Computational Model of Neutron Star Atmospheres

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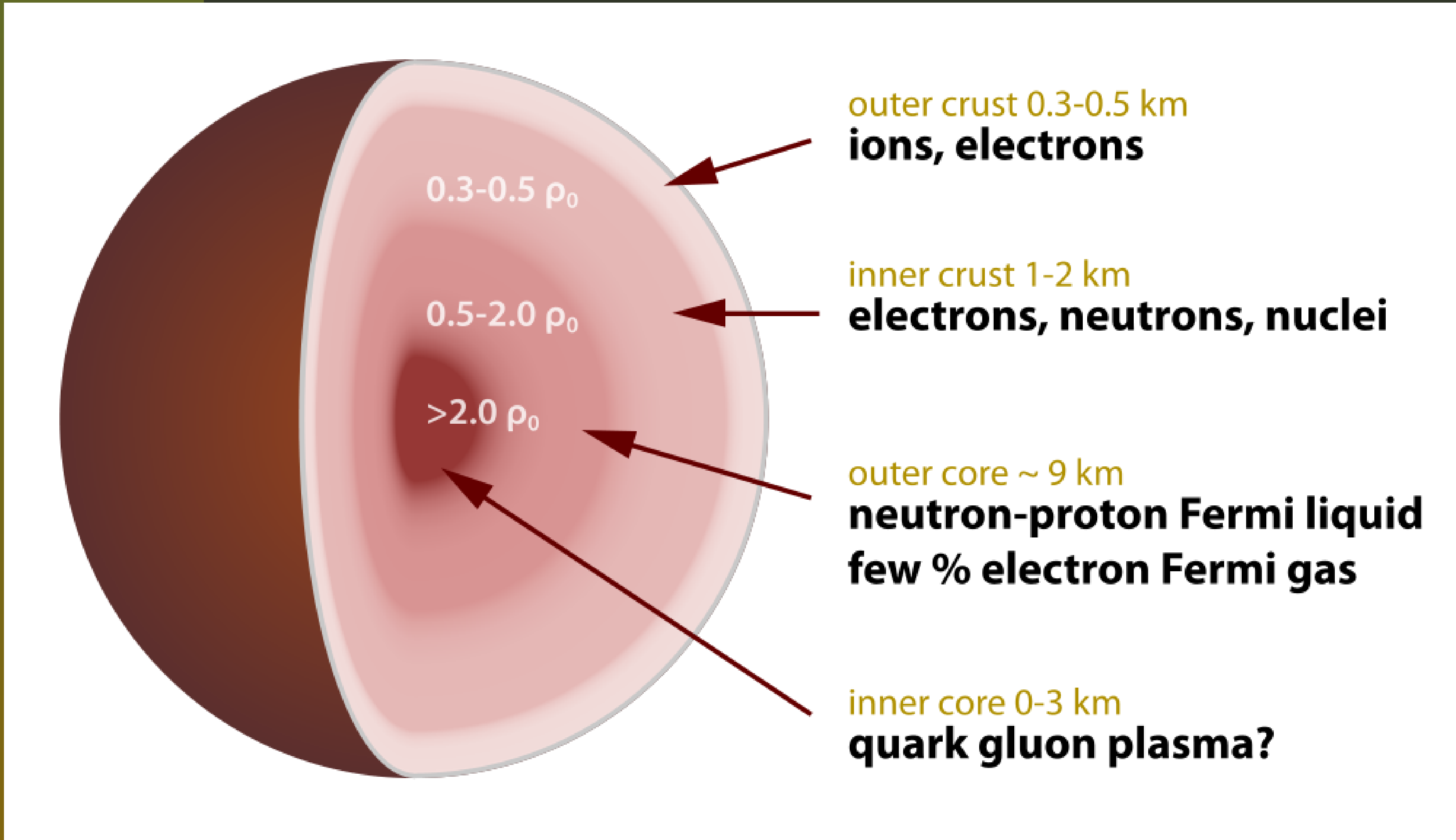
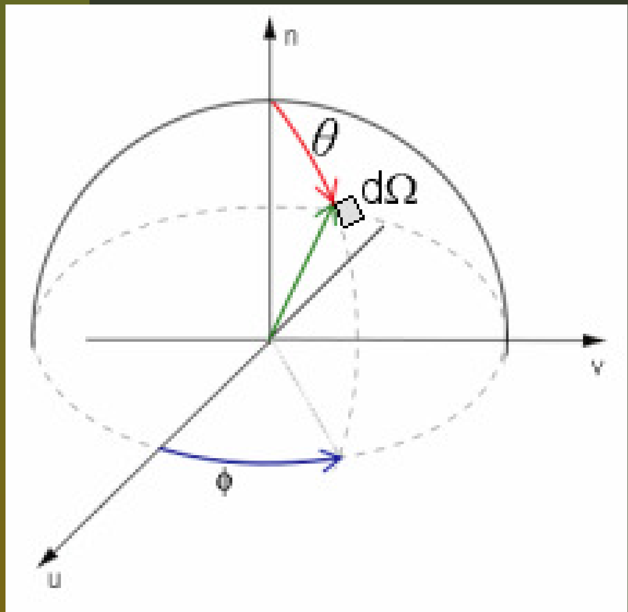
Neutron Star Atmosphere Basics

