Center for Astrophysics

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MEMORANDUM

To: Distribution

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REV-01

Subject: Optical path through the imperfect corner cube

This is a calculation done using MathCad (a math and symbolic algebra environment for the IBM-PC) to double-check the work done in TM90-07 (which was done the hard way, using MILO). That memo gives the deflection of the beam emerging from a misaligned corner cube (not all of the faces perpendicular), to first order in the misalignment angles ε_i of the corner cube, for any plausible input ray direction (*i.e.* within the corner cube aperture). In that memo, the three angles ε_i between pairs of mirror surface normals are assembled into a vector $\vec{\varepsilon}$, which permits a simpler presentation of the result.

This memo also contains a calculation of the optical path length travelled through the misaligned corner cube, again to first order in the corner cube misalignment and again for any plausible input ray direction. I then demonstrate that the result is equal to the sum of (1) the distance to a plane which is perpendicular to the input ray and which contains the corner cube apex, and (2) the distance from a plane which is perpendicular to the deflected output ray and which contains the corner cube apex. The equivalence holds for all plausible directions of the input ray, to first order in the misalignment angles ε_i.

<u>Distribution</u>

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Optical path through the imperfect corner cube
Double-checking the deflection calc of TM90-07

$$k_0 = \begin{pmatrix} k_x \\ k_y \\ k_z \end{pmatrix} \qquad n_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad n_2 = \begin{pmatrix} \varepsilon_3 \\ 1 \\ 0 \end{pmatrix} \qquad n_3 = \begin{pmatrix} -\varepsilon_2 \\ \varepsilon_1 \\ 1 \end{pmatrix}$$

Input k vector (unit vector, all components <0 so that it points toward the apex) and unit normals of the three mirror faces

The three mirror surfaces are defined as the set of points R_i that satisfy

$$R_i \cdot n_i = 0$$

NB: this implies that the apex is at the origin.

Mirror reflection operator for normal vec n.1

$$B_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - 2 \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \vec{k} - 2 \cdot \vec{n_1} \cdot \left(\vec{n_1} \cdot \vec{k} \right) = B_1 \cdot \vec{k}$$

Mirror reflection operator for normal vec n.2

$$B_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - 2 \cdot \begin{pmatrix} \varepsilon_3^2 & \varepsilon_3 & 0 \\ \varepsilon_3 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -2\cdot\varepsilon_3 & 0 \\ -2\cdot\varepsilon_3 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + O\left(\varepsilon_3^2\right)$$

Mirror reflection operator for normal vec n.3

$$B_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - 2 \cdot \begin{bmatrix} \varepsilon_2^2 & -\varepsilon_2\cdot\varepsilon_1 & -\varepsilon_2 \\ -\varepsilon_2\cdot\varepsilon_1 & \varepsilon_1^2 & \varepsilon_1 \\ -\varepsilon_2 & \varepsilon_1 & 1 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 2\cdot\varepsilon_2 \\ 0 & 1 & -2\cdot\varepsilon_1 \\ 2\cdot\varepsilon_2 & -2\cdot\varepsilon_1 & -1 \end{pmatrix} + O\left(\varepsilon^2\right)$$

For an "n.1 -then- n.2 -then- n.3" path through the corner cube, the total operator is

$$\begin{pmatrix} 1 & 0 & 2\cdot\varepsilon_2 \\ 0 & 1 & -2\cdot\varepsilon_1 \\ 2\cdot\varepsilon_2 & -2\cdot\varepsilon_1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -2\cdot\varepsilon_3 & 0 \\ -2\cdot\varepsilon_3 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -2\cdot\varepsilon_3 & 2\cdot\varepsilon_2 \\ 2\cdot\varepsilon_3 & -1 & -2\cdot\varepsilon_1 \\ -2\cdot\varepsilon_2 & 2\cdot\varepsilon_1 & -1 \end{pmatrix} + O\left(\varepsilon^2\right)$$

