Chapter II

Geodesics on S^3

In this chapter we study the geodesic vector field on the tangent bundle of the 3-sphere. We examine its relation to the Kepler vector field, which governs the motion of two bodies in $\mathbf{R}^3$ under gravitational attraction. We give a method to regularize the flow of the Kepler vector field for all negative energies at once and not energy surface by energy surface.

1 The geodesic and Delaunay vector fields

In this section we find the geodesic vector field on the 3-sphere and give an explicit formula for its flow. Rescaling time gives the Delaunay vector field, which we need when regularizing the Kepler vector field, (see section 3.4).

We begin by discussing the geodesic vector field. Suppose that $\langle\ ,\ \rangle$ is the Euclidean inner product on $\mathbf{R}^4$. This induces a Riemannian metric g on $\mathbf{R}^4$ defined by $g(x)^\sharp(y)z = \langle y, z\rangle$ where $x \in \mathbf{R}^4$ and $y, z \in T_x\mathbf{R}^4 = \mathbf{R}^4$. Pulling back the canonical symplectic 2-form Ω on $T^*\mathbf{R}^4$ by the map $g^\sharp$ (see appendix A section 2) we obtain the symplectic form $\omega_4 = -d\langle y, dx\rangle$ on $T\mathbf{R}^4$. On $(T\mathbf{R}^4, \omega_4)$ consider the Hamiltonian function

$$\mathcal{H} : T\mathbf{R}^4 \to \mathbf{R} : (x, y) \to \tfrac{1}{2}\langle y, y\rangle. \tag{1}$$

Since an integral curve of the Hamiltonian vector field $X_{\mathcal{H}}$ satisfies

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= 0, \end{cases}$$

it is a straight line on $T\mathbf{R}^4$, except when $y = 0$ where it is a point. Hence $X_{\mathcal{H}}$ describes the *motion* of a particle in $T\mathbf{R}^4$ which is not subject to any *force*. To constrain this free particle so that it moves on the 3-sphere

$$S^3 = \left\{ x \in \mathbf{R}^4 \,\middle|\, \langle x, x\rangle = 1 \right\},$$

we add a force $\lambda(x, \dot{x})x$ which is normal to S^3 at the point x. The particle subjected to this force moves according to

$$\ddot{x} = \lambda(x, \dot{x})\, x. \tag{2}$$

Differentiating the defining equation of S^3 twice gives

$$\langle x, \ddot{x}\rangle + \langle \dot{x}, \dot{x}\rangle = 0. \tag{3}$$

Substituting (2) into (3) and using $\langle x, x\rangle = 1$ gives $\lambda(x, \dot{x}) = -\langle \dot{x}, \dot{x}\rangle$. Hence the *motion* of the free particle constrained to S^3 is governed by the second order equation

$$\ddot{x} = -\langle \dot{x}, \dot{x}\rangle x \tag{4}$$

subject to the constraints $\langle x, x\rangle = 1$ and $\langle \dot{x}, x\rangle = 0$. Written as a first order equation on the tangent bundle

$$TS^3 = \Big\{(x, y) \in T\mathbf{R}^4 \big| \langle x, x\rangle = 1 \ \& \ \langle x, y\rangle = 0\Big\}$$

of S^3, the constrained system (4) becomes

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= -\langle y, y\rangle x. \end{cases} \tag{5}$$

This defines a vector field Y on TS^3. Note that TS^3 is an invariant manifold of (5), thought of as a vector field on $T\mathbf{R}^4$, since the initial conditions $\langle x, x\rangle = 1$ and $\langle x, y\rangle = 0$ are preserved under its flow.

▷ The above discussion is not at all Hamiltonian. What we want to do is to show that Y is a Hamiltonian vector field on the phase space (TS^3, Ω_4), where Ω_4 is a suitable symplectic form.

(1.1) **Proof**: To do this, we use modified Dirac brackets, (see appendix A section 4). On the open subset $M = T\mathbf{R}^4 - (\{0\} \times \mathbf{R}^4)$ of $T\mathbf{R}^4$ consider the constraint functions

$$c_1 : M \to \mathbf{R} : (x, y) \to \tfrac{1}{2}(\langle x, x\rangle - 1) \ \text{ and } \ c_2 : M \to \mathbf{R} : (x, y) \to \langle x, y\rangle.$$

Let $\{\, , \,\}$ be the standard Poisson bracket on $C^\infty(T\mathbf{R}^4)$, the space of smooth functions on the symplectic manifold $(T\mathbf{R}^4, \omega_4)$, (see appendix A section 4). Since the matrix

$$(\{c_i, c_j\}) = \begin{pmatrix} 0 & \langle x, x\rangle \\ -\langle x, x\rangle & 0 \end{pmatrix}$$

is invertible on M with inverse

$$(C_{ij}) = \frac{1}{\langle x, x\rangle}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and since 0 is a regular value of the constraint map

$$\mathscr{C} : M \to \mathbf{R}^2 : m \to \big(c_1(m), c_2(m)\big),$$

the constraint manifold $TS^3 = \mathscr{C}^{-1}(0)$ is a *cosymplectic* submanifold of $(M, \omega_4|M)$. In other words, $\Omega_4 = \omega_4|TS^3$ is a symplectic form on TS^3. For $F \in C^\infty(M)$ let

$$F^* = F - \sum_{i,j=1}^{2}(\{F, c_i\} + F_i)C_{ij}c_j,$$

where the F_i lie in the ideal generated by c_1 and c_2. Define a Poisson bracket $\{\,,\,\}_{TS^3}$ on $C^\infty(TS^3)$ by

$$\{F|T^+S^3, G|T^+S^3\}_{TS^3} = \{F^*, G^*\}|TS^3.$$

Note that the Hamiltonian vector field $X_{F|TS^3}$ of the Hamiltonian F constrained to TS^3 is the Hamiltonian vector field X_{F^*} restricted to TS^3.

