Digital Signal Processing
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Homework 1

ECE 310 (Spring 2020)
Assigned: 01/29 - Due: 02/05

Topics covered in this homework are: complex numbers, discrete-time (DT) sequences, and properties of DT systems. Homework will be graded for (1) completion and (2) one randomly picked problem will be graded. Submissions will be using gradescope. Please solve problems on your own in order to maximally benefit from this homework.

Problem 1: Complex Numbers

Consider the following complex numbers:

(a) $2e^{-j\pi/6} + 3e^{j\pi/3}$

(b) $\dfrac{(\sqrt{2} - j\sqrt{2})^{2n}}{(\sqrt{8} + j\sqrt{8})^n}$

Evaluate and represent these numbers in both Cartesian and polar forms.

Solution: For a complex number z, the Cartesian form is $z = a + \jmath b$ for $a, b \in \mathbb{R}$ and the polar form is $z = |z|e^{j\angle z}$ or $z = |z|e^{\jmath \arg z}$ for a different notation of the exponent of the complex exponential (phase of z). In both forms, $\jmath = \sqrt{-1}$. Additionally, $a = \mathrm{Re}\{z\} = |z|\cos(\angle z)$, $b = \mathrm{Im}\{z\} = |z|\sin(\angle z)$, $|z| = \sqrt{a^2 + b^2}$ and $\angle z = \arg z = \tan^{-1}\frac{b}{a}$, where $\mathrm{Re}\{\cdot\}$ and $\mathrm{Im}\{\cdot\}$ denote the real and imaginary parts of z, respectively.

(a)

$$
\begin{aligned}
z = 2e^{-j\pi/6} + 3e^{j\pi/3} &= 2\left(\cos(\pi/6) - \jmath\sin(\pi/6)\right) + 3(\cos(\pi/3) + \jmath\sin(\pi/3)) \\
&= 2\left(\frac{\sqrt{3}}{2} - \jmath\frac{1}{2}\right) + 3\left(\frac{1}{2} + \jmath\frac{\sqrt{3}}{2}\right) \\
&= \left(\sqrt{3} + \frac{3}{2}\right) + \jmath\left(\frac{3\sqrt{3}}{2} - 1\right) && \text{(Cartesian form)} \\
&= 3.6056e^{\jmath 0.4592} && \text{(Polar form)}
\end{aligned}
$$

(b)

$$
\begin{aligned}
z = \frac{(\sqrt{2} - j\sqrt{2})^{2n}}{(\sqrt{8} + j\sqrt{8})^n} &= \frac{\left(2e^{-\jmath\frac{\pi}{4}}\right)^{2n}}{\left(4e^{\jmath\frac{\pi}{4}}\right)^{n}} \\
&= \frac{4^n e^{-\jmath\frac{\pi}{2}n}}{4^n e^{\jmath\frac{\pi}{4}n}} \\
&= e^{-\jmath\frac{3\pi}{4}n} && \text{(Polar form)} \\
&= \cos\left(\frac{3\pi}{4}n\right) - \jmath\sin\left(\frac{3\pi}{4}n\right) && \text{(Cartesian form)}
\end{aligned}
$$

Problem 2: **Complex Numbers**

Determine all roots of $z^4 = 1$ on the complex plane.

Solution: Note that

$$1 = e^{\jmath 2\pi k}, \quad k \in \mathbb{Z}$$

Substituting to the equation we obtain:

$$\begin{aligned} z^4 &= e^{\jmath 2\pi k} \\ z &= e^{\jmath \frac{2\pi k}{4}} = e^{\jmath \frac{\pi}{2} k}, \quad k \in \mathbb{Z} \end{aligned}$$

Because the polynomial is of degree 4 (the polynomial in the left-hand side of the equation $z^4 - 1 = 0$), we expect four different solutions for the given complex equation. These solutions correspond to $k = 0, 1, 2, 3$ (in general, for the equation $z^N = 1$ the solutions correspond to $e^{\jmath \frac{2\pi k}{N}}$ for $k = 0, 1, 2, \ldots, N-1$, which constitute the so-called *N-th roots of unity*): $e^{\jmath \frac{\pi}{2} 0}, e^{\jmath \frac{\pi}{2} 1}, e^{\jmath \frac{\pi}{2} 2}, e^{\jmath \frac{\pi}{2} 3}$ or $1, j, -1, -j$.

