Topics covered in this homework are: DT response of an analog system, DFT properties, the DFT and its properties. Homework is due at 5:00 PM on Wednesdays. Homework will be graded for (1) completion and (2) one randomly picked problem will be graded. Submissions will be using gradescope. Please solve problems on your own in order to maximally benefit from this homework.

Problem 1:

Consider the communication link shown below: The channel frequency response $H_c(\Omega)$ is shown

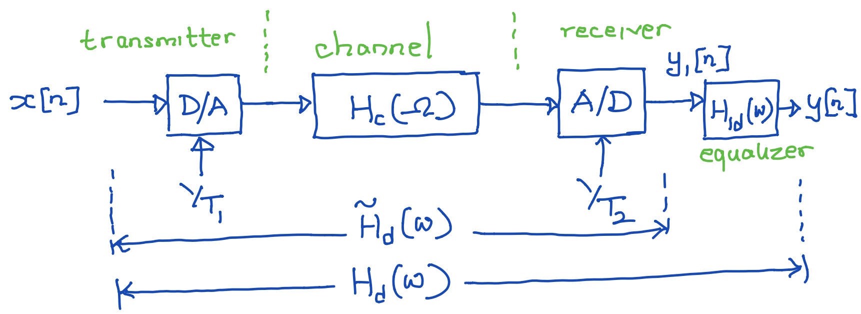

Figure 1: A communication system.

below: We wish to design this link, i.e., select the values of the D/A and A/D sample rates and

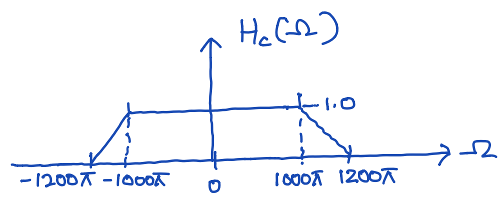

Figure 2: The channel frequency response.

the equalizer frequency response $H_{1d}(\omega)$, such that the end-to-end equivalent discrete-time impulse response $h[n] = \delta[n]$ so that the final output $y[n] = x[n]$. In other words, the receiver is able to recover the transmitted data $x[n]$ in spite of the distortion introduced by the channel due to the roll-off in its frequency response (see Fig. 2). In practice, the channel will also introduce noise, which can be accounted for via the use of material from ECE 313. But, we will ignore noise in this problem to keep things simple. In the following, assume that the D/A is ideal.

(a) What are the D/A and A/D sample rates $\frac{1}{T_1}$ and $\frac{1}{T_2}$ such that the entire flat portion of the channel response, i.e. $[-1000\pi, 1000\pi]$, is fully utilized and the equalizer has a constant frequency response, $H_{1d} = c$, to satisfy the end-to-end impulse response as $h[n] = \delta[n]$? Also, report the value for c.

(b) If the D/A sample rate is fixed at $\frac{1}{T_1} = 1.2\,\text{kS/s}$ so that the entire channel bandwidth is utilized, find an appropriate value of the A/D sample rate $\frac{1}{T_2}$ and the equalizer frequency response

$H_{1d} = c$ where c is a constant, such that the end-to-end impulse response $h[n] = \delta[n]$.

(c) If $\frac{1}{T_1} = \frac{1}{T_2} = 1.2\,\text{kS/s}$, which is a typical scenario, find the equalizer frequency response $H_{1d}(\omega)$ needed to obtain an end-to-end impulse response $h[n] = \delta[n]$.

