Unit 1 Lecture Notes: Introduction to Mathematical Programming

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Day 1: Motivation & Modeling Foundations

Introductions

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- Who am I?
- Student introductions: What year and what field? Research interests?
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Course Orientation & Motivation

- Why optimization in economics?
  - Many economic problems are decision problems under constraints where some agent is pursuing an objective and faces constraints.
  - Examples:
    * A farmer allocates land and water across crops.
    * A firm chooses output facing costs and regulations.
    * A planner designs climate policy subject to resource limits and production technologies.
    * A consumer maximizes utility subject to income.
- Analytical solutions not always possible → need **numerical/computational methods**. Closed form solutions do not exist.
  - We will learn to optimize both analytically and numerically.
- Mathematical programming provides a structured, computer-implementable way to model these decisions.

What is a Mathematical Program?

- **General Form**:

$$\max_x f(x) \quad \text{s.t. } g_i(x) \in S_1,\ x \in S_2$$

- Components:
  - **Decision variables**: choices (e.g., acres planted, production levels).
  - **Objective function**: what we optimize (profit, cost, utility).
  - **Constraints**: resource limits, technology, budgets.
    - ∗ These constraints can be that functions of x lie within some boundary or that the individual x's lie in some boundary

Example (crop mix):

- Variables: acres in wheat (x_1), corn (x_2).
- Objective: maximize profit.
- Constraints: land, labor, water.

Uses of Mathematical Programming (McCarl §1.3)

`What does McCarl say are the uses of math programming?`

1. **Problem Insight Construction**
   - State problem carefully and really understand the moving parts
   - Clarify objectives, constraints, and trade-offs.
   - Example: Writing down a water allocation model forces us to quantify resource limits. It also forces one to write down the functional relationships between variables, which may be a simplification or approximation of a physical or natural phenomenon: hydrology, population growth functions, atmospheric models.
2. **Numerical Applications**
   - **Prescription of Solutions**
     - Goal: Recommend an **optimal plan** given data, objectives, and constraints.
     - Example: Farm crop mix — how many acres of wheat, corn, soy to maximize profit?

- **Prediction of Consequences**
    - Goal: Forecast **outcomes under scenarios**.
    - Example: If fertilizer availability drops 20%, what happens to optimal output and profit?
- **Demonstration of Sensitivity**
    - Goal: Show how solutions **shift with parameters**.
    - Example: How does optimal irrigation use change as water prices rise?

3. **Algorithm Development**
    - solution algorithms… not that important for most of us.

Linear Programming (LP) Structure

- **Standard LP Form**:

$$\max \ \pi' x \quad \text{s.t. } Ax \leq b, \ x \geq 0$$

- Characteristics:
    - Linear objective.
    - Linear constraints.
    - Nonnegativity.

Can also be written in matrix notation.

Economic interpretation:

- π: profit per unit.
- A: resource use matrix.
- b: resource endowment.

Model Building Process (McCarl framework)

1. Identify **decision variables**.
2. State the **objective**.
3. Identify and formulate **constraints**.
4. Collect data.
5. Translate into computer-readable form.
6. Solve and interpret.

Farm LP Example

A farmer has 500 acres of land available and is deciding how to allocate it between wheat and corn. Each acre of wheat yields a profit of \$200, requires 3 hours of labor, and 4 units of fertilizer. Each acre of corn yields a profit of \$300, requires 4 hours of labor, and 3 units of fertilizer. The farm has at most 1,800 hours of labor available and 2,000 units of fertilizer.

Formulate this situation as a linear programming problem. Clearly define the decision variables, write down the objective function representing total profit, and specify the constraints that capture the land, labor, fertilizer, and nonnegativity restrictions.

Decision variables.
Let W = acres of wheat, C = acres of corn.