Problem 3: **Complex Signals**

Suppose that $x[n]$ is a sequence of complex numbers (complex signal). The *conjugate symmetric part* of $x[n]$ is defined as $x_{\text{cs}}[n] = \frac{1}{2}(x[n] + x^*[-n])$, where $x^*[n]$ is the complex conjugate of $x[n]$. Let $x^r_{\text{cs}}[n] = \text{Re}\{x_{\text{cs}}[n]\}$ and $x^i_{\text{cs}}[n] = \text{Im}(x_{\text{cs}}[n])$. Argue that $x^r_{\text{cs}}[n] = x^r_{\text{cs}}[-n]$ and $x^i_{\text{cs}}[n] = -x^i_{\text{cs}}[-n]$, i.e., $x^r_{\text{cs}}[n]$ has *even symmetry* and $x^i_{\text{cs}}[n]$ has *odd symmetry*.

Solution: Let $x[n] = x^r[n] + \jmath x^i[n]$, where $x^r[n] = \text{Re}\{x[n]\}$ and $x^i[n] = \text{Im}\{x[n]\}$. Then,

$$\begin{aligned} x_{\text{cs}}[n] &= \frac{1}{2}(x[n] + x^*[-n]) \\ &= \frac{1}{2}\left(x^r[n] + \jmath x^i[n] + x^r[-n] - \jmath x^i[-n]\right) \\ &= \frac{x^r[n] + x^r[-n]}{2} + \jmath \frac{x^i[n] - x^i[-n]}{2}. \end{aligned}$$

Therefore,

$$\begin{aligned} x^r_{\text{cs}}[n] &= \frac{x^r[n] + x^r[-n]}{2}, \\ x^i_{\text{cs}}[n] &= \frac{x^i[n] - x^i[-n]}{2}. \end{aligned}$$

Clearly, $x^r_{\text{cs}}[n] = x^r_{\text{cs}}[-n]$ and $x^i_{\text{cs}}[n] = -x^i_{\text{cs}}[-n]$ hold.

Problem 4: **Basic Discrete-Time Signals**

Sketch the following signals:

1. $n^2(u[n+2] - u[n-4])$

2. $u[-n+2]u[n+6]$
3. $\sin\left(\frac{\pi}{4}n\right)\delta[n-1]+\cos(\frac{\pi}{7}(2n-1))\delta[n-4]$
4. $\cos(3n), n=0,1,\ldots,10$

where $u[n]$ is the unit step signal and $\delta[n]$ is the unit sample signal.

Solution:

1. $n^2(u[n+2]-u[n-4])$:

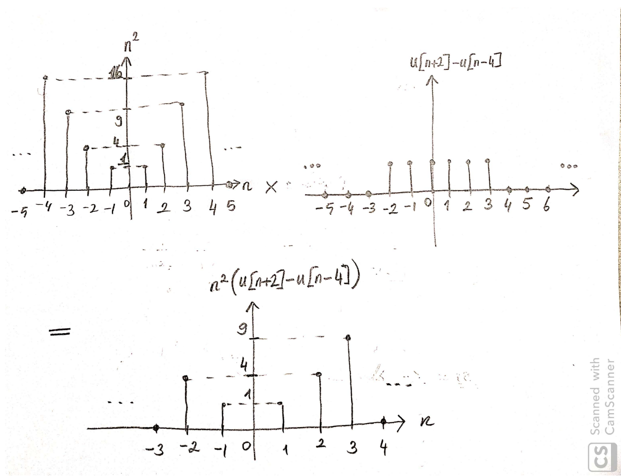

Figure 1: Sketch of 4.1

2. $u[-n+2]u[n+6]$:

