# Proof Portfolio 3

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**Theorem 1.** *If $x$ and $y$ are positive real numbers, then $2\sqrt{xy} \leq x + y$.*

*Proof.* We know that any real number squared has to be greater or equal to 0. Thus, $0 \leq (x-y)^2$. To begin, add $4xy$ on both sides:

$$\begin{aligned} 4xy &\leq (x-y)^2 + 4xy \\ &= x^2 - 2xy + y^2 + 4xy, \text{ expanding the square term} \\ &= x^2 + 2xy + y^2, \text{ collect the } xy \text{ terms} \\ &= (x+y)^2, \text{ factor} \end{aligned}$$

Since $x$ and $y$ must be positive, $4xy$ is positive. Also, since $(x+y)^2$ is positive, we can take the square roots of both sides. So, $2\sqrt{xy} \leq (x+y)$. Therefore if $x$ and $y$ are positive real numbers, then $2\sqrt{xy} \leq (x+y)$. □

**Theorem 2.** *Given that $A \subseteq X$, and $B \subseteq X$, then $A \subseteq B$, $A \cap B^{\complement} = \emptyset$, and $A^{\complement} \cup B = X$ are equivalent.*

*Proof.* First, it needs to be shown that $(A \subseteq B)$ is equivalent to $(A \cap B^{\complement} = \emptyset)$. To begin we need to show that if $(A \subseteq B)$ then $(A \cap B^{\complement} = \emptyset)$. Assume that $A \subseteq B$ and $x \in A$, then it follows that $x \in B$. Since $x \in B$ and $x$ is in the universal set $X$, then $x \notin B^{\complement}$. Since every item in $A$ is not in $B^{\complement}$, the intersection of $A$ and $B^{\complement}$ is empty. Likewise, it needs to be shown that the converse is true. First assume $(A \cap B^{\complement} = \emptyset)$, then by definition of intersection $\forall y \in A, y \notin B^{\complement}$. Since $y \notin B^{\complement}$ but $y \in X$, then $y \in B$. So since every element in A is also in $B$, then $A \subseteq B$. Therefore, if $A \cap B^{\complement} = \emptyset$, then $A \subseteq B$. Thus, $(A \subseteq B) \equiv (A \cap B^{\complement} = \emptyset)$.

Now examine $A \cap B^{\complement} = \emptyset$, so applying the compliment on each side $(A \cap B^{\complement})^{\complement} = \emptyset^{\complement}$. Applying De Morgan's Law on the left side we get $A^{\complement} \cup (B^{\complement})^{\complement} = A^{\complement} \cup B$. On the right side of the equation the complement of the empty set is the universal set, which in this case is $X$. So, if $A \cap B^{\complement} = \emptyset$, then $A^{\complement} \cup B = X$. Now to show that the converse is true, assume that $A^{\complement} \cup B = X$. Applying the complement of each side, it can be shown that $(A^{\complement} \cup B)^{\complement} = X^{\complement}$. Again using De Morgan's Law on the right and the fact that the complement of $X$ equals the $\emptyset$, it is shown that $A \cap B^{\complement} = \emptyset$. So $A \cap B^{\complement} = \emptyset$ and $A^{\complement} \cup B = X$ are equivalent.

Therefore by the transitive property, $A \subseteq B$, $A \cap B^{\complement} = \emptyset$, and $A^{\complement} \cup B = X$ are equivalent. □

**Theorem 3.** *Every prime except* 2 *or* 3 *has the form* $6q+1$ *or* $6q+5$ *where* $q \in \mathbb{Z}$.

*Proof.* Since this theorem only examines prime numbers we can restrict out domain to natural numbers. This theorem can also be written as a contrapositive as follows: if a natural number $k$ is not in the form $6q+1$ or $6q+5$, then $k$ is either 2 or 3 or a composite number. Any natural number can be written as $6q+1$ $6q+2$, $6q+3$, $6q+4$, $6q+5$ or $6q+6$, where $q \in \mathbb{Z}$ and $q \geq 0$. So, $k$ has to equal $6q+2$, $6q+3$, $6q+4$, or $6q+6$.

