Homework 4

Topics covered in this homework are: inverse z-transform, system analysis via the z-transform and system transfer functions.

Problem 1: ROCs and inverse z-transforms

Find all possible ROCs for the following z-transforms and determine the associated inverse z-transform for each case.

(a) $\dfrac{z^2 - 2z}{z^2 + 4z + 3}$

(b) $\dfrac{1}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{3}z^{-1})}$

Solution:

(a) To find the inverse z-transform, we can expand the z-transform (denoted as $X(z)$) in terms of partial fractions as

$$\begin{aligned} X(z) &= \frac{z^2 - 2z}{z^2 + 4z + 3} \\ &= \frac{1 - 2z^{-1}}{1 + 4z^{-1} + 3z^{-2}} \\ &= \frac{1 - 2z^{-1}}{(1 + z^{-1})(1 + 3z^{-1})} \\ &= \frac{A}{1 + z^{-1}} + \frac{B}{1 + 3z^{-1}} \end{aligned} \tag{1}$$

where the factors A and B can be computed as

$$\begin{aligned} A &= X(z)(1 + z^{-1})|_{z=-1} = \left.\frac{1 - 2z^{-1}}{1 + 3z^{-1}}\right|_{z=-1} = -\frac{3}{2}, \\ B &= X(z)(1 + 3z^{-1})|_{z=-3} = \left.\frac{1 - 2z^{-1}}{1 + z^{-1}}\right|_{z=-3} = \frac{5}{2}. \end{aligned} \tag{2}$$

Therefore we have:

$$\boxed{X(z) = -\frac{3}{2}\underbrace{\left(\frac{1}{1 + z^{-1}}\right)}_{X_1(z)} + \frac{5}{2}\underbrace{\left(\frac{1}{1 + 3z^{-1}}\right)}_{X_2(z)}.} \tag{3}$$

We can observe from Eq. (1) that the poles are $z = -1$ and $z = -3$, so there will be three possibles ROCs:

- Case 1: ROC: $\{z : |z| < 1\}$, both $X_1(z)$ and $X_2(z)$ in Eq. (3) are z-transforms of left-sided sequences, the associated inverse z-transform is:

$$\boxed{x[n] = \frac{3}{2}(-1)^n u[-n-1] - \frac{5}{2}(-3)^n u[-n-1].} \tag{4}$$

- Case 2: ROC: $\{z : 1 < |z| < 3\}$, $X_1(z)$ is the z-transform of a right-sided sequence and $X_2(z)$ is the z-transform a left-sided sequence, the associated inverse z-transform is:

$$\boxed{x[n] = -\frac{3}{2}(-1)^n u[n] - \frac{5}{2}(-3)^n u[-n-1].} \tag{5}$$

- Case 3: ROC: $\{z : |z| > 3\}$, both $X_1(z)$ and $X_2(z)$ in Eq. (3) are z-transforms of right-sided sequences, the associated inverse z-transform is:

$$\boxed{x[n] = -\frac{3}{2}(-1)^n u[n] + \frac{5}{2}(-3)^n u[n].} \tag{6}$$

Remark: One can expand $X(z)$ in another way as

$$\begin{aligned}\frac{X(z)}{z} &= \frac{z-2}{z^2+4z+3} \\ &= \frac{A}{z+1} + \frac{B}{z+3}\end{aligned} \tag{7}$$

where the constants A and B can be computed as $A = -\frac{3}{2}$ and $B = \frac{5}{2}$, then $X(z)$ can be expressed as

$$\begin{aligned}X(z) &= A\left(\frac{z}{z+1}\right) + B\left(\frac{z}{z+3}\right) \\ &= -\frac{3}{2}\left(\frac{z}{z+1}\right) + \frac{5}{2}\left(\frac{z}{z+3}\right),\end{aligned} \tag{8}$$

which results the same as Eq. (3). The following analysis will be identical.

