Project 1 - Game theory

INFO-F409

Julien Baudru - ULB - 000460130

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Contents

1 Introduction

This first project of the Learning Dynamics course focuses mainly on Nash equilibrium, pure strategy, mixed strategy and Bayesian game analysis.

2 Hawks and doves

The hawk-dove game is defined as follows, where the first cell element is the payoff of the column player and the second is the payoff of the row player :

	Hawk	Dove
Hawk	$\frac{V-D}{2}$, $\frac{V-D}{2}$	V, 0
Dove	0, V	$\frac{V}{2} - T$, $\frac{V}{2} - T$

To find the Nash equilibria (**NE**) for this game, we will vary the values to the variables V, D and T. Thus, for each configuration of the game, we will compute the best answers for each player based on the other player's choice. With these results we can define the Nash equilibrium for each of the game variants.

2.1 Computations

The best answers (**B**) for the first player will be marked in blue and the best answers for the second player will be marked in red .

2.1.1 V > D and T > -$\frac{V}{2}$

Player 1 to player 2 : **B(H)** = {H} and **B(D)** = {H}
Player 2 to player 1 : **B(H)** = {H} and **B(D)** = {H}

	Hawk	Dove
Hawk	$\frac{V-D}{2}$, $\frac{V-D}{2}$	V, 0
Dove	0, V	$\frac{V}{2} - T$, $\frac{V}{2} - T$

NE = u(H,H). *Hawk* is the dominant strategy for this configuration.

2.1.2 V > D and T < -$\frac{V}{2}$

Player 1 to player 2 : **B(H)** = {H} and **B(D)** = {D}
Player 2 to player 1 : **B(H)** = {H} and **B(D)** = {D}

	Hawk	Dove
Hawk	$\frac{V-D}{2}$, $\frac{V-D}{2}$	V, 0
Dove	0, V	$\frac{V}{2} - T$, $\frac{V}{2} - T$

NE = u(H,H) and u(D,D). In this configuration each player want to play the same strategy as his opponent.

2.1.3 D > V and T > -$\frac{V}{2}$

Player 1 to player 2 : **B(H)** = {D} and **B(D)** = {H}
Player 2 to player 1 : **B(H)** = {D} and **B(D)** = {H}

	Hawk	Dove
Hawk	$\frac{V-D}{2}$, $\frac{V-D}{2}$	V, 0
Dove	0, V	$\frac{V}{2} - T$, $\frac{V}{2} - T$

NE = u(H,D) and u(D,H). In this configuration each player want to play the opposite strategy to his opponent.

2.1.4 D > V and T < -$\frac{V}{2}$

Player 1 to player 2 : **B(H)** = {D} and **B(D)** = {D}
Player 2 to player 1 : **B(H)** = {D} and **B(D)** = {D}

	Hawk	Dove
Hawk	$\frac{V-D}{2}$, $\frac{V-D}{2}$	V, 0
Dove	0, V	$\frac{V}{2}-T$, $\frac{V}{2}-T$

NE = u(D,D). *Dove* is the dominant strategy for this configuration.

2.2 Mixed Strategy

Let p be the probability that the first player plays *Hawk* and *p-1* the probability that he plays *Dove*, as this game is symmetrical, we must calculate the probabilities for only one of the two players. Thus the payoffs for the pure *Dove* and *Hawk* strategies are :

$U_{Hawk} = p * \frac{V-D}{2} + (1-p) * V$
$U_{Dove} = p * 0 + (1-p) * (\frac{V}{2} - T)$

In order to find the Nash equilibrium for the mixed strategy, we need to make the two payoffs above equal, i.e. solve the equation for p :

$$\begin{aligned}
p * \frac{V-D}{2} + (1-p) * V &= p * 0 + (1-p) * (\frac{V}{2} - T) \\
p * \frac{V-D}{2} + V - (p * V) &= (\frac{V}{2} - T) - p * (\frac{V}{2} - T) \\
V - (\frac{V}{2} - T) &= -p * (\frac{V}{2} - T) + p * (V) - p * (\frac{V-D}{2}) \\
\frac{V}{2} + T &= p * [(\frac{-V}{2} + T) + V + (\frac{-V+D}{2})] \\
\frac{V+2T}{2} &= p * (\frac{D+2T}{2}) \\
p &= \frac{V+2T}{D+2T}
\end{aligned}$$

For arbitrary values 2, 3 and 1 (c.f. Q3) given to the variables V, D and T, we find that p is 0.8 which means that the Nash equilibrium for the mixed strategy will appear when both players play 80% of the time *Hawks* and 20% of the time *Dove* (Because: $1-p = 1-0.8 = 0.2$).

