Linear Programming

0.0.1 Linear Programming (LP) Structure

- **Standard LP Form**:

$$\max \ \pi' x \quad \text{s.t. } Ax \leq b, \ x \geq 0$$

- Characteristics:
  - Linear objective.
  - Linear constraints.
  - Nonnegativity.

Can also be written in matrix notation.

Economic interpretation:

- π: profit per unit.
- A: resource use matrix.
- b: resource endowment.

0.0.2 Model Building Process (McCarl framework)

1. Identify **decision variables**.
2. State the **objective**.
3. Identify and formulate **constraints**.
4. Collect data.
5. Translate into computer-readable form.
6. Solve and interpret.

0.0.3 Farm LP Example

A farmer has 500 acres of land available and is deciding how to allocate it between wheat and corn. Each acre of wheat yields a profit of $200, requires 3 hours of labor, and 4 units of fertilizer. Each acre of corn yields a profit of $300, requires 4 hours of labor, and 3 units of fertilizer. The farm has at most 1,800 hours of labor available and 2,000 units of fertilizer.

Formulate this situation as a linear programming problem. Clearly define the decision variables, write down the objective function representing total profit, and specify the constraints that capture the land, labor, fertilizer, and nonnegativity restrictions.

Decision variables.
Let W = acres of wheat, C = acres of corn.

0.0.4 Scalar (algebraic) form

$$
\begin{aligned}
\max_{W,C} \quad & 200\,W + 300\,C \\
\text{s.t.} \quad & W + C \le 500 && \text{(land)} \\
& 3W + 4C \le 1800 && \text{(labor)} \\
& 4W + 3C \le 2000 && \text{(fertilizer)} \\
& W,\ C \ge 0\ .
\end{aligned}
$$

0.0.5 Matrix (compact) form

$$
\begin{aligned}
\max_{x \in \mathbb{R}^2_{\ge 0}} \quad & c^\top x \\
\text{s.t.} \quad & Ax \le b,
\end{aligned}
\qquad \text{with} \quad x = \begin{bmatrix} W \\ C \end{bmatrix},\ c = \begin{bmatrix} 200 \\ 300 \end{bmatrix},\ A = \begin{bmatrix} 1 & 1 \\ 3 & 4 \\ 4 & 3 \end{bmatrix},\ b = \begin{bmatrix} 500 \\ 1800 \\ 2000 \end{bmatrix}.
$$

0.1 Summary

- Optimization = decision making under constraints.
- Mathematical programming provides a general framework: decision variables, objective, constraints.
- Linear programs (LPs): linear objective + linear constraints + nonnegativity.
- Example: Farm resource allocation with land, labor, fertilizer.

0.2 Day 2: Linear Programming in Practice

0.2.1 Review

- General form of a mathematical program.
- Structure of a linear program:
  - Linear objective
  - Linear constraints
  - Nonnegativity
- Wheat–Corn farm allocation example in scalar and matrix notation.

0.2.2 7 Assumptions of Linear Programming

Linear programming models rely on a set of assumptions that make them tractable but also limit their realism. McCarl & Spreen (Ch. 2.4) identify **seven important assumptions**. The first three involve the *appropriateness of the formulation*; the last four describe *mathematical properties* of the LP model.

0.2.2.1 1. Objective Function Appropriateness

- The objective function is assumed to be the **sole criterion** for evaluating solutions.

- This means the decision maker's preferences can be fully represented by a single linear function (e.g., profit, cost, utility).

- In practice, decisions may depend on multiple objectives (profit, risk, leisure), but LP assumes one dominates.

0.2.2.2 2. Decision Variable Appropriateness

- All relevant decision variables must be included, and each must be **fully controllable** by the decision maker.

- Omitting key variables or including variables outside the decision maker's control invalidates the formulation.

0.2.2.3 3. Constraint Appropriateness

- Constraints must **accurately and completely capture** the limits faced by the decision maker:
  - They fully describe resource, technological, and institutional limits.

  - Resources within a constraint are **homogeneous** and freely substitutable among activities.

  - No constraint should arbitrarily rule out feasible choices.

  - Constraints cannot be bent outside the model.

0.2.2.4 4. Proportionality

- Contributions of activities to the objective function are **proportional** to their level.

- Likewise, resource use is proportional: doubling an activity doubles its input use.

- This rules out fixed costs, economies of scale, or price effects that depend on output level.

0.2.2.5 5. Additivity

- Total contributions to the objective and resource use are the **sum of individual contributions**.
- No interactions among variables are allowed (e.g., no multiplicative terms).

0.2.2.6 6. Divisibility

- Decision variables can take on **fractional values**.
- This assumes continuous activities (e.g., acres of land).
- When variables must be integer (e.g., number of tractors), integer programming is required instead.

0.2.2.7 7. Certainty

- All parameters (objective coefficients, resource availability, input-output coefficients) are **known with certainty**.
- LP is thus a deterministic model.
- In practice, parameters are often estimated, and uncertainty can be explored with sensitivity or stochastic programming.

