A. Sea $f(n) : n \in [0, .., N-1]$ se define el par transformada-antitransformada discreta de Fourier 1-D:

$$F(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f(n) e^{\frac{-2\pi i n k}{N}}, \qquad k = 0, ..., N-1$$

$$F^{-1}(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} F(k) e^{\frac{2\pi i n k}{N}}, \qquad n = 0, ..., N-1$$

(Nota: F es la DFT y F^{-1} es la $IDFT$)

Demostrar las siguientes propiedades de la DFT:

1. $F^{-1}(F(k)) = f(n)$
2. $F(f * g) = F(f).F(g)$
3. $F(k) = F(k+N)$
4. Si $f(n)$ es real, entonces $F(k) = F^*(-k)$(simétrica conjugada).
5. $|F(k)| = |F(-k)|$
6. $F^*(N-k) = F(k)$
7. $F^*(\frac{N}{2}+k) = F(\frac{N}{2}-k)$ para $k = 0, 1, .., \frac{N}{2}-1$.

A) 1) $F^{-1}(F(k))(n)$

$$= F^{-1}\left(\frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} f(m) e^{\frac{-2\pi i m k}{N}}\right)(n)$$

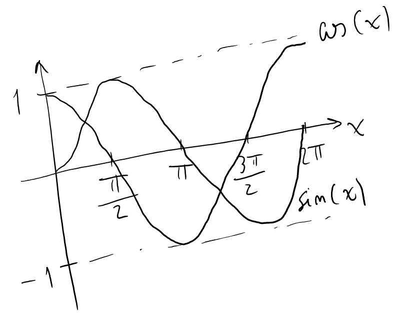
$$= \left(\frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \left(\frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} f(m) e^{\frac{-2\pi i m k}{N}}\right) e^{\frac{2\pi i n k}{N}}\right)(n)$$

$$= \left(\frac{1}{N} \sum_{k,m=0}^{N-1} f(m) e^{\frac{2\pi i n k}{N} - \frac{2\pi i m k}{N}}\right)(n)$$

$$= \left(\frac{1}{N} \sum_{k,m=0}^{N-1} f(m) e^{i\frac{2\pi k}{N}(n-m)}\right)(n)$$

$$= \left(\frac{1}{N} \sum_{k=0}^{N-1} \sum_{m=0}^{N-1} f(m) e^{\frac{i2\pi k}{N}(n-m)}\right)(n)$$

$$= \left(\frac{1}{N} \sum_{m=0}^{N-1} f(m) \sum_{k=0}^{N-1} e^{\frac{i2\pi k}{N}(n-m)}\right)(n) = (*)$$



Even: $f(x) = f(-x)$ Ex: $\cos(x)$

Odd: $-f(x) = f(-x)$ Ex: $\sin(x)$

s

Analicemos qué pasa con la suma interna:

$$\sum_{k=0}^{N-1} e^{\frac{i2\pi k}{N}(n-m)} = \sum_{k=0}^{N-1} \left(e^{i\frac{2\pi(n-m)}{N}}\right)^k$$

Sumamos $\boxed{\sum_{k=0}^{N} a^k = \frac{1-a^{N+1}}{1-a}}$. En este caso: $i2\pi(n-m)$

$\sum_{K=0}$ ...

Si miramos $\boxed{\sum_{K=0}^{N} a^K = \frac{1-a^N}{1-a}}$. En es...

$$\sum_{K=0}^{N-1}\left(e^{\frac{i2\pi(n-m)}{N}}\right)^K = \frac{1-e^{\frac{i2\pi(n-m)}{N}N}}{1-e^{\frac{i2\pi(n-m)}{N}}} = \frac{1-e^{i2\pi(n-m)}}{1-e^{\frac{i2\pi(n-m)}{N}}} = (1)$$

