Digital Signal Processing
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Homework 8

ECE 310 (Spring 2020)
Assigned: 03/25 - Due: 04/01

Topics covered in this homework are: Sampling, Analog Frequency response of DSP system. Homework is due at 5:00 PM on Wednesdays. Homework will be graded for (1) completion and (2) three randomly picked problems will be graded. Submissions will be using gradescope. Please solve problems on your own in order to maximally benefit from this homework.

Problem 1:

Consider the discrete time system shown below. The CTFT $X_a(\Omega)$ of $x_a(t)$ and the frequency response $H_d(\omega)$ are also shown. Given that $T_1 = T_2 = \dfrac{1}{8 \times 10^3}$ s.

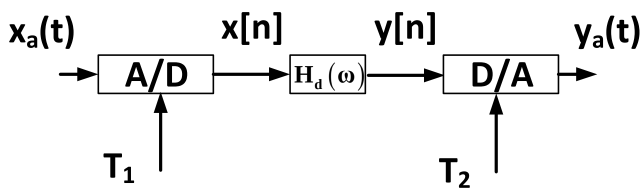
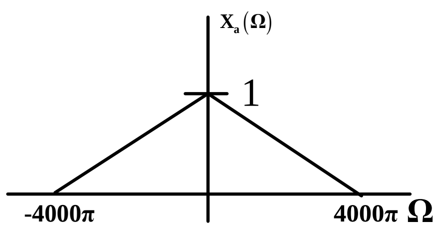
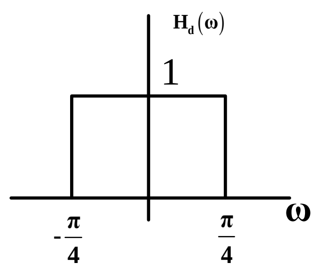


Figure 1: DSP system for Problem 1





1. Sketch $X_d(\omega)$ and $Y_d(\omega)$ over $|\omega| \leq \pi$.
2. Assume the D/A is ideal. Sketch the spectrum $Y_a(\Omega)$

Solution:

1. Given the sample rate T_1, the maximum frequency in the ω domain should be $4000\pi \times \dfrac{1}{T_1} = \dfrac{\pi}{2} < \pi$. Therefore, there is no aliasing. The DTFT of $x[n]$ as $X_d(\omega)$ will be the following:

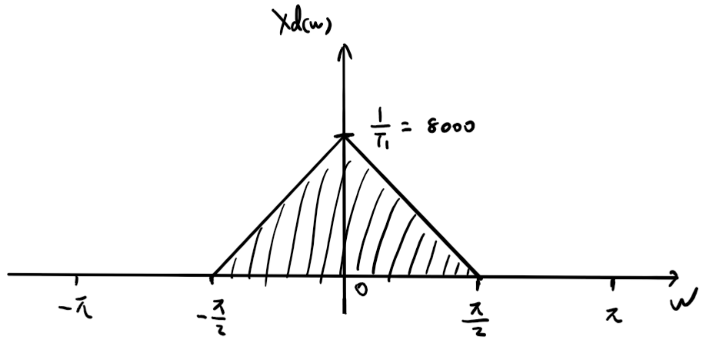


Figure 2: Problem 1: The sketch of $X_d(\omega)$.

Then we pass $x[n]$ through the filter $H_d(\omega)$, which is an ideal low-pass filter with the maximum frequency $\frac{\pi}{4}$, therefore we get the following frequency response $Y_d(\omega)$ of $y[n]$ as

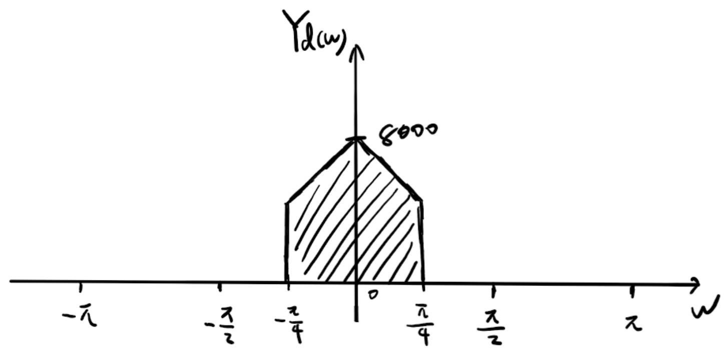


Figure 3: Problem 1: The sketch of $Y_d(\omega)$.