Scalar (algebraic) form

$$\begin{aligned} \max_{W,C} \quad & 200\,W + 300\,C \\ \text{s.t.} \quad & W + C \le 500 && \text{(land)} \\ & 3W + 4C \le 1800 && \text{(labor)} \\ & 4W + 3C \le 2000 && \text{(fertilizer)} \\ & W,\ C \ge 0\ . \end{aligned}$$

Matrix (compact) form

$$\begin{aligned} \max_{x \in \mathbb{R}^2_{\ge 0}} \quad & c^\top x \\ \text{s.t.} \quad & Ax \le b, \end{aligned} \qquad \text{with} \quad x = \begin{bmatrix} W \\ C \end{bmatrix},\ c = \begin{bmatrix} 200 \\ 300 \end{bmatrix},\ A = \begin{bmatrix} 1 & 1 \\ 3 & 4 \\ 4 & 3 \end{bmatrix},\ b = \begin{bmatrix} 500 \\ 1800 \\ 2000 \end{bmatrix}.$$

Summary

- Optimization = decision making under constraints.
- Mathematical programming provides a general framework: decision variables, objective, constraints.
- Linear programs (LPs): linear objective + linear constraints + nonnegativity.
- Example: Farm resource allocation with land, labor, fertilizer.

Day 2: Linear Programming in Practice

Review

- General form of a mathematical program.
- Structure of a linear program:
  - Linear objective
  - Linear constraints
  - Nonnegativity
- Wheat–Corn farm allocation example in scalar and matrix notation.

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7 Assumptions of Linear Programming

Linear programming models rely on a set of assumptions that make them tractable but also limit their realism. McCarl & Spreen (Ch. 2.4) identify **seven important assumptions**. The first three involve the *appropriateness of the formulation*; the last four describe *mathematical properties* of the LP model.

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1. Objective Function Appropriateness

- The objective function is assumed to be the **sole criterion** for evaluating solutions.
- This means the decision maker's preferences can be fully represented by a single linear function (e.g., profit, cost, utility).
- In practice, decisions may depend on multiple objectives (profit, risk, leisure), but LP assumes one dominates.

2. Decision Variable Appropriateness

- All relevant decision variables must be included, and each must be **fully controllable** by the decision maker.
- Omitting key variables or including variables outside the decision maker's control invalidates the formulation.

3. **Constraint Appropriateness**

   - Constraints must **accurately and completely capture** the limits faced by the decision maker:
     - They fully describe resource, technological, and institutional limits.
     - Resources within a constraint are **homogeneous** and freely substitutable among activities.
     - No constraint should arbitrarily rule out feasible choices.
     - Constraints cannot be bent outside the model.

4. **Proportionality**

   - Contributions of activities to the objective function are **proportional** to their level.
   - Likewise, resource use is proportional: doubling an activity doubles its input use.
   - This rules out fixed costs, economies of scale, or price effects that depend on output level.

5. **Additivity**

   - Total contributions to the objective and resource use are the **sum of individual contributions**.
   - No interactions among variables are allowed (e.g., no multiplicative terms).

6. **Divisibility**

   - Decision variables can take on **fractional values**.
   - This assumes continuous activities (e.g., acres of land).
   - When variables must be integer (e.g., number of tractors), integer programming is required instead.

7. Certainty

- All parameters (objective coefficients, resource availability, input-output coefficients) are **known with certainty**.
- LP is thus a deterministic model.
- In practice, parameters are often estimated, and uncertainty can be explored with sensitivity or stochastic programming.

Teaching Note

These assumptions both **enable LP to be solvable** and **limit realism**. They provide a natural segue to later topics in the course:

- Multi-objective programming (relax objective function assumption)
- Integer programming (relax divisibility)
- Stochastic programming (relax certainty)
- Nonlinear programming (relax proportionality and additivity)

