Digital Signal Processing ECE 310 (Spring 2020)
Katselis/Radhakrishnan/Shanbhag **Homework 2** Assigned: 02/05 - Due: 02/12

Topics covered in this homework are: System Properties, Convolution. Homework is due at 5:00 PM on Wednesdays. Homework will be graded for (1) completion and (2) one randomly picked problem will be graded. Submissions will be using gradescope. Please solve problems on your own in order to maximally benefit from this homework.

Problem 1:

Determine if the following systems are **linear** or **non-linear**, **causal** or **non-causal**, **shift-invariant** or **shift-varying**.

1. $y[n-2] = x^2[n] + 3y[n]$
   Solution:

   (i) Linearity: For parts (i) and (ii) the system is assumed to have zero initial conditions (initial rest). Consider inputs $x_1[n]$ and $x_2[n]$,

   $$y_1[n-2] = x_1^2[n] + 3y_1[n]$$

   $$y_2[n-2] = x_2^2[n] + 3y_2[n]$$

   Let $x[n] = x_1[n] + x_2[n]$,

   $$\Rightarrow y[n-2] = (x_1[n] + x_2[n])^2 + 3y[n]$$

   Since $(x_1[n] + x_2[n])^2 \neq x_1^2[n] + x_2^2[n]$, the system is **not linear**.

   (ii) Causality: The way the equation is written, the system is **not causal**.

   (iii) Time/Shift-Invariance: Let $z[n] = y[n-2] - 3y[n]$,

   $$z[n] = x^2[n]$$

   Let,

   $$x_1[n] = x[n-n_0] \Rightarrow z_1[n] = x^2[n-n_0]$$
   $$z_1[n] = z[n-n_0] = y[n-n_0-2] - 3y[n-n_0]$$

   The system is **shift-invariant**.

2. $y[n] = y[n-1] + \sum_{m=-\infty}^{n} x[m]$
   Solution:

   (i) Linearity:

   $$x_1[n] \to y_1[n] = y_1[n-1] + \sum_{m=-\infty}^{n} x_1[m] \quad (1)$$

   $$x_2[n] \to y_2[n] = y_2[n-1] + \sum_{m=-\infty}^{n} x_2[m] \quad (2)$$

   $$x[n] = ax_1[n] + bx_2[n]$$

Multiply Eqn.(1) by a and Eqn.(2) by b and add:

$$ay_1[n] + by_2[n] = ay_1[n-1] + by_2[n-1] + \sum_{m=-\infty}^{n} (ax_1[m] + bx_2[m])$$

The system is **linear**.

(ii) <u>Causality</u>: Causal as written.

(iii) <u>Shift-Invariance</u>: Let $z[n] = y[n] - y[n-1]$,

$$\Rightarrow z[n] = \sum_{m=-\infty}^{n} x[m]$$

Let $x_1[n] = x[n-n_0]$,

$$\Rightarrow z_1[n] = \sum_{m=-\infty}^{n} x_1[m] = \sum_{m=-\infty}^{n} x[m-n_0]$$

Let $k = m - n_0$

$$\Rightarrow z_1[n] = \sum_{k=-\infty}^{n-n_0} x[k]$$

$$\Rightarrow z_1[n] = z[n-n_0] = y[n-n_0] - y[n-n_0-1]$$

$$\Rightarrow y[n-n_0] = y[n-n_0-1] + \sum_{k=-\infty}^{n-n_0} x[k]$$

System is **shift-invariant**.

3. $y[n] = x[n^2]$
Solution:

(i) <u>Linearity</u>: **Linear** system. Can be verified following the similar procedure as in part 2.

(ii) <u>Causality</u>: $n = -2 \Rightarrow y[-2] = x[4]$. System is **not causal**.

(iii) <u>Shift-Invariance</u>: Let $x_1[n] = x[n-n_0]$

$$\Rightarrow y_1[n] = x_1[n^2] = x[n^2 - n_0] \neq y[n-n_0]$$

System is **shift-varying**.
Counter example: Let $x[n] = \delta[n]$

$$y[n] = x[n^2] = \delta[n]$$

$$x_1[n] = \delta[n-2] \Rightarrow y_1[n] = x_1[n^2] = \delta[n^2-2] = 0$$

$$\text{Not } y[n-2] = \delta[n-2] \neq 0$$

Note: try $x_1[n] = \delta[n-1]$.

