Center for Astrophysics

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MEMORANDUM

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From: R.D. Reasenberg

Subject: False alarms in experiments that search for rare events

Introduction

There exist many experiments in which the object is to look for a rare event by repeatedly making a measurement, *i.e.*, looking in multiple channels. Traditionally, a serious problem for such experiments is false alarms. A "3 σ event" may be rare, but if 10,000 measurements are made, one expects to see about 13 of them under the assumption that the errors are Gaussian. The statistical problem is much the same independent of whether one is looking for gravitational waves detected by an aluminum bar or planets around distant stars detected by an astrometric instrument. A further complication comes from the non-Gaussian noise found empirically to corrupt many experiments. This latter complication is not addressed here.

In this memorandum, we consider the question: At what level must we set the detection threshold for the individual measurements in order that the experiment have the required reliability? The answer to this question need not be very precise. Experiment uncertainties are often not known to better than 10%.

Analysis

Let x be a measured quantity which has a zero mean and a Gaussian distribution with a standard deviation of σ. Let P(B) be the probability that x is greater than $B\sigma$ for a given measurement in which the sought-after rare event is absent. We assume that this probability is small. If there are N measurements, then the probability of $x > B\sigma$ in at least one of them is

$$P_N(B) = 1 - (1-P(B))^N \approx NP(B)\left(1-\frac{N-1}{2}P(B)\right. + ... \qquad (1)$$

To an excellent approximation, we are free to keep just the first term of the series since we are considering reliable experiments for which NP(B) is small.

It is convenient to talk about an experiment in terms of confidence limits, which are often translated to the number of standard deviations from the mean that a result represents. When this terminology is used, a Gaussian distribution is implicit. We may determine A, the number of standard deviations that characterizes the entire experiment, according to

$$P(A) = P_N(B) \approx NP(B) \qquad (2)$$

For a Gaussian probability density function (PDF), the cumulative probability function P(A) has an asymptotic approximation

$$P(A) \approx \frac{1}{A\sqrt{2\pi}} e^{-A^2/2} (1-A^{-2} + \ldots) \tag{3}$$

(See Abramowitz and Stegun, page 298.) By combining Equations 2 and 3, dropping small terms, and taking a natural log, we obtain.

$$-\frac{A^2}{2} - \ln A + \ln(1-A^{-2}) = \ln N - \frac{B^2}{2} - \ln(B) + \ln(1 - B^{-2}) \tag{4}$$

and so

$$B^2 = A^2+2\ln N+2\ln(A/B)+2\ln\frac{1-B^{-2}}{1-A^{-2}} \tag{5}$$

We can obtain a useful first approximation to the solution to Equ. 5 by neglecting the terms on the right that contain B.

$$B_1 = \sqrt{A^2+2\ln N} \tag{6}$$

Better approximations can be obtained iteratively as was done in Table 1, which gives some useful examples. However, as can be seen from the table, there is little reason to calculate the corrections to B_1. "Real world" considerations are likely to dwarf the difference between B_1 and B_3, and the remaining corrections are even smaller.

Conclusion

In an experiment with N measurements, an "A standard deviation null result" requires that a false alarm in a single measurement be a "B standard deviation event." A convenient approximation for B exists and, for reliable experiments, this approximation is sufficiently accurate for all but the most demanding purposes.

Table 1

A	N	$2\ln N$	B_1	Δ_1	B_2	Δ_2	B_3
--	---------	--------	------	-------	------	------	------
2	100	9.21	3.63	-1.19	3.47	.42	3.53
2	10000	18.42	4.74	-1.72	4.55	.48	4.60
3	100	9.21	4.27	-0.70	4.18	.12	4.20
3	10000	18.42	5.24	-1.11	5.13	.16	5.14

In the table, we have used the following notation:

$$B_1 = \sqrt{A^2 + 2\ln N}$$
$$B_2 = \sqrt{B_1^2 + \Delta_1}$$
$$B_3 = \sqrt{B_2^2 + \Delta_2}$$

$$\Delta_1 = 2\ln(A/B_1)$$
$$\Delta_2 = 2\ln\left[\frac{1 - B_1^{-2}}{1 - A^{-2}}\right]$$

Reference

"Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables," National Bureau of Standards Applied Mathematics Series 55, M. Abramowitz and I.A. Stegun Editors, May 1968.

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