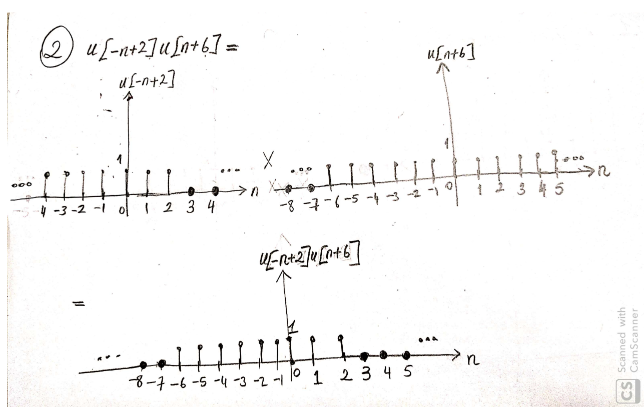

Figure 2: Sketch of 4.2

3. Using the sifting property,

$$\sin\left(\frac{\pi}{4}n\right)\delta[n-1]+\cos(\frac{\pi}{7}(2n-1))\delta[n-4]=\sin\left(\frac{\pi}{4}1\right)\delta[n-1]+\cos\left(\frac{\pi}{7}7\right)\delta[n-4]$$

$$= \sin\left(\frac{\pi}{4}\right)\delta[n-1] + \cos(\pi)\,\delta[n-4] = \frac{\sqrt{2}}{2}\delta[n-1] - \delta[n-4]$$

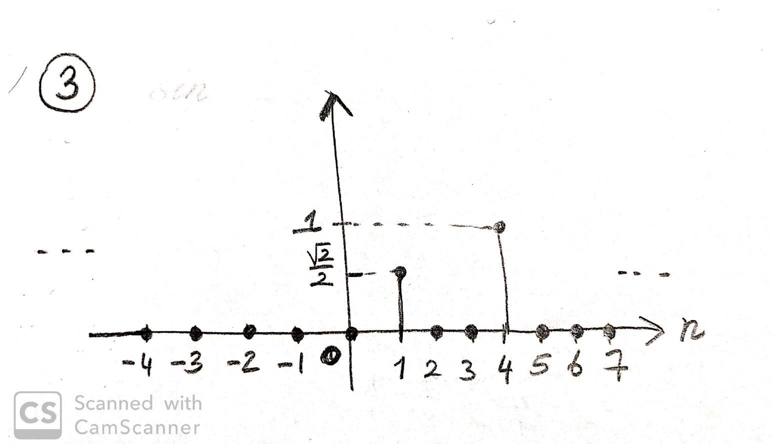

Figure 3: Sketch of 4.3

4. $\cos(3n), n = 0, 1, \ldots, 10$:

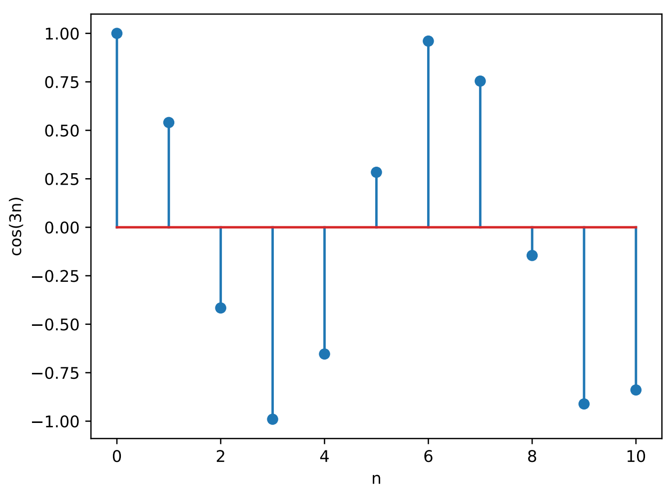

Figure 4: Sketch of 4.4

Problem 5: Finite-Length Signals

Express the signal $\{x[n]\} = \{\cdots, 0, 2, 0, -3, 1, \underset{\uparrow}{4}, 0, 3, 1, -2, 0, 4, 0 \cdots\}$ in terms of the unit sample (impulse) $\delta[n]$. Here, $\cdots$ denotes zeros.

Solution: Any discrete-time signal of finite-or infinite duration (or length) can be expressed as a (finite or infinite) sum of shifted and modulated unit samples:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k].$$

From left to right, the given (finite-length) signal can be rewritten as:

$$x[n] = 2\delta[n+4] - 3\delta[n+2] + \delta[n+1] + 4\delta[n] + 3\delta[n-2] + \delta[n-3] - 2\delta[n-4] + 4\delta[n-6].$$

Problem 6: Properties of Discrete-Time Systems

Consider the following discrete-time systems:

(a) $y[n] = x[n] + 2x[n-1] + 3x[n-3]$

(b) $y[n] = \cos(x[n])$

(c) $y[n] = x[n]e^{\jmath\frac{\pi}{4}n}$

(d) $y[n] = x[0]x[n] + x[1]x[n+1]$

Determine if the systems are: (a) linear (b) time-invariant.