Si tomamos $x=(n-m)$

$$(1) = \frac{1-e^{i2\pi x}}{1-e^{\frac{i2\pi x}{N}}} = \frac{e^{i\pi x}\left(e^{-i\pi x}-e^{i\pi x}\right)}{e^{\frac{i\pi x}{N}}\left(e^{-\frac{i\pi x}{N}}-e^{\frac{i\pi x}{N}}\right)}$$

odd

$$= e^{i\pi x\frac{N-1}{N}}\frac{\left(\cancel{\cos(-\pi x)}+i\sin(-\pi x)\right)-\left(\cancel{\cos(\pi x)}+i\sin(\pi x)\right)}{\left(\cancel{\cos\left(-\frac{\pi x}{N}\right)}+i\sin\left(-\frac{\pi x}{N}\right)\right)-\left(\cancel{\cos\left(\frac{\pi x}{N}\right)}+i\sin\left(\frac{\pi x}{N}\right)\right)} =$$

even

$$= e^{i\pi x\frac{N-1}{N}}\frac{\sin(\pi x)}{\sin\left(\frac{\pi x}{N}\right)}$$

Es decir, si volvemos al reemplazo para atrás:

$$\sum_{K=0}^{N-1} e^{i\frac{2\pi(n-m)}{N}K} = e^{i\pi(n-m)\frac{N-1}{N}}\frac{\sin(\pi(n-m))}{\sin\left(\pi\frac{(n-m)}{N}\right)}$$

Pero observemos que $(n-m)\in\mathbb{Z}$, por lo que $\sin(\pi(n-m))=0$

(obs: tenemos que ver qué pasa cuando $n=m$ en un caso separado)

$$\sum_{K=0}^{N-1} e^{i\frac{2\pi(n-m)K}{N}} = 0 \quad \forall n,m \,/\, n\neq m$$

Si $n=m$, tenemos que $\sum = N$. Por ende

$$\circledast = \frac{1}{N}\sum_{m=0}^{N-1} f(m)\,N\,\delta(n-m) = f(n) \;//$$

2) $F(f*g) = F(f)\cdot F(g)$

$$f*g(n) = \sum_{k=-\infty}^{+\infty} f(k)\, g(n-k)$$

$$F((f*g)(m))(k) = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} (f*g)(m)\, e^{-i\frac{2\pi}{N}mk}$$

$$F(f)(k) = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} f(m)\, e^{-i\frac{2\pi}{N}mk}$$

$$F(g)(k) = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} g(m)\, e^{-i\frac{2\pi}{N}mk}$$

$$(F(f)\cdot F(g))(k) = \frac{1}{N}\left(\sum_{m=0}^{N-1} f(m)\, e^{-i\frac{2\pi}{N}mk}\right)\left(\sum_{m=0}^{N-1} g(m)\, e^{-i\frac{2\pi}{N}mk}\right)$$

$$= \frac{1}{N}\left(\sum_{n=0}^{N-1} f(n)\, e^{-\frac{i2\pi nk}{N}}\right)\left(\sum_{m=0}^{N-1} g(m)\, e^{-\frac{i2\pi mk}{N}}\right)$$

$$= \frac{1}{N}\left(\sum_{n=0}^{N-1} f(n)\, e^{-\frac{i2\pi nk}{N}} \sum_{m=0}^{N-1} g(m)\, e^{-\frac{i2\pi mk}{N}}\right)$$

$$= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} f(n)\,g(m)\, e^{\underbrace{-\frac{i2\pi nk}{N}-\frac{i2\pi mk}{N}}_{-\frac{i2\pi k}{N}(n+m)}}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} f(n)\,g(m)\, e^{-\frac{i2\pi k(n+m)}{N}}$$

$$F((f*g)(m))(k) = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} (f*g)(m)\, e^{\frac{i2\pi mk}{N}}$$

$$= \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} \left(\sum_{j=-\infty}^{+\infty} \underbrace{f(j)}_{\text{sólo}\neq 0 \text{ cuando } j\in\{0,\dots,N-1\}} g(m-j)\right) e^{-\frac{i2\pi mk}{N}}$$

$$= \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} \left(\sum_{j=0}^{N-1} f(j)\, g(m-j)\right) e^{-\frac{i2\pi mk}{N}}$$

