Homework 3

Topics covered in this homework are: Difference equations, z-transforms, and z-transform properties. Homework is due at 5:00 PM on Wednesdays. Homework will be graded for (1) completion and (2) one randomly picked problem will be graded. Submissions will be using gradescope. Please solve problems on your own in order to maximally benefit from this homework.

Problem 1:

The following difference equation:

$$y[n] = ay[n-1] + x[n] \tag{1}$$

with initial conditions $y[-1] = c$ and $x[n] = b\delta[n]$ has the solution:

$$y[n] = a^{n+1}c + a^n bu[n] \tag{2}$$

Now answer the following. Provide reasons.

(a) What is the impulse response $h[n]$ of the system in (1)?
Solution: Since (2) is the solution to (1) when $x[n] = b\delta[n]$, we obtain $h[n] = a^{n+1}c + a^n u[n]$ by setting $b = 1$ in (2).

(b) Is this a causal system?
Solution: No. By inspecting (2), we find that $y[n] \neq 0$ for $n < 0$ when $x[n] = b\delta[n]$, i.e., the input $x[n]$ 'appears' at $n = 0$, i.e., the system anticipates this input.

(c) Obtain an expression for the response $y[n]$ when $x[n] = 2b\delta[n]$. Based on your answer, what can you say about the linearity of the system?
Solution: From (2), $y[n] = a^{n+1}c + 2ba^n u[n]$ if $x[n] = 2b\delta[n]$. It is clear this is a non-linear system since $y[n]$ does not scale by a factor of 2 when the input $x[n]$ is scaled up by a factor of 2, i.e., the homogeneity property is not satisfied. Note however, that the system is linear if a broader definition of linearity were to be adopted in which the system is linear both to the input $x[n]$ (zero-state linearity) and to the initial state $y[-1]$ (zero-input linearity).

(d) Repeat Part (a)-(c) if the system is at *initial rest*.
Solution: Under initial rest $y[-1] = 0$, and therefore, $y[n] = h[n] = ba^n u[n]$. This *maybe* causal system since its impulse response $h[n] = 0$ for $n < 0$ though we need to prove this for an arbitrary $x[n]$ in order to be assert its causality. Also, this system *maybe* linear since it does satisfy the homogeneity property though to prove linearity we need to show additivity/-superposition applies for arbitrary inputs.

(e) Obtain an expression for the response $y[n]$ when $x[n] = b\delta[n-1]$ under initial rest conditions. What can you say about the time-invariance of this system?
Solution: Under initial rest conditions and $x[n] = b\delta[n-1]$, $y[n] = ba^{n-1}u[n-1]$. This can be shown by calculating a few terms of $y[n]$, e.g., $y[0] = 0$, $y[1] = b$, $y[2] = ab$ and so on. Thus, the system *maybe* a time-invariant system. In fact, a linear constant coefficient difference equation (LCCDE) such as the first-order one in (1) does describe an LTI system under zero initial conditions. Hopefully, this problem provides some intuition as to why this might be the case.

Problem 2: DE vs. Block-diagram

Sketch a block diagram of the system described by the difference equation:

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n-1] \tag{3}$$

Assume the input $x[n]$ is real-valued.
Solution: See below:

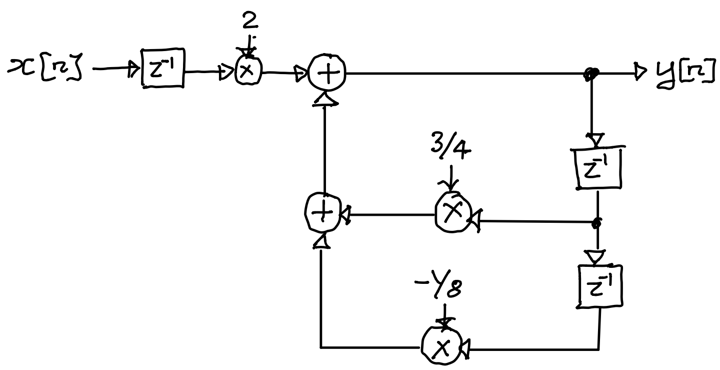

Figure 1: Block diagram of DE in (3).

