*Topics covered in this homework are: Continuous-Time Fourier TRansform(CTFT) and Discrete-Time Fourier Transform(DTFT). Homework is due at 5:00 PM on Wednesdays. Homework will be graded for (1) completion and (2) one randomly picked problem will be graded. Submissions will be using gradescope. Please solve problems on your own in order to maximally benefit from this homework.*

**Problem 1**:

Determine if the following systems are BIBO Stable. Give brief justification. For each case in which the system is determined to be unstable, find a bounded real-valued input that will produce an unbounded output. Does DTFT exist for parts (3) - (6) ?

1. System with input-output relationship, $y[n] = x[n]sin^2(x[n])$

   **Solution:** $\forall n \in \mathbb{Z}$, we have $0 \leq \sin^2(x[n]) \leq 1$. Thus, for any input $|x[n]| < M,\ \forall n \in \mathbb{Z}$ where $M \in \mathbb{R}$, we have a bounded output as follows:

   $$y[n] = x[n]\sin^2(x[n]) < M\sin^2(M) < M$$

   Thus, the system is BIBO stable.

2. System with input-output relationship, $y[n] = nx[n]$

   **Solution:** For any bounded input $|x[n]| < M$ where $M \in \mathbb{R}$ and $n \in \mathbb{Z}$,

   $$y[n] = nx[n] < nM$$

   However, as $n \to \infty$, $nM \to \infty$. Thus, the system is not BIBO stable. Input example: $x[n] = a\,,\ a \in \mathbb{R} \implies |y[n]| = |na| \to \infty$ as $n \to \infty$ or $-\infty$.

3. Causal LSI system with transfer function $H(z) = \dfrac{z-7}{z^2+\dfrac{1}{9}}$

   **Solution:** ROC: ($|z| > \frac{1}{3}$) contains the unit circle $\Rightarrow$ BIBO stable

4. Anticausal LSI system with transfer function $H(z) = \dfrac{z-7}{z^2+\dfrac{1}{9}}$

   **Solution:** ROC ($|z| < \frac{1}{3}$) doesn't contain the unit circle $\Rightarrow$ NOT BIBO stable. Input example: for $x[n] = \delta[n]$, output will have two elements of the form $(\frac{j}{3})^n u[-n-1]$ and $(-\frac{j}{3})^n u[-n-1]$ (neglecting the exact scales and time shifts) which are not bounded.

5. $H(z) = \dfrac{z}{(z-0.7)(z^2+z+1)}$, $h[n]$ is LSI and two-sided

   **Solution:** ROC ($0.7 < |z| < 1$) doesn't contain the unit circle $\Rightarrow$ NOT BIBO stable. Input example: for $x[n] = \sin(2n\pi/3)u[n]$, $X(z) = \dfrac{\sqrt{\dfrac{2}{3}}z^2}{z^2+z+1}$ and $Y(z) = \dfrac{\sqrt{\dfrac{2}{3}}z^3}{(z-0.7)(z^2+z+1)^2}$. Therefore, $y[n]$ contains a term $ne^{j\frac{2}{3}n\pi}$, which is unbounded.

6. Causal LSI system with transfer function $H(z) = \dfrac{z^2}{z^2-\sqrt{2}z+1}$

**Solution:**

$$H(z) = \frac{z^2}{z^2 - \sqrt{2}z + 1} = \frac{z^2}{(z - (\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}))(z - (\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2}))}$$

has poles at $z = (\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2})$ and $z = (\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2})$ which are on the unit circle. Hence, the system is not BIBO stable. Input example: $x[n] = \delta[n]$ which will produce $y[n] = \frac{\sqrt{2}}{2}e^{j(\pi/4)n}u[n] - \frac{\sqrt{2}}{2}e^{-j(\pi/4)n}u[n]$ using partial fractions.

**Problem 2**:

The input $x[n] = 2^n\left(u[n] - 3u[n-1]\right)$ to an unknown LSI system produces output $y[n] = (3^n - 2^n)u[n]$.

