Dynamic Programming Part B

1 Functional Equations and Value Function Properties

Dynamic programming problems often lead to **functional equations**—equations in which the *unknown* is a function rather than a scalar. The **Bellman equation** is the most important example in economics. Instead of solving for a scalar like x, we are solving for an entire function $V(\cdot)$ that satisfies a relationship involving itself.

1.1 What Is a Functional Equation?

- A **functional equation** relates the value of a function at one point to its value at another point.
- In its simplest form, it can look like:

$$V(s) = F(s, V(g(s)))$$

- where the function V appears on both sides.

- We are not trying to find a single number, but rather a function $V(\cdot)$ that makes this equation true for every s.
- Functional equations appear whenever we describe **recursive behavior** — situations where "the value of something today depends on the value of something tomorrow."

A functional equation specifies a relationship between a function and its transformed version.

In dynamic optimization:

$$V(s) = \max_{x \in X(s)} \{\pi(s, x) + \beta V(f(s, x))\}$$

where:

- $V(s)$ is the **value function**, giving the maximum attainable value from state s.
- $\pi(s, x)$ is the **current payoff** (or profit, utility).
- $f(s, x)$ gives the **next-period state**.
- $\beta \in (0, 1)$ is the **discount factor**.

The equation says: *The value of being in state s today equals the current payoff plus the discounted value of the next state, assuming optimal choice x.*

1.1.1 Functional Equations in Economics

Example: A Simple Savings Problem

Suppose an agent chooses consumption c_t and next-period assets a_{t+1}.

$$
\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)
$$

$$
\text{s.t. } c_t + a_{t+1} = (1+r)a_t + y_t, \quad a_t \geq 0
$$

where:

- $u(c)$ is the utility function (e.g., $u(c) = \log(c)$),
- r is the interest rate,
- y_t is income,
- $\beta \in (0,1)$ is the discount factor.

Then the Bellman equation is

$$
V(a_t) = \max_{a_{t+1} \geq 0} \{u((1+r)a_t + y - a_{t+1}) + \beta V(a_{t+1})\}.
$$

FOC (interior) with respect to a_{t+1}

Let $c_t = (1+r)a_t + y - a_{t+1}$. Then

$$
\frac{\partial}{\partial a_{t+1}} \Big[u((1+r)a_t + y - a_{t+1}) + \beta V(a_{t+1}) \Big] = -u'(c_t) + \beta V'(a_{t+1}) = 0,
$$

The agent trades off **current utility** $u(c)$ versus the **future value** $\beta V(a')$.

Firm Investment Problem

A firm with capital stock k_t chooses investment i_t:

$$
\max_{i_t} \sum_{t=0}^{\infty} \beta^t \pi(k_t, i_t) \quad \text{s.t.} \quad k_{t+1} = (1-\delta)k_t + i_t, \quad k_t \geq 0.
$$

The recursive form:

$$
V(k_t) = \max_{i_t \geq 0} \{\pi(k_t, i_t) + \beta V((1-\delta)k_t + i_t)\}.
$$

The function $V(k_t)$ gives the value of capital k_t as current profit plus discounted future (optimal) value of next-period capital. Again, the value function appears on both sides.

FOC with respect to i_t:

$$
\frac{\partial}{\partial i_t} \Big[\pi(k_t, i_t) + \beta V((1-\delta)k_t + i_t) \Big] = \pi_i(k_t, i_t) + \beta V'((1-\delta)k_t + i_t) = 0.
$$

Resource Extraction Problem

A resource owner decides how much to extract x_t from stock s_t:

$$
V(s_t) = \max_{0 \leq x_t \leq s_t} \{p x_t - c(x_t) + \beta V(s_t - x_t)\}.
$$

Here:

- The **state** is remaining stock s_t.
- The **control** is extraction x_t.
- The **transition** is $s_{t+1} = s_t - x_t$.