Problem 3:

An LTI system is described by its system transfer function $H(z) = \sum_n h[n]z^{-n} = z + z^{-1}$. Find its output $y[n]$ when the input is $x[n] = 2(-\frac{1}{2} + j\frac{\sqrt{3}}{2})^n$ using:

(a) the eigenfunction property of LTI systems.
Solution: Since $x[n] = bz^n$ is a complex exponential it is an eigenfunction of an LTI system. Therefore, the output $y[n]$ will be a scaled and phase-shifted version of $x[n]$ given by:

$$y[n] = x[n]H\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) = -2\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)^n \tag{4}$$

(b) the convolution sum.
Solution: The expression for $H(z)$ tells us that $h[n] = \delta[n+1] + \delta[n-1]$. Applying the convolution sum, we get:

$$y[n] = h[-1]x[n+1] + h[1]x[n-1] \tag{5}$$

$$= 2\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)^{n+1} + 2\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)^{n-1} \tag{6}$$

$$= -2\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)^n \left(-\frac{1}{2} + j\frac{\sqrt{3}}{2} + \frac{1}{-\frac{1}{2} + j\frac{\sqrt{3}}{2}}\right) = -2\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)^n \tag{7}$$

Problem 4:

Determine the z-transform $X(z)$ of the following sequences and sketch the pole-zero plot:

(a) $x[n] = (\frac{1}{3})^n u[-n]$
Solution:

$$X(z) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{3}\right)^n u[-n] z^{-n} = \sum_{n=0}^{\infty} (3z)^n = \frac{1}{1-3z}, \quad \text{ROC} : \{z : |z| < \frac{1}{3}\} \tag{8}$$

(b) $x[n] = (\frac{1}{3})^n (u[n] - u[n-5])$
Solution:

$$X(z) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{3}\right)^n (u[n] - u[n-5]) z^{-n} = \sum_{n=0}^{4} \left(\frac{1}{3z}\right)^n = \frac{1 - \left(\frac{1}{3z}\right)^5}{1 - \frac{1}{3}z^{-1}}, \quad \text{ROC} : \{z : |z| \neq 0\} \tag{9}$$

(c) $x[n] = (\frac{1}{2})^{n-3} u[n-2]$
Solution:

$$x[n] = (\frac{1}{2})^{n-3} u[n-2] = 2(\frac{1}{2})^{n-2} u[n-2] \tag{10}$$

Hence, $x[n] = 2x_1[n-2]$ where $x_1[n] = (\frac{1}{2})^n u[n]$ with a z-transform of $X_1(z) = \frac{z}{z-\frac{1}{2}}$ and an ROC:$\{z : |z| > \frac{1}{2}$. Using the delay property $x_1[n-2] \Longleftrightarrow z^{-2} X_1(z)$, we get:

$$X(z) = (2z^{-2}) X_1(z) = \frac{4}{2z^2 - z}, \quad \text{ROC} : \{z : |z| > \frac{1}{2}\} \tag{11}$$

(d) $x[n] = \alpha^{|n|}$ with $0 < |\alpha| < 1$
Solution:

$$X(z) = \sum_{n=-\infty}^{\infty} \alpha^{|n|} z^{-n} = \sum_{n=-\infty}^{-1} \alpha^{-n} z^{-n} + \sum_{n=0}^{\infty} \alpha^n z^{-n} \tag{12}$$

$$= -1 + \sum_{n=0}^{\infty} (\alpha z)^n + \sum_{n=0}^{\infty} \left(\frac{\alpha}{z}\right)^n = -1 + \frac{1}{1-\alpha z} + \frac{1}{1-\frac{\alpha}{z}} \tag{13}$$

$$= \frac{z(1-\alpha^2)}{(1-\alpha z)(z-\alpha)}, \quad \text{ROC} : \{z : |\alpha| < |z| < \frac{1}{|\alpha|}\} \tag{14}$$

