Center for Astrophysics

Harvard College Observatory
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MEMORANDUM

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Subject: Restoration of Prereduced Normal Equations

The technique of partial prereduction is nothing more than a numerical shortcut in the handling of large parameter sets (see reference 1). Indeed, the staged processing is fully equivalent to the usual direct inversion technique, except that some of the work is skipped at the expense of some of the (less interesting) results, particularly the estimates and variances of the reduced parameters. If the intermediate quantities in the prereduction are saved, however, the full results can still be obtained by filling in the missing calculations. This memorandum will lay out the steps for recovering some of the suppressed results. In particular, it is shown that the variances of the reduced parameters are easily obtained and that the method creates neither a computational nor a storage burden.

By the notation of reference 1, we have the normal equations partitioned between two subsets of the parameters. The equation

$$BX = U \tag{1}$$

becomes

$$\begin{pmatrix} C & F \\ F^T & D \end{pmatrix} \begin{pmatrix} Y \\ Z \end{pmatrix} = \begin{pmatrix} V \\ W \end{pmatrix} \tag{2}$$

where the superscript T denotes a transposed matrix and where Y represents the "interesting" and Z the "uninteresting" parameter estimates. The reduction consists of obtaining the equations

$$\overline{C}Y = \overline{V} \tag{3}$$

where

$$\begin{aligned} \overline{C} &= C - FD^{-1}F^T \\ \overline{V} &= V - FD^{-1}W \end{aligned} \tag{4}$$

Equations (1-4) here correspond to equations (1-5) of reference 1. References 2 and 3 cover the prediction of postfit statistics in the presence of *a priori* constraints and partial prereduction, and that development need not be repeated here. The predictions of the postfit RMS were never, in fact, suppressed, since the corrections due to prereduction were described in reference 1. However, obtaining the postfit RMS of the data alone, excluding the *a priori* information, requires extra corrections as shown in the references 2 and 3.

It can be shown that inverting $\overline{C}$ gives the same covariances as would be obtained by solving (1) directly. Thus, by virtue of inverting D, we have effectively removed the corresponding parameters from further processing and saved about half the necessary computation if the Y/Z partition is about even. The savings are even greater if the reduced equations (3) are to be combined with others or used in multiple-parameter-set experiments. We can, however, reverse the process. Again, in the notation of equations (6) and (7) of reference 1, we can define

$$\bar{Z} = D^{-1}W$$

$$\bar{F} = FD^{-1} \tag{5}$$

whence

$$Z = \bar{Z} - \bar{F}^T Y \tag{6}$$

Note that matrices B, C, D, $\bar{C}$, and their inverses are all symmetric and, thus, equal to their own transposes.

We can do more. Returning to Equations (1) and (2), let us write

$$B^{-1} = \begin{pmatrix} \tilde{C} & \tilde{F} \\ \tilde{F}^T & \tilde{D} \end{pmatrix} \tag{7}$$

By the definition of matrix inversion, we have four equations from multiplying B by B^{-1} componentwise.

$$\tilde{C}C = I - \tilde{F}F^T$$

$$\tilde{C}F = -\tilde{F}D \tag{8}$$

$$\tilde{F}^T C = -\tilde{D}F^T$$

$$\tilde{D}D = I - \tilde{F}^T F$$

where I is the identity matrix (consisting of ones on the diagonal). From these equations and equation (4) it can easily be shown, as previously noted, that $\tilde{C}$ is the inverse of $\overline{C}$, that is,

$$\begin{aligned} \tilde{C}\overline{C} &= \tilde{C}C - \tilde{C}FD^{-1}F^T \\ &= \tilde{C}C + \tilde{F}DD^{-1}F^T \\ &= \tilde{C}C + \tilde{F}F^T \\ &= I \end{aligned} \tag{9}$$

Further, from (8) we get

$$\begin{aligned} \tilde{D} &= D^{-1} - \tilde{F}^T F D^{-1} \\ &= D^{-1} + D^{-1}F^T\tilde{C}FD^{-1} \\ &= D^{-1} + \overline{F}^T\overline{C}^{-1}\overline{F} \end{aligned} \tag{10}$$

Also, from (8) we get

$$\begin{aligned} \tilde{F} &= -\,\tilde{C}FD^{-1} \\ &= -\,\overline{C}^{-1}\overline{F} \end{aligned} \tag{11}$$

Equations (9-11) provide the entirety of B^{-1} and, thereby, the covariances, formal uncertainties, and correlations for all parameters. In principle, a comparison of the formal uncertainties with the diagonal of B would also yield the parameter masking factors, but in practice B might not be available all in one place. Indeed, for practical purposes it may be argued that $\widetilde{F}$ and $\widetilde{D}$ are uninteresting and not worth the computation. Moreover, since D^{-1} is not needed except to calculate $\widetilde{D}$, it constitutes a burden in terms of storage alone. However, it is computationally simple to get at least the diagonal elements of $\widetilde{D}$ by, first, saving just the diagonal of D^{-1} and, second, taking a column at a time of F and pre- and post-multiplying $\overline{C}^{-1}$. Recently, the PEP prereduction facility was augmented to save the diagonal of D^{-1} along with $\bar{Z}$ and $\bar{F}$, which had previously been the only quantities saved toward the restoration of "uninteresting" parameters. At the same time, the facility for restoring solutions for the "uninteresting" parameters was modified to take advantage of the new vector, if present, to restore the formal uncertainties.

References

1. Reasenberg, R.D., "Partial Prereduction of the Normal Equations," TM75-3, 1975 Jul 29.

2. Chandler, J.F., "Extension of PEP Normal Equation Error Statistics," unnumbered memorandum, 1978 Jul 17.

3. Chandler, J.F., "Correcting the predicted χ^2 for *a priori* constraints," TM88-06, 1988 Dec 30.

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