- $T_{NS} \sim 10^6\, K$
- $M_{NS} \simeq 1.4 - 3.0\, M_{\odot}$
- $R_{NS} \sim km$
- The large density gives rise to strong gravitational fields.
- Thus the atmosphere of the neutron star is crunched to cm scales.
- The high temperatures ionize the atmosphere; leaving a dense plasma of ions and electrons. For our purposes, we assume the atmosphere is hydrogenic.

Neutron Star Interior



Specific Intensity



Imagine a collection of photons propagating in some arbitrary direction $\hat{k}$ with frequency range $(\nu,\ \nu + d\nu)$. Each photon has a direction. The distribution of photons through an arbitrary surface $dA_\perp$, in time $(t, t + dt)$, has a direction $d\theta$ and $d\phi$, measured from the plane of the surface. Thus, the direction can be described by taking the area element in spherical polar coordinates and dividing by r^2,

$$d\Omega = \sin\theta\, d\theta\, d\phi$$

Therefore, the energy in the beam passing through $dA_\perp$ in time dt per unit frequency per unit solid angle defines the specific intensity of the beam,

$$I_\nu = \frac{dE}{dt\, dA_\perp\, d\nu\, d\Omega} = \frac{1}{\mu} \frac{dE}{dt\, dA\, d\nu\, d\Omega}$$

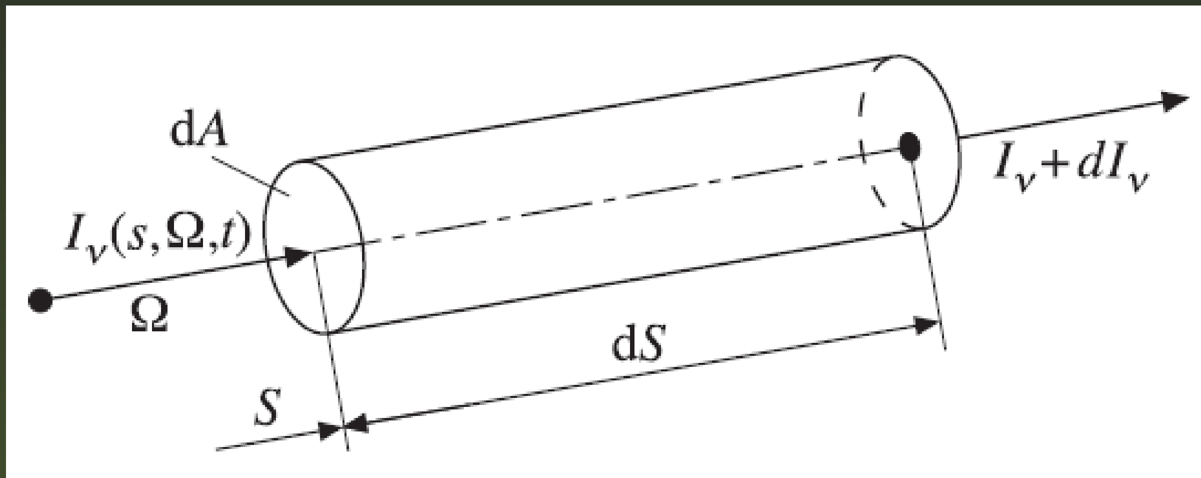
where $\mu = \cos\theta$.

Flux

We know flux is the change in power per unit area; naturally, we can define the specific flux F_ν in terms of specific intensity I_ν.

$$F_\nu \equiv \frac{dE}{dt\,dA\,d\nu} \;\Leftrightarrow\; F_\nu = \int_{4\pi} d\Omega\,\mu\,I_\nu$$

Radiative Transfer Equation (RTE)



By defintion, I_ν is the change in radiative energy between s and $s + ds$.