In the first case, $6q+2$ can be written as the product of two natural numbers, $2(3q+1)$. So $k$ is either 2 or a composite number. The next case, $6q+3 = 3(2q+1)$, can either be written as 3 or a multiple of three. Likewise $6q+4 = 2(3q+2)$, and $6q+6 = 6(q+1)$, so these are composite. So $k$ is composite numbers or 2 or 3. Thus, every prime number except for 2 or 3 will have the form $6q+1$ or $6q+5$

□

**Theorem 4.** $\sqrt[4]{13}$ *is irrational.*

*Proof.* Assume the negation that $\sqrt[4]{13}$ is rational, then $\sqrt[4]{13} = \frac{p}{q}$, where $p \in \mathbb{Z}$, $q \in \mathbb{N}$ and $\frac{p}{q}$ is fully reduced. Therefore,

$$\begin{aligned} \sqrt[4]{13} &= \frac{p}{q} \\ 13 &= \frac{p^4}{q^4} \\ 13q^4 &= p^4 \end{aligned}$$

Notice that $p^4$ is divisible by two integers 13 and $q^4$. Since $p^4$ is divisible by 13 and 13 is prime, then $p$ must be divisible by 13. Thus, $p$ can be written as $13k$, where $k \in \mathbb{Z}$. From above, it can be shown that $q^4 = \frac{p^4}{13} = \frac{(13k)^4}{13} = 13^3k^4$. Thus, 13 is a divisor of $q^4$, and 13 is a divisor of $q$. Since 13 is a divisor of both $p$ and $q$, then $\frac{p}{q}$ can be reduced, which is a contradiction. So $\sqrt[4]{13}$ must be irrational.

□

**Theorem 5.** $\sum_{i=1}^{n} \frac{1}{\binom{i+1}{2}} = 2 - \frac{2}{n+1}$ *for* $n \geq 2$.

*Proof.* This can be proved through induction. The initial case, $n = 2$, can be evaluated as follows

$$\begin{aligned}\sum_{i=1}^{2} \frac{1}{\binom{i+1}{2}} &= \frac{1}{\binom{2}{2}} + \frac{1}{\binom{3}{2}} \\ &= \frac{1}{1} + \frac{1}{3} \\ &= \frac{4}{3}\end{aligned}$$

This is equivalent to the formula where $2 - \frac{2}{n+1} = 2 - \frac{2}{3} = \frac{4}{3}$. Thus the theorem holds where $n = 2$.

For the inductive step, it must be shown, if $\sum_{i=1}^{k} \frac{1}{\binom{i+1}{2}} = 2 - \frac{2}{k+1}$ where $k \geq 2$, then $\sum_{i=1}^{k+1} \frac{1}{\binom{i+1}{2}} = 2 - \frac{2}{(k+1)+1}$. This can begin as follows:

$$\begin{aligned}\sum_{i=1}^{k+1} \frac{1}{\binom{i+1}{2}} &= \sum_{i=1}^{k} \frac{1}{\binom{i+1}{2}} + \frac{1}{\binom{k+1+1}{2}} \\ &= 2 - \frac{2}{k+1} + \frac{1}{\binom{k+1+1}{2}}, \text{ by the inductive hypothesis} \\ &= 2 - \frac{2}{k+1} + \frac{1}{\frac{(k+2)!}{2!k!}}, \text{ the definition of a binomial coefficient} \\ &= 2 - \frac{2}{k+1} + \frac{2}{(k+2)(k+1)} \\ &= 2 - \frac{2}{k+1} - \frac{2}{k+2} + \frac{2}{k+1}, \text{ by partial fraction decomposition} \\ &= 2 - \frac{2}{k+2} \\ &= 2 - \frac{2}{(k+1)+1}, \text{ which proves the inductive step}\end{aligned}$$

Thus, $\sum_{i=1}^{n} \frac{1}{\binom{i+1}{2}} = 2 - \frac{2}{n+1}$ holds for $n \geq 2$. □

**Theorem 6.** *Given that* $x_0 = 1, x_1 = 3$, *and* $x_n = 6x_{n-1} - 8x_{n-2}$ *for* $n \geq 2$, *then* $x_n = 2^{n-1} + \frac{4^n}{2}$ *for* $n \geq 0$.

