# Linear Programming

### 0.0.1 Linear Programming (LP) Structure

- **Standard LP Form**:

$$\max\ \pi' x \quad \text{s.t. } Ax \le b,\ x \ge 0$$

- Characteristics:
  - Linear objective.
  - Linear constraints.
  - Nonnegativity.

Can also be written in matrix notation.

**Economic interpretation:**

- $\pi$: profit per unit.
- $A$: resource use matrix.
- $b$: resource endowment.

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### 0.0.2 Model Building Process (McCarl framework)

1. Identify **decision variables**.
2. State the **objective**.
3. Identify and formulate **constraints**.
4. Collect data.
5. Translate into computer-readable form.
6. Solve and interpret.

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### 0.0.3 Farm LP Example

A farmer has 500 acres of land available and is deciding how to allocate it between wheat and corn. Each acre of wheat yields a profit of $200, requires 3 hours of labor, and 4 units of fertilizer. Each acre of corn yields a profit of $300, requires 4 hours of labor, and 3 units of fertilizer. The farm has at most 1,800 hours of labor available and 2,000 units of fertilizer.

Formulate this situation as a linear programming problem. Clearly define the decision variables, write down the objective function representing total profit, and specify the constraints that capture the land, labor, fertilizer, and nonnegativity restrictions.

**Decision variables.**
Let $W$ = acres of wheat, $C$ = acres of corn.

### 0.0.4 Scalar (algebraic) form

$$
\begin{aligned}
\max_{W,C} \quad & 200\,W + 300\,C \\
\text{s.t.} \quad & W + C \le 500 && \text{(land)} \\
& 3W + 4C \le 1800 && \text{(labor)} \\
& 4W + 3C \le 2000 && \text{(fertilizer)} \\
& W,\ C \ge 0\ .
\end{aligned}
$$

### 0.0.5 Matrix (compact) form

$$
\begin{aligned}
\max_{x \in \mathbb{R}^2_{\ge 0}} \quad & c^\top x \\
\text{s.t.} \quad & Ax \le b,
\end{aligned}
\qquad \text{with} \quad x = \begin{bmatrix} W \\ C \end{bmatrix},\ c = \begin{bmatrix} 200 \\ 300 \end{bmatrix},\ A = \begin{bmatrix} 1 & 1 \\ 3 & 4 \\ 4 & 3 \end{bmatrix},\ b = \begin{bmatrix} 500 \\ 1800 \\ 2000 \end{bmatrix}.
$$

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## 0.1 Summary

- Optimization = decision making under constraints.
- Mathematical programming provides a general framework: decision variables, objective, constraints.
- Linear programs (LPs): linear objective + linear constraints + nonnegativity.
- Example: Farm resource allocation with land, labor, fertilizer.

## 0.2 Day 2: Linear Programming in Practice

### 0.2.1 Review

- General form of a mathematical program.
- Structure of a linear program:
  - Linear objective
  - Linear constraints
  - Nonnegativity
- Wheat–Corn farm allocation example in scalar and matrix notation.

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### 0.2.2 7 Assumptions of Linear Programming

Linear programming models rely on a set of assumptions that make them tractable but also limit their realism. McCarl & Spreen (Ch. 2.4) identify **seven important assumptions**. The first three involve the *appropriateness of the formulation*; the last four describe *mathematical properties* of the LP model.

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#### 0.2.2.1 1. Objective Function Appropriateness

- The objective function is assumed to be the **sole criterion** for evaluating solutions.
- This means the decision maker's preferences can be fully represented by a single linear function (e.g., profit, cost, utility).
- In practice, decisions may depend on multiple objectives (profit, risk, leisure), but LP assumes one dominates.

#### 0.2.2.2 2. Decision Variable Appropriateness

- All relevant decision variables must be included, and each must be **fully controllable** by the decision maker.
- Omitting key variables or including variables outside the decision maker's control invalidates the formulation.

#### 0.2.2.3 3. Constraint Appropriateness

- Constraints must **accurately and completely capture** the limits faced by the decision maker:
  - They fully describe resource, technological, and institutional limits.
  - Resources within a constraint are **homogeneous** and freely substitutable among activities.
  - No constraint should arbitrarily rule out feasible choices.
  - Constraints cannot be bent outside the model.