$$\Delta E_{21} = [I_\nu(s + ds) - I_\nu(s)]\, d\nu\, d\Omega\, dt\, dA_\perp$$

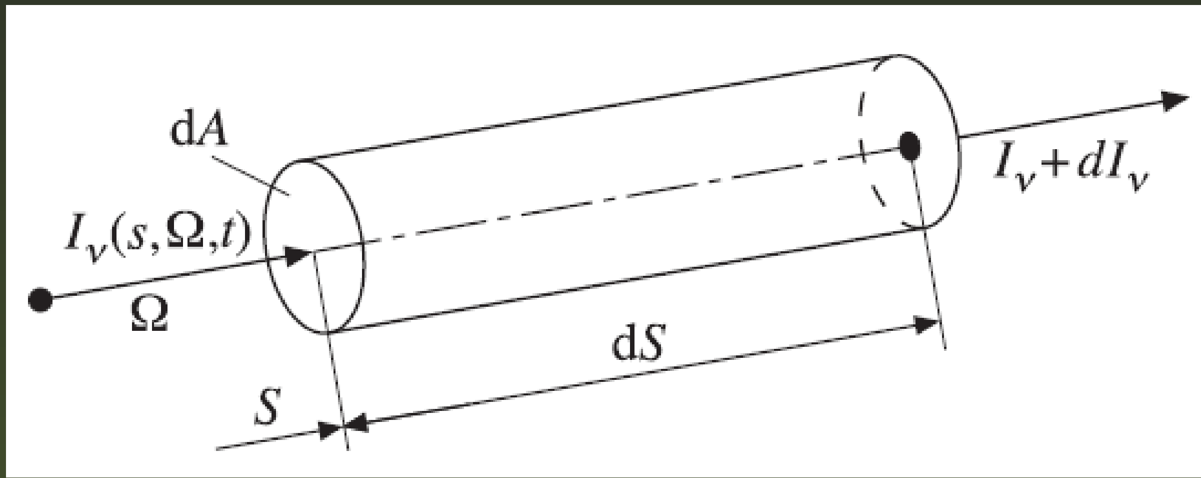
The energy added to the beam is descibed by the emission coefficient ζ_ν.

$$\Delta E_{emit} = \zeta_\nu\, d\nu\, d\Omega\, dt\, dA_\perp\, ds$$

And the energy removed from the beam is described by the extinction coefficient χ_ν

$$\Delta E_{extinct} = -\chi_\nu\, I_\nu\, d\nu\, d\Omega\, dt\, dA_\perp\, ds$$

RTE cont...



Expanding $I_\nu(s + ds)$ in a Taylor series yeilds,

$$\frac{dI_\nu}{ds} d\nu\, d\Omega\, dt\, dA_\perp\, ds = (\zeta_\nu - \chi_\nu\, I_\nu) d\nu\, d\Omega\, dt\, dA_\perp\, ds$$

where $\Delta E_{21} = \Delta E_{emit} + \Delta E_{extinct}$. Thus, the RTE takes the form,

$$\frac{dI_\nu}{ds} = -\chi_\nu\, I_\nu + \zeta_\nu$$

Plane-Parallel Geometry

A simplification is made for radiative transfer in neutron star atmospheres. It is a results of the small ratio between the radius of the neutron star and the depth of the atmosphere. Thus, we say the medium only varies along the z-axis, where $ds = dz/\mu$

$$\pm\mu\frac{\partial I_\nu^\pm}{\partial z} = -\chi_\nu(z)\, I_\nu^\pm + \zeta_\nu(z)$$

where, $I_\nu = I_\nu(z, \pm\mu)$ for $\mu \in [0, 1]$.

Note: We've assumed two things here,

1. The medium is in steady state, i.e. $\frac{\partial I_\nu}{\partial t} = 0$.
2. The medium is locally isotropic, i.e. $\chi_\nu \neq \chi_\nu(\mu)$ and $\zeta_\nu \neq \zeta_\nu(\mu)$.

Optical Depth

A natural scale to define is optical depth τ_ν,

$$d\tau_\nu = -\chi_\nu \, dz$$

or better yet, $\tau_{\mu\nu}$, to include the angular dependence

$$d\tau_{\mu\nu} = -\frac{1}{\mu} \, \chi_\nu \, dz$$

Thus, the RTE now takes the "simple" form,

$$\frac{dI^{\pm}_{\mu\nu}}{d\tau_{\mu\nu}} = \pm \left(I^{\pm}_{\mu\nu} - S_\nu \right)$$

where, $S_\nu \equiv \zeta_\nu / \chi_\nu$.