1. Compute the z-transform of $x[n]$ and $y[n]$. Identify the region of convergence of $X(z)$ and $Y(z)$.

   **Solution:**

$$\begin{aligned}
Y(z) &= \frac{1}{1-3z^{-1}} - \frac{1}{1-2z^{-1}}\ ,\ |z| > 3 \\
&= \frac{z^{-1}}{(1-2z^{-1})(1-3z^{-1})}\ ,\ |z| > 3 \\
X(z) &= \frac{1}{1-2z^{-1}} - \frac{6z^{-1}}{1-2z^{-1}}\ ,\ |z| > 2 \\
&= \frac{1-6z^{-1}}{1-2z^{-1}}\ ,\ |z| > 2
\end{aligned}$$

2. Compute the transfer function $H(z)$. Identify the region of convergence

   **Solution:**

$$\begin{aligned}
H(z) &= \frac{z^{-1}}{(1-3z^{-1})(1-6z^{-1})}\ ,\ \text{subject to } |z| > 3 \\
&= \frac{1}{3}\left(\frac{1}{1-6z^{-1}} - \frac{1}{1-3z^{-1}}\right)\ ,\ \text{subject to } |z| > 3
\end{aligned}$$

   One-sided, causal solution (ROC: $|z| > 6$):

$$h[n] = \left(-\frac{1}{3}(3)^n + \frac{1}{3}(6)^n\right)u[n]$$

   Two-sided solution (ROC: $3 < |z| < 6$):

$$h[n] = -\frac{1}{3}(3)^n u[n] - \frac{1}{3}(6)^n u[-n-1]$$

3. Is the impulse response $h[n]$ unique? Can $h[n]$ represent a stable system? Can it represent a causal system?

   **Solution:** The solution is not unique. There are two possible answers as shown above. The system is unstable for both solutions. Therefore, even if the system is known to be unstable, there are two possible solutions. If the system is given to be causal, then the ROC is $|z| > 6$ and there is a unique, one-sided solution for $h[n]$.

**Problem 3**:

Compute the CTFT of the following sequences

1. $x(t) = \delta(5t - 2)$

   **Solution:**

2. $x(t) = e^{-3t}u(t)$

   **Solution:**

3. $x(t) = e^{-3t} * \delta(5t - 2)$

   **Solution:**

4. $x(t) = 4sin(2000\pi t)$

   **Solution:**

**Problem 4**:

Determine the DTFT of the sequences given below. For parts 1 and 2 plot the magnitude and phase of DTFT.

1. $x[n] = u[n + 3] - u[n - 5]$

**Solution:**

$$
\begin{aligned}
X_d(\omega) &= \sum_{n=-3}^{4} e^{-j\omega n} \\
&= \frac{e^{j\omega 3}\left(1-e^{-j8\omega}\right)}{1-e^{-j\omega}} \\
&= \frac{e^{j\omega 3}-e^{-j\omega 5}}{1-e^{-j\omega}} \\
&= \frac{e^{-j\omega}\left(e^{j\omega 4}-e^{-j\omega 4}\right)}{e^{-j\omega/2}(e^{j\omega/2}-e^{-j\omega/2})} \\
&= e^{-j\omega/2}\frac{sin\,(4\omega)}{sin\,(\omega/2)} \\
|X_d(\omega)| &= \left|\frac{sin\,(4\omega)}{sin\,(\omega/2)}\right| \\
\angle X_d(\omega) &= \begin{cases} -\frac{w}{2}, & \text{if } \frac{sin(4\omega)}{sin(\omega/2)} > 1 \\ -\frac{w}{2}+\pi & \text{if } \frac{sin(4\omega)}{sin(\omega/2)} < 1 \\ 0 & \text{if } \frac{sin(4\omega)}{sin(\omega/2)} = 0 \end{cases}
\end{aligned}
$$

