Problem Set: Dynamic Programming Continuous Time

1 Problem 1

Backstop technology and optimal switching (continuous time)

Consider a planner who must meet a constant flow demand D using either:

i. extraction from an exhaustible stock $S(t)$ at rate $x(t) \geq 0$ with convex extraction cost, or
ii. an unlimited "backstop" technology $y(t) \geq 0$ with constant unit cost B.

The planner minimizes the present value of total costs with discount rate $\rho > 0$.

- State: $S(t) \geq 0$ with $S(0) = S0$.
- Controls: extraction $x(t) \geq 0$ and backstop use $y(t) \geq 0$.
- Flow constraint: $x(t) + y(t) = D$ for all $t \geq 0$.
- Stock dynamics: $\dot{S}(t) = -x(t)$.
- Extraction cost: $C(x) = 1/2cx^2$ with $c > 0$.
- Backstop cost: $By(t)$ with $B > 0$.

Thus the planner's problem is

$$\min_{\{x(t),y(t)\}_{t\geq 0}} \int_0^\infty e^{-\rho t}\left[\tfrac{1}{2}c\,x(t)^2 + B\,y(t)\right]dt \quad \text{s.t.} \quad \dot{S}(t) = -x(t),\ S(0) = S_0,\ x(t) + y(t) = D,\ x(t), y(t) \geq 0.$$

a) Eliminate $y(t)$ using the flow constraint and write the problem with one control $x(t) \in [0, D]$. Write the current-value Hamiltonian $H(S, x, \lambda)$. Clearly state the state equation, control bounds, and the transversality condition.

b) Derive the first-order necessary conditions (Pontryagin):

- control condition $\partial H/\partial x = 0$ for an interior solution,
- co-state equation $\dot{\lambda} = \rho\lambda - \partial H/\partial S$,
- state equation $\dot{S} = -x$,

Show that (for interior x) the optimal policy satisfies

$$x(t) = \frac{B - \lambda(t)}{c}, \qquad \dot{\lambda}(t) = \rho\,\lambda(t).$$

c) Solve the co-state ODE and show that $\lambda(t) = \lambda_0 e^{\rho t}$.

Argue that $0 < x(t) < D$ followed by a switching time T^* at which $x(T^*) = 0$ and the backstop fully supplies demand thereafter ($y(t) = D$ for $t \geq T^*$).

d) Assuming the interior $0 < x(t) < D$ until the switch, derive closed-form expressions for the switching time T^* defined by $\lambda(T^*) = B$.

e) Comparative statics. For the pre-switch phase ($t < T^*$), characterize how $x(t)$ and T^* change with each parameter c, B, ρ, and S_0. Provide economic intuition for each sign.