#### 0.2.2.4 4. Proportionality

- Contributions of activities to the objective function are **proportional** to their level.
- Likewise, resource use is proportional: doubling an activity doubles its input use.
- This rules out fixed costs, economies of scale, or price effects that depend on output level.

#### 0.2.2.5 5. Additivity

- Total contributions to the objective and resource use are the **sum of individual contributions**.
- No interactions among variables are allowed (e.g., no multiplicative terms).

#### 0.2.2.6 6. Divisibility

- Decision variables can take on **fractional values**.
- This assumes continuous activities (e.g., acres of land).
- When variables must be integer (e.g., number of tractors), integer programming is required instead.

#### 0.2.2.7 7. Certainty

- All parameters (objective coefficients, resource availability, input-output coefficients) are **known with certainty**.
- LP is thus a deterministic model.
- In practice, parameters are often estimated, and uncertainty can be explored with sensitivity or stochastic programming.

### 0.2.3 Teaching Note

These assumptions both **enable LP to be solvable** and **limit realism**. They provide a natural segue to later topics in the course:

- Multi-objective programming (relax objective function assumption)
- Integer programming (relax divisibility)
- Stochastic programming (relax certainty)
- Nonlinear programming (relax proportionality and additivity)

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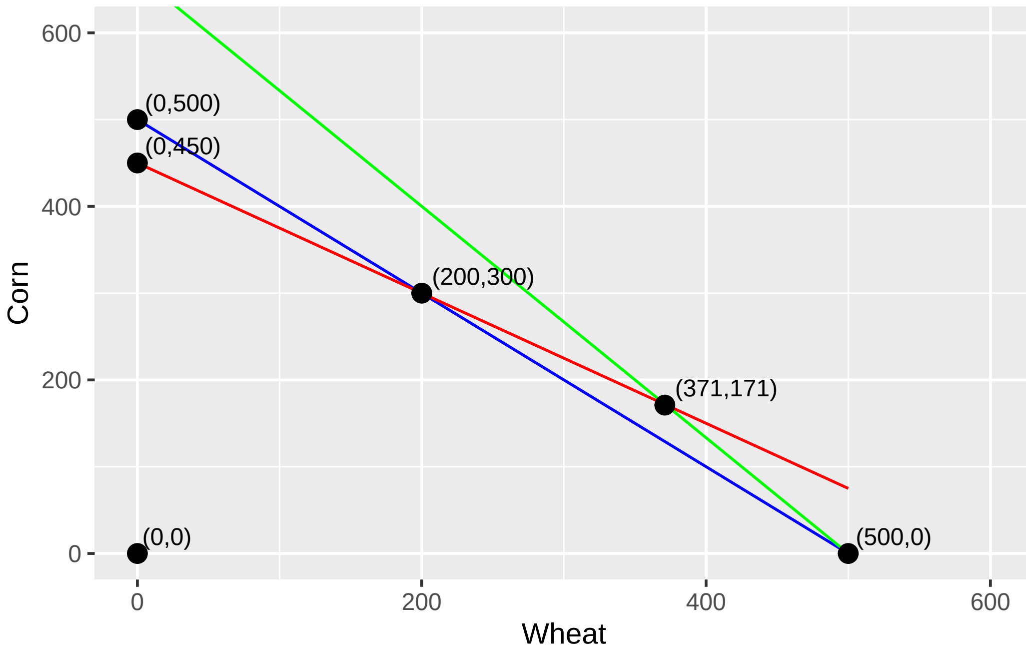
### 0.2.4 Graphical Solution Method (2 variables)

**Step 1. Draw the constraints.**

- Land: $W + C \leq 500$
- Labor: $3W + 4C \leq 1800$
- Fertilizer: $4W + 3C \leq 2000$
- Nonnegativity: $W, C \geq 0$
- Plot based on endpoints. set one var 0 and devote all resources to that var.




**Step 2. Identify the feasible region.**

- Intersection of all constraints in the $(W, C)$ plane.
- Polygon bounded by lines.

