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MEMORANDUM

To: Distribution

From: J. F. Chandler

Subject: <u>The light-time correction to partial derivatives in PEP</u>

It is well known among PEP users that instantaneous "Doppler partials" computed in the COMPAR link are all off by roughly a part in 10^4. There is, however, no simple correction to the problem because of the way partials of velocities are stored during the calculations, as we shall see. Indeed, velocities are fundamentally mishandled throughout the COMPAR link.

Consider a moving observer and a point target labeled with the vectors $\vec{o}$ and $\vec{p}$ respectively in an inertial reference frame. For complete generality suppose there is a moving radar transmitter at s. Then a single observation at $\vec{o}$ occurring at time t_1, will have been reflected from the target at time t_2 and emitted at time t_3 with

$$
\begin{aligned}
t_2 &= t_1 - |\vec{o}(t_1)-\vec{p}(t_2)| \\
t_3 &= t_2 - |\vec{s}(t_3)-p(t_2)|
\end{aligned}
\qquad (1)
$$

(We shall use units such that $c = 1$.)

Henceforth, the vectors $\vec{o}$, $\vec{p}$, and $\vec{s}$ will be implicitly tagged with the times indicated in Equation 1 and a velocity (denoted by "•") will represent the derivative with respect to the implied argument, e.g., $\dot{\vec{p}} = d\vec{p}/dt_2$. Then let

$$
\begin{aligned}
\vec{a} &\equiv \vec{o} - \vec{p} \\
\vec{b} &\equiv \vec{s} - \vec{p}
\end{aligned}
\qquad (2)
$$

Here we encounter the difficulty in the PEP treatment of velocities: how should we define $\dot{\vec{a}}$ and $\dot{\vec{b}}$? The present choice is to use the corresponding differences of true velocities, e.g., $\dot{\vec{a}} = \dot{\vec{o}}(t_1)-\dot{\vec{p}}(t_2)$, but that means that $\dot{\vec{a}}$ and $\dot{\vec{b}}$ are not time derivatives of $\vec{a}$ and $\vec{b}$. A more useful choice, the one that we shall develop here, is to define $\vec{a}$ and $\vec{b}$ as functions of t_1 through Equation (1) and let

$$\dot{\vec{a}} \equiv \frac{d\vec{o}(t_1)}{dt_1} - \frac{d\vec{p}(t_2)}{dt_1}$$
$$\dot{\vec{b}} \equiv \frac{d\vec{s}(t_3)}{dt_1} - \frac{d\vec{p}(t_2)}{dt_1} \tag{3}$$

That is,

$$\dot{\vec{a}} = \dot{\vec{o}} - \frac{dt_2}{dt_1}\dot{\vec{p}}$$
$$\dot{\vec{b}} = \frac{dt_2}{dt_1}\left(\frac{dt_3}{dt_2}\dot{\vec{s}}-\dot{\vec{p}}\right) \tag{4}$$

Thus, we need the quantities dt_2/dt_1 and dt_3/dt_2. From Equations (1) and (2) we have

$$\frac{dt_2}{dt_1} = 1 - \frac{da}{dt_1} = 1 - \hat{a}\cdot\frac{d\vec{a}}{dt_1} = 1 - \hat{a}\cdot\left(\dot{\vec{o}}-\frac{dt_2}{dt_1}\dot{\vec{p}}\right) \tag{5}$$

where $a \equiv |\vec{a}|$, so

$$\frac{dt_2}{dt_1} = \frac{1 - \hat{a}\cdot\dot{\vec{o}}}{1 - \hat{a}\cdot\dot{\vec{p}}} \tag{6}$$

and similarly

$$\frac{dt_3}{dt_2} = \frac{1 + \hat{b}\cdot\dot{\vec{p}}}{1 + \hat{b}\cdot\dot{\vec{s}}} \tag{7}$$

The purely classical total time delay is just the sum of the two distances, $\tau = a+b$, and the frequency shift can be found easily by counting the emitted and received cycles. Note: PEP actually computes the general relativistic delay and frequency, but the procedure requires the classical values as first approximations. If N_3 and N_1 represent cumulative counts at $\vec{s}$ and $\vec{o}$ then dN_3/dt_3 and dN_1/dt_1 are the respective frequencies. Conservation of cycles then requires that $dN_3 = dN_1$ for corresponding transmission and reception intervals. Thus,

$$\frac{\Delta f}{f} = \frac{dt_3}{dt_1} - 1 \tag{8}$$

From this and Equation (1) it is easy to see that

$$\begin{aligned}\frac{\Delta f}{f} &= -\dot{a} - \dot{b} \\ &= -\hat{a}\cdot\dot{\vec{a}} - \hat{b}\cdot\dot{\vec{b}}\end{aligned} \tag{9}$$