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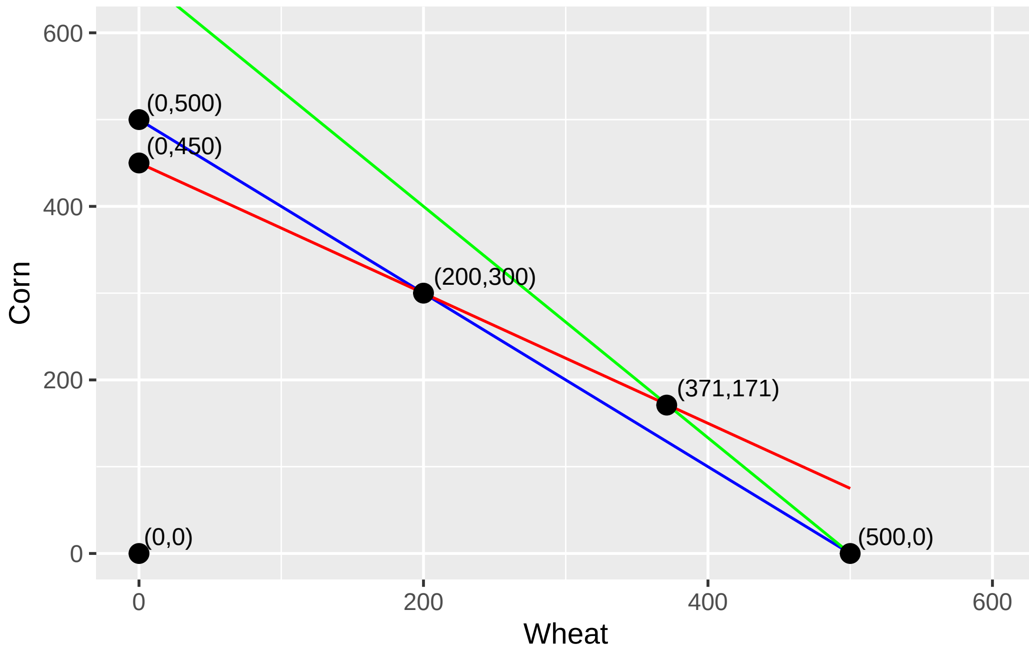
2. Graphical Solution Method (2 variables)

Step 1. Draw the constraints.

- Land: $W + C \leq 500$
- Labor: $3W + 4C \leq 1800$
- Fertilizer: $4W + 3C \leq 2000$
- Nonnegativity: $W, C \geq 0$
- Plot based on endpoints. set one var 0 and devote all resources to that var.



Step 2. Identify the feasible region.

- Intersection of all constraints in the (W, C) plane.
- Polygon bounded by lines.

Step 3. Plot the objective function.

- Profit $= 200W + 300C$.
- Show isoprofit lines:
  - Suppose we plot \$6000 profit. All wheat no corn, then all corn no wheat.
  - lines of constant profit slope $-\frac{200}{300} = -\frac{2}{3}$.

Step 4. Locate the optimum.

- Slide the isoprofit line outward until the last point of contact with the feasible region.
- Optimum is always at a **corner point** (fundamental theorem of LP).

Simplex Method

Why We Need It

- The graphical method only works for **two variables**.

- Real problems may involve **hundreds or thousands** of variables.
- Key geometric fact:
  - The feasible region of an LP is a **convex polytope**.
  - The **optimal solution lies at a vertex (corner point)**. What about the problem makes this a fact?
- The simplex method provides a **systematic way** to move from vertex to vertex until the best one is found.

Core Idea

- Start at a **basic feasible solution (BFS)** — a corner point of the feasible region.
- At each step:
  1. Compute **reduced costs** (how much the objective improves if a variable increases from 0).
  2. Identify an **entering variable** (the candidate to increase).
  3. Determine which constraint binds first — this sets the **leaving variable**.
  4. **Pivot** to a new BFS.
- Stop when no variable can improve the objective — this is optimal.

Intuition

- Simplex is like **walking along the edges** of the feasible polygon.
- At each corner, ask: *"If I move along this edge, does profit go up?"*
- Continue until no edge yields improvement.
- The same logic applies in higher dimensions, even though we cannot draw the polytope.

