Digital Signal Processing | ECE 310 (Spring 2020)
Shanbhag/Radhakrishnan/Katselis | **Homework 12** | Assigned: 04/24 - Due: 05/06

Topic covered in this homework is: fast Fourier transforms, digital filter structures, sample-rate conversion, and generalized linear-phase. Homework will be graded for (1) completion and (2) Three randomly picked problems will be graded. Submissions will be using gradescope. Please solve problems on your own in order to maximally benefit from this homework.

Problem 1: Radix-3 Decimation-in-Time (DIT) FFT

Assume $N = 3^v$ for the following parts:

(a) Derive a radix-3 DIT algorithm, i.e., express $\{X[k]\}_{k=0}^{N-1}$ in terms of three $\frac{N}{3}$-point DFT computations.

(b) Sketch the first stage of the signal flow-graph (SFG) of a $N = 9$ radix-3 FFT algorithm completely and trace the connections from the input to the output $X[0]$. Calculate the total number of complex multiply-adds needed. For simplicity, assume all twiddle factors require one complex multiplication even if they are equal to unity or -1.

(c) For a $N = 3^v$-point radix-3 FFT obtain: i) the number of stages; ii) the number of radix-3 butterflies per stage; and iii) the number of complex multiply-adds per butterfly. Using the results from i), ii) and iii) find the total computational complexity, i.e., the number of complex multiply-adds, of an $N = 3^v$-point radix-3 FFT? For simplicity, assume all twiddle factors require one complex multiplication even if they are equal to unity or -1.

(d) The DFT of a length-26 sequence $\{x[n]\}_{n=0}^{25}$ needs to be computed with a frequency resolution $\Delta\omega < \frac{\pi}{13}$. Compare the number of complex multiply-adds that would be required when using a radix-2 FFT and a radix-3 FFT.

Problem 2: Throughput and Energy Consumption of an FFT Core

You have at your disposal a $N = 8$-point radix-2 DIT-based FFT core built from real multipliers and 2-input real adders with the following parameters: Delay ($T_m = 0.5\,\text{ns}$; $T_a = 0.25\,\text{ns}$) and energy consumption ($E_m = 400\,\text{fJ}$; $E_a = 100\,\text{fJ}$).

(a) Sketch the block diagram of a radix-2 butterfly unit in terms of complex multipliers and complex adders and compute its delay T_{BF} and energy consumption E_{BF}.

(b) If one butterfly unit of Part (a) is available, how much time T_{FFT} does it take to compute an 8-point FFT? What is its energy consumption E_{FFT}?

(c) If infinite number of butterfly units of Part(a) are available, how much time does it take to compute an 8-point FFT? What is its energy consumption E_{FFT}?

Problem 3: Overlap-Add Method

You have at your disposal one $N = 256$-point radix-2 FFT core that can compute one 256-point FFT in $T_{FFT} = 24\,\text{ns}$ and while consuming $E_{FFT} = 7\,\text{nJ}$. Assume that you have an infinite number of complex multipliers and 2-input complex adders. Furthermore, a complex multiplier takes $T_{CM} = 2\,\mathrm{ns}$ time and consumes $E_{CM} = 4\,\mathrm{pJ}$ of energy, and a 2-input complex adder takes $T_{CA} = 1\,\mathrm{ns}$ of time and consumes $E_{CA} = 1\,\mathrm{pJ}$ of energy. This FFT core and the available arithmetic units is to be used to implement a linear convolution $y[n] = x[n] * h[n]$ of an infinite-length signal $x[n]$ and a $M = 120$-tap filter with impulse response $h[n]$ using the overlap-add method. Now answer the following:

(a) Determine the maximum length L of the finite-length segments $x_m[n]$ that would be needed.

(b) Determine the total delay T_{256} and energy E_{256} needed to compute 256 samples of $y[n]$.

(c) Estimate the delay and energy to compute 256 samples of the output $y[n]$ directly in the time-domain using a *transversal filter structure*. Compare your answer with the one obtained in Part (c).