(b) Similarly, we can expand the z-transform (denoted as $X(z)$) as

$$X(z) = \frac{1}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})} = \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1-\frac{1}{3}z^{-1}} \tag{9}$$

where the factors A and B are:

$$\begin{aligned}A &= X(z)\left(1-\frac{1}{2}z^{-1}\right)\Big|_{z=\frac{1}{2}} = \frac{1}{1-\frac{1}{3}z^{-1}}\Big|_{z=\frac{1}{2}} = 3, \\ B &= X(z)\left(1-\frac{1}{3}z^{-1}\right)|_{z=\frac{1}{3}} = \frac{1}{1-\frac{1}{2}z^{-1}}\Big|_{z=\frac{1}{3}} = -2.\end{aligned} \tag{10}$$

Therefore we have:

$$\boxed{X(z) = \underbrace{\frac{3}{1-\frac{1}{2}z^{-1}}}_{X_1(z)} - \underbrace{\frac{2}{1-\frac{1}{3}z^{-1}}}_{X_2(z)}.} \tag{11}$$

From Eq. (9) we observe that the poles are $z = \frac{1}{2}$ and $z = \frac{1}{3}$, so there will be three possible ROCs:

- Case 1: ROC: $\{z : |z| < \frac{1}{3}\}$, both $X_1(z)$ and $X_2(z)$ in Eq. (11) are z-transform of left-sided sequences, the associated inverse z-transform is:

$$\boxed{x[n] = -3\left(\frac{1}{2}\right)^n u[-n-1] + 2\left(\frac{1}{3}\right)^n u[-n-1].} \tag{12}$$

- Case 2: ROC: $\{z : \frac{1}{3} < |z| < \frac{1}{2}\}$, here in Eq. (11), $X_1(z)$ is the z-transform of a left-sided sequence and $X_2(z)$ is the z-transform of a right-sided sequence, the associated inverse z-transform is:

$$\boxed{x[n] = -3\left(\frac{1}{2}\right)^n u[-n-1] - 2\left(\frac{1}{3}\right)^n u[n].} \tag{13}$$

- Case 3: ROC: $\{z : |z| > \frac{1}{2}\}$, both $X_1(z)$ and $X_2(z)$ in Eq. (11) are z-transforms of right-sided sequences, the associated inverse z-transform is given as

$$\boxed{x[n] = 3\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{3}\right)^n u[n].} \tag{14}$$

Problem 2: ***z*-transform properties and inverse *z*-transform**

Evaluate the convolution of the sequences $h[n] = (1/3)^n u[n]$ and $x[n] = 2^n u[-n]$ using z-transform properties and the inverse z-transform.

Solution: First we compute the z-transforms of $h[n]$ and $x[n]$:

$$\begin{aligned}
H(z) &= \mathcal{Z}\{h[n]\} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{3}\right)^n u[n] z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{3z}\right)^n = \frac{1}{1-\frac{1}{3}z^{-1}}, \quad \text{ROC: } \{z : |z| > \frac{1}{3}\}. \\
X(z) &= \mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} 2^n u[-n] z^{-n} = \sum_{n=-\infty}^{0} 2^n z^{-n} = \sum_{n=0}^{\infty} 2^{-n} z^n = \sum_{n=0}^{\infty} \left(\frac{1}{2z^{-1}}\right)^n \\
&= \frac{1}{1-\frac{1}{2z^{-1}}} = -\frac{2z^{-1}}{1-2z^{-1}}, \quad \text{ROC: } \{z : |z| < 2\}.
\end{aligned} \tag{15}$$

According to the convolution property, the z-transform $Y(z)$ of the convolutions of $h[n]$ and $x[n]$ is:

$$Y(z) = \mathcal{Z}\{x[n] * h[n]\} = H(z)X(z) = -\frac{2z^{-1}}{(1-\frac{1}{3}z^{-1})(1-2z^{-1})}, \quad \text{ROC: } \{z : \frac{1}{3} < |z| < 2\}. \tag{16}$$

To clarify, since there is no pole-zero cancellation, the ROC of $Y(z)$ should be the intersection of the ROCs of $X(z)$ and $H(z)$, that is, if $y[n] = x[n] * h[n]$, then $\text{ROC}_Y = \text{ROC}_X \cap \text{ROC}_H$. Therefore, one can expand $Y(z)$ as

$$Y(z) = \frac{A}{1-\frac{1}{3}z^{-1}} + \frac{B}{1-2z^{-1}}, \quad \text{ROC: } \{z : \frac{1}{3} < |z| < 2\}. \tag{17}$$