The functional equation states that the value of the resource today equals profit from extraction plus the discounted value of what remains for tomorrow.

1.2 The Bellman Operator

The Bellman equation tells us that the value of being in a given state today, $V(s)$, equals the best possible current payoff plus the discounted value of what happens next.

We can think of this process as an **operator** — a kind of "machine" that takes a guess about the value function and produces a new, updated guess.

1.2.1 How it works

Start with any function $V(s)$ that tells you what the value might be for each state.
Then define a new function:

$$(\mathcal{T}V)(s) = \max_{x \in X(s)} \big\{\pi(s,x) + \beta V(f(s,x))\big\}.$$

What this says in words:

> "Given my current guess about how valuable future states are (that's V), what would be the total value of making the best decision today?"

So $\mathcal{T}$ takes the *old* value function and gives you a *new* one that's a little closer to the truth.
It's a way to **think one step ahead**.

1.2.2 The Fixed Point Idea

The true value function, $V^*(s)$, is the one that **doesn't change** when we apply this operation again:

$$V^*(s) = (\mathcal{T}V^*)(s).$$

In other words, if you already know the correct V^*, thinking one step ahead doesn't change your beliefs because you are already correct about the future.

Why this matters

- The operator gives us a **recipe for computing** V^*:
  - start with a guess and keep applying $\mathcal{T}$ repeatedly.
  - Each time, you're improving your estimate of the value of being in each state.
- Economically, this process mirrors **learning or planning**:
  - we evaluate today's decisions using our expectations of tomorrow, adjust, and repeat until everything is internally consistent.

Visual Intuition

- If you plotted $V_0(s)$ (your first guess) and then $V_1(s) = \mathcal{T}V_0(s)$, the curves would move closer and closer together until they line up at $V^*(s)$.
- That's what it means for the Bellman equation to be a **fixed point** — a steady state in your expectations about value.

In the context of economics

- For a **consumer**, $\mathcal{T}$ means re-evaluating how much future consumption is worth.
- For a **firm**, it means updating the expected profitability of holding or investing capital.

- For a **resource owner**, it means revising how valuable it is to leave part of the stock for tomorrow.

In all cases, the Bellman operator captures the logic of **forward-looking behavior**:
today's value depends on how optimally we plan for tomorrow.

Each application of $\mathcal{T}$ corresponds to "thinking one step further ahead."

- Starting with any initial guess $V_0(s)$,
- Repeatedly applying $\mathcal{T}$, $V_{k+1} = \mathcal{T}V_k$,
- Converges to the true value function V^* under mild conditions.

This is **Value Function Iteration (VFI)**.

1.3 Existence and Uniqueness: Why the Bellman Equation Has a Single Solution

Once we define the Bellman operator $\mathcal{T}$ - the rule that takes a guess about future value and updates it - we can ask two key questions:

1. Does this process always lead to a stable value function?
2. Will it always settle on the **same** function, no matter where we start?

The answer is yes — as long as future payoffs are **discounted** (so $\beta < 1$).

The reason is the **contraction mapping property**.

A **contraction mapping** is a transformation that *pulls things closer together* every time you apply it.

$$||\mathcal{T}V_1 - \mathcal{T}V_2|| \leq \gamma \, ||V_1 - V_2||$$

Imagine taking two different guesses about the value function, say $V_1(s)$ and $V_2(s)$. When we apply the Bellman operator to both, the resulting functions $\mathcal{T}V_1$ and $\mathcal{T}V_2$ are **closer to each other** than the originals.

Each round of updating reduces the distance between our guesses — eventually, all sequences converge to the same point, the **true value function** V^*.

Discounting is what makes this work. Because future rewards are multiplied by $\beta < 1$, any disagreement about the future is automatically **shrunk** when we think one step ahead.

Small differences in how we value the future can't explode backward into large differences today — they fade over time.

This gives the Bellman operator its *gravitational pull* toward a single stable value function.

