Center for Astrophysics

Harvard College Observatory
Smithsonian Astrophysical Observatory

MEMORANDUM

To: Distribution — 1988 December 30 — 88-06

From: J. F. Chandler JFC

Subject: Correcting the predicted χ^2 for *a priori* constraints

It has been shown that the postfit χ^2 may be predicted for the solution of a set of normal equations including arbitrary *a priori* constraints, provided the constraint matrix is available (see Ref. 1). By Equation 12 of Ref. 1, we have

$$S(\hat{x}) = S(0) - \hat{x}^T u + \hat{x}^T(2u^{(a)} - B^{(a)}\hat{x}) \tag{1}$$

where S is the χ^2 function, $\hat{x}$ is the vector of the parameter adjustments from the solution, u is the prefit "right-hand side," B is the coefficient matrix, and the superscript (a) refers to the *a priori* part of the equations. In short, the predicted χ^2 is just the prefit value less the "norm-square" of the solution plus a correction term based on the *a priori* constraint. In the case where the normal equations are prereduced, this simple picture becomes rather more complicated.

The technique of "partial prereduction" (see Ref. 2) involves partitioning the parameter set into "interesting" and "uninteresting" parts. As a result, the normal equations are similarly split, and the uninteresting part can be solved once and, in effect, projected out of the equations. By the partition, we have

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad B = \begin{pmatrix} C & F \\ F^T & D \end{pmatrix} \quad \hat{x} = \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} \tag{2}$$

Accordingly, we have two subsets of the equations

$$u_1 = C\hat{x}_1 + F\hat{x}_2 \tag{3}$$

$$u_2 = F^T\hat{x}_1 + D\hat{x}_2 \tag{4}$$

The two subsets can be combined to eliminate $\hat{x}_2$ by writing

$$\overline{u}_1 = \overline{C}\hat{x}_1 \tag{5}$$

where

$$\begin{aligned} \overline{u}_1 &\equiv u_1 - FD^{-1}u_2 \\ \overline{C} &\equiv C - FD^{-1}F^T \end{aligned} \tag{6}$$

In the absence of *a priori* constraints, the predicted χ^2 is given by

$$S(\hat{x}) = S(0) - \hat{x}_1{}^T u_1 - \hat{x}_2{}^T u_2 = S(0) - \hat{x}_1{}^T \overline{u}_1 - \overline{x}_2{}^T u_2 \tag{7}$$

where $\overline{x}_2 \equiv D^{-1}u_2$. In other words, the predicted χ^2 is calculated in the usual way from the reduced solution and is then corrected by a term obtained in the course of prereduction. If, however, we include the *a priori* correction and eliminate $\hat{x}_2$ as before, we get

$$S(\hat{x}) = S(0) - \hat{x}_1{}^T\overline{u}_1 - \overline{x}_2{}^T u_2 + \hat{x}_1{}^T(2\tilde{u}_1 - \tilde{C}\hat{x}_1) + \overline{x}_2{}^T(2u_2{}^{(a)} - D^{(a)}\overline{x}_2) \tag{8}$$

where

$$\tilde{u}_1 \equiv u_1{}^{(a)} - F^{(a)}\overline{x}_2 - FD^{-1}u_2{}^{(a)} + FD^{-1}D^{(a)}\overline{x}_2 \tag{9}$$

$$\tilde{C} \equiv C^{(a)} - F^{(a)}D^{-1}F^T - FD^{-1}F^{(a)T} + FD^{-1}D^{(a)}D^{-1}F^T \tag{10}$$

Clearly, if the *a priori* constraints involved only the interesting parameters, we would choose to apply them after the prereduction had been finished, and, further, an *a priori* constraint tying an interesting parameter to an uninteresting one is a contradiction in terms. Thus, by the nature of prereduction, the only *a priori* constraints added beforehand must involve only the uninteresting parameters, and $u_1^{(a)}$, $C^{(a)}$, and $F^{(a)}$ must all be zero. Therefore, the first two terms of (9) and the first three of (10) vanish.

Operationally, then, the prediction of χ^2 described by Equation 7 must be enhanced in two ways. First, the correction due to prereduction [the last term in (7)] must be decremented by the last term in (8), and, second, the prereduced *a priori* constraint represented by $\tilde{u}_1$ and $\tilde{C}$ must be included as in (1), along with any directly applied constraints involving the interesting parameters. Forming $\tilde{u}_1$ and $\tilde{C}$ entails the following procedure:

1. Read FD^{-1} into memory (it is formed and saved in the course of prereduction; indeed, it may still be in memory).

   Arrays: B-matrix

2. Read $u_2^{(a)}$ and form $-FD^{-1}u_2^{(a)}$. This will require applying two levels of pointers to map the saved *a priori* $u^{(a)}$ into $u_2^{(a)}$. The first set of pointers is part of the *a priori* dataset, and the second is a byproduct of the prereduction. Also form $\bar{x}_2{}^T u_2^{(a)}$ ($\bar{x}_2$ is still in memory).

   Arrays: buffer, vector 1 for $u_2^{(a)}$, vector 2 for $-FD^{-1}u_2^{(a)}$

3. Read $D^{(a)}$ one row at a time, and form $FD^{-1}D^{(a)}$ one column at a time. The same double mapping is needed as in step 2. Also form $FD^{-1}D^{(a)}\bar{x}_2$ and $\tilde{C}$, combining the former with $FD^{-1}u_2^{(a)}$ to get $\tilde{u}_1$. Note that a buffer large enough for the full normal equations can hold both an F-sized and a C-sized matrix. Further, form one element of $D^{(a)}\bar{x}_2$ at a time and compute $\bar{x}_2{}^T D^{(a)}\bar{x}_2$.

Arrays: buffer, vector 1 for $D^{(a)}$, buffer for $FD^{-1}D^{(a)}$, B-matrix for $\tilde{C}$, vector 2 for $\tilde{u}_1$

4. Adjust the prereduction χ^2 correction.

5. Save $\tilde{u}_1$ and $\tilde{C}$.

Once the mechanism for applying the correction for $\tilde{u}_1$ and $\tilde{C}$ is in place, it can be used equally well for *a priori* information imbedded in ordinary saved normal equations. In particular, when prereduced normal equations are combined and resaved, the imbedded *a priori* information should be propagated.

References

1. Chandler, J. F., "Extension of PEP Normal Equation Error Statistics," TM dated July 17, 1978.

2. Reasenberg, R. D., "Partial Prereduction of the Normal Equations," TM75-3.

Distribution

R. W. Babcock
R. W. King
J. D. Phillips
M. I. Ratner
R. D. Reasenberg
I. I. Shapiro

JFC:nm