4. $y[n] = x[-n]$
   Solution:

   (i) Linearity:
$$x_1[n] \to y_1[n] = x_1[-n]$$
$$x_2[n] \to y_2[n] = x_2[-n]$$
   Let $x[n] = ax_1[n] + bx_2[n]$,
$$\Rightarrow y[n] = x[-n] = ax_1[-n] + bx_2[-n]$$
$$\Rightarrow y[n] = ay_1[n] + by_2[n]$$
   System is **linear**.

   (ii) Causality: Note $y[-2] = x[2] \Rightarrow$ **not causal**.

   (iii) Shift-Invariance:
$$x_1[n] = x[n-n_0]$$
$$y_1[n] = x_1[-n] = x[-n-n_0]$$
   But $y[n-n_0] = x[-(n-n_0)] = x[-n+n_0]$, system is **shift-varying**.

5. $y[n] = x[n] + u[n]$
   Solution:

   (i) Linearity:
$$x_1[n] \to y_1[n] = x_1[n] + u[n]$$
$$x_2[n] \to y_2[n] = x_2[n] + u[n]$$
   Let $x[n] = x_1[n] + x_2[n] \Rightarrow y[n] = x[n] + u[n]$,
$$\Rightarrow y[n] = x_1[n] + x_2[n] + u[n] \neq y_1[n] + y_2[n]$$
$$\Rightarrow \textit{System is } \textbf{not linear}$$

   (ii) Causality: The system is **Causal**.

   (iii) Shift-Invariance:
$$x_1[n] = x[n-n_0]$$
$$\Rightarrow y_1[n] = x_1[n] + u[n] = x[n-n_0] + u[n] \neq y[n-n_0]$$
$$\Rightarrow \textit{System is } \textbf{shift-varying}.$$

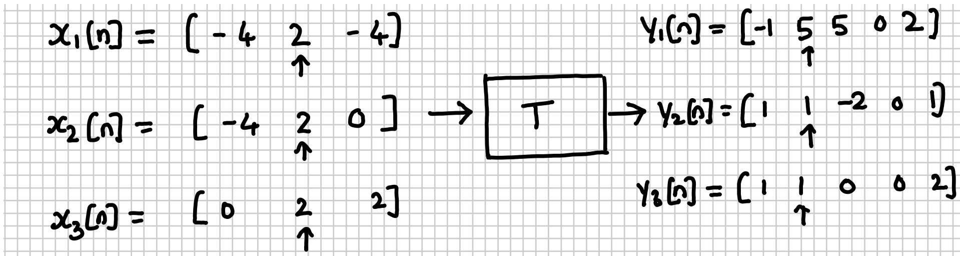

Figure 1: Figure for Problem 2

Problem 2:

Consider the **linear** system T shown below. The outputs $y_k[n], k = 1, 2, 3$ of the system for three inputs $x_k[n], k = 1, 2, 3$ are shown below.

1. Find the output of the system when the input $x[n] = \delta[n]$.
2. Can the system be Shift-Invariant?

Solution:
Note $\delta[n] = \frac{1}{2}x_3[n] - [\frac{1}{4}x_2[n] - \frac{1}{4}x_1[n]]$.
Since system is linear:

$$h[n] = \frac{1}{2}y_3[n] - [\frac{1}{4}y_2[n] - \frac{1}{4}y_1[n]]$$

$$\boxed{h[n] = [0\ 1.5\ 1.75\ 0\ 1.25]}$$

$$\delta[n-1] = \frac{1}{4}(x_2[n] - x_1[n]) = [0\ 0\ -1]$$

$$\frac{1}{4}(y_2[n] - y_1[n]) = [0.5\ -1\ -1.75\ 0\ -0.25] \neq h[n-1]$$

System is **shift-varying**.

Problem 3:

Consider two systems in cascade as shown below.

