JOURNAL OF ENVIRONMENTAL ECONOMICS AND MANAGEMENT 11, 180–190 (1984)

The Effects of the Risk of Fire on the Optimal Rotation of a Forest[1]

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Received February 2, 1983; revised July 1983

The effects of the risk of fire or other unpredictable catastrophe on the optimal rotation period of a forest stand are investigated. It is demonstrated that when fires occur in a time-independent Poisson process, and cause total destruction, the policy effect of the fire risk is equivalent to adding a premium to the discount rate that would be operative in a risk-free environment. Other cases are also investigated and in each a modified form of the Faustmann formula is derived and a "marginal" economic interpretation given.

1. INTRODUCTION

One of the oldest theoretical problems in forestry is to determine the best age at which to cut down a tree or stand of trees. A formulation of the problem, using the notions of capital theory was given by Faustmann [2] in 1849. Pearse [4] presented a more modern formulation of the problem and derived the solution in terms of basic economic concepts. Clark [1] also discussed the problem and related it to other bioeconomic management problems. In the model used by all of these authors it is assumed that a stand will continue to grow until it reaches maturity, unless it is cut down by the forester. The possibility of destruction of the stand by fire or other natural causes is not considered. The optimal rotation period when the risk of destruction is present has been considered by Martell [3] and Routledge [6] using discrete time models. The main concern of both authors is the practical problem of determining the optimal rotation, rather than giving an economic interpretation to the results obtained. Martell [3] uses dynamic programming to obtain the optimum solution, while Routledge uses a modified form of the Faustmann formula in discrete time. In this paper the problem is formulated and solved in continuous time. Economic interpretations of the results are discussed.

In Section 2 the model is presented. Fires (or other catastrophes) are assumed to occur in a homogeneous Poisson process and to result in total destruction. In Section 3 the optimal rotation to maximize the long-run average yield is determined and the economic implications of the presence of risk discussed. In Section 4 the optimal rotation to maximize discounted expected yield is determined. A numerical example is given showing how the value of fire protection can be determined. In Section 5 two extensions are considered. First, the assumption that the catastrophe results in *total* destruction is relaxed. Second, the case when the probability of fire depends on the age of the stand is considered.

[1]Research supported by NSERC Grant A-7252.

0095-0696/84 $3.00

2. THE MODEL

The commercial value of a tree, or more realistically, a stand of trees is customarily expressed as *stumpage value*. The stumpage value is the value of the cut timber at some center minus the costs of logging and delivery. Typically the stumpage value tends to increase with the age of the trees, for not only does the timber volume in the stand increase with age, but also the quality of the timber tends to improve. Furthermore the unit costs of logging and handling tend to be lower for larger-sized trees. However, trees do not grow forever. Once biological maturity is achieved, growth ceases. Ultimately decay sets in. Thus the stumpage value will at first increase to a maximum and subsequently decrease.

Let $V(t)$ denote the stumpage value of a stand of trees of age t. It is reasonable to suppose that $V(t)$ is zero for $t < t_0$ (trees of age less than t_0 have no commercial value) and is increasing for $t_0 \leqslant t \leqslant t_1$. It will be assumed that once trees are removed from the site by logging, a new stand will start to grow immediately. This will be the case if the site is replanted with new trees. Let c_1 denote the costs associated with this replanting.

It is well-known (see e.g., Clark [1, p. 261]) that to maximize the long-run average economic yield from such a resource, the optimal rotation period, T, is given by the solution to

$$V'(T) = \frac{V(T) - c_1}{T}, \tag{1}$$

and that to maximize the discounted economic yield, or present value of the resource, the optimal rotation period is given by the solution to

$$V'(T) = \frac{\delta(V(T) - c_1)}{1 - e^{-\delta T}} \tag{2}$$

where δ is the instantaneous discount rate. This equation is known as the *Faustmann formula*.

