Assignment 1

MATH 239 Fall 2025

Due Thursday September 18 at 11:59pm

Answer the following questions using the multiple choice checkboxes in Crowdmark. No written response or justification required.

A-1-1. [1 mark] What is the degree of the polynomial $\binom{n}{k}$ (as a polynomial in n)?

A-1-2. [1 mark] What proportion of the set of all permutations of $\{1, \ldots, 8\}$ consists of permutations in which every odd number is immediately followed by an even number and every even number is immediately followed by an odd number (except for the last number, which has no number after it)?

A-1-3. [1 mark] What is the number of integer solutions to the equation $x_1 + x_2 + x_3 = 6$, where $x_i \geq 0$?

A-1-4. [1 mark] Is the function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(x) = x + 1$ invertible? (Recall in MATH 239 we define $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.)

Submit written answers to the following questions on Crowdmark. We encourage you to look at the document "How to write solutions" on LEARN (under Content → Course Information) for advice on how to present your solutions. Your written solution to each question should fit in at most 2.5 pages; see the "How to write solutions" document for more details on page length.

A-1-5. **This assignment question will be taken up in the tutorials. You may bring partial solutions to the tutorials to discuss with fellow students. If you collaborated with other students on this question, list the names of the students you worked with in your submission.**

[5 marks] Fix $n \in \mathbb{N}$. Give a bijection between the set $\{(a, b) : 1 \leq a \leq b \leq n \text{ and } a, b \in \mathbb{N}\}$ and the set $\{1, 2, \ldots, n(n+1)/2\}$. Prove that your function is well-defined, injective, and surjective; or equivalently, prove that your function is well-defined and invertible.

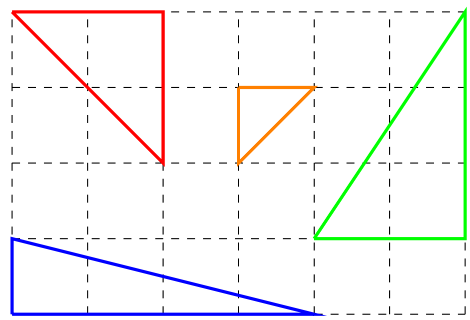
A-1-6. [5 marks] Let n, a be positive integers such that $n \geq a \geq 3$. Give a *combinatorial* proof of the following identity:

$$\binom{n}{a}\binom{a}{1}\binom{a-1}{2} = 3\binom{n}{3}\binom{n-3}{a-3}.$$

In particular, *do not* use factorials here – a solution using factorials is *not* a combinatorial proof and will get 0 marks.

A-1-7. [6 marks] Let $m, n \geq 1$. Consider an $m \times n$ grid divided into 1×1 squares. We want to know how many right-angle triangles can be formed in the grid using grid lines for 2 of the 3 sides.

An example 4×6 grid with 4 possible triangles is drawn here.



Give an expression for the number of such triangles that can be formed in an $m \times n$ grid. By counting the number of triangles in two different ways, prove the following identity:

$$\binom{n+1}{2}\binom{m+1}{2} = \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}(n-i)(m-j).$$