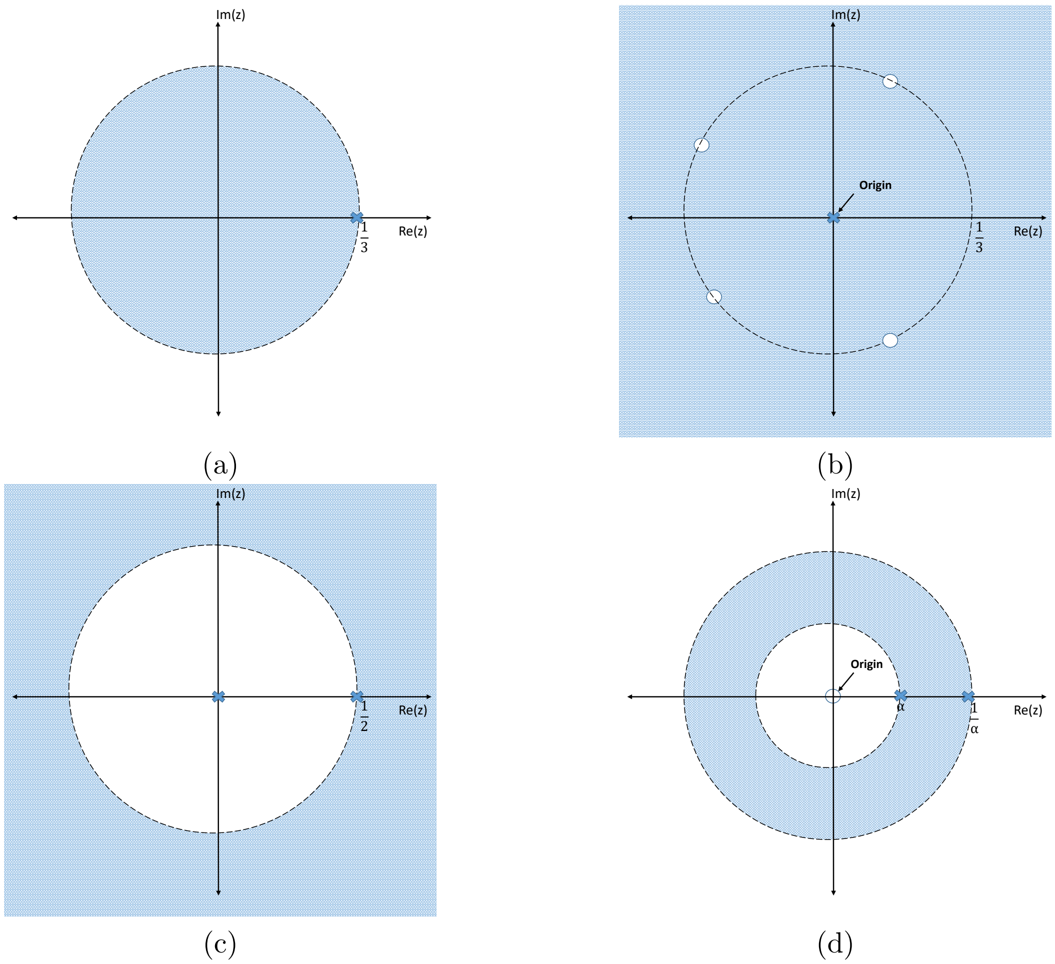

Figure 2: Pole-Zero Diagrams Problem 4

Problem 5:

Given the z-transform pair

$$x[n] = a^n u[n] \Longleftrightarrow X(z) = \frac{z}{z-a}, \quad \text{with ROC:} |z| > |a| \tag{15}$$

employ only the z-transform properties to obtain $y[n]$ from $Y(z)$ and its ROC:

(a) $Y(z) = \frac{a^2 z}{1-az}$ ROC:$|z| < |a^{-1}|$
Solution: Using properties of time-reversal and time-shift

$$y[n] = a^{-n+1} u[-n-1] \tag{16}$$

(b) if $Y(z) = \frac{z-1}{z+\frac{3}{4}}$ and $y[n]$ is a causal sequence
Solution: Rewrite $Y(z)$ as:

$$Y(z) = \frac{z}{z+\frac{3}{4}} - \frac{1}{z+\frac{3}{4}} \tag{17}$$

applying linearity and time-shift properties:

$$y[n] = (-\frac{3}{4})^n u[n] - (-\frac{3}{4})^{n-1} u[n-1] \tag{18}$$

(c) $Y(z) = \log(\frac{z+\frac{1}{2}}{z})$ ROC:$|z| > |\frac{1}{2}|$
Solution: applying differentiation property:

$$-z\frac{\partial Y(z)}{\partial z} = \frac{\frac{1}{2}}{z+\frac{1}{2}} = \frac{z^{-1}}{2}\left[\frac{z}{z+\frac{1}{2}}\right] \Longleftrightarrow ny[n] \tag{19}$$

From (15), term in brackets corresponds to the time-domain sequence:

$$(-\frac{1}{2})^n u[n] \tag{20}$$

Applying linearity and time-shift properties:

$$\frac{z^{-1}}{2}\left[\frac{z}{z+\frac{1}{2}}\right] \Longleftrightarrow \frac{1}{2}(-\frac{1}{2})^{n-1}u[n-1] = ny[n] \tag{21}$$

$$y[n] = \frac{1}{2n}(-\frac{1}{2})^{n-1}u[n-1] \tag{22}$$

(d) $Y(z) = \frac{2z}{z+\frac{3}{\sqrt{2}}(1+j)}$ ROC:$|z| > |3|$
Solution: Applying the modulation property:

$$y[n] = 2(e^{j\frac{\pi}{4}})^n(-3)^n u[n] \tag{23}$$