Calculating χ_ν

- The extinction coefficient is calculated from the mechanics of three-body collisions and quantum mechanics.
- For a fully ionized, non-magnetic plasma, the most important radiative process is free-free absorption.
- In this process, a photon is absorbed when an electron changes its trajectory in the Coulomb field of a proton.
- The extinction coefficient, for the process we are interested in, has been derived before by Hummer in 1988,

$$\chi_\nu = N_e \, N_i \, \bar{g}_{f\,f}(T,\nu) \, \frac{Z^2 \, e^6}{3 \, \pi \, h \, \nu^3 \, m_e^2 \, c} \left(\frac{2 \, \pi \, m_e}{3 \, k_B \, T} \right)^{\frac{1}{2}} \left(1 - e^{\frac{-h \, \nu}{k_B \, T}} \right)$$

Local Thermodynamic Equilibrium

In practice, the two fundamental approximations made are

1. The matter and radiation are not in equilibrium; otherwise, $I_\nu = B_\nu$, where B_ν is Planck's law.
2. The matter is, locally, in equilibrium. Therefore, due to LTE,

$$\zeta_\nu = \chi_\nu \, B_\nu \;\Leftrightarrow\; S_\nu = B_\nu$$

Constraints

We need to impose constraints on our model, otherwise, T and ρ are free parameters. Constraint:

1. Hydrostatic Equilibrium: The fluid equation for an element of atmosphere is

$$\rho \frac{d\tilde{v}}{dt} = -\rho \tilde{g} - \vec{\nabla} p$$

   In equilibrium, $\vec{\nabla} p = -\rho \tilde{g}$.

2. Radiative Equilibrium: The radition emitted, from each layer in the atmosphere, is constant in time. Therefore,

$$\frac{\partial F}{\partial z} = 0 \;\Rightarrow\; \int d\nu \int_{4\pi} d\Omega\, \mu \frac{\partial I_\nu}{\partial z} = \int d\nu \int_{4\pi} d\Omega\, \chi_\nu \left(I_\nu - B_\nu \right) = 0$$

Equation of State

We treat the atmosphere as an ideal gas of protons and electron. Thus, the pressure is

$$P = P_e + P_p = n_e\, k_B\, T + n_p\, k_B\, T$$

For ionized hydrogen, $\rho = m_p\, n_p + m_e\, n_e \approx m_p\, n_p$. Thus, the pressure is

$$P = \frac{2\,\rho\, k_B\, T}{m_p}$$

Recap: Three relations

1. Hydrostatic equilibrium.
2. Radiative equilibrium.
3. Ideal gas.

The radiative equilibrium condition constrains the temperature of the atmosphere, and (1) and (3) relate density and temperature.

Thompson Depth Grid

We have defined an "auxilary" depth scale τ_{th}, related to Thompson scattering.

$$d\tau_{th} = -\chi_{th}\, dz$$

where,

$$\chi_{th} = n_e\, \sigma_{th} = \frac{\rho}{m_p} \frac{8\pi}{3} \left(\frac{e^2}{m_e\, c^2} \right)^2$$

$$\therefore \tau_{th} \propto \rho$$

Due to large ranges of density throughout the atmosphere, the Thompson depth is spaced logarithmically to get even sampling at all depths. To keep numerical integration simple,

$$\log \left(\frac{\tau_{th,d+1}}{\tau_{th,d}} \right) = const.$$

Energy and Angular Grid

The typical energies associated with neutron star atmospheres are of order keV. Thus, we define a logarithmically spaced energy grid (much like the Thompson grid). Again,

$$E_{k+1} = E_k \times 10^{\Delta}$$

where, Δ is the grid spacing and is constant.

Note:

$$\int dE\, f(E) = \int d(\log E)\, E\, f(E) \approx \sum_{k=0}^{K-1} h_k\, f(E_k)$$

where h_k are the integration weights.

To define the angular grid points, we used the Gaussian-Legendre quadrature. Here, the abscissas are our μ grid points and the weights are the integration weights. Here,

$$\int d\mu\, f(\mu) \approx \sum_{m=0}^{M-1} w_m\, f(\mu_m)$$

Units

The units for our data have been choosen carefully.

- Energy $\sim keV$
- Density $\sim 10^{14}\,\frac{g}{cm^3}$
- Temperature $\sim 10^6\,K$
- Intensity $\sim 10^{22}\,\frac{erg}{keV\ cm^2\ s\ sr}$

For example, Planck's law can be written as,

$$B_E = 5.04 \times \frac{E_1^3}{\exp\left(11.6\,\frac{E_1}{T_6}\right) - 1}\left(\frac{10^{22}\,erg}{keV\ cm^2\ s\ sr}\right)$$

where, $E_1 \equiv \left(\frac{E}{keV}\right)$ and $T_6 \equiv \left(\frac{T}{10^6\ K}\right)$ are unitless quantities.

