

# Natural Capital: From Metaphor to Measurement

## 1 Introduction

Fenichel and Abbott (2014) address the challenge of moving from the *metaphor* of natural capital to its *measurement*.

While economists and ecologists agree that natural resources function as forms of capital, they note that practical valuation methods lag far behind theory.

Their contribution is to provide a **capital-theoretic approach** to valuing natural capital that:

- Links to **Jorgenson's theory of capital** in private markets.
- Applies under **imperfect or non-optimizing institutions**.
- Yields **accounting prices** consistent with welfare economics.

## 2 Conceptual Foundation

### 2.1 What is Natural Capital?

Natural capital is the stock of natural resources that yields flows of ecosystem services and welfare benefits over time. Examples include fish stocks, forests, wetlands, and groundwater.

Like produced capital, natural capital:

- Stores wealth.
- Generates productive flows.
- Has an intertemporal trade-off between use today and conservation for tomorrow.

However, markets for many forms of natural capital are missing or incomplete, making their valuation difficult.

## 3 Theoretical Framework

### 3.1 State Dynamics

Let  $s(t)$  denote the natural capital stock, and  $x(t)$  represent human behavior or extraction effort. The stock evolves as

$$\dot{s} = G(s) - f(s, x(s))$$

where:

- $G(s)$  is the ecological growth function,
- $f(s, x)$  represents human extraction or degradation of the stock,
- $x(s)$  is a behavioral *feedback rule* determined by institutions  $\Omega$ .

The feedback rule  $x(s)$  captures the *economic program*—society's behavior conditional on its institutions, not necessarily the socially optimal policy. Dasgupta calls this “kakotopia”.

Drop the time argument because we assume time autonomy.

### 3.2 Welfare and the Value Function

Let  $W(s, x)$  represent instantaneous social welfare (the flow of net benefits derived from the stock). The present value of welfare is

$$V(s_t) = \int_t^\infty e^{-\delta(\tau-t)} W(s(\tau), x(s(\tau))) d\tau$$

where  $\delta$  is the social discount rate.

Assume an economic program  $x(s)$  is given, so  $V$  depends only on the stock  $s$  and the system is time autonomous. Then  $W^*(s) = W(s, x(s))$ .

The **accounting price** or shadow price of natural capital is defined as

$$p_t = \frac{\partial V(s_t)}{\partial s_t}$$

which measures the change in welfare from a marginal increase in the stock of natural capital in situ - the benefit to society of having a little more stock.

Differentiating  $V(s_t)$  with respect to time gives two equivalent forms.

#### **i** Dynamics of Natural Capital Value

To obtain the time derivative  $\dot{V} = \frac{dV}{dt}$ , we must differentiate this *integral with a time-dependent lower limit* and with  $t$  also appearing in the integrand through the exponential term and the state variable  $s(t)$ .

Using **Leibniz's rule** for differentiation under the integral sign:

$$\frac{d}{dt} \int_{a(t)}^{b(t)} F(\tau, t) d\tau = F(b(t), t) \frac{db}{dt} - F(a(t), t) \frac{da}{dt} + \int_{a(t)}^{b(t)} \frac{\partial F(\tau, t)}{\partial t} d\tau$$

we set

$$F(\tau, t) = e^{-\delta(\tau-t)} W(s(\tau), x(s(\tau))), \quad a(t) = t, \quad b(t) = \infty.$$

Then  $\frac{db}{dt} = 0$  (upper limit fixed) and  $\frac{da}{dt} = 1$  (lower limit moves with  $t$ ).