**Step 3. Plot the objective function.**

- Profit $= 200W + 300C$.
- Show isoprofit lines:
  - Suppose we plot $6000 profit. All wheat no corn, then all corn no wheat.
  - lines of constant profit slope $-\frac{200}{300} = -\frac{2}{3}$.

**Step 4. Locate the optimum.**

- Slide the isoprofit line outward until the last point of contact with the feasible region.
- Optimum is always at a **corner point** (fundamental theorem of LP).

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### 0.2.5 Simplex Method

#### 0.2.5.1 Why We Need It

- The graphical method only works for **two variables**.
- Real problems may involve **hundreds or thousands** of variables.
- Key geometric fact:
  - The feasible region of an LP is a **convex polytope**.

  - The **optimal solution lies at a vertex (corner point)**. What about the problem makes this a fact?
- The simplex method provides a **systematic way** to move from vertex to vertex until the best one is found.

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#### 0.2.5.2 Core Idea

- Start at a **basic feasible solution (BFS)** — a corner point of the feasible region.

- At each step:
  1. Compute **reduced costs** (how much the objective improves if a variable increases from 0).
  2. Identify an **entering variable** (the candidate to increase).
  3. Determine which constraint binds first — this sets the **leaving variable**.
  4. **Pivot** to a new BFS.
- Stop when no variable can improve the objective — this is optimal.

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### 0.2.6 Intuition

- Simplex is like **walking along the edges** of the feasible polygon.
- At each corner, ask: *"If I move along this edge, does profit go up?"*
- Continue until no edge yields improvement.
- The same logic applies in higher dimensions, even though we cannot draw the polytope.

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### 0.2.7 Simplex Pivot — Tiny Worked Example

We use the smallest LP that still shows the mechanics:

**Problem**

$$\begin{aligned} &\max z = 3x_1 + 2x_2 \\ &\text{s.t. } x_1 + x_2 \leq 4, \quad x_1 \leq 2, \quad x_1, x_2 \geq 0. \end{aligned}$$

**Standard form (add slacks $s_1, s_2$)**

$$\begin{aligned} x_1 + x_2 + s_1 &= 4, \\ x_1 + s_2 &= 2, \\ z - 3x_1 - 2x_2 &= 0, \qquad s_1, s_2 \geq 0. \end{aligned}$$

- Slack variables are added to "$\leq$" constraints to convert them into equalities, making the LP system compatible with the simplex algorithm.

- They measure the unused portion of a resource — e.g., if a land constraint is $x_1 + x_2 \leq 500$ and only 400 acres are used, the slack variable equals 100.
- Slack variables always have a zero coefficient in the objective function, since they do not directly contribute to profit or cost.
- Each slack variable typically appears with a coefficient of +1 in one constraint and 0 elsewhere, so they "fill the gap" between resource availability and resource use.
- At optimality, a nonzero slack indicates an unused resource. Checking which slack variables are positive helps interpret whether constraints are binding or loose.

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### 0.2.8 Initial tableau and choice of entering/leaving variables

**Initial Basic Feasible Solution (BFS):** $x_1 = x_2 = 0 \Rightarrow s_1 = 4, s_2 = 2, z = 0$.

**Tableau**

| | $x_1$ | $x_2$ | $s_1$ | $s_2$ | RHS |
|---|---|---|---|---|---|
| $s_1$ | 1 | 1 | 1 | 0 | 4 |
| $s_2$ | 1 | 0 | 0 | 1 | 2 |
| $z$ | $-3$ | $-2$ | 0 | 0 | 0 |

- **Entering variable:** look at the objective row; most negative reduced cost is under $x_1$ (coefficient $-3$) $\rightarrow$ **enter** $x_1$.
- **Leaving variable (ratio test):** divide RHS by the positive entries in the $x_1$ column:
  Row $s_1$: $4/1 = 4$, Row $s_2$: $2/1 = 2$. Minimum is $2 \rightarrow$ **leave** $s_2$.
- **Pivot element:** the entry at row $s_2$, column $x_1$ (which is 1).

We will **pivot on that 1**, swapping $s_2 \leftrightarrow x_1$.

### 0.2.9 Row operations (make pivot column a unit vector)

Goal: pivot column ($x_1$) should become $(0, 1, 0)^\top$.

