Center for Astrophysics

Harvard College Observatory
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MEMORANDUM

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Subject: A Test of General Relativity Through the Geodesic Precession of the Moon's Orbit

One consequence of General Relativity is that the orbital plane of a planetary satellite precesses about the plane of the planet's orbit. This effect, known as geodesic precession, was pointed out by de Sitter in 1916, and the salient feature is that the precession rate does not depend on the satellite's semi-major axis, but only on the planet's. In the case of Earth satellites (such as the Moon), the rate is about 2 arcseconds per century (see, for example, Slade 1973). There is also, of course, an advance of perigee analogous to Mercury's perihelion advance, but the latter effect is orders of magnitude smaller because it depends on Earth's mass rather than the Sun's. Naturally, the geodesic precession is not separately observable, but only as an excess over the purely Newtonian effects of the Sun, Earth, and planets. Indeed, the Moon's node makes a full cycle in 18.6 years, which means the total precession rate is about 7×10^{6} arcseconds per century. Nevertheless, detecting the excess depends only on making observations precise enough in conjunction with a sufficiently accurate model of the solar system. This memorandum will describe an attempt to isolate the phenomenon of geodesic precession and provide, thereby, an "independent" test of General Relativity.

In the Planetary Ephemeris Program (PEP), the equations of motion are cast in terms of the Newtonian accelerations plus relativistic terms with coefficients composed of β and γ from the usual Parametrized Post-Newtonian (PPN) formalism. There is also an arbitrary scale factor λ multiplying all the relativistic terms, and all three of these parameters are programmed in variational equations so that any of them can be estimated by least-squares analysis of observation residuals. Thus, for PEP, a departure of β or γ from unity is equivalent to a first-order violation of General Relativity, and a departure of λ from unity means a breakdown of the PPN formalism. The purpose of including λ in the parameter set is obvious: to be sensitive to "plausible" departures from the nominal physical model. In the same way, we can profitably add other _ad hoc_ terms to the equations of motion with adjustable scale factors nominally zero, such as an extra precession of the Moon's node. The extra precession rate and its standard deviation would quantify a possible deviation from General Relativity or place bounds on the deviation. The implementation of the _ad hoc_ precession is, of course, arbitrary, but to match as closely as possible the properties of the target phenomenon, the geodesic precession, it must be (a) in the plane of the ecliptic, (b) independent of the Moon's orbital elements, and (c) constant to the extent that Earth's orbit is a fixed size. For convenience (and because Earth's orbit precesses very slowly about the solar-system invariable plane), we may let the _ad hoc_ precession take place in the plane of the mean ecliptic of 1950.0. Similarly, we may ignore variations in the size of Earth's orbit over the time span of available data and simply let the precession rate be constant. Thus, if $\underline{x}$ is a vector in equatorial coordinates, then $T\underline{x}$ is the same vector in ecliptic coordinates, where

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos E & \sin E \\ 0 & -\sin E & \cos E \end{pmatrix} \qquad (1)$$

and E is the mean obliquity of the ecliptic. The _ad hoc_ precession matrix in ecliptic coordinates R* is particularly simple:

$$R^* = \begin{pmatrix} \cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2}$$

where ω is the constant <u>ad</u> <u>hoc</u> rate and t is the time elapsed from some reference epoch. From Equations (1) and (2) we get the precession matrix in equatorial coordinates

$$R = T^+R^*T = \begin{pmatrix} \cos\omega t & -\sin\omega t\cos E & -\sin\omega t\sin E \\ \sin\omega t\cos E & \cos\omega t\cos^2 E+\sin^2 E & (\cos\omega t-1)\sin E\cos E \\ \sin\omega t\sin E & (\cos\omega t-1)\sin E\cos E & \cos\omega t\sin^2 E+\cos^2 E \end{pmatrix} \tag{3}$$

where "+" denotes a transposed matrix. In the case of a rotation matrix (such as T) the transpose is also the inverse. It is useful here to resolve R into symmetric and antisymmetric components by introducing two new matrices depending only on the obliquity.

$$P = \begin{pmatrix} 0 & -\cos E & -\sin E \\ \cos E & 0 & 0 \\ \sin E & 0 & 0 \end{pmatrix} \tag{4}$$

$$Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sin^2 E & -\sin E\cos E \\ 0 & -\sin E\cos E & \cos^2 E \end{pmatrix} \tag{5}$$

Thus, we get

$$R = I\cos\omega t + P\sin\omega t + Q(1-\cos\omega t) \tag{6}$$

where I is the identity matrix. Equation (6) is particularly convenient to implement because the matrices I, P, and Q are all time-independent. In PEP, the matrix R is formed each time the Moon coordinates are requested from the interpolator while P and Q are formed only once.