*Proof.* When $n = 0$ , $x_0 = 2^{0-1} + \frac{4^0}{2} = 1$. When $n = 1$ , $x_1 = 2^{1-1} + \frac{4^1}{2} = 3$. This is consistent with our given values. Thus the initial cases hold.

So to prove the inductive step, it needs to be shown that if $x_k = 2^{k-1} + \frac{4^k}{2}$ for $0 \leq k \leq n$, then $x_{n+1} = 2^{(n+1)-1} + \frac{4^{n+1}}{2}$. From the assumptions, we know that:

$$\begin{aligned}
x_{n+1} &= 6x_n - 8x_{n-1} \\
&= 6\left(2^{n-1} + \frac{4^n}{2}\right) - 8\left(2^{n-2} + \frac{4^{n-1}}{2}\right), \text{ by the inductive assumption} \\
&= 6 \cdot 2^{n-1} + \frac{6 \cdot 4^n}{2} - 8 \cdot 2^{n-2} - \frac{8 \cdot 4^{n-1}}{2} \\
&= 3 \cdot 2^n + 3 \cdot 4^n - 2 \cdot 2^n - 4^n \\
&= (3-2) \cdot 2^n + (3-1) \cdot 4^n, \text{ group the } 2^n \text{ and } 4^n \text{ terms} \\
&= 2^n + 2 \cdot 4^n \\
&= 2^{(n+1)-1} + \frac{4^{n+1}}{2}
\end{aligned}$$

Since the inductive step holds, then $x_n = 2^{n-1} + \frac{4^n}{2}$ for $n \geq 0$. □

**Theorem 7.** *If $a, b \in \mathbb{Z}$ and $a > b > 0$, then $\gcd(a, b) = \gcd(a - b, b)$.*

*Proof.* First, assume that $\gcd(a, b) = d$. By definition of greatest common divisor $d|a$ and $d|b$. Since $d$ is a divisor of $a$ and $b$, $dk_1 = a$ and $dk_2 = b$, where $k_1, k_2 \in \mathbb{Z}$. Notice that since $a > b > 0$ then $d > 0$ and $k_1 > k_2$. So $a - b = dk_1 - dk_2 = d(k_1 - k_2)$. So $d|(a - b)$. Since $d|a$ and $d|(a - b)$, $d$ is a divisor of $a$ and $a - b$ but not necessarily the greatest. So, $\gcd(a - b, a) \geq \gcd(a, b)$.

Now start with $\gcd(a-b, a)$. By definition of greatest common divisor, $\gcd(a-b, a)|a$ and $\gcd(a-b, a)|a-b$. So $\gcd(a - b, a)k_1 = a - b$ and $\gcd(a - b, a)k_2 = a$. So,

$$\begin{aligned} b &= a - \gcd(a - b, a)k_1 \\ &= \gcd(a - b, a)k_2 - \gcd(a - b, a)k_1 \\ &= \gcd(a - b, a)(k_2 - k_1) \end{aligned}$$

This shows that $\gcd(a - b, a)|b$ and we know from before that $\gcd(a - b, a)|a$. So $\gcd(a - b, a)$ is a divisor of $a$ and $b$ but not necessary the greatest. Thus, $\gcd(a - b, a) \leq \gcd(a, b)$.

