Math 105 - 035 9/23/17

INVERSE FUNCTIONS AND DOMAIN + RANGE (SECTION 2.1 AND 2.5)

Domain and Range (Section 2.1)

We'll briefly recap domain and range since it is on Monday's quiz and I understand that some of you were confused when it came to determining what the range of a function is.

Domain.

Definition 1. The domain of a function is defined as *all* possible real values that can be used as inputs in the function.

First, we'll go over finding the domain. We can do this by following this general process:

(1) Look at the function and determine where the function could have some issues (dividing by 0, negative numbers under an *even* root, etc.)
(2) Solve for the x-values that cause these issues,which gives us the x-values that *can't* be in the domain.
(3) By determining what x-values can't be in the domain, we can deduce what values *can* be in the domain.

Example 1: $y = f(x) = \frac{10x}{x^2-4}$

In this example, we don't notice any issues in the numerator (top of the fraction). For the most part, the numerator won't have issues unless we have some expression that could create issues (such as a negative number under an even root).

However, there will be an issue in the denominator because it is possible for $x^2 - 4 = 0$. We've completed step 1 of our process, so now let's move onto step 2. When is $x^2 = 4$?

$$x^2 - 4 = 0$$
$$x^2 = 4$$
$$x = \pm 2$$

Note: When taking an even root of a number, we must always consider the positive and negative case ($-2 \cdot -2 = 4$ and $2 \cdot 2 = 4$ so both -2 and 2 are solutions)

We now know for what x values our function has issues, so we can deduce what values of x belong to the domain.

Answer: x can be any real number other than -2 and 2. $x \neq -2, 2$ or $(-\infty, -2) \cup (2, \infty)$

Range.

Definition 2. The range of a function is defined as all the possible real values that the output of the function can be.

The range is a little harder to determine since sometimes it requires us to look at the graph, but there are times when the graph doesn't tell us everything.

To determine the range, we have to look at the function and see how it might behave. We also have to consider the domain of the function.

Example 2:

$$f(x) = \begin{cases} 2x & \text{if } x \leq 1 \\ \sqrt{5 - \frac{1}{x}} & \text{if } x > 1 \end{cases}$$

Let's look at this piecewise function. Let's find the domain of the piecewise function. When $x \leq 1$, $f(x) = 2x$ and this function has no issues on the interval it's defined on, which is $(-\infty, 1]$.

This helps us find the range; We know that as x gets really small (really negative), we'll approach negative infinity. If we plug in $x = 1$ (since this is the endpoint of the domain for this part of the function), we'll get $f(1) = 2$.

We can say that the range for the first part of the piecewise function is $-\infty < f(x) \leq 2$.

Now for $x > 1$,$f(x) = \sqrt{5 - \frac{1}{x}}$. We know that we can not have a negative number under an even root, so let's see if it can be negative.

Let's test out some points such as: 2, 20, 200, and 2000.

$$\mathbf{x = 2:}\sqrt{5 - \frac{1}{2}} = \sqrt{4.5}$$
$$\mathbf{x = 20:}\sqrt{5 - \frac{1}{20}} = \sqrt{4.95}$$
$$\mathbf{x = 200:}\sqrt{5 - \frac{1}{200}} = \sqrt{4.995}$$
$$\mathbf{x = 2000:}\sqrt{5 - \frac{1}{2000}} = \sqrt{4.9995}$$

You can notice that we get closer and closer to $\sqrt{5}$ as x gets bigger and bigger.

You can try going in the opposite direction (get closer and closer to 1) and you'll see that $f(x)$ gets closer and closer to $\sqrt{4} = 2$. By observing the behavior of this function, we can see that $\sqrt{4} < f(x) < \sqrt{5}$.

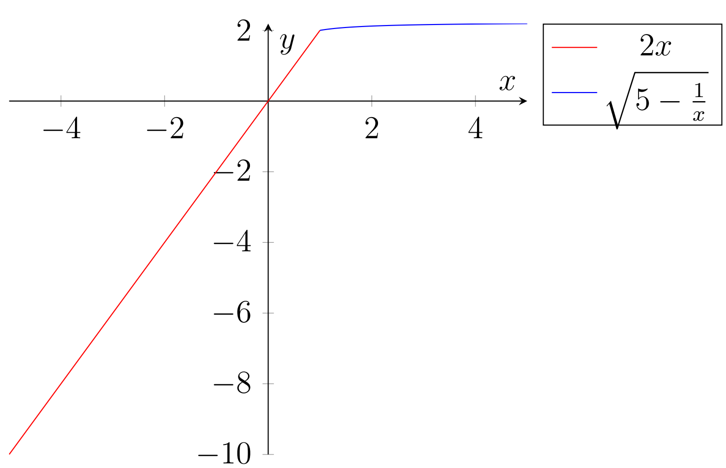
The range for the second part of the piecewise function is $\sqrt{4} = 2 < f(x) < \sqrt{5}$

Now, we combine the domain and range of both parts to get our complete and final answer:

Domain: All real numbers ($-\infty < x < \infty$)

Range: $(-\infty, \sqrt{5})$ or $-\infty < f(x) < \sqrt{5}$

Below is the graph of the piecewise function for you to compare



Inverse Functions (Section 2.5)

As I mentioned in class, we aren't going in detail when it comes to finding an inverse. We just need to know what an inverse function is, what it does, and one rule when it comes to composing a function with its inverse (and vice versa)

First, let's define what a function is again.