1) **Normalize pivot row** (already 1, so no change):

$$(s_2)\ :\ [\,1\ \ 0\ \ 0\ \ 1\,|\,2\,].$$

2) **Zero out the other entries in the $x_1$ column:**

- Row $s_1$: $(s_1) \leftarrow (s_1) - 1 \cdot (s_2)$

$$[\,1\ \ 1\ \ 1\ \ 0\,|\,4\,] - [\,1\ \ 0\ \ 0\ \ 1\,|\,2\,] = [\,0\ \ 1\ \ 1\ \ -1\,|\,2\,].$$

- Row $z$: $(z) \leftarrow (z) + 3 \cdot (s_2)$

$$[\,-3\ \ -2\ \ 0\ \ 0\,|\,0\,] + 3 \cdot [\,1\ \ 0\ \ 0\ \ 1\,|\,2\,] = [\,0\ \ -2\ \ 0\ \ 3\,|\,6\,].$$

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### 0.2.10 New tableau (after one pivot)

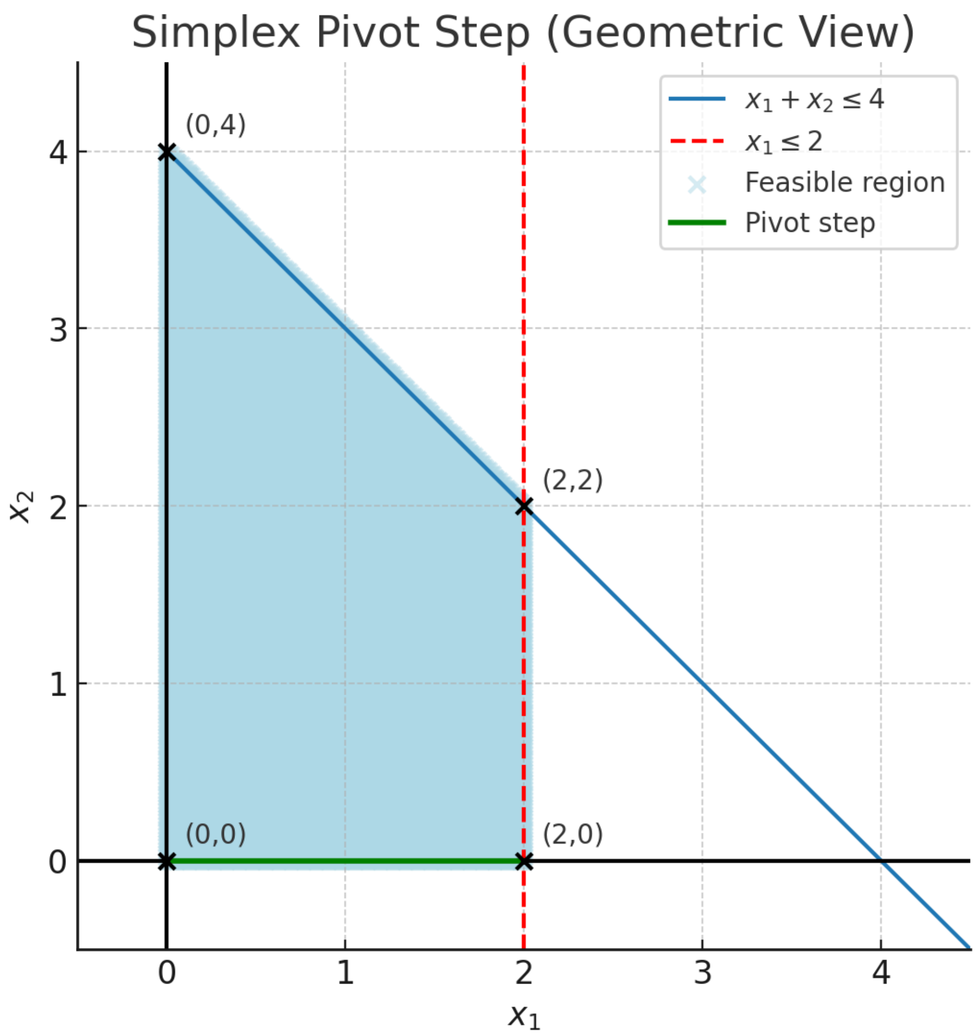
| | $x_1$ | $x_2$ | $s_1$ | $s_2$ | RHS |
|---|---|---|---|---|---|
| $s_1$ | 0 | 1 | 1 | $-1$ | 2 |
| $x_1$ | 1 | 0 | 0 | 1 | 2 |
| $z$ | 0 | $-2$ | 0 | 3 | 6 |

- **Basis after pivot:** $\{\, s_1,\ x_1 \,\}$.