Clark [1, p. 259] notes that Eq. (2) can be reexpressed as

$$V'(T) = \delta[V(T) - c_1] + \delta[V(T) - c_1] \sum_{j=1}^{\infty} e^{-j\delta T}. \tag{3}$$

On the left-hand side, $V'(T)$ represents the marginal increase in stumpage value over the time period T to $T + 1$ if the stand is not cut. The first term on the right-hand side represents the interest that can be earned over the same period on the net revenue earned from cutting at time T. The second term on the right represents the marginal increase, over the same time period, in the present value of the stream of future revenues from trees grown on the site after cutting using the rotation period T. This is known as the *site value* (or *land value* or *soil-expectation value*). Thus condition (3) is that, at the optimal rotation age T, the marginal increase in the value of standing trees equals the sum of the opportunity costs of the investment tied up in the standing trees and the site.

We now consider the possibility of the destruction of the stand through uncontrolled catastrophes such as fire, disease, or insect infestation. For the sake of brevity we shall refer to catastrophes as "fires," (fire being the most likely agent of destruction), the possibility of other causes being understood.

Suppose that fires occur in a Poisson process at rate λ (Ross [5, p. 14]) i.e., that fires occur independently of one another, and randomly in time at an average rate of λ per unit time. Note that we are assuming that the probability of fire is independent of the age of the stand and that the destruction is total, no part of the stumpage being salvageable (we shall relax these assumptions in Section 5). Let c_2 denote the cost of clearing and replanting the site after a fire. In the next two sections we shall determine optimal rotation periods to maximize (a) the expected long-run average economic yield, and (b) the expected discounted economic yield.

3. OPTIMAL ROTATION PERIOD TO MAXIMIZE EXPECTED LONG-RUN AVERAGE YIELD

Let $X_1, X_2, \ldots$ denote the times *between* successive destructions of the stand, either by fire or by logging. If the policy is to cut the stand whenever it reaches age T, the distribution of the random variables $\{X_n\}$ is

$$\begin{aligned} F_X(t) &= 1 - e^{-\lambda t}, \quad && t < T \\ &= 1, && t \geqslant T \end{aligned} \tag{4}$$

where $F_X(t) = \mathrm{pr}\{X_n \leqslant t\}$. Furthermore the random variables X_n, $n = 1, 2, \ldots$ are independent and so the times of destruction form a *renewal process* (Ross [5, p. 31]). The net economic return (stumpage minus replanting costs) associated with a time X_n between destructions is

$$\begin{aligned} Y_n &= -c_2, && \text{if } X_n < T \\ &= V(T) - c_1, \quad && \text{if } X_n = T. \end{aligned} \tag{5}$$

The process $\{(X_n, Y_n)\}$ is a *renewal reward process* (Ross [5, p. 51]). Let

$$Y(t) = \sum_{n=1}^{N(t)} Y_n$$

where $N(t)$ is the number of renewals (destructions) in time period 0 to t, then the *average yield* is $Y(t)/t$. It can be shown (Ross, [5, p. 52]) that as $t \to \infty$

(a) $Y(t)/t \to E(Y)/E(X)$ with probability 1, and

(b) $E(Y(t))/t \to E(Y)/E(X)$,

where X and Y are generic random variables with the distribution of the X_n and Y_n, respectively. Thus the *long-run average* yield from the forest site can be expressed as $E(Y)/E(X)$.

