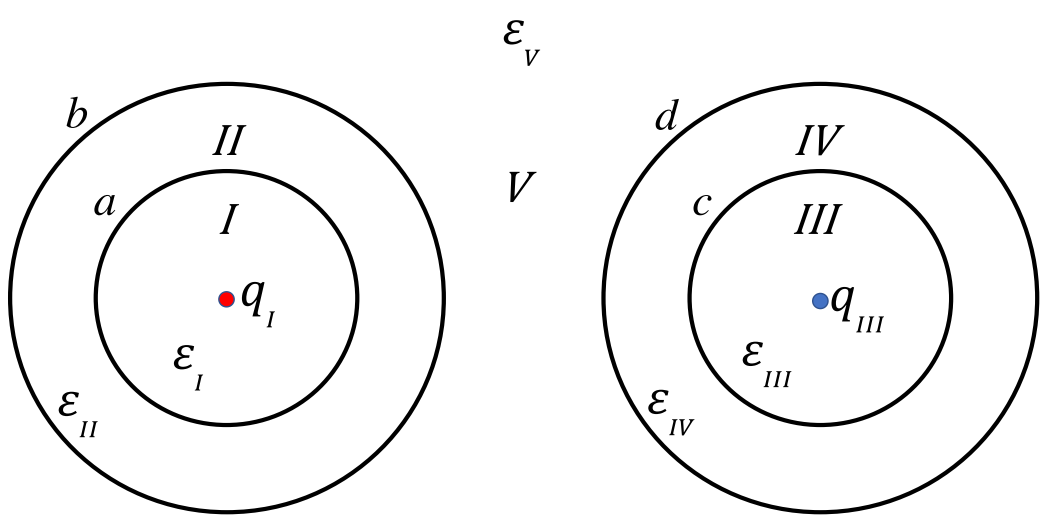
1 Governing integral equations for two-region continuum electrostatics problem

1.1 Problem schematic



1.2 Two-region intergal equation using classical continuum theory

$$
\left[\begin{array}{cccc|cccc}
\frac{1}{2}I + K^a_{I,a} & -V^a_{I,a} & & & & & & \\
\frac{1}{2}I - K^a_{II,a} & (\frac{\epsilon_I}{\epsilon_{II}})V^a_{II,a} & K^a_{II,b} & -V^a_{II,b} & & & & \\
-K^b_{II,a} & (\frac{\epsilon_I}{\epsilon_{II}})V^b_{II,a} & \frac{1}{2}I + K^b_{II,b} & -V^b_{II,b} & & & & \\
 & & \frac{1}{2}I - K^b_{V,b} & (\frac{\epsilon_{II}}{\epsilon_V})V^b_{V,b} & & & -K^b_{V,d} & (\frac{\epsilon_{IV}}{\epsilon_V})V^b_{V,d} \\
\hline
 & & & & \frac{1}{2}I + K^c_{III,c} & -V^c_{III,c} & & \\
 & & & & \frac{1}{2}I - K^c_{IV,c} & (\frac{\epsilon_{III}}{\epsilon_{IV}})V^c_{IV,c} & K^c_{IV,d} & -V^c_{IV,d} \\
 & & & & -K^d_{IV,c} & (\frac{\epsilon_{III}}{\epsilon_{IV}})V^d_{IV,c} & \frac{1}{2}I + K^d_{IV,d} & -V^d_{IV,d} \\
 & & -K^d_{V,b} & (\frac{\epsilon_{II}}{\epsilon_V})V^d_{V,b} & & & \frac{1}{2}I - K^d_{V,d} & (\frac{\epsilon_{IV}}{\epsilon_V})V^d_{V,d}
\end{array}\right]
\begin{bmatrix} \phi_a \\ \frac{\partial \phi_a}{\partial n} \\ \phi_b \\ \frac{\partial \phi_b}{\partial n} \\ \phi_c \\ \frac{\partial \phi_c}{\partial n} \\ \phi_d \\ \frac{\partial \phi_d}{\partial n} \end{bmatrix}
=
\begin{bmatrix} \sum_{i=1}^{N_a} \frac{q_i}{\epsilon_I} G^a_{I,i} \\ 0 \\ 0 \\ 0 \\ \sum_{i=1}^{N_c} \frac{q_i}{\epsilon_{III}} G^c_{III,i} \\ 0 \\ 0 \\ 0 \end{bmatrix}
\tag{1}
$$

Integral operators:

Integral operators are defined as:

$$
\text{Single layer operator}: \quad V^s_{i,\Gamma} \frac{\partial \phi_\Gamma}{\partial n} = \int_\Gamma G_i(r_s; r') \frac{\partial \phi_\Gamma}{\partial n(r')}(r') dA', \tag{2}
$$

and

$$
\text{Double layer operator}: \quad K^s_{i,\Gamma} \phi_\Gamma = \int_\Gamma \frac{\partial G_i}{\partial n(r')}(r_s; r') \phi_\Gamma(r') dA', \tag{3}
$$

which respectively, calculate the potential at the surface s due to a monopole or dipole charge density on surface Γ, given the Green's function $G_i(r; r')$.