Since $\gcd(a - b, a) \leq \gcd(a, b)$ and $\gcd(a - b, a) \geq \gcd(a, b)$, $\gcd(a - b, a) = \gcd(a, b)$. □

**Theorem 8.** *If $n \in \mathbb{N}$, then $1 + (-1)^n(2n-1)$ is a multiple of 4. Also, if $x$ is a multiple of 4, then $\exists n \in \mathbb{N}$ such that $x = 1 + (-1)^n(2n-1)$.*

*Proof.* First assume that $n$ is a natural number. If $n$ is a positive even number, $n$ can be written as $n = 2k, k \in \mathbb{N}$. So,

$$\begin{aligned} 1 + (-1)^{2k}(2(2k)-1) &= 1 + \left((-1)^2\right)^k (4k-1), \text{ -1 raised to an even power equals 1} \\ &= 1 + (1)(4k-1) \\ &= 4k, \text{ which is a multiple of 4} \end{aligned}$$

Thus $1 + (-1)^{2n}(2n-1)$ is a multiple of 4 when $n$ is even.

Now when $n$ is odd, we know that $n = 2k+1, k \in \mathbb{Z}$. Similar to the last part we can show that

$$\begin{aligned} 1 + (-1)^{2k+1}(2(2k+1)-1) &= 1 + (-1)^{2k}(-1)^1(4k+2-1) \\ &= 1 + (1)(-1)(4k+1) \\ &= 1 - 4k - 1 \\ &= -4k, \text{ which is a multiple of 4} \end{aligned}$$

Thus $1 + (-1)^{2n}(2n-1)$ is a multiple of 4 for all $n$.

Now it must be shown that the converse is true. Assume that $x$ is a multiple of 4, so $x = 4k, k \in \mathbb{Z}$. When $x > 0$, there exist a $n$ such that $n = \frac{x}{2} = 2k$. Since $x > 0$ then $n > 0$. So,

$$\begin{aligned} 1 + (-1)^n(2n-1) &= 1 + (-1)^{2k}(2(2k)-1) \\ &= 1 + 1 \cdot (4k-1) \\ &= 4k \\ &= x \end{aligned}$$

Thus, if $x$ is multiple of 4 and greater than 0, there exists a $n = \frac{x}{2}$ such that $x = 1 + (-1)^n(2n-1)$.

Now for the case when $x \leq 0$. As we have already stated $x = 4k, k \in \mathbb{Z}$, because $x$ is a multiple of 4. Let $n = \frac{-x}{2} + 1 = -2k + 1$. Since $x \leq 0$ then $n > 1$ so $n$ is positive. It follows that:

$$\begin{aligned} 1 + (-1)^n(2n-1) &= 1 + (-1)^{-2k+1}(2(-2k+1)-1) \\ &= 1 + (-1)^{-2k}(-1)^1(-4k+2-1) \\ &= 1 + (1)(-1)(-4k+1) \\ &= 4k \\ &= x \end{aligned}$$

So for all $x$ that are a multiple of 4 and less than 0, there exist a $n = \frac{-x}{2} + 1$, such that $x = 1 + (-1)^n(2n-1)$.

Thus, for all $x$ that are a multiple of 4, there exist a natural number, such that $x = 1 + (-1)^n(2n-1)$. □

**Theorem 9.** *If $h(x) = g(f(x))$ and $h(x)$ is bijective then $f(x)$ is injective and $g(x)$ is surjective.*

*Proof.* Assume that the negation is true, $h(x) = g(f(x))$ and $h(x)$ is bijective, and $f(x)$ is not injective or $g(x)$ is not surjective. Also say that domains and co-domains of the functions are: $f : X \to Y$, $g : Y \to Z$, and $h : X \to Z$.

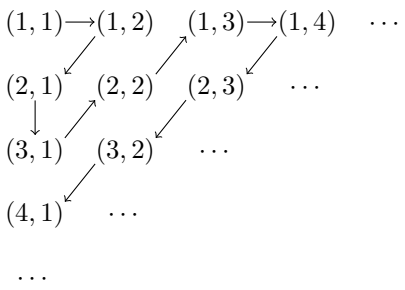
We have two options $f(x)$ is not injective or $g(x)$ is not surjective. So to start assume that $f(x)$ is not injective. By the definition of injective, there exists an $x_1, x_2 \in X$, such that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$. Since $g(x)$ is a function we know that $g(f(x_1)) = g(f(x_2))$, which can be rewritten as $h(x_1) = h(x_2)$. But it was already shown that $x_1 \neq x_2$, so $h(x)$ can not be injective and thus is not bijective.