Solution:

(a) An ideal D/A will have the following frequency response:

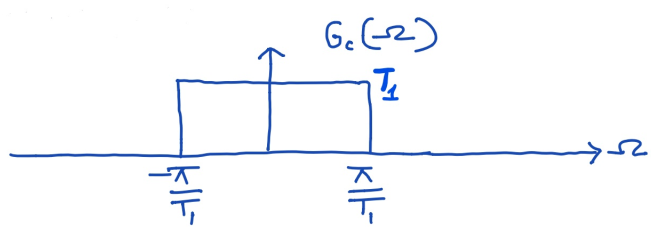

Figure 3: Frequency response of an ideal D/A

Choosing $\frac{\pi}{T_1} = 1000\pi \implies \frac{1}{T_1} = 1000 = 1\ kS/s$, the cascade by D/A and channel has the frequency response:

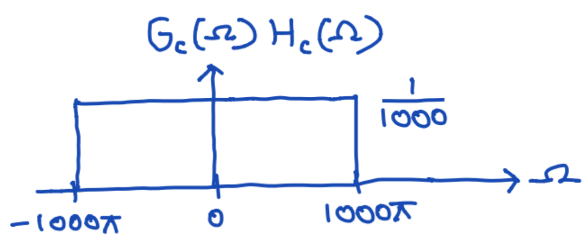

Figure 4: Frequency response of the cascade of D/A and channel

The A/D sample rate $\frac{1}{T_2} = 2\frac{1000\pi}{2\pi} = 1\ kS/s$ will generate an $\tilde{H}_d(\omega)$:

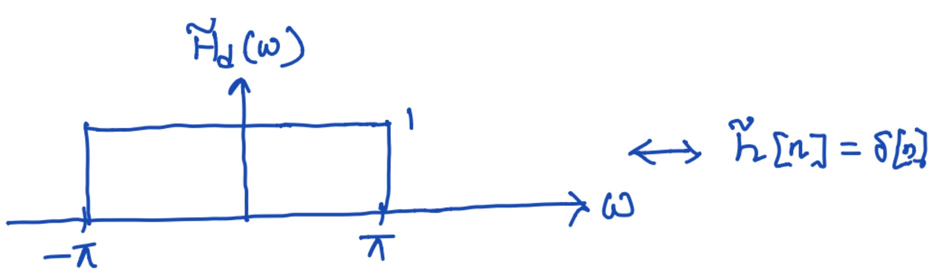

Figure 5: Frequency response of the respective $\tilde{H}_d(\omega)$

$H_{1d}(\omega) = \leftrightarrow h_1[n] = \delta[n]$ so that $y[n] = x[n]$.

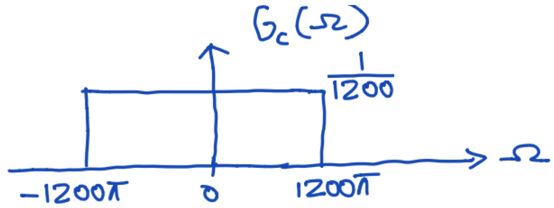

Figure 6: Frequency response of the D/A for $\frac{1}{T_1} = 1.2\ kS/s$

(b) With $\frac{1}{T_1} = 1.2kS/s$, the D/A frequency response is indicated in Fig.6 and $G_c(\Omega)H_c(\Omega) = H_c(\Omega)$.

Now, we need to choose $\frac{1}{T_2}$ such that $\tilde{H}_d(\omega) = c^{-1} \leftrightarrow c^{-1}\delta[n] = \tilde{h}[n]$.

This requires a controlled level of aliasing so that with aliasing $\tilde{H}_d(\omega)$ is flat with respect to ω, i.e.,

