Homework 7

Problem 1:
Solution:

$$H(z) = \frac{z^{-1} - 0.3}{1 - 0.3z^{-1}}$$
$$= z^{-1}\frac{1 - 0.3z}{1 - 0.3z^{-1}}$$

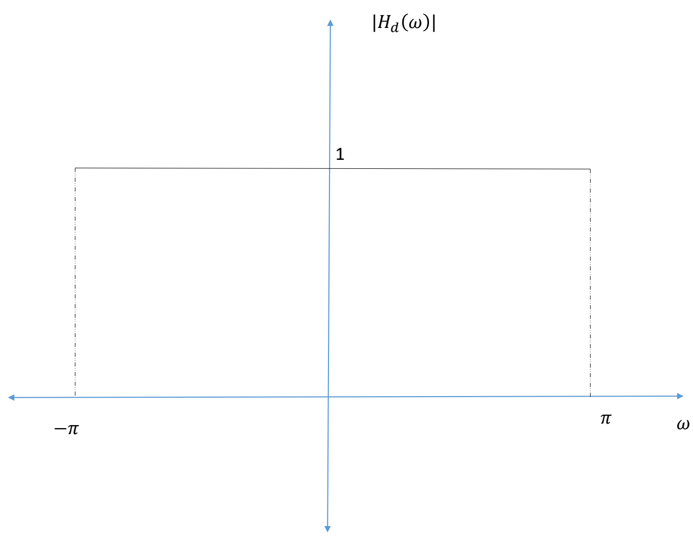
Then $H_d(\omega)$ is as follows:

$$H_d(\omega) = H(z)|_{z=e^{j\omega}} = e^{-j\omega}\frac{1 - 0.3e^{j\omega}}{1 - 0.3e^{-j\omega}}$$
$$= e^{-j\omega}\frac{(1 - 0.3e^{-j\omega})^*}{1 - 0.3e^{-j\omega}}$$
$$= e^{-j\omega}\frac{c^*}{c}$$

where $c \in \mathbb{C}$. Therefore,

$$|H_d(\omega)| = |e^{-j\omega}\frac{c^*}{c}| = |e^{-j\omega}|\frac{|c^*|}{|c|} = 1, \forall\omega$$

Hence, $H_d(\omega)$ is an all-pass filter.



Problem 2:
Solution:
The sampling rate is $T = \frac{1}{2000}$. The sample points are $nT = \frac{n}{2000}$. As $\cos(\frac{\pi}{4}n) = \cos\left((\frac{\pi}{4} + 2\pi k)n\right) = \cos\left((-\frac{\pi}{4} + 2\pi k)n\right)$ so some possible choices for signal frequency can be,

$$\left(\frac{\pi}{4} + 2k\pi\right) = \frac{\Omega_0}{2000} \rightarrow \Omega_0 = 2000\left(\frac{\pi}{4} + 2k\pi\right), \forall k \in \mathbb{Z}$$
$$\left(-\frac{\pi}{4} + 2k\pi\right) = \frac{\Omega_0}{2000} \rightarrow \Omega_0 = 2000\left(-\frac{\pi}{4} + 2k\pi\right), \forall k \in \mathbb{Z}$$

Based on the above expressions, two examples of the candidate solutions are $\Omega_0 = 500\pi$ and $\Omega_0 = 3500\pi$.

Problem 3:

Solution:

1. The continuous-time Fourier transform of $x_a(t)$ is as follows:

$$x_a(t) = \cos(150\pi t) \longleftrightarrow X_a(\Omega) = \pi[\delta(\Omega - 150\pi) + \delta(\Omega + 150\pi)]$$

The discrete-time Fourier transform of $x[n]$ is as follows:

$$x[n] = x_a(nT_s) = \cos(150\pi n T_s)$$
$$\mathbb{F}\{x[n]\} = X_d(\omega) = \pi \sum_{k=-\infty}^{\infty} \left[\delta\left(\omega - 150\pi T_s + 2\pi k\right) + \delta\left(\omega + 150\pi T_s + 2\pi k\right)\right]$$

For $T_s = 10^{-3}$ seconds,

$$X_d(\omega) = \pi \sum_{k=-\infty}^{\infty} \left[\delta\left(\omega - 0.15\pi + 2\pi k\right) + \delta\left(\omega + 0.15\pi + 2\pi k\right)\right]$$