2. In order to get the CTFT $Y_a(\Omega)$ of the output signal, we need to apply the window function within the range of $[-\pi, \pi]$ and the height (magnitude) of the window function is T_2, then we replace ω by $\Omega = \dfrac{\omega}{T_2}$ and the result will be

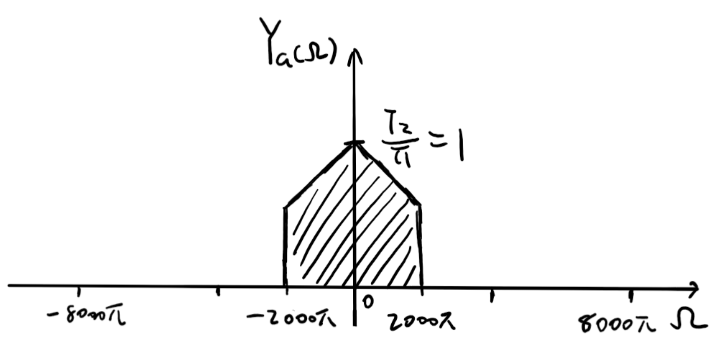


Figure 4: Problem 1: The sketch of $Y_a(\Omega)$.

which is the CTFT $Y_a(\Omega)$.

Problem 2:

Consider the DSP system shown in fig. 1. The output $y[n]$ of the filter $H_d(\omega)$ is described by $y[n] = x[n] - \sqrt{2}x[n-1] + x[n-2]$. Given $x(t) = cos(500\pi t) + cos(1000\pi t)$. Compute $y_a(t)$ for:

1. $T_1 = T_2 = \dfrac{1}{1500}$s
2. $T_1 = T_2 = \dfrac{1}{750}$s.

Solution:

1. The input signal after sampling by sampling period $T_1 = \dfrac{1}{1500}s$ should be

$$\begin{aligned} x[n] &= x_a(nT_1) = \cos(500\pi nT_1) + \cos(1000\pi nT_1) \\ &= \cos\left(\frac{\pi}{3}n\right) + \cos\left(\frac{2\pi}{3}n\right). \end{aligned} \tag{1}$$

Next, in order to get the output analog signal $y_a(t)$, there are two ways:

Method 1: Using the difference equation directly. Let $x[n] = x_1[n] + x_2[n]$ where $x_1[n] = \cos\left(\dfrac{\pi}{3}n\right)$ and $x_2[n] = \cos\left(\dfrac{2\pi}{3}n\right)$. Then according to the linearity of the difference equation, we can compute the output $y_2[n]$ and $y_1[n]$ for $x_1[n]$ and $x_2[n]$, respectively. That is,

$$\begin{aligned} y_1[n] &= x_1[n] - \sqrt{2}x_1[n-1] + x_1[n-2] \\ &= \cos\left(\frac{\pi}{3}n\right) - \sqrt{2}\cos\left(\frac{\pi}{3}(n-1)\right) + \cos\left(\frac{\pi}{3}(n-2)\right) \\ &= -(\sqrt{2}-1)\left[\frac{1}{2}\cos\left(\frac{\pi}{3}n\right) + \frac{\sqrt{3}}{2}\sin\left(\frac{\pi}{3}n\right)\right] \\ &= -(\sqrt{2}-1)\cos\left(\frac{\pi}{3}n - \frac{\pi}{3}\right). \\ y_2[n] &= x_1[n] - \sqrt{2}x_1[n-1] + x_1[n-2] \\ &= \cos\left(\frac{2\pi}{3}n\right) - \sqrt{2}\cos\left(\frac{2\pi}{3}(n-1)\right) + \cos\left(\frac{2\pi}{3}(n-2)\right) \\ &= (\sqrt{2}+1)\left[\frac{1}{2}\cos\left(\frac{2\pi}{3}\right) - \frac{\sqrt{3}}{2}\sin\left(\frac{2\pi}{3}\right)\right] \\ &= (\sqrt{2}+1)\cos\left(\frac{2\pi}{3} + \frac{\pi}{3}\right). \end{aligned} \tag{2}$$

where we use the fact $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$. Therefore, the discrete signal $y[n]$ after filtering should be

$$\begin{aligned} y[n] &= y_1[n] + y_2[n] \\ &= -(\sqrt{2}-1)\cos\left(\frac{\pi}{3}n - \frac{\pi}{3}\right) + (\sqrt{2}+1)\cos\left(\frac{2\pi}{3} + \frac{\pi}{3}\right). \end{aligned} \tag{3}$$

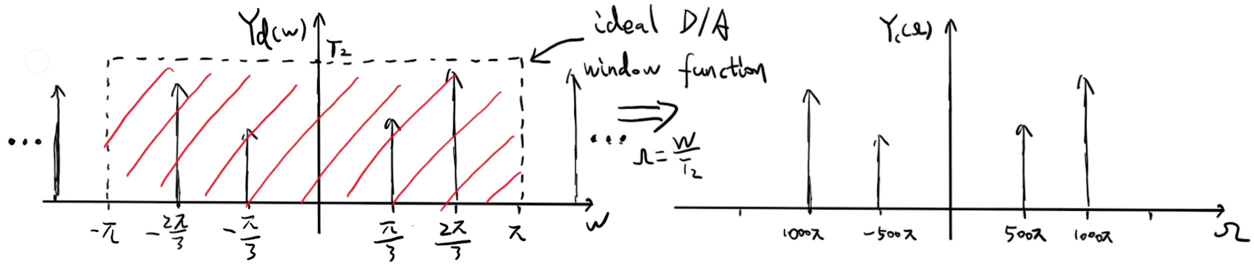


Figure 5: Problem 2: The illustration of an ideal D/A in part 1, note here we ignore the magnitude of the different frequency components, e.g. $(\sqrt{2}-1)$ and $(\sqrt{2}+1)$ in Eq. (3).