2. $x[n] = 2\delta[n+1] - 2\delta[n]$

**Solution:**

$$
\begin{aligned}
X_d(\omega) &= 2(e^{j\omega}-1) \\
&= 2e^{j\frac{\omega}{2}}\left(e^{j\frac{\omega}{2}}-e^{-j\frac{\omega}{2}}\right) \\
&= 4jsin(\frac{\omega}{2})e^{j\frac{\omega}{2}} \\
&= 4sin(\frac{\omega}{2}) \quad e^{j\left(\frac{\omega}{2}+\frac{\pi}{2}\right)} \\
|X_d(\omega)| &= \left|4sin(\frac{\omega}{2})\right| \\
\angle X_d(\omega) &= \begin{cases} -\frac{w+\pi}{2}, & \text{if } 4sin(\frac{\omega}{2}) > 1 \\ -\frac{w-\pi}{2} & \text{if } 4sin(\frac{\omega}{2}) < 1 \\ 0 & \text{if } 4sin(\frac{\omega}{2}) = 0 \end{cases}
\end{aligned}
$$

3. $x[n] = \left(\frac{1}{3}\right)^n u[n-4]$

**Solution:**

$$
\begin{aligned}
x[n] &= \left(\frac{1}{3}\right)^4 \quad \left(\frac{1}{3}\right)^{n-4} u[n-4] \\
\rightarrow X_d(\omega) &= \left(\frac{1}{3}\right)^4 \frac{e^{j\omega}}{e^{j\omega}-\frac{1}{3}} \quad e^{-j4\omega}
\end{aligned}
$$

4. $x[n] = (1-n)\left(\frac{1}{3}\right)^n u[n]$

   **Solution:**
$$x[n] = \left(\frac{1}{3}\right)^n u[n] - n\left(\frac{1}{3}\right)^n u[n] = y[n] - ny[n]$$
   Using differentiation property:
$$\rightarrow X(\omega) = Y(\omega) - j\frac{dY(\omega)}{d\omega} = \frac{1}{1-\frac{1}{3}e^{-j\omega}} - j(-j)\frac{-(1/3)e^{-j\omega}}{(1-\frac{1}{3}e^{-j\omega})^2}$$
$$= \frac{1}{1-\frac{1}{3}e^{-j\omega}} + \frac{1}{3}\frac{e^{-j\omega}}{(1-\frac{1}{3}e^{-j\omega})^2}$$

5. $x[n] = (1-n)3^n u[n]$

   **Solution:** This does not converge. Hence Fourier Transform does not exist.

6. $x[n] = \left(\frac{1}{3}\right)^n cos\left(\frac{\pi(n-5)}{2}\right) u[n]$

   **Solution:**
$$x[n] = \left(\frac{1}{3}\right)^n cos\left(\frac{\pi(n-5)}{2}\right) u[n] = \left(\frac{1}{3}\right)^n \frac{1}{2}\left(e^{j\frac{\pi(n-5)}{2}} + e^{-j\frac{\pi(n-5)}{2}}\right) u[n]$$
$$= \frac{1}{2}\left(e^{-j\frac{5\pi}{2}}\right)\left(\frac{1}{3}e^{j\frac{\pi}{2}}\right)^n u[n] + \frac{1}{2}\left(e^{j\frac{5\pi}{2}}\right)\left(\frac{1}{3}e^{-j\frac{\pi}{2}}\right)^n u[n]$$
$$= \frac{-j}{2}\left(\frac{1}{3}e^{j\frac{\pi}{2}}\right)^n u[n] + \frac{j}{2}\left(\frac{1}{3}e^{-j\frac{\pi}{2}}\right)^n u[n]$$
$$\rightarrow X(\omega) = \frac{j}{2}\left(\frac{-1}{1-\frac{j}{3}e^{-jw}} + \frac{1}{1+\frac{j}{3}e^{-jw}}\right)$$