#### **Step 1: Evaluate the boundary term**

At the lower limit  $\tau = t$ ,

$$F(a(t), t) = e^{-\delta(t-t)} W(s(t), x(s(t))) = W(s(t), x(s(t))).$$

Therefore the first boundary contribution is

$$-F(a(t), t) \frac{da}{dt} = -W(s(t), x(s(t))).$$

#### **Step 2: Differentiate the integrand with respect to $t$**

Since the exponential term depends on  $t$ , we have

$$\frac{\partial F(\tau, t)}{\partial t} = \frac{\partial}{\partial t} \left( e^{-\delta(\tau-t)} \right) W(s(\tau), x(s(\tau))) = \delta e^{-\delta(\tau-t)} W(s(\tau), x(s(\tau))).$$

### Step 3: Substitute into Leibniz's rule

Combining the pieces:

$$\frac{dV}{dt} = -W(s(t), x(s(t))) + \int_t^\infty \delta e^{-\delta(\tau-t)} W(s(\tau), x(s(\tau))) d\tau.$$

Recognizing the integral term as  $\delta V(s_t)$  from equation (2), we obtain

$$\boxed{\frac{dV}{dt} = \delta V - W(s, x)}$$

which is **equation (4)** in the paper.

#### Intuition

- The first term,  $-W(s, x)$ , represents the *dividend* or current flow of welfare being “paid out” at time  $t$ .
- The second term,  $\delta V$ , represents the *required return* on the total value of capital, analogous to an interest rate on wealth.

Thus the change in total capital value equals the “required return” minus the current flow — exactly as in standard capital theory, but now applied to natural capital.

From the definition of  $V$ :

$$\dot{V} = \delta V - W(s, x)$$

Although  $V$  is defined as an integral, it ultimately depends on the *current state*  $s(t)$ . That is,  $V = V(s(t))$ .

So by the **chain rule** for a composite function:

$$\frac{dV}{dt} = \frac{\partial V}{\partial s} \frac{ds}{dt} = p_t \dot{s}$$

using equation (3) in the paper. Note that equation (3) just shows the shadow price definition.

Setting these equal yields the **current-value Hamiltonian**:

$$\delta V = W(s, x) + p(s) \dot{s} = H(s, p)$$

This expression defines the current return on total capital value.

Rearranging terms,

$$V(s) = \frac{W(s, x) + p(s) \dot{s}}{\delta}$$

Differentiating wrt to the stock  $s$ ,

$$p(s) = \frac{\partial}{\partial s} \left( \frac{W(s, x) + p(s) \dot{s}}{\delta} \right) = \frac{W_s(s, x) + p(s) \frac{\partial \dot{s}}{\partial s} + \dot{s} \frac{\partial p(s)}{\partial s}}{\delta}$$

note that  $\dot{p} = \frac{\partial p(s(t))}{\partial s(t)} \dot{s}$ , so

$$\delta p = W_s(s, x) + p(s) \frac{\partial \dot{s}}{\partial s} + \dot{p}$$

and substituting  $\dot{s} = G(s) - f(s, x)$  and rearranging in terms of  $p(s)$

$$p = \frac{W_s(s, x) + \dot{p}}{\delta - \frac{\partial \dot{s}}{\partial s}} = \frac{W_s(s, x) + \dot{p}}{\delta - [G_s(s) - f_s(s, x)]}$$

This mirrors Jorgenson's (1963) model for produced capital:

- $W_s(s, x)$ : marginal benefit (marginal ecosystem service flow)
- $\dot{p}$ : capital gain or loss (price appreciation)
- $\delta$ : discount rate
- $G_s(s) - f_s(s, x)$ : net rate of natural capital productivity

Hence, the **price of natural capital** equals the adjusted marginal benefit divided by an *effective* discount rate that accounts for natural growth and extraction.

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## 4 Economic Interpretation

Symbol	Meaning	Economic Interpretation
$W_s$	Marginal ecosystem service flow	Incremental welfare from a unit increase in stock
$G_s$	Marginal natural growth	Productivity of the stock
$f_s$	Marginal human impact	How stock size affects extraction
$\dot{p}$	Price appreciation	Scarcity change or capital gain
$\delta$	Discount rate	Social rate of time preference
$p$	Accounting price	Shadow value of natural capital

If  $G_s > 0$ , the stock reproduces itself, lowering the effective discount rate and increasing  $p$ .

If  $f_s > 0$ , a larger stock encourages more extraction, reducing  $p$ .

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## 5 From Theory to Measurement

Equation (10) provides a measurement strategy:

$$p = \frac{W_s + \dot{p}}{\delta - (G_s - f_s)}$$

Most components are measurable or estimable:

- $W_s$ : from market data or nonmarket valuation.
- $G(s)$  and  $G_s(s)$ : from ecological production models.
- $f(s, x)$ : from harvest or degradation data.
- $\delta$ : from social or public accounting rates.