2.3 Displaying or escalating

It becomes more interesting to display when the expected payoff of this strategy is higher than the payoff of the escalating strategy, i.e. $U_{Dove} > U_{Hawk}$. By the same operations as in the first question, we obtain $p = \frac{V+2T}{D+2T}$, so the graph of the different equilibria is the following :

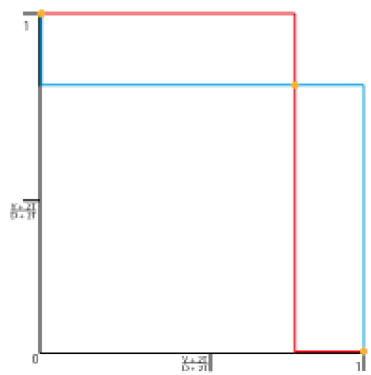

Figure 1: Mixed and pure Nash equilibra

2.4 NashPy

Using the jupyter notebook (with NashPy) provided for this project with the values V = 2, D = 3 and T = 1, we notice that the result returned is identical to that calculated at the end of the first question. The result provided is the following, for the Nash equilibrium for the mixed strategy both players will play Hawks 80% of the time and Dove the rest of the time.

3 Which social dilemma

In this exercise, the information available to the player A are incomplete. He does not know which of the three games he will face, he only knows that each game has the same probability of being presented to him, which is $\frac{1}{3}$. Note that the three games are the prisoner's dilemma, the snowdrift game and the stag-hunt game.

The player B knows which game he is going to play. To know the Nash equilibrium of this exercise we must first find the best answers for each of the two players. However, here as the informations are incomplete for player A, we will build a table for each of the strategy combinations of player B. This table is constructed as follows :

$$u(C, CCC) = \frac{1}{3} * (2 + 5 + 2) = 3 \tag{1}$$
$$u(C, DCC) = \frac{1}{3} * (0 + 5 + 2) = \frac{7}{3} \tag{2}$$
$$u(C, DDC) = \frac{1}{3} * (0 + 0 + 2) = \frac{2}{3} \tag{3}$$
$$u(C, DDD) = \frac{1}{3} * (0 + 0 + 1) = \frac{1}{3} \tag{4}$$
$$u(C, CDD) = \frac{1}{3} * (2 + 0 + 1) = 1 \tag{5}$$
$$u(C, CDC) = \frac{1}{3} * (2 + 0 + 2) = \frac{4}{3} \tag{6}$$
$$u(C, CCD) = \frac{1}{3} * (2 + 5 + 1) = \frac{8}{3} \tag{7}$$
$$u(C, DCD) = \frac{1}{3} * (0 + 5 + 1) = 2 \tag{8}$$
$$u(D, CCC) = \frac{1}{3} * (5 + 2 + 5) = 4 \tag{9}$$
$$u(D, DCC) = \frac{1}{3} * (1 + 2 + 5) = \frac{8}{3} \tag{10}$$
$$u(D, DDC) = \frac{1}{3} * (1 + 1 + 5) = \frac{7}{3} \tag{11}$$
$$u(D, DDD) = \frac{1}{3} * (1 + 1 + 0) = \frac{2}{3} \tag{12}$$
$$u(D, CDD) = \frac{1}{3} * (5 + 1 + 0) = 2 \tag{13}$$
$$u(D, CDC) = \frac{1}{3} * (5 + 1 + 5) = \frac{11}{3} \tag{14}$$
$$u(D, CCD) = \frac{1}{3} * (5 + 2 + 0) = \frac{7}{3} \tag{15}$$
$$u(D, DCD) = \frac{1}{3} * (1 + 2 + 0) = 1 \tag{16}$$