Now from (4)

$$E(X) = \int_0^\infty t\, dF_X(t) = \int_0^T \lambda t e^{-\lambda t}\, dt + Te^{-\lambda T} = \frac{1 - e^{-\lambda T}}{\lambda} \tag{6}$$

after integrating by parts. From (5)

$$E(Y) = \int_0^T - c_2 \cdot \lambda e^{-\lambda t}\,dt + \left(V(T) - c_1\right)e^{-\lambda T}$$
$$= \left[V(T) - c_1\right]e^{-\lambda T} - c_2(1 - e^{-\lambda T}) \tag{7}$$

and thus the long-run average yield, $\bar{Y}$ is

$$\bar{Y} = \frac{E(Y)}{E(X)} = \frac{\lambda\left(V(T) - c_1\right)e^{-\lambda T}}{1 - e^{-\lambda T}} - \lambda c_2. \tag{8}$$

The optimal cutting age, or rotation period, can be determined by setting the derivative of (8) with respect to T equal to zero. This gives

$$V'(T) = \frac{\lambda\left(V(T) - c_1\right)}{1 - e^{-\lambda T}}. \tag{9}$$

This is exactly the same as (2) with the discount rate δ replaced by the average rate of occurrence of fires λ. Thus *when there is a risk of fires present the optimal long-run average yield policy is to act as if there were no fire risk but to seek to maximize the present value (using the Faustmann formula) with a discount rate equal to the average rate at which fires occur.* The presence of a fire risk thus *decreases* the optimal rotation period.

The solution provides a very clear example of how the phenomenon of time discounting in part arises as a response to risk. In the next section it will be seen how the presence of risk modifies a time-preference rate determined in the absence of risk (say, as the marginal productivity of capital in the economy as a whole, in a certain world) through the addition of a premium.

4. OPTIMAL ROTATION PERIOD TO MAXIMIZE EXPECTED DISCOUNTED YIELD

It is not difficult to show that the optimal policy to maximize expected discounted yield is to cut the stand whenever it reaches some age, T. In this section we shall see how the optimal cutting age, or rotation period, T, can be determined.

If the stand is cut only when it reaches age T, the expected discounted yield (or present value) J, is

$$J = E\left\{\sum_{n=1}^{\infty} e^{-\delta(X_1 + X_2 + \cdots + X_n)}Y_n\right\} \tag{10}$$

where X_n and Y_n are as defined in Section 3, and δ is the discount rate (determined, say, as the marginal productivity of capital in the economy as a whole). Since the random variables X_i ($i = 1, 2, \ldots$) are independent, (10) can be written as

$$\begin{aligned} J &= \sum_{n=1}^{\infty} E\left(e^{-\delta(X_1 + \cdots + X_{n-1})}\right) \cdot E\left(e^{-\delta X_n} Y_n\right) \\ &= \sum_{n=1}^{\infty} \prod_{i=1}^{n-1} E\left(e^{-\delta X_i}\right) \cdot E\left(e^{-\delta X_n} Y_n\right) \\ &= E\left(e^{-\delta X} Y\right) / \left[1 - E\left(e^{-\delta X}\right)\right]. \end{aligned} \tag{11}$$

Furthermore

$$\begin{aligned} E\left(e^{-\delta X}\right) &= \int_0^{\infty} e^{-\delta t}\, dF_X(t) \\ &= \int_0^T e^{-\delta t} \lambda e^{-\lambda t}\, dt + e^{-\delta T} e^{-\lambda T} \\ &= \left(\lambda + \delta e^{-(\lambda+\delta)T}\right) / (\lambda + \delta) \end{aligned} \tag{12}$$

and

$$\begin{aligned} E\left(e^{-\delta X} Y\right) &= \int_0^T (-c_2) e^{-\delta t} \lambda e^{-\lambda t}\, dt + e^{-\delta T}\left(V(T) - c_1\right) e^{-\lambda T} \\ &= \left(V(T) - c_1\right) \cdot e^{-(\lambda+\delta)T} - \lambda c_2\left(1 - e^{-(\lambda+\delta)T}\right) / (\lambda + \delta). \end{aligned} \tag{13}$$

Thus from (11), (12), and (13),

$$J = \frac{(\lambda + \delta)\left(V(T) - c_1\right) e^{-(\lambda+\delta)T}}{\delta\left(1 - e^{-(\lambda+\delta)T}\right)} - \frac{\lambda}{\delta} c_2. \tag{14}$$