Simplex Pivot — Tiny Worked Example

We use the smallest LP that still shows the mechanics:

Problem

$$\max z = 3x_1 + 2x_2$$
$$\text{s.t. } x_1 + x_2 \le 4, \quad x_1 \le 2, \quad x_1, x_2 \ge 0.$$

Standard form (add slacks s_1, s_2)

$$\begin{aligned} x_1 + x_2 + s_1 &= 4, \\ x_1 + s_2 &= 2, \\ z - 3x_1 - 2x_2 &= 0, \qquad s_1, s_2 \ge 0. \end{aligned}$$

- Slack variables are added to "$\le$" constraints to convert them into equalities, making the LP system compatible with the simplex algorithm.

- They measure the unused portion of a resource — e.g., if a land constraint is $x_1+x_2 \le 500$ and only 400 acres are used, the slack variable equals 100.
- Slack variables always have a zero coefficient in the objective function, since they do not directly contribute to profit or cost.
- Each slack variable typically appears with a coefficient of +1 in one constraint and 0 elsewhere, so they "fill the gap" between resource availability and resource use.
- At optimality, a nonzero slack indicates an unused resource. Checking which slack variables are positive helps interpret whether constraints are binding or loose.

Initial tableau and choice of entering/leaving variables

Initial Basic Feasible Solution (BFS): $x_1 = x_2 = 0 \Rightarrow s_1 = 4, s_2 = 2, z = 0.$

Tableau

	x_1	x_2	s_1	s_2	RHS
s_1	1	1	1	0	4
s_2	1	0	0	1	2
z	-3	-2	0	0	0

- **Entering variable:** look at the objective row; most negative reduced cost is under x_1 (coefficient -3) $\to$ **enter** x_1.
- **Leaving variable (ratio test):** divide RHS by the positive entries in the x_1 column: Row s_1: $4/1 = 4$, Row s_2: $2/1 = 2$. Minimum is 2 $\to$ **leave** s_2.

- **Pivot element:** the entry at row s_2, column x_1 (which is 1).

We will **pivot on that 1**, swapping $s_2 \leftrightarrow x_1$.

Row operations (make pivot column a unit vector)

Goal: pivot column (x_1) should become $(0, 1, 0)^\top$.

1) **Normalize pivot row** (already 1, so no change):

$$(s_2) \;:\; [\,1 \;\; 0 \;\; 0 \;\; 1 \mid 2\,].$$

2) **Zero out the other entries in the x_1 column:**

- Row s_1: $(s_1) \leftarrow (s_1) - 1 \cdot (s_2)$

$$[\,1 \;\; 1 \;\; 1 \;\; 0 \mid 4\,] - [\,1 \;\; 0 \;\; 0 \;\; 1 \mid 2\,] = [\,0 \;\; 1 \;\; 1 \;\; -1 \mid 2\,].$$

