Digital Signal Processing | ECE 310 (Spring 2020)
Katselis/Radhakrishnan/Shanbhag | **Homework 6** | Assigned: 03/04 - Due: 03/11

Topics covered in this homework are: DTFT properties, Fourier analysis of LTI systems and frequency response (magnitude and phase). Homework is due at 5:00 PM on Wednesdays. Homework will be graded for (1) completion and (2) one randomly picked problem will be graded. Submissions will be using gradescope. Please solve problems on your own in order to maximally benefit from this homework.

Problem 1:

It is known that DTFT $X_d(\omega)$ is a real function of ω. Now answer the following:

(a) What property does $x[n]$ need to satisfy? Provide reasons.

(b) Given that $x[n] = 0, \forall |n| > 3$, $x[-3] = -j$, $x[-2] = 1 + \frac{1}{2}j$, $x[0] = 1$, $x[1] = 2 + j$, determine the remaining values of $x[n]$.

(c) If $x[n]$ is a real-valued sequence, determine the values of $X_d(\frac{3\pi}{4})$, $X_d(-\frac{\pi}{4})$, $X_d(-\frac{5\pi}{4})$ and $X_d(\frac{7\pi}{4})$ if $X_d(-\frac{3\pi}{4}) = -2$ and $X_d(\frac{\pi}{4}) = 3.5$.

Solution:
(a)

$$x[n] \leftrightarrow X_d(\omega),$$
$$X_d(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n},$$
$$X_d^*(\omega) = \sum_{n=-\infty}^{\infty} x^*[n]e^{j\omega n} = \sum_{n=-\infty}^{\infty} x^*[-n]e^{-j\omega n} = \mathcal{F}\{x^*[-n]\}.$$

$\therefore X_d(\omega) = X_d^*(\omega) \Rightarrow x[n] = x^*[-n]$. i.e., $x[n]$ is a conjugate-symmetric sequence. Note the duality: a real-valued $X_d(\omega)$ implies that $x[n]$ is conjugate-symmetric whereas and a real-valued $x[n]$ implies that its DTFT $X_d(\omega)$ is conjugate-symmetric.

(b) Given $x[-3], x[-2], x[0], x[1]$, need to find $x[-1], x[2], x[3]$. Using the conjugate symmetry property of $x[n]$ from (a),

$$x[-1] = x^*[1] = 2 - j,$$
$$x[2] = x^*[-2] = 1 - \frac{1}{2}j,$$
$$x[3] = x^*[-3] = j.$$

(c) Part (a) showed that $X_d(\omega) = X_d^*(\omega) \Rightarrow x[n] = x^*[-n]$. In addition, if $x[n]$ is real, then

$x^*[-n] = x[-n]$. Thus, $x[n] = x[-n]$, i.e., $x[n]$ is an even sequence. For an even sequence $x[n]$:

$$\begin{aligned} X_d(\omega) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} x[-n]e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} x[n]e^{j\omega n} = X_d(-\omega) \end{aligned}$$

i.e., $X_d(\omega) = X_d(-\omega)$ is an even function of ω. A simpler way to derive this result is to recall that if $x[n]$ is real-valued then $X_d(\omega) = X_d^*(-\omega)$ (symmetry property) and if in addition $X_d(\omega)$ is real-valued as in this problem, i.e., $X_d^*(-\omega) = X_d(-\omega)$, then $X_d(\omega) = X_d(-\omega)$. Thus, given $X_d(-\frac{3\pi}{4}) = -2$ and $X_d(\frac{\pi}{4}) = 3.5$, find:

1. $X_d(\frac{3\pi}{4}) = X_d(-\frac{3\pi}{4}) = -2$,
2. $X_d(-\frac{\pi}{4}) = X_d(\frac{\pi}{4}) = 3.5$,
3. $X_d(-\frac{5\pi}{4}) = X_d(\frac{5\pi}{4}) = X_d(-\frac{3\pi}{4}) = -2$,
4. $X_d(\frac{7\pi}{4}) = X_d(-\frac{\pi}{4}) = 3.5$,

where 3. and 4. uses the periodicity property of $X_d(\omega)$.

Problem 2:

Consider a causal LTI system described by the following difference equation:

$$y[n] + \frac{1}{2}y[n-4] = x[n] - x[n-2] \tag{1}$$

(a) Determine the frequency response $H_d(\omega)$.

