5.5 Bézier Curves

Pierre Bézier at Renault and Paul de Casteljau at Citroën independently developed the ***Bézier curve*** for CAD/CAM operations, in the 1970s. These parametrically defined polynomials are a class of approximating splines. Bézier curves are the basis of the entire Adobe PostScript drawing model that is used in the software products Adobe Illustrator, Macromedia Freehand, and Fontographer. Bézier curves continue to be the primary method of representing curves and surfaces in computer graphics (CAD/CAM, computer-aided geometric design).

In Casteljau's original development, Bézier curves were defined implicitly by a recursive algorithm (see Property 1 below). The development of the properties of Bézier curves will be facilitated by defining them explicitly in terms of *Bernstein polynomials*.

Definition 5.5. ***Bernstein polynomials*** of degree N are defined by

$$B_{i,N}(t) = \binom{N}{i} t^i (1-t)^{N-i},$$

for $i = 0, 1, 2, \ldots, N$, where $\binom{N}{i} = \dfrac{N!}{i!\,(N-i)!}$. ▲

In general, there are $N+1$ Bernstein polynomials of degree N. For example, the Bernstein polynomials of degrees 1, 2, and 3 are

$$B_{0,1}(t) = 1 - t,\ B_{1,1}(t) = t; \tag{1}$$

$$B_{0,2}(t) = (1-t)^2,\ B_{1,2}(t) = 2t(1-t),\ B_{2,2}(t) = t^2; \quad \text{and} \tag{2}$$

$$B_{0,3}(t) = (1-t)^3,\ B_{1,3}(t) = 3t(1-t)^2,\ B_{2,3}(t) = 3t^2(1-t),\ B_{3,3}(t) = t^3; \tag{3}$$

respectively.

Properties of Bernstein Polynomials

Property 1. Recurrence Relation

Bernstein polynomials can be generated in the following way. Set $B_{0,0}(t) = 1$ and $B_{i,N}(t) = 0$ for $i < 0$ or $i > N$, and use the recurrence relation

$$B_{i,N}(t) = (1-t)B_{i,N-1}(t) + tB_{i-1,N-1}(t) \quad \text{for } i = 1, 2, 3, \ldots, N-1. \tag{4}$$

Property 2. Nonnegative on [0, 1]

The Bernstein polynomials are nonnegative over the interval [0, 1] (see Figure 5.21).

Property 3. The Bernstein polynomials form a partition of unity

(5) $$\sum_{i=0}^{N} B_{i,N}(t) = 1$$

Substituting $x = t$ and $y = 1 - t$ into the binomial theorem

$$(x+y)^N = \sum_{i=0}^{N} \binom{N}{i} x^i y^{N-i}$$

yields

$$\sum_{i=0}^{N} \binom{N}{i} x^i y^{N-i} = (t + (1-t))^N = 1^N = 1.$$

Property 4. Derivatives

(6) $$\frac{d}{dt} B_{i,N}(t) = N(B_{i-1,N-1}(t) - B_{i,N-1}(t))$$

Formula (6) is established by taking the derivative of the Bernstein polynomial in Definition 5.5.

$$\begin{aligned}
\frac{d}{dt} B_{i,N}(t) &= \frac{d}{dt} \binom{N}{i} t^i (1-t)^{N-i} \\
&= \frac{iN!}{i!\,(N-i)!} t^{i-1}(1-t)^{N-i} - \frac{(N-i)N!}{i!\,(N-i)!} t^i (1-t)^{N-i-1} \\
&= \frac{N(N-1)!}{(i-1)!\,(N-i)!} t^{i-1}(1-t)^{N-i} - \frac{N(N-1)!}{i!\,(N-i-1)!} t^i (1-t)^{N-i-1} \\
&= N\left(\frac{(N-1)!}{(i-1)!\,(N-i)!} t^{i-1}(1-t)^{N-i} - \frac{(N-1)!}{i!\,(N-i-1)!} t^i (1-t)^{N-i-1}\right) \\
&= N\big(B_{i-1,N-1}(t) - B_{i,N-1}(t)\big)
\end{aligned}$$

Property 5. Basis

The Bernstein polynomials of order N ($B_{i,N}(t)$ for $i = 0, 1, \ldots, N$) form a basis of the space of all polynomials of degree less than or equal to N.