- **Current solution:** $x_1 = 2,\ x_2 = 0,\ s_1 = 2,\ s_2 = 0,\ z = 6.$

- The column labels show that $x_1$ **entered** (blue row) and $s_2$ **left**.

**Next step (if continuing):** the most negative in the $z$-row is under $x_2$ $(-2)$, so $x_2$ would enter next; the algorithm would pivot again and reach the optimum at $(x_1, x_2) = (2, 2)$ with $z = 10$.




### 0.2.11 Excel Solver

Excel implements the simplex method in the solver add-on. See LP_land_alloc.xlxs

### 0.2.12 Summary

- Linear programming problems have:
  - **Decision variables** (choices to make),
  - **Objective function** (profit, cost, etc.),
  - **Constraints** (resource limits, requirements).
- Graphical method (2 variables) shows:
  - Feasible region = convex polygon.
  - Optimum occurs at a **corner point**.
- **Fundamental theorem of LP:** optimum is always at a vertex of the feasible region.
- **Simplex method** generalizes:
  - Moves systematically from one **basic feasible solution (BFS)** to another.
  - Uses **entering and leaving variables** to pivot.
  - Stops when no further improvement is possible.
- **Slack variables** convert inequalities to equalities and measure unused resources.
- Tableau row operations implement pivots (targeted Gaussian elimination).

## 0.3 Sensitivity Analysis

### 0.3.1 Review

- LPs make 7 assumptions. What are a few?
- How do LPs algorithms solve? What is the fundamental theorem of LP?

### 0.3.2 Motivation

- LPs give an **optimal solution** and an **objective value**.
- But: parameters (profits, resources, input requirements) are rarely known with certainty.
- We need to know how **robust** the solution is.

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### 0.3.3 What is Sensitivity Analysis?

- Also called **post-optimality analysis**.
- Asks: *how much can model parameters change before the current solution changes?*
- Focus on three categories:
    1. RHS (resources)

    2. Objective coefficients (profits/costs)

    3. Technical coefficients (input-output relationships)

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### 0.3.4 RHS (Right-Hand Side) Ranging

- Suppose resource availability changes:

$$b_{new} = b_{old} + \Delta r$$

- Current solution remains optimal **as long as basic variables stay $\geq$ 0**.
- Interpretation:
    - Within the allowable range, the **shadow price** is valid.
    - Outside the range, the optimal activity mix shifts.

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### 0.3.5 Objective Coefficient Ranging

- How much can a profit coefficient change before the basis changes?
- For **nonbasic variables**: reduced costs must remain $\geq 0$.
- For **basic variables**: check feasibility of objective row with new coefficient.

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### 0.3.6 Technical Coefficient Changes

- What if technology or input requirements change?
- Example: wheat now needs 3.5 instead of 3 labor hours per acre.
- Sensitivity analysis uses shadow prices to approximate the effect on objective value.
- Interpretation: tighter labor constraints may shift which crop dominates.

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### 0.3.7 The 100% Rule

The 100% Rule provides a way to evaluate whether simultaneous changes in objective function coefficients or right-hand side (RHS) values will preserve the current optimal basis — that is, whether the current solution remains optimal without re-solving the problem.

This rule applies separately to:

- Changes in objective function coefficients (i.e., the $c_i$'s), and
- Changes in the RHS values of constraints (i.e., the $b_j$'s).

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Objective Function Coefficients

Suppose multiple objective coefficients change. For each decision variable $x_i$, let:

- $\Delta c_i$ be the change in the objective coefficient.
- $AL_i, AU_i$ be the allowable decrease and increase, respectively, as reported in the sensitivity analysis.