0.2.3 Teaching Note

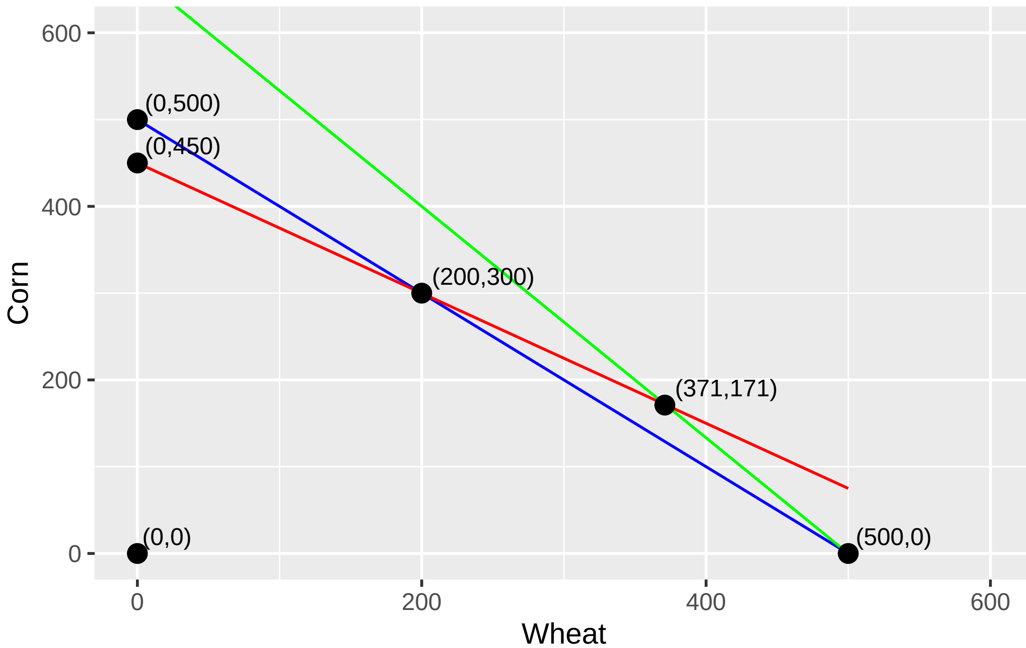
These assumptions both **enable LP to be solvable** and **limit realism**. They provide a natural segue to later topics in the course:

- Multi-objective programming (relax objective function assumption)
- Integer programming (relax divisibility)
- Stochastic programming (relax certainty)
- Nonlinear programming (relax proportionality and additivity)

0.2.4 Graphical Solution Method (2 variables)

Step 1. Draw the constraints.

- Land: $W + C \leq 500$
- Labor: $3W + 4C \leq 1800$
- Fertilizer: $4W + 3C \leq 2000$
- Nonnegativity: $W, C \geq 0$
- Plot based on endpoints. set one var 0 and devote all resources to that var.



Step 2. Identify the feasible region.

- Intersection of all constraints in the (W, C) plane.
- Polygon bounded by lines.

Step 3. Plot the objective function.

- Profit $= 200W + 300C$.
- Show isoprofit lines:
  - Suppose we plot $6000 profit. All wheat no corn, then all corn no wheat.
  - lines of constant profit slope $-\frac{200}{300} = -\frac{2}{3}$.

Step 4. Locate the optimum.

- Slide the isoprofit line outward until the last point of contact with the feasible region.
- Optimum is always at a **corner point** (fundamental theorem of LP).

0.2.5 Simplex Method

0.2.5.1 Why We Need It

- The graphical method only works for **two variables**.
- Real problems may involve **hundreds or thousands** of variables.
- Key geometric fact:
  - The feasible region of an LP is a **convex polytope**.

  - The **optimal solution lies at a vertex (corner point)**. What about the problem makes this a fact?
- The simplex method provides a **systematic way** to move from vertex to vertex until the best one is found.

0.2.5.2 Core Idea

- Start at a **basic feasible solution (BFS)** — a corner point of the feasible region.

- At each step:
  1. Compute **reduced costs** (how much the objective improves if a variable increases from 0).
  2. Identify an **entering variable** (the candidate to increase).
  3. Determine which constraint binds first — this sets the **leaving variable**.
  4. **Pivot** to a new BFS.
- Stop when no variable can improve the objective — this is optimal.

0.2.6 Intuition

- Simplex is like **walking along the edges** of the feasible polygon.
- At each corner, ask: *"If I move along this edge, does profit go up?"*
- Continue until no edge yields improvement.
- The same logic applies in higher dimensions, even though we cannot draw the polytope.

0.2.7 Simplex Pivot — Tiny Worked Example

We use the smallest LP that still shows the mechanics:

Problem

$$\begin{aligned} &\max z = 3x_1 + 2x_2 \\ &\text{s.t. } x_1 + x_2 \le 4, \quad x_1 \le 2, \quad x_1, x_2 \ge 0. \end{aligned}$$

Standard form (add slacks s_1, s_2)

$$\begin{aligned} x_1 + x_2 + s_1 &= 4, \\ x_1 + s_2 &= 2, \\ z - 3x_1 - 2x_2 &= 0, \qquad s_1, s_2 \ge 0. \end{aligned}$$

- Slack variables are added to "$\le$" constraints to convert them into equalities, making the LP system compatible with the simplex algorithm.

- They measure the unused portion of a resource — e.g., if a land constraint is $x_1 + x_2 \le 500$ and only 400 acres are used, the slack variable equals 100.
- Slack variables always have a zero coefficient in the objective function, since they do not directly contribute to profit or cost.
- Each slack variable typically appears with a coefficient of +1 in one constraint and 0 elsewhere, so they "fill the gap" between resource availability and resource use.
- At optimality, a nonzero slack indicates an unused resource. Checking which slack variables are positive helps interpret whether constraints are binding or loose.

0.2.8 Initial tableau and choice of entering/leaving variables

Initial Basic Feasible Solution (BFS): $x_1 = x_2 = 0 \Rightarrow s_1 = 4, s_2 = 2, z = 0$.