Initial Temperature Profile

We need an initial guess for the temperature profile. In an atmosphere where extinction does not depend on frequency, it can be shown that,

$$T_6 = T_{eff} \left[\frac{3}{4} \left(\tau + \frac{2}{3} \right) \right]^{1/4}$$

but, χ_ν has a strong dependence on frequency. So we need an average value representative of optical depth, the Rosseland mean χ_R,

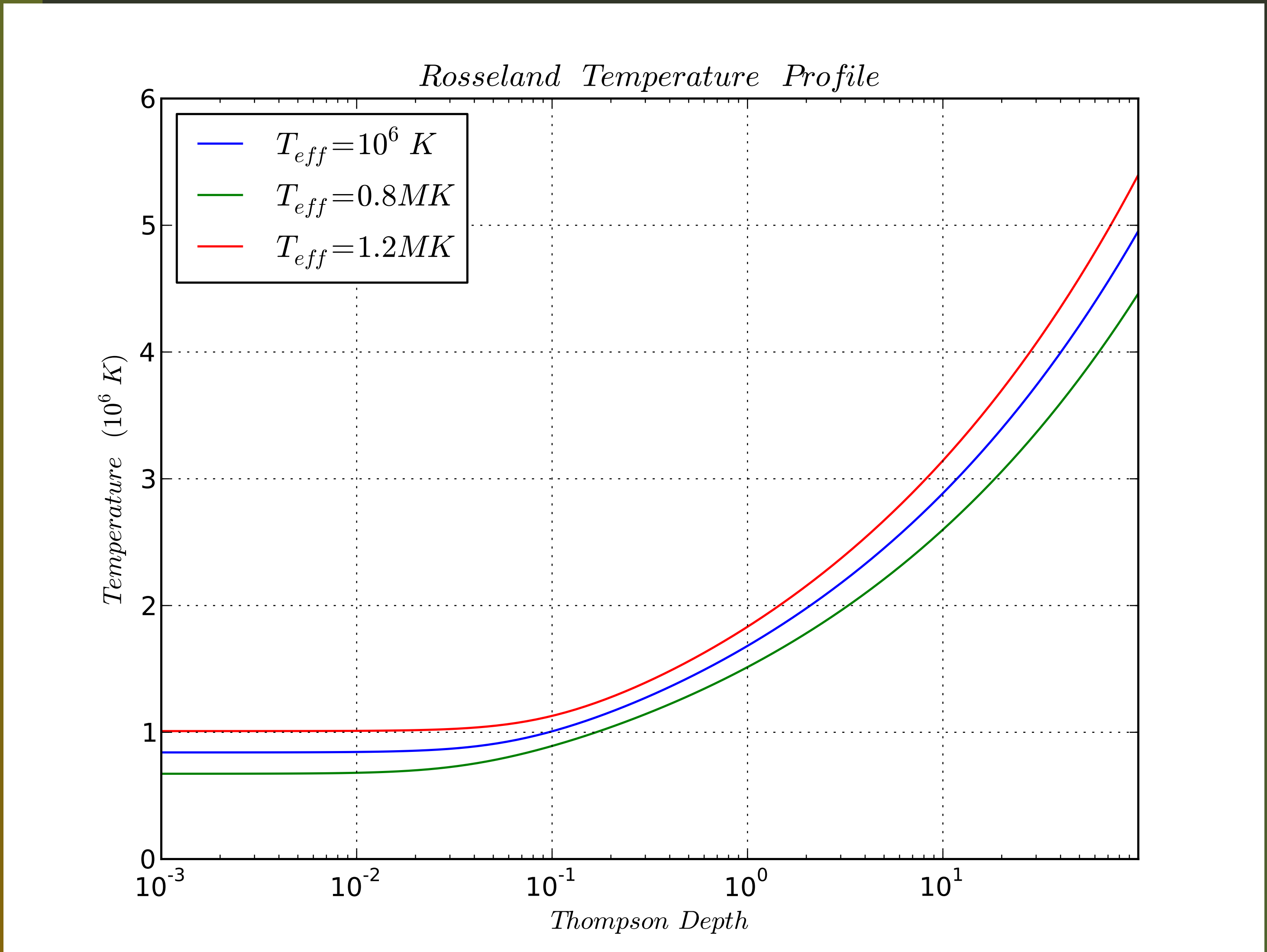
$$\frac{1}{\chi_R} = \frac{\pi}{4\,\sigma_{SB}\,T^3} \int_0^\infty d\nu \left(\frac{1}{\chi_\nu} \right) \frac{\partial B_\nu}{\partial T}$$