**Problem 4 part 1,2 plots:**

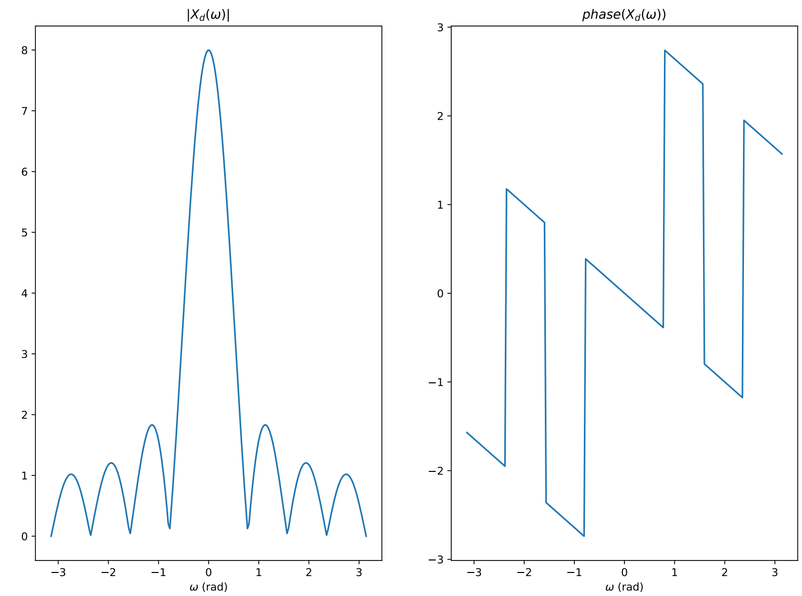


Figure 1: Magnitude and phase responses for Problem 4-1

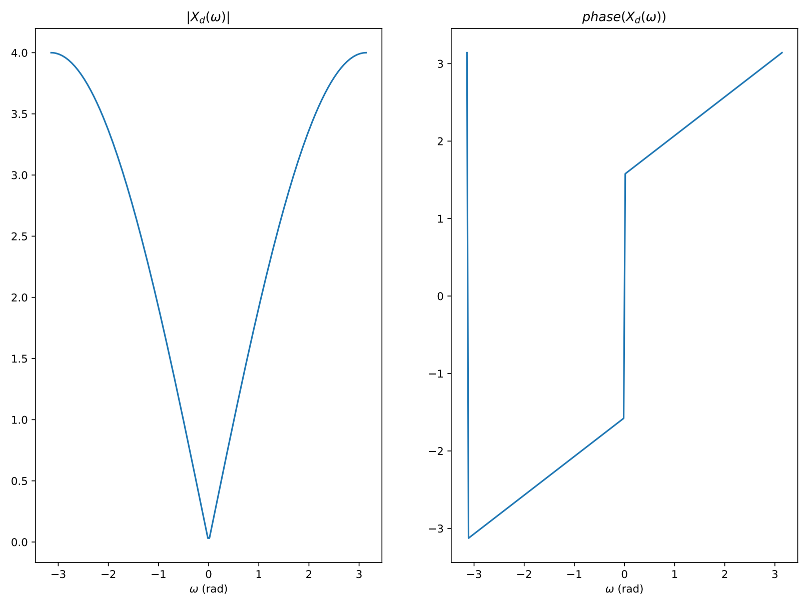


Figure 2: Magnitude and phase responses for Problem 4-2

**Problem 5**:

Let $x[n]$ be an arbitrary signal not necessarily real-valued with DTFT $X_d(\omega)$. Express the DTFT of the following signals in terms of $X_d(\omega)$.