Turning now to the partial derivatives of the precessed $\underline{x}' = R\underline{x}$, we observe

$$\frac{\partial R}{\partial \omega} = tS \tag{7}$$

$$S \equiv (Q - I)\sin \omega t + P \cos \omega t \tag{8}$$

$$\frac{\partial \underline{x}'}{\partial \omega} = tS\underline{x} \ . \tag{9}$$

Although Equation (9) is, in a sense, the desired result, we must remember that we are now dealing with the precessed coordinates $\underline{x}'$. Thus, we should recast Equation (9) is terms of $\underline{x}'$. With that in mind, we note the following identities

$$\begin{aligned} P^{+} &= -P \\ PP &= Q - I \\ PQ &= 0 \\ QQ &= Q \ . \end{aligned} \tag{10}$$

By substituting $R^{+}\underline{x}'$ for $\underline{x}$ in Equation (9) and making use of Equations (10), we get the simple result

$$\begin{aligned} \frac{\partial \underline{x}'}{\partial \omega} &= tSR^{+}\underline{x}' \\ &= tP\underline{x}' \ . \end{aligned} \tag{11}$$

A similar result obtains for the velocity. Although the Moon's velocity doesn't enter directly into the computation of the observables of interest (Lunar laser ranging), it is instructive to derive the formulas here. We begin with the precessed position vector $\underline{x}'$ and note that its time derivative has two contributions, one from the unprecessed velocity and one from the precession rate itself. Noting that the matrices R and S contain ω and t symmetrically, we see that ω and t can be interchanged in Equation (7). Then making the same substitutions as in Equations (11) we get

$$\dot{\underline{x}}' = R\dot{\underline{x}} + \omega P\underline{x}' \tag{12}$$

$$\dot{\underline{x}} = R^{+}\dot{\underline{x}}' - \omega R^{+}P\underline{x}' \tag{13}$$

Finally, using Equations (7) and (11) for the partial derivatives of R and $\underline{x}'$, we get

$$\frac{\partial \dot{\underline{x}}'}{\partial \omega} = tS\dot{\underline{x}} + P\underline{x}' + \omega tPP\underline{x}' \tag{14}$$

$$\frac{\partial \dot{\underline{x}}'}{\partial \omega} = tP\dot{\underline{x}}' + P\underline{x}' \tag{15}$$

This completes the derivation of the coordinate transformation and its partial derivatives. Inspection of Equations (11) and (15) reveals that the procedure for computing the partial derivatives is rather simple because the matrix P is constant, and the vectors $\underline{x}'$ and $\dot{\underline{x}}'$ will inevitably be available when their partial derivatives are needed. Moreover, with the nominal value of 0 for the precession rate ω, the computation of $\underline{x}'$ and $\dot{\underline{x}}'$ becomes even simpler, since they are then equal to $\underline{x}$ and $\dot{\underline{x}}$, respectively.

The implementation of this <u>ad hoc</u> precession in PEP has been subjected to a number of tests. First of all, the consistency of the partial derivatives has been checked by the usual difference method wherein both observables and partials are calculated at two slightly offset values of the parameter of interest. Provided that within that range the contributions of the third and higher-order derivatives are negligible compared to that of the first, this test gives an accurate check of the consistency of the observable(s) and partials. The results of the test confirm the consistency of the partials at a level ranging from the digital noise limit to 100 times the noise with a relative accuracy of 10^{-5} or better. Additional tests were performed to check the gross behavior of the precession implementation, first, by specifying a huge rate such that the precession amounted to 2π radians in 2000 days and verifying that the effect made a full cycle in that period and, second, by temporarily substituting 0 for the obliquity of the ecliptic within the implementation and verifying that the _ad hoc_ precession acted in the proper sense in right ascension and not at all in declination.

A more detailed test was performed by comparing covariances calculated in PEP with analytically-derived covariances. To derive the analytical form we may take the Moon's orbit to be an ellipse of low eccentricity slowly precessing as in Figure 1. Applying the usual Keplerian description to this orbit, we have $\theta \approx \ell$, $\ell = n(t-t_o)$, $n = (GM/a^3)^{1/2}$, $P = \omega t$, where ℓ is the mean anomaly, t_o is the orbit reference epoch, P the precession angle, and ω the precession rate.