The inverse z-transform of $Y(z)$ is then given as

$$y[n] = \mathcal{Z}^{-1}\{Y(z)\} = A\left(\frac{1}{3}\right)^n u[n] - B2^n u[-n-1] \tag{18}$$

where the factors A and B are:

$$\begin{aligned} A &= Y(z)(1-\frac{1}{3}z^{-1})|_{z=\frac{1}{3}} = -\left.\frac{2z^{-1}}{1-2z^{-1}}\right|_{z=\frac{1}{3}} = \frac{6}{5}, \\ B &= Y(z)(1-2z^{-1})|_{z=2} = -\left.\frac{2z^{-1}}{1-\frac{1}{3}z^{-1}}\right|_{z=2} = -\frac{6}{5}. \end{aligned} \tag{19}$$

Therefore the convolution $y[n]$ is:

$$\boxed{y[n] = \frac{6}{5}\left[\left(\frac{1}{3}\right)^n u[n] + 2^n u[-n-1]\right].} \tag{20}$$

Remark: One can check the inverse z-transform in Eq. (20) by performing the convolution sum (which is tedious!) as

$$\begin{aligned} y[n] = h[n] * x[n] &= \sum_{m=-\infty}^{\infty} x[m]h[n-m] \\ &= \sum_{m=-\infty}^{\infty} 2^m u[-m]\left(\frac{1}{3}\right)^{n-m} u[n-m] \\ &= \left(\frac{1}{3}\right)^n \sum_{m=-\infty}^{\infty} 6^m u[-m]u[n-m]. \end{aligned} \tag{21}$$

To simplify Eq. (21) we need to find the m that make the term $u[-m]u[n-m]$ to be non-zero, that is $m \leq 0$ and $m \leq n$. Therefore we need to discuss the following two cases:

- For $n < 0$, the convolution in Eq. (21) can be simplified as

$$y[n] = \left(\frac{1}{3}\right)^n \sum_{m=-\infty}^{n} 6^m = \left(\frac{1}{3}\right)^n \sum_{m=-n}^{\infty}\left(\frac{1}{6}\right)^m = \left(\frac{1}{3}\right)^n \frac{\left(\frac{1}{6}\right)^{-n}}{1-\frac{1}{6}} = \left(\frac{6}{5}\right)2^n. \tag{22}$$

- For $n \geq 0$, the Eq. (21) can be simplified as

$$y[n] = \left(\frac{1}{3}\right)^n \sum_{m=-\infty}^{0} 6^m = \sum_{m=0}^{\infty} 6^{-m} = \left(\frac{1}{3}\right)^n \frac{1}{1-\frac{1}{6}} = \frac{6}{5}\left(\frac{1}{3}\right)^n. \tag{23}$$

Therefore, we can express the convolution as

$$\boxed{y[n] = \frac{6}{5}\left(\frac{1}{3}\right)^n u[n] + \left(\frac{6}{5}\right) 2^n u[-n-1],} \tag{24}$$

which is the same as the result of the inverse z-transform in Eq. (20).

Problem 3: z-transform properties and difference equations

Consider the system described as

$$y[n] = \sum_{k=-\infty}^{n} kx[k]$$

(a) Find a difference equation for this system. Is this a constant coefficient difference equation?

(b) Take the z-transform to both sides of the difference equation to express $Y(z)$ in terms of $X(z)$.

(c) Let $y[n] = \sum_{k=0}^{n} k2^{-k},\ \ n \geq 0$. Use your answer in part (b) to find $Y(z)$ and the corresponding ROC.

Solution:

(a) According to the system description, for n and $n-1$ we have:

$$y[n] = \sum_{k=-\infty}^{n} kx[k], \quad y[n-1] = \sum_{k=-\infty}^{n-1} kx[k]. \tag{25}$$

By checking the difference between $y[n]$ and $y[n-1]$ we have:

$$\boxed{y[n] = y[n-1] + nx[n],} \tag{26}$$

which is the difference equation of this system, and it is *not* a constant coefficient since the n in the term $nx[n]$ is not a constant.