Solution:

(a) (i) **Linearity**: The system S is linear if for any constants $a, b \in \mathbb{R}$, the following holds:

$$S\{ax_1[n] + bx_2[n]\} = aS\{x_1[n]\} + bS\{x_2[n]\}$$

where $x_1[n], x_2[n]$ are two input signals. For the given system, we have

$$S\{ax_1[n] + bx_2[n]\} = (ax_1[n]+bx_2[n])+2(ax_1[n-1]bx_2[n-1])+3(ax_1[n-3]+bx_2[n-3])$$

$$= a(x_1[n]+2x_1[n-1]+3x_1[n-3])+b(x_2[n]+2x_2[n-1]+3x_2[n-3]) = aS\{x_1[n]\}+bS\{x_2[n]\}$$

Thus, the system is **linear**.

(ii) **Time-invariance**: The system S is time-invariant if for any constant n_0, the following holds:

$$S\{x[n-n_0]\} = y[n-n_0]$$

For the given system, if we denote $x_{n_0}[n] = x[n-n_0]$,

$$S\{x_{n_0}[n]\} = x_{n_0}[n]+2x_{n_0}[n-1]+3x_{n_0}[n-3] = x[n-n_0]+2x[n-n_0-1]+3x[n-n_0-3] = y[n-n_0]$$

Thus, the system is **time-invariant**.

(b) Applying the same tests:

(i) **Linearity**:

$$S\{ax_1[n] + bx_2[n]\} = \cos(ax_1[n]+bx_2[n]) \neq a\cos(x_1[n])+b\cos(x_2[n]) = aS\{x_1[n]\}+bS\{x_2[n]\}$$

The system is **not linear**.

(ii) **Time-invariance:**

$$S\{x[n-n_0]\} = cos(x[n-n_0]) = y[n-n_0]$$

The system is **time-invariant**.

(c) (i) **Linearity**:

$$S\{ax_1[n] + bx_2[n]\} = (ax_1[n]+bx_2[n])e^{j\frac{\pi}{4}n} = ax_1[n]e^{j\frac{\pi}{4}n}+bx_2[n]e^{j\frac{\pi}{4}n} = aS\{x_1[n]\}+bS\{x_2[n]\}$$

The system is **linear**.

(ii) **Time-invariance:**

$$S\{x[n-n_0]\} = x[n-n_0]e^{j\frac{\pi}{4}n} \neq x[n-n_0]e^{j\frac{\pi}{4}(n-n_0)} = y[n-n_0]$$

The system is **time-varying**.

(d) (i) **Linearity**: Let $y[n] = S\{x[n]\}$ be the description of a discrete-time system S. If $y[n]$ is the output of S to input $x[n]$ and S is linear, then the output of S to input $cx[n]$ for any constant $c \in \mathbb{C}$ should be $cy[n]$.
For the given system, consider $x[n] = \delta[n]$. Then,

$$y[n] = \delta[0]\delta[n] + \delta[1]\delta[n+1] = \delta[n].$$

Consider now the input $\tilde{x}[n] = 2x[n] = 2\delta[n]$. Then,

$$\tilde{y}[n] = 4\delta[0]\delta[n] + 4\delta[1]\delta[n+1] = 4\delta[n] \neq 2\delta[n] = 2y[n].$$

Therefore, the system is **nonlinear**.

(ii) **Time-Invariance**: Setting $n - n_0$ for n in the given input-output equation we have:

$$y[n-n_0] = x[0]x[n-n_0] + x[1]x[n-n_0+1].$$

On the other hand, applying as input to the system the signal $\tilde{x}[n] = x[n-n_0]$, the obtained output is

$$\tilde{y}[n] = \tilde{x}[0]\tilde{x}[n] + \tilde{x}[1]\tilde{x}[n+1] = x[-n_0]x[n-n_0] + x[1-n_0]x[n-n_0+1] \neq y[n-n_0].$$

Recall that for time-invariance we need $\tilde{y}[n] = y[n-n_0]$ for **any** $n_0 \in \mathbb{Z}$ and an arbitrary input signal $x[n]$. Hence, the system is **time-varying**.