Now the second part of the proof we need to assume that $g(x)$ is not surjective. Since $g(x)$ is not surjective, there exists a $z' \in Z$ such that $\forall y \in Y$, $g(y) \neq z'$. By the definition of a function $y = f(x)$ every $x \in X$ has to map to a $y \in Y$, but the last statement showed that $\forall y \in Y$, $g(y) \neq z'$. Thus there exists a $z' \in Z$ such that $\forall x \in X, g(f(x)) = h(x) \neq z'$. However this shows that $h(x)$ is not surjective and thus is not bijective. So this shows that it is a contradiction.

Thus, if $h(x) = g(f(x))$ and $h(x)$ is bijective then $f(x)$ is injective and $g(x)$ is surjective. □

**Theorem 10.** $\mathbb{N} \times \mathbb{N}$ *is countable.*

*Proof.* We can form a table of ordered pairs. We can start with $(1, 1)$. Moving one element to right increments the second term by one. Moving one element down increments the first term by one. This will arrange the one set $N = 1, 2, 3, \ldots$ horizontal and the second set of $1, 2, 3, \ldots$ vertical. Thus the ordered pair $(a, b)$ would be placed at the $b$ column and the $a$ row. The table will look like this:




Now define the $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. The first term $f(1)$ would start at the top left corner, and then every term after that will follow the arrows. Then the second term would follow the arrow so $f(2) = (1, 2)$. So $f(n)$ is the $nth$ element in this sequence. $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ would start as follows:

$$\begin{aligned} f(1) &= (1, 1) \\ f(2) &= (1, 2) \\ f(3) &= (2, 1) \\ f(4) &= (3, 1) \\ f(5) &= (2, 2) \\ &\cdots \end{aligned}$$

This is an injective function because no element is repeated. Since the path never overlaps itself, if $f(x_1) = f(x_2)$, then $(c_1, r_1) = (c_2, r_2)$. Thus $c_1 = c_2$ and $r_1 = r_2$, and they must come from the same row and column in the table. So $x_1 = x_2$, and function $f$ is injective. A

This is a surjective function because it maps to all $\mathbb{N} \times \mathbb{N}$. Any element of $\mathbb{N} \times \mathbb{N}$ can be defined as $(x, y)$ such that $x, y \in \mathbb{N}$. So to find this in the sequence you would just have to go to the element on the $x$ row and the $y$ column.

So thus this is a bijective function, and $\mathbb{N} \times \mathbb{N}$ is countable. □

# 1 Reflection Questions

- I am most proud of Theorem 2. I do not think that it is my best proof, but I have put the most work on Theorem 2. Every draft, I have found something that I can improve on that proof.
- For the most part the general ideas have stayed the same between drafts, and I did not have to throw out any proofs. However, I have spent time refining details of the proof. I tried to work on making the proof more precise and providing more information. Specifically Theorem 3, I had to revise it several times to make it clearer and more specific.
- I thought Theorem 9 was the most difficult to work on. Theorem 9 could apply to any function, so it had to rely on the definitions and the wording had to be pretty specific. I actually enjoyed solving this one because of this challenge.

  Also Theorem 7 on the greatest common divisor was annoying to work on, because I had the half of the proof, which showed the less than portion. However, I could not figure out the other half. So it seemed like I was so close but could not pull across the finish line.
- I think that induction is super cool because you can prove something infinite in two steps. The basic idea of induction is like climbing an infinite ladder. If you are starting on the first rung, it looks like it is impossible to reach the top. But if you know that it is easy just to climb one step up the ladder anywhere on the ladder, then you can complete the proof. This step is called the inductive step. Thus you can prove that you can climb the infinite ladder. An induction proof allows you prove something that is infinite in just two steps, the base case and the inductive step.

I have neither given or received nor have I tolerated others use of unauthorized aid. Jon Bayert