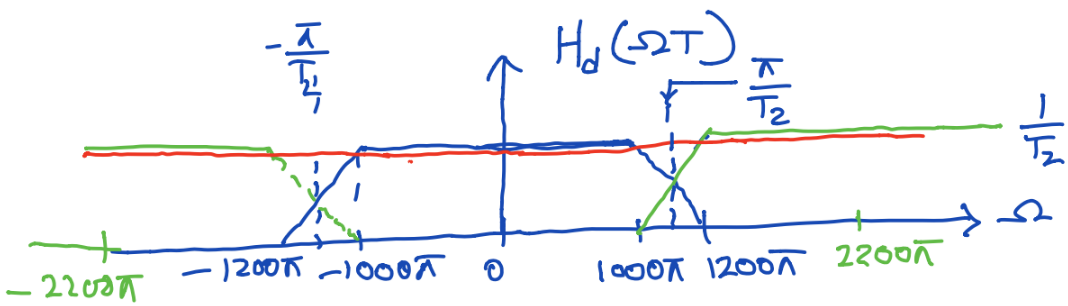

Figure 7: Frequency response of $\tilde{H}_d$

where $\frac{\pi}{T_2} = 1100\pi \implies \frac{1}{T_2} = 1100 = 1.1kS/s$ and therefore $H_{1d}(\omega) = \frac{1200}{1100} \leftrightarrow h_1[n] = \frac{1200}{1100}\delta[n]$ i.e., $c = \frac{1200}{1100}$.

(c) For $\frac{1}{T_1} = \frac{1}{T_2} = 1.2\ kS/s$, $\tilde{H}_d(\omega)$ is obtained as

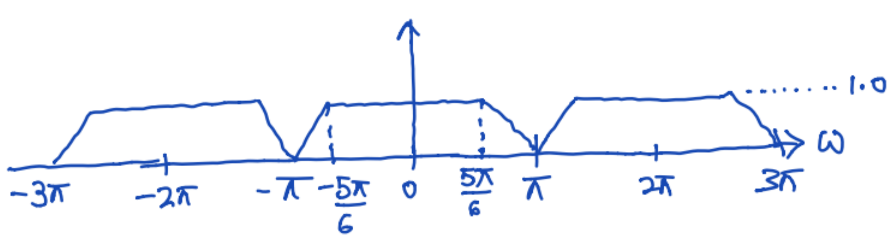

Figure 8: Frequency response of $\tilde{H}_d$ for $\frac{1}{T_1} = \frac{1}{T_2} = 1.2\ kS/s$

The equalizer frequency response that 'flattens' the channel is shown in Fig. 9:

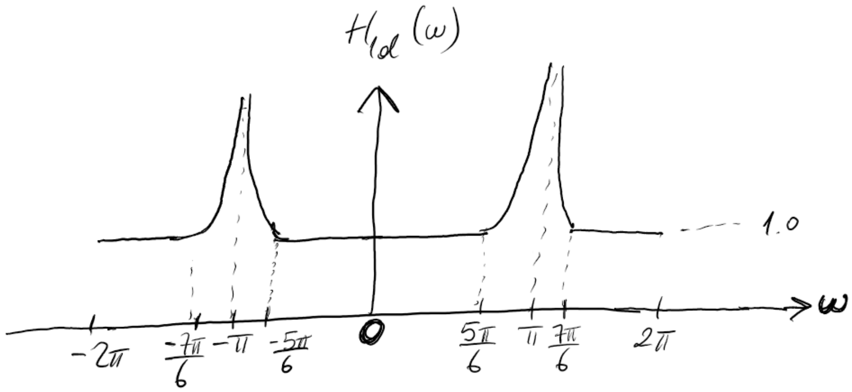

Figure 9: Frequency response of $\tilde{H}_{1d}$ to 'flatten' the channel

Problem 2:

Consider the length-32 sequence $\{x[n]\}_{n=0}^{31}$. A sequence of length-64 $\{y[n]\}_{n=0}^{63}$ is generated by setting $y[2n] = x[n]$ $(n = 0, \ldots, 31)$, i.e., for even values of n, and $y[n] = 0$ for odd values of n. Determine $\{Y[k]\}_{k=0}^{63}$ in terms of $\{X[k]\}_{k=0}^{31}$.