Let's trace the ray through and calc optical path. At start:

$$r_1 = \begin{pmatrix} 0 \\ y_1 \\ z_1 \end{pmatrix} \qquad k_0 = \begin{pmatrix} k_x \\ k_y \\ k_z \end{pmatrix}$$

First bounce:

$$k_1 = B_1 \cdot k_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} k_x \\ k_y \\ k_z \end{pmatrix} = \begin{pmatrix} -k_x \\ k_y \\ k_z \end{pmatrix}$$

Propagate to 2nd bounce: for some L.1, the ray hits the second mirror

$$r_2 = r_1 + L_1 \cdot k_1 \qquad r_2 \cdot n_2 = 0$$

The solution is $L_1 = \dfrac{-y_1}{k_y - k_x \cdot \varepsilon_3}$ and

$$r_2 = \begin{pmatrix} 0 \\ y_1 \\ z_1 \end{pmatrix} + \frac{-y_1}{k_y - k_x \cdot \varepsilon_3} \cdot \begin{pmatrix} -k_x \\ k_y \\ k_z \end{pmatrix}$$

Second bounce

$$k_2 = B_2 \cdot k_1 = \begin{pmatrix} 1 & -2\cdot\varepsilon_3 & 0 \\ -2\cdot\varepsilon_3 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -k_x \\ k_y \\ k_z \end{pmatrix} = \begin{pmatrix} -k_x - 2\cdot\varepsilon_3\cdot k_y \\ -k_y + 2\cdot\varepsilon_3\cdot k_x \\ k_z \end{pmatrix}$$

Propagate to 3rd bounce

$r_3 = r_2 + L_2 \cdot k_2 \qquad r_3 \cdot n_3 = 0$

$$L_2 = \left(\frac{-k_y}{k_z^2}\cdot\varepsilon_1 + \frac{k_x}{k_z^2}\cdot\varepsilon_2 - \frac{1}{k_z}\right)\cdot z_1 + \left(\frac{1}{k_y} + \frac{1}{k_y^2}\cdot k_x\cdot\varepsilon_3 + \frac{1}{k_z}\cdot\varepsilon_1\right)\cdot y_1 + O\left(\varepsilon^2\right)$$

$$k_3 = B_3 \cdot k_2 = \begin{pmatrix} -k_x - 2\cdot\varepsilon_3\cdot k_y + 2\cdot\varepsilon_2\cdot k_z \\ -k_y + 2\cdot k_x\cdot\varepsilon_3 - 2\cdot\varepsilon_1\cdot k_z \\ -k_z + 2\cdot\varepsilon_1\cdot k_y - 2\cdot\varepsilon_2\cdot k_x \end{pmatrix}$$

$$r_3 = \begin{bmatrix} \frac{z_1}{k_z}\cdot k_x + 2\cdot\frac{z_1}{k_z}\cdot\varepsilon_3\cdot k_y - k_x\cdot\frac{y_1}{k_z}\cdot\varepsilon_1 - 2\cdot y_1\cdot\varepsilon_3 + k_x\cdot\frac{z_1}{k_z^2}\cdot\varepsilon_1\cdot k_y - \frac{z_1}{k_z^2}\cdot\varepsilon_2\cdot k_x^2 \\ -y_1 + \frac{z_1}{k_z}\cdot k_y - 2\cdot\frac{z_1}{k_z}\cdot k_x\cdot\varepsilon_3 - k_y\cdot\frac{y_1}{k_z}\cdot\varepsilon_1 + \frac{z_1}{k_z^2}\cdot\varepsilon_1\cdot k_y^2 - k_y\cdot\frac{z_1}{k_z^2}\cdot\varepsilon_2\cdot k_x \\ y_1\cdot\varepsilon_1 - \frac{z_1}{k_z}\cdot\varepsilon_1\cdot k_y + \frac{z_1}{k_z}\cdot\varepsilon_2\cdot k_x \end{bmatrix}$$

So the total path within the corner cube (between the first and third bounces) is

$$L_1 + L_2 = \frac{-z_1}{k_z} - z_1\cdot\frac{\left(\varepsilon_1\cdot k_y - \varepsilon_2\cdot k_x\right)}{k_z^2} + \varepsilon_1\cdot\frac{y_1}{k_z} + O\left(\varepsilon^2\right)$$