	CCC	DCC	DDC	DDD	CDD	CDC	CCD	DCD
C	3	$\frac{7}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	1	$\frac{4}{3}$	$\frac{8}{3}$	2
D	4	$\frac{8}{3}$	$\frac{7}{3}$	$\frac{2}{3}$	2	$\frac{11}{3}$	$\frac{7}{3}$	1

In the table on the previous page, the best answers of player A in response to the different combinations of player B are noted in blue. Below, for each of the three games, the player B's best answers are marked in red :

	C	D
C	2	5
D	0	1

(a) Prisonners dilemma

	C	D
C	5	2
D	0	1

(b) Stag-Hunt game

	C	D
C	2	5
D	1	0

(c) Snowdrift game

Now that we have the best answers for both players, we will look for the strategies for which both players have their best answers. For example, for *u(D,CCC)*, we need player A to have his best answer in strategy D and player B to have his best answers in each game in strategy C.

Thus, we find that the Nash equilibrium for this exercise lies in the strategies *u(C,DCD)* and *u(D,DDC)*.

4 Sequential truel

4.1 Diagram

The diagram below shows the different possible histories for this extensive game. In this diagram player A starts shooting first. The blue nodes symbolize the chances moves, player x will hit his target with a probability equal to p_x and will miss with a probability of $1 - p_x$. The red node on the right of the diagram represents the situation in which all the players have successively missed their target, so the round is restarted under the same conditions as initially. Finally the green nodes represent the payoff for players A, B and C. If player x survives then his payoff will be 1 otherwise it will be 0.