Solution:

$$X[k] = \sum_{n=0}^{31} x[n]e^{-j\frac{2\pi}{32}kn} = \sum_{n=0}^{31} x[n]e^{-j\frac{\pi}{16}kn} \quad (k = 0, \ldots, 31) \tag{1}$$

$$Y[k] = \sum_{n=0}^{63} y[n]e^{-j\frac{2\pi}{64}kn} = \sum_{n=0}^{63} y[n]e^{-j\frac{\pi}{32}kn} \quad (k = 0, \ldots, 63) \tag{2}$$

We know that $y[2n] = x[n]$ and $y[2n+1] = 0$, $n = 0, \ldots, 31$. Hence, substituting $n = 2m$ in (2), we get:

$$Y[k] = \sum_{m=0}^{31} y[2m]e^{-j\frac{\pi}{32}k(2m)} = \sum_{m=0}^{31} x[m]e^{-j\frac{\pi}{16}km} \quad (k = 0, \ldots, 63) \tag{3}$$

Note: (1) $\equiv$ (3), for $k = 0, \ldots, 31$, i.e, $\{Y[k]\}_{k=0}^{31} = \{X[k]\}_{k=0}^{31}$

To find $\{Y[k]\}_{k=32}^{63}$, substitute $k = l + 32$ with $l = 0, \ldots, 31$ in (3):

$$\begin{aligned} Y[l+32] &=_{m=0}^{31} x[m]e^{-j\frac{\pi}{16}m(l+32)} \quad (l = 0, \ldots, 31) \\ &= \sum_{m=0}^{31} x[m]e^{-j\frac{\pi}{16}lm}e^{-j2\pi} \\ &= \sum_{m=0}^{31} x[m]e^{-j\frac{\pi}{16}lm} = X[l] \end{aligned} \tag{4}$$

$$\implies \{Y[k]\}_{k=32}^{63} = \{X[k]\}_{k=0}^{31}$$

Hence, $Y[k] = X[< k >_{32}]$, $k = (0, \ldots, 63)$.

Problem 3:

Given two sequences $\{x_1[n]\}_{n=0}^{31}$ and $\{x_2[n]\}_{n=0}^{31}$, a new sequence $\{y[n]\}_{n=0}^{63}$ is composed by interleaving $x_1[n]$ and $x_2[n]$ as follows: $y[n] = x_1[m]$ when $n = 2m$ i.e., n is even, otherwise $y[n] = x_2[m]$ when $n = 2m + 1$, i.e., n is odd, where $m = 0, \ldots, 31$. If $\{X_1[k]\}_{k=0}^{31}$ is the DFT of $\{x_1[n]\}_{n=0}^{31}$ and $\{X_2[k]\}_{k=0}^{31}$ is the DFT of $\{x_2[n]\}_{n=0}^{31}$, express the DFT $\{Y[k]\}_{k=0}^{63}$ of $\{y[n]\}_{n=0}^{63}$ in terms of the $\{X_1[k]\}_{k=0}^{31}$ and $\{X_2[k]\}_{k=0}^{31}$.

Solution:

$$X_1[k] = \sum_{n=0}^{31} x_1[n]e^{-j\frac{2\pi}{32}kn} = \sum_{n=0}^{31} x_1[n]e^{-j\frac{\pi}{16}kn} \quad (k = 0, \ldots, 31) \tag{5}$$

$$X_2[k] = \sum_{n=0}^{31} x_2[n] e^{-j\frac{\pi}{16}kn} \quad (k = 0, \ldots, 31) \tag{6}$$

$$Y[k] = \sum_{n=0}^{63} y[n] e^{-j\frac{2\pi}{64}kn} = \sum_{n=0}^{63} y[n] e^{-j\frac{\pi}{32}kn} \quad (k = 0, \ldots, 63) \tag{7}$$

By setting $n = 2m$ and $n = 2m+1$ ($m = 0, \ldots, 31$) we sum the even and odd terms in (7) separately:

$$\begin{aligned} Y[k] &= \sum_{m=0}^{31} y[2m] e^{-j\frac{\pi}{32}k(2m)} + \sum_{m=0}^{31} y[2m+1] e^{-j\frac{\pi}{32}k(2m+1)} \\ &= \sum_{m=0}^{31} x_1[m] e^{-j\frac{\pi}{16}km} + \sum_{m=0}^{31} x_2[m] e^{-j\frac{\pi}{16}km} e^{-j\frac{\pi}{32}k} \quad (k = 0, \ldots, 63) \end{aligned} \tag{8}$$