(b) Will the impulse response $h[n]$ be real-valued or complex? Provide reasons.

(c) Determine $y[n]$ when the input is $x[n] = \sin(\frac{\pi}{4}n)$.

Solution:
(a) Taking the DTFT of both sides of the difference equation in Eqn.(1), we get:

$$\begin{aligned} Y_d(\omega)[1 + \frac{1}{2}e^{-j4\omega}] &= X_d(\omega)[1 - e^{-j2\omega}] \\ \therefore H_d(\omega) = \frac{Y_d(\omega)}{X_d(\omega)} &= \frac{[1 - e^{-j2\omega}]}{[1 + \frac{1}{2}e^{-j4\omega}]} \end{aligned}$$

(b) If $h[n]$ is real-valued, then $H_d(\omega) = H_d^*(-\omega)$, i.e., $H_d(\omega)$ would be conjugate-symmetric. Let us check this:

$$H_d(\omega) = \frac{[1 - e^{-j2\omega}]}{[1 + \frac{1}{2}e^{-j4\omega}]}$$
$$H_d^*(\omega) = \frac{[1 - e^{j2\omega}]}{[1 + \frac{1}{2}e^{j4\omega}]}$$
$$H_d^*(-\omega) = \frac{[1 - e^{-j2\omega}]}{[1 + \frac{1}{2}e^{-j4\omega}]} = H_d(\omega)$$

So, $h[n]$ is indeed a real-valued sequence. In general, an LCCDE with real-valued coefficients corresponds will lead to a real-valued $h[n]$ and therefore have a conjugate-symmetric frequency response.

(c) If $x[n] = sin(\frac{\pi}{4}n)$, then $x[n] = \frac{e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n}}{2j}$. Using the eigenfunction property of LTI systems,

$$\begin{aligned} y[n] &= \frac{1}{2j}H_d(\frac{\pi}{4})e^{j\frac{\pi}{4}n} - \frac{1}{2j}H_d(-\frac{\pi}{4})e^{-j\frac{\pi}{4}n} \\ &= \frac{1}{2j}[|H_d(\frac{\pi}{4})|e^{j(\frac{\pi}{4}n + \angle H_d(\frac{\pi}{4}))} - |H_d(-\frac{\pi}{4})|e^{-j(\frac{\pi}{4}n - \angle H_d(-\frac{\pi}{4}))}] \end{aligned}$$

Since $h[n]$ is real, $|H_d(\omega)| = |H_d(-\omega)|$ and $\angle H_d(\omega) = -\angle H_d(-\omega)$, therefore,

$$\begin{aligned} y[n] &= \frac{|H_d(\frac{\pi}{4})|}{2j}[e^{j(\frac{\pi}{4}n + \angle H_d(\frac{\pi}{4}))} - e^{-j(\frac{\pi}{4}n + \angle H_d(\frac{\pi}{4}))}] \\ &= |H_d(\frac{\pi}{4})|sin(\frac{\pi}{4}n + \angle H_d(\frac{\pi}{4})) \end{aligned}$$

Next, find $|H_d(\omega)|$ and $\angle H_d(\omega)$:

$$|H_d(\omega)| = |\frac{1 - e^{-j2\cdot\frac{\pi}{4}}}{1 + \frac{1}{2}e^{-j4\cdot\frac{\pi}{4}}}| = |\frac{1 - e^{-j\frac{\pi}{2}}}{1 + \frac{1}{2}e^{-j\pi}}| = |\frac{1 + j}{1 - \frac{1}{2}}| = |2(1 + j)| = 2\sqrt{2}$$
$$\angle H_d(\omega) = \angle 2(1 + j) = tan^{-1}(1) = \frac{\pi}{4}$$
$$\therefore y[n] = 2\sqrt{2}sin(\frac{\pi}{4}n + \frac{\pi}{4}) = 2\sqrt{2}sin(\frac{\pi}{4}(n + 1))$$

Problem 3:

Consider the LTI system with frequency response:

$$H_d(\omega) = 1 + \sin(2\omega) - j\cos(2\omega) \tag{2}$$

Determine the response $y[n]$ when:

(a) $x[n] = 3 + 2\sin(\frac{\pi}{4}n) + 4e^{-j\frac{\pi}{2}n}$.

(b) $x[n] = 1, \forall n$.

(c) the input is a 2-sample delayed version of $x[n]$ in part (a).