1.3 Two region SLIC/PCM Intergal Equation:

$$
\left[\begin{array}{cccc|cccc}
\frac{1}{2}I + K^{a}_{I,a} & -V^{a}_{I,a} & & & & & & \\
\frac{1}{2}I - K^{a}_{II,a} & (\frac{f^{a}_{I,II}}{1+f^{a}_{I,II}})V^{a}_{II,a} & K^{a}_{II,b} & -V^{a}_{II,b} & & & & \\
-K^{b}_{II,a} & (\frac{f^{a}_{I,II}}{1+f^{a}_{I,II}})V^{b}_{II,a} & \frac{1}{2}I + K^{b}_{II,b} & -V^{b}_{II,b} & & & & \\
 & & \frac{1}{2}I - K^{b}_{V,b} & (\frac{d_{I,II}}{d^{II}_{b}})V^{b}_{V,b} & & & -K^{b}_{V,d} & (\frac{d_{III,IV}}{d^{IV}_{d}})V^{b}_{V,d} \\
\hline
 & & & & \frac{1}{2}I + K^{c}_{III,c} & -V^{c}_{III,c} & & \\
 & & & & \frac{1}{2}I - K^{c}_{IV,c} & (\frac{f^{c}_{III,IV}}{1+f_{III,IV}}^{c})V^{c}_{IV,c} & K^{c}_{IV,d} & -V^{c}_{IV,d} \\
 & & & & -K^{d}_{IV,c} & (\frac{f^{c}_{III,IV}}{1+f^{c}_{III,IV}})V^{d}_{IV,c} & \frac{1}{2}I + K^{d}_{IV,d} & -V^{d}_{IV,d} \\
 & & -K^{d}_{V,b} & (\frac{d_{I,II}}{d^{II}_{b}})V^{d}_{V,b} & & & \frac{1}{2}I - K^{d}_{V,d} & (\frac{d_{III,IV}}{d^{IV}_{d}})V^{d}_{V,d}
\end{array}\right]
\begin{bmatrix} \phi_a \\ \frac{\partial \phi_a}{\partial n} \\ \phi_b \\ \frac{\partial \phi_b}{\partial n} \\ \phi_c \\ \frac{\partial \phi_c}{\partial n} \\ \phi_d \\ \frac{\partial \phi_d}{\partial n} \end{bmatrix}
=
\begin{bmatrix} \sum_{i=1}^{N_a} \frac{q_i}{\epsilon_I} G^{a}_{I,i} \\ 0 \\ 0 \\ 0 \\ \sum_{i=1}^{N_c} \frac{q_i}{\epsilon_{III}} G^{c}_{III,i} \\ 0 \\ 0 \\ 0 \end{bmatrix}
\tag{4}
$$

,

where,

$$
\begin{aligned}
d_{i,k} &= -\frac{q_i^{tot}}{\epsilon_k} \\
d_\Gamma^i &= \int_\Gamma \frac{\partial \phi_i}{\partial n}(r_\Gamma) dA \\
f_{i,k}^\Gamma &= \frac{\epsilon_i}{\epsilon_k - \epsilon_i} - h\big(E_{n,i}^\Gamma\big) \\
h(x) &= \alpha \tanh(\beta x - \gamma) + \mu \\
V_{i,\Gamma}^\Gamma \sigma_{eff,\Gamma} &= V_{i,\Gamma}^\Gamma \frac{\partial \phi_i}{\partial n} - \big(\frac{1}{2} I + K_{i,\Gamma}^\Gamma\big)\phi_i \\
E_{n,i}^\Gamma &= \sum_{m=1}^{N_{q,i}} q_m \Big(-\frac{\partial G_i}{\partial n}\Big) - K'^{\,\Gamma}_{i,\Gamma} \sigma_{eff,\Gamma} \\
K'^{\,s}_{i,\Gamma} \phi_\Gamma &= \int_\Gamma \frac{\partial G_i}{\partial n(r_s)}(r_s; r')\phi_\Gamma(r') dA'
\end{aligned}
$$