Define the proportion of the allowable range used for each change:

$$r_i = \begin{cases} \frac{|\Delta c_i|}{AL_i} & \text{if } \Delta c_i < 0 \\ \frac{|\Delta c_i|}{AU_i} & \text{if } \Delta c_i > 0 \end{cases}$$

Then, compute: $\sum_{i \in \text{changed vars}} r_i$. If: $\sum r_i \leq 1$, then the current optimal basis remains optimal, although the optimal value of the objective function will generally change. If the sum exceeds 1, the basis may change, and re-solving the problem is required.

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### 0.3.8 Wheat–Corn Example (Excel Solver Report):

#### 0.3.8.1 Excel top panel: Variable Cells

Interpretation of columns:

- **Final Value** – the optimal level of the decision variable (e.g., acres of wheat or corn).
- **Reduced Cost** – for nonbasic variables (value = 0), how much the objective coefficient must improve before the variable would enter the solution. Zero if the variable is positive in the solution.
- **Objective Coefficient** – the profit (or cost) per unit used in the objective function.
- **Allowable Increase / Decrease** – the range over which the objective coefficient can change without altering the current optimal basis (solution structure).

Interpretation of results

- Corn: optimal solution plants 450 acres of corn. Reduced cost = 0 because it's in the basis.
- Wheat: optimal solution plants 0 acres of wheat. Reduced cost = -25 means if wheat's profit increased by more than \$25/acre (from 200 → 225), it would enter the solution.

Ranges:

- Wheat: profit can increase up to +25 before wheat enters.
- Corn: profit can fall as much as 33.3 (300 → 266.7) before solution changes.
- Interpretation: Corn dominates under current prices. Wheat only becomes attractive if its relative profit improves substantially.

#### 0.3.8.2 Excel bottom panel: Constraints

Interpretation of columns:

- **Final Value** – the amount of the resource actually used at the solution.

- **Constraint R.H. Side** – the available amount of the resource (the right-hand side of the inequality).
- **Shadow Price** – the marginal value of relaxing the constraint (increase in objective if RHS increases by 1 unit), valid only within the allowable range.
- **Allowable Increase / Decrease** – the range over which the shadow price remains valid and the current basis stays optimal.

Interpretation of results:

- Land: only 450 acres used out of 500. Slack = 50 acres. Shadow price = 0 because land is not binding. You can increase land indefinitely without improving profit (since labor is the true bottleneck).
- Labor: fully used (1800/1800). Shadow price = 75 means each additional labor hour would increase profit by $75, valid for up to +200 extra hours.
- Fertilizer: only 1350 units used of 2000. Slack = 650. Shadow price = 0 because it's not binding.

**Economic Takeaways for Discussion**

- Which resource is scarce? Labor.
- How much should the agent be willing to pay for an additional unit of labor? 75 per labor hour. This is also the opportunity cost of labor on this operation.
- Why does wheat drop out of the solution? Because relative to corn it uses too much labor per profit dollar.
- Policy thought experiment: If labor availability were increased by 200 hours, profit would rise by $15,000 (200 × 75).

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#### 0.3.9 Economic Interpretation

- **Shadow prices**: marginal value of resources.
- **Allowable ranges**: robustness of those shadow prices.
- **Managerial use**:
  - Identify which resources are most binding.
  - Assess which profit coefficients are critical.
  - Evaluate new technologies or policy changes.

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#### 0.3.10 Summary

- Sensitivity analysis extends LP results beyond a single point solution.
- Provides insight into:
  - Resource valuation (RHS changes),
  - Profit robustness (objective changes),
  - Technology shifts (coefficient changes).

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### 0.4 Getting Started with R & RStudio

#### 0.4.1 RStudio Orientation (2–3 min tour)

- **Source Pane (top-left):** where you edit scripts (`.R`) and notebooks (`.qmd`, `.Rmd`).
- **Console (bottom-left):** runs commands immediately (`>` prompt).
- **Environment/History (top-right):** objects in memory (data, vectors, functions).
- **Files/Plots/Packages/Help (bottom-right):** manage files, see plots, install packages, read docs.

**Workflow tips**

- Create a **Project** (`File → New Project…`) for the course; it pins your working directory.
- Put scripts in a `code/` folder, data in `data/`, and outputs in `out/`.
- Use **scripts** for anything you might need to re-run. Avoid one-off console work for assignments.