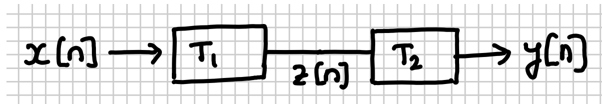

Figure 2: Figure for Problem 3

1. Assume T_1 and T_2 are linear. Will the overall system be linear?
2. Assume $z[n] = e^{x[n]}$ and $y[n] = log(z[n])$. Are T_1 and T_2 linear? What can you say about the overall system?
3. Assume $z[n] = x[n]e^{j\frac{\pi n}{4}}$ and $y[n] = z[n]e^{-j\frac{\pi n}{4}}$. Are T_1 and T_2 Shift Invariant? What can you say about the overall system?

Solution:
Consider the inputs $x_1[n], x_2[n]$. T_1 and T_2 are linear.

$$T_1\{x_1[n]\} = z_1[n] \; and \; T_1\{x_2[n]\} = z_2[n]$$

$$T_2\{z_1[n]\} = y_1[n] \; and \; T_2\{z_2[n]\} = y_2[n]$$

(1) Let $x[n] = ax_1[n] + bx_2[n]$,

$$z[n] = T_1\{[x[n]\} = az_1[n] + bz_2[n]$$

$$y[n] = T_2\{z[n]\} = T_2\{az_1[n] + bz_2[n]\}$$

$$y[n] = ay_1[n] + by_2[n]$$

$\Rightarrow$ **Overall system is linear.**

(2) Let $x_1[n]$ and $x_2[n]$ be two inputs,

$$z_1[n] = e^{x_1[n]} \; and \; z_2[n] = e^{x_2[n]}$$

Let $x[n] = x_1[n] + x_2[n]$,

$$z[n] = e^{x_1[n]+x_2[n]} \neq e^{x_1[n]} + e^{x_2[n]}$$

T_1 **is not linear**.
Can similarly show T_2 is not linear.

Overall System: $y[n] = log(z[n]) = log\left\{e^{x[n]}\right\}$

$\Rightarrow y[n] = x[n] \Rightarrow$ **Linear**.

Note: The non-linearity of the first system is cancelled by the second system.

(3) Let $x_1[n] = x[n - n_0]$ and we know $z[n] = x[n]e^{\jmath \frac{\pi}{4} n}$,

$$\Rightarrow z_1[n] = x_1[n]e^{\jmath \frac{\pi}{4} n} = x[n - n_0]e^{\jmath \frac{\pi}{4} n} \neq z[n - n_0]$$

$\Rightarrow T_1$ is **not SI**.
Similarly can show T_2 is not SI

Overall system: $z[n] = x[n]e^{\jmath \frac{\pi}{4} n}$ and $y[n] = z[n]e^{-\jmath \frac{\pi}{4} n}$,

$$\Rightarrow y[n] = x[n]$$

$\Rightarrow$ Overall system is **SI**.

Problem 4:

Determine if the following systems with given impulse response are causal and(stable?).

1. $h[n] = \left(\frac{1}{2}\right)^n u[n]$
2. $h[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n-1]$
3. $h[n] = u[n+5] - u[n-5]$
4. $h[n] = \left(\frac{1}{4}\right)^{|n|}$
5. $h[n] = sin(\pi n)u[n]$

Solution:

(1) $h[n] = (\frac{1}{2})^n u[n]$
$h[n] = 0$ and $n < 0 \Rightarrow$ **causal**
$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} (\frac{1}{2})^n = \frac{1}{1-\frac{1}{2}} = 2 \Rightarrow h[n]$ is **stable**.