For $T_s = 2 * 10^{-3}$ seconds,

$$X_d(\omega) = \pi \sum_{k=-\infty}^{\infty} \left[\delta\left(\omega - 0.3\pi + 2\pi k\right) + \delta\left(\omega + 0.3\pi + 2\pi k\right)\right]$$

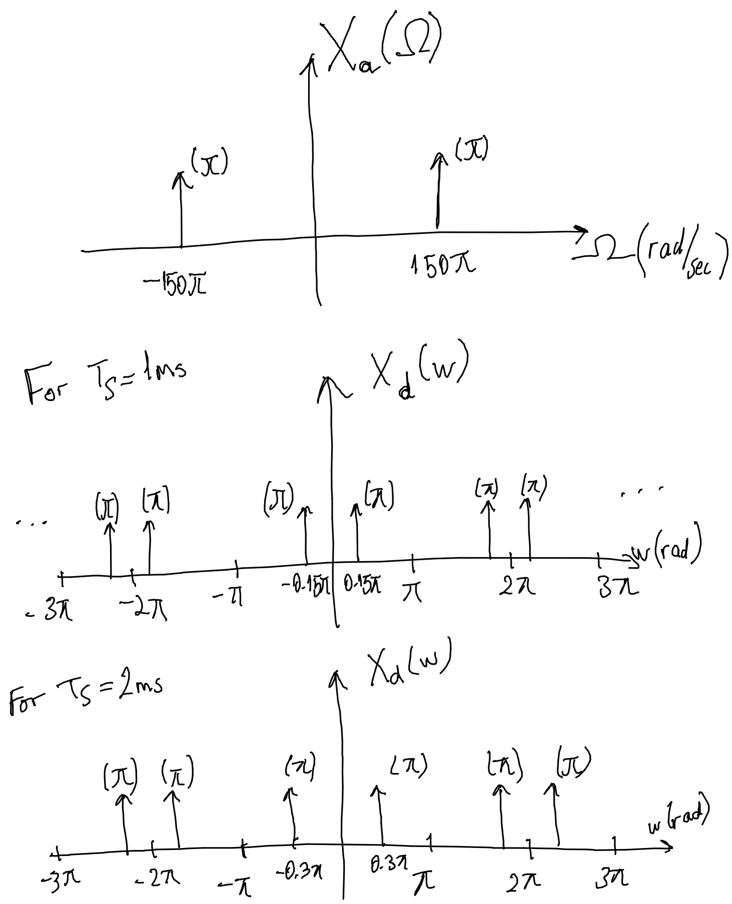


Figure 1: Problem 3 Part 1

2. The maximum sampling period $T_{s,max}$ such that no aliasing occurs in the sampling process is

$$T_{s,\max} = \frac{1}{2f_{\max}} = \frac{1}{2 \times 75} = \frac{1}{150} = 6.\overline{6}\text{ms}$$

Problem 4:
Solution:
The relation between $X_a(\Omega)$ and $X_d(\omega)$ is as follows:

$$X_d(\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_a\left(\frac{\omega + 2\pi k}{T_s}\right)$$