Tableau

	x_1	x_2	s_1	s_2	RHS
s_1	1	1	1	0	4
s_2	1	0	0	1	2
z	-3	-2	0	0	0

- **Entering variable:** look at the objective row; most negative reduced cost is under x_1 (coefficient -3) $\rightarrow$ **enter** x_1.
- **Leaving variable (ratio test):** divide RHS by the positive entries in the x_1 column:
  Row s_1: $4/1 = 4$, Row s_2: $2/1 = 2$. Minimum is $2 \rightarrow$ **leave** s_2.
- **Pivot element:** the entry at row s_2, column x_1 (which is 1).

We will **pivot on that 1**, swapping $s_2 \leftrightarrow x_1$.

0.2.9 Row operations (make pivot column a unit vector)

Goal: pivot column (x_1) should become $(0, 1, 0)^\top$.

1) **Normalize pivot row** (already 1, so no change):

$$(s_2) \;:\; [\,1\;\;0\;\;0\;\;1\,|\,2\,].$$

2) **Zero out the other entries in the x_1 column:**

- Row s_1: $(s_1) \leftarrow (s_1) - 1 \cdot (s_2)$

$$[\,1\;\;1\;\;1\;\;0\,|\,4\,] - [\,1\;\;0\;\;0\;\;1\,|\,2\,] = [\,0\;\;1\;\;1\;\;-1\,|\,2\,].$$

- Row z: $(z) \leftarrow (z) + 3 \cdot (s_2)$

$$[\,-3\;\;-2\;\;0\;\;0\,|\,0\,] + 3 \cdot [\,1\;\;0\;\;0\;\;1\,|\,2\,] = [\,0\;\;-2\;\;0\;\;3\,|\,6\,].$$

0.2.10 New tableau (after one pivot)

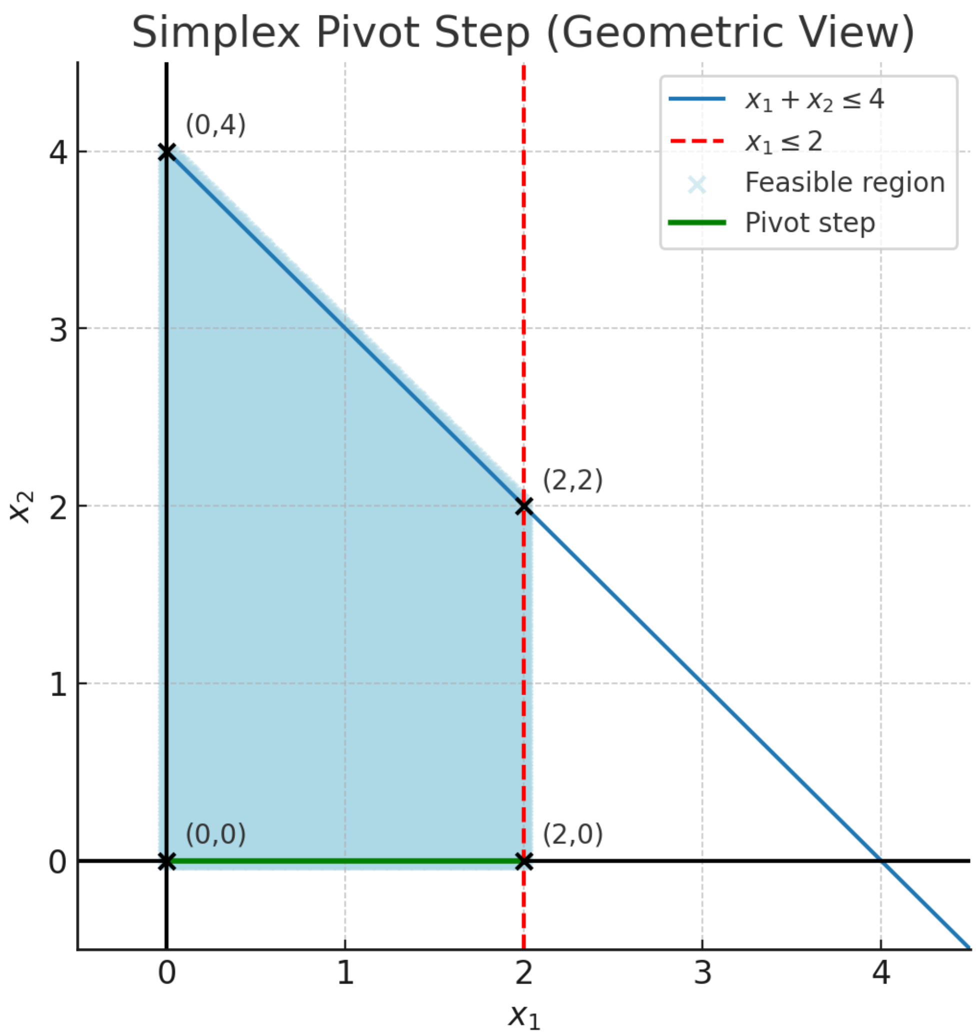
	x_1	x_2	s_1	s_2	RHS
s_1	0	1	1	-1	2
x_1	1	0	0	1	2
z	0	-2	0	3	6

- **Basis after pivot:** $\{\, s_1,\ x_1 \,\}$.

- **Current solution:** $x_1 = 2,\ x_2 = 0,\ s_1 = 2,\ s_2 = 0,\ z = 6.$

- The column labels show that x_1 **entered** (blue row) and s_2 **left**.