(2) $h[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n-1]$
Note $h[n] \neq 0$ for $n < 0$, for example,

$$h[-2] = 2^{-2}u[2-1] = \frac{1}{4}$$

$\Rightarrow$ System is **not causal**.

$$\begin{aligned}\sum_{n=-\infty}^{\infty}|h[n]| &= \sum_{n=-\infty}^{-1}2^n + \sum_{n=0}^{\infty}(\frac{1}{2})^n \\ &= -1+1+\sum_{n=-\infty}^{-1}2^n + \sum_{n=0}^{\infty}(\frac{1}{2})^n \\ &= -1+\sum_{n=-\infty}^{0}2^n + \sum_{n=0}^{\infty}(\frac{1}{2})^n \\ &= -1+\sum_{n=0}^{\infty}(\frac{1}{2})^n + \sum_{n=0}^{\infty}(\frac{1}{2})^n\end{aligned}$$

$\Rightarrow \sum|h[n]| = -1 + 2\cdot\frac{1}{1-\frac{1}{2}} \Rightarrow \sum|h[n]| = 3 \; < \infty \;\Rightarrow$ **Stable**

(3) $h[n] = u[n+5] - u[n-5]$
$h[n] \neq 0$ for $n < 0 \Rightarrow$ **Not causal**
$\sum_{n=-\infty}^{\infty}|h[n]| = \sum_{n=-\infty}^{\infty}|u[n+5]-u[n-5]| = 10 < \infty \Rightarrow$ **Stable**

(4) $h[n] = \left(\frac{1}{4}\right)^{|n|}$
$h[n] \neq 0, n < 0 \Rightarrow$ **Not causal**

$$\begin{aligned}\sum_{n=-\infty}^{\infty}|h[n]| &= \sum_{n=-\infty}^{-1}(\frac{1}{4})^{-n} + \sum_{n=0}^{\infty}(\frac{1}{4})^n \\ &= -1+\sum_{n=-\infty}^{0}(\frac{1}{4})^{-n} + \sum_{n=0}^{\infty}(\frac{1}{4})^n \\ &= -1+\sum_{n=0}^{\infty}(\frac{1}{4})^n + \sum_{n=0}^{\infty}(\frac{1}{4})^n\end{aligned}$$

$\Rightarrow \sum|h[n]| = -1 + 2\cdot\frac{1}{1-\frac{1}{4}} \Rightarrow \sum|h[n]| = \frac{5}{3} \; < \infty \;\Rightarrow$ **Stable**

(5) $h[n] = sin(\pi n)u[n]$
$h[n] = 0$ for $n < 0 \Rightarrow$ **Causal**
$\sum_{n=-\infty}^{\infty}|h[n]| = \sum_{n=0}^{\infty}|(sin(\pi n)| = 0 \;\Rightarrow$ **Stable**

If we had $h[n] = cos(\pi n)u[n]$ then
$h[n] = 0$ for $n < 0 \Rightarrow$ **Causal**
$\sum_{n=-\infty}^{\infty}|h[n]| = \sum_{n=0}^{\infty}|(cos(\pi n)| \to \infty$ as $n \to \infty \;\Rightarrow$ **Not stable**

Problem 5:

Compute the convolution $x[n] * h[n]$ for $x[n]$ and $h[n]$ shown below.

1. $x[n] = (\delta[n] - \delta[n-1]), h[n] = (0.5)^n u[n]$
2. $x[n] = \{\underset{\uparrow}{1}, 1, 1, 1, 1\}, h[n] = \{1, \underset{\uparrow}{2}, 3\}$
3. $x[n] = u[n-10], h[n] = cos(n)u[n]$
Solution:

(1) $x[n] = (\delta[n] - \delta[n-1]), h[n] = (0.5)^n u[n]$

$$\begin{aligned} y[n] = x[n] * h[n] &= (\delta[n] - \delta[n-1]) \times h[n] \\ &= h[n] - h[n-1] \\ &= (\frac{1}{2})^n - (\frac{1}{2})^{n-1} \ for \ n \geq 0 \end{aligned}$$

$\Rightarrow y[n] = (\frac{1}{2})^{n-1} \ [\frac{1}{2} \ - 1] = -\frac{1}{2} \cdot (\frac{1}{2})^{n-1} \ for \ n \geq 0$

(2) $x[n] = \{\underset{\uparrow}{1}, 1, 1, 1, 1\}, h[n] = \{1, \underset{\uparrow}{2}, 3\}$