Definition 3. A function is a relationship between an independent variable (input), x, and a dependent variable (output), y, such that for each input there is *exactly one* output.

An inverse function is almost the exact same thing!

The inverse of a function is still the relationship between the quantities represented by x and y, except that y is now our independent variable (input) and x is our dependent variable (output). Like I said in class, we swap the input and output of the function.

This inverse function is denoted by f^{-1}. Please remember that $f^{-1} \neq \frac{1}{f(x)}$!!

Regular function: $y = f(x)$

Now assuming that $f(x)$ is invertible (which means that a function has an inverse and its inverse is also a function).

Inverse function: $x = f^{-1}(y)$

In the first case, x is the input and y is the output. In the second case, y is our input and x is our output.

Let's think about this using a real world example.

Example 3: Let's say we have a vending machine in our classroom and it contains the following five candies/snacks: Starbursts, Jolly Ranchers, Snickers, Gummy Bears, and Oreos. Our vending machine has five buttons labeled 1 through 5. The relationship between the candy and the buttons is defined as follows: Button 1 gives Starbursts, Button 2 gives Jolly Ranchers, Button 3 gives Snickers...etc. We call this function c = f(b), where c is the candy/snack and b is the button.

Assuming $f(x)$ is invertible, write out it's inverse function and determine what $f^{-1}(oreos)$ is.

Answer: Since the regular function has buttons as its input, this means that for the inverse function we will have buttons as an output and candy/snacks as an input.

Inverse function: $f^{-1}(c) = b$

Now, let's evaluate $f^{-1}(oreos)$

For the original function, we were told that when we hit button 5, we get oreos. So now if we swap in the input and output, we can say that if we received oreos, that must mean we hit button 5!

$f^{-1}(oreos) = 5$

Here are two additional concepts that are important to keep in mind when it comes to functions and their inverses:

Concept 1: The domain of a function ,f, is the range of its inverse function, f^{-1}, and the range of the function f is the domain of its inverse function, f^{-1}.

This makes sense because when we take the inverse of a function, we swap the inputs and outputs of the regular function.

Concept 2: When we compose a function with it's inverse or vice versa, we're left with the input of the inside function.

$$f^{-1}(f(x)) = x$$

$$f(f^{-1}(y)) = y$$

In other words, a function and its inverse "undo" each other.

To conclude inverses, I will do problem 40 on pg 103 of your text books.

Example 4: Suppose that $j(x) = h^{-1}(x)$ and that both j and h are defined for all values of x. Let $h(4) = 2$ and $j(5) = -3$. Evaluate if possible:

Note that $j(x)$ is the inverse of h

(a) $j(h(4))$
Answer: Note that we can use concept 2 for this problem. In this problem, we are composing the inverse of h, h^{-1}, with h. This can be rewritten as $h^{-1}(h(4))$. Using concept 2, the answer to this is 4.

However, let's step through the problem so that we can believe concept 2 is true.

We known that $h(4) = 2$ so this means that when we use the value 4 for the input of 4, our output is 2.

This results in the input of $h^{-1}(x)$ being changed to 2 instead of $h(4)$.

Now we're left with $h^{-1}(2) = ?$. This asks the following: When the output of h is 2, what is our input? If we refer back to what the problem tells us, we know the answer is 4.

$$h(4) = 2 \rightarrow h^{-1}(h(4)) = h^{-1}(2) = 4$$

(b) $j(4)$

Answer: This is asking that when the output of h is 4, what is the input? We cannot find the answer since we don't know what input value is used to get 4 as an output of h using the information given to us.

(c) $h(j(4))$

Answer: Using concept 2, the answer is 4.

(d) $j(2)$

Answer: Similar to before, this is asking what the input of $h(x)$ is when the output is 2. The problem gives us this information so we know that the answer is 4.

(e) $h^{-1}(-3)$

Answer: This is not possible. We don't know what input for $h(x)$ gives us an output of -3.

(f) $j^{-1}(-3)$

Answer: This can be written as $h(-3)$, which we observe is 5. $j(5) = -3$ gives us this answer. $j(5) = -3$ tells us that when the output of $h(x) = 5$, the the input (x) is -3. Final answer: 5

(g) $h(5)$

Answer: This is not possible. We are not given information to evaluate this.

(h) $(h(-3))^{-1}$

Answer: This can be rewritten as: $\frac{1}{h(-3)}$ Using the same logic in (f), this means that $(\frac{1}{h(-3} = \frac{1}{5}$.

(i) $(h(2))^{-1}$

Answer: This can be rewritten as: $\frac{1}{h(2)}$. However, we don't know what $h(2)$ is, so we cannot evaluate this part.