Note the similarity between (14) and (8). The optimal rotation period can be obtained by setting the derivative of J with respect to T equal to zero. This gives

$$V'(T) = \frac{(\lambda + \delta)\left(V(T) - c_1\right)}{1 - e^{-(\lambda+\delta)T}}. \tag{15}$$

This is exactly the Faustmann formula (2), with the discount rate δ replaced by $\lambda + \delta$. Thus *the effect on the rotation period of a risk of destruction by fire is the same as that of an increase in the discount rate by an amount equal to the average rate at which fires occur. The presence of risk effectively adds a premium to the risk-free time-preference rate determined exogeneously.* The effect is to shorten the rotation period.

The result in Section 3 can be regarded as a special case of this result, because the present value approaches the long-run average yield as the discount rate δ approaches zero.

The result (15) has a "marginal" interpretation similar to that of the Faustmann formula (2) given by Clark [1, p. 259]. Equation (15) can be written as

$$V'(T) = (\lambda + \delta)\left(V(T) - c_1\right) + \frac{e^{-(\lambda+\delta)}(\lambda + \delta)\left(V(T) - c_1\right)}{1 - e^{-(\lambda+\delta)T}}, \tag{16}$$

which, using (14) can be reexpressed as

$$V'(T) - \lambda\big(V(T) - c_1 + c_2\big) = \delta\big(V(T) - c_1\big) + \delta J \tag{17}$$

where J is the site value. We shall show that the left-hand side is the *expected* marginal increase in the stumpage value of standing trees (net of possible clearing and replanting costs) given that the stand is *not cut* over unit time, while as before, the right-hand side is the sum of the opportunity cost of the investment held in standing trees and in the site. To see this consider a small time interval of infinitesimal duration h, and reexpress (17) as

$$V'(T)h(1 - \lambda h) - \big[V(T) - c_1 + c_2\big]\lambda h = \delta h\big[V(T) - c_1\big] + \delta hJ + o(h). \tag{18}$$

Over this time interval the probability of a stand of age T being destroyed by fire is λh. Also $V'(T)h$ is the growth in stumpage value of the stand conditional on it not being destroyed by fire. Furthermore $[V(T) - c_1 + c_2]$ is the *loss* in stumpage value (net of clearing and replanting costs) given that the stand is destroyed by fire. Thus from conditional expectation it can be seen that the left-hand side represents the expected marginal increase in the value of the stand (net of possible replanting costs) given that the stand is not cut. On the right-hand side the first term represents the interest that could be earned over the time period of length h on the net revenue realized from cutting the stand at age T and the second term represents the interest that could be earned on the revenue which would be realized from selling the cleared site. Thus the condition (17) is to cut at the age T at which the expected marginal increment to the value of the stand which would be realized through not cutting equals the marginal revenue that could be earned through cutting. Viewed in this light the optimal policy can be seen to be an example of an *infinitesimal look-ahead stopping rule* (Ross [5, p. 188]).

It is interesting to note that the equation (15) which determines the optimal rotation does not involve the cost, c_2, of clearing and replanting after a fire. Basically this is because one cannot control (through the rotation time, T) the probability of a fire occurring. The probability of a fire in an infinitesimal time period $(t, t + h)$ is constant at λh *regardless of the age of the stand*. In Section 5 we shall see that when the probability of fire depends on the age of the stand the optimal rotation period depends on c_2. In this case, through controlling T, one can control the probability of a fire.

