Digital Signal Processing
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Homework 10

ECE 310 (Spring 2020)
Assigned: 04/08 - Due: 04/15

Problem 1: DFT and DTFT

Let $\{X[k]\}_{k=0}^{50}$ and $X_d(\omega)$ respectively be the 51-point DFT and DTFT of a *real-valued* sequence $\{x[n]\}_{n=0}^{17}$ that is zero-padded to length 51. Determine all the correct relationships in the following and justify your answer.

1. $X[49] = X_d(-\frac{4\pi}{51})$.
2. $X[2] = X_d^*(-\frac{4\pi}{51})$
3. $X[1] = X_d(\frac{104\pi}{51})$
4. $X[25] = X_d(\pi)$

Solution:
(a) For $\{X[k]\}_{k=0}^{50}$, recall that values $0 \leq k < N = 51$ map to values of $0 \leq \omega < 2\pi$ for $X_d(\omega)$. Also recall that $X_d(\omega)$ is 2π-periodic and $\omega = \frac{w\pi k}{N} + 2\pi m \; \forall m \in \mathbb{Z}$. So:

$$X[19] = X_d(\frac{98\pi}{51}) = X_d(\frac{98\pi}{51} - 2\pi) = X_d(-\frac{4\pi}{51})$$

Therefore, the relation ship is correct.

(b) Recall that if $x[n]$ is a real-valued sequence, then $X_d(\omega) = X_d^*(-\omega)$. So:

$$X[2] = X_d(\frac{4\pi}{51}) = X_d^*(-\frac{4\pi}{51})$$

Therefore, the relationship is correct.

(c)

$$X[1] = X_d(\frac{2\pi}{51}) = X_d(\frac{2\pi}{51} + 2\pi) = X_d(\frac{104\pi}{51})$$

Therefore, the relationship is correct.

(d)

$$X[25] = X_d(\frac{50\pi}{51}) \neq X_d(\frac{\pi}{51})$$

Therefore, the relationship is not correct.

Problem 2: DFT of a Cosine

A continuous-time signal $x_c(t) = \cos(24\pi t)$ is sampled at a rate of 120 Hz for 5 seconds to produce a discrete-time signal $x[n]$ with length $L = 600$.

1. Let $X[k]$ be the L-point DFT of $x[n]$. At what value(s) of k will $X[k]$ have the greatest magnitude?

2. Suppose that $x[n]$ is zero-padded to a total length of $N = 1024$. At what value(s) of k does the N-point DFT have the greatest magnitude?

Solution:
(a) To convert the signal to discrete-time, we use $x[n] = x_c(nT) = cos(\frac{\pi}{5}n)$. However, note that we are only sampling for five seconds, so the signal is truncated and finite: $\{x[n]\}_{n=0}^{599}$. If $x[n]$ was not truncated, the DTFT of a cosine would be a pair of shifted delta functions for every period. Since there is truncation, the DTFT now has the deltas replaced with sinc-like functions.
Given the resemblance between the DTFT of the cosine and truncated cosine, notice that the greatest magnitudes for both DTFTs are at the same values of ω:

$$\frac{\pi}{5} + 2\pi m,\ -\frac{\pi}{5} + 2\pi m,\ \forall m \in \mathbb{Z}$$

For $X[k]$, values of $0 \le k < N$ map to values of $0 \le \omega < 2\pi$. So the values of ω between 0 and π with the greatest magnitude are $\frac{\pi}{5}$ and $\frac{9\pi}{5}$. Using the fact that $\omega = \frac{2\pi k}{N}$, we can find the integer values of k where $X[k]$ has the greatest magnitude:

$$\frac{\pi}{5} = \frac{2\pi k}{600} \rightarrow k = 60$$
$$\frac{9\pi}{5} = \frac{2\pi k}{600} \rightarrow k = 540$$

(b) When $N = 1024$, the solved values of k are no longer integers. Round to the nearest integer:

$$\frac{\pi}{5} = \frac{2\pi k}{1024} \rightarrow k = 102.4 \approx 102$$
$$\frac{9\pi}{5} = \frac{2\pi k}{1024} \rightarrow k = 921.6 \approx 922$$

Problem 3: Circular and Linear Convolution

Consider the two finite-length sequences:

$$x = \{\underset{\uparrow}{-1}, 2, -3, 4, -5\} \text{ and } h = \{\underset{\uparrow}{1}, 1, 1\}$$

1. Compute the linear convolution $x * h$.
2. Compute the circular convolution $x \circledast_5 h$.
3. What is the smallest value of N so that the N-point circular convolution is equal to the linear convolution?