Property 5 states than any polynomial of degree less than or equal to N can be written uniquely as a linear combination of the Bernstein polynomials of order N. The concept of a basis of a vector space is introduced in Chapter 11.

Given a set of *control points*, $\{\mathbf{P}_i\}_{i=0}^{N}$, a Bézier curve of degree N is now defined as a weighted sum of the Bernstein polynomials of degree N.

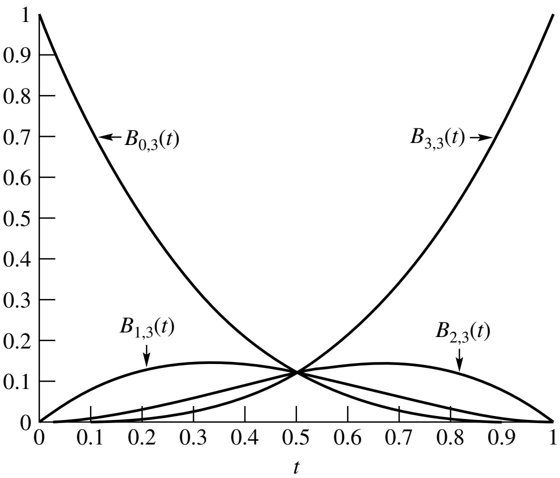

Figure 5.21 Bernstein polynomials of degree three.

Definition 5.6. Given a set of control points $\{\mathbf{P}_i\}_{i=0}^N$, where $\mathbf{P}_i = (x_i, y_i)$, a ***Bézier curve of degree*** N is

$$\mathbf{P}(t) = \sum_{i=0}^{N} \mathbf{P}_i B_{i,N}(t), \tag{7}$$

where $B_{i,N}(t)$, for $i = 0, 1, \ldots, N$, are the Bernstein polynomials of degree N, and $t \in [0, 1]$. ▲

In formula (7) the control points are ordered pairs representing x- and y-coordinates in the plane. Without ambiguity the control points can be treated as vectors and the corresponding Bernstein polynomials as scalars. Thus formula (7) can be represented parametrically as $\mathbf{P}(t) = (x(t), y(t))$, where

$$x(t) = \sum_{i=0}^{N} x_i B_{i,N}(t) \quad \text{and} \quad y(t) = \sum_{i=0}^{N} y_i B_{i,N}(t), \tag{8}$$

and $0 \le t \le 1$. The function $\mathbf{P}(t)$ is said to be a vector-valued function, or equivalently, the range of the function is a set of points in the xy-plane.

Example 5.16. Find the Bézier curve which has the control points $(2, 2)$, $(1, 1.5)$, $(3.5, 0)$, and $(4, 1)$.

Substituting the x- and y-coordinates of the control points and $N = 3$ into formula (8) yields

$$x(t) = 2B_{0,3}(t) + 1B_{1,3}(t) + 3.5B_{2,3}(t) + 4B_{3,3}(t) \tag{9}$$

$$y(t) = 2B_{0,3}(t) + 1.5B_{1,3}(t) + 0B_{2,3}(t) + 1B_{3,3}(t). \tag{10}$$

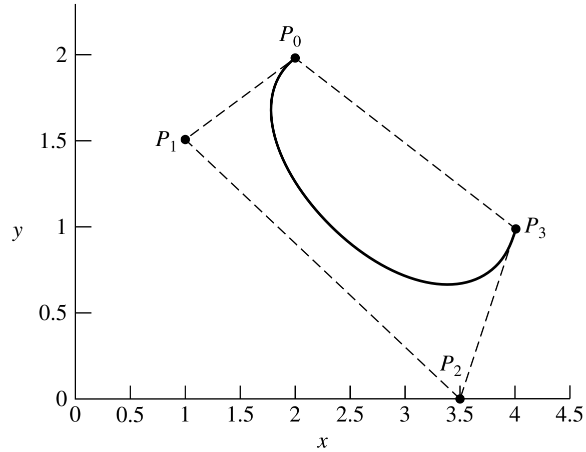

Figure 5.22 Bézier curve of degree three and convex hull of control points.