We introduce unit vectors $\hat{r}$ and $\hat{\theta}$ such that $\bar{r} = r\hat{r}$ and $\hat{\theta} = \frac{\partial \hat{r}}{\partial \theta}$. Clearly, the effect of the precession projected onto the orbital plane is purely tangential, and so, neglecting the out-of-plane component, we have

$$\frac{\partial \bar{r}}{\partial \omega} = \hat{\theta} tr \cos i \quad . \tag{16}$$

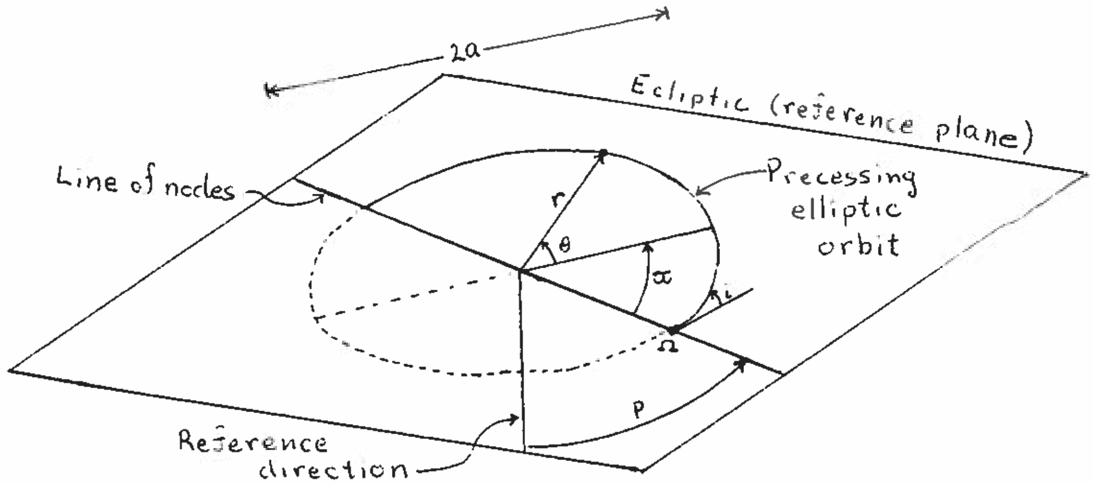

Fig. 1

For the semi-major axis sensitivity (which we may include as a check on the method) we write

$$\frac{\partial \vec{r}}{\partial a} = \frac{\vec{r}}{a} + \frac{\partial \vec{r}}{\partial \theta}\,\frac{\partial \theta}{\partial \ell}\,\frac{\partial \ell}{a} = \frac{\vec{r}}{a} - \frac{3n}{2a}(t - t_\circ)\,\frac{\partial \vec{r}}{\partial \theta}\,\frac{\partial \theta}{\partial \ell}\ . \tag{17}$$

Clearly, to first order in the eccentricity e, we have

$$r \approx a\,(1 - e\cos\theta)$$

$$\frac{\partial \vec{r}}{\partial \theta} = r\,\frac{\partial \hat{r}}{\partial \theta} + \frac{\partial r}{\partial \theta}\hat{r} \approx r\hat{\theta} + ae\sin\theta\,\hat{r} \tag{18}$$

$$\frac{\partial \vec{r}}{\partial a} \approx \frac{\vec{r}}{a} - \frac{3}{2}\,\frac{n}{a}(t - t_\circ)\,\frac{\partial \theta}{\partial \ell}(r\hat{\theta} + ae\sin\theta\,\hat{r})\ .$$

Now, introducing an observing site with geocentric position $\vec{s}$ we define $\vec{R} = \vec{r}-\vec{s}$ and note that, to first order in $\frac{s}{r}$, $R = r-\hat{r}\cdot\vec{s}$. We now define a small quantity $\epsilon = -\hat{\theta}\cdot\vec{s}/a$, the observing parallax. Neglecting planetary aberration, we get the round trip delay $D = 2R$ with

$$\frac{\partial D}{\partial \omega} = 2\epsilon at\cos i$$

$$\frac{\partial D}{\partial a} = 2 - 2e\left\{\cos\theta + \frac{3}{2}n\,(t - t_\circ)\frac{\partial \theta}{\partial \ell}\sin\theta\right\} - 3\epsilon n\frac{\partial \theta}{\partial \ell}(t - t_\circ)\ . \tag{19}$$