The challenge is that  $\dot{p}$  (price change) is not directly observable.

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## 6 7. Numerical Approximation

Fenichel and Abbott employ **function approximation methods** from computational economics (Miranda and Fackler, 2002).

They approximate the natural capital price function  $p(s)$  using basis functions:

$$p(s) \approx \sum_{n=0}^{N-1} \beta_n \phi_n(s)$$

where  $\phi_n(s)$  are **Chebyshev polynomials**.

Then

$$\dot{p}(s) = \sum_{n=0}^{N-1} \beta_n \phi'_n(s)$$

They select a set of collocation points  $\{s_i\}$  and solve for coefficients  $\beta$  such that equation (10) holds at each point.

This numerical *collocation method* recovers a continuous approximation for the accounting price function.

### i Chebyshev Polynomial Derivative

From the trigonometric form  $T_n(x) = \cos(n\theta)$  with  $x = \cos \theta$ , we can use the chain rule to find the derivative.

1. Differentiate  $T_n(x)$  with respect to  $x$ :

$$\frac{dT_n(x)}{dx} = \frac{d}{dx} [\cos(n\theta)] = -n \sin(n\theta) \frac{d\theta}{dx}.$$

Since  $\frac{d\theta}{dx} = -\frac{1}{\sqrt{1-x^2}}$  (from  $x = \cos(\theta) \rightarrow \theta = \cos^{-1}(x)$ ), this gives

$$T'_n(x) = n \frac{\sin(n\theta)}{\sqrt{1-x^2}}.$$

Substituting  $\sin(n\theta) = \sqrt{1-x^2} U_{n-1}(x)$  (a trigonometric identity) gives a more compact polynomial expression.

2. Therefore, the derivative can be written in terms of **Chebyshev polynomials of the second kind**,  $U_{n-1}(x)$ :

$$T'_n(x) = n U_{n-1}(x),$$

where  $U_n(x)$  satisfies the recurrence:

$$U_0(x) = 1, \quad U_1(x) = 2x, \quad U_{n+1}(x) = 2x U_n(x) - U_{n-1}(x).$$

## 7 8. Application: Gulf of Mexico Reef Fish

- **Stock:** Gulf reef fish biomass.
- **Dynamics:** Estimated using ecological models for  $G(s)$  and harvest functions for  $f(s, x)$ .
- **Institutions:** Real-world fishery management with quotas and enforcement.

- **Result:** An estimated accounting price (shadow value) per pound of fish biomass under actual management, not an idealized optimum.

The example demonstrates the feasibility of integrating ecological data and economic dynamics to recover the value of natural capital assets.

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## 8 9. Broader Implications

- Moves the concept of *natural capital* from metaphor to **operational measurement**.
- Integrates **economic welfare theory** with **biophysical models**.
- Works even when institutions are **inefficient** or management is **non-optimal**.
- Enables **wealth accounting** that includes both produced and natural assets.

Ignoring the adjustment terms in the denominator of equation (10) can lead to serious misvaluation—analogous to mispricing a durable asset by ignoring depreciation or appreciation.

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## 9 10. Key Equations Summary

$$\begin{aligned}\dot{s} &= G(s) - f(s, x) \\ V(s) &= \int_0^{\infty} e^{-\delta t} W(s, x) dt \\ p &= \frac{\partial V}{\partial s} \\ \dot{p} &= \delta p - W_s - p(G_s - f_s) \\ p &= \frac{W_s + \dot{p}}{\delta - (G_s - f_s)}\end{aligned}$$


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## 10 11. Discussion Questions

1. What is the difference between an **accounting price** and a **market price** of natural capital?
  2. Why is the method robust to *non-optimal* institutional settings?
  3. How does the term  $(G_s - f_s)$  modify the effective discount rate?
  4. In what sense is  $p_t$  a *shadow value* of natural capital?
  5. How would you apply this method to a renewable groundwater basin or a forest ecosystem?
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## 11 References

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