Then in order to get the output analog signal $y_a(t)$, we can follow the same way as in Problem 1. That is, we first draw the DTFT of $y[n]$ as $Y_d(\omega)$ in the left plot of Fig. 5. Then the ideal D/A is equivalent to applying a window function $[-\pi,\ \pi]$ on the frequency response, next we replace ω by $\Omega = \dfrac{\omega}{T_2}$ and we get the CTFT of $y_a(t)$ as the right plot of Fig. 5. We can see that $y_a(t)$ will be equal to $y[n]$ in Eq. (3) if we replace n by $\dfrac{t}{T_2}$. That is,

$$\begin{aligned} y_a(t) &= y\left[\frac{t}{T_2}\right] \\ &= -(\sqrt{2}-1)\cos\left(\frac{\pi t}{3T_2}-\frac{\pi}{3}\right)+(\sqrt{2}+1)\cos\left(\frac{2\pi t}{3T_2}+\frac{\pi}{3}\right) \\ &= \boxed{-(\sqrt{2}-1)\cos\left(500\pi t-\frac{\pi}{3}\right)+(\sqrt{2}+1)\cos\left(1000\pi t+\frac{\pi}{3}\right).} \end{aligned} \tag{4}$$

Method 2: Computing the transfer function H_d: Another way is to follow the frequency domain. To do this, we need to first compute the transfer function $H(w)$. By applying the z-transform on the difference equation we have:

$$Y(z) = X(z) - \sqrt{2}z^{-1}x(Z) + z^{-2}x(Z) \Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{z^2-2z+1}{z^2}. \tag{5}$$

Next, we need to compute the magnitude response and phase response of the filter $H_d(\omega)$. We do this by plugging $z = \exp(j\omega)$ into the transfer function as

$$\begin{aligned} H(e^{j\omega}) &= \frac{e^{2j\omega}-\sqrt{2}e^{j\omega}+1}{e^{2j\omega}} \\ &= e^{-j\omega}(e^{j\omega}+e^{-j\omega}-\sqrt{2}) \\ &= e^{-j\omega}(2\cos(w)-\sqrt{2}). \end{aligned} \tag{6}$$

Therefore, since the input signal has two frequency components at $\omega_1 = \dfrac{\pi}{3}$ and $\omega_2 = \dfrac{2\pi}{3}$, the corresponding filter response is

$$H(e^{j\frac{\pi}{3}}) = -(\sqrt{2}-1)e^{-j\frac{\pi}{3}}, \quad H(e^{j\frac{2\pi}{3}}) = -(\sqrt{2}+1)e^{-j\frac{2\pi}{3}}. \tag{7}$$

Then it is not hard to get $y[n]$ as

$$y[n] = -(\sqrt{2}-1)\cos\left(\frac{\pi}{3}n - \frac{\pi}{3}\right) + (\sqrt{2}+1)\cos\left(\frac{2\pi}{3} + \frac{\pi}{3}\right). \tag{8}$$

which matches the result in Eq. (3). The following thing will be the same as in the method 1.

2. Similarly, the input signal after sampling is

$$\begin{aligned} x[n] = x_a(nT_1) &= \cos\left(\frac{2\pi}{3}n\right) + \cos\left(\frac{4\pi}{3}n\right) \\ &= \cos\left(\frac{2\pi}{3}n\right) + \cos\left(-\frac{2\pi}{3}n\right) \\ &= 2\cos\left(\frac{2\pi}{3}n\right). \end{aligned} \tag{9}$$

Here we follow the method 2 in the previous part. Again, the filter response is given as

$$H(e^{j\omega}) = e^{-j\omega}(2\cos(w) - \sqrt{2}) \Rightarrow H(e^{j\frac{2\pi}{3}}) = -(\sqrt{2}+1)e^{-j\frac{2\pi}{3}}. \tag{10}$$

Therefore the y[n] is given as

$$y[n] = 2(\sqrt{2}+1)\cos\left(\frac{2\pi}{3}n + \frac{\pi}{3}\right). \tag{11}$$

To get the output $y_a(t)$, similar to Fig. 5 we draw the DTFT of $y[n]$ as $Y_d(\omega)$ and apply the window function, that is

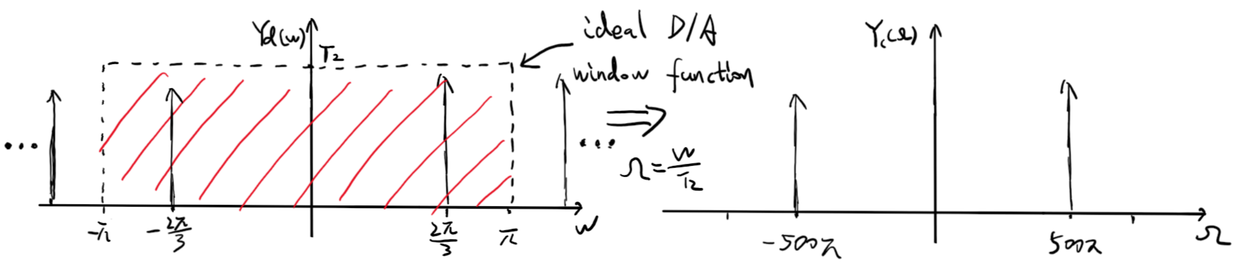


Figure 6: Problem 2: The illustration of an ideal D/A in part 2.