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#### 0.4.2 Preview

- Next class: **duality** — formalize the relationship between primal and dual problems.
- Show how shadow prices emerge naturally from the dual formulation.

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# 1 Duality in Linear Programming

## 1.1 Review

- Sensitivity analysis evaluates robustness of LP solutions to parameter changes.
- Shadow prices measure the marginal value of resources.
- Ranging analysis indicate how much parameters can change before the solution structure shifts.
- The 100% Rule helps assess simultaneous changes in parameters.
- Excel Solver provides a convenient way to solve LPs and get sensitivity reports.

## 1.2 Motivation for Duality

- Duality provides a **theoretical foundation** for shadow prices.
- Every LP (the **primal**) has a corresponding **dual** LP.
- Solutions to the dual give the **shadow prices** of the primal constraints.
- Duality reveals deep connections between resources and values.
- Duality also aids in sensitivity analysis and economic interpretation.

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## 1.3 Canonical primal–dual pair

**Primal (max form, $\leq$ constraints):**

$$\begin{aligned} \max_{x\geq 0} \quad & c^\top x \\ \text{s.t.} \quad & Ax \leq b \end{aligned}$$

**Dual (min form, $\geq$ constraints):**

$$\begin{aligned} \min_{y\geq 0} \quad & b^\top y \\ \text{s.t.} \quad & A^\top y \geq c \end{aligned}$$

**Rules of transformation (quick guide):**

- One **dual variable** per **primal constraint**; one **dual constraint** per **primal variable**.
- Primal "$a_i^T x \leq b_i$" with $x \geq 0 \to$ dual variable $y_i \geq 0$. Relaxing the RHS makes the feasible set bigger, so the dual variable (shadow price) must be nonnegative.
- Primal "$a_i^T x \geq b_i$" $\to$ dual variable $y_i \leq 0$. Tightening the RHS makes the feasible set bigger (since it's "$\geq$"), so the associated price flips sign.
- Primal "$a_i^T x = b_i$" $\to$ dual variable $y$ **free** (unrestricted in sign). An equality can cut the feasible set in either direction, so the shadow price can be positive or negative.
- If a primal variable is **free**, the corresponding dual constraint is an **equality**; if primal variable has sign $x_j \leq 0$, flip inequality accordingly.

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## 1.4 Wheat–Corn primal and its dual

**Primal (wheat–corn):**

$$\begin{aligned} \max\ & 200\,x_W + 300\,x_C \\ \text{s.t.}\ & x_W + x_C \le 500 && \text{(land)} \\ & 3x_W + 4x_C \le 1800 && \text{(labor)} \\ & 4x_W + 3x_C \le 2000 && \text{(fertilizer)} \\ & x_W,\ x_C \ge 0\ . \end{aligned}$$

Introduce dual variables $y_1, y_2, y_3 \ge 0$ for (land, labor, fertilizer).

**Dual:**

$$\begin{aligned} \min\ & 500\,y_1 + 1800\,y_2 + 2000\,y_3 \\ \text{s.t.}\ & y_1 + 3y_2 + 4y_3 \ge 200 && \text{(wheat)} \\ & y_1 + 4y_2 + 3y_3 \ge 300 && \text{(corn)} \\ & y_1, y_2, y_3 \ge 0\ . \end{aligned}$$

**Economic meanings:**

$y_1$= land rent (per acre), $y_2$= wage (per labor hour), $y_3$= fertilizer value (per unit).

### 1.4.1 Solve directly by intuition

A low-cost feasible choice is to try just one variable:

- Set $y_1 = 0$, $y_3 = 0$. The corn constraint gives $4y_2 \ge 300 \Rightarrow y_2 \ge 75$.
- Check wheat: $3y_2 \ge 200 \Rightarrow y_2 \ge 66.67$. So $y_2 = 75$ satisfies both.
- Dual objective: $500(0) + 1800(75) + 2000(0) = 135{,}000$.
- Could adding anything to $y_1$ or $y_3$ lower the cost?
- No, because both constraints are already satisfied and any increase would only raise the objective.
- Feasibility check:
  - Wheat: $0 + 3(75) + 4(0) = 225 \ge 200$
  - Corn: $0 + 4(75) + 3(0) = 300 \ge 300$