(d) if $\{x[n]\}_{n=0}^{\infty} = \{1, -2, 3, \frac{1}{2}, 2, -7, \frac{1}{4}, \frac{1}{9}, \ldots\}$ and $\{h[n]\}_{n=0}^{2} = \{1, -1, 1\}$ use the overlap-add method to calculate the linear convolution $h[n] * x[n]$ using segments of length $L = 4$.

Problem 4: Filter Structures

We wish to implement an IIR filter with poles at $z = \frac{2}{3}e^{\pm j\frac{2\pi}{3}}, -\frac{2}{3}$ and zeros at $z = 2, -3$. Assume that each real multiplier takes $T_m = 2\,\mathrm{ns}$ and each 2-input real adder takes $T_a = 1\,\mathrm{ns}$ to complete. In the following, no complex multipliers or adders are to be used.

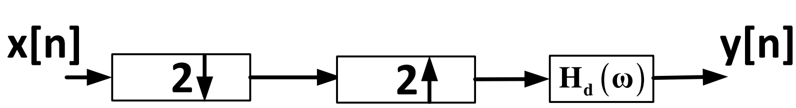
(a) Write down the transfer function $H(z)$ of this IIR filter in terms of first and second order terms with real-valued coefficients.

(b) Sketch the block diagrams of Direct Form-I, II, and the transpose structure and calculate their throughputs.

(c) Sketch the block diagram of a 2-stage cascade form structure using a cascade of two Direct Form II structures, and calculate its throughput.

Problem 5:

Consider the system shown above. The frequency response $H_d(\omega)$ is given by,



$$H_d(\omega) = \begin{cases} 1 & |\omega| \leq \pi/2 \\ 0 & otherwise \end{cases}$$

Find the output $y[n]$ for the following input sequences,

1. $x[n] = cos(\frac{\pi}{4}n)$
2. $x[n] = cos(\frac{3\pi}{4}n)$

Problem 6:

Given an input $x[n]$, let $w[n]$ and $v[n]$ be given by,

$$w[n] = x[nD]$$

$$v[n] = \begin{cases} x[n/L] & n = Lk \\ 0 & otherwise. \end{cases}$$

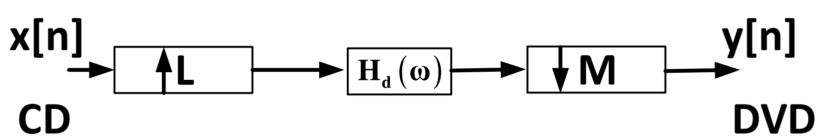
Given that,

$$X_d(\omega) = \begin{cases} 1 - \frac{4|\omega|}{\pi} & |\omega| \leq \pi/4 \\ 0 & \pi/4 \leq |\omega| \leq \pi \end{cases}$$

1. Sketch $V_d(\omega)$ for $L = 3$.
2. Sketch $W_d(\omega)$ for $D = 4$.

Problem 7:

You desire to transfer music stored on CD to a DVD. CD players operate at a rate of 44.1 kHz, while DVD players operate at 48 kHz. Consider the following system to perform this operation:



Find the smallest possible values for L and M, and sketch the frequency response $H_d(\omega)$ to perform this conversion.

Problem 8:

The frequency response of a generalized linear phase (GLP) filter can be expressed as $H_d(\omega) = R(\omega)e^{j(\alpha-\beta\omega)}$ where $R(\omega)$ is a real function and α and β are constants. For each of the following LTI systems, described by their impulse response, transfer function, or difference equation, determine whether it is a GLP filter. If it is, determine $R(\omega)$, α, and β, and indicate whether it is also a strictly linear phase filter.

(a) $y[n] = -x[n] + x[n-2]$

(b) $y[n] = 0.4y[n-1] + 0.4x[n]$

(c) $y[n] = 2x[n] + x[n-1] + x[n-2] + 2x[n-3]$

(d) $\{h[n]\}_{n=0}^{2} = \{4, 5, 6\}$

(e) $\{h[n]\}_{n=0}^{2} = \{3, 4, 3\}$

(f) $H(z) = -1 + 4z^{-1} + z^{-2}$