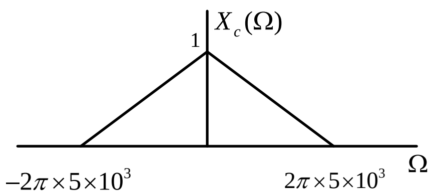
In this way, we can see the output $y_a(t)$ should be

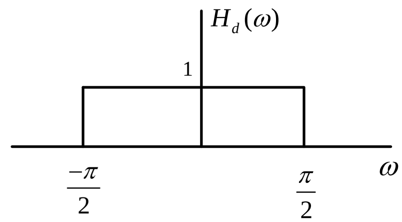
$$\boxed{y_a(t) = y\left[\frac{t}{T_2}\right] = 2(\sqrt{2}+1)\cos\left(500\pi t + \frac{\pi}{3}\right).} \tag{12}$$

Remark: Note that in the second part, the component $\cos(1000\pi t)$ causes aliasing, therefore it cannot be distinguished with the component $\cos(500\pi t)$ and the information has been lost in the output.

Problem 3:

For the system shown in Fig. (1), the CTFT of $x_a(t)$ and the frequency response $H_d(\omega)$ are shown below,





Sketch and label the CTFT of $y_a(t)$ for each of the following cases:

(a) $1/T_1 = 1/T_2 = 10^4$

(b) $1/T_1 = 1/T_2 = 2\times10^4$

(c) $1/T_1 = 2\times10^4, 1/T_2 = 10^4$

(d) $1/T_1 = 10^4, 1/T_2 = 2\times10^4$

Solution:

We show the plots in Fig. 7 and Fig. 8. In order to get the CTFT of the output signal $y_a(t)$, we can derive the frequency response at each intermediate stage step by step. First we get the DTFT of $x[n]$ which is sampled by T_1; then we apply the filter $H_d(\omega)$; next we apply the Ideal D/A, that is to apply a window function between the frequency range $[-\pi,\pi]$, and replace ω by $\Omega = \dfrac{\omega}{T_2}$. The resulting frequency response $Y_d(\Omega)$ will be the CTFT of the output signal $y_a(t)$.

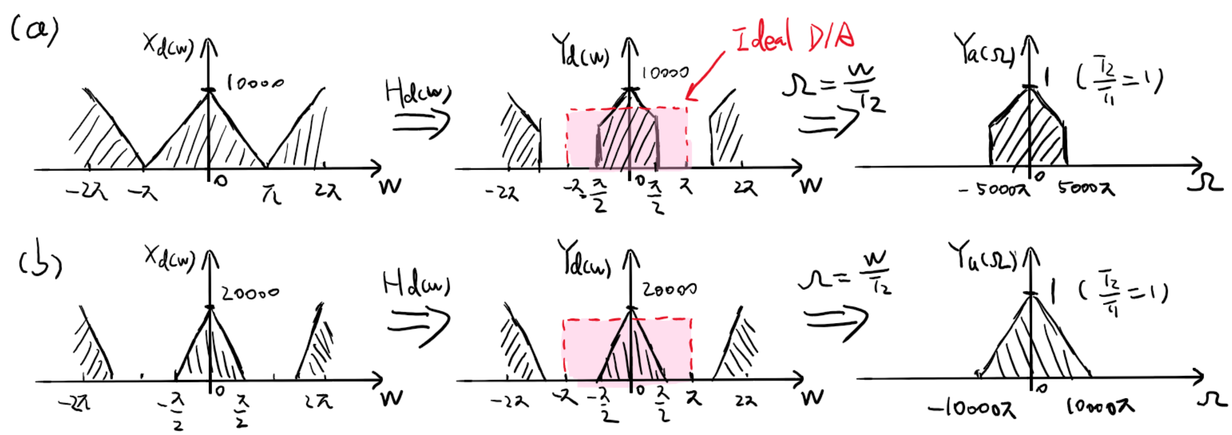


Figure 7: Problem 3: The plots of (a) and (b).