Applying these remarks to the unconstrained Hamiltonian $\mathcal{H}$ (1) gives

$$\begin{aligned}
\mathcal{H}^* &= \mathcal{H} - \sum_{i,j}(\{\mathcal{H}, c_i\} + \mathcal{H}_i)C_{ij}c_j \\
&= \tfrac{1}{2}\langle y, y\rangle + \frac{1}{\langle x, x\rangle}\Big\langle\big(\langle x, y\rangle - \mathcal{H}_1, \langle y, y\rangle - \mathcal{H}_2\big), \big(-\langle x, y\rangle, \tfrac{1}{2}(\langle x, x\rangle - 1)\big)\Big\rangle \\
&= \tfrac{1}{2}\big(\langle x, x\rangle\langle y, y\rangle - \langle x, y\rangle^2\big),
\end{aligned}$$

where we have chosen

$$\mathcal{H}_1 = \langle x, y\rangle(1 - \tfrac{1}{2}\langle x, x\rangle) \quad \text{and} \quad \mathcal{H}_2 = -\langle y, y\rangle(\langle x, x\rangle - 1).$$

From Hamilton's equations on $(T\mathbf{R}^4, \omega_4)$ it follows that the integral curves of $X_{\mathcal{H}^*}$ satisfy

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = A(x, y)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\langle x, y\rangle & \langle x, x\rangle \\ -\langle y, y\rangle & \langle x, y\rangle \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}. \tag{6}$$

Using (6) and the definition of TS^3, it is easy to see that the integral curves of $X_{\mathcal{H}^*}|TS^3$ satisfy (5). Because $X_{\mathcal{H}|TS^3} = X_{\mathcal{H}^*}|TS^3$, the geodesic vector field on TS^3 is the Hamiltonian vector field X_H on (TS^3, Ω_4) corresponding to the Hamiltonian function

$$H = \mathcal{H}^*|TS^3 : TS^3 \to \mathbf{R} : (x, y) \to \tfrac{1}{2}\langle y, y\rangle. \tag{7}$$

Note that H is the free particle Hamiltonian on $T\mathbf{R}^4$ restricted to TS^3. Thus the integral curves of the geodesic vector field X_H on TS^3 satisfy

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= -\langle y, y\rangle x. \end{cases} \qquad \square \tag{8}$$

▷ To find the flow of the geodesic vector field X_H, we first look for integrals (= conserved quantities) of the vector field $X_{\mathcal{H}^*}$. From the construction of the Hamiltonian $\mathcal{H}^*$ on $T\mathbf{R}^4$, we know that TS^3 is an invariant manifold of $X_{\mathcal{H}^*}$. Therefore the functions

$$f_1(x, y) = \tfrac{1}{2}\langle x, x\rangle \text{ and } f_2(x, y) = \langle x, y\rangle \tag{9}$$

are integrals of $X_{\mathcal{H}^*}$. A calculation shows that $f_3(x, y) = \frac{1}{2}\langle y, y\rangle$ is also an integral of $X_{\mathcal{H}^*}$. The integrals $\{f_1, f_2, f_3\}$ span a Lie subalgebra of $(C^\infty(T\mathbf{R}^4), \{\,,\,\})$, which is isomorphic to $\mathrm{sl}(2, \mathbf{R})$ since

$$\{f_1, f_2\} = 2f_1, \ \{f_1, f_3\} = f_2, \text{ and } \{f_3, f_2\} = -2f_3.$$

Because the functions f_i are constant along the integral curves of $X_{\mathcal{H}^*}$, so is the matrix $A(x, y)$ (6). Since $A^2(x, y) = -2\,\mathcal{H}^*(x, y)\, I_2$ and $\mathcal{H}^*(x, y) \geq 0$, the flow of $X_{\mathcal{H}^*}$ is

$$\begin{aligned}\varphi_t^{\mathcal{H}^*}(x, y) &= \exp tA(x, y)\begin{pmatrix} x \\ y \end{pmatrix} \\ &= \left(\cos\left(t\sqrt{2\mathcal{H}^*}\right) I_2 + \left(\sin\left(t\sqrt{2\mathcal{H}^*}\right)/\sqrt{2\mathcal{H}^*}\right) A(x, y)\right)\begin{pmatrix} x \\ y \end{pmatrix}.\end{aligned}$$

Restricting $\varphi_t^{\mathcal{H}^*}$ to the invariant manifold TS^3 gives

$$\varphi_t^H(x, y) = \begin{pmatrix} \cos\left(t\sqrt{2H}\right) & \sin\left(t\sqrt{2H}\right)/\sqrt{2H} \\ -\sqrt{2H}\sin\left(t\sqrt{2H}\right) & \cos\left(t\sqrt{2H}\right) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \tag{10}$$

which is the flow of the geodesic vector field X_H on TS^3. □

Clearly, all of the integral curves of X_H on the level set $H^{-1}(h)$ with $h > 0$ are periodic ▷ of *period* $2\pi/\sqrt{2h}$. In fact, when $y \neq 0$, the image of the integral curve $t \to \varphi_t^H(x, y)$ under the bundle projection to S^3 is the geodesic

$$\gamma_{(x,y)} : \mathbf{R} \to S^3 : t \to x\left(\cos\left(t\sqrt{2H}\right)\right) + y\left(\sin\left(t\sqrt{2H}\right)/\sqrt{2H}\right). \tag{11}$$

(1.2) **Proof**: To see that $\gamma_{(x,y)}$ is a geodesic on S^3 it suffices to show that

1. $\gamma_{(x,y)}$ is parametrized up to an affine transformation by arc length.
2. The *acceleration* $\ddot{\gamma}_{(x,y)}$ has no tangential component.

From (8) it follows that 2) holds. 1) holds because γ is parametrized. Another argument to prove 1) goes as follows. Differentiating (11) gives

$$\langle \dot{\gamma}_{(x,y)}, \dot{\gamma}_{(x,y)} \rangle = 2H\sin^2\left(t\sqrt{2H}\right)\langle x, x\rangle + \cos^2\left(t\sqrt{2H}\right)\langle y, y\rangle = 2H,$$

which is a constant of motion. This constant is nonzero, since $y \neq 0$. □

The explicit formula (10) for the flow of the geodesic vector field gives no *qualitative* information about how the integral curves are organized into invariant manifolds. To understand the invariant manifolds, it is useful to explain the role of the obvious symmetry of the problem, namely, the group SO(4) of rigid motions of the 3-sphere. This will be done in section 2.