$y[n] = \{\underset{\uparrow}{1}, 1, 1, 1, 1\}$

$\{3, \underset{\uparrow}{2}, 1\}$

$y[-1] = 1, y[0] = 3, y[1] = 6, y[2] = 6, y[3] = 6, y[4] = 5, y[5] = 3$

(3) $x[n] = u[n-10], h[n] = cos(n)u[n]$

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} u[k-10] \cdot cos(n-k)u[n-k] \\ &= \sum_{k=10}^{\infty} cos(n-k)u[n-k] \\ &= \sum_{k=10}^{n} cos(n-k) \\ &= \frac{1}{2} \sum_{k=10}^{n} (e^{j(n-k)} + e^{-j(n-k)}) \end{aligned}$$

Put: $n - k = m \Rightarrow y[n] = \frac{1}{2} \sum_{m=0}^{n-10} (e^{jm} + e^{-jm})$

$\Rightarrow y[n] = \frac{1}{2}[\frac{1+e^{j(n-9)}}{1-e^{j}} + \frac{1+e^{-j(n-9)}}{1-e^{-j}}]$ for $n \geq 10$

Problem 6:

Let $x[n] = \left(\frac{1}{3}\right)^n u[n], \quad h[n] = \left(\frac{2}{3}\right)^n u[n], \quad w[n] = u[n] - u[n-10]$. Compute:

1. $y_1[n] = x[n] * h[n]$
2. $y_2[n] = x[n] * w[n]$
3. Use results obtained in parts (1) and (2) to compute $y_3[n] = x[n] * \left(\frac{1}{5}h[n-1] + \frac{1}{10}w[n+9]\right)$

   Solution:

   (1)

$$\begin{aligned} y_1[n] &= x[n] * h[n] \\ &= \sum_{k=-\infty}^{\infty} (\frac{1}{3})^k u[k] \cdot (\frac{2}{3})^{n-k} u[n-k] \\ &= \sum_{k=0}^{\infty} (\frac{1}{3})^k (\frac{2}{3})^n (\frac{2}{3})^{-k} u[n-k] \\ &= (\frac{2}{3})^n \sum_{k=0}^{n} (\frac{1}{3})^k (\frac{2}{3})^{-k} \\ &= (\frac{2}{3})^n \sum_{k=0}^{n} (\frac{1}{2})^k \end{aligned}$$

$\Rightarrow y_1[n] = 2(\frac{2}{3})^n (1 - (\frac{1}{2})^{n+1})$ for $n \geq 0$

   (2)

$$\begin{aligned} y_2[n] &= x[n] * w[n] \\ &= \sum_{k=-\infty}^{\infty} (\frac{1}{3})^k u[k] \cdot (u[n-k] - u[n-k-10]) \\ &= \sum_{k=0}^{\infty} (\frac{1}{3})^k (u[n-k] - u[n-k-10]) \end{aligned}$$

<u>Note</u>: $y[n] = 0$ for $n < 0$
For $0 \leq n \leq 9$:

$$\begin{aligned} y_2[n] &= \sum_{k=0}^{\infty} (\frac{1}{3})^k u[n-k] \\ &= \sum_{k=0}^{n} (\frac{1}{3})^k \\ &= \frac{3}{2}[1 - (\frac{1}{3})^{n+1}] \end{aligned}$$

For $n \geq 9$:

$$\begin{aligned} y_2[n] &= \sum_{k=0}^{n} (\frac{1}{3})^k - \sum_{k=0}^{n-10} (\frac{1}{3})^k \\ &= \sum_{k=n-9}^{n} (\frac{1}{3})^k \\ &= \frac{3}{2}(1 - (\frac{1}{3})^{10})(\frac{1}{3})^{n-9} \end{aligned}$$

(3) Note: $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

$$\Rightarrow y[n-m] = \sum_{k=-\infty}^{\infty} x[k]h[n-m-k]$$

Also $y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$

$$\Rightarrow y[n-m] = \sum_{k=-\infty}^{\infty} h[k]x[n-m-k]$$

$$\Rightarrow y_3[n] = \frac{1}{5} y_1[n-1] + \frac{1}{10} y_2[n+9]$$