Substituting the Bernstein polynomials of degree three, found in formula (3), into formulas (9) and(10) yields

$$x(t) = 2(1-t)^3 + 3t(1-t)^2 + 10.5t^2(1-t) + 4t^3 \tag{11}$$

$$y(t) = 2(1-t)^3 + 4.5t(1-t)^2 + t^3. \tag{12}$$

Simplifying formulas (11) and (12) yields

$$\mathbf{P}(t) = (2 - 3t + 10.5t^2 - 5.5t^3,\ 2 - 1.5t - 3t^2 + 3.5t^3),$$

where $0 \le t \le 1$. ■

The functions $x(t)$ and $y(t)$ in formulas (11) and (12) are polynomials and are continuous and differentiable over the interval $0 \le t \le 1$. Thus the graph of the Bézier curve $\mathbf{P}(t)$ is a continuous and differentiable curve in the xy-plane (see Figure 5.22), where $0 \le t \le 1$. *Note.* $\mathbf{P}(0) = (2, 2)$ and $\mathbf{P}(1) = (4, 1)$. The graph of the curve starts at the first control point (2, 2) and ends at the last control point (4, 1).

Properties of Bézier Curves

Property 1. The points $\mathbf{P}_0$ and $\mathbf{P}_1$ are on the curve $P(t)$

Substituting $t = 0$ into Definition 5.5 yields

$$B_{i,N}(0) = \begin{cases} 1 & \text{for } i = 0 \\ 0 & \text{for } i \neq 0. \end{cases}$$

Similarly, $B_{i,N}(1) = 1$ for $i = N$ and is zero for $i = 0, 1, \ldots, N-1$. Substituting these results into Definition 5.6 yields

$$\mathbf{P}(0) = \sum_{i=0}^{N} \mathbf{P}_i B_{i,N}(0) = \mathbf{P}_0 \quad \text{and} \quad \mathbf{P}(1) = \sum_{i=0}^{N} \mathbf{P}_i B_{i,N}(1) = \mathbf{P}_N.$$

Thus the first and last points in the sequence of control points, $\{\mathbf{P}_i\}_{i=0}^{N}$, are the endpoints of the Bézier curve. *Note*. The remaining control points are not necessarily on the curve.

In Example 5.16 there were four control points and the resulting components $x(t)$ and $y(t)$ were third-degree polynomials. In general, when there are $N + 1$ control points the resulting components will be polynomials of degree N. Since polynomials are continuous and have continuous derivatives of all orders, it follows that the Bézier curve in Definition 5.6 will be continuous and have derivatives of all orders.

Property 2. $\mathbf{P}(t)$ is continuous and has derivatives of all orders on the interval [0, 1]

The derivative of $\mathbf{P}(t)$, with respect to t, is

$$\begin{aligned}\mathbf{P}'(t) &= \frac{d}{dt}\sum_{i=0}^{N} \mathbf{P}_i B_{i,N}(t) \\ &= \sum_{i=0}^{N} \mathbf{P}_i \frac{d}{dt} B_{i,N}(t) \\ &= \sum_{i=0}^{N} \mathbf{P}_i N(B_{i-1,N-1}(t) - B_{i,N-1}(t))\end{aligned}$$

(Property 4 of Bernstein polynomials). Setting $t = 0$ and substituting $B_{i,N}(0) = 1$ for $i = 0$ and $B_{i,N}(0) = 0$ for $i \geq 1$ (Definition 5.5) into the right-hand side of the expression for $P'(t)$ and simplifying yields

$$\begin{aligned}\mathbf{P}'(0) &= \sum_{i=0}^{N} \mathbf{P}_i N(B_{i-1,N-1}(0) - B_{i,N-1}(0)) \\ &= N(\mathbf{P}_1 - \mathbf{P}_0).\end{aligned}$$

Similarly, $\mathbf{P}'(1) = N(\mathbf{P}_N - \mathbf{P}_{N-1})$. In other words, the tangent lines to a Bézier curve at the endpoints are parallel to the lines through the endpoints and the adjacent control points. The property is illustrated in Figure 5.23.