We now discuss the Delaunay vector field. Let

$$T^+S^3 = \left\{(x, y) \in TS^3 \middle| y \neq 0\right\}$$

be the tangent bundle of S^3 less its zero section. Clearly T^+S^3 is an open subset of TS^3. Therefore $\Omega_4 | T^+S^3$ is a symplectic form $\widetilde{\Omega}_4$ on T^+S^3. On $(T^+S^3, \widetilde{\Omega}_4)$ consider the *Delaunay Hamiltonian*

$$\widetilde{\mathcal{H}} : T^+S^3 \subseteq T\mathbf{R}^4 \to \mathbf{R} : (x, y) \to -\tfrac{1}{2}\frac{\mu^2}{\langle y, y\rangle}. \tag{12}$$

The integral curves of the *Delaunay vector field* $X_{\widetilde{\mathcal{H}}}$ satisfy

$$\begin{cases} \dfrac{dx}{dt} & = \dfrac{\mu^2}{\langle y, y\rangle^2}\, y \\ \dfrac{dy}{dt} & = -\dfrac{\mu^2}{\langle y, y\rangle}\, x. \end{cases} \tag{13}$$

(1.3) **Proof**: As with the geodesic Hamiltonian, we will treat $(\widetilde{\mathcal{H}}, T^+S^3, \widetilde{\Omega}_4)$ as a constrained Hamiltonian system. Give M, which is $T\mathbf{R}^4$ with the coordinate planes $\{0\} \times \mathbf{R}^4$ and $\mathbf{R}^4 \times \{0\}$ removed, the symplectic form $\widetilde{\omega}_4 = \omega_4|M$. Consider the unconstrained system $(\widetilde{H}, M, \widetilde{\omega}_4)$ with unconstrained Hamiltonian

$$\widetilde{H} : M \subseteq T\mathbf{R}^4 \to \mathbf{R} : (x, y) \to -\tfrac{1}{2}\,\frac{\mu^2}{\langle y, y\rangle}.$$

Now constrain this system to T^+S^3, which is a submanifold of M defined by the constraint functions

$$c_1 : M \to \mathbf{R} : (x, y) \to \tfrac{1}{2}\big(\langle x, x\rangle - 1\big), \quad c_2 : M \to \mathbf{R} : (x, y) \to \langle x, y\rangle,$$

and the condition $\langle y, y\rangle > 0$. For every $(x, y) \in T^+S^3$ the 2×2 matrix $(\{c_i, c_j\})$ is invertible with inverse

$$(C_{ij}) = \tfrac{1}{\langle x, x\rangle}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Thus T^+S^3 is a cosymplectic submanifold of $(M, \widetilde{\omega}_4)$ with symplectic form $\Omega_4 = \widetilde{\omega}_4|TS^3$. Since T^+S^3 is an open subset of TS^3, T^+S^3 is a symplectic manifold with symplectic form $\widetilde{\Omega}_4 = \Omega_4|T^+S^3$. Therefore we may use the modified Dirac procedure to compute the Delaunay vector field $X_{\widetilde{\mathcal{H}}}$. Towards this goal let

$$\begin{aligned} \widetilde{\mathcal{H}}^* &= \widetilde{\mathcal{H}} - \sum_{i,j}\big(\{\widetilde{\mathcal{H}}, c_i\} + \widetilde{\mathcal{H}}_i\big)C_{ij}c_j \\ &= -\tfrac{1}{2}\tfrac{\mu^2}{\langle y, y\rangle} - \tfrac{1}{\langle x, x\rangle}\Big\langle\Big(-\tfrac{\mu^2\langle x, y\rangle}{\langle y, y\rangle^2} + \widetilde{\mathcal{H}}_1, \tfrac{\mu^2}{\langle y, y\rangle} + \widetilde{\mathcal{H}}_2\Big), \Big(-\langle x, y\rangle, \tfrac{1}{2}(\langle x, x\rangle - 1)\Big)\Big\rangle \\ &= -\tfrac{1}{2}\,\frac{\mu^2}{\langle y, y\rangle} - \tfrac{1}{2}\langle x, y\rangle^2 + \tfrac{1}{2}\,\frac{\mu^2}{\langle y, y\rangle}\big(\langle x, x\rangle - 1\big), \end{aligned}$$

where we have chosen the functions

$$\widetilde{\mathcal{H}}_1(x, y) = \Big(\frac{\mu^2}{\langle y, y\rangle^2} - \tfrac{1}{2}\langle x, x\rangle\Big)\langle x, y\rangle \quad \text{and} \quad \widetilde{\mathcal{H}}_2(x, y) = \frac{\mu^2}{\langle y, y\rangle}\big(\langle x, x\rangle - 1\big)$$

to lie in the ideal generated by c_1 and c_2. On $(M, \widetilde{\omega}_4)$ the Hamiltonian vector field $X_{\widetilde{\mathcal{H}}^*}$ has integral curves which satisfy

$$\begin{cases} \dfrac{dx}{dt} & = \dfrac{\partial \widetilde{\mathcal{H}}^*}{\partial y} = \dfrac{\mu^2}{\langle y, y\rangle^2}\, y - \langle x, y\rangle\, x + \dfrac{\mu^2}{\langle y, y\rangle^2}\big(\langle x, x\rangle - 1\big) y \\ \dfrac{dy}{dt} & = -\dfrac{\partial \widetilde{\mathcal{H}}^*}{\partial x} = -\dfrac{\mu^2}{\langle y, y\rangle}\, x + \langle x, y\rangle\, y. \end{cases}$$

Since $X_{\widetilde{\mathcal{H}}} = X_{\widetilde{H}^*|T^+S^3} = X_{\widetilde{H}^*}|T^+S^3$, the integral curves of $X_{\widetilde{\mathcal{H}}}$ satisfy (13). $\square$

Before we show that the Delaunay vector field $X_{\widetilde{\mathcal{H}}}$ is a time rescaling of the geodesic vector field X_H, we discuss the general concept of rescaling. Let X be a smooth vector field on a smooth manifold M with flow φ_t and let f be a smooth real valued function on M. Consider the vector field $f \cdot X$, defined by $(f \cdot X)(m) = f(m)X(m)$ for $m \in M$. Define a new time scale s by

$$\frac{dt}{ds} = f(\varphi_t(m)), \tag{14}$$

that is,

$$s(t) = \int_0^t \frac{dt}{f(\varphi_t(m))}.$$

We assume that the right hand side of the above equation exists and that the function s is invertible. In the new time $t = t(s)$ the integral curve $t \to \gamma(t)$ of X with $\gamma(0) = m$ satisfies

$$\frac{d\gamma(t(s))}{ds} = \frac{d\varphi_{t(s)}(m)}{ds} = \frac{d\varphi_{t(s)}(m)}{dt}\,\frac{dt}{ds} = f(\varphi_{t(s)}(m))\, X(\varphi_{t(s)}(m)),$$

using (14). Therefore $s \to \gamma(t(s))$ is an integral curve of $f \cdot X$ with $\gamma(t(0)) = \gamma(0) = m$, since $s(0) = 0$. We say that $f \cdot X$ is a *time rescaling* of X with *time rescaling function* f.