If applying $\mathcal{T}$ repeatedly always pulls guesses closer together, there must be one and only one function that can't be improved upon — that's the **fixed point**:

$$V^*(s) = \mathcal{T}V^*(s).$$

Mathematically, this follows from the **Contraction Mapping Theorem**, but economically, you can think of it as saying:

There is one internally consistent way to value the future that agrees with itself when we plan forward.

1.3.1 How this helps us in practice

Because $\mathcal{T}$ is a contraction, we can find V^* by **value function iteration**:

1. Start with any initial guess $V_0(s)$ — even something crude.
2. Apply $\mathcal{T}$ to get an updated guess $V_1 = \mathcal{T}V_0$.

3. Keep repeating: $V_{n+1} = \mathcal{T} V_n$.

Each iteration gets us closer to the truth.
No calculus tricks, no global search — just forward iteration guided by economic logic.

1.3.2 The big takeaway

- Discounting and diminishing returns make the future "well-behaved."

- Together they guarantee that the Bellman equation has **one** solution, and that repeated forward-looking reasoning will find it.
- This is why we can compute dynamic equilibria with confidence: as long as the problem is discounted and well-behaved, there is a single, stable value function waiting to be found.

1.4 Euler Equations and the Envelope Condition

Dynamic optimization gives two complementary characterizations of optimal behavior:

- **Euler equation (FOC in the control):** how the agent trades off current vs. future returns when choosing x.
- **Envelope condition (FOC "in the state"):** how the lifetime value V changes with the state s.

The envelope condition is what lets us **eliminate messy derivatives of the policy function** and express the Euler equation in terms of primitives and V' only.

We work with the same notation as above:

$$V(s) = \max_{x \in X(s)} \{\pi(s,x) + \beta V(f(s,x))\}, \quad 0 < \beta < 1,$$

with optimal policy $x^*(s)$.

1.4.1 From Bellman to First-Order Conditions

Assume an interior, differentiable solution for intuition (we add bounds/KKT below).

- **Stationarity (optimum in x):**

$$\pi_x(s, x^*(s)) + \beta \, V'(f(s, x^*(s))) \, f_x(s, x^*(s)) = 0.$$

- **Envelope (optimum in s):**

$$V'(s) = \pi_s(s, x^*(s)) + \beta \, V'(f(s, x^*(s))) \, f_s(s, x^*(s)).$$

Why envelope works: when differentiating the max wrt s, the chain-rule term involving $x_s^*(s)$ vanishes because the stationarity condition sets the derivative wrt x to zero at the optimum. Intuitively, a marginal change in s doesn't induce a first-order change through the (already optimized) $x^*(s)$.

1.4.2 The (One-Step-Ahead) Euler Equation

Write the time-t Bellman equation at (s_t, x_t) with $s_{t+1} = f(s_t, x_t)$:

$$\pi_x(s_t, x_t) + \beta \, V'(s_{t+1}) \, f_x(s_t, x_t) = 0.$$

This is the **Euler equation**: the current marginal payoff from x_t equals the discounted marginal value of how x_t moves the state into the future.

We can **eliminate** $V'(s_{t+1})$ using the envelope condition at $t+1$:

$$V'(s_{t+1}) = \pi_s(s_{t+1}, x_{t+1}) + \beta\, V'(s_{t+2})\, f_s(s_{t+1}, x_{t+1}),$$

which yields a purely **primitive** intertemporal tradeoff once substituted back. In many applications π does not depend directly on s (only through feasibility), making this especially clean.

1.4.2.1 Consumption–Savings (simple, separable)

Let's illustrate with the **savings–consumption problem**.

