Center for Astrophysics

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MEMORANDUM

To: Distribution 9 April 1987 87-03

From: R. D. Reasenberg

Subject: De Sitter Precession of the Earth-Moon Gyroscope

Bertotti, Ciufolini, and Bender (Phys. Rev. Lett., 58, 1062, 1987) discuss the de Sitter (or geodetic) precession of the lunar orbit and claim that present data and analyses have confirmed this effect to 10%. Our initial studies with PEP dispute this claim. However, we have used an approximate description of the effect to calculate the elements of the sensitivity matrix, $A = \partial z / \partial \alpha$ where z is the vector of observations and α is the vector of parameters: we take the effect as causing the lunar orbit, as it was integrated by PEP, to precess at the 19.2 mas/y rate of the de Sitter precession. The actual lunar ephemeris contains no such approximation. Bender (private communication) notes that our approach ties the solar perturbation signature into the precession which is clearly not correct. Less clear is how serious this approximation (error?) is for our analysis. The purpose of this memorandum is to describe a simple and accurate approach to a numerically integrable set of variational equations for obtaining A.

We consider two sets of inertial coordinates:

1) X applicable throughout the solar system (solar-system frame)

2) χ applicable in the vicinity of the Earth-Moon system (local frame).

These are connected by a rotation

$$X = R\chi \tag{1}$$

$$\dot{X} = \dot{R}\chi + R\dot{\chi} \tag{2}$$

where we are free to make $R(t_i) = I$, I is the identity matrix, and t_i is the epoch of the present step of the numerical integration of the variational equations.

In the local frame, the laws of motion for the Moon with respect to the Earth are simple, i.e., this is a locally inertial coordinate system. The lunar equations of motion take the form

$$\chi = \int\int \ddot{\chi}(\chi, \alpha, t) \tag{3}$$

where the integrations are with respect to time. However, in PEP we integrate in the solar-system frame

$$X = \int\int \ddot{X}(X, \alpha, t) = \int \dot{X} \ . \tag{4}$$

where the functional form of the integrand differs (slightly) from that of Eq. (3). By inserting Eq. (2) in Eq. (4) we obtain

$$X = \int \dot{R}\chi + \int R \int \ddot{\chi} \tag{5}$$

which is the starting point for investigating the variational equations. If δ is an ad hoc coefficient of the de Sitter precession (and thus one of the α), then

$$\begin{aligned} \frac{\partial X}{\partial \delta} &= \int \frac{\partial \dot{R}}{\partial \delta}\chi + \int \dot{R}\, \frac{\partial \chi}{\partial \delta} + \int \frac{\partial R}{\partial \delta} \int \ddot{\chi} \\ &+ \int R \int \frac{\partial \ddot{\chi}}{\partial \delta} + \int R \int \frac{\partial \ddot{\chi}}{\partial \chi}\, \frac{\partial \chi}{\partial \delta} \ . \end{aligned} \tag{7}$$

At every step of the integration, we may choose a new set of local coordinates χ differing from the global coordinates X only by a translation: at the epoch of the integration step, R=I and $\partial R/\partial\delta = 0$. However, at that epoch, the velocities of a point in the two systems differ since $\dot{R} \neq 0$. Thus, the third term on the right of Eq. (7) is zero. Since the "de Sitter force" does not exist in the local frame, there can be no direct dependence of $\ddot{\chi}$ on δ and $\partial\ddot{\chi}/\partial\delta = 0$. The remaining three terms are real

$$\frac{\partial X}{\partial \delta} = \int \frac{\partial \dot{R}}{\partial \delta} \chi + \int \dot{R} \; \frac{\partial \chi}{\partial \delta} + \iint \frac{\partial \ddot{\chi}}{\partial \chi} \; \frac{\partial \chi}{\partial \delta} \; . \tag{8}$$

We will return to this equation later and show that the second term is too small to consider.

The rate of the de Sitter precession is given by Misner, Thorne, and Wheeler (Gravitation, 1973, p. 1119)

$$\Omega = \frac{3}{2} \; V_B \times \nabla U_B \tag{9}$$

where V_B is the velocity of the Earth-Moon center of mass (EMB) with respect to the Sun, U_B is the potential at the EMB due to bodies other than the Earth and Moon, and "natural units" are assumed. To obtain a numerical value for $|\Omega|$ we note that $V_B \approx 10^{-4}C$ and ∇U_B is dominated by the solar contribution of about $10^{-4}/(\mathrm{year}/2\pi)$: $|\Omega| \approx 3\pi 10^{-8} y^{-1} = 9.4 \times 10^{-8} y^{-1} = 1.94 \times 10^{-2}$ arcsec $y^{-1} = 2.57 \times 10^{-10} d^{-1} = 3.2 \times 10^{-11} (d/8)^{-1}$.

We next construct the matrix $\dot{R}$. For a small rotation vector θ, the corresponding matrix is

$$R(\theta) = \begin{pmatrix} 1 & -\theta_3 & \theta_2 \\ \theta_3 & 1 & -\theta_1 \\ -\theta_2 & \theta_1 & 1 \end{pmatrix} \tag{10}$$

where terms of order $\theta_i \theta_j$ are neglected. If $\dot{\theta} = \delta\Omega$, then

$$\dot{R} = \delta \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix} \tag{11}$$

and

$$\frac{\partial \dot{R}}{\partial \delta} = \frac{1}{\delta} \dot{R} \; . \tag{12}$$

In integrating Eq. (8), as is the case for all non-IC variational equations,

$$\left. \frac{\partial \chi}{\partial \delta} \right|_{start} = 0 \tag{13}$$

The driving term (first term) of Eq. (8) is of order $3\times10^{-10}a$ per day where a is the semimajor axis of the lunar orbit. Integrating just this term for 20 years yields $1.9\times10^{-6}a$. Thus, the second term in Eq. (8) is six orders smaller than the first and can be neglected. In the third term, the gravity gradient matrix is of order $2\omega^2$, where $\omega=2\pi/P$ and P is the lunar orbital period. Thus, this term is significant (as always).

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