Demystifying Fortunes Formula

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Introduction: A Simple Game

Let us start with a game.

- Suppose that each person in this room has a bankroll of W_0 to wager on a sequence of n independent flips of a biased coin.
- The coin is biased such that the probability of flipping heads and tails are p and q, where $\frac{1}{2} < p \leq 1$ and $q = 1 - p$.

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- The coin is biased such that the probability of flipping heads and tails are p and q, where $\frac{1}{2} < p \leq 1$ and $q = 1 - p$.
- You may bet anywhere between 0 and W_0 on each flip, with each resulting in either a gain or loss of the amount bet.
- The objective is to have the largest bankroll in the room after the final flip. *How should you play?*

Suppose that $(W_0, n, p, q) = (\$25, 300, 0.6, 0.4)$ for the sake of simplicity.

Relevance to Finance

- There are two cardinal problems to quantitative finance and gambling to consistently generate profit:
  1. Identifying a set of profitable opportunities
  2. Optimally sizing your positions
- The second problem is typically seen as the hardest; humans are particularly bad at risk management

Case Study: Observed Betting Patterns on a Biased Coin

- In a study, 61 students in economics and finance were offered our game [1].
- The final amount of their bankroll (up to a cap of $250) was to be written to them afterwards. .
- To the right, we can see a summary of the results of the study.

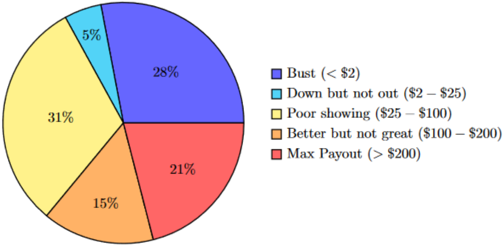

Figure: Summary of Case Study

Fortunes Formula

The Kelly Criterion

The Kelly Criterion is a formula which accepts known probabilities and payoffs as inputs and outputs the proportion of total wealth to bet in order to achieve the maximum exponential growth rate of your wealth.

$$f^* = \frac{bp - q}{b}.$$

where p and q are the probabilities of success and failure, and b is the net odds (i.e., the ratio of what we stand to win and lose).

The next bit of the presentation will be dedicated to motivating this result.

A Brief History

There are three main actors in the historical development of the Kelly Criterion:

1. Claude Shannon
    - Wrote "A Mathematical Theory of Communication" in 1948 [2]
    - The father of Information Theory
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3. Edward Thorp
    - Worked with Shannon to make the Kelly Criterion operational for gambling and investing
    - Developed one of the first wearable computers for roulette

A First Idea: Expectation Maximization

One intuitive start would be to attempt to maximize the expected value of our wealth. Suppose we bet an amount x of our current bankroll W_k, then our expectation after the next roll is given by

$$\mathbb{E}[W_{k+1}] = p(x + x) + q(-x) = (2p - q)x = (3p - 1)x.$$

As we can see, the expectation is strictly increasing, which means that the maximum would be $x = W_k$.

Although this strategy may be optimal for a singular bet, it fails to consider the fact that we wish to optimize for repeated bets. The risk of ruin would be too high as we would expect to lose our bankroll in $\frac{1}{q}$ flips.

A Second Idea: Betting a Constant Amount

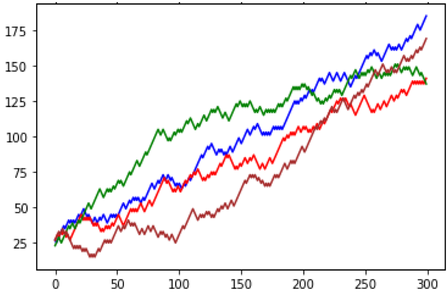
Another intuitive consideration would be to try betting a constant amount each time. Suppose we bet an amount x for each flip, then our expectation after n flips can be found as follows

$$\mathbb{E}[W_n] = \mathbb{E}\left[\sum_{k=0}^{n} W_k\right] = \sum_{k=0}^{n} \mathbb{E}[W_k] = W_0 + (3p-1)nx.$$

If one were to attempt to maximize this expectation with respect to x, then they would run into similar issues to the prior idea.

A Second Idea: Betting a Constant Amount

Suppose that we consider betting a constant of \$2 from the bankroll given in the case study. A few potential realizations are given as follows:



As we can see, our results average out to linear growth over time, which begs the question: *can we do better?*

The Final Idea: The Exponential Growth Rate

We can resolve the issue to the prior two ideas by considering the exponential growth rate. If we make the assumption that we can compound our bankroll over time, then we can make the assumption that

$$W_n = W_0 e^{rn}.$$

With this assumption in mind, we can start to reframe our objective as maximizing the exponential rate r, as the following quantity

$$r = \frac{1}{n} \log\left(\frac{W_n}{W_0}\right).$$

Now we have the proper machinery to go about our derivation.

Derivation

Suppose that we have a total of m heads out of our n flips. If we are to bet a fraction f on heads each time, we would have a a final wealth given by

$$W_n = W_0(1+f)^m(1-f)^{n-m}.$$

Now we can define the exponential rate of growth per flip by

$$g(f) = \frac{1}{n}\log\left(\frac{W_n}{W_0}\right) = \frac{m}{n}\log(1+f) + \frac{n-m}{n}\log(1-f).$$

Next we need to take the expectation of the exponential growth rate (defined by $G(f)$) as follows

$$G(f) = \mathbb{E}[g(f)] = p\log(1+f) + q\log(1-f).$$

Derivation

To find the extrema of $G(f)$ we must take the first derivative with respect to f:

$$G'(f) = \frac{p}{1+f} + \frac{q}{1-f} = \frac{p-q-f}{1-f^2} = 0.$$

Therefore the value of f that maximizes $G(f)$ is $f^* = p - q$.
We can also ensure that our extrema is a maxima since

$$G''(f) = \frac{-f^2 + 2f(p-q) - 1}{(1-f^2)^2} < 0$$

.

Derivation

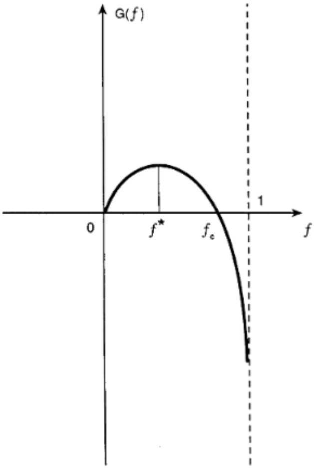


Fig. 1.

Limitations of the Model

When using the Kelly Criterion in practice, there are a few assumptions that are important to keep in mind:

1. Your time-frame is large (i.e., $n \to \infty$)
2. Wealth has a log utility to you
3. Your wealth is infinitely divisible
4. You can handle the volatility implied with Kelly betting
5. You have full knowledge of your edge

References

[1] Victor Haghani and Richard Dewey. “Rational Decision-Making under Uncertainty: Observed Betting Patterns on a Biased Coin”. In: *SSRN* (2016). URL: https://papers.ssrn.com/sol3/papers.cfm?abstract_id=2856963.

[2] Claude Shannon. “A Mathematical Theory of Communication”. In: *The Bell System Technical Journal* (1948). URL: https://people.math.harvard.edu/~ctm/home/text/others/shannon/entropy/entropy.pdf.

[3] John Kelly. “A New Interpretation of Information Rate”. In: (1956). URL: https://www.princeton.edu/~wbialek/rome/refs/kelly_56.pdf.

Ending and Q&A

Thank you :)