Next step (if continuing): the most negative in the z-row is under x_2 (-2), so x_2 would enter next; the algorithm would pivot again and reach the optimum at $(x_1, x_2) = (2, 2)$ with $z = 10$.



0.2.11 Excel Solver

Excel implements the simplex method in the solver add-on. See LP_land_alloc.xlxs

0.2.12 Summary

- Linear programming problems have:
  - **Decision variables** (choices to make),
  - **Objective function** (profit, cost, etc.),
  - **Constraints** (resource limits, requirements).
- Graphical method (2 variables) shows:
  - Feasible region = convex polygon.
  - Optimum occurs at a **corner point**.
- **Fundamental theorem of LP:** optimum is always at a vertex of the feasible region.
- **Simplex method** generalizes:
  - Moves systematically from one **basic feasible solution (BFS)** to another.
  - Uses **entering and leaving variables** to pivot.
  - Stops when no further improvement is possible.
- **Slack variables** convert inequalities to equalities and measure unused resources.
- Tableau row operations implement pivots (targeted Gaussian elimination).

0.3 Sensitivity Analysis

0.3.1 Review

- LPs make 7 assumptions. What are a few?
- How do LPs algorithms solve? What is the fundamental theorem of LP?

0.3.2 Motivation

- LPs give an **optimal solution** and an **objective value**.
- But: parameters (profits, resources, input requirements) are rarely known with certainty.
- We need to know how **robust** the solution is.

0.3.3 What is Sensitivity Analysis?

- Also called **post-optimality analysis**.
- Asks: *how much can model parameters change before the current solution changes?*
- Focus on three categories:
    1. RHS (resources)

    2. Objective coefficients (profits/costs)

    3. Technical coefficients (input-output relationships)

0.3.4 RHS (Right-Hand Side) Ranging

- Suppose resource availability changes:

$$b_{new} = b_{old} + \Delta r$$

- Current solution remains optimal **as long as basic variables stay $\geq$ 0**.
- Interpretation:
    - Within the allowable range, the **shadow price** is valid.
    - Outside the range, the optimal activity mix shifts.

0.3.5 Objective Coefficient Ranging

- How much can a profit coefficient change before the basis changes?
- For **nonbasic variables**: reduced costs must remain ≥ 0.
- For **basic variables**: check feasibility of objective row with new coefficient.

0.3.6 Technical Coefficient Changes

- What if technology or input requirements change?
- Example: wheat now needs 3.5 instead of 3 labor hours per acre.
- Sensitivity analysis uses shadow prices to approximate the effect on objective value.
- Interpretation: tighter labor constraints may shift which crop dominates.

0.3.7 The 100% Rule

The 100% Rule provides a way to evaluate whether simultaneous changes in objective function coefficients or right-hand side (RHS) values will preserve the current optimal basis — that is, whether the current solution remains optimal without re-solving the problem.

This rule applies separately to:

- Changes in objective function coefficients (i.e., the c_i's), and
- Changes in the RHS values of constraints (i.e., the b_j's).

Objective Function Coefficients

Suppose multiple objective coefficients change. For each decision variable x_i, let:

- Δc_i be the change in the objective coefficient.
- AL_i, AU_i be the allowable decrease and increase, respectively, as reported in the sensitivity analysis.

Define the proportion of the allowable range used for each change:

$$r_i = \begin{cases} \frac{|\Delta c_i|}{AL_i} & \text{if } \Delta c_i < 0 \\ \frac{|\Delta c_i|}{AU_i} & \text{if } \Delta c_i > 0 \end{cases}$$

Then, compute: $\sum_{i \in \text{changed vars}} r_i$. If: $\sum r_i \leq 1$, then the current optimal basis remains optimal, although the optimal value of the objective function will generally change. If the sum exceeds 1, the basis may change, and re-solving the problem is required.

0.3.8 Wheat–Corn Example (Excel Solver Report):

0.3.8.1 Excel top panel: Variable Cells

Interpretation of columns:

- **Final Value** – the optimal level of the decision variable (e.g., acres of wheat or corn).
- **Reduced Cost** – for nonbasic variables (value = 0), how much the objective coefficient must improve before the variable would enter the solution. Zero if the variable is positive in the solution.
- **Objective Coefficient** – the profit (or cost) per unit used in the objective function.
- **Allowable Increase / Decrease** – the range over which the objective coefficient can change without altering the current optimal basis (solution structure).

Interpretation of results

- Corn: optimal solution plants 450 acres of corn. Reduced cost = 0 because it's in the basis.
- Wheat: optimal solution plants 0 acres of wheat. Reduced cost = -25 means if wheat's profit increased by more than $25/acre (from 200 → 225), it would enter the solution.

Ranges:

- Wheat: profit can increase up to +25 before wheat enters.
- Corn: profit can fall as much as 33.3 (300 → 266.7) before solution changes.
- Interpretation: Corn dominates under current prices. Wheat only becomes attractive if its relative profit improves substantially.

0.3.8.2 Excel bottom panel: Constraints

Interpretation of columns:

- **Final Value** – the amount of the resource actually used at the solution.

- **Constraint R.H. Side** – the available amount of the resource (the right-hand side of the inequality).
- **Shadow Price** – the marginal value of relaxing the constraint (increase in objective if RHS increases by 1 unit), valid only within the allowable range.
- **Allowable Increase / Decrease** – the range over which the shadow price remains valid and the current basis stays optimal.