For $k = 0, \ldots, 31$,

$$Y[k] = X_1[k] + e^{-j\frac{\pi}{32}k} X_2[k] = Y[k+32] \tag{9}$$

$$Y[k] = X_1[< k >_{32}] + e^{-j\frac{\pi}{32}k} X_2[< k >_{32}] \quad k = (0, \ldots, 63) \tag{10}$$

Problem 4:

Given the sequence $\{x[n]\}_{n=0}^{3} = \{2, 0, 6, 4\}$ with DFT $\{X[k]\}_{k=0}^{3} = \{X[0], X[1], X[2], X[3]\}$:

(a) Express the DFT $\{\hat{X}[k]\}_{k=0}^{3}$ of the sequence $\{\hat{x}[n]\}_{n=0}^{3} = \{-2, 1, 0, 3\}$ in terms of $\{X[k]\}_{k=0}^{3}$.

(b) validate the relationship derived in part (a) by computing $\{X[k]\}_{k=0}^{3}$ and $\{\hat{X}[k]\}_{k=0}^{3}$ directly from their corresponding time-domain sequences.

Solution:

(a) Note that $\hat{x}[n]$ can be obtained from $x[n]$ by applying the following sequence of transformations:

1) $x_1[n] = e^{j\pi n} x[n] = \{2, 0, 6, -4\}$ (modulation)
2) $x_2[n] = \frac{1}{2} x_1[n] = \{1, 0, 3, -2\}$ (scaling)
3) $\hat{x}[n] = x_2[< n-1 >_4] = \{-2, 1, 0, 3\}$ (time-shift)

Hence, using the modulation property

$$x_1[n] = e^{j\frac{2\pi}{4}(2n)} x[n] \longleftrightarrow X_1[k] = X[< k-2 >_4]$$
$$X_1[k] = \{X[2], X[3], X[0], X[1]\}$$

Next, from scaling property

$$x_2[n] = \frac{1}{2}x_1[n] \longleftrightarrow X_2[k] = \frac{1}{2}X_1[k]$$
$$X_2[k] = \{\frac{X2}{2}, \frac{X3}{2}, \frac{X0}{2}, \frac{X1}{2}\}$$

Finally, from time-shift property

$$\hat{x}[n] = x_2[< n-1 >_4] \longleftrightarrow \hat{X}[k] = e^{-j\frac{2\pi}{4}k}X_2[k]$$
$$\hat{X}[k] = \{\frac{X[2]}{2}, e^{-j\frac{\pi}{2}}\frac{X[3]}{2}, e^{-j\pi}\frac{X[0]}{2}, e^{-j\frac{3\pi}{2}}\frac{X[1]}{2}\} = \{\frac{X[2]}{2}, -j\frac{X[3]}{2}, -\frac{X[0]}{2}, j\frac{X[1]}{2}\}$$

(b) Finding $X[k]$ first:

$$X[k] = \sum_{k=0}^{3} x[n]e^{-j\frac{2\pi}{4}kn} = \sum_{k=0}^{3} x[n]e^{-j\frac{\pi}{2}kn} = 2 + 0e^{-j\frac{\pi}{2}k} + 6e^{-j\pi k} + 4e^{-j\frac{3\pi}{2}k}$$

Then $X[0] = 12, X[1] = 4(-1+j), X[2] = 4, X[3] = 4(-1-j)$.
Calculating $\hat{X}[k]$ similarly:

$$\hat{X}[k] = -2 + 1e^{-j\frac{\pi}{2}k} + 0e^{-j\pi k} + 3e^{-j\frac{3\pi}{2}k}$$

we have $\hat{X}[0] = 2, \hat{X}[1] = 2(-1+j), \hat{X}[2] = -6, \hat{X}[3] = 2(-1-j)$
Using the relationship between $X[k]$ and $X[k]$ derived in part (a), we can compute $\hat{X}[k]$ as follows:

$$\hat{X}[0] = \frac{1}{2}X[2] = 2, \quad \hat{X}[1] = -jX[3]/2 = 2(-1+j), \quad \hat{X}[2] = -X[0]/2 = -6, \quad \hat{X}[3] = jX[1]/2 = 2(-1-j)$$

Problem 5:

Consider two length-4 sequences $\{x[n]\}_{n=0}^{3} = \{1, 0, -1, 0\}$ and $h[n]_{n=0}^{3} = \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}\}$.