Solution: We are given $H_d(\omega) = 1 + sin(2\omega) - jcos(2\omega)$.
(a) If $x[n] = 3 + 2sin(\frac{\pi}{4}n) + 4e^{-j\frac{\pi}{4}n}$, first we check if $h[n]$ is real-valued:

$$H_d^*(\omega) = 1 + sin(2\omega) - jcos(2\omega)$$
$$H_d^*(-\omega) = 1 - sin(2\omega) - jcos(2\omega) \neq H_d(\omega)$$

$\therefore$ $h[n]$ is not real-valued. Need to express $x[n]$ in terms of sums of eigenfunctions ($e^{j\omega_0 n}$) of LTI system:

$$x[n] = 3e^{j0n} + 2\frac{(e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n})}{2j} + 4e^{-j\frac{\pi}{2}n}$$

There are three terms. Each is in the form of $Ae^{j\omega_0 n}$, and will contribute a term of the form $Ae^{j\omega_0 n} \times H_d(\omega_0)$.

$$\therefore y[n] = 3H_d(0) + \frac{1}{j}[e^{j\frac{\pi}{4}n}H_d(\frac{\pi}{4})] - \frac{1}{j}[e^{-j\frac{\pi}{4}n}H_d(-\frac{\pi}{4})] + 4e^{-j\frac{\pi}{2}n}H_d(-\frac{\pi}{2})$$
$$H_d(0) = 1 - j = \sqrt{2}e^{-j\frac{\pi}{4}}$$
$$H_d(\frac{\pi}{4}) = 1 + sin(\frac{\pi}{2}) - jcos(\frac{\pi}{2}) = 2$$
$$H_d(-\frac{\pi}{4}) = 1 - sin(\frac{\pi}{2}) - jcos(\frac{\pi}{2}) = 0$$
$$H_d(-\frac{\pi}{2}) = 1 - sin(\pi) - jcos(\pi) = 1 + j = \sqrt{2}e^{j\frac{\pi}{4}}$$
$$\therefore y[n] = 3\sqrt{2}e^{-j\frac{\pi}{4}} + \frac{2}{j}e^{j\frac{\pi}{4}n} + 4\sqrt{2}e^{-j(\frac{\pi}{2}n - \frac{\pi}{4})}$$

Note that $y[n]$ is complex-valued since $x[n]$ is real-valued and $h[n]$ is complex-valued.
(b) Given $x[n] = 1, \forall n$, find $X_d(\omega)$.

$$X_d(\omega) = 2\pi\delta(\omega), |\omega| < \pi$$

Then,

$$\begin{aligned} Y_d(\omega) &= H_d(\omega)X_d(\omega) \\ &= H_d(\omega)2\pi\delta(\omega) \end{aligned}$$

Taking inverse DTFT and employing the sifting property of $\delta(\omega)$, we get

$$\begin{aligned} y[n] &= 2\pi\int_{-\pi}^{\pi} Y_d(\omega)e^{j\omega n}d\omega \\ &= 2\pi\int_{-\pi}^{\pi} H_d(\omega)2\pi\delta(\omega)e^{j\omega n}d\omega \\ &= \int_{-\pi}^{\pi}(H_d(\omega)e^{j\omega n})\delta(\omega)d\omega \\ &= H_d(0) = 1 - j \\ \therefore y[n] &= 1 - j, \ \forall n \end{aligned}$$

To confirm this result, note one can obtain $h[n]$ by inspection:

$$\begin{aligned}
H_d(\omega) &= 1 + sin(2\omega) - jcos(2\omega) = 1 - je^{j2\omega} \\
&\Rightarrow h[n] = \delta[n] - j\delta[n+2] \\
y[n] &= \sum_{k=-\infty}^{\infty} (\delta[k] - j\delta[k+2]) \times x[n-k] \\
&\because x[n-k] = 1,\ \forall n, k \\
\therefore y[n] &= \sum_{k=-\infty}^{\infty} \delta[k] - j \sum_{k=-\infty}^{\infty} \delta[k+2] = 1 - j
\end{aligned}$$

