Final Review

1 Overview

This module has covered a variety of dynamic programming methods and applications. The final review will cover key concepts and techniques from the module, including:

- Formulating dynamic programming problems
- Understanding the components of the dynamic program: state and control variables, transition functions, and payoff/reward functions, salvaging value functions
- Interpreting policy functions and value functions
- Types of dynamic programming problems: finite vs infinite horizon, discrete vs continuous state and control spaces
- Karush-Kuhn-Tucker (KKT) conditions for unconstrained optimization in dynamic programming: feasibility, stationarity, complementary slackness
- Envelope conditions and their role in characterizing optimal policies
- Solving Bellman equations using value function iteration (conceptual)

2 Discrete time: Fishery Problem

We study a single-stock fishery with:

- continuous state s (biomass),
- control (harvest) h
- next period's stock is $s' = G(s - h)$.
- Per-period payoff is $\pi(h) = p\,h - c\,h$ (constant net price for simplicity).
- Discount factor $\beta \in (0, 1)$.

The planner's problem is to choose a harvest policy $h(s)$ to maximize the present value of profits.

$$\max_h \sum_{t=0}^{\infty} \beta^t \pi(h_t)$$

subject to the stock transition equation $s_{t+1} = G(s_t - h_t)$.

The infinite-horizon Bellman equation (time-stationary) is

$$V(s) = \max_h \{\pi(h) + \beta V(s')\} = \max_h \{p\,h - c\,h + \beta V(G(s - h))\}$$

where $s' = G(s - h)$ and $G()$ is some twice differentiable growth function.

2.1 KKT system

- Feasibility: $0 \leq h \leq s$.
- Stationarity (interior): $\pi_h(s, h) - \beta V'(s')\, G'(s - h) = 0$ with $s' = G(s - h)$.
- Complementary slackness:
  - $h \geq 0$, $\mu_1 \geq 0$, $\mu_1 h = 0$ (no negative harvest)
  - $h \leq s$, $\mu_2 \geq 0$, $\mu_2(s - h) = 0$ (cannot harvest more than stock)

With $\pi_h(h) = p - c$, the interior FOC is

$$p - c - \beta V'(s')\, G'(s - h) = 0$$

2.2 Envelope condition

The envelope condition is not formally a KKT condition, but it has a clear economic interpretation and can help simplify the stationarity condition and solve for a policy function. We derive the envelope condition by differentiating the Bellman wrt to s. Importantly, a marginal change in the state has no direct effect on the optimal control, $\frac{\partial h^*}{\partial s} = 0$. Mathematically, when you differentiate the Bellman wrt to s, you get a cluster of terms that contain the stationarity condition, which is equal to 0 at the optimim. Also, in this case, π does not depend directly on s:

$$V'(s) = 0 + \beta V'(s')\, G'(s - h^*(s)).$$

Note that the second term of the envelope condition is the same as in the stationary condition above. Combining the two (interior) yields a classic shadow-value condition:

$$p - c = V'(s) \quad \text{(if interior)}.$$

Interpretation: harvest up to the point where the marginal net revenue equals the marginal value of stock - the additional future discounted revenue from the growth of the stock (the shadow price or "user cost"). With bounds, policies tilt to no-harvest at very low s (stock rebuilding) and binding-harvest (take most of the stock) when s is very high.

What happens if we increase the price p? The marginal net revenue increases, so the optimal harvest $h^*(s)$ increases for each stock level s. This leads to lower stock levels in the long run, as higher harvests reduce the biomass available for future periods.

What if we increase the cost c? The marginal net revenue decreases, leading to lower optimal harvests $h^*(s)$ for each stock level s. This results in higher stock levels in the long run, as reduced harvests allow the biomass to recover and grow over time.

3 Value function iteration

We can solve the Bellman equation numerically using value function iteration:

1. Initialize a guess for the value function $V_0(s)$ over a grid of stock levels s.
2. For each stock level s in the grid, solve the maximization problem to find the optimal harvest $h^*(s)$ and compute the updated value function:

$$V_{n+1}(s) = \max_h \{p\, h - c\, h + \beta V_n(G(s - h))\}$$

3. Repeat step 2 until convergence, i.e., until $|V_{n+1}(s) - V_n(s)|$ is below a specified tolerance for all s

4 Problem set 4: Continuous Time Optimal Control

Backstop technology and optimal switching (continuous time)

Consider a planner who must meet a constant flow demand D using either:

i. extraction from an exhaustible stock $S(t)$ at rate $x(t) \geq 0$ with convex extraction cost, or
ii. an unlimited "backstop" technology $y(t) \geq 0$ with constant unit cost B.

The planner minimizes the present value of total costs with discount rate $\rho > 0$.

- State: $S(t) \geq 0$ with $S(0) = S0$.
- Controls: extraction $x(t) \geq 0$ and backstop use $y(t) \geq 0$.

- Flow constraint: $x(t) + y(t) = D$ for all $t \geq 0$.
- Stock dynamics: $\dot{S}(t) = -x(t)$.
- Extraction cost: $C(x) = 1/2cx^2$ with $c > 0$.
- Backstop cost: $By(t)$ with $B > 0$.