Interpretation of results:

- Land: only 450 acres used out of 500. Slack = 50 acres. Shadow price = 0 because land is not binding. You can increase land indefinitely without improving profit (since labor is the true bottleneck).
- Labor: fully used (1800/1800). Shadow price = 75 means each additional labor hour would increase profit by $75, valid for up to +200 extra hours.
- Fertilizer: only 1350 units used of 2000. Slack = 650. Shadow price = 0 because it's not binding.

Economic Takeaways for Discussion

- Which resource is scarce? Labor.
- How much should the agent be willing to pay for an additional unit of labor? 75 per labor hour. This is also the opportunity cost of labor on this operation.
- Why does wheat drop out of the solution? Because relative to corn it uses too much labor per profit dollar.
- Policy thought experiment: If labor availability were increased by 200 hours, profit would rise by $15,000 (200 × 75).

0.3.9 Economic Interpretation

- **Shadow prices**: marginal value of resources.
- **Allowable ranges**: robustness of those shadow prices.
- **Managerial use**:
  - Identify which resources are most binding.
  - Assess which profit coefficients are critical.
  - Evaluate new technologies or policy changes.

0.3.10 Summary

- Sensitivity analysis extends LP results beyond a single point solution.
- Provides insight into:
  - Resource valuation (RHS changes),
  - Profit robustness (objective changes),
  - Technology shifts (coefficient changes).

0.4 Getting Started with R & RStudio

0.4.1 RStudio Orientation (2–3 min tour)

- **Source Pane (top-left):** where you edit scripts (`.R`) and notebooks (`.qmd`, `.Rmd`).
- **Console (bottom-left):** runs commands immediately (`>` prompt).
- **Environment/History (top-right):** objects in memory (data, vectors, functions).
- **Files/Plots/Packages/Help (bottom-right):** manage files, see plots, install packages, read docs.

Workflow tips

- Create a **Project** (`File → New Project…`) for the course; it pins your working directory.
- Put scripts in a `code/` folder, data in `data/`, and outputs in `out/`.
- Use **scripts** for anything you might need to re-run. Avoid one-off console work for assignments.

0.4.2 Preview

- Next class: **duality** — formalize the relationship between primal and dual problems.
- Show how shadow prices emerge naturally from the dual formulation.

---

1 Duality in Linear Programming

1.1 Review

- Sensitivity analysis evaluates robustness of LP solutions to parameter changes.
- Shadow prices measure the marginal value of resources.
- Ranging analysis indicate how much parameters can change before the solution structure shifts.
- The 100% Rule helps assess simultaneous changes in parameters.
- Excel Solver provides a convenient way to solve LPs and get sensitivity reports.

1.2 Motivation for Duality

- Duality provides a **theoretical foundation** for shadow prices.
- Every LP (the **primal**) has a corresponding **dual** LP.
- Solutions to the dual give the **shadow prices** of the primal constraints.
- Duality reveals deep connections between resources and values.
- Duality also aids in sensitivity analysis and economic interpretation.

1.3 Canonical primal–dual pair

Primal (max form, $\leq$ constraints):

$$\begin{aligned} \max_{x \geq 0} \quad & c^\top x \\ \text{s.t.} \quad & Ax \leq b \end{aligned}$$

Dual (min form, $\geq$ constraints):

$$\begin{aligned} \min_{y \geq 0} \quad & b^\top y \\ \text{s.t.} \quad & A^\top y \geq c \end{aligned}$$

Rules of transformation (quick guide):

- One **dual variable** per **primal constraint**; one **dual constraint** per **primal variable**.
- Primal "$a_i^T x \leq b_i$" with $x \geq 0 \rightarrow$ dual variable $y_i \geq 0$. Relaxing the RHS makes the feasible set bigger, so the dual variable (shadow price) must be nonnegative.
- Primal "$a_i^T x \geq b_i$" $\rightarrow$ dual variable $y_i \leq 0$. Tightening the RHS makes the feasible set bigger (since it's "$\geq$"), so the associated price flips sign.
- Primal "$a_i^T x = b_i$" $\rightarrow$ dual variable y **free** (unrestricted in sign). An equality can cut the feasible set in either direction, so the shadow price can be positive or negative.
- If a primal variable is **free**, the corresponding dual constraint is an **equality**; if primal variable has sign $x_j \leq 0$, flip inequality accordingly.

1.4 Wheat–Corn primal and its dual

Primal (wheat–corn):

$$\begin{aligned} \max\ & 200\,x_W + 300\,x_C \\ \text{s.t.}\ & x_W + x_C \le 500 && \text{(land)} \\ & 3x_W + 4x_C \le 1800 && \text{(labor)} \\ & 4x_W + 3x_C \le 2000 && \text{(fertilizer)} \\ & x_W,\ x_C \ge 0\ . \end{aligned}$$

Introduce dual variables $y_1, y_2, y_3 \ge 0$ for (land, labor, fertilizer).

Dual:

$$\begin{aligned} \min\ & 500\,y_1 + 1800\,y_2 + 2000\,y_3 \\ \text{s.t.}\ & y_1 + 3y_2 + 4y_3 \ge 200 && \text{(wheat)} \\ & y_1 + 4y_2 + 3y_3 \ge 300 && \text{(corn)} \\ & y_1, y_2, y_3 \ge 0\ . \end{aligned}$$

Economic meanings:

y_1= land rent (per acre), y_2= wage (per labor hour), y_3= fertilizer value (per unit).

