Quiz 4 Solutions

1. Find the derivatives of the following functions

a) $f(x) = \tan(x)\sin(x) + e^{-x}\cos(x)$

$f'(x) = \left(\tan(x)\frac{d}{dx}[\sin(x)] + \frac{d}{dx}[\tan(x)]\sin(x)\right) +$
$\left(\frac{d}{dx}(e^{-x})\cos(x) + e^{-x}\frac{d}{dx}[\cos(x)]\right)$

$f'(x) = \left(\tan(x)\cos(x) + \sec^2(x)\sin(x)\right) + \left(-e^{-x}\cos(x) + e^{-x}(-\sin(x))\right)$

$f'(x) = \tan(x)\cos(x) + \sec^2(x)\sin(x) - e^{-x}\cos(x) - e^{-x}\sin(x)$

b) $f(x) = \dfrac{\pi \tan(x)}{\sqrt{e}\, x^4}$; Need to use quotient rule

$f'(x) = \dfrac{\pi}{\sqrt{e}}\dfrac{d}{dx}\left[\dfrac{\tan(x)}{x^4}\right] = \dfrac{\pi}{\sqrt{e}}\left[\dfrac{x^4(\sec^2(x)) - \tan(x)(4x^3)}{(x^4)^2}\right]$

$f'(x) = \dfrac{\pi}{\sqrt{e}}\left(\dfrac{x^4\sec^2(x) - 4x^3\tan(x)}{x^8}\right)$

c) Sorry for asking you to take this derivative.

$f(x) = \dfrac{\sin(x)\cos(x) - \tan(x)\cos(x)}{x^2\sin(x)}$

Need to use the quotient rule and then the product rule

$f(x) = \dfrac{\sin(x)\cos(x) - \tan(x)\cos(x)}{x^2\sin(x)}$

Let's calculate the derivative of the numerator and denominator

$$\frac{d}{dx}[\sin(x)\cos(x) - \tan(x)\cos(x)] = \left[\sin(x)\frac{d}{dx}(\cos(x)) + \frac{d}{dx}(\sin(x))\cos(x)\right) - \left(\frac{d}{dx}(\tan(x))\cos(x) + \tan(x)\frac{d}{dx}(\cos(x))\right)$$

$$= (-\sin(x)\sin(x) + \cos(x)\cos(x)) - (\sec^2(x)\cos(x) - \tan(x)\sin(x))$$

$$\frac{d}{dx}[\sin(x)\cos(x) - \tan(x)\cos(x)] = -\sin^2(x) + \cos^2(x) - \sec^2(x)\cos(x) + \tan(x)\sin(x)$$

$$\frac{d}{dx}[x^2\sin(x)] = \frac{d}{dx}(x^2)\sin(x) + x^2\frac{d}{dx}(\sin(x))$$
$$= 2x\sin(x) + x^2\cos(x)$$

$$\frac{d}{dx}\left[\frac{\sin(x)\cos(x) - \tan(x)\cos(x)}{x^2\sin(x)}\right] \Rightarrow$$

$$\frac{x^2\sin(x)(-\sin^2(x) + \cos^2(x) - \sec^2(x)\cos(x) + \tan(x)\sin(x)) - (\sin(x)\cos(x) - \tan(x)\cos(x))(2x\sin(x) + x^2\cos(x))}{x^4\sin^2(x)}$$

2. $f(x) = x^3e^x$

a) To find when it is increasing, we need to take the first derivative and see when it is > 0

$f'(x) = 3x^2e^x + x^3e^x$

$3x^2e^x + x^3e^x > 0$

$x^2e^x(3+x) > 0$

x^2e^x is always > 0 so we don't bother looking at that term.

$3+x > 0 \Rightarrow x > -3$

Thus, $f(x)$ is increasing when $x > -3$

b) For what interval(s) is $f(x)$ concave up?