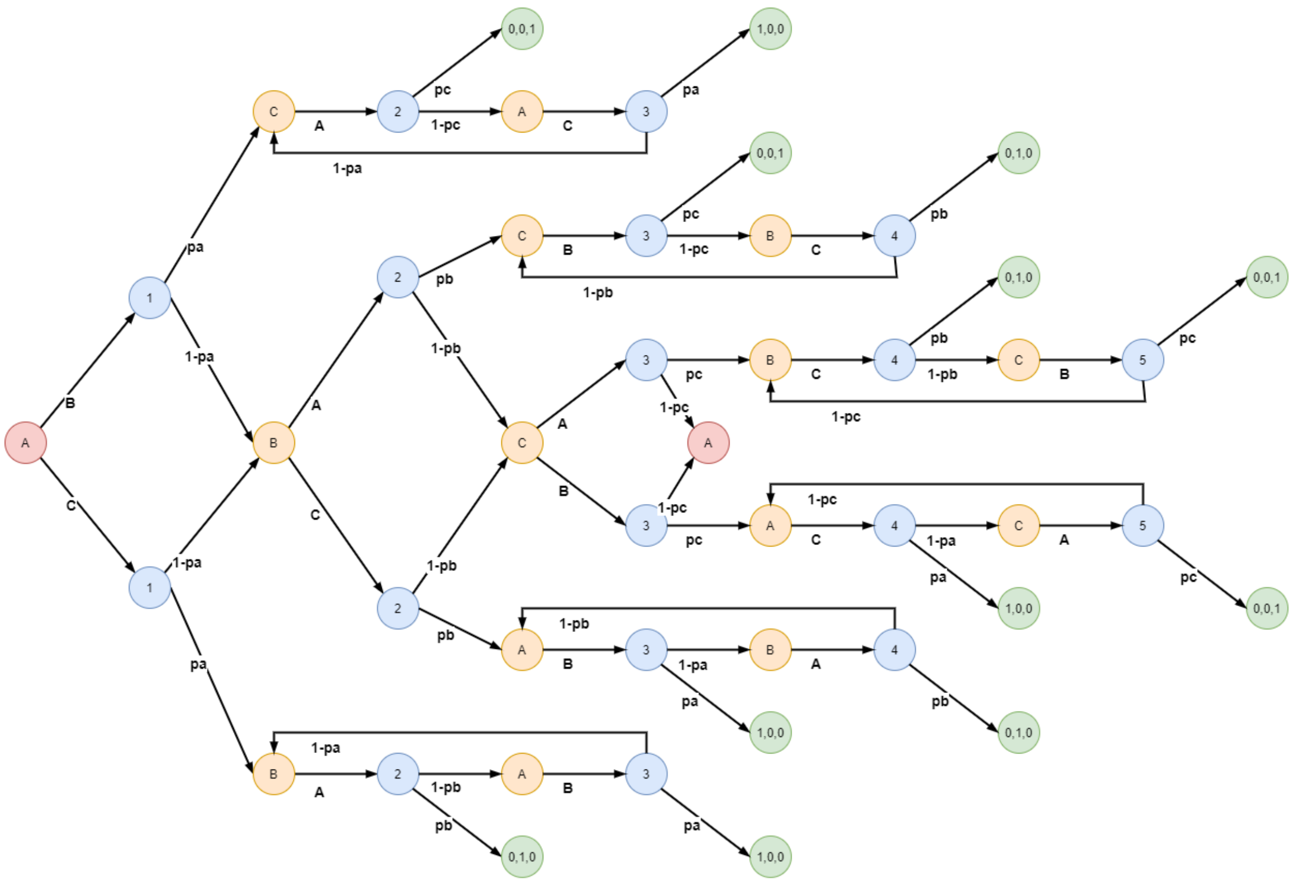

Figure 2: Diagram for the sequential game

We notice that we are here in a type of game with infinitely long stories. Indeed, it is the case for the six loops present in the graph above as well as for the case where each player misses consecutively their target until we come back to the first player.

Using logic, we can see that if $p_c < p_b$ then player C's chances of survival will increase compared to the case where $p_c > p_b$. Indeed, in the first case, player C will tend to choose to shoot first at player B, while in the other case player C will be targeted first. Therefore, the player with the lowest probability of hitting his target will have the best chance of surviving for a long time, so we can say that weakness is strength.

4.2 Subgame perfect equilibria

To find the results below, the use of backward induction is necessary.

4.2.1 $p_a > p_b > p_c$

In this configuration, the **S.P.E.** is reached when A shoots at B and hits its target, then C shoots at A and misses its target and finally A shoots at C and hits it.

$S.P.E = (B, p_a, A, 1 - p_c, C, p_a)$

4.2.2 $p_a > p_c > p_b$

In this configuration, the **S.P.E.** is reached when A shoots at C and hits its target, then B shoots at A and misses its target and finally A shoots at B and hits it.

$S.P.E = (C, p_a, A, 1 - p_b, B, p_a)$

4.2.3 $p_b > p_a > p_c$

In this configuration, the **S.P.E.** is reached when A shoots at B and hits its target, then C shoots at A and misses its target and finally A shoots at C and hits it.

$S.P.E = (B, p_a, A, 1 - p_c, C, p_a)$

4.2.4 $p_b > p_c > p_a$

In this configuration, the **S.P.E.** is reached when A shoots at C and misses its target, then B shoots at C and hits its target and finally A shoots at B and hits it.

$S.P.E = (C, 1 - p_a, C, p_b, B, p_a)$

4.2.5 $p_c > p_a > p_b$

In this configuration, the **S.P.E.** is reached when A shoots at C and hits its target, then B shoots at A and misses its target and finally A shoots at B and hits it.

$S.P.E = (C, p_a, A, 1 - p_b, B, p_a)$

4.2.6 $p_c > p_b > p_a$

In this configuration, the **S.P.E.** is reached when A shoots at C and misses its target, then B shoots at C and hits its target and finally A shoots at B and hits it.

$S.P.E = (C, 1 - p_a, C, p_b, B, p_a)$

4.3 Logic behind A's equilibrium action

Concerning the logic behind player A's actions, we notice that when player A has the lowest probability of hitting his target, it will be preferable for him to miss his initial target (the one with the highest probability) and let the next player with a better chance of hitting it shoot. In other cases, the player A will shoot (successfully) at the player with the highest probability of successfully shooting.