1. $y[n] = x^*[-n+4]$

   **Solution:** Let $z[n] = x^*[n]$ and $w[n] = z[-n]$. Then, $y[n] = w[n-4]$. Also,

$$Z_d(\omega) = \sum_{n=-\infty}^{\infty} x^*[n]e^{-j\omega n} = \left(\sum_{n=-\infty}^{\infty} x[n]e^{j\omega n}\right)^* = X_d^*(-\omega)$$

   Using this result, time reversal property, $W_d(\omega) = Z_d(-\omega)$, and time shift property, $Y_d(\omega) = W_d(\omega)e^{-j4\omega}$, we have

$$Y_d(\omega) = X_d^*(\omega)e^{-j4\omega}$$

2. $y[n] = x[n-2]cos\left(\frac{\pi n}{4}\right)$

   **Solution:** Let $z[n] = x[n-2]$, then $y[n] = z[n]cos(\frac{\pi n}{4})$.
   Using time shift property, $Z_d(\omega) = X_d(\omega)e^{-j2\omega}$
   Using modulation property, $Y_d(\omega) = \frac{1}{2}\left(Z_d(\omega - \frac{\pi}{4}) + Z_d(\omega + \frac{\pi}{4})\right)$
   Combining the above,

$$Y_d(\omega) = \frac{1}{2}\left(X_d(\omega - \frac{\pi}{4})e^{-j2(\omega-\frac{\pi}{4})} + X_d(\omega + \frac{\pi}{4})e^{-j2(\omega+\frac{\pi}{4})}\right)$$

**Problem 6**:

Determine the signal $x[n]$ corresponding to each of the following cases.

1. $X_d(\omega) = 2 + 3e^{-j\omega} + 2e^{-3j\omega} - e^{-5j\omega}$

   **Solution:** Using linearity and time-shift properties of the DTFT:

$$X_d(\omega) = 2 + 3e^{-j\omega} + 2e^{-3j\omega} - e^{-3j\omega} \leftrightarrow 2\delta[n] + 3\delta[n-1] + 2\delta[n-3] - \delta[n-5]$$

2. Assume $x[n]$ is real-valued.

**Solution:**

$$
\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(\omega)e^{j\omega n}d\omega \\
&= \frac{1}{2\pi}\left(\int_{-\pi}^{-9\pi/10} 2e^{j\omega n}d\omega + \int_{-9\pi/10}^{-8\pi/10} e^{j\omega n}d\omega + \int_{8\pi/10}^{9\pi/10} e^{j\omega n}d\omega + \int_{9\pi/10}^{\pi} 2e^{j\omega n}d\omega\right) \\
&= \frac{1}{2\pi}\left(\int_{8\pi/10}^{9\pi/10} (e^{j\omega n} + e^{-j\omega n})d\omega + \int_{9\pi/10}^{\pi} 2(e^{j\omega n} + e^{-j\omega n})\right) \\
&= \frac{1}{\pi}\left(\int_{8\pi/10}^{9\pi/10} cos(\omega n)d\omega + \int_{9\pi/10}^{\pi} 2cos(\omega n)d\omega\right), \quad n \neq 0 \\
&= \frac{1}{\pi n}\left[\left(sin\left(\frac{9\pi}{10}n\right) - sin\left(\frac{8\pi n}{10}\right)\right) + \left(2sin\left(\pi n\right) - 2sin\left(\frac{9\pi n}{10}\right)\right)\right], \quad n \neq 0 \\
&= \frac{-1}{\pi n}\left[sin\left(\frac{9\pi}{10}n\right) + sin\left(\frac{8\pi}{10}n\right)\right]
\end{aligned}
$$

for $n = 0$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(\omega)d\omega = \frac{1}{2\pi}\left(\frac{6\pi}{10}\right) = \frac{3}{10}$$

Hence,

$$x[n] = 2\delta[n] - \frac{9}{10}sinc\left(\frac{9\pi}{10}n\right) - \frac{8}{10}sinc\left(\frac{8\pi}{10}n\right)$$

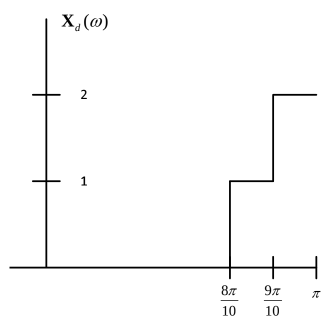


Figure 3: Figure for Problem 6-2