If the stumpage value, net of replanting costs, $V(T) - c_1$, is known, the optimal rotation period for given λ and δ can be determined by solving (15) numerically. A graphical method given by Clark [1, p. 260] is to plot the "relative growth rate" in the net stumpage value

$$\frac{V'(T)}{V(T) - c_1}$$

TABLE I

Stumpage Values/Acre of a Typical Stand of British Columbia Douglas Fir[a]

Age, t (years)	30	40	50	60	70	80	90	100	110	120
Net stumpage value $V(t) - c$ (dollars/acre)	0	43	143	303	497	650	805	913	1000	1075

[a]From Pearse [4]. Values expressed in 1967 dollars.

and the family of curves

$$\frac{\lambda + \delta}{1 - e^{-(\lambda+\delta)T}}$$

as functions of T, on the same graph.

For the data on a typical 1-acre stand of British Columbia Douglas Fir (Pearse [4]), given in Table I, the optimal rotation period for various *annual* discount rates ($i = e^{\delta} - 1$) has been calculated when (a) there is no risk of fire ($\lambda = 0$), (b) when fires occur at rate $\lambda = 0.02$ (on average one per 50 years) and (c) when fires occur at rate $\lambda = 0.05$ (on average one per 20 years). The results are given in Table II. It can be seen that the effect of the risk of fire on rotation time is greatest when the discount rate is small.

Once the optimal cutting policy has been determined a cost–benefit analysis of fire protection can be carried out (see Martell [3]). Suppose that fire protection measures reduce the instantaneous probability of fire from λ_2 to λ_1, and that the cost of the protection is incurred continuously at the rate of p dollars per unit time. We ask, for what values of p is the protection worthwhile? The total discounted cost of protection is

$$\int_0^\infty pe^{-\delta t}\,dt = \frac{p}{\delta}$$

and the change in expected present value is

$$J_{\max}(\lambda_1;\delta) - J_{\max}(\lambda_2;\delta)$$

TABLE II

Optimal Rotation Periods (Cutting Ages) for British Columbia Douglas Fir for Various Annual Discount Rates (i) and Various (Homogeneous) Average Rates of Fire (λ)

Annual discount rate, i		0	0.03	0.05	0.10	0.15	0.20
Rate of fires	$\lambda = 0$	100	70	63	49	43	40
	0.02	77	63	56	46	41	38
	0.05	63	54	49	43	40	37

where $J_{\max}(\lambda; \delta)$ is given by (14) evaluated at its maximum (determined by (15)). A threshold value p_0 for costs is obtained by equating costs with benefits

$$\frac{p_0}{\delta} = J_{\max}(\lambda_1; \delta) - J_{\max}(\lambda_2; \delta)$$

i.e.,

$$p_0 = (\lambda_2 - \lambda_1)c_2 + \frac{(\lambda_1 + \delta)(V(T_1) - c_1)e^{-(\lambda_1+\delta)T_1}}{1 - e^{-(\lambda_1+\delta)T_1}} - \frac{(\lambda_2 + \delta)(V(T_2) - c_1)e^{-(\lambda_2+\delta)T_2}}{1 - e^{-(\lambda_2+\delta)T_2}} \tag{19}$$

where T_1 and T_2 are the values of T which solve (15) with $\lambda = \lambda_1$ & $\lambda = \lambda_2$, respectively. Fire protection will be worthwhile if and only if for a given discount rate δ the cost per unit time of reducing the instantaneous probability of fire from λ_2 to λ_1 is no greater than p_0.

It can be seen from (19) that unlike the optimum cutting age the value of fire protection p_0 depends on the cost, c_2, of clearing and replanting after fire. For the sake of numerical illustration we shall assume that for the B. C. Douglas Fir this cost is \$20 per acre. Table III gives the value of fire protection, p_0 for various *annual* discount rates ($i = 0$ in Table III(a), $i = 0.05$ in Table III(b) and $i = 0.10$ in Table III(c)).

