty} k = 0.8$

$\Rightarrow \boxed{k = 0.8}$

6.

t	4	6
g	100	1000

) g(t) where g(t) is an exponential function

$g(t) = ab^t$

Take the ratio of the given points from the table to solve for b

$\frac{g(6)}{g(4)} = \frac{\cancel{a}b^6}{\cancel{a}b^4} = b^2 = \frac{1000}{100} = 10$

$\Rightarrow b^2 = 10 \Rightarrow b = \sqrt{10}$

Use one pt to solve for a: $g(4) = a(\sqrt{10})^4 = 100$

Pg 9

$$a\left((10)^{1/2}\right)^4 = 100$$

$$a(10^2) = 100$$

$$a(100) = 100 \Rightarrow a = 1$$

$$\boxed{N = g(t) = (\sqrt{10})^t}$$

b) Look at Quiz 9 - 1c for solution

6b.) Look at where $f(x)$ is defined and use pts

$$F(x) = \begin{cases} ① & , -3 \le x \le 1 \\ ② & , 1 \le x \le 5 \end{cases}$$

① This is a linear function

$(1,-2)$, $(-3,-1)$

$$\Rightarrow m = \frac{-2-(-1)}{1-(-3)} = -\frac{1}{4}$$

Then use pt-slope form: $y - y_0 = m(x - x_0)$

$$y-(-1) = -\frac{1}{4}(x-(-3)) \Rightarrow y = -\frac{1}{4}(x+3)-1$$

② Same process

$$(1,1), (5,3) \Rightarrow m = \frac{3-1}{5-1} = \frac{2}{4} = \frac{1}{2}$$

$$y-1 = \frac{1}{2}(x-1) \Rightarrow y = \frac{1}{2}(x-1)+1$$

$$F(x) = \begin{cases} -\frac{1}{4}(x+3)-1 & , -3 \le x < 1 \\ \frac{1}{2}(x-1)+1 & , 1 \le x \le 5 \end{cases}$$

c) $G(w) = \sqrt{w+1}$ with domain $(0, 8)$
$H(w) = F(G(w))$

Domain of $H(w)$ = domain of $G(w)$ unless this causes issues with $F(w)$.

Use $G(w)$ to find out what you're plugging into $F(w)$

$G(0) = \sqrt{1} = 1$ $\quad$ $G(8) = \sqrt{8+1} = \sqrt{9} = 3$

So we're looking at values between $(1,3)$; we plug these into $F(w)$

$\Rightarrow$ $F(1) = 1$, $F(3) = 2$

$\Rightarrow$ Range of $H(w)$: $(1,2)$

Range of $H^{-1}(y)$? Since we know the domain of $H(w)$, we know the range of $H^{-1}(y)$

Range of $H^{-1}(y)$ = domain of $H(w)$
$= (0, 8)$

7. Solutions on Quiz 9 problem 2

Pg 11

8. $y = q(x) = \frac{ax}{1+ax}$ $a>0$

DO NOT SWAP Y AND X

$y = \frac{ax}{1+ax}$

$\boxed{\text{Just solve for } x}$

$(1+ax)y = ax$

$y + yax = ax$

$y = ax - yax$

$y = x(a - ay)$

Get anything with X in it on one side

$\Rightarrow x = \frac{y}{a-ay} = q^{-1}(y)$

9.

a) Invertible; D(t) is only positive and defined with domain: (0, 10) and range: [0, 9)

Since the range of D(t) is $0 \le D(t) \le 9$, we plug these values into S(d)

S(d) is invertible for $0 \le d \le 10$
since it's a quadratic func. with positive inputs
(if we included negative inputs, it might not be invertible; check graph of $-d^2 - 10d + 100$

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b) $S(D(t))=25$

First, need to find when $S(d)=25$

$$S(d)=-d^2-10d+100=25$$
$$\rightarrow d^2+10d-75=0$$
$$(d+15)(d-5)$$

$\Rightarrow D(t)=5$ or $D(t)=-15$

$D(t)$ is never negative (based on graph) so we ignore $D(t)=-15$

$D(t)=5$; Based graph $D(t)=5$ at $t=3$ or 3.1