(b) According to the difference equation in Eq. (26), we apply z-transform on the both sides:

$$\begin{aligned} Y(z) &= z^{-1}Y(z) + \mathcal{Z}\{nx[n]\} \\ &= z^{-1}Y(z) - z\frac{dX(z)}{dz} \end{aligned} \tag{27}$$

where in Eq. (27) we use the multiplication by n property (or derivative property): $nx[n] \xrightarrow{\text{z-transform}} -z\frac{dX(z)}{dz}$. Therefore the z-transform $Y(z)$ can be expressed as

$$Y(z)(1-z^{-1}) = -z\frac{dX(z)}{dz} \ \Rightarrow \boxed{Y(z) = -\frac{z}{1-z^{-1}}\frac{dX(z)}{dz} = -\frac{z^2}{z-1}\frac{dX(z)}{dz}.} \tag{28}$$

(c) One can inspect that, when $y[n] = \sum_{k=0}^{n} k2^{-k}$, the input $x[n]$ and the corresponding z-transform $X(z)$ can be expressed as

$$x[n] = \left(\frac{1}{2}\right)^n u[n] \;\Rightarrow\; X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}, \quad \text{ROC: } \{z : |z| > \frac{1}{2}\}. \tag{29}$$

In this way, the z-transform $Y(z)$ in Eq. (28) can be expressed as

$$\begin{aligned} Y(z) &= -\frac{z^2}{z-1}\frac{dX(z)}{dz} \\ &= -\frac{z^2}{z-1}\frac{d}{dz}\left(\frac{1}{1-\frac{1}{2}z^{-1}}\right) \\ &= -\frac{z^2}{z-1}\left(\frac{-1}{(z-\frac{1}{2})^2}\right) \\ &= \boxed{\frac{z^2}{(z-1)(z-\frac{1}{2})^2}} \end{aligned} \tag{30}$$

To determine the ROC of $Y(z)$, one can observe that the poles of $Y(z)$ in Eq. (30) are $p_1 = 1$ and $p_2 = \frac{1}{2}$, where p_2 is a pole of order 2. Then according to the definition of this syetem in Eq. (25), the system is *causal*, therefore the ROC of $Y(z)$ should be the exterior of a circle with radius the largest pole magnitude, i.e. $\{z : |z| > \max\{|p_1|,\ |p_2|\}\}$. Thus we can determine the ROC of $Y(z)$ as $\text{ROC}_Y : \{z : |z| > 1\}$.

Problem 4: System Cascades

Two systems with unit-pulse responses

$$h_1[n] = u[n] + \left(\frac{1}{4}\right)^n u[n], \qquad h_2[n] = \delta[n] - \delta[n-1]$$

are in serial connection.

(a) For each of the individual systems, as well as for the overall system, determine whether they are BIBO stable.

(b) Determine the unit pulse response of the overall system.

(c) Find the difference equation for the overall system.

Solution:

(a) Idea: To check if a system is BIBO stable, one can check if the ROC of the z-transform of the system includes the unit circle.

- For the system $h_1[n]$, the z-transform $H_1(z)$ is given as

$$\begin{aligned} H_1(z) &= \sum_{n=-\infty}^{\infty} u[n]z^{-n} + \sum_{n=-\infty}^{\infty} \left(\frac{1}{4}\right)^n u[n]z^{-n} \\ &= \sum_{n=0}^{\infty} z^{-n} + \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n z^{-n} \\ &= \frac{1}{1-z^{-1}} + \frac{1}{1-\frac{1}{4}z^{-1}} \\ &= \frac{(1-\frac{1}{4}z^{-1}) + (1-z^{-1})}{(1-z^{-1})(1-\frac{1}{4}z^{-1})} \\ &= \frac{2-\frac{5}{4}z^{-1}}{(1-z^{-1})(1-\frac{1}{4}z^{-1})}. \end{aligned} \tag{31}$$

  For the ROC of $H_1(z)$, since this system is causal and no zero-pole cancellation occurs in Eq. (31), we have ROC_{H_1}: $\{z : |z| > 1\}$. Therefore, due to the fact the unit circle is *not* included in the ROC, this system is *not* BIBO stable.

- For the system $h_2[n]$, the z-transform $H_2(z)$ is given as

$$H_2(z) = \sum_{n=-\infty}^{\infty} \delta[n]z^{-n} - \sum_{n=-\infty}^{\infty} \delta[n-1]z^{-n} = 1 - z^{-1}. \tag{32}$$

  The corresponding ROC is ROC_{H_2}: $\{z : z \neq 0\}$. Therefore, since the unit circle is included in the ROC, this system is BIBO stable.