(a) Compute the DFTs $\{X[k]\}_{k=0}^{3}$ and $\{H[k]\}_{k=0}^{3}$.

(b) Compute the circular convolution $y[n] = x[n] \circledast h[n]$ directly in the sequence domain.

(c) Compute the circular convolution $y[n] = x[n] \circledast h[n]$ in the frequency domain using DFT and IDFT. Compare your answer with the one in part (b).

Solution:

(a) Calculate $X[k]$ as follows:

$$X[k] = \sum_{k=0}^{3} x[n]e^{-j\frac{2\pi}{4}kn} = \sum_{k=0}^{3} x[n]e^{-j\frac{\pi}{2}kn}$$

Since $x[n] = \{1, 0, -1, 0\}$, we get $X[k] = 1 - e^{-j\pi k}$ for $k = 0, 1, 2, 3$. Hence $\{X[k]\}_{k=0}^{3} = \{0, 2, 0, 2\}$. Similarly,

$$H[k] = \sum_{k=0}^{3} h[n]e^{-j\frac{2\pi}{4}kn} = \frac{k=0}{3} h[n]e^{-j\frac{\pi}{2}kn}$$

Since $h[n] = \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}\}$, we get $H[k] = 1 + \frac{1}{2}e^{-j\frac{\pi}{2}k} + \frac{1}{4}e^{-j\pi k} + \frac{1}{8}e^{-j\frac{3\pi}{2}k}$ for $k = 0, 1, 2, 3$, and thus $\{H[k]\}_{k=0}^{3} = \{\frac{15}{8}, \frac{3}{4} - j\frac{3}{8}, \frac{5}{8}, \frac{3}{4} + j\frac{3}{8}\}$

(b) Circular convolution $y[n] = x[n] \circledast h[n]$ is given by:

$$y[n] = \sum_{k=0}^{3} x[k]h[< n - k >_4]$$

where $x[k] = \{1, 0, -1, 0\}$ and

$$h[< -k >_4] = \{1, 1/8, 1/4, 1/2\}, \quad h[< 1 - k >_4] = \{1/2, 1, 1/8, 1/4\},$$
$$h[< 2 - k >_4] = \{1/4, 1/2, 1, 1/8\}, \quad h[< 3 - k >_4] = \{1/8, 1/4, 1/2, 1\}$$

Therefore,

$$y[0] = \sum_{k=0}^{3} x[k]h[< -k >_4] = 1 - 1/4 = 3/4$$

$$y[1] = \sum_{k=0}^{3} x[k]h[< 1 - k >_4] = 1/2 - 1/8 = 3/8$$

$$y[2] = \sum_{k=0}^{3} x[k]h[< 2 - k >_4] = 1/4 - 1 = -3/4$$

$$y[3] = \sum_{k=0}^{3} x[k]h[< 3 - k >_4] = 1/8 - 1/2 = -3/8$$

$$y[n] = \{3/4, 3/8, -3/4, -3/8\}$$

(c) Calculate $Y[k] = H[k]X[k]$ as follows:

$$Y[0] = H[0]X[0] = 0, \quad Y[1] = H[1]X[1] = (\frac{3}{2} - j\frac{3}{4})$$

$$Y[2] = H[2]X[2] = 0, \quad Y[3] = H[3]X[3] = (\frac{3}{2} + j\frac{3}{4})$$

Computing IDFT of $Y[k]$, we get

$$\begin{aligned} y[n] &= \frac{1}{4}\sum_{k=0}^{3} Y[k]e^{j\frac{\pi}{2}kn} \\ &= \frac{1}{4}\left[(\frac{3}{2} - j\frac{3}{4})e^{j\frac{\pi}{2}n} + (\frac{3}{2} + j\frac{3}{4})e^{j\frac{3\pi}{2}n}\right] \end{aligned}$$