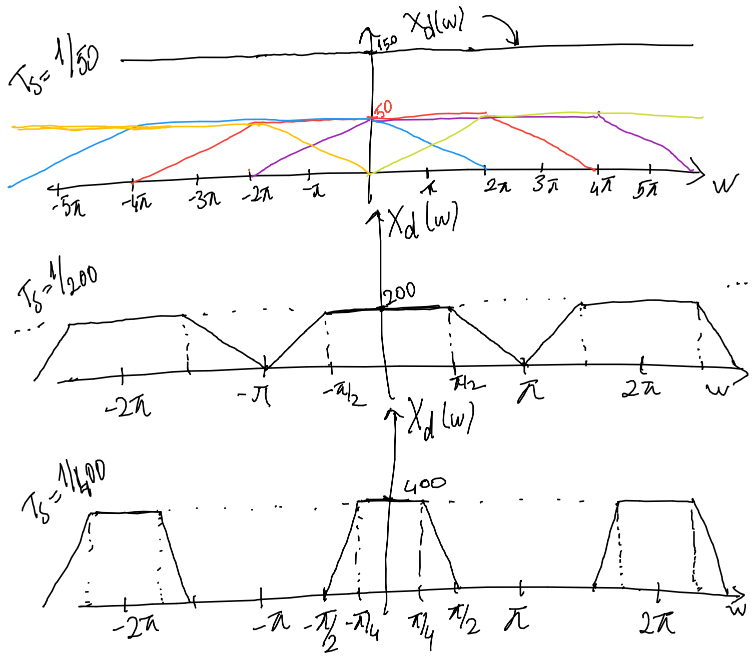


Figure 2: Problem 4

Problem 5:
Solution:
By Parseval's Theorem for the CTFT,

$$E_a = \int_{-\infty}^{\infty} |x_a(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X_a(\Omega)|^2 d\Omega = \frac{1}{2\pi}\int_{-\Omega_{max}}^{\Omega_{max}} |X_a(\Omega)|^2 d\Omega$$

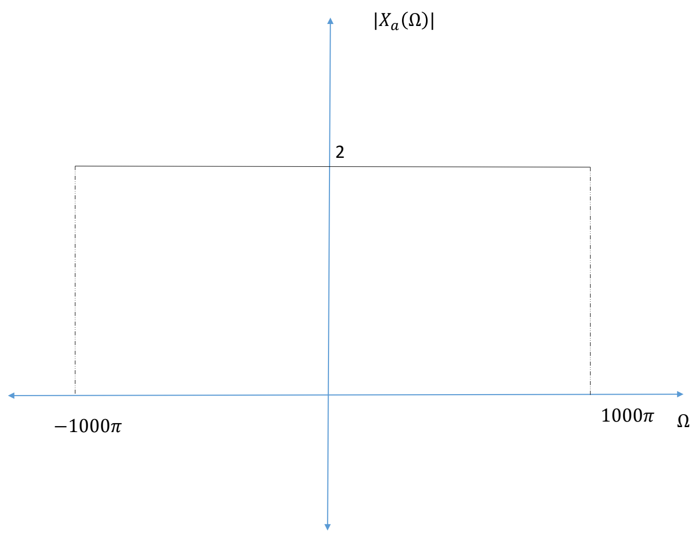
By $X_d(\omega) = \frac{1}{T_s}\sum_{-\infty}^{\infty} X_a(\frac{\omega+2k\pi}{T_s})$, the interval $[-\pi,\pi]$. Focusing on $[-\pi,\pi]$ corresponds to $k=0$,

$$X_d(\omega) = \begin{cases} \frac{1}{T_s}X_a(\frac{w}{T_s}) & |w| \leq \Omega_{max}T_s \\ 0 & \Omega_{max}T_s < |\omega| \leq \pi \end{cases}$$

when sampled with $\Omega_s \geq 2\Omega_{max}$(or $F_s \geq 2F_{max}$). By Parseval's Theorem for the DTFT,

$$\begin{aligned} E_d = \sum_{n=-\infty}^{+\infty} |x[n]|^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |X_d(\omega)|^2 d\omega \\ &= \frac{1}{2\pi}\int_{-\Omega_{max}T_s}^{\Omega_{max}T_s} \frac{1}{T_s^2}|X_a(\frac{\omega}{T_s})|^2 d\omega \\ &= \frac{1}{2\pi T_s}\int_{-\Omega_{max}T_s}^{\Omega_{max}T_s} \frac{1}{T_s}|X_a(\frac{\omega}{T_s})|^2 d\omega \\ &= \frac{1}{2\pi T_s}\int_{-\Omega_{max}}^{\Omega_{max}} |X_a(\Omega')|^2 d\Omega' = \frac{1}{T_s}E_a \end{aligned}$$

Verification:



$$E_a = \frac{1}{2\pi} \times 2^2 \times 2 \times 1000 \times \pi = 4000$$

Sampling at Nyquist rate: $\Omega_s = 2\Omega_{max}$ or $\frac{2\pi}{T_{Nyq}} = 2 \times 1000 \times \pi \implies T_{Nyq} = \frac{1}{1000}$ seconds.

$$\begin{aligned} E_d &= \frac{1}{2\pi}\left(\frac{2}{T_s}\right)^2 \times 2 \times \Omega_{max} \times T_s \\ &= \frac{1}{2\pi} \times 4 \times 10^6 \times (2\pi) = \frac{4000}{\frac{1}{10^3}} \\ &= \frac{E_a}{T_s} \end{aligned} \tag{1}$$