Property 3. $\mathbf{P}'(0) = N(\mathbf{P}_1 - \mathbf{P}_0)$ and $\mathbf{P}'(1) = N(\mathbf{P}_N - \mathbf{P}_{N-1})$

The final property is based on the concept of a ***convex set***. A subset C of the xy-plane is said to be a convex set, provided that all the points on the line segment joining any two points in C are also elements of the set C. For example, a line segment or a circle and its interior are convex sets, while a circle without its interior is not a convex set. The convex set concept extends naturally to higher-dimension spaces.

Definition 5.7. The ***convex hull*** of a set C is the intersection of all convex sets containing C. ▲

Figure 5.22 shows the convex hull (the indicated quadrilateral and its interior) of the control points for the Bézier curve from Example 5.16. In the xy-plane the convex hull of a set of points, $\{\mathbf{P}_i\}_{i=0}^{N}$, may be *visualized* by placing pins at each point and placing a rubber band around the resulting configuration.

A sum $\sum_{i=0}^{N} m_i \mathbf{P}_i$ is said to be a *convex combination* of the points $\{\mathbf{P}_i\}_{i=0}^{N}$, provided that the set of coefficients $m_0, m_1, \ldots, m_N$ are nonnegative and $\sum_{i=0}^{N} m_i = 1$. A convex combination of points must necessarily be a subset of the convex hull of the set of points. It follows from properties 2 and 3 of the Bernstein polynomials that the Bézier curve in formula (7) is a convex combination of the control points. Therefore, the graph of the curve must lie in the convex hull of the control points.

Property 4. The Bézier curve lies in the convex hull of its set of control points

The properties indicate that the graph of a Bézier curve of degree N is a continuous curve, bounded by the convex hull of the set of control points, $\{\mathbf{P}_i\}_{i=0}^{N}$, and that the curve begins and ends at points $\mathbf{P}_0$ and $\mathbf{P}_N$, respectively. Bézier observed that the graph is sequentially *pulled* toward each of the remaining control points $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_{N-1}$. For example, if the control points $\mathbf{P}_1$ and $\mathbf{P}_{N-1}$ are replaced by the control points $\mathbf{Q}_1$ and $\mathbf{Q}_{N-1}$, which are farther away (but in the same direction) from the respective endpoints, then the resulting Bézier curve will more closely approximate the tangent line near the endpoints. Figure 5.23 illustrates the pulling and tangent effects using the Bézier curve $\mathbf{P}(t)$ from Example 5.16 and the curve $\mathbf{Q}(t)$ with control points $(2, 2)$, $(0, 1)$, $(3, -1)$, and $(4, 1)$. Clearly, $\mathbf{Q}_1$, $\mathbf{P}_1$, and $\mathbf{P}_0 = \mathbf{Q}_0$, and $\mathbf{Q}_2$, $\mathbf{P}_2$, and $\mathbf{P}_3 = \mathbf{Q}_3$ are collinear, respectively.

The effectiveness of Bézier curves lies in the ease with which the shape of the curve can be modified (mouse, keyboard, or other graphical interface) by making small adjustments to the control points. Figure 5.24 shows four Bézier curves, of different degrees, with the corresponding sets of control points sequentially connected to form polygonal paths. The reader should observe that the polygonal paths provide a rough *sketch* of the resulting Bézier curves. Changing the coordinates of any one control point, say $\mathbf{P}_k$, will change the shape of the entire curve over the parameter interval $0 \le t \le 1$. The changes in the shape of the curve will be somewhat localized, since the Bernstein polynomial $B_{k,N}$, corresponding to the control point $\mathbf{P}_k$ (formula (7)) , has