The household chooses next period's assets a_{t+1} each period:

$$V(a_t) = \max_{a_{t+1} \geq 0} \Big\{ u(c_t) + \beta V(a_{t+1}) \Big\},$$

subject to the budget constraint

$$c_t + a_{t+1} = (1+r)a_t + y.$$

Substitute $c_t = (1+r)a_t + y - a_{t+1}$, so

$$V(a_t) = \max_{a_{t+1} \geq 0} \{ u((1+r)a_t + y - a_{t+1}) + \beta V(a_{t+1}) \}.$$

The First-Order Condition (Euler Equation)

Take the derivative of the Bellman RHS with respect to a_{t+1}:

$$-u'(c_t) + \beta V'(a_{t+1}) = 0,$$

which gives

$$u'(c_t) = \beta V'(a_{t+1}).$$

This is the **Euler condition** in implicit form: the marginal utility today equals the discounted shadow value of next period's assets.

The Envelope Condition

Differentiate the Bellman equation with respect to a_t, treating a_{t+1} as constant (by the envelope theorem):

$$V'(a_t) = u'(c_t)(1+r).$$

Intuitively: a marginal increase in assets a_t raises consumption by $(1+r)$, increasing utility by $u'(c_t)(1+r)$.

> **i** Why We Can "Treat a_{t+1} as Constant"
>
> $$\frac{dV(a_t)}{da_t} = u'(c_t)\big[(1+r) - \frac{da^*_{t+1}(a_t)}{da_t}\big] + \beta V'(a_{t+1}) \frac{da^*_{t+1}(a_t)}{da_t}.$$
>
> Now, note that the **first-order condition** for optimal a_{t+1} is:
>
> $$-u'(c_t) + \beta V'(a_{t+1}) = 0.$$
>
> Rearrange and substitute this into the derivative:
>
> $$\frac{dV(a_t)}{da_t} = u'(c_t)(1+r) - \left[u'(c_t) - \beta V'(a_{t+1})\right] \frac{da^*_{t+1}(a_t)}{da_t}.$$

The term in brackets equals zero by the FOC, so the entire last term drops out:

$$V'(a_t) = u'(c_t)(1+r).$$

We can use the envelope condition one period ahead

$$V'(a_{t+1}) = u'(c_{t+1})(1+r).$$

Then substitute it into the original FOC wrt a_{t+1}

$$u'(c_t) = \beta(1+r)u'(c_{t+1}).$$

This is the **standard consumption Euler equation**, expressing intertemporal optimality purely in terms of marginal utilities and the return $1+r$.

Economic interpretation

The Euler equation equates the *marginal benefit* and *marginal cost* of saving.

- The left-hand side, $u'(c_t)$, is the **marginal utility cost** of giving up one unit of consumption today.
- The right-hand side, $\beta(1+r)u'(c_{t+1})$, is the **discounted marginal utility benefit** of the extra consumption made possible tomorrow by saving that unit and earning the return $(1+r)$.

In equilibrium, these two must be equal — meaning that the household is indifferent (at the margin) between consuming one more unit today or saving it to consume tomorrow.

Interpreting the ratio of marginal utilities

Rearranging gives

$$\frac{u'(c_{t+1})}{u'(c_t)} = \frac{1}{\beta(1+r)}.$$

- The **left-hand side** is the *intertemporal marginal rate of substitution* (IMRS) — the rate at which the household is willing to trade future consumption for current consumption.
- The **right-hand side** is the *intertemporal price ratio* implied by markets — the relative price of tomorrow's consumption in terms of today's, discounted by β and adjusted for the gross interest rate $(1+r)$.

In equilibrium, these two ratios are equal: the household's willingness to substitute consumption across periods equals the market's rate of return on savings.

Intuition

- If the interest rate r rises, the right-hand side decreases.
  To restore equality, the ratio $u'(c_{t+1})/u'(c_t)$ must fall — meaning c_{t+1}/c_t rises.
  $\rightarrow$ **Higher interest rates encourage saving and future consumption.**
- If β falls (the agent becomes more impatient), the right-hand side decreases.
  $\rightarrow$ **The household consumes more today and less tomorrow.**

This condition captures the **core intertemporal tradeoff** at the heart of dynamic economic behavior: balancing impatience, returns, and the curvature of utility (risk aversion or diminishing marginal utility).