Consider the special case where the time rescaling f is a smooth integral of X. In this case we can integrate (14) to obtain $s = 1/f(m)\, t$. Hence the flow

$$\psi : \mathbf{R} \times M \to M : (s, m) \to \psi_s(m)$$

of the rescaled vector field $f \cdot X$ is $\varphi_{sf(m)}$. Specializing further, suppose that X_G is a Hamiltonian vector field on a symplectic manifold (M, ω) with flow φ_t^G. Let ϑ be the time one map of the flow of the rescaled vector field $f \cdot X_G$, where $f : M \to \mathbf{R}$ is a smooth integral of X_G. Then

$$\vartheta^*\omega = \omega + df \wedge dG. \tag{15}$$

(1.4) **Proof**: Let

$$\varphi^G : \mathbf{R} \times M \to M : (s, m) \to \varphi^G(s, m) = \varphi_s^G(m) = \varphi_m^G(s).$$

By definition $\vartheta(m) = \varphi^G(f(m), m)$. We compute the tangent to ϑ as follows. For $v_m, w_m \in T_m M$

$$\begin{aligned} T_m\vartheta\, v_m &= \left(\frac{d}{ds}\bigg|_{s=0} \varphi_m^G\right)(f(m))v_m + T_m\varphi_{f(m)}^G v_m \\ &= (df(m)v_m)\, X_G(\varphi_{f(m)}^G(m)) + T_m\varphi_{f(m)}^G v_m. \end{aligned}$$

Consequently,

$$\begin{aligned} \vartheta^*\omega(m)(v_m, w_m) = {}& (df(m)v_m)\, \omega(\varphi_{f(m)}^G(m))\Big(X_G(\varphi_{f(m)}^G(m)), T_m\varphi_{f(m)}^G w_m \Big) \\ & - (df(m)w_m)\, \omega(\varphi_{f(m)}^G(m))\Big(X_G(\varphi_{f(m)}^G(m)), T_m\varphi_{f(m)}^G v_m \Big) \\ & + \omega(\varphi_{f(m)}^G(m))\Big(T_m\varphi_{f(m)}^G v_m, T_m\varphi_{f(m)}^G w_m \Big), \end{aligned}$$

using the definition of pull back and

$$\omega(\varphi^G_{f(m)}(m)\big(X_G(\varphi^G_{f(m)}(m)),\, X_G(\varphi^G_{f(m)}(m))\big) = 0.$$

Therefore

$$\begin{aligned}
\omega(\vartheta(m))\big(T_m\vartheta\, v_m, T_m\vartheta\, w_m\big) &= \\
&= (df(m)v_m)\, dG(\varphi^G_{f(m)}(m))T_m\varphi^G_{f(m)} w_m - (df(m)w_m)\, dG(\varphi^G_{f(m)}(m))T_m\varphi^G_{f(m)} v_m \\
&\qquad + \omega(m)(v_m, w_m), \\
&\qquad\qquad \text{since } X_G \,\lrcorner\, \omega = dG \text{ and } \varphi^G_{f(m)} \text{ is symplectic} \\
&= (df(m)v_m)\, dG(m)w_m - (df(m)w_m)\, dG(m)v_m + \omega(m)(v_m, w_m), \\
&\qquad\qquad \text{since } ((\varphi^G_{f(m)})^*G = G \\
&= (df \wedge dG)(m)(v_m, w_m) + \omega(m)(v_m, w_m). \quad \square
\end{aligned}$$

From (15) we see that ϑ is symplectic if and only if $df \wedge dG = 0$, that is, f is *locally* a smooth function of G. Thus the rescaled vector field $f \cdot X_G$ is a Hamiltonian vector field $X_{\widehat{G}}$ for some smooth function $\widehat{G}$ on M if there is a smooth function $\mathscr{G} : \mathbf{R} \to \mathbf{R}$ such that $\widehat{G} = \mathscr{G} \circ G$. When this is the case, the time rescaling function f is $\mathscr{G}' \circ G$.

We now apply these observations about time rescaling to the Delaunay vector field $X_{\widetilde{\mathscr{H}}}$. The Delaunay Hamiltonian

$$\widetilde{\mathscr{H}} : T^+S^3 \subseteq T\mathbf{R}^4 \to \mathbf{R} : (x, y) \to -\tfrac{1}{2}\,\frac{\mu^2}{\langle y, y\rangle}$$

is a smooth function of the geodesic Hamiltonian

$$H : T^+S^3 \subseteq T\mathbf{R}^4 \to \mathbf{R} : (x, y) \to \tfrac{1}{2}\,\langle y, y\rangle.$$

More precisely, $\widetilde{\mathscr{H}} = \mathscr{G} \circ H$, where $\mathscr{G} : \mathbf{R}_> \to \mathbf{R} : z \to -\frac{\mu^2}{4z}$. Thus $X_{\widetilde{\mathscr{H}}} = f \cdot X_H$ where

$$f = \mathscr{G}' \circ H : T^+S^3 \to \mathbf{R} : (x, y) \to \frac{\mu^2}{\langle y, y\rangle^2}.$$

Because the flow of the geodesic vector field X_H is

$$\varphi^H_t(x, y) = \begin{pmatrix} \cos \nu t & \frac{1}{\nu}\sin \nu t \\ -\nu \sin \nu t & \cos \nu t \end{pmatrix},$$

where $\nu^2 = \langle y, y\rangle$, it follows that the flow of the Delaunay vector field is

$$\varphi^{\widetilde{\mathscr{H}}}_t(x, y) = \begin{pmatrix} \cos \frac{\mu^2}{\nu^3} t & \frac{1}{\nu}\sin \frac{\mu^2}{\nu^3} t \\ -\nu \sin \frac{\mu^2}{\nu^3} t & \cos \frac{\mu^2}{\nu^3} t \end{pmatrix}. \tag{16}$$

Note that on the energy surface $\widetilde{\mathscr{H}}^{-1}\left(-\frac{1}{2}\,\mu^2/\nu^3\right)$ all the integral curves of the Delaunay vector field $X_{\widetilde{\mathscr{H}}}$ are periodic of period $2\pi\nu^3/\mu^2$.

2 The SO(4) momentum mapping

In this section we construct the momentum mapping associated to the SO(4) symmetry of the geodesic vector field on (TS^3, Ω_4) and study its geometric properties.