Thus,

$$y[0] = \frac{3}{4}, \quad y[1] = \frac{1}{4}\left[(\frac{3}{2} - j\frac{3}{4})(j) + (\frac{3}{2} + j\frac{3}{4})(-j)\right] = \frac{3}{8}$$

$$y[2] = \frac{1}{4}\left[-(\frac{3}{2} - j\frac{3}{4}) - (\frac{3}{2} + j\frac{3}{4})\right] = -\frac{3}{4}, \quad y[3] = \frac{1}{4}\left[(\frac{3}{2} - j\frac{3}{4})(-j) + (\frac{3}{2} + j\frac{3}{4})(j)\right] = -\frac{3}{8}$$

which is identical to the result obtained in part (b).

Problem 6:

Consider the two sequences $\{x[n]\}_{n=0}^{2} = \{1, -1, 1\}$ and $h[n]_{n=0}^{1} = \{-1, 1\}$.

(a) Compute the linear convolution $y[n] = x[n] * h[n]$ directly in the sequence domain.

(b) Compute the linear convolution $y[n] = x[n] * h[n]$ in the frequency domain using DFT and IDFT after appropriately zero-padding the length-4 sequences. Compare your answer with the answer in part (a).

Solution:

(a) Define $X(z) = 1 - z^{-1} + z^{-2}$ and $H(z) = -1 + z^{-1}$. Hence,

$$Y(z) = H(z)X(z) = -1 + 2z^{-1} - 2z^{-2} + z^{-3}$$

$$y[n] = \{-1, 2, -2, 1\}$$

(b) We want $y[n]$ a length 4 sequence using DFTs. Hence, $x[n]$ and $h[n]$ are zero-padded as follows:

$$x_{zp}[n] = \{1, -1, 1, 0\} \quad h_{zp}[n] = \{-1, 1, 0, 0\}$$

Then, we find $X_{zp}[k]$ as:

$$X_{zp}[k] = \sum_{k=0}^{3} x_{zp}[n] e^{-j\frac{2\pi}{4}kn} = \sum_{k=0}^{3} x_{zp}[n] e^{-j\frac{\pi}{2}kn} = 1 - e^{-j\frac{\pi}{2}k} + e^{-j\pi k}$$

$$\{X_{zp}[k]\}_{k=0}^{3} = \{1, j, 3, -j\}$$

Similarly, find $H_{zp}[k]$:

$$H_{zp}[k] = \sum_{k=0}^{3} h_{zp}[n] e^{-j\frac{2\pi}{4}kn} = \sum_{k=0}^{3} h_{zp}[n] e^{-j\frac{\pi}{2}kn} = -1 + e^{-j\frac{\pi}{2}k}$$

$$\{H_{zp}[k]\}_{k=0}^{3} = \{0, -1-j, -2, -1+j\}$$

Therefore,

$$Y[0] = X_{zp}[0] H_{zp}[0] = 0$$
$$Y[1] = X_{zp}[1] H_{zp}[1] = 1 - j$$
$$Y[2] = X_{zp}[2] H_{zp}[2] = -6$$
$$Y[3] = X_{zp}[3] H_{zp}[3] = 1 + j$$

Computing the IDFT of $Y[k]$:

$$y[n] = \frac{1}{4} \sum_{k=0}^{3} Y[k] e^{j\frac{2\pi}{4}kn} = \frac{1}{4} \sum_{k=0}^{3} Y[k] e^{j\frac{\pi}{2}kn} = \frac{1}{4}\left[(1-j)e^{j\frac{\pi}{2}n} - 6e^{j\pi n} + (1+j)e^{j\frac{3\pi}{2}n}\right]$$

Then, $y[n] = \{-1, 2, -2, 1\}$ which matches the result in part (a).