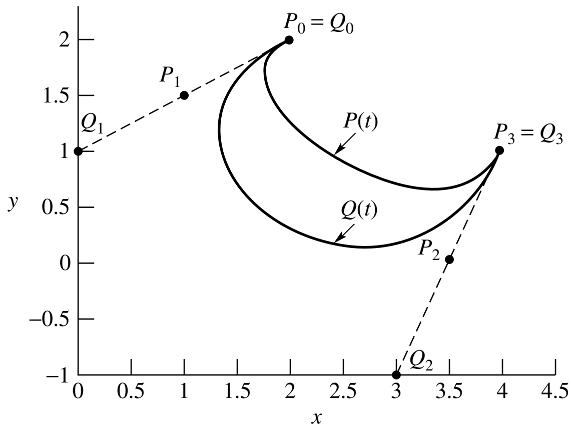

Figure 5.23 $P(t)$, $Q(t)$ and control points.

a maximum at the parameter value $t = k/N$. Thus the majority of the change in the shape of the graph of the Bézier curve will occur near the point $\mathbf{P}(k/N)$. Consequently, creating a curve of a specified shaped requires a relatively small number of changes to the original set of control points.

In practice, curves are produced using a sequence of Bézier curves sharing common endpoints. This process is analogous to that used in the creation of cubic splines. In that case it was necessary to use a sequence of cubic polynomials to avoid the oscillatory behavior of polynomials of high degree. Property 4 shows that the oscillatory behavior of higher-degree polynomials is not a problem with Bézier curves. Since changing one control point changes the shape of a Bézier curve, it is simpler to break the process into a series of Bézier curves and minimize the number of changes in the control points.

Example 5.17. Find the composite Bézier curve for the four sets of control points

$$\{(-9, 0), (-8, 1), (-8, 2.5), (-4, 2.5)\}, \qquad \{(-4, 2.5), (-3, 3.5), (-1, 4), (0, 4)\}$$
$$\{(0, 4), (2, 4), (3, 4), (5, 2)\}, \qquad \{(5, 2), (6, 2), (20, 3), (18, 0)\}$$

Following the process outlined in Example 5.16 yields

$$\begin{aligned}
\mathbf{P}_1(t) &= (-9 + 3t - 3t^2 + 5t^3, 3t + 1.5t^2 - 2t^3)\\
\mathbf{P}_2(t) &= (-4 + 3t + 3t^2 - 2t^3, 2.5 + 3t - 1.5t^2)\\
\mathbf{P}_3(t) &= (6t - 3t^2 + 2t^3, 4 - 2t^3)\\
\mathbf{P}_4(t) &= (5 + 3t + 39t^2 - 29t^3, 2 + 3t^2 - 5t^3).
\end{aligned}$$

The graph of the composite Bézier curve and corresponding control points is shown in Figure 5.25. ■

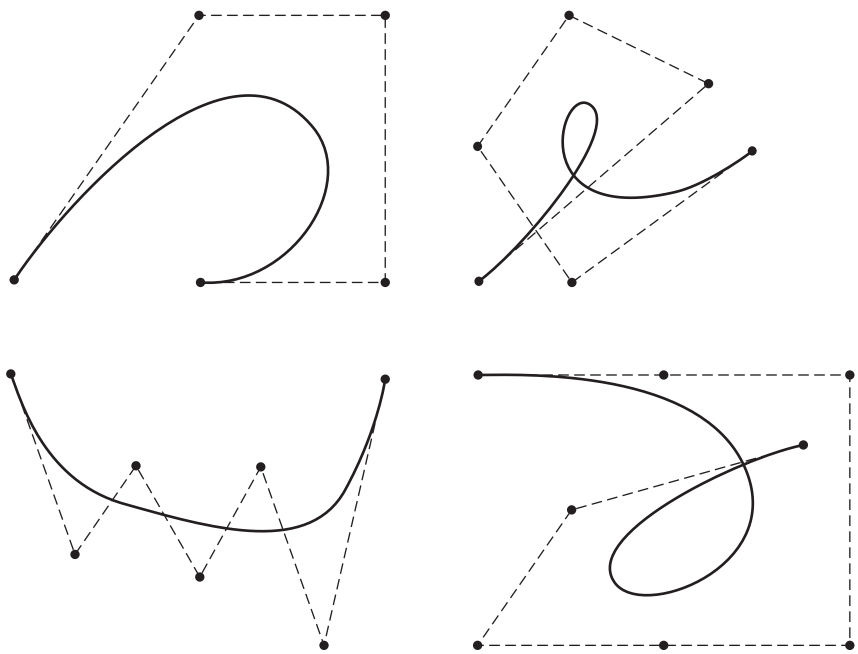

Figure 5.24 Bézier curves and polygonal paths.