Thus the planner's problem is

$$\min_{\{x(t),y(t)\}_{t\geq 0}} \int_0^\infty e^{-\rho t}\left[\tfrac{1}{2}c\,x(t)^2 + B\,y(t)\right]dt \quad \text{s.t.} \quad \dot{S}(t) = -x(t),\ S(0) = S_0,\ x(t) + y(t) = D,\ x(t), y(t) \geq 0.$$

4.1 a)

Eliminate $y(t)$ using the flow constraint and write the problem with one control $x(t) \in [0, D]$. Write the current-value Hamiltonian $H(S, x, \lambda)$. Clearly state the state equation, control bounds, and the transversality condition.

The flow constraint $x(t) + y(t) = D$ allows us to express $y(t)$ in terms of $x(t)$:

$$y(t) = D - x(t).$$

Substituting this into the objective function, we get:

$$\min_{\{x(t)\}_{t\geq 0}} \int_0^\infty e^{-\rho t}\left[\tfrac{1}{2}c\,x(t)^2 + B(D - x(t))\right]dt$$

The current-value Hamiltonian is given by:

$$H(S, x, \lambda) = \tfrac{1}{2}c\,x^2 + B(D - x) + \lambda(-x)$$

where $\lambda(t)$ is the co-state variable associated with the state equation $\dot{S}(t) = -x(t)$.

4.2 b)

Derive the first-order necessary conditions (Pontryagin):

- control condition $\partial H/\partial x = 0$ for an interior solution,
- co-state equation $\dot{\lambda} = \rho\lambda - \partial H/\partial S$,
- state equation $\dot{S} = -x$,

Show that (for interior x) the optimal policy satisfies

$$x(t) = \frac{B - \lambda(t)}{c}, \qquad \dot{\lambda}(t) = \rho\,\lambda(t).$$

The first-order necessary conditions are:

1. **Control condition**: For an interior solution, we set the derivative of the Hamiltonian with respect to the control variable x to zero:
$$\frac{\partial H}{\partial x} = c\,x - B - \lambda = 0 \implies c\,x = B + \lambda \implies x(t) = \frac{B - \lambda(t)}{c}.$$
2. **Co-state equation**: The co-state equation is given by:
$$\dot{\lambda} = \rho\lambda - \frac{\partial H}{\partial S}.$$
Since H does not explicitly depend on S, we have $\frac{\partial H}{\partial S} = 0$, leading to:
$$\dot{\lambda} = \rho\lambda.$$
3. **State equation**: The state equation is given by:
$$\dot{S} = -x.$$

4.3 c)

Solve the co-state ODE and show that $\lambda(t) = \lambda_0 e^{\rho t}$.

Argue that $0 < x(t) < D$ followed by a switching time T^* at which $x(T^*) = 0$ and the backstop fully supplies demand thereafter ($y(t) = D$ for $t \geq T^*$).

The co-state ODE is:

$$\dot{\lambda} = \rho\lambda.$$

This is a first-order linear ordinary differential equation, so rarranging terms and integrating both sides, we have:

$$\int \frac{1}{\lambda} d\lambda = \int \rho dt.$$

Integrating gives:

$$\ln|\lambda| = \rho t + C,$$

where C is the constant of integration. Exponentiating both sides, we get:

$$\lambda(t) = e^C e^{\rho t} = \lambda_0 e^{\rho t},$$

where $\lambda_0 = e^C$ is the initial value of the co-state variable at time $t = 0$.

- This means the shadow price of the resource grows exponentially over time at the rate ρ.
- $\lambda(t)$ represents the in situ value of the resource stock.
- Because the resource is exhaustible, the scarcity value increases over time.
- The discount rate ρ reflects the planner's time preference, influencing how future costs are valued relative to present costs. Higher ρ leads to a faster increase in $\lambda(t)$, so the costate must rise more quickly to justify delaying extraction. This is the Hotelling rule in resource economics.

4.4 d)

Assuming the interior $0 < x(t) < D$ until the switch, derive closed-form expressions for the switching time T^* defined by $\lambda(T^*) = B$.

From the expression for $\lambda(t)$, we have:

$$\lambda(T^*) = \lambda_0 e^{\rho T^*} = B.$$

Solving for T^*, we get:

$$T^* = \frac{1}{\rho} \ln\left(\frac{B}{\lambda_0}\right).$$

4.5 e)

Comparative statics. For the pre-switch phase ($t < T^*$), characterize how $x(t)$ and T^* change with each parameter c, B, ρ. Provide economic intuition for each sign.

1. **Effect of c (extraction cost coefficient)**:
   - As c increases, the optimal extraction rate $x(t) = \frac{B-\lambda(t)}{c}$ decreases for a given $\lambda(t)$. This is because higher extraction costs make it less attractive to extract the resource.
   - The switching time T^* increases with c because higher costs delay the point at which it becomes optimal to switch to the backstop technology.
2. **Effect of B (backstop cost)**:
   - As B increases, the optimal extraction rate $x(t)$ increases for a given $\lambda(t)$. A higher backstop cost makes extraction more attractive relative to using the backstop technology.
   - The switching time T^* decreases with B because a higher backstop cost makes it optimal to switch to the backstop technology sooner since you are depleting the stock faster.
3. **Effect of ρ (discount rate)**:

- As ρ increases, the optimal extraction rate $x(t)$ increases since $\lambda(t) = \lambda_0 e^{\rho t}$. A higher discount rate places more weight on present costs, making extraction more attractive.
- The switching time T^* decreases with ρ because a higher discount rate increases the growth rate of $\lambda(t)$, leading to an earlier switch to the backstop technology.