Problem Set: Dynamic Programming Discrete Time

1 Problem 1

Consider a firm facing deterministic demand $\{D_t\}_{t=0}^{T-1}$ over a finite horizon T. Let i_t be the inventory at the start of period t and y_t the production decision in period t. Inventory evolves as

$$i_{t+1} = i_t + y_t - D_t, \quad i_0 \text{ given}, \quad i_T = 0.$$

The firm can produce any nonnegative amount $y_t \geq 0$ in each period with a production cost:

$$C_t(y_t) = \frac{c_2}{2} y_t^2 + c_1 y_t$$

where $c_2 > 0$ and $c_1 \geq 0$. Holding inventory incurs a cost:

$$H_t(i_t) = \frac{h}{2} i_t^2$$

where $h > 0$. The firm discounts future costs by a factor $\beta \in (0, 1]$.

a. Formulate the firm's problem as a dynamic program. Clearly define the state and action variables, the transition equation, the per-period cost function, and the Bellman equation.

b. Derive the first-order condition for the firm's optimization problem in period t given state i_t. What is the optimal production decision y_t^* as a function of the value function $V_{t+1}(i_{t+1})$?

c. What happens to the optimal production decision y_t^* as the inventory holding cost h increases? Provide an economic intuition for this result.

2 Problem 2

A firm supplies a homogeneous material to a deterministic market over horizon T (finite or infinite). The firm controls:

- extraction of primary material $x_t \geq 0$ from an exhaustible ore stock s_t
- processing of recyclable scrap $u_t \geq 0$ collected from past sales

Let total sales be $y_t = x_t + q_t$, where $q_t = \eta u_t$ is secondary output from recycling with yield $\eta \in (0, 1]$.

Technology and stocks

- Ore stock dynamics:
$$s_{t+1} = s_t - x_t, \quad s_0 > 0, \quad s_t \geq 0.$$
- Scrap availability: A fraction $\theta \in [0, 1]$ of last period's sales becomes recoverable scrap at the start of period t. Let a_t denote available scrap in period t. u_t is the amount of scrap used in production in period t. The stock of recycled input evolves as,
$$a_{t+1} = a_t - u_t + \theta y_t, \qquad a_0 \geq 0$$

Demand and prices

- Inverse demand is linear:

$$P(y_t) = \alpha - \beta y_t \qquad \alpha > 0, \beta > 0.$$

Costs

- Primary extraction cost (convex):

$$C_p(x_t) = \kappa_p x_t + \frac{\gamma_p}{2} x_t^2, \qquad \gamma_p > 0,\ \kappa_p \geq 0.$$

- Recycling cost (convex and strictly higher marginal intercept):

$$C_r(u_t) = \kappa_r u_t + \frac{\gamma_r}{2} u_t^2, \qquad \gamma_r > 0,\ \kappa_r > \kappa_p.$$

  Thus, recycling is technologically feasible but (weakly) more expensive than using raw ore at the margin.
- The firm discounts with factor $\beta \in (0, 1]$.

1. Formulate the dynamic program including:
   - State variables, control variables, transition equations.
   - One-period profit $\pi_t(\cdot)$ and the Bellman equation.
   - Specify natural nonnegativity and capacity constraints (e.g., $x_t \leq s_t$, $u_t \leq a_t$).
2. Derive first-order (KKT) conditions for x_t and u_t in period t (assume interior solution).
   - Express the conditions using the shadow values $V_{t+1}^{(s)}$ and $V_{t+1}^{(a)}$ of the ore and scrap states.
   - Provide the pricing rules equating marginal revenue to marginal costs plus shadow values.
3. Interpretation.
   - Explain when the firm will substitute toward recycling (high ore shadow value, large θ, high η) despite higher direct processing cost.
   - Discuss how increasing $h \equiv 0$ here (no inventory cost) vs. introducing storage costs for scrap (set $\delta < 1$ or add a holding cost on a_t) would affect the recycling decision.
4. Numerical solution (finite horizon).
   - Implement a numerical solution of the dynamic program for $T = 10$ periods using value function iteration.
   - Use parameter values:
     - $\alpha = 100$,
     - $\beta = 0.95$,
     - $\kappa_p = 10$,
     - $\kappa_r = 20$,
     - $\gamma_p = 1$,
     - $\gamma_r = 2$,
     - $\eta = 0.8$,
     - $\theta = 0.5$,
     - $s_0 = 100$,
     - $a_0 = 0$.
   - Plot optimal paths of x_t, u_t, y_t, s_t, a_t over time.
5. Steady state (infinite horizon, optional).
   - Assume $\delta = 1$ (one-period scrap lag, $u_t \leq \theta y_{t-1}$) and consider a stationary policy with constant y, x, u.
   - Derive conditions under which a mixed supply steady state ($x > 0$, $u > 0$) is optimal vs. corner solutions ($x > 0, u = 0$ or $x = 0, u > 0$).
   - Characterize how the steady state shifts with $\theta, \eta, \kappa_r - \kappa_p$, and β.

Hints

- Write one-period profit as

$$\pi_t = P(y_t)\, y_t - C_p(x_t) - C_r(u_t), \quad y_t = x_t + \eta u_t.$$

- With state $X_t = (s_t, a_t)$, your Bellman equation (finite horizon) is

$$V_t(X_t) = \max_{x,u\geq 0} \left\{ \pi_t + \beta V_{t+1}(s_t - x, a_t - u_t + \theta y_t) \right\}$$

  subject to $x \leq s_t$, $u \leq a_t$.

- In the FOCs, note $\partial y_t / \partial x_t = 1$ and $\partial y_t / \partial u_t = \eta$. Marginal revenue is $MR_t(y_t) = P(y_t) + P\text{'}(y_t) y_t = \alpha - 2\beta y_t$ for the linear demand given.
- The ore stock's shadow value acts like an opportunity cost added to extraction's direct marginal cost; similarly for scrap.