- Row z: $(z) \leftarrow (z) + 3 \cdot (s_2)$

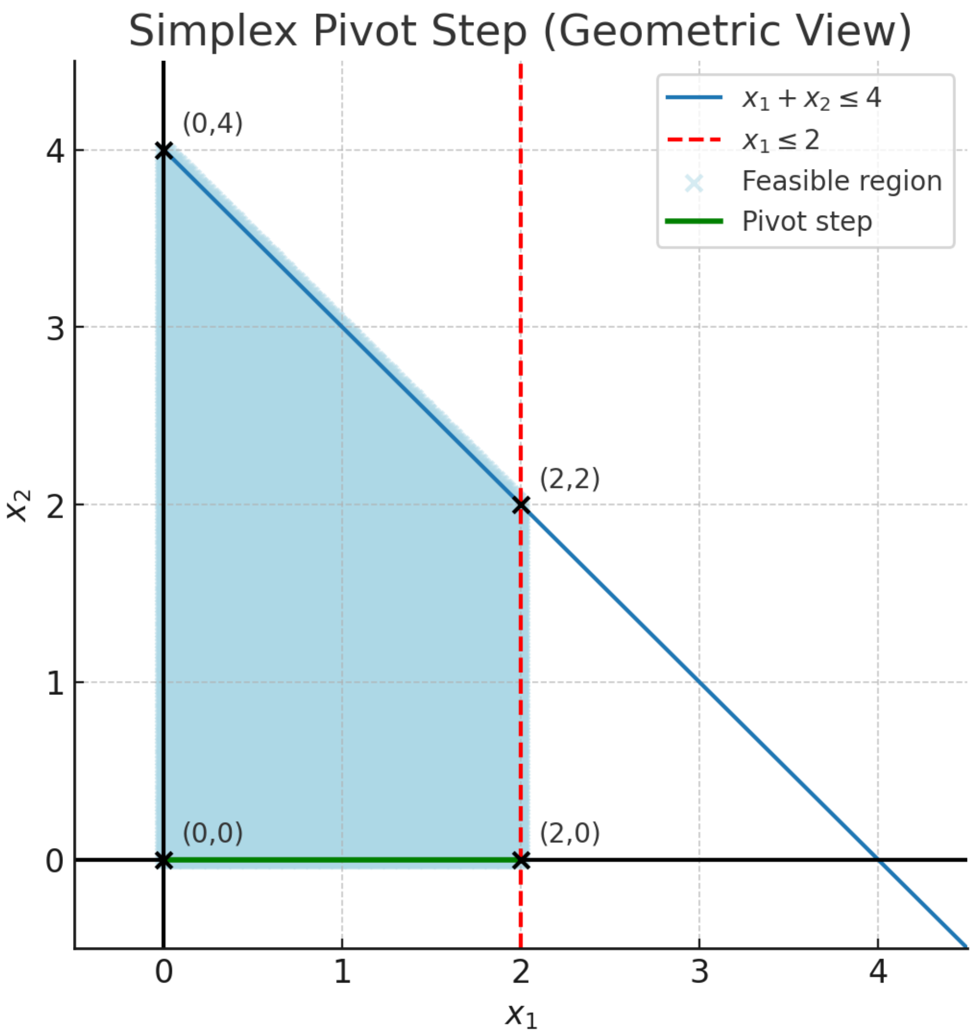
$$[\,-3 \;\; -2 \;\; 0 \;\; 0 \mid 0\,] + 3 \cdot [\,1 \;\; 0 \;\; 0 \;\; 1 \mid 2\,] = [\,0 \;\; -2 \;\; 0 \;\; 3 \mid 6\,].$$

New tableau (after one pivot)

	x_1	x_2	s_1	s_2	RHS
s_1	0	1	1	-1	2
x_1	1	0	0	1	2
z	0	-2	0	3	6

- **Basis after pivot:** $\{\, s_1,\; x_1 \,\}$.
- **Current solution:** $x_1 = 2,\; x_2 = 0,\; s_1 = 2,\; s_2 = 0,\; z = 6.$
- The column labels show that x_1 **entered** (blue row) and s_2 **left**.

Next step (if continuing): the most negative in the z-row is under x_2 (-2), so x_2 would enter next; the algorithm would pivot again and reach the optimum at $(x_1, x_2) = (2, 2)$ with $z = 10$.



Excel Solver

Excel implements the simplex method in the solver add-on. See LP_land_alloc.xlxs

Summary

- Linear programming problems have:

- **Decision variables** (choices to make),
- **Objective function** (profit, cost, etc.),
- **Constraints** (resource limits, requirements).

- Graphical method (2 variables) shows:
  - Feasible region = convex polygon.
  - Optimum occurs at a **corner point**.
- **Fundamental theorem of LP:** optimum is always at a vertex of the feasible region.
- **Simplex method** generalizes:
  - Moves systematically from one **basic feasible solution (BFS)** to another.
  - Uses **entering and leaving variables** to pivot.
  - Stops when no further improvement is possible.
- **Slack variables** convert inequalities to equalities and measure unused resources.
- Tableau row operations implement pivots (targeted Gaussian elimination).