The Bézier curves in Example 5.17 do not meet *smoothly* at the common endpoints. To have two Bézier curves $\mathbf{P}(t)$ and $\mathbf{Q}(t)$ meet smoothly would require that $\mathbf{P}_N = \mathbf{Q}_0$ and $\mathbf{P}'(\mathbf{P}_N) = \mathbf{Q}'(\mathbf{Q}_0)$. Property 3 indicates that it is sufficient to require that the control points $\mathbf{P}_{N-1}$, $\mathbf{P}_N = \mathbf{Q}_0$, and $\mathbf{Q}_1$ be collinear. To illustrate, consider the Bézier curves $\mathbf{P}(t)$ and $\mathbf{Q}(t)$ of degree three with the control point sets

$$\{(0,3), (1,5), (2,1), (3,3)\} \quad \text{and} \quad \{(3,3), (4,5), (5,1), (6,3)\},$$

respectively. Clearly, the control points $(2, 1)$, $(3, 3)$, and $(4, 5)$ are collinear. Again, following the process outlined in Example 5.16:

$$\mathbf{P}(t) = (3t, 3 + 6t - 18t^2 + 12t^3) \tag{13}$$

$$\mathbf{Q}(t) = (3 + 3t, 3 + 6t - 18t^2 + 12t^3) \tag{14}$$

and

$$\mathbf{P}'(t) = (3, 6 - 36t + 36t^2) \quad \text{and} \quad \mathbf{Q}'(t) = (3, 6 - 36t + 36t^2).$$

Substituting $t = 1$ and $t = 0$ into $\mathbf{P}'(t)$ and $\mathbf{Q}'(t)$, respectively, yields

$$\mathbf{P}'(1) = (3, 6) = \mathbf{Q}'(0)$$

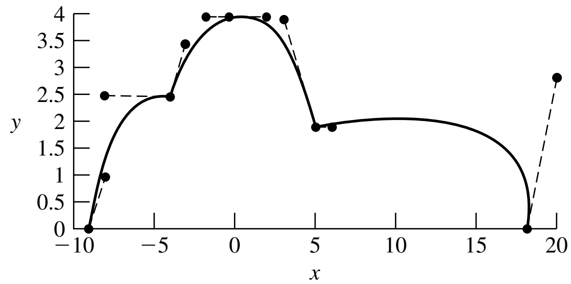

Figure 5.25 Composite Bźier curves.

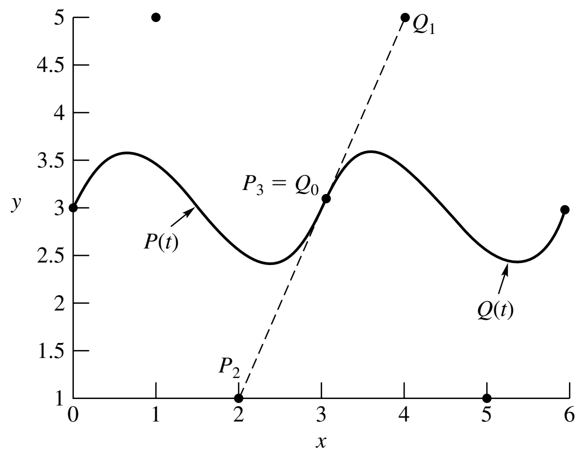

Figure 5.26 Matching derivatives at Bézier curves at common endpoints.

The graphs of $\mathbf{P}(t)$ and $\mathbf{Q}(t)$ and the smoothness at the common endpoint are shown in Figure 5.26.

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