Yet another simpler solution is to note that $x[n] = 1 = e^{j0n} \Rightarrow y[n] = H_d(0)e^{j0n} = H_d(0) = 1 - j$.
(c) $x[n] = 3 + sin(\frac{\pi}{4}(n-2)) + 4e^{-j\frac{\pi}{2}(n-2)}$
The simplest method is to realize that this is a time-invariant system. Hence, $y[n]$ will be a 2-sample delayed version of $y[n]$ in part(a), i.e.,

$$\begin{aligned}
y[n] &= 3\sqrt{2}e^{-j\frac{\pi}{4}} + \frac{2}{j}e^{j\frac{\pi}{4}(n-2)} + 4\sqrt{2}e^{-j(\frac{\pi}{2}n - \frac{5\pi}{4})} \\
&= 3\sqrt{2}e^{-j\frac{\pi}{4}} - 2e^{j\frac{\pi}{4}n} + 4\sqrt{2}e^{-j(\frac{\pi}{4}n - \frac{5\pi}{4})}
\end{aligned}$$

The other method is to use the time-shift property of DTFT and modify $H_d(\omega)$ as
$\tilde{H}_d(\omega) = H_d(\omega)e^{-j2\omega}$ and use $x[n]$ from Part (a). Then,

$$\begin{aligned}
\tilde{H}_d(0) &= H_d(0) = 1 - j = \sqrt{2}e^{-j\frac{\pi}{4}} \\
\tilde{H}_d(\frac{\pi}{4}) &= H_d(\frac{\pi}{4})e^{-j\frac{\pi}{2}} = -2j \\
\tilde{H}_d(-\frac{\pi}{4}) &= H_d(-\frac{\pi}{4})e^{j\frac{\pi}{2}} = 0 \\
\tilde{H}_d(-\frac{\pi}{2}) &= H_d(-\frac{\pi}{2})e^{j\pi} = \sqrt{2}e^{j\frac{5\pi}{4}}
\end{aligned}$$

Then,

$$\begin{aligned}
y[n] &= 3\tilde{H}_d(0) + \frac{e^{j\frac{\pi}{4}n}}{j}\tilde{H}_d(\frac{\pi}{4}) - \frac{e^{-j\frac{\pi}{4}n}}{j}\tilde{H}_d(-\frac{\pi}{4}) + 4e^{-j\frac{\pi}{2}n}\tilde{H}_d(-\frac{\pi}{2}) \\
&= 3\sqrt{2}e^{-j\frac{\pi}{4}} - 2e^{j\frac{\pi}{4}n} + 0 + 4\sqrt{2}e^{-j(\frac{\pi}{2}n - \frac{5\pi}{4})}
\end{aligned}$$

which is identical to the answer obtained by simply delaying the output in part(a) by 2 samples.

Problem 4:

The magnitude $H_d(\omega)|$ of an LTI system is given by:

$$|H_d(\omega)| = |2cos(2\omega)| \tag{3}$$

A complex sinusoid $x[n] = ae^{j\omega_1 n}$ $(a > 0)$ is fed at the input to the LTI system after multiplying it by $e^{j\omega_o n}$ where the modulation frequency $-\pi < \omega_o < \pi$ is under user control. A peak detector at the output estimates the magnitude of the output $y[n]$. As ω_o is swept from 0 to π, it is observed that the magnitude of the output nulls out when $\omega_o = \frac{\pi}{8}$ and $\omega_o = \frac{5\pi}{8}$ and its peak magnitude is 6.

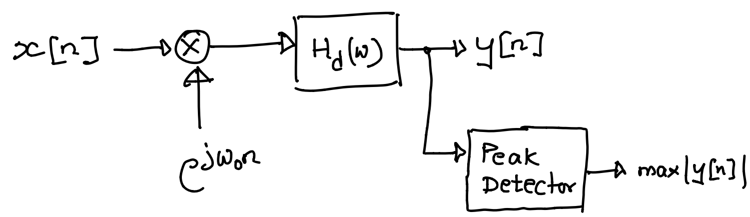

Figure 1: System set-up for Problem 4.

(a) What are the possible values of ω_1 and the amplitude a of $x[n]$?

(b) If $x[n] = 4e^{j\pi n}$ and ω_o is swept from 0 to $-\pi$, at what values of ω_o will the output magnitude peak and what will be its peak magnitude?

(c) If $x[n] = 4e^{j\pi n}$ and ω_o is swept from 0 to $-\pi$, at what values of ω_o will the output magnitude null out?