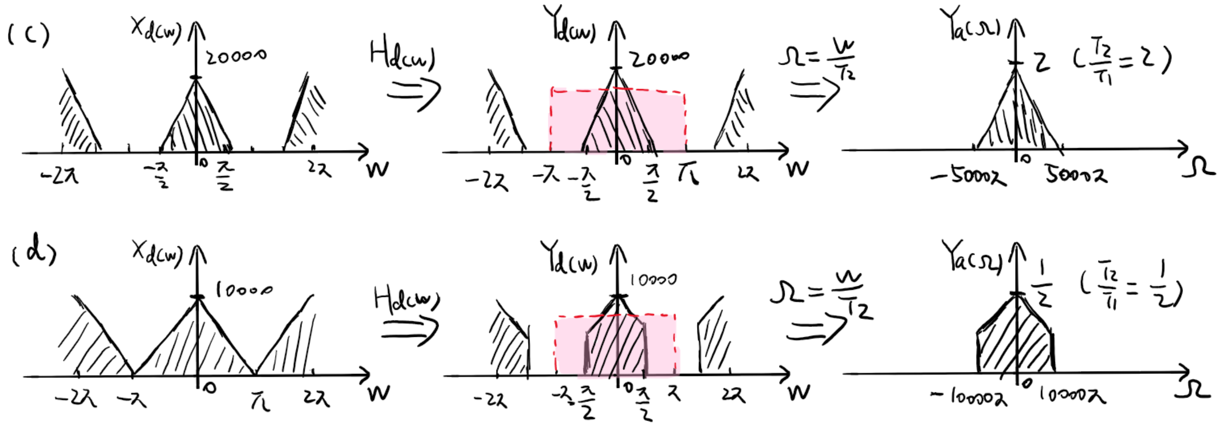


Figure 8: Problem 3: The plots of (c) and (d).

Problem 4:

In communication systems a message signal, $x(t)$, needs to be transmitted over long distance. This is achieved by taking help of a high frequency carrier signal, $x_c(t)$. This process is called modulation and is given by:

$$\begin{aligned} x_a(t) &= 2cos\left(\Omega_c t\right) x(t) \\ &= \left(e^{j\Omega_c t} + e^{-j\Omega_c t}\right) x(t) \end{aligned}$$

and is illustrated in Fig. 9 below.

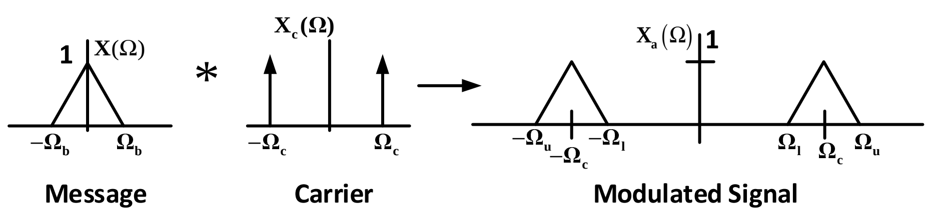


Figure 9: DSP system

At the receiver, the incoming signal must be demodulated. The demodulation can be implemented digitally as shown in Fig. 10. Generally the carrier frequency is large compared to signal bandwidth. This requires the sampling frequency to be large. In this problem we will look at implementing digital demodulation by sampling at lower than Nyquist frequency. Consider the system in Fig. 10. Assume $\Omega_C = 2\pi 100000$ rad/s, $\Omega_b = 2\pi 5000$ rad/s.

1. What is the sampling period T such that no aliasing occurs at the output of the A/D converter.
2. Assume $T = \dfrac{1}{150000}$s. Sketch $X_d(\omega), Y_{1d}(\omega)$ in $(-2\pi, 2\pi)$. Please note: H_{1d} is periodic with period 2π.
3. Let $\omega_0 = \dfrac{2\pi}{3}$. Sketch $Y_{2d}(\omega)$ in $(-2\pi, 2\pi)$.

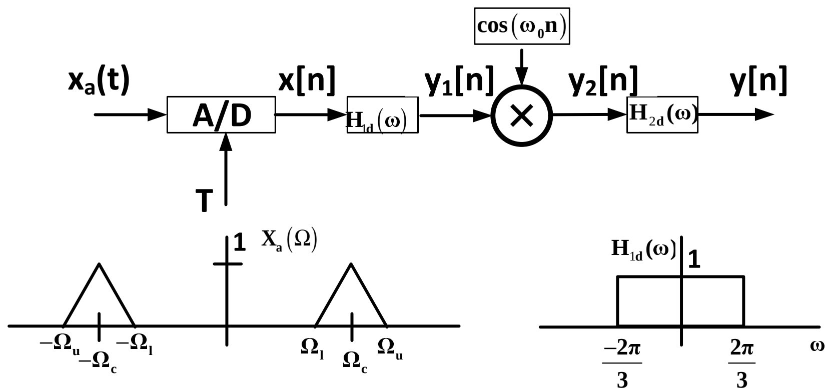


Figure 10: DSP system

4. Sketch the frequency response of $H_{2d}(\omega)$ so that the spectrum of $y[n]$ is equivalent to the signal obtained by demodulating $x_a(t)$.