It can be seen, for example, that when the per annum discount rate is 5% (Table III(b)), it is worth spending up to \$1.24 per acre per year on protection measures which would reduce the probability of fire from $\lambda_2 = 0.05$ (on average 1 per 20 years) to $\lambda_1 = 0.01$ (on average 1 per 100 years). If there were no time discounting the value of this degree of fire protection would be much higher—\$5.50 per acre per

TABLE III

This Table Shows the Value, p_0, of the Protection Measures (in Dollars per Acre per Year) Which Would Reduce the Instantaneous Probability of Fires from λ_2 to λ_1

Probability of fire λ_2, with *no* protection	Probability of fire λ_1, *with* protection		
	0	0.01	0.02
	(a) $i = 0$ (no time discounting)		
0.01	4.32		
0.02	6.73	2.41	
0.05	9.82	5.50	3.09
	(b) $i = 0.05$		
0.01	0.49		
0.02	0.90	0.41	
0.05	1.73	1.24	0.83
	(c) $i = 0.10$		
0.01	0.23		
0.02	0.46	0.23	
0.05	1.10	0.87	0.64

(a) Is when there is no time discounting ($i = 0$), (b) is for discounting at the rate of 5% per annum ($i = 0.05$), and (c) is for discounting at the rate of 10% per annum ($i = 0.10$).

year (Table III(a)). If on the other hand the per annum discount rate was 10% the value of this degree of protection would only be \$0.87 per acre per year.

When the discount rate is high the difference between the last two terms on the r.h.s. of (19) is small and in consequence the first term representing clearing/replanting costs becomes more important in determining p_0. (This can be clearly seen numerically in Table III(c). For example, of the value \$0.87 for the degree of protection above, the first term on r.h.s. of (19) has value 0.80 while the difference between the last two terms is 0.07.) The meaning of this is that when discount rates are high the importance of fire protection *in the long run* is not so much to increase the stumpage yield, but to reduce the money spent on clearing/replanting of the site after fi·es. This is not the case for lower discount rates.

5. SOME EXTENSIONS OF THE MODEL

The model can be extended in a number of ways. In this section two such extensions are considered. First, we consider the case when the destruction through fire or other loss agents is only partial.

Suppose that after a fire, in a stand of age t, a *random* proportion, K_t, of the stumpage value is salvageable. Suppose that the random variable K_t has mean $\bar{k}(t)$ and that the clearing and replanting costs are c_2 regardless of the proportion salvageable. The net return, Y_n for the nth rotation is then

$$\begin{aligned} Y_n &= K_{X_n} V(X_n) - c_2 \qquad && \text{if } X_n < T \\ &= V(T) - c_1 && \text{if } X_n = T. \end{aligned}$$

Now

$$\begin{aligned} E(e^{-\delta X}Y) &= E_X\big(E(e^{-\delta X}Y|X)\big) \\ &= \int_0^T e^{-\delta x}\big(\bar{k}(x)V(x) - c_2\big)\lambda e^{-\lambda x}\,dx + \big[V(T) - c_1\big]e^{-(\lambda+\delta)T}. \end{aligned}$$

Thus using (11) and (12) the expected present value is

$$J = \frac{\lambda+\delta}{\delta}\,\frac{\big[(V(T) - c_1)e^{-(\lambda+\delta)T} + \phi(T)\big]}{1 - e^{-(\lambda+\delta)T}} - \frac{\lambda c_2}{\delta} \tag{20}$$

where $\quad \phi(T) = \int_0^T \lambda\bar{k}(x)V(x)e^{-(\lambda+\delta)x}\,dx.$

Setting the derivative with respect to T equal to zero gives

$$V'(T) + e^{(\lambda+\delta)T}\phi'(T) = \frac{\lambda+\delta}{1 - e^{-(\lambda+\delta)T}}\big[V(T) - c_1 + \phi(T)\big]. \tag{21}$$

This is much like (15), with the stumpage value, $V(T)$ replaced by $V(T) + \phi(T)$. Note however, that in this case the presence of the fire risk not only has the effect of changing the effective discount rate, but also of changing the effective stumpage value curve V in a way that depends on λ.