Now, for a series of observations of the delay D, the normal equations coefficient matrix is given by

$$B = \sum_i \frac{1}{\sigma_i^2} \begin{pmatrix} \left(\frac{\partial D_i}{\partial \omega}\right)^2 & \\ \frac{\partial D_i}{\partial a}\frac{\partial D_i}{\partial \omega} & \left(\frac{\partial D_i}{\partial a}\right)^2 \end{pmatrix} \tag{20}$$

where σ_i represents the uncertainty of the ith observation. The summation in Equation (20) can be approximated by an integral.

$$B = \frac{1}{\sigma^2 \Delta t} \int_{t_1}^{t_2} \begin{pmatrix} \left(\frac{\partial D_i}{\partial \omega}\right)^2 & \\ \frac{\partial D_i}{\partial a}\frac{\partial D_i}{\partial \omega} & \left(\frac{\partial D_i}{\partial a}\right)^2 \end{pmatrix} dt \tag{21}$$

where σ and Δt are the (constant or mean) uncertainty and interval between observations. Dropping terms with odd powers of ϵ and high powers of e, we get

$$\begin{aligned} \sigma^2 \Delta t B_{11} &= \frac{4}{3}\epsilon^2 a^2 \cos^2 i\left[t^3\right] \\ \sigma^2 \Delta t B_{12} &= 2\epsilon a \cos i\left[t^2\right] - 6n\epsilon^2 a\left[\frac{t^3}{3} - \frac{t_o t^2}{2}\right]\cos i \\ \sigma^2 \Delta t B_{22} &= 4\left[t\right] + 3\left(\epsilon^2 + \frac{e^2}{2}\right)n^2\left[\left(t - t_o\right)^3\right] \end{aligned} \tag{22}$$

where the square brackets denote increment from t_1 to t_2. The results of these computations are summarized in Table 1 along with the corresponding solutions from PEP. The non-geocentric PEP solutions are based on observations with varying parallax and so are presented along with theoretical results for bracketing constant values $\epsilon = 1/60$ and $\epsilon = 1/600$. The theoretical uncertainties for the precession rate ω do, in fact, bracket the corresponding PEP results although the semi-major axis uncertainties disagree by about 16%.

Table 1. Comparison of parameter sensitivities between PEP solutions and analytic model for Moon range observation series

Source	t_1 (d)	t_2 (d)	Δt (d)	σ_ω ($10^{-13}\frac{rad}{d}$)	σ_a (10^{-16} AU)	$\lvert\epsilon\rvert$
LLR	469	4850	1.43	2.8	3.2	$\leq$ 1/60
theory	"	"	"	1.5	3.8	1/60
theory	"	"	"	15.0	3.8	1/600
dummy	400	1900	8.29	56.8	55.2	$\leq$ 1/60
theory	"	"	"	15.0	46.0	1/60
theory	"	"	"	140.0	47.0	1/600
dummy	"	"	"	----	59.5	0
theory	"	"	"	----	48.0	0

All dates are relative to JD 2440001. The lunar reference orbit is t_0 = 400, a = 1/400 AU, e = 0.074, i = 5°, n = 0.23 rad/d, and all observations are assumed to have an uncertainty of 1 ns.

The results of solutions in PEP with the whole available data set are disappointing for their lack of consistency. Table 2 summarizes these results.

Table 2. Solutions for <u>ad</u> <u>hoc</u> lunar precession in 10^{-11} rad/d.

Data/parameters	Solving for Earth precession	Not solving for Earth precession
radio without prec + inner optical/ I.C.'s to Mars	-4.3 ± 0.5	-4.7 ± 0.5
radio with prec/ I.C.'s to Mars	2.7 ± 0.7	-4.5 ± 0.5
radio without prec + inner optical/ I.C.'s to Uranus	-3.3 ± 0.5	-3.4 ± 0.5
radio with prec/ I.C.'s to Uranus	2.9 ± 0.7	-3.1 ± 0.5

Some data sets include optical data for Sun, Mercury, and Venus ("inner optical"); some include a combination of LLR, planetary radar, and spacecraft tracking, all with Earth precession partials ("radio with prec") or without ("radio without prec"). The parameter sets included either initial conditions for Mercury through Mars or Mercury through Uranus in addition to the "usual" set.

References

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