Recall that SO(4) is the Lie group of *orthogonal* linear mappings of $(\mathbf{R}^4, \langle\ ,\ \rangle)$ into itself with determinant 1. Consider the linear action of SO(4) on $\mathbf{R}^4$ defined by

$$\varphi : \mathrm{SO}(4) \times \mathbf{R}^4 \to \mathbf{R}^4 : (A, x) \to Ax.$$

This action lifts to an action of SO(4) on $(T\mathbf{R}^4, \omega_4)$ defined by

$$\Phi : \mathrm{SO}(4) \times T\mathbf{R}^4 \to T\mathbf{R}^4 : \big(A, (x, y)\big) \to (Ax, Ay).$$

▷ Φ preserves the 1-form $\theta = \langle y, dx\rangle$ on $T\mathbf{R}^4$.

(2.1) **Proof**: We compute

$$\Phi_A^*\theta = \langle Ay, dAx\rangle = \langle Ay, Adx\rangle = \langle A^t Ay, dx\rangle = \langle y, dx\rangle = \theta.$$

The second to last equality follows because $A \in \mathrm{SO}(4)$. □

Thus the action Φ is symplectic, for

$$\Phi_A^*\omega_4 = -\Phi_A^*(d\theta) = -d(\Phi_A^*\theta) = -d\theta = \omega_4.$$

▷ To show that Φ is a Hamiltonian action, we must verify that for every $a \in \mathrm{so}(4)$ (= the Lie algebra of SO(4)) the vector field

$$\begin{aligned} X^a(x, y) &= \frac{d}{dt}\Big|_{t=0} \Phi_{\exp ta}(x, y) = \frac{d}{dt}\Big|_{t=0} \big((\exp ta)x, (\exp ta)y\big) \\ &= (ax, ay) = (X_a(x), ay), \end{aligned}$$

which is the infinitesimal generator of Φ in the direction a, is a Hamiltonian vector field on $(T\mathbf{R}^4, \omega_4)$.

(2.2) **Proof**: From the momentum lemma (see appendix B ((3.6))) it follows that $X^a = X_{J^a}$ where

$$J^a : T\mathbf{R}^4 \to \mathbf{R} : (x, y) \to \theta(x, y)X_a(x) = \langle ax, y\rangle. \quad \square \tag{17}$$

Thus the action Φ has momentum mapping $J : T\mathbf{R}^4 \to \mathrm{so}(4)^*$ defined by

$$J(x, y)a = J^a(x, y) = \langle ax, y\rangle. \tag{18}$$

Choose a basis $\{e_{ij}\}_{1 \le i < j \le 4}$ of so(4) where the $(k, \ell)^{th}$ entry of the 4×4 matrix e_{ij} is

$$\begin{cases} -1, & \text{if } (k, \ell) = (i, j) \\ 1, & \text{if } (k, \ell) = (j, i) \\ 0, & \text{otherwise.} \end{cases} \tag{19}$$

Then

$$J^{e_{ji}}(x, y) = \langle x, e_{ij}y\rangle = x_i y_j - x_j y_i = S_{ij}(x, y). \tag{20}$$

▷ The mapping J is coadjoint equivariant.

(2.3) **Proof**: We compute

$$\begin{aligned} J(\Phi_A(x,y))a &= J(Ax,Ay)a = \langle aAx, Ay\rangle = \langle A^{-1}aAx, y\rangle \\ &= J(x,y)(Ad_{A^{-1}}a) = Ad^t_{A^{-1}}(J(x,y))a. \quad \square \end{aligned}$$

Since Φ_A maps TS^3 into itself for every $A \in \mathrm{SO}(4)$, Φ restricts to an action $\widehat{\Phi}$ on TS^3 given by

$$\widehat{\Phi} : \mathrm{SO}(4) \times TS^3 \to TS^3 : \big(A, (x,y)\big) \to (Ax, Ay). \tag{21}$$

For every $a \in \mathrm{so}(4)$ the infinitesimal generator X^a of the SO(4) action Φ leaves TS^3 invariant because

$$\begin{aligned} \frac{d}{dt}\langle x,x\rangle &= 2\langle x,\dot{x}\rangle = 2\langle x, ax\rangle = 0, \\ \frac{d}{dt}\langle x,y\rangle &= \langle \dot{x},y\rangle + \langle x,\dot{y}\rangle = \langle x, ay\rangle + \langle ax, y\rangle = 0, \\ \frac{d}{dt}\langle y,y\rangle &= 2\langle y,\dot{y}\rangle = 2\langle y, ay\rangle = 0, \end{aligned}$$

since $a^t = -a$. Therefore $X^a|TS^3$ is a vector field on TS^3. $\widehat{\Phi}$ preserves the symplectic form Ω_4 on TS^3 because

$$\widehat{\Phi}^*_A\Omega_4 = \widehat{\Phi}^*_A(\omega_4|TS^3) = (\Phi^*_A\omega_4)|TS^3 = \omega_4|TS^3 = \Omega_4.$$

Claim: The action $\widehat{\Phi}$ on (TS^3, Ω_4) is Hamiltonian with momentum mapping

$$\widetilde{\mathcal{J}} = J|TS^3 : TS^3 \subseteq T\mathbf{R}^4 \to \mathrm{so}(4)^*. \tag{22}$$

(2.4) **Proof**: Because X^a leaves TS^3 invariant and $\Omega_4 = \omega_4|TS^3$, it follows that $X^a|TS^3 = X_{J^a}|TS^3$. Thus $X^a|TS^3$ is the infinitesimal generator of $\widehat{\Phi}$ on TS^3 in the direction a. $\square$

▷ So far the SO(4) symmetry is not related to the geodesic flow on TS^3. But note, the Hamiltonian $\mathcal{H}^*$ is preserved by the action Φ, that is, for every $A \in \mathrm{SO}(4)$

$$\mathcal{H}^*(\Phi_A(x,y)) = \tfrac{1}{2}\big(\langle Ax,Ax\rangle\langle Ay,Ay\rangle - \langle Ax,Ay\rangle^2\big) = \mathcal{H}^*(x,y).$$

▷ Thus the function J^a is an integral of the vector field $X_{\mathcal{H}^*}$ for every $a \in \mathrm{so}(4)$.