Solution:

1. According to the Nyquist's sampling theorem, the maximum frequency is $[-\Omega_u,\ \Omega_u]$ where $\Omega_u = \Omega_c + \Omega_b = 210000\pi$ rad/s. Then in order to avoid aliasing, the sampling period T satisfies

$$\frac{2\pi}{T} \geq \Omega_u, \quad \Rightarrow \quad \boxed{T \leq \frac{\pi}{\Omega_u} = \frac{1}{210000}\text{s.}} \tag{13}$$

2. In this case, $T = \frac{1}{150000}s > \frac{1}{210000}$, therefore there exists aliasing. We compute the frequency Ω_l, Ω_c and Ω_u (which are the left, center and right frequency in Fig. 9) in ω as

$$\begin{aligned} &\Omega_c = 200000\pi \quad \Rightarrow \quad \omega_c = \Omega_c T = \frac{4\pi}{3}, \\ &\Omega_l = \Omega_c - \Omega_b = 190000\pi \quad \Rightarrow \quad \omega_l = \Omega_l T = \frac{4\pi}{3} - \frac{\pi}{15}, \\ &\Omega_u = \Omega_c + \Omega_b = 210000\pi \quad \Rightarrow \quad \omega_u = \Omega_u T = \frac{4\pi}{3} + \frac{\pi}{15}. \end{aligned} \tag{14}$$

Since aliasing happens, we need to figure out those copies that have components within the range $[-2\pi,\ 2\pi]$. Since the center frequency is $\omega_c = \frac{4\pi}{3}$, we can observe that there will be two components of copies centered on $\frac{4\pi}{3} - 2\pi = -\frac{2\pi}{3}$ and $-\frac{4\pi}{3} + 2\pi = \frac{2\pi}{3}$. Therefore, we can draw the DTFT $X_d(\omega)$ as following

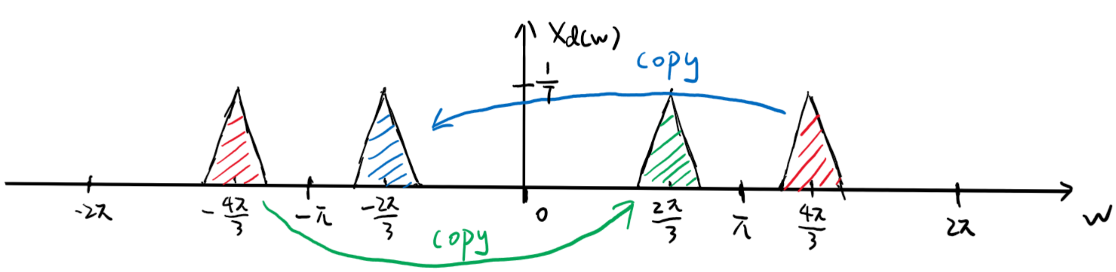


Figure 11: Problem 4: The sketch of $X_d(\omega)$. Different color represents different copy.

Next we need to apply the filter $H_{1d}(\omega)$, which is a window function within the range $[\frac{-2\pi}{3}, \frac{2\pi}{3}]$. We sketch the DTFT $Y_{1d}(\omega)$ as following

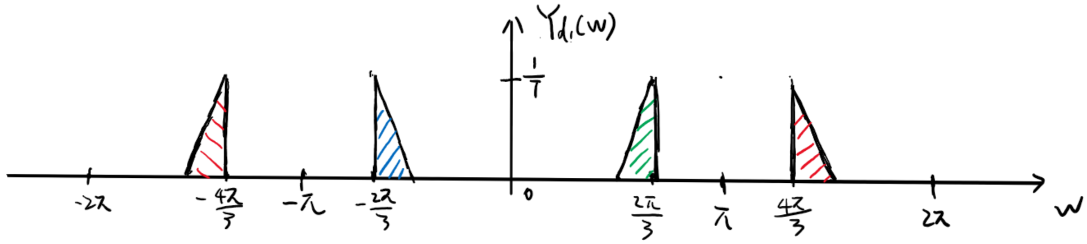


Figure 12: Problem 4: The sketch of $Y_{1d}(\omega)$. Different color represents different copy in Fig. 11.

Note that the window function is periodic, so it should be applied to each 2π block, and the resulting $Y_{1d}(\omega)$ is still periodic by period 2π.