and

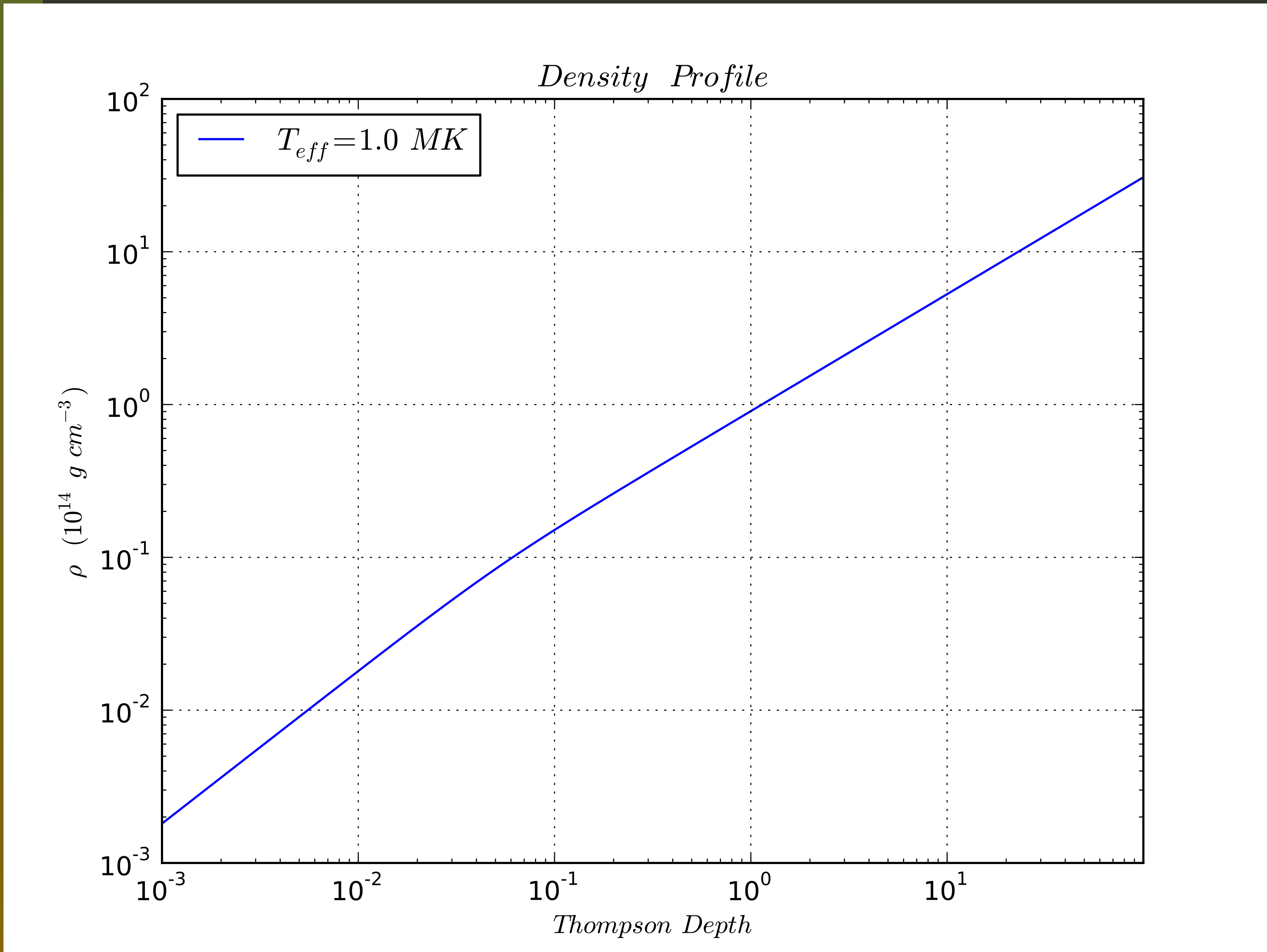
$$d\tau_R = -\chi_R\, dz$$

Using τ_R, the temperature profile is calculated in a recursive process until convergence.

Initial Temperature Profile – Plots



Plots cont...



Density Profile
T_{eff}=1.0 MK
ρ (10^{14} g cm^{-3})
Thompson Depth
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}