- For the overall system $h[n]$, due to the cascade connection we have $h[n] = h_1[n] * h_2[n]$, and the z-transform is given as

$$\begin{aligned} H(z) &= \mathcal{Z}\{h_1[n] * h_2[n]\} \\ &= H_1(z)H_2(z) \quad \text{(convolution property)} \\ &= \left[\frac{2-\frac{5}{4}z^{-1}}{(1-z^{-1})(1-\frac{1}{4}z^{-1})}\right]\left(1-z^{-1}\right) \\ &= \frac{2-\frac{5}{4}z^{-1}}{1-\frac{1}{4}z^{-1}}. \end{aligned} \tag{33}$$

  Since a pole-zero cancellation occurs (the term $1 - z^{-1}$ is eliminated), one can use the fact that, the ROC of $H(z)$ should be a superset of the intersection of ROC_{H_1} and ROC_{H_2}, that is, $\text{ROC}_H \supseteq \text{ROC}_{H_1} \cap \text{ROC}_{H_2} = \{z : |z| > 1\}$. Therefore, we can determine the ROC of the overall system as ROC_H: $\{z : |z| > \frac{1}{4}\}$, and the system is BIBO stable since the unit circle is included.

(b) According to the z-transform and the corresponding ROC of the overall system:

$$H(z) = \frac{2-\frac{5}{4}z^{-1}}{1-\frac{1}{4}z^{-1}} = \frac{2}{1-\frac{1}{4}z^{-1}} - \frac{5}{4}\left(\frac{z^{-1}}{1-\frac{1}{4}z^{-1}}\right), \quad \text{ROC: } \{z : |z| > \frac{1}{4}\}. \tag{34}$$

We can then perform the inverse z-transform as

$$\begin{aligned} h[n] &= z^{-1}\left\{\frac{2}{1-\frac{1}{4}z^{-1}} - \frac{5}{4}\left(\frac{z^{-1}}{1-\frac{1}{4}z^{-1}}\right)\right\} \\ &= z^{-1}\left\{\frac{2}{1-\frac{1}{4}z^{-1}}\right\} - z^{-1}\left\{\frac{5}{4}\left(\frac{z^{-1}}{1-\frac{1}{4}z^{-1}}\right)\right\} \\ &= \boxed{2\left(\frac{1}{4}\right)^{n} u[n] - \frac{5}{4}\left(\frac{1}{4}\right)^{n-1} u[n-1],} \end{aligned} \tag{35}$$

which is the unit pulse response of the overall system.

(c) We can express the transfer function $H(z)$ of the overall system as

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2-\frac{5}{4}z^{-1}}{1-\frac{1}{4}z^{-1}} \tag{36}$$

where $X(z)$ and $Y(z)$ are the z-transforms of the input and output, respectively. Therefore from Eq. (36) we can derive the relation between $Y(z)$ and $X(z)$ as

$$Y(z)\left(1-\frac{1}{4}z^{-1}\right) = X(z)\left(2-\frac{5}{4}z^{-1}\right). \tag{37}$$

To determine the difference equation we can apply the inverse z-transform on both sides of Eq. (37) as

$$\boxed{y[n] - \frac{1}{4}y[n-1] = x[n] - \frac{5}{4}x[n-1],} \tag{38}$$

which is the difference equation of the overall system.

Problem 5: System Analysis

Consider the system described by the following difference equation (or LCCDE) with zero initial conditions:

$$y[n] = \frac{1}{2}y[n-2] + x[n] - x[n-1], \quad \text{for } n = 0, 1, 2, \ldots$$

(a) Find the transfer function and its ROC.

(b) Find the impulse response of the system.

(c) Determine the output $y[n]$ to the input $x[n] = (1/4)^n u[n]$.

Solution:

(a) We can take the z-transform on both sides of the difference equation and get

$$Y(z) = \left(\frac{1}{2}\right)z^{-2}Y(z) + X(z) - z^{-1}X(z) \;\Rightarrow\; Y(z)\left(1-\frac{1}{2}z^{-2}\right) = X(z)(1-z^{-1}). \tag{39}$$

Then the transfer function $H(z)$ is given as

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{1 - \frac{1}{2}z^{-2}} = \boxed{\frac{1 - z^{-1}}{(1 - \frac{\sqrt{2}}{2}z^{-1})(1 + \frac{\sqrt{2}}{2}z^{-1})}}. \tag{40}$$

To determine the ROC, we observe that the system has pole $z = \pm\frac{\sqrt{2}}{2}$ and is *causal* given the difference equation and the zero initial condition. Therefore the ROC of this system is ROC_H: $\{z : |z| > \frac{\sqrt{2}}{2}\}$.