3. In this part, the signal $y_1[n]$ is multiplied by $\cos(\omega_0 n)$ and get $y_2[n]$, which can be expressed as

$$\begin{aligned} y_2[n] = y_1[n]\cos(\omega_0 n) = y_1[n](\frac{e^{-j\omega_0 n} + e^{j\omega_0 n}}{2}) \\ = \left(\frac{1}{2}\right) y_1[n]e^{-j\omega_0 n} + \left(\frac{1}{2}\right) y_1[n]e^{j\omega_0 n}. \end{aligned} \tag{15}$$

Therefore if we compute DTFT on both sides we get the DTFT $Y_{2d}(\omega)$ as

$$Y_{2d}(\omega) = \left(\frac{1}{2}\right) Y_{1d}(\omega + \omega_0) + \left(\frac{1}{2}\right) Y_{1d}(\omega - \omega_0). \tag{16}$$

Therefore, the $Y_{2d}(\omega)$ should be the sum of the input $Y_{1d}(\omega)$ shifted by ω_0 and $-\omega_0$. Here $\omega = \frac{2\pi}{3}$. Then we can sketch $Y_{2d}(\omega)$ as following

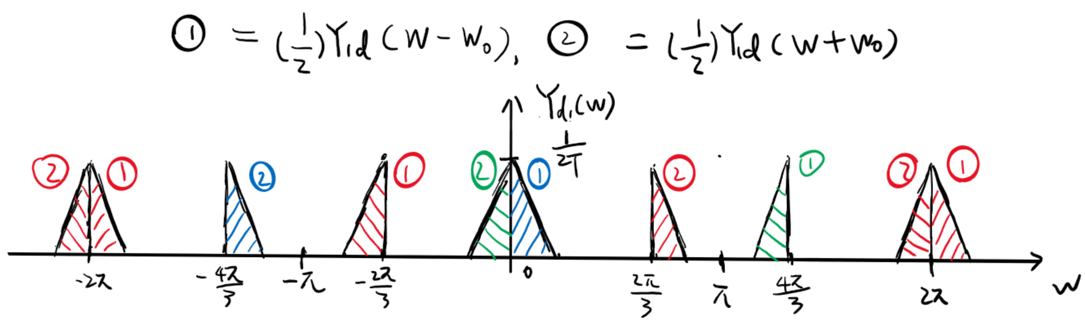


Figure 13: Problem 4: The sketch of $Y_{2d}(\omega)$, we first sketch the two shifted version and combine them together, the different color represents different copy in Fig. 12.

It is not hard to check that $Y_{2d}(\omega)$ is still periodic by period 2π.

4. In order to recover the frequency components of the original message, we can observe that the components within the range $[-\frac{\pi}{15}, \frac{\pi}{15}]$ is what we need, therefore it is sufficient to apply a window function within the range $[-\frac{2\pi}{3}, \frac{2\pi}{3}]$ with magnitude 2 (since the magnitude in $Y_{2d}(\omega)$ is $\frac{1}{2T}$). Then we can design the $H_{2d}(\omega)$ as following

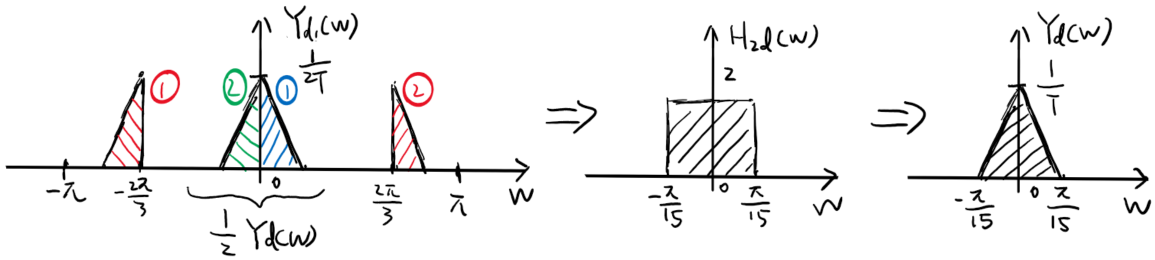


Figure 14: Problem 4: The sketch of the filter $H_{2d}(\omega)$.

Actually, any window function with the maximum frequency less than $\frac{2\pi}{3}$ is feasible.

Problem 5:

A speech signal $x_a(t)$ is assumed to be bandlimited to 15kHz. It is desired to filter this signal with a bandpass filter that will pass the frequencies between 500Hz and 5kHz by using a digital filter $H_d(\omega)$ shown in Fig. 1.

1. Determine the Nyquist sampling rate for the input signal.
2. Sketch the frequency response $H_{d,1}(\omega)$ for the necessary discrete-time filter, when sampling at the Nyquist rate.
3. Find the largest sampling period T for which the overall system comprising A/D, digital filter response ($H_{d,2}(\omega)$), and D/A realize the desired band pass filter.
4. For the system using T from part (3), sketch the necessary $H_d(\omega)$.