Let's assume constant relative risk aversion

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}, \quad \sigma > 0,$$

where σ is the **coefficient of relative risk aversion** (and $1/\sigma$ is the **intertemporal elasticity of substitution**).

Then

$$u'(c) = c^{-\sigma}.$$

Plug this into the Euler equation:

$$u'(c_t) = \beta(1+r)u'(c_{t+1}) \quad \Rightarrow \quad c_t^{-\sigma} = \beta(1+r)c_{t+1}^{-\sigma}.$$

Rearrange for the **growth rate of consumption**:

$$\frac{c_{t+1}}{c_t} = \left[\,\beta(1+r)\,\right]^{1/\sigma}.$$

Interpretation

- If $\beta(1+r) = 1$, then $c_{t+1} = c_t$: consumption is *constant* over time.
- If $\beta(1+r) > 1$, the household values future consumption relatively more $\rightarrow$ **consumption grows** over time.
- If $\beta(1+r) < 1$, the household is relatively impatient $\rightarrow$ **consumption declines** over time.

Economic parameters: - β captures *patience*: higher $\beta \rightarrow$ slower consumption decline (more saving). - σ governs *willingness to smooth consumption*: higher σ (more curvature) $\rightarrow$ less sensitivity of growth to interest rates.

This simple form makes the Euler condition directly testable and provides a foundation for empirical work in both **macroeconomics** and **household finance**.

1.4.2.2 Renewable Resource (fishery; matches your notes)

Bellman: $V(s) = \max_{0\le h\le s}\{p\,h - c\,h + \beta V(G(s-h))\}$ with $s' = G(s-h)$.

- **Euler (interior):**
$$(p-c) - \beta\,V'(s')\,G'(s-h) = 0.$$
- **Envelope:**
$$V'(s) = \beta\,V'(s')\,G'(s-h).$$

Combine them to eliminate $V'(s')$:
$$p - c = V'(s).$$

Interpretation: harvest until marginal net revenue equals the **shadow value** of the stock (the user cost). With bounds, you get the usual "escapement" logic: at low s, $h^* = 0$ (rebuilding); at high s, h^* hits the upper bound $h = s$ (if profitable).

1.5 Analytical Example: Hotelling Resource Extraction

We now work through a fully analytical example that ties together the **Euler equation** and the **Envelope (shadow value)** condition.

1.5.1 Setup

A social planner (or resource owner) chooses extraction $\{q_t\}$ to maximize the discounted value of profits from a nonrenewable resource.

$$\max_{\{q_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t [p\,q_t - c(q_t)]$$

subject to the **resource stock constraint**

$$S_{t+1} = S_t - q_t, \qquad S_t \geq 0,$$

with given S_0 and discount factor $0 < \beta < 1$.

1.5.2 Bellman Equation

Let $V(S)$ be the value of having S units of the resource left:

$$V(S) = \max_{0 \leq q \leq S} \left\{ p\, q - c(q) + \beta\, V(S - q) \right\}.$$

1.5.3 First-Order and Envelope Conditions

1.5.3.1 (a) Euler (stationarity in q)

$$p - c'(q_t) = \beta\, V'(S_{t+1}).$$

The planner extracts up to the point where the **current net price** equals the **discounted shadow value of remaining stock**.

1.5.3.2 (b) Envelope (marginal value of the stock)

Differentiating the Bellman equation with respect to S and using the envelope theorem:

$$V'(S_t) = \beta\, V'(S_{t+1}).$$

1.5.4 Combine Euler and Envelope

Shift the envelope one period forward:

$$V'(S_{t+1}) = \beta\, V'(S_{t+2}).$$

Substitute into the Euler:

$$p - c'(q_t) = \beta\, V'(S_{t+1}) = \beta^2 V'(S_{t+2}) = \cdots$$

so $V'(S_t)$ grows at the rate $1/\beta$.