1.4.1 Solve directly by intuition

A low-cost feasible choice is to try just one variable:

- Set $y_1 = 0$, $y_3 = 0$. The corn constraint gives $4y_2 \ge 300 \Rightarrow y_2 \ge 75$.
- Check wheat: $3y_2 \ge 200 \Rightarrow y_2 \ge 66.67$. So $y_2 = 75$ satisfies both.
- Dual objective: $500(0) + 1800(75) + 2000(0) = 135{,}000$.
- Could adding anything to y_1 or y_3 lower the cost?
- No, because both constraints are already satisfied and any increase would only raise the objective.
- Feasibility check:
  - Wheat: $0 + 3(75) + 4(0) = 225 \ge 200$
  - Corn: $0 + 4(75) + 3(0) = 300 \ge 300$

Let's verify in excel

1.5 Duality theorems

1.5.1 Weak duality

For any feasible solution x and y,

$$c^\top x \le b^\top y$$

- From the primal constraints: $Ax \le b$
- From the dual constraints: $A^\top y \ge c$
- Premultiply the dual constraint by $x^\top \ge 0$: $x^\top A^\top y \ge x^\top c$ and note that $x^\top A^\top y = (Ax)^\top y$
- And post multiply the primal constraint by y: $(Ax)^\top y \le b^\top y$.
- Then putting the two together gives $c^\top x \le x^\top A^\top y = (Ax)^\top y \le b^\top y$.

Wheat-corn example:

- Primal feasible: $x_W = 0, x_C = 450 \Rightarrow Z = 135{,}000$.

- Dual feasible: $y_1 = 0, y_2 = 75, y_3 = 0 \Rightarrow W = 500(0) + 1800(75) + 2000(0) = 135{,}000$.
- Weak duality holds: $135{,}000 \leq 135{,}000$.

1.5.2 Strong duality

If the primal and dual have optimal solutions, so does the other, and $c^\top x^* = b^\top y^*$.

- At the optimum, the primal objective (profit) equals the dual objective (resource value).
- The upper bound becomes tight: best possible activity plan = best possible valuation of resources.
- Market analogy:
  - Primal: maximize profit given resource costs.
  - Dual: minimize resource costs to support given profits.
  - At equilibrium, profit = cost of resources.
- Wheat–corn example:
  - Primal optimum: $Z^* = 135{,}000$.
  - Dual optimum: $500(0) + 1800(75) + 2000(0) = 135{,}000$

1.5.3 Complementary slackness conditions

The **complementary slackness conditions** link the primal and dual solutions. They provide the bridge between activity levels and resource prices:

- For each **primal constraint** $a_i^\top x \leq b_i$ with dual variable $y_i \geq 0$:

  $$y_i\,(b_i - a_i^\top x) = 0$$

  This means that either:
  - The constraint is **binding** ($a_i^\top x = b_i$) and then $y_i \geq 0$,

  - Or the constraint is **slack** ($a_i^\top x < b_i$) and then $y_i = 0$.
- For each **primal variable** $x_j \geq 0$ with dual inequality $a_j^\top y \geq c_j$:

  $$x_j\,(a_j^\top y - c_j) = 0$$

  This means that either:
  - The activity is **produced** ($x_j > 0$) and then its dual inequality binds exactly ($a_j^\top y = c_j$),

  - Or the activity is **not produced** ($x_j = 0$) and then the dual inequality can be slack ($a_j^\top y > c_j$).

2 Summary

- Every LP has a corresponding dual LP.
- Dual variables represent shadow prices of primal constraints.
- Weak duality: primal objective $\leq$ dual objective for any feasible solutions.
- Strong duality: at optimality, primal objective = dual objective.
- Complementary slackness links primal and dual solutions, indicating which constraints and variables are binding.
- Duality provides a theoretical foundation for sensitivity analysis and economic interpretation of LP results.

3 Input–Output Modeling

3.1 Motivation

- Economies are networks of interdependent sectors.

- The output of one sector (e.g., steel) may be an input for another (e.g., car manufacturing).
- Understanding these linkages helps us analyze how a change in demand or policy ripples through the entire economy.
- Input–output (I–O) analysis was developed by **Wassily Leontief** (Nobel Prize, 1973).

3.2 Basic Input–Output Structure

- Consider an economy with n sectors.
- **Notation:**
  - x_j: total output of sector j.
  - a_{ij}: units of good i required as an intermediate input to produce one unit of good j.
  - f_j: final demand for good j (consumers, government, exports).
- **Balance equation (sector j):**

$$x_j = \sum_i a_{ij} x_j + f_j$$

- In vector notation:

$$x = Ax + f$$

where A is the matrix of technical coefficients.

> **Productivity Condition in Input–Output Models**
>
> For the Leontief model
>
> $$x = Ax + f \quad \Rightarrow \quad (I - A)x = f,$$
>
> the matrix $(I - A)$ must be invertible for a unique solution x to exist.
> This requires that the **technical coefficient matrix A be productive**.
>
> - For each sector j, the column sum
>
> $$\sum_i a_{ij} < 1$$
>
>   means that producing one unit of output j uses **less than one unit of total inputs**.
> - Leaves some output available for **final demand** ($f_j > 0$).
> - If $\sum_i a_{ij} = 1$, the industry just produces to sell to other industries.
> - If $\sum_i a_{ij} > 1$, the industry is not productive - uses more resources than it produces.
>
> **More formally:**
> The **spectral radius** (largest eigenvalue) of A satisfies $\rho(A) < 1$.
> This guarantees $(I - A)^{-1}$ exists and the Leontief inverse is well-defined.

3.3 The Leontief Model

- Rearrange the balance equation:

$$x - Ax = f \quad \Rightarrow \quad (I - A)x = f$$

- Provided $(I - A)$ is invertible:

$$x = (I - A)^{-1} f$$

- The matrix $(I - A)^{-1}$ is called the **Leontief inverse**.