(2.5) **Proof**: For every $a \in \mathrm{so}(4)$ we have $\Phi^*_{\exp ta}\mathcal{H}^* = \mathcal{H}^*$. Therefore

$$0 = \frac{d}{dt}\Big|_{t=0} \Phi^*_{\exp ta}\mathcal{H}^* = L_{X^a}\mathcal{H}^* = L_{X_{J^a}}\mathcal{H}^* = -L_{X_{\mathcal{H}^*}}J^a. \quad \square$$

From the fact that Φ preserves both the Hamiltonian $\mathcal{H}^*$ and the manifold TS^3, it follows that $\widehat{\Phi}$ preserves the geodesic Hamiltonian $H = \mathcal{H}^*|TS^3$. Therefore for every $a \in \mathrm{so}(4)$ the function $J^a|TS^3$ is an integral of the geodesic vector field X_H. $\square$

In order to study the geometry of the momentum mapping $\widetilde{\mathcal{J}}$ (22), we transform it into an easier to understand mapping, (see (27) below). We begin by recalling that the 4×4 skew symmetric matrices $\{e_{ij}\}_{1\le i<j\le 4}$ (19) form a basis for the Lie algebra $(\mathrm{so}(4), [\,,\,])$. The vectors $\{e^*_{ij}\}_{1\le i<j\le 4}$, where $e^*_{ij} = e^t_{ij}$, form the standard dual basis for $\mathrm{so}(4)^*$. The Lie bracket $\{\,,\,\}_{\mathrm{so}(4)^*}$ on $\mathrm{so}(4)^*$ is defined by

$$\{e^*_{ij}, e^*_{\ell k}\}_{\mathrm{so}(4)^*} = -\sum_{m,n} c^{mn}_{ij,\ell k}\, e^*_{mn},$$

where

$$[e_{ij}, e_{\ell k}] = \sum_{m,n} c^{mn}_{ij,\ell k}\, e_{mn}.$$

Let $\vartheta : \bigwedge^2 \mathbf{R}^4 \to \mathrm{so}(4)$ be the linear map defined by

$$\vartheta(u \wedge v)w = \langle w, v\rangle\, u - \langle w, u\rangle\, v, \tag{23}$$

for $u, v, w \in \mathbf{R}^4$. Using the basis $\{e_i \wedge e_j\}_{1\le i<j\le 4}$ of $\bigwedge^2\mathbf{R}^4$, we see that $\vartheta(e_i \wedge e_j) = e_{ij}$. Thus ϑ is an isomorphism. Consequently the mapping

$$\vartheta^t : \mathrm{so}(4)^* \to (\textstyle\bigwedge^2\mathbf{R}^4)^* = \bigwedge^2(\mathbf{R}^4)^*$$

sends e^*_{ij} to $e^*_i \wedge e^*_j$. Since

$$\begin{aligned} \vartheta^t(e^*_{ij})(x, y) &= (e^*_i \wedge e^*_j)(x, y) = e^*_i(x)\, e^*_j(y) - e^*_i(y) e^*_j(x) = x_i y_j - x_j y_i \\ &= S_{ij}(x, y) = \langle x, e_{ij}(y)\rangle, \quad \text{for every } x, y \in \mathbf{R}^4, \end{aligned} \tag{24}$$

$(\bigwedge^2\mathbf{R}^4)^*$ is the space $\mathcal{S}$ of homogeneous quadratic functions on $T\mathbf{R}^4$ which is spanned by $\{S_{ij}\}_{1\le i<j\le 4}$. As a subspace of $C^\infty(T\mathbf{R}^4)$, $\mathcal{S}$ has a Poisson bracket $\{\,,\,\}_{\mathcal{S}}$ which is induced from the standard Poisson bracket $\{\,,\,\}$ on the space of smooth functions on $(T\mathbf{R}^4, \omega_4)$. In other words, for every $(x, y) \in T\mathbf{R}^4$

$$\{S_{ij}, S_{\ell k}\}_{\mathcal{S}}(x, y) = \omega_4\big(X_{S_{ij}}(x, y), X_{S_{\ell k}}(x, y)\big), \tag{25}$$

where $X_{S_{rs}}$ is the Hamiltonian vector field on $(T\mathbf{R}^4, \omega_4)$ corresponding to the Hamiltonian function S_{rs}. A calculation using (25) gives table 2.1.

$\{A, B\}$	S_{12}	S_{13}	S_{14}	S_{23}	S_{24}	S_{34}	B
S_{12}	0	S_{23}	S_{24}	$-S_{13}$	$-S_{14}$	0	
S_{13}	$-S_{23}$	0	S_{34}	S_{12}	0	$-S_{14}$	
S_{14}	$-S_{24}$	$-S_{34}$	0	0	S_{12}	S_{13}	
S_{23}	S_{13}	$-S_{12}$	0	0	S_{34}	$-S_{24}$	
S_{24}	S_{14}	0	$-S_{12}$	$-S_{34}$	0	S_{23}	
S_{34}	0	S_{14}	$-S_{13}$	S_{24}	$-S_{23}$	0	
A							

Table 2.1 The Poisson bracket on $\mathcal{S}$.

Because the functions $f_1 = \frac{1}{2}\langle x, x\rangle$, $f_2 = \langle x, y\rangle$, and $f_3 = \frac{1}{2}\langle y, y\rangle$ are invariant under the SO(4) action Φ on $T\mathbf{R}^4$, the function J^a (17) is an integral of X_{f_i} for every $a \in \mathrm{so}(4)$.

In other words, $\{f_i, J^a\} = 0$ for $i = 1, 2, 3$ and $a \in \mathrm{so}(4)$. Thus the Lie algebra $(\mathrm{sl}(2, \mathbf{R}), \{\, , \,\})$ spanned by $\{f_i\}_{1 \le i \le 3}$ and the Lie algebra $(\mathcal{S}, \{\, , \,\}_{\mathcal{S}})$ are *dual pairs* in the Lie algebra of homogeneous quadratic functions on $T\mathbf{R}^4$ with Poisson bracket $\{\, , \,\}$. In other words, they have the following properties:

1. They centralize $\mathcal{H}^*$, that is, $\{\mathcal{H}^*, f_i\} = 0 = \{\mathcal{H}^*, S_{jk}\}$.
2. They centralize each other, that is, $\{f_i, S_{jk}\} = 0$.

$\triangleright$ We now show that the Lie algebras $(\mathcal{S}, \{\, , \,\}_{\mathcal{S}})$ and $(\mathrm{so}(4)^*, \{\, , \,\}_{\mathrm{so}(4)^*})$ are isomorphic.