Define $\lambda_t \equiv V'(S_t)$ as the **shadow price** or **resource rent**. Then:

$$\lambda_{t+1} = \frac{\lambda_t}{\beta}.$$

If $\beta = (1 + r)^{-1}$, this gives the classic **Hotelling rule**:

$$\lambda_{t+1} = (1 + r)\lambda_t.$$

That is, the *resource rent* (or in equilibrium, the **net price**) must grow at the rate of interest.

1.5.5 Linear-Cost Example

Let $c(q) = \frac{1}{2}\gamma q^2$ with $\gamma > 0$.
Then the FOC is

$$p - \gamma q_t = \beta V'(S_{t+1}).$$

Using the envelope $V'(S_t) = \beta V'(S_{t+1})$, we can eliminate V':

$$p - \gamma q_t = V'(S_t).$$

Differentiate over time to get the *Euler equation in observable variables*:

$$(p - \gamma q_{t+1}) = \frac{1}{\beta}(p - \gamma q_t).$$

If $\beta = (1+r)^{-1}$,

$$p - \gamma q_{t+1} = (1+r)(p - \gamma q_t),$$

so the **net price (marginal rent)** rises at the interest rate.

1.5.6 6. Economic Interpretation

Condition	Meaning	Intuition
$p - c'(q_t) = \beta V'(S_{t+1})$	*Euler:* marginal profit = discounted shadow value	Extract until current profit = opportunity cost of future scarcity
$V'(S_t) = \beta V'(S_{t+1})$	*Envelope:* shadow price grows at $1/\beta$	Keeping one more unit today increases lifetime value by the discounted rent
$\lambda_{t+1} = (1+r)\lambda_t$	*Hotelling rule*	In equilibrium, the resource rent (or net price) grows at the interest rate

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1.5.7 Policy Intuition

- When the interest rate r is **high**, future rents are heavily discounted → faster extraction.
- When r is **low** or β high, future rents matter more → slower extraction.
- The rule does *not* depend on the level of stock S_t; it is purely an **arbitrage condition** between leaving the resource in the ground vs. extracting and investing the proceeds.

> **Note**
>
> Why Must Rents Grow at the Interest Rate?
> The Hotelling rule says that in equilibrium,
>
> $$\lambda_{t+1} = (1+r)\lambda_t,$$
>
> or equivalently that the **net price (resource rent)** grows at the rate of interest.
> Let's unpack *why* this must be true.

The Resource as an Asset

Think of the remaining stock S_t as an **asset**: each unit in the ground is a claim on a future profit when extracted.

- If you **extract one more unit now**, you earn today's net revenue
  current rent $= p - c'(q_t) = \lambda_t$.
- If you **leave it in the ground**, you keep the option to sell it in the future, when scarcity will be higher and rent will be larger.

Thus, each unit of resource is an asset that yields **capital gains** (through rising rent) but **no dividends** (unless extracted).

No-Arbitrage Condition

An owner can hold wealth in two forms:

1. **Financial asset** yielding interest rate r.

2. **Resource asset** whose value (the rent λ_t) appreciates over time.

If both are riskless, **arbitrage** implies they must yield the same return.

Otherwise:

- If $\lambda_{t+1}/\lambda_t > 1 + r$:
  holding the resource yields higher return $\rightarrow$ everyone hoards $\rightarrow$ no extraction.
- If $\lambda_{t+1}/\lambda_t < 1 + r$:
  financial assets dominate $\rightarrow$ everyone extracts and invests proceeds.

Only when $\lambda_{t+1}/\lambda_t = 1 + r$ can both be held in equilibrium.

That is exactly the **Hotelling condition**:

$$\frac{\lambda_{t+1} - \lambda_t}{\lambda_t} = r.$$