Now let's try a geometrical construction to arrive at this same answer. Define plane A as the plane normal to the input k.0 vector and passing through the apex (origin):

$R_A \cdot k_0 = 0$

and likewise for plane B and the outgoing k.3 vector

$R_B \cdot k_3 = 0$

The distance from the first bounce to plane A is

$-r_1 \cdot k_0 = -y_1 \cdot k_y - z_1 \cdot k_z$ because kx, ky, kz < 0, this is positive

The distance from the last bounce to plane B is

$r_3 \cdot k_3 =$

small pieces of r.3 times big part of k.3:

$$\begin{bmatrix} 2\cdot\frac{z_1}{k_z}\cdot\varepsilon_3\cdot k_y - k_x\cdot\frac{y_1}{k_z}\cdot\varepsilon_1 - 2\cdot y_1\cdot\varepsilon_3 + k_x\cdot\frac{z_1}{k_z^2}\cdot\varepsilon_1\cdot k_y - \frac{z_1}{k_z^2}\cdot\varepsilon_2\cdot k_x^2 \\ -2\cdot\frac{z_1}{k_z}\cdot k_x\cdot\varepsilon_3 - k_y\cdot\frac{y_1}{k_z}\cdot\varepsilon_1 + \frac{z_1}{k_z^2}\cdot\varepsilon_1\cdot k_y^2 - k_y\cdot\frac{z_1}{k_z^2}\cdot\varepsilon_2\cdot k_x \\ y_1\cdot\varepsilon_1 - \frac{z_1}{k_z}\cdot\varepsilon_1\cdot k_y + \frac{z_1}{k_z}\cdot\varepsilon_2\cdot k_x \end{bmatrix} \cdot \begin{pmatrix} -k_x \\ -k_y \\ -k_z \end{pmatrix}$$

plus the big pieces of r.3 times the big AND small parts of k.3

$$\begin{bmatrix} \frac{z_1}{k_z}\cdot k_x \\ -y_1 + \frac{z_1}{k_z}\cdot k_y \\ 0 \end{bmatrix} \cdot \begin{pmatrix} -k_x - 2\cdot\varepsilon_3\cdot k_y + 2\cdot\varepsilon_2\cdot k_z \\ -k_y + 2\cdot k_x\cdot\varepsilon_3 - 2\cdot\varepsilon_1\cdot k_z \\ -k_z + 2\cdot\varepsilon_1\cdot k_y - 2\cdot\varepsilon_2\cdot k_x \end{pmatrix}$$

gives

$$r_3\cdot k_3 = \left(\frac{1}{k_z}\cdot\varepsilon_1 + k_y\right)\cdot y_1 + \left[\frac{k_x}{k_z^2}\cdot\varepsilon_2 - \frac{k_y}{k_z^2}\cdot\varepsilon_1 - \frac{1}{k_z}\cdot\left(k_x^2 + k_y^2\right)\right]\cdot z_1 + O\left(\varepsilon^2\right)$$

The sum of these two distances is

$$r_3\cdot k_3 - r_1\cdot k_0 = \frac{-z_1}{k_z} + \frac{\left(\varepsilon_2\cdot k_x - \varepsilon_1\cdot k_y\right)}{k_z^2}\cdot z_1 + \frac{y_1}{k_z}\cdot\varepsilon_1$$

the path through the corner cube was

$$L_1 + L_2 = \frac{-z_1}{k_z} - z_1\cdot\frac{\left(\varepsilon_1\cdot k_y - \varepsilon_2\cdot k_x\right)}{k_z^2} + \varepsilon_1\cdot\frac{y_1}{k_z} + O\left(\varepsilon^2\right)$$

Thus the path thru a perfect corner cube is the distance to plane A and back; but in an imperfect corner cube, we must adjust the reference plane for the returning beam to account for the deflection in order to get the correct distance. This gives a better idea how to conceptualize what the measured distance is in a gauge with an imperfect corner cube.