3) $F(f)(k) = F(f)(k+N)$

$$\cancel{\frac{1}{\sqrt{N}}} \; f(m)\, e^{-\frac{i2\pi mk}{N}} = \cancel{\frac{1}{\sqrt{N}}} \sum_{m=0}^{N-1} f(m)\, e^{-\frac{i2\pi m(k+N)}{N}}$$

$$\frac{1}{\sqrt{N}}\sum_{m=0}^{N-1} f(m)\, e^{\frac{-i2\pi mk}{N}} = \frac{1}{\sqrt{N}}\sum_{m=0} f(m)\, e$$

$$\sum_{m=0}^{N-1} f(m)\left[e^{\frac{-i2\pi mk}{N}} - e^{\frac{-i2\pi m(k+N)}{N}}\right] = 0$$

$$\sum_{m=0}^{N-1} f(m)\left[\cancel{\cos\left(-\frac{2\pi mk}{N}\right)} + i\,\text{sen}\cancel{\left(-\frac{2\pi mk}{N}\right)} - \underbrace{\cancel{\cos\left(-\frac{2\pi m(k+N)}{N}\right)}} - i\,\text{sen}\cancel{\left(-\frac{2\pi m(k+N)}{N}\right)}\right] = 0$$

$$\cos\left(-\frac{2\pi mk}{N} - \frac{2\pi m \cancel{N}}{\cancel{N}}\right)$$

$$\cos\left(-\frac{2\pi mk}{N} - 2\pi m\right)$$

periodo!

$$\cos\left(-\frac{2\pi mk}{N}\right)$$

$$0 = 0$$

4) $$F^*(-k) = \left[\frac{1}{\sqrt{N}}\sum_{m=0}^{N-1} f(m)\, e^{\frac{2\pi imk}{N}}\right]^*$$

$$= \frac{1}{\sqrt{N}}\sum_{m=0}^{N-1} f(m)\,\underbrace{\left[e^{\frac{2\pi imk}{N}}\right]^*}$$

~~Dmq~~: $$e^{-\frac{2\pi imk}{N}} = \left[e^{\frac{2\pi imk}{N}}\right]^*$$

$$\left[e^{\frac{2\pi imk}{N}}\right]^* = \cos\left(\frac{2\pi imk}{N}\right) - i\,\text{sen}\left(\frac{2\pi imk}{N}\right)$$

$$\underbrace{\cos\left(-\frac{2\pi imk}{N}\right) - \cos\left(\frac{2\pi imk}{N}\right)}_{\text{even}} + i\left[\underbrace{\text{sen}\left(-\frac{2\pi imk}{N}\right) - \text{sen}\left(\frac{2\pi imk}{N}\right)}_{\text{odd.}}\right] = 0$$

5) $$|F(k)| = |F(-k)|$$

Quiero ver el $\leq$ y $\geq$

$$|F(k)| = \left|\frac{1}{\sqrt{N}}\sum_{m=0}^{N-1} f(m)\, e^{\frac{-i2\pi mk}{N}}\right|$$

7)

$$|F(k)| = \left|\frac{1}{\sqrt{N}}\sum_{m=0}^{N-1} f(m)\, e^{\frac{-i2\pi mk}{N}}\right|$$

$$= \frac{1}{\sqrt{N}}\sum_{m=0}^{N-1}\left|f(m)\, e^{\frac{-i2\pi mk}{N}}\right|$$

Vale porque es una descomposición en una base ortonormal.

$$= \frac{1}{\sqrt{N}}\sum_{m=0}^{N-1}|f(m)|\left|e^{\frac{-i2\pi mk}{N}}\right|$$

Qvq: $\left|e^{\frac{-i2\pi mk}{N}}\right| = \left|e^{\frac{i2\pi mk}{N}}\right|$

Pero esto vale por estar en ángulos exactamente opuestos.