(b) To find the impluse response $h[n]$, we can expand the transfer function $H(z)$ in partial fractions as

$$\begin{aligned} H(z) &= \frac{1 - z^{-1}}{(1 - \frac{\sqrt{2}}{2}z^{-1})(1 + \frac{\sqrt{2}}{2}z^{-1})} \\ &= \frac{A}{1 - \frac{\sqrt{2}}{2}z^{-1}} + \frac{B}{1 + \frac{\sqrt{2}}{2}z^{-1}} \end{aligned} \tag{41}$$

where the factors A and B are determined as

$$\begin{aligned} A &= H(z)\left(1 - \frac{\sqrt{2}}{2}z^{-1}\right)\Bigg|_{z=\frac{\sqrt{2}}{2}} = \frac{1 - z^{-1}}{1 + \frac{\sqrt{2}}{2}z^{-1}}\Bigg|_{z=\frac{\sqrt{2}}{2}} = \frac{1 - \sqrt{2}}{2}, \\ B &= H(z)\left(1 + \frac{\sqrt{2}}{2}z^{-1}\right)\Bigg|_{z=-\frac{\sqrt{2}}{2}} = \frac{1 - z^{-1}}{1 - \frac{\sqrt{2}}{2}z^{-1}}\Bigg|_{z=-\frac{\sqrt{2}}{2}} = \frac{1 + \sqrt{2}}{2}. \end{aligned} \tag{42}$$

Therefore, the impulse response $h[n]$ can be given as

$$\begin{aligned} h[n] &= A\left(\frac{\sqrt{2}}{2}\right)^n u[n] + B\left(-\frac{\sqrt{2}}{2}\right)^n u[n] \\ &= \boxed{\left(\frac{1 - \sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)^n u[n] + \left(\frac{1 + \sqrt{2}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right)^n u[n].} \end{aligned} \tag{43}$$

(c) To determine the output $y[n]$, we can first get its z-transform $Y(z)$ and take the inverse z-transform. Let us first compute the z-transform of $x[n]$:

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty}\left(\frac{1}{4}\right)^n u[n]z^{-n} = \sum_{n=0}^{\infty}\left(\frac{1}{4z}\right)^n = \frac{1}{1 - \frac{1}{4}z^{-1}}, \quad \text{ROC: } \{z : |z| > \frac{1}{4}\}. \tag{44}$$

According to the convolution property, the z-transform of the output $y[n]$ is:

$$\begin{aligned} Y(z) &= z\left\{x[n] * h[n]\right\} \\ &= X(z)H(z) \quad \text{(convolution property)} \\ &= \frac{1 - z^{-1}}{(1 - \frac{\sqrt{2}}{2}z^{-1})(1 + \frac{\sqrt{2}}{2}z^{-1})(1 - \frac{1}{4}z^{-1})} \end{aligned} \tag{45}$$

To determine the ROC, since there is no pole-zero cancellation in Eq. (45), the ROC should be the intersection of ROC_H and ROC_X, that is, $\mathrm{ROC}_Y = \{z : |z| > \frac{\sqrt{2}}{2}\} \cap \{z : |z| > \frac{1}{4}\} = \{z : |z| > \frac{\sqrt{2}}{2}\}$. Therefore, we can express $Y(z)$ as

$$Y(z) = \frac{A}{1 - \frac{\sqrt{2}}{2}z^{-1}} + \frac{B}{1 + \frac{\sqrt{2}}{2}z^{-1}} + \frac{C}{1 - \frac{1}{4}z^{-1}} \tag{46}$$

where the factors A, B and C are:

$$\begin{aligned}
A &= Y(z)\left(1 - \frac{\sqrt{2}}{2}z^{-1}\right)\Bigg|_{z=\frac{\sqrt{2}}{2}} = \frac{1 - z^{-1}}{(1 + \frac{\sqrt{2}}{2}z^{-1})(1 - \frac{1}{4}z^{-1})}\Bigg|_{z=\frac{\sqrt{2}}{2}} = \frac{2(1-\sqrt{2})}{4-\sqrt{2}}, \\
B &= Y(z)\left(1 + \frac{\sqrt{2}}{2}z^{-1}\right)\Bigg|_{z=-\frac{\sqrt{2}}{2}} = \frac{1 - z^{-1}}{(1 - \frac{\sqrt{2}}{2}z^{-1})(1 - \frac{1}{4}z^{-1})}\Bigg|_{z=-\frac{\sqrt{2}}{2}} = \frac{2(1+\sqrt{2})}{4+\sqrt{2}}, \\
C &= Y(z)\left(1 - \frac{1}{4}z^{-1}\right)\Bigg|_{z=\frac{1}{4}} = \frac{1 - z^{-1}}{(1 - \frac{\sqrt{2}}{2}z^{-1})(1 + \frac{\sqrt{2}}{2}z^{-1})}\Bigg|_{z=\frac{1}{4}} = -\frac{3}{7}.
\end{aligned} \tag{47}$$