- Interpretation: it captures both the **direct** and **indirect** requirements needed to meet final demand f.

- Example: An increase in demand for cars requires more steel, which in turn requires more mining, etc.

3.4 Worked Example (3-sector economy)

Suppose three sectors: Agriculture (Ag), Manufacturing (M), and Services (S).

- **Technical coefficient matrix** the fraction of output from each sector used as input by others:
  - Rows = input sectors (what is used).
  - Columns = output sectors (what is produced).
  - Entry a_{ij} = units of input i needed per unit of output j. Ag needs 30% of its own output, 20% from M, 10% from S, etc.

$$A = \begin{bmatrix} 0.3 & 0.2 & 0.1 \\ 0.1 & 0.4 & 0.2 \\ 0.2 & 0.1 & 0.3 \end{bmatrix}$$

Input / Output	Agriculture	Manufacturing	Services
Agriculture	0.3	0.2	0.1
Manufacturing	0.1	0.4	0.2
Services	0.2	0.1	0.3

- Final demand vector:

$$f = \begin{bmatrix} 100 \\ 150 \\ 200 \end{bmatrix}$$

- Solve:

$$x = (I - A)^{-1} f$$

- Interpretation: The required outputs x will be larger than f because each sector's output must also supply intermediates to others.

```r
# Technical coefficients (3×3)
A <- matrix(c(0.3,0.2,0.1,
              0.1,0.4,0.2,
              0.2,0.1,0.3), nrow = 3, byrow = TRUE)

I3 <- diag(3)

# Leontief inverse (you can also pre-specify L as below)
L <- solve(I3 - A)

# Base final demand
```

```r
f <- matrix(c(100,150,200), ncol = 1)

# Compute gross outputs
x <- L %*% f
round(x, 5)
```

```
         [,1]
[1,] 336.7347
[2,] 455.1020
[3,] 446.9388
```

3.5 The Leontief Inverse and Multipliers

The matrix $(I - A)^{-1}$ is called the **Leontief inverse**.
Each element of this matrix has an important interpretation: it measures the **total requirement** of one sector's output needed to deliver one unit of final demand in another sector.

- The entry (i, j) of $(I - A)^{-1}$ tells us: "How many units of sector i are needed, directly and indirectly, to produce one additional unit of final demand for sector j."
- **Direct requirements:** given by A.
  - Example: if $a_{12} = 0.2$, producing 1 unit of sector 2 requires 0.2 units of sector 1 directly.
- **Indirect requirements:** captured through higher powers of A.
  - A^2 shows inputs needed two steps back (e.g., steel needed to make machine tools, which then make cars).
  - A^3 shows three-step linkages, and so on.
- The series expansion makes this clear (Nuemann):

$$(I - A)^{-1} = I + A + A^2 + A^3 + \cdots$$

  So the Leontief inverse accumulates all **direct and indirect linkages**.

3.5.1 Multipliers

- A **multiplier** measures the total effect on the economy of a 1-unit increase in final demand for a given sector.
- In practice, the multiplier for sector j is the **column sum** of the jth column of $(I - A)^{-1}$.
  – It tells us how much total output across all sectors must rise to support 1 more unit of f_j.
- Sector-specific multipliers can also be read row by row:
  – The i, j entry shows the effect on sector i of a unit shock to demand in sector j.

3.5.2 Example (2-sector economy)

Suppose

$$A = \begin{bmatrix} 0.2 & 0.1 \\ 0.3 & 0.2 \end{bmatrix}, \quad (I - A)^{-1} = \begin{bmatrix} 1.27 & 0.16 \\ 0.48 & 1.24 \end{bmatrix}.$$

- A 1-unit increase in demand for sector 1 requires 1.27 units of sector 1 output and 0.48 units of sector 2 output in total.

- Column sums: $1.27 + 0.48 = 1.75$.
  - Interpretation: a 1-unit increase in final demand for sector 1 leads to 1.75 total units of output economy-wide (a multiplier of 1.75).

- For sector 2, the multiplier is $0.16 + 1.24 = 1.40$.

Takeaway: The Leontief inverse is not just an algebraic trick — it directly encodes the multipliers that make input–output analysis such a powerful tool for tracing the effects of shocks.

3.6 Assumptions of the Input–Output Model

- **Fixed input coefficients (Leontief technology)**
  - Each sector uses inputs in fixed proportions.
  - Example: if it takes 0.3 units of steel to make a car, every car always uses 0.3 units — no substitution is allowed.
- **Constant returns to scale**
  - Doubling output requires exactly double the inputs.
  - There are no economies or diseconomies of scale.
- **Linearity**
  - Input requirements are linear in output.
  - The balance equation $x = Ax + f$ is valid for all production levels.
- **Homogeneous outputs**
  - Each sector produces a single, uniform good.
  - All uses of that good (whether as intermediate or final demand) are treated as identical.
- **No supply constraints**
  - Any level of final demand f can, in principle, be met — resources are not limited.
  - The model solves for required gross outputs without considering feasibility of factors like labor, land, or capital.
- **Static technology and demand**
  - The input coefficients A are assumed constant over time (no innovation or efficiency change within the model run).
  - The model is comparative static — it compares equilibria before and after shocks, not dynamic adjustment paths.
- **Closed vs. open model treatment**
  - In the "open" model, households are part of final demand.
  - In the "closed" model, households are included as a sector that both consumes and supplies labor.