Solution:

(a) Linear convolution is defined as $(x * h)[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$

$$\begin{aligned} y[n] &= (x * h)[n] \\ &= \{\underset{\uparrow}{-1}, -1+2, -1+2-3, 2-3+4, -3+4-5, 4-5, -5\} \\ &= \{\underset{\uparrow}{-1}, 1, -2, 3, -4, -1, -5\} \end{aligned}$$

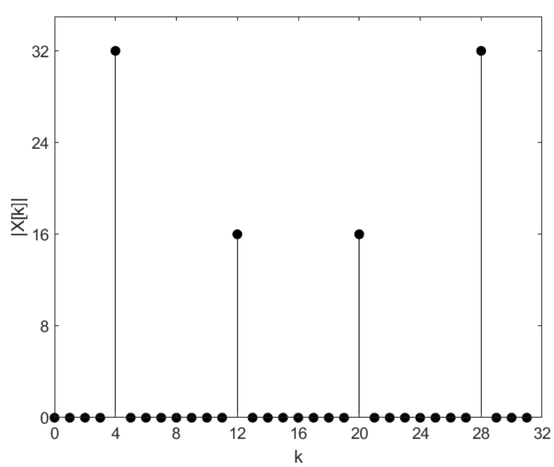
(b) Circular convolution is defined as $(x \circledast_5 h)[n] = \sum_{m=0}^{N-1} x[m]h[\langle n-m \rangle_N]$. To perform length-5 circular convolution, we will first need to zero pad $\{h\}_{n=0}^{2}$ to length 5. We define the sequence $\{\tilde{h}[n]\}_{n=0}^{4} = \{1, 1, 1, 0, 0\}$. Now we can construct the circular convolution matrix for $\tilde{h}$.

$$\begin{aligned} y[n] &= (x \otimes_5 h)[n] \\ &= \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ -3 \\ 4 \\ -5 \end{bmatrix} \\ &= \{\underset{\uparrow}{-2}, -4, -2, 3, -4\} \end{aligned}$$

(c) Let $L = 5$ and $M = 3$ be the length of x and of h, respectively. Then the length of $y = x * h$ is $N = L + M - 1 = 7$. For N-point circular convolution to be equal to linear convolution, we will need to pad x and h to length $N = 7$.

Problem 4: DFT for the Sum of Cosines

Suppose that the signal $x_c(t) = A_0 \cos(\Omega_0 t) + A_1 \cos(\Omega_1 t)$ is sampled at a rate of 64 kHz for 1/2 msec. The DFT of the obtained signal is provided by the plot.



1. Assume that Ω_0 and Ω_1 are both less than the Nyquist frequency. Find A_0, A_1, Ω_0, and Ω_1.

2. Suppose that $x_c(t)$ was instead sampled at 128 kHz for 1/2 msec. Sketch the new DFT magnitude plot and clearly label all nonzero values.

Solution:
(a) Each pair of peaks in the graph corresponds to a cosine. Assume that the pair $k = 4$ and$k = 28$ are the DFTs of $A_0 cos(\Omega_0 t)$ and the pair $k = 12$ and$k = 20$ are the DFTs of $A_1 cos(\Omega_1 t)$. To find the frequencies of the continuous-time signal, recall that $\omega = \frac{2\pi k}{N}$ and $\Omega = \frac{\omega}{T}$. So:

$$\Omega_0 = \frac{2\pi k}{NT} = \frac{2\pi 4}{32T} = 16\pi \times 10^3$$
$$\Omega_0 = \frac{2\pi k}{NT} = \frac{2\pi 12}{32T} = 48\pi \times 10^3$$

where $N = 32$ and $\frac{1}{T} = 64$ kHz. To find the amplitudes A_0 and A_1, we first take the DFT of one discrete cosine $x[n] = A_0 cos(\Omega_0 n)$:

$$X_0[k] = A_0 \frac{sin(\frac{N}{2}(\frac{2\pi k}{N} - \omega_0))}{2sin(\frac{1}{2}(\frac{2\pi k}{N} - \omega_0))} e^{-j\frac{N-1}{2}(\frac{2\pi k}{N} - \omega_0)} + A_0 \frac{sin(\frac{N}{2}(\frac{2\pi k}{N} - \omega_0))}{2sin(\frac{1}{2}(\frac{2\pi k}{N} - \omega_0))} e^{-j\frac{N-1}{2}(\frac{2\pi k}{N} + \omega_0)}$$

And find its magnitude, where $\omega_0 = \frac{\pi}{4}$:

$$|X_0[k]| = |A_0 \frac{sin(\frac{N}{2}(\frac{2\pi k}{N} - \omega_0))}{2sin(\frac{1}{2}(\frac{2\pi k}{N} - \omega_0))}| + |A_0 \frac{sin(\frac{N}{2}(\frac{2\pi k}{N} - \omega_0))}{2sin(\frac{1}{2}(\frac{2\pi k}{N} - \omega_0))}|$$

We know by graph that there are peaks for $|X_0[k]|$ at $|X_0[4]| = |X_0[28]| = 32$. Using L'Hopital's rule, we find $|X_0[4]| = |X_0[28]| = \frac{A_0 N}{2}$. So, $A_0 = 2$. Repeating the same steps to solve for A_1 and we get $A_1 = 1$.

(b) We now have a sampling rate of $\frac{1}{T} = 128$kHz and $N = 64$. Using the answers from the last part, we can find each peak's new location and magnitude.

$$k_0 = \frac{\Omega_0 NT}{2\pi} = 4$$
$$k_1 = \frac{\Omega_1 NT}{2\pi} = 12$$
$$|X[k_0]| = \frac{A_0 N}{2} = 64$$
$$|X[k_1]| = \frac{A_1 N}{2} = 32$$

Note that the spectral copies of the DTFT in interval $(\pi, 2\pi]$ will also be present in the DFT, locates at $k_0 = 60$ and $k_1 = 52$ respectively.