Therefore we can apply the inverse z-transform on $Y(z)$:

$$\begin{aligned}
y[n] &= A\left(\frac{\sqrt{2}}{2}\right)^n u[n] + B\left(-\frac{\sqrt{2}}{2}\right)^n u[n] + C\left(\frac{1}{4}\right)^n u[n] \\
&= \boxed{\left(\frac{2(1-\sqrt{2})}{4-\sqrt{2}}\right)\left(\frac{\sqrt{2}}{2}\right)^n u[n] + \left(\frac{2(1+\sqrt{2})}{4+\sqrt{2}}\right)\left(-\frac{\sqrt{2}}{2}\right)^n u[n] - \left(\frac{7}{3}\right)\left(\frac{1}{4}\right)^n u[n],}
\end{aligned} \tag{48}$$

which is the output $y[n]$.

Problem 6: ***z*-transform properties and difference equations**

Let $X(z) = e^{1/z}$ for a right-sided signal $x[n]$ starting at $n = 0$ and having initial value $x[0] = 1$.

(a) Take the derivative of $X(z)$ and use z-transform properties to obtain a recursion for $x[n]$.

(b) Find the inverse z-transform by solving the recursion for $x[n]$ with initial condition $x[0] = 1$.

(c) Obtain the inverse z-transform via the power series method.

Solution:

(a) The derivative of $X(z) = \exp\left(\frac{1}{z}\right)$ is given as:

$$\frac{dX(z)}{dz} = \frac{d}{dz}\left(\exp\left(\frac{1}{z}\right)\right) = \exp\left(\frac{1}{z}\right)\frac{d}{dz}\left(\frac{1}{z}\right) = -z^{-2}\exp\left(\frac{1}{z}\right). \tag{49}$$

Then recall the multiplication by n property (derivative property), that is $nx[n] \xrightarrow{\text{z-transform}} -z\left(\frac{dX(z)}{dz}\right)$, therefore we apply the z-transform on $nx[n]$ and have

$$\mathcal{Z}\{nX[n]\} = -z\left(\frac{dX(z)}{dz}\right) = z^{-1}\underbrace{\exp\left(\frac{1}{z}\right)}_{X(z)} = z^{-1}X(z) \tag{50}$$

To find a recursion for $x[n]$, then we notice that $z^{-1}X(z)$ should be the z-transform of $x[n-1]$, therefore we have:

$$\mathcal{Z}\{nx[n]\} = z^{-1}X(z) = \mathcal{Z}\{x[n-1]\} \Rightarrow \boxed{nx[n] = x[n-1],} \tag{51}$$

which is the recursion for $x[n]$.

(b) Based on the recursion of $x[n]$ in Eq. (51) and the initial condition $x[0] = 1$, we observe that:

$$x[1] = \frac{x[0]}{1} = 1,\ x[2] = \frac{x[1]}{2} = \frac{1}{2},\ x[3] = \frac{x[2]}{3} = \frac{1}{2\times 3},\ \ldots,\ x[n] = \frac{x[n-1]}{n} = \frac{1}{n!} \tag{52}$$

In this way, the inverse z-transform of $X(z)$ can be expressed as:

$$\boxed{x[n] = \frac{1}{n!}u[n].} \tag{53}$$

(c) The Taylor expansion of $X(z) = \exp\left(\frac{1}{z}\right)$ is given as

$$X(z) = \exp\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty}\left(\frac{1}{n!}\right)z^{-n} = \sum_{n=-\infty}^{\infty}\underbrace{\left(\frac{1}{n!}u[n]\right)}_{x[n]}z^{-n}. \tag{54}$$

By inspecting the definition of z-transform, we can observe that the corresponding $x[n]$ is

$$\boxed{x[n] = \frac{1}{n!}u[n],} \tag{55}$$

which is the same result as in Eq. (53).