Solution: $|H_d(\omega)| = |2cos(2\omega)|$

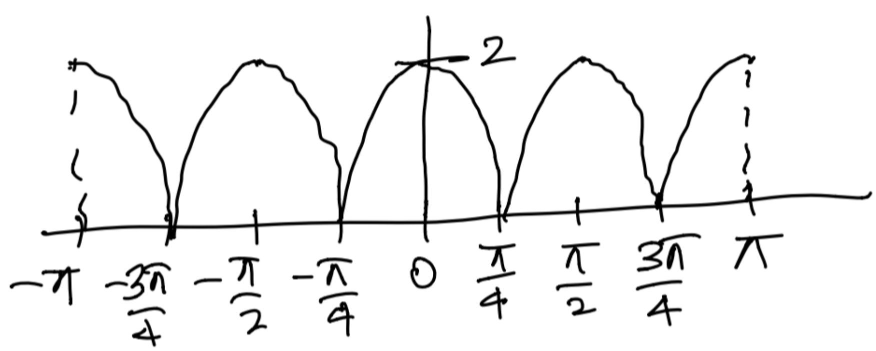

Figure 2: Magnitude diagram of $H_d(\omega)$

(a) Using the modulation property of DTFT:

$$e^{j\omega_0 n}x[n] \leftrightarrow X_d(\omega - \omega_0)$$

as ω_0 is swept from 0 to $+\pi$, $X_d(\omega)$ moves (modulated) by ω_0 to the right (increasing ω).
$\therefore$ If

$$X_d(\omega - \frac{\pi}{8}) \times H_d(\omega) = 0,$$

and

$$H_d(-\frac{3\pi}{4}) = H_d(-\frac{\pi}{4}) = H_d(\frac{\pi}{4}) = H_d(\frac{3\pi}{4}) = 0,$$

then ω_1 can take the following values:

$$\omega_1 = -\frac{3\pi}{4} - \frac{\pi}{8} = -\frac{7\pi}{8} = -\frac{\pi}{4} - \frac{5\pi}{8}$$
$$\omega_1 = -\frac{\pi}{4} - \frac{\pi}{8} = -\frac{3\pi}{8} = \frac{\pi}{4} - \frac{5\pi}{8}$$
$$\omega_1 = \frac{\pi}{4} - \frac{\pi}{8} = \frac{\pi}{8} = \frac{3\pi}{4} - \frac{5\pi}{8}$$
$$\omega_1 = \frac{3\pi}{4} - \frac{\pi}{8} = \frac{5\pi}{8} = \frac{5\pi}{4} - \frac{5\pi}{8}$$

Since

$$x[n] = ae^{j\omega_1 n}$$
$$y[n] = ae^{j\omega_1 n}H_d(\omega_1)$$
$$|y[n]| = |a||H_d(\omega_1)|$$
$$max|y[n]| = |a|max|H_d(\omega_1)|$$
$$6 = |a| \times 2$$
$$\therefore |a| = 3$$

(b) Following similar arguments as in part(a),

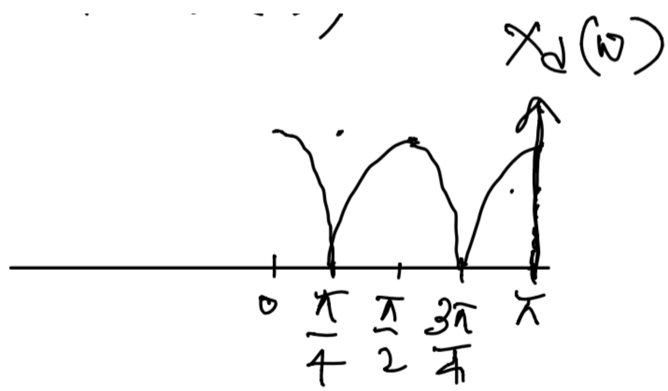

Figure 3: Diagram of $X_d(\omega)$

$\therefore$ $y[n]$ will peak at $\omega_0 = 0,\ \frac{\pi}{2},\ and\ -\pi$ as $X_d(\omega)$ is modulated to lower frequencies. The peak output will be

$$4 \times 2 = 8$$

(c) $y[n]$ will null out when $X_d(\omega)$ is shifted left by $\frac{\pi}{4}$ and $\frac{3\pi}{4}$. Hence,

$$\omega_0 = -\frac{\pi}{4}\ and\ -\frac{3\pi}{4}.$$

Problem 5:

Which of the following sequences are eigenfunctions of stable LTI systems:

(a) $x[n] = e^{j\frac{2\pi}{3}n}$.

