Ammendment: A Simple Spatial Equilibrium Model (2 regions, 1 good)

1 A Simple Spatial Equilibrium Model (2 regions, 1 good)

We have two regions, A and B, that can produce and consume a single homogeneous commodity and trade with iceberg-like per-unit transport costs c_{AB} from A to B and c_{BA} from B to A.

1.1 Primitives

- Linear supply in region $i \in \{A, B\}$: $p_{s,i} = a_i + b_i Q_{s,i}$ with $b_i > 0$
- Linear demand in region i: $p_{d,i} = e_i - f_i Q_{d,i}$ with $f_i > 0$
- Trade flows $T_{ij} \geq 0$ for $i, j \in \{A, B\}$ (allow $i = j$ to represent internal, i.e., local use)
- Per-unit transport costs $c_{ij} \geq 0$

Mass balance in each region requires

$$Q_{s,i} = \sum_j T_{ij}, \quad Q_{d,i} = \sum_j T_{ji}.$$

A standard welfare formulation (Takayama–Judge/Samuelson) maximizes total surplus minus transport costs:

$$\max_{Q_{s,\cdot}, Q_{d,\cdot}, T_{\cdot\cdot}} \sum_i \left(\int_0^{Q_{d,i}} p_{d,i}(q)\, dq - \int_0^{Q_{s,i}} p_{s,i}(q)\, dq \right) - \sum_i \sum_j c_{ij} T_{ij},$$

subject to the two balance equations and $T_{ij} \geq 0$. With linear supply and demand, this objective is quadratic and the model is a convex QP . This class of models traces back to Samuelson (1952) and Takayama & Judge (1971).

1.2 KKT/equilibrium conditions (economic meaning)

Let p_i denote the (equilibrium) price in region i. Complementarity gives (i) if $T_{ij} > 0$ then $p_j = p_i + c_{ij}$ (trade equalizes destination price to origin price plus transport cost); (ii) if $T_{ij} = 0$ then $p_j \leq p_i + c_{ij}$. When a region both supplies and consumes internally, p_i also equals its local supply and demand prices, $p_{s,i} = p_{d,i}$.

1.3 Worked numeric example

Choose parameters:

- Supply: $p_{s,A} = 10 + 0.5Q_{s,A}$, $p_{s,B} = 20 + 0.5Q_{s,B}$
- Demand: $p_{d,A} = 50 - Q_{d,A}$, $p_{d,B} = 60 - Q_{d,B}$
- Transport costs: $c_{AB} = 5$, $c_{BA} = 7$

Guess and verify the trading pattern $A \to B$ only (so $T_{AB} > 0$, $T_{BA} = 0$). Then the KKT conditions imply

$$p_A = p_{s,A} = p_{d,A}, \qquad p_B = p_A + c_{AB}.$$

Hence

- Region A: $p_A = 10 + 0.5Q_{s,A}$ and $p_A = 50 - Q_{d,A} \Rightarrow Q_{s,A} = 2(p_A - 10),\ Q_{d,A} = 50 - p_A$.
- Region B: $p_B = p_A + 5 = 20 + 0.5Q_{s,B} \Rightarrow Q_{s,B} = 2(p_A - 15)$ and $Q_{d,B} = 60 - p_B = 55 - p_A$.
- Market clearing: $Q_{s,A} + Q_{s,B} = Q_{d,A} + Q_{d,B}$.

Solve for p_A:

$$2(p_A - 10) + 2(p_A - 15) = (50 - p_A) + (55 - p_A) \ \Rightarrow\ 4p_A - 50 = 105 - 2p_A \ \Rightarrow\ p_A = \frac{155}{6} \approx 25.833.$$

Back out quantities:

- $Q_{s,A} = 2(25.833 - 10) = 31.667$
- $Q_{s,B} = 2(25.833 - 15) = 21.667$
- $Q_{d,A} = 50 - 25.833 = 24.167$
- $Q_{d,B} = 55 - 25.833 = 29.167$
- Flow $A \to B$: $T_{AB} = Q_{s,A} - Q_{d,A} = 7.5$ (matches $Q_{d,B} - Q_{s,B} = 7.5$)

Prices:

- $p_A = 25.833$, $p_B = p_A + 5 = 30.833$, and indeed $p_{s,B} = 20 + 0.5 \cdot 21.667 = 30.833$.

No backhaul is profitable: $p_A \le p_B + c_{BA}$ is $25.833 \le 30.833 + 7$, which holds strictly, so $T_{BA} = 0$ is consistent with complementarity. This solution satisfies the spatial equilibrium relationships and illustrates how transport costs wedge regional prices while equalizing them net of c_{ij} where flows are positive.

1.4 Notes

- You can equivalently write the model with trade balance variables T_{ii} for internal use and treat $Q_{s,i}, Q_{d,i}$ as sums of flows; the solver returns regional supplies, demands, and bilateral shipments, with regional prices in the duals.
- In teaching, this two-region case is great for hand-solving by guessing the active trade arcs, imposing $p_j = p_i + c_{ij}$ on those arcs, and using market clearing to solve the linear system.