Under the presently implemented scheme in PEP, this simple expression can not be used. Instead, PEP uses an expansion to third order in v/c. If necessary, a fourth term could be included for the added accuracy, but this approach is much more cumbersome in dealing with partial derivatives of the Doppler shift. From Equation (8) we can, instead, write

$$\frac{\partial}{\partial\beta}\left(\frac{\Delta f}{f}\right) = \frac{\partial}{\partial\beta}\left(\frac{dt_3}{dt_1}\right) = \frac{dt_3}{dt_2}\,\frac{\partial}{\partial\beta}\left(\frac{dt_2}{dt_1}\right) + \frac{\partial}{\partial\beta}\left(\frac{dt_3}{dt_2}\right)\frac{dt_2}{dt_1} \tag{10}$$

Then, using Equation (6) we get

$$\begin{aligned}\frac{\partial}{\partial\beta}\left(\frac{dt_2}{dt_1}\right) &= \frac{-1}{1-\hat{a}\;\dot{\vec{p}}}\left\{\hat{a}\cdot\frac{\partial\dot{\vec{o}}}{\partial\beta} + \frac{\partial\hat{a}}{\partial\beta}\cdot\dot{\vec{o}} - \frac{dt_2}{dt_1}\left(\hat{a}\cdot\frac{\partial\vec{p}}{\partial\beta} + \frac{\partial\hat{a}}{\partial\beta}\cdot\dot{\vec{p}}\right)\right\} \\ &= \frac{-1}{1-\hat{a}\cdot\dot{\vec{p}}}\left\{\hat{a}\cdot\vec{A} + \frac{\partial\hat{a}}{\partial\beta}\cdot\dot{\vec{a}}\right\}\end{aligned} \tag{11}$$

where $\vec{A} \equiv \frac{\partial\dot{\vec{o}}}{\partial\beta} - \frac{dt_2}{dt_1}\,\frac{\partial\dot{\vec{p}}}{\partial\beta}$. Similarly, from Equation (7) we have

$$\frac{\partial}{\partial\beta}\left(\frac{dt_3}{dt_2}\right) = \frac{-1}{1+\hat{b}\cdot\dot{\vec{s}}}\left\{\hat{b}\cdot\vec{B} + \frac{\partial b}{\partial\beta}\cdot\dot{\vec{b}}\right\}\frac{1}{\frac{dt_2}{dt_1}} \tag{12}$$

where $\vec{B} \equiv \frac{dt_2}{dt_1}\left(\frac{\partial\dot{\vec{s}}}{\partial\beta}\,\frac{dt_3}{dt_2} - \frac{\partial\dot{\vec{p}}}{\partial\beta}\right)$. We can easily show that

$$\frac{\partial\hat{a}}{\partial\beta}\cdot\dot{\vec{a}} = \dot{a}\;\frac{\partial\vec{a}}{\partial\beta} \tag{13}$$

so, putting Equations (11-13) into Equation (10) we get

$$\frac{\partial}{\partial\beta}\left(\frac{\Delta f}{f}\right) = -\left(\frac{\frac{dt_3}{dt_2}}{1-\hat{a}\cdot\dot{\vec{p}}}\left(\hat{a}\cdot\vec{A}+\dot{a}\;\frac{\partial\vec{a}}{\partial\beta}\right) + \frac{1}{1+\hat{b}\cdot\dot{\vec{s}}}\left(\hat{b}\cdot\vec{B}+\dot{b}\cdot\frac{\partial\vec{b}}{\partial\beta}\right)\right) \tag{14}$$

This is equivalent to the present formulation in PEP if all the correction factors are set to unity, that is,

$$\frac{\partial}{\partial\beta}\left(\frac{\Delta f}{f}\right) \approx -\left[\hat{a}\cdot\left[\frac{\partial\dot{\vec{o}}}{\partial\beta} - \frac{\partial\dot{\vec{p}}}{\partial\beta}\right] + \dot{\hat{a}}\cdot\frac{\partial\vec{a}}{\partial\beta} + \hat{b}\cdot\left[\frac{\partial\dot{\vec{s}}}{\partial\beta} - \frac{\partial\dot{\vec{p}}}{\partial\beta}\right] + \dot{\hat{b}}\cdot\frac{\partial\vec{b}}{\partial\beta}\right] \tag{15}$$

The formulations described in this memorandum have been implemented in PEP as far as Equation (7) in the form of a subroutine VLRTRD. The necessary correction factors are computed and stored in common block RTRDVL, and $\dot{\vec{a}}$ and $\dot{\vec{b}}$ are formed and stored in the XSITEP arrays. VLRTRD should be called (after the unit vectors have been set up) by any subroutine that uses velocities in the calculation of observables. At present, only the TRNSIT link uses the feature, but the RADAR, OPTIC, and FERMTR links also use velocities.

As a preliminary implementation of Equations (9-15) the new formulation should at first be used only if JCT(67) = 1. It should be clear that the results are not the same, either in the Doppler itself or in the partials. The situation is further complicated by the existence (at present) of two parallel sets of code in the RADAR link chosen by the value of ICT(43). Unless the so called "old code" vanishes soon, it must be changed either to incorporate the revised Doppler calculations or to bypass them gracefully. The next step in implementation must be the proper calculation of $\vec{A}$ and $\vec{B}$ (stored in DEREM) using the correction factors left by VLRTRD. Eventually, the old formulation can be removed.

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