Short Characteristic Solution to RTE

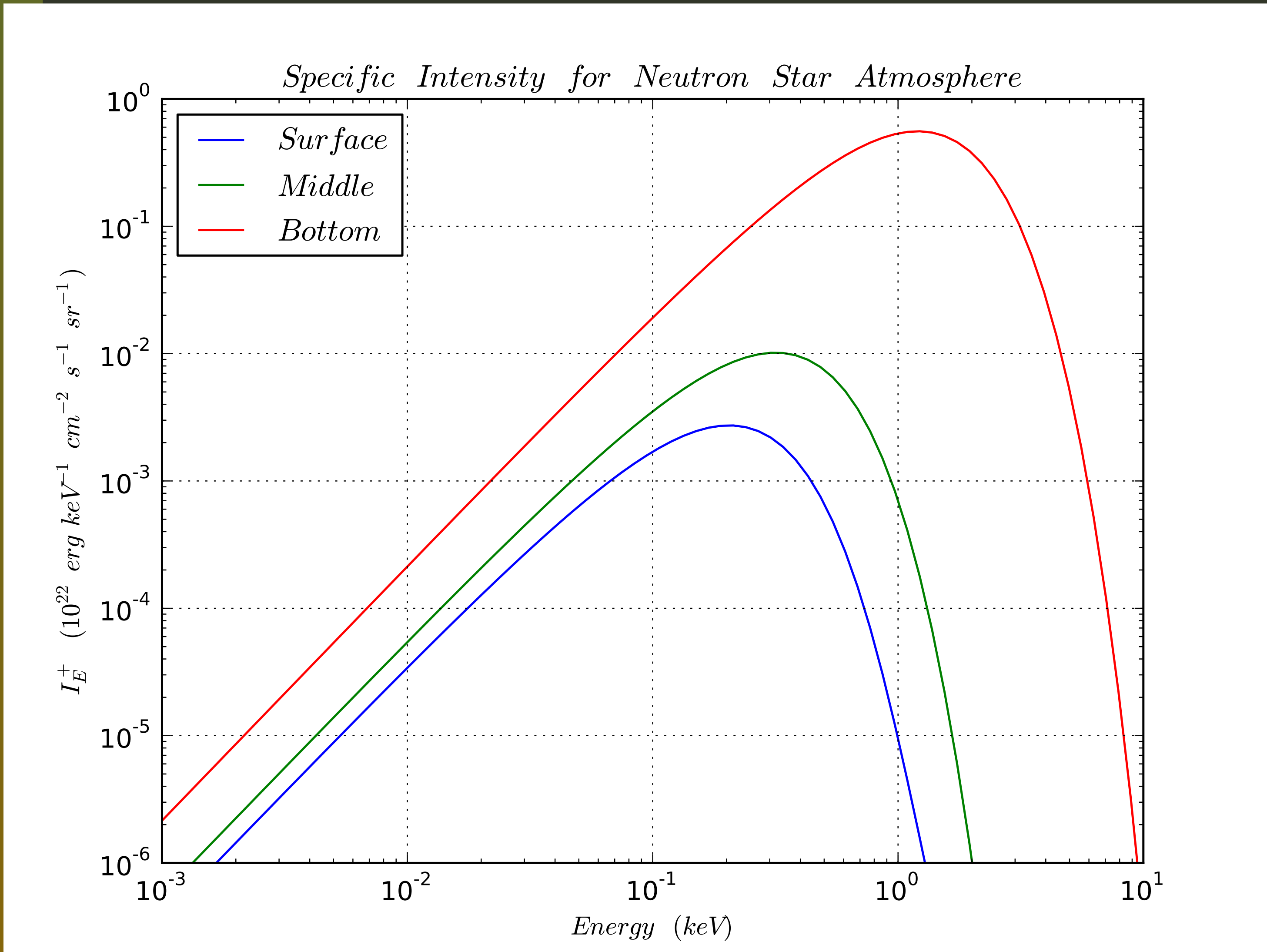
With the initial temperature profile, we can solve the RTE. It can be shown that,

$$I_{di}^{\pm} = e^{-\Delta\tau_{di}^{\pm}}\, I_{d\pm1,\,i}^{\pm} + \int_{\tau_{d\pm1,\,i}}^{\tau_{di}} e^{\mp(\tau_{\mu\nu}-\tau_{di})} S_\nu \, d\tau_{\mu\nu}$$