6) $F^*(N-k) = F(k)$

$$F^*(N-k) = \frac{1}{\sqrt{N}}\sum_{m=0}^{N-1}\left\{f(m)\, e^{\frac{-i2\pi m(N-k)}{N}}\right\}^*$$

Si $f(m) \in \mathbb{R}$:

$$= \frac{1}{\sqrt{N}}\sum_{m=0}^{N-1} f(m) \underbrace{\left\{e^{-\frac{i2\pi m(N-k)}{N}}\right\}^*}_{(1)}$$

$$(1) = \cos\left(-\frac{2\pi m(N-k)}{N}\right) - i\,\mathrm{sen}\left(-\frac{2\pi m(N-k)}{N}\right)$$

$$= \cos\left(-2\pi m + \frac{2\pi mk}{N}\right) - i\,\sin\left(-2\pi m + \frac{2\pi mk}{N}\right)$$

$$\overset{\text{period}}{=} \cos\left(\frac{2\pi mk}{N}\right) - i\,\mathrm{sen}\left(\frac{2\pi mk}{N}\right)$$

$$\overset{\text{sin impar}}{=} \cos\left(\frac{2\pi mk}{N}\right) + i\,\mathrm{sen}\left(\frac{-2\pi mk}{N}\right)$$

$$\overset{\text{cos par}}{=} \cos\left(\frac{-2\pi mk}{N}\right) + i\,\mathrm{sen}\left(\frac{-2\pi mk}{N}\right)$$

$$\overset{\text{euler}}{=} e^{\frac{-i2\pi mk}{N}}$$

7) $F^*\left(\frac{N}{2}+k\right) = F\left(\frac{N}{2}-k\right) \qquad k \in \{0, \ldots, \frac{N}{2}-1\}$

$$F^*\left(\frac{N}{2}+k\right) = \frac{1}{\sqrt{N}}\sum_{m=0}^{N-1} \underbrace{\{ f(m) \underbrace{e^{\frac{-i2\pi m\left(\frac{N}{2}+k\right)}{N}}}_{(1)} \}^*}_{(2)}$$

Asumiendo $f(m)$ real,

$(2) = f(m) \; (1)^*$

$$(1)^* = \cos\left(\frac{-2\pi m\left(\frac{N}{2}+k\right)}{N}\right) - i\,\text{sen}\left(-\pi m - \frac{2\pi m k}{N}\right)$$

$$= \cos\left(\frac{-2\pi m \frac{N}{2}}{N} + \frac{-2\pi m k}{N}\right) - i\,\text{sen}(\text{—})$$

$$= \cos\left(-\pi m - \frac{2\pi m k}{N}\right) - i\,\text{sen}\left(-\pi m - \frac{2\pi m k}{N}\right)$$

$$= \cos\left(\frac{2\pi m k}{N} + \pi m\right) + i\,\text{sen}\left(\frac{2\pi m k}{N} + \pi m\right)$$

$$F\left(\frac{N}{2}-k\right) = \frac{1}{\sqrt{N}}\sum_{m=0}^{N-1} f(m)\, e^{\frac{-i2\pi m\left(\frac{N}{2}-k\right)}{N}}$$

$$e^{\frac{-i2\pi m\left(\frac{N}{2}-k\right)}{N}} = \cos\left(\frac{-2\pi m\left(\frac{N}{2}-k\right)}{N}\right) + i\,\text{sen}\left(\frac{-2\pi m\left(\frac{N}{2}-k\right)}{N}\right)$$

$$= \cos\left(-\pi m + \frac{2\pi m k}{N}\right) + i\,\text{sen}\left(-\pi m + \frac{2\pi m k}{N}\right)$$

$$= \cos\left(\frac{2\pi m k}{N} - \pi m\right) + i\,\text{sen}\left(\frac{2\pi m k}{N} - \pi m\right)$$

<u>Qvq</u>: $\cos\left(\frac{2\pi m k}{N} + \pi m\right) = \cos\left(\frac{2\pi m k}{N} - \pi m\right)$

$$\text{sen}\left(\frac{2\pi m k}{N} + \pi m\right) = \text{sen}\left(\frac{2\pi m k}{N} - \pi m\right)$$

Lo cual vale pues ambas funciones tienen

2π

Lo cual vale pues ... ,
periodo 2π.