3.7 Input–Output Analysis in Practice

The input–output (I–O) model is usually used in a **calibration + shock** framework.
Rather than solving an optimization problem, we work with observed data to trace how the economy responds to changes in demand or technology.

1. Calibration (Base Year)

- Construct the **technical coefficient matrix** A from observed data:

$$a_{ij} = \frac{\text{input of } i \text{ used by } j}{\text{total output of } j}$$

  - Data typically come from national input–output tables.

- Observe base-year **final demand** f (household consumption, government, exports).

- Solve the identity

$$x = Ax + f \quad \Rightarrow \quad x = (I - A)^{-1} f$$

  to check consistency with observed outputs x.

- The calibrated model reproduces the base year exactly.

2. Apply a Shock

- Change final demand f:
  - Example: +10 units of exports of manufacturing.

- Or change technical coefficients A:
  - Example: new technology reduces electricity needed in agriculture.

- Or both.

3. Recompute Outputs

- Solve again:

$$x' = (I - A)^{-1} f'$$

  where f' is the new final demand (and possibly new A).

- The difference $x' - x$ shows the **direct and indirect effects** of the shock.

4. Interpret Results

- Which sectors expand the most?

- What are the **multipliers** (change in total output per unit change in final demand)?

- How do shocks propagate across the economy?

Example (illustrative):

- Suppose $f = (100, 150, 200)^\top$ and manufacturing exports rise by 20. And,

$$(I - A)^{-1} = \begin{bmatrix} 1.6 & 0.6 & 0.4 \\ 0.4 & 1.9 & 0.6 \\ 0..5 & 0.4 & 1.6 \end{bmatrix}$$

- Recalculate x'.

- Agriculture and services expand too, even though their final demand did not change — because they supply inputs to manufacturing.

```r
# Shock: +20 to Manufacturing final demand
df <- matrix(c(0,20,0), ncol = 1)
f_prime <- f + df

# New outputs after shock
x_prime <- L %*% f_prime

# Change in outputs
dx <- x_prime - x

list(
  x = round(x, 5),
  x_prime = round(x_prime, 5),
  delta_x = round(dx, 5)
)
```

```
$x
         [,1]
[1,] 336.7347
[2,] 455.1020
[3,] 446.9388

$x_prime
         [,1]
[1,] 348.9796
[2,] 493.4694
[3,] 455.9184

$delta_x
         [,1]
[1,] 12.24490
[2,] 38.36735
[3,]  8.97959
```

```r
# Manufacturing column of L (column 2)
L_col_M <- L[,2, drop=FALSE]
check_delta <- 20 * L_col_M

cbind(
  `20 * L[,2]` = round(check_delta, 5),
  `delta x`    = round(dx, 5)
)
```

```
         [,1]     [,2]
[1,] 12.24490 12.24490
[2,] 38.36735 38.36735
[3,]  8.97959  8.97959
```

This "calibrate + shock" procedure is the standard way input–output analysis is used in policy and research.

3.8 Relation to LP

We can recast the input–output system as a linear program. This allows us to apply the LP tools we've already studied (primal, dual, shadow prices) to questions of production and pricing.

- **Primal problem (quantities).**
  Decision variables are sector outputs x_j. The planner's problem might be to:
  - *Minimize cost* of meeting a fixed vector of final demands f, subject to input–output balance, or
  - *Maximize welfare* given resource or technology constraints.

  The balance condition $x = Ax + f$ can be rearranged as

  $$Ax + f \leq x,$$

  meaning that each sector's gross output must be at least enough to cover its intermediate input requirements (Ax) plus final demand (f).

- **Objective.**
  A typical choice is to minimize the total value of inputs required to satisfy demand. For example:

  $$\min \; c^\top x$$

  where c is a vector of unit costs. Alternatively, if outputs are valued in welfare terms, the objective can be to maximize $v^\top f$.

- **Constraints.**
  The inequalities $Ax + f \leq x$ ensure feasibility: sectors cannot promise more intermediate and final goods than they can produce.

- **Dual problem (prices).**
  The dual associates a variable p_i with each balance constraint. These p_i can be interpreted as **commodity prices** consistent with general equilibrium:
  - If a commodity is scarce (constraint binding), its price is positive.
  - If there is surplus, its shadow price falls to zero.

- **Strong duality.**
  At the optimum,

  $$\text{Value of outputs} = \text{Value of inputs}.$$

  In other words, the total revenue from selling goods at equilibrium prices equals the total cost of producing them using intermediate and primary inputs. This is the familiar condition for a competitive equilibrium in input–output economics.

Draw the parallels:

- *Primal* = "quantities world": how much each sector produces to satisfy demands.
- *Dual* = "prices world": what set of commodity prices makes those quantities consistent with equilibrium.
- Duality ensures these two views are perfectly consistent.

3.9 Applications in Economics

- **Environmental / Resource Economics**
  - Trace carbon, water, or energy embodied in consumption.

  - Assess land-use change from shifts in demand.
- **Policy Analysis**
  - Regional impact studies (IMPLAN, RIMS II).

  - Tariff or trade shock analysis.
- **Research extensions**
  - Hybrid I–O and LP models for energy systems planning.

  - Coupling I–O with CGE (computable general equilibrium) for price and income effects.

3.10 Classroom Activity

- Provide students with a small 2×2 A matrix and demand vector f.

- Ask them to compute:
  1. $(I - A)$

  2. $(I - A)^{-1}$

  3. $x = (I - A)^{-1} f$
- Discussion prompts:
  - Which sector has the larger multiplier?

  - What happens to required outputs if one coefficient a_{ij} increases (e.g., manufacturing requires more agriculture)?