**Let's verify in excel**

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## 1.5 Duality theorems

### 1.5.1 Weak duality

For any feasible solution $x$ and $y$,

$$c^\top x \le b^\top y$$

- From the primal constraints: $Ax \le b$
- From the dual constraints: $A^\top y \ge c$
- Premultiply the dual constraint by $x^\top \ge 0$: $x^\top A^\top y \ge x^\top c$ and note that $x^\top A^\top y = (Ax)^\top y$
- And post multiply the primal constraint by $y$: $(Ax)^\top y \le b^\top y$.
- Then putting the two together gives $c^\top x \le x^\top A^\top y = (Ax)^\top y \le b^\top y$.

Wheat-corn example:

- Primal feasible: $x_W = 0, x_C = 450 \Rightarrow Z = 135{,}000$.

- Dual feasible: $y_1 = 0, y_2 = 75, y_3 = 0 \Rightarrow W = 500(0) + 1800(75) + 2000(0) = 135{,}000$.
- Weak duality holds: $135{,}000 \le 135{,}000$.

### 1.5.2 Strong duality

If the primal and dual have optimal solutions, so does the other, and $c^\top x^* = b^\top y^*$.

- At the optimum, the primal objective (profit) equals the dual objective (resource value).
- The upper bound becomes tight: best possible activity plan = best possible valuation of resources.
- Market analogy:
  - Primal: maximize profit given resource costs.
  - Dual: minimize resource costs to support given profits.
  - At equilibrium, profit = cost of resources.
- Wheat–corn example:
  - Primal optimum: $Z^* = 135{,}000$.
  - Dual optimum: $500(0) + 1800(75) + 2000(0) = 135{,}000$

### 1.5.3 Complementary slackness

The **complementary slackness conditions** link the primal and dual solutions. They provide the bridge between activity levels and resource prices:

- For each **primal constraint** $a_i^\top x \le b_i$ with dual variable $y_i \ge 0$:

$$y_i \, (b_i - a_i^\top x) = 0$$

  This means that either:
  - The constraint is **binding** ($a_i^\top x = b_i$) and then $y_i \ge 0$,

  - Or the constraint is **slack** ($a_i^\top x < b_i$) and then $y_i = 0$.
- For each **primal variable** $x_j \ge 0$ with dual inequality $a_j^\top y \ge c_j$:

$$x_j \, (a_j^\top y - c_j) = 0$$

  This means that either:
  - The activity is **produced** ($x_j > 0$) and then its dual inequality binds exactly ($a_j^\top y = c_j$),

  - Or the activity is **not produced** ($x_j = 0$) and then the dual inequality can be slack ($a_j^\top y > c_j$).

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## 1.6 Verify with the wheat–corn solution

From your Solver/`lpSolve` runs, the primal optimum is (x_W=0,, x_C=450) with profit (Z^* = 135{,}000). The shadow prices were (y_1=0) (land), (y_2=75) (labor), (y_3=0) (fertilizer).

- **Dual feasibility check:**
  - Wheat: (y_1 + 3y_2 + 4y_3 = 0 + 225 + 0 = 225  200)
  - Corn: (y_1 + 4y_2 + 3y_3 = 0 + 300 + 0 = 300  300)
- **Dual objective:** (500(0) + 1800(75) + 2000(0) = 135{,}000 = Z^*) (strong duality).
- **Complementary slackness:**
  - Land slack (= 50 > 0  y_1=0).
  - Fertilizer slack (= 650 > 0  y_3=0).
  - Labor binding ((=0,slack)  y_2>0).
  - Corn produced ((x_C>0) ) its dual constraint binds at equality (y_1+4y_2+3y_3=300).
  - Wheat not produced ((x_W=0) ) its dual constraint may be slack (here 225 > 200).

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### 1.6.1 6) Managerial interpretation

- Dual variable (y_i) is the **marginal value** of resource (i): increase RHS by 1 unit → objective rises by (y_i) (within its allowable range).
- In the example, labor is the bottleneck: each extra hour adds **$75** of profit until another resource becomes binding.
- Duality links directly to **sensitivity**: RHS ranges = where these prices remain valid.