Solution:

1. According to the sampling theorem, since the maximum frequency is $f_{\max} = 15$kHz, the Nyquist sampling rate will be

$$\boxed{f_N = 2f_{\max} = 30\text{kHz}.} \tag{17}$$

2. Let $f_1 = 500$Hz and $f_2 = 5$kHz. When sampling at the Nyquist rate, the corresponding frequency range of the bandpass filter in ω will be

$$\omega_1 = \frac{2\pi f_1}{f_N} = \frac{\pi}{30}, \quad \omega_2 = \frac{2\pi f_2}{f_N} = \frac{\pi}{3}. \tag{18}$$

Therefore we can sketch $H_{d,1}(\omega)$ as

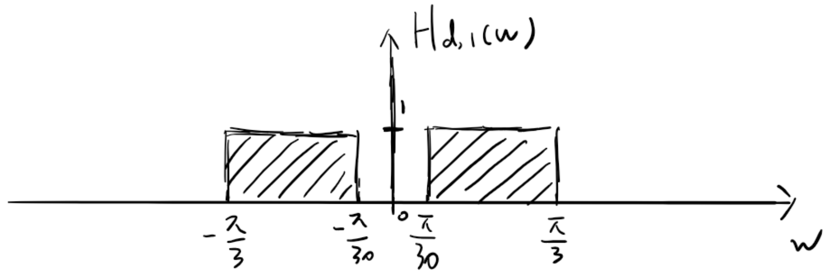


Figure 15: Problem 5: The sketch of the bandpass filter $H_{d,1}(\omega)$.

3. Here we give an illustration in Fig. 16. Given the input speech signal with maximum frequency $\Omega_{\max} = 2\pi f_{\max}$. When we increase the sampling period T as $T \geq \frac{1}{f_N}$, there exists aliasing (Case 1 and Case 2 in Fig. 16). But as long as the aliasing has no impact on the frequency range of the bandpass filter, that is,

$$\omega_2 < 2\pi - \Omega_{\max}T, \tag{19}$$

then the overall system realizes the desired band pass filter (Case 1). When

$$\omega_2 > 2\pi - \Omega_{\max}T, \tag{20}$$

The aliasing has impacts within the range of the bandpass filter as well as the overall system (Case 2). Therefore, the largest sampling period T is given as

$$\omega_2 = 2\pi f_2 T = 2\pi - \Omega_{\max} T \quad \Rightarrow \quad \boxed{T = \frac{1}{f_2 + f_{\max}} = \frac{1}{20000}\text{s.}} \tag{21}$$

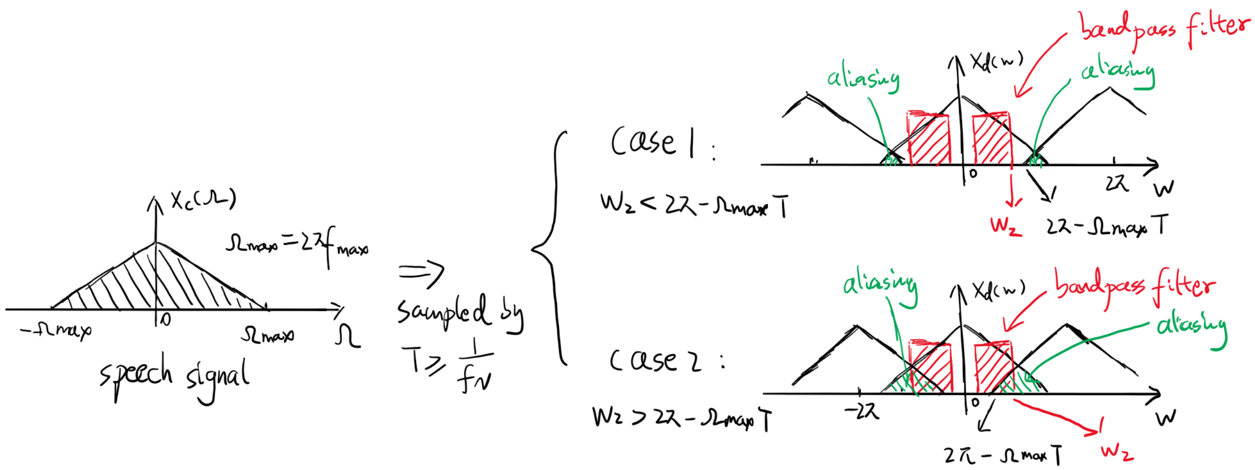


Figure 16: Problem 5: Illustration of the part 3.

4. Under the sampling period in part 3 as $T = \frac{1}{20000}$s, for the frequency range of the bandpass filter we have

$$\omega_1 = 2\pi f_1 T = \frac{\pi}{50}, \quad \omega_2 = 2\pi f_2 T = \frac{\pi}{2}. \tag{22}$$

Then we can sketch the bandpass filter $H_d(\omega)$ as following

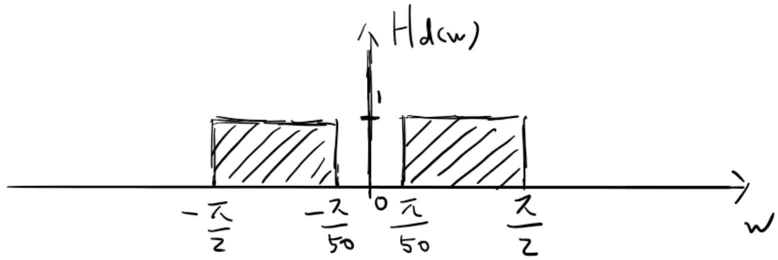


Figure 17: Problem 5: The sketch of the bandpass filter $H_d(\omega)$.