(2.6) **Proof**: From the definition of S_{ij} (24) we obtain

$$dS_{ij}(x, y) = \langle e_{ij}(y), dx\rangle - \langle e_{ij}(x), dy\rangle.$$

Since $\omega_4^\flat(dx) = -\frac{\partial}{\partial y}$ and $\omega_4^\flat(dy) = \frac{\partial}{\partial x}$, we find that

$$X_{S_{ij}}(x, y) = \omega_4^\flat(dS_{ij})(x, y) = -\langle e_{ij}(x), \frac{\partial}{\partial x}\rangle - \langle e_{ij}(y), \frac{\partial}{\partial y}\rangle.$$

Therefore

$$\begin{aligned} \{S_{ij}, S_{\ell k}\}_{\mathcal{S}}(x, y) &= (X_{S_{\ell k}} \lrcorner\, dS_{ij})(x, y) = -\langle e_{\ell k}(x), e_{ij}(y)\rangle + \langle e_{\ell k}(y), e_{ij}(x)\rangle \\ &= -\langle x, (e_{ij}e_{\ell k} - e_{\ell k}e_{ij})y\rangle = -\langle x, [e_{ij}, e_{\ell k}]y\rangle \\ &= -\vartheta^t([e_{ij}, e_{\ell k}]^*)(x, y) = \vartheta^t(\{e_{ij}^*, e_{\ell k}^*\}_{\mathrm{so}(4)^*})(x, y). \end{aligned} \tag{26}$$

The last equality follows by definition of the Poisson bracket $\{\, , \,\}_{\mathrm{so}(4)^*}$. Hence ϑ^t is a Lie algebra isomorphism. □

On $\bigwedge^2 \mathbf{R}^4$ define an inner product

$$K : \bigwedge^2 \mathbf{R}^4 \times \bigwedge^2 \mathbf{R}^4 \to \mathbf{R} : (u \wedge v, x \wedge y) \to \det\begin{pmatrix} \langle u, x\rangle & \langle u, y\rangle \\ \langle v, x\rangle & \langle v, y\rangle \end{pmatrix}.$$

Since $\{e_i \wedge e_j\}_{1 \le i < j \le 4}$ is an orthonormal basis of $(\bigwedge^2 \mathbf{R}^4, K)$, we may identify $\bigwedge^2 \mathbf{R}^4$ with $(\bigwedge^2 \mathbf{R}^4)^*$.

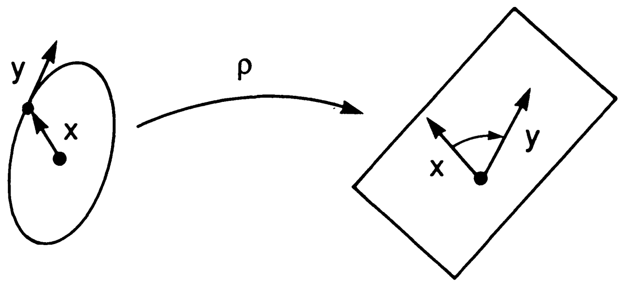

Figure 2.1. The mapping ρ.

Instead of studying the momentum mapping $\widetilde{\mathcal{J}}$ (22) we study the mapping

$$\rho : T^+S^3 \subseteq T\mathbf{R}^4 \to \bigwedge^2 \mathbf{R}^4 : (x, y) \to x \wedge y = \sum_{1 \le i < j \le 4} S_{ij}(x, y)\, e_i \wedge e_j, \tag{27}$$

which is nothing but $K^{\flat} \circ \vartheta^t \circ \mathcal{J}$. The S_{ij} are the *Plücker coordinates* of the oriented 2-plane spanned by $\{x, y\}$ corresponding to the 2-vector $x \wedge y$. In other words, S_{ij} is the 2×2 minor formed from the i^{th} and j^{th} columns of the 2×4 matrix with rows x and y, that is, $S_{ij} = \det\begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix}$. Because

$$0 = (x \wedge y) \wedge (x \wedge y) = (S_{12}S_{34} - S_{13}S_{24} + S_{14}S_{23})\, e_1 \wedge e_2 \wedge e_3 \wedge e_4,$$

the Plücker coordinates of $x \wedge y$ satisfy *Plücker's equation*

$$S_{12}S_{34} - S_{13}S_{24} + S_{14}S_{23} = 0. \tag{28}$$

Let C be the set of all nonzero 2-vectors on $\mathbf{R}^4$ whose Plücker coordinates satisfy (28). By
▷ definition $\rho(T^+S^3) \subseteq C$. Actually, C is the image of ρ.

(2.7) **Proof**: Suppose that $\theta \in C$. Then θ is decomposable, that is, there are vectors $u, v \in \mathbf{R}^4$ such that $\theta = u \wedge v$. To see this, let (S_{ij}) be the Plücker coordinates of θ. Since $\theta \neq 0$ not every S_{ij} is zero. Suppose that S_{12} is nonzero. Let $u = (1, 0, -S_{23}/S_{12}, -S_{24}/S_{12})$ and $v = (0, S_{12}, S_{13}, S_{14})$. Using Plücker's equation (28) it is easy to check that the Plücker coordinates of the 2-vector $u \wedge v$ are (S_{ij}). Therefore $\theta = u \wedge v$. A similar argument, which we omit, works in the other cases. Let $\{x, y\}$ be an orthonormal basis of the 2-plane spanned by $\{u, v\}$. Then $u \wedge v = \lambda\, x \wedge y$ for some nonzero λ. Therefore $\rho(x, \lambda y) = \theta$. □

For $h > 0$ let

$$H^{-1}(h) = \left\{ (x, y) \in T^+S^3 \subseteq T\mathbf{R}^4 \,\middle|\, \tfrac{1}{2}\langle y, y\rangle = h \right\}$$

be the h-level set of the geodesic Hamiltonian H (7). Consider the mapping

$$\rho_h : H^{-1}(h) \subseteq T^+S^3 \to C \subseteq \bigwedge^2 \mathbf{R}^4 : (x, y) \to x \wedge y, \tag{29}$$

which is the restriction of ρ (27) to $H^{-1}(h)$. From the identity

$$\sum_{1 \le i < j \le 4} (x_i y_j - x_j y_i)^2 = \langle x, x\rangle\langle y, y\rangle - \langle x, y\rangle^2 \tag{30}$$

we see that the image of ρ_h is contained in the submanifold C_h of C defined by

$$\sum_{1 \le i < j \le 4} S_{ij}^2 = 2h. \tag{31}$$

▷ C_h is diffeomorphic to $S^2_{\sqrt{h/2}} \times S^2_{\sqrt{h/2}}$.