(b) $x[n] = 3^n$ and the system is also causal.

(c) $x[n] = 3^n u[n]$.

(d) $x[n] = cos(\omega_o n)$.

Solution:
(a)

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\ &= \sum_{k=-\infty}^{\infty} h[k]e^{j\frac{2\pi}{3}(n-k)} \\ &= e^{j\frac{2\pi}{3}n} \sum_{k=-\infty}^{\infty} h[k]e^{-j\frac{2\pi}{3}k} \end{aligned}$$

Since $h[k]$ is stable, the unit circle lies in its ROC, hence the summation above converges.
$\therefore y[n] = e^{j\frac{2\pi}{3}n}H_d(\frac{2\pi}{3})$ and $x[n] = e^{j\frac{2\pi}{3}n}$ is an eigenfunction.
(b) LTI system is causal, i.e., $h[n] = 0, \forall n < 0$. It is also stable. Hence,

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\ &= \sum_{k=0}^{\infty} h[k]x[n-k] (h[n] \textit{ is causal}) \\ &= \sum_{k=0}^{\infty} h[k]3^{n-k} \\ &= 3^n \sum_{k=0}^{\infty} h[k]3^{-k} \end{aligned}$$

Since $h[k]$ is causal and stable, $Z = 3 \subseteq ROC_H$, hence the summation also converges.
$\therefore 3^n$ is an eigenfunction of a stable and causal LTI system.

(c)

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k]3^{n-k}u[n-k] \\ &= \sum_{k=-\infty}^{n} h[k]3^{n-k} \\ &= \sum_{k=-\infty}^{n} h[k]3^{n-k} \\ &= 3^n \sum_{k=-\infty}^{n} h[k]3^{-k} \end{aligned}$$

Since the summation depends upon index n, $3^n u[n]$ is not an eigenfunction of a stable LTI system.

(d)

$$\begin{aligned} x[n] &= cos(\omega_o n) \\ &= \frac{e^{j\omega_o n} + e^{-j\omega_o n}}{2} \end{aligned}$$

Since $e^{j\omega_o n}$ is an eigenfunction of an LTI system:

$$\begin{aligned} y[n] &= \frac{1}{2}e^{j\omega_o n}H_d(\omega_o) + \frac{1}{2}e^{-j\omega_o n}H_d(-\omega_o) \\ &\neq H_d(\omega_o)cos(\omega_o n) \end{aligned}$$

So $cos(\omega_o n)$ is not an eigenfunction of an LTI system in general. However, as we saw in Problem 1 part(c), if $H_d(\omega) = H_d(-\omega)$, which would occur if $h[n] = h[-n]$ and if $h[n]$ is real-valued, then

$$y[n] = H_d(\omega_o)cos(\omega_o n)$$

and $cos(\omega_o n)$ would be an eigenfunction of such a special case.

Problem 6:

Solve Problem 3 from HWK5.

Solution:

(a)$x(t) = \delta(5t-2)$

$$\begin{aligned} X(\Omega) &= \int_{-\infty}^{\infty} \delta(5t-2)e^{-j\Omega t}dt \\ &= \frac{1}{5}e^{-j\frac{2}{5}\Omega} \end{aligned}$$

(b)$x(t) = e^{-3t}u(t)$

$$\begin{aligned} X(\Omega) &= \int_{-\infty}^{\infty} e^{-3t}u(t)e^{-j\Omega t}dt \\ &= \int_{0}^{\infty} e^{-3t}e^{-j\Omega t}dt \\ &= \frac{1}{j\Omega + 3} \end{aligned}$$

(c) $x(t) = e^{-3t} * \delta(5t-2)$

$$x(t) = \int_{-\infty}^{\infty} e^{-3(t-\tau)}\delta(5\tau - 2)d\tau = \frac{1}{5}e^{-3(t-\frac{2}{5})}$$

The CTFT of this problem does not exist because the CTFT of the signal $x(t) = \frac{1}{5}e^{-3(t-\frac{2}{5})}$ does not exist.

(d) $x(t) = 4sin(2000\pi t)$

$$\begin{aligned} X(\Omega) &= \int_{-\infty}^{\infty} 4sin(2000\pi t)e^{-j\Omega t}dt \\ &= -j4\pi[\delta(\Omega - 2000\pi) + \delta(\Omega + 2000\pi)] \end{aligned}$$