We need to find when $f''(x) > 0$.

$$f'(x) = 3x^2e^x + x^3e^x$$
$$f''(x) = (6xe^x + 3x^2e^x) + (3x^2e^x + x^3e^x) > 0$$
$$f''(x) = e^x(6x + 3x^2 + 3x^2 + x^3)$$
$$f''(x) = e^x(x^3 + 6x^2 + 6x)$$
$$f''(x) = xe^x(x^2 + 6x + 6)$$

This is a bit harder. We need for the product of these two terms, xe^x and x^2+6x+6, to be positive. This when both terms are positive or both are negative.

$xe^x > 0 \Rightarrow x > 0$ (since e^x is always > 0)

$x^2 + 6x + 6 > 0$

Let's find the roots of this quadratic function

Quadratic formula: $\frac{-6 \pm \sqrt{36-24}}{2} = \frac{-6 \pm \sqrt{12}}{2} = \frac{-3 \pm \sqrt{4 \cdot 3}}{2}$

$= \frac{-3 \pm 2\sqrt{3}}{2} = -3 \pm \sqrt{3}$

Thus the roots are:

$x_1 = -3 + \sqrt{3}$ $\quad$ $x_2 = -3 - \sqrt{3}$

The next step is to find out if our 2nd derivative is positive between these two roots or negative between these roots. We can also determine whether it is positive (or negative) before and after the interval between the roots.

To recap, we're looking at point inside the interval

$(-3-\sqrt{3}, -3+\sqrt{3})$

A point to the right of the interval and a point to the left

$x = -3$ is inside the interval

Let $g(x) = x^2 + 6x + 6$

$g(-3) = (-3)^2 + 6(-3) + 6 = 9 - 18 + 6 = -3$

This means $g(x)$ is negative within this interval

$g(x) < 0$ on the interval $(-3-\sqrt{3}, -3+\sqrt{3})$

$x = -1$ is a point to the right

$g(-1) = (-1)^2 + 6(-1) + 6 = 1$

so $g(x) > 0$ on the interval $(-3+\sqrt{3}, \infty)$

$x = -5$ is a point to the left

$g(-5) = (-5)^2 + 6(-5) + 6 = 25 - 30 + 6 = 1$

so $g(x) > 0$ on the interval $(-\infty, -3-\sqrt{3})$

So far we know that we need
$f''(x) = xe^x(x^2+6x+6) > 0$

① $xe^x > 0$ when $x > 0$
② $xe^x < 0$ when $x < 0$

③ $(x^2+6x+6) < 0$ when $-3-\sqrt{3} < x < -3+\sqrt{3}$
④ $(x^2+6x+6) > 0$ when $x < -3-\sqrt{3}$ and $x > -3+\sqrt{3}$

Now, we need the product of these terms to be > 0, which is when both terms < 0 and both terms > 0

$xe^x < 0$ on $(-\infty, 0)$
$x^2+6x+6 < 0$ on $(-3-\sqrt{3}, -3+\sqrt{3})$

Therefore, we take the interval where both are negative: $(-3-\sqrt{3}, -3+\sqrt{3})$

$xe^x > 0$ on $(0, \infty)$
$x^2+6x+6 > 0$ on $(-\infty, -3-\sqrt{3})$ and $(-3+\sqrt{3}, \infty)$

Therefore, we take the interval where both are positive: $(0, \infty)$

Answer: $(-3-\sqrt{3}, -3+\sqrt{3}) \cup (0, \infty)$

c) Equation of tangent line at $x=-2$

The equation of the tangent line at a pt is a linear function:

$$y = mx + b \quad , \quad m = \text{derivative at that pt}$$

$$f'(x) = 3x^2 e^x + x^3 e^x$$
$$f'(-2) = 3(-2)^2 e^{-2} + (-2)^3 e^{-2} = 12e^{-2} - 8e^{-2} = 4e^{2}$$

$$y = 4e^{-2}x + b$$

We need the point on the graph where the tangent line and graph intersect

$$f(-2) = -8e^{-2}$$

Using this point, we have:

$$-8e^{-2} = 4e^{-2}(-2) + b$$
$$-8e^{-2} = -8e^{-2} + b \implies b = 0$$

Thus,

$y = 4e^{-2}x$ is the equation of the tangent line