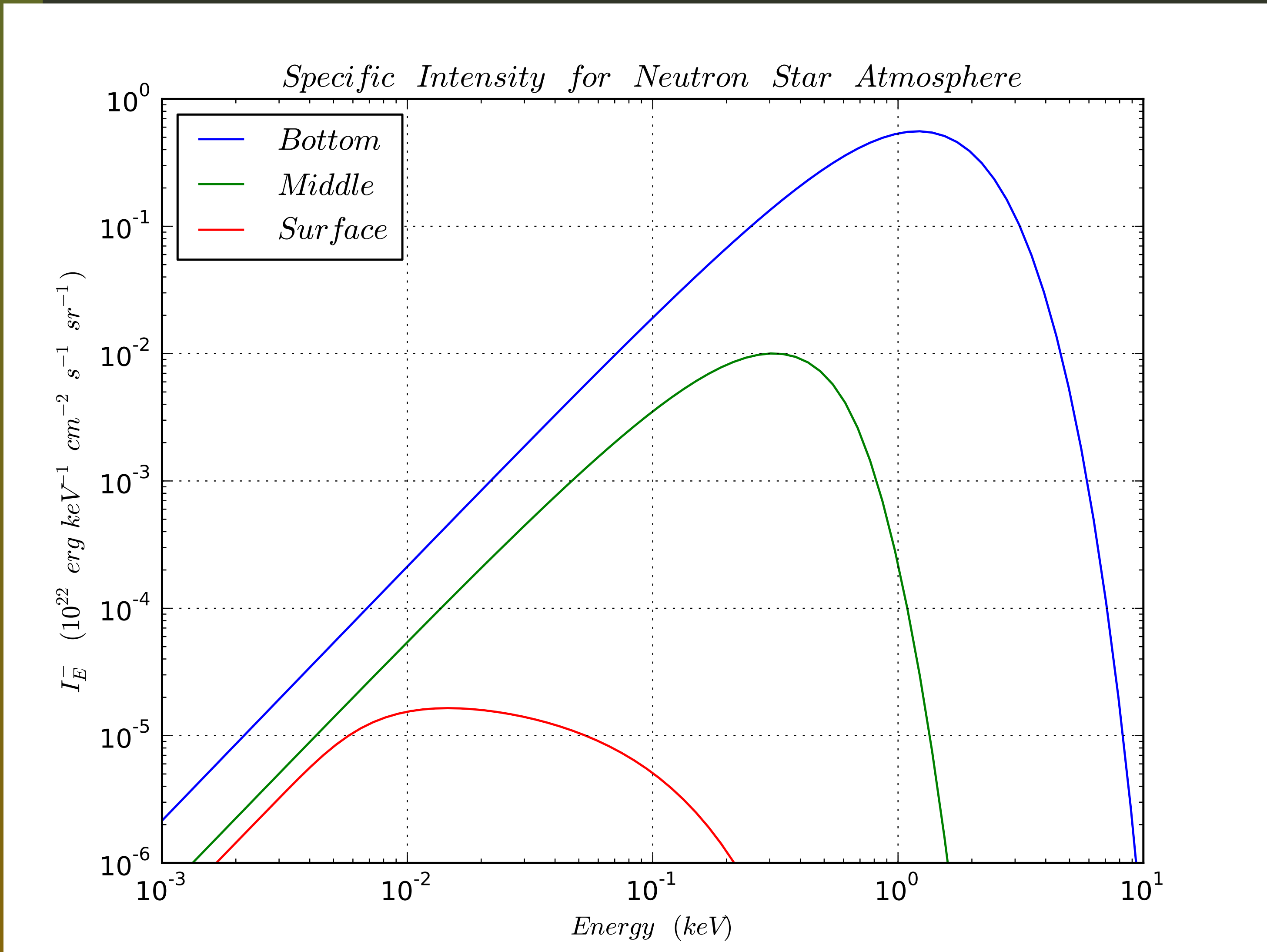
The short-characteristics solution approximates the specific intensity by expanding the source function S_ν in a Taylor series and calculating the above integral analytically.

At the lower boundary, $I_{\mu\nu}^{+} = B_\nu(\tau_{D-1})$, because the radiation from the deeper layer are in thermal equilibrium with the matter. At the surface, $I_{\mu\nu}^{-} = 0$, because we assume that there is no incoming radiation.

Specific Intensity – Plots



Plots cont...



$Specific \;\; Intensity \;\; for \;\; Neutron \;\; Star \;\; Atmosphere$
$Bottom$
$Middle$
$Surface$
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
$I_E^- \;\; (10^{22} \; erg \; keV^{-1} \; cm^{-2} \; s^{-1} \; sr^{-1})$
$Energy \;\; (keV)$

Temperature Corrections

Thus far, we have satisfied the constraint for matter to be in LTE. However, the constaint for steady-state has yet to be satisfied. If the flux is not constant throughout the atmosphere, excess deposits of energy will cause LTE to break down.

The Unsóld-Lucy temperature correction scheme uses the 0^{th}, 1^{st}, and 2^{nd} moment of specific intensity to update the temperature profile. Where,

$$J_\nu = \frac{1}{4\pi} \int d\nu \int_\Omega d\Omega \, I_{\mu\nu}$$

$$\bar{F}_\nu = \frac{1}{4\pi} \int d\nu \int_\Omega d\Omega \, \mu \, I_{\mu\nu}$$

$$H_\nu = \frac{1}{4\pi} \int d\nu \int_\Omega d\Omega \, \mu^2 \, I_{\mu\nu}$$

are the 0^{th}, 1^{st}, and 2^{nd} moments, respectively.

Best Practices for Computing

Matt and I have made it our goal to practice good scientific computing. The procedure for which was proposed in "Best Practices for Scientific Computing" (2012).

To do this we have:

- Setup strict rules to keep the code style throughout the program consistent.
- Documented the code using Doxygen.
- Setup a versioning system using an online repository called BitBucket.
- We use an open-source software configuration tool called SCons to build the program (a much better option than GNU make).
- We compile our source files using python wrapper functions for easy testing.

What's Next?

- Complete debugging procedure.
- Include opacities for other dynamic processes.
- Implement a procedure for magnetic atmosphere (possibly).
- Run the program on a Graphics Processing Unit (GPU).

Questions?