A "marginal" interpretation of (21) can be obtained as before. Using (20), (21) can be rewritten as

$$V'(T) + e^{(\lambda+\delta)T}\phi'(T) = (\lambda + \delta)\big(V(T) - c_1\big) + \delta J - \lambda c_2. \tag{22}$$

For a small time interval of infinitesimal duration h it follows that

$$\begin{aligned} &V'(T)h(1 - \lambda h) + \big[\big(\bar{k}(T)V(T) - c_2\big) - \big(V(T) - c_1\big)\big]\lambda h \\ &\quad = \delta h\big[V(T) - c_1\big] + \delta h J + o(h) \end{aligned} \tag{23}$$

since $\phi'(T) = \bar{k}(T)V(T) \cdot \lambda e^{-(\lambda+\delta)T}$.

As before the left-hand side represents the increase in expected value of the stand given that it is not cut. The new term $(\bar{k}(T)V(T) - c_2)$ is the expected revenue that would be salvaged if a fire were to occur during the short time interval. The right-hand side represents the sum of interest earned on the realized stumpage value and on the potential site value, given that the stand is cut at age T.

The second extension to the model considered, is when the rate of fires depends on the age of the stand. We suppose that for a stand of age t the probability of a fire in an infinitesimal time interval (of length h) is $\lambda(t)h$. This is an example of a *nonhomogeneous Poisson process* (Ross [5, p. 24]). We define a function, m

$$m(t) = \int_0^t \lambda(s)\,ds.$$

It can then be shown that the time between successive destructions in an unharvested stand has distribution function $F(x) = 1 - e^{-m(x)}$. If a stand is cut whenever it reaches age T, the distribution of the times between successive destructions, $X_1, X_2, \ldots,$ etc. has distribution function

$$\begin{aligned} F(x) &= 1 - e^{-m(x)}, &\quad x &< T \\ &= 1, & x &\geqslant T. \end{aligned}$$

Thus using (11), after evaluating $E(e^{-\delta X})$ and $E(e^{-\delta X}Y)$, gives

$$J = \frac{\big(V(T) - c_1\big)e^{-(m(T)+\delta T)} - c_2\big(1 - e^{-(m(T)+\delta T)}\big)}{\delta \int_0^T e^{-(m(x)+\delta x)}\,dx} + c_2. \tag{24}$$

Setting dJ/dT equal to zero, gives

$$V'(T) + \big(\lambda(T) + \delta\big)\big(V(T) - c_1 + c_2\big) + \frac{e^{-(m(T)+\delta T)}\big(V(T) - c_1 + c_2\big)}{\int_0^T e^{-(m(x)+\delta x)}\,dx} - c_2. \tag{25}$$

This equation, analogous to (16), can be used to determine the optimal rotation. The effect of a risk of fire is more complicated than the simple change in the effective discount rate which arose in the homogeneous case. For example, the optimal T now depends on the cost, c_2, of clearing and replanting after a fire. This is because, through controlling T, one can influence the probability of a fire, which was not so in the homogeneous case.

A "marginal" interpretation of (25) can be obtained by substituting (24) into it, to give

$$V'(T) = (\lambda(T) + \delta)(V(T) - c_1) + \delta J + \lambda(T)c_2. \tag{26}$$

It follows that

$$\begin{aligned} &V'(T)h(1 - \lambda(T)h) - [V(T) - c_1 + c_2]\lambda(T)h \\ &\quad = \delta h[V(T) - c_1] + \delta hJ + o(h). \end{aligned} \tag{27}$$

This is the same as (18) with λ replaced by $\lambda(T)$. Again the optimal harvest occurs when the expected marginal increase in the value of the stumpage equals the opportunity cost of the investment held in standing trees and in the site.

ACKNOWLEDGMENTS

I thank Darrell Errico, David Gilbert, and Howard Stauffer of the British Columbia Ministry of Forests for their contributions in formulating the problem and discussing the results.

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