(2.8) **Proof**: Adding and subtracting one half times (28) from one quarter times (31) and using the variables

$$\begin{aligned} \xi_1 &= \tfrac{1}{2}(S_{12} + S_{34}) & \eta_1 &= \tfrac{1}{2}(S_{12} - S_{34}) \\ \xi_2 &= \tfrac{1}{2}(S_{13} - S_{24}) & \eta_2 &= \tfrac{1}{2}(S_{13} + S_{24}) \\ \xi_3 &= \tfrac{1}{2}(S_{14} + S_{23}) & \eta_3 &= \tfrac{1}{2}(S_{14} - S_{23}), \end{aligned} \tag{32}$$

we obtain

$$\xi_1^2 + \xi_2^2 + \xi_3^2 = h/2 \quad \text{and} \quad \eta_1^2 + \eta_2^2 + \eta_3^2 = h/2. \qquad \square \tag{33}$$

We now investigate the geometry of the map ρ_h.

Claim: For every $h > 0$, the map $\rho_h : H^{-1}(h) \to C_h$ (29) is a surjective submersion each of whose fibers is a single oriented orbit of the geodesic vector field X_H of energy h.

(2.9) **Proof**: To show that ρ_h is surjective, suppose that $S = (S_{ij}) \in C_h$. Since C_h is contained in $C = \rho(T^+S^3)$, there is an $(x, y) \in T^+S^3$ such that $\rho(x, y) = S$. But $2h = \sum_{1\le i<j\le 4} S_{ij}^2$ since $S \in C_h$. From (30), the definition of S_{ij} (24), and the fact that $(x, y) \in T^+S^3$, we find that $\frac{1}{2}\langle y, y\rangle = h$. Hence $(x, y) \in H^{-1}(h)$.

To show that ρ_h is a submersion, we must verify that the rank of $T_{(x,y)}\rho_h$ is 4 for every $(x, y) \in H^{-1}(h)$, because C_h is 4-dimensional. Towards this goal, let

$$V_{(x,y)} = \operatorname{span}\{X_{S_{ij}}(x, y)\}_{1\le i<j\le 4},$$

where $X_{S_{ij}}$ is the Hamiltonian vector field on $(T\mathbf{R}^4, \omega_4)$ corresponding to the Hamiltonian function

$$S_{ij} : T\mathbf{R}^4 \to \mathbf{R} : (x, y) \to x_i y_j - x_j y_i.$$

Since $S_{ij}|T^+S^3$ is an integral of of the geodesic vector field X_H on T^+S^3, we see that

$$V_{(x,y)} \subseteq \ker dH(x, y) = T_{(x,y)}H^{-1}(h)$$

for every $(x, y) \in T^+S^3$. Now

$$\left(T_{(x,y)}\rho_h\right)|V_{(x,y)} = \left(dS_{ij}(x, y)X_{S_{\ell k}}(x, y)\right) = (\{S_{ij}, S_{\ell k}\}_{\mathcal{S}}) = \widetilde{P}. \tag{34}$$

The 6×6 matrix $\widetilde{P}$ is conjugate to the matrix

$$P = \begin{pmatrix} (\{\xi_i, \xi_j\}_{\mathcal{S}}) & 0 \\ 0 & (\{\eta_i, \eta_j\}_{\mathcal{S}}) \end{pmatrix} = \begin{pmatrix} (\sum_k \varepsilon_{ijk}\xi_k) & 0 \\ 0 & (\sum_k \varepsilon_{ijk}\eta_k) \end{pmatrix},$$

using (33) and table 2.1. Here ε_{ijk} is 0 if $\{i, j, k\}$ is a proper subset of $\{1, 2, 3\}$, is $+1$ if $\{i, j, k\}$ is an even permutation of $1, 2, 3$, and is -1 otherwise. But $S = (S_{ij}) = \rho_h(x, y) \in C_h$. By (33) each of the 3×3 skew symmetric matrices $(\sum_k \varepsilon_{ijk}\xi_k)$ and $(\sum_k \varepsilon_{ijk}\eta_k)$ is nonzero and hence each has rank 2. Therefore, the rank of $T_{(x,y)}\rho_h$ is 4 for every $(x, y) \in H^{-1}(h)$. Thus ρ_h is a submersion.

Given $S = (S_{ij}) \in C_h$, the fiber $W = \rho_h^{-1}(S)$ is a union of orbits of the geodesic vector field X_H of energy h because $S_{ij}|T^+S^3$ are integrals of X_H. By definition of ρ_h (29), W is the set of all ordered pairs $\{x, y\}$ of orthogonal vectors in $\mathbf{R}^4$ such that $\langle x, x\rangle = 1$, $\langle y, y\rangle = 2h$ and the 2-plane Π spanned by $\{x, y\}$ has Plücker coordinates (S_{ij}). Since any two such bases of Π are related by a *counterclockwise* rotation in Π, we find that

$$W = \left\{(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta) \in H^{-1}(h)\middle|\theta \in [0, 2\pi]\right\}.$$

Therefore W is a unique oriented orbit of X_H traced out by an integral curve of X_H. $\square$

Corollary: C_h is the space of orbits of positive energy h of the geodesic vector field X_H on T^+S^3 with orbit mapping $\rho_h : H^{-1}(h) \rightarrow C_h$.

(2.10) **Proof**: The corollary follows immediately from the definition of orbit space (see appendix B section 2) and the claim. □

The goal of the following discussion is to construct a symplectic form on C_h. We begin by defining a Poisson bracket $\{\, ,\, \}$ on the space $C^\infty(\mathcal{S})$ of smooth functions on the Lie algebra $(\mathcal{S}, \{\, ,\, \}_{\mathcal{S}})$. For $f, g \in C^\infty(\mathcal{S})$ let

$$\{f, g\} = \sum_{\substack{1 \le i < j \le 4 \\ 1 \le \ell < k \le 4}} \frac{\partial f}{\partial S_{ij}} \frac{\partial g}{\partial S_{\ell k}} \{S_{ij}, S_{\ell k}\}_{\mathcal{S}}. \tag{35}$$

As is shown in example 1 of appendix A section 4, $(C^\infty(\mathcal{S}), \{\, ,\, \}_{\mathcal{S}})$ is a Lie algebra. On $C^\infty(\mathcal{S})$ define a multiplication $\cdot$ by $(f \cdot g)(S) = f(S)g(S)$ for every $S \in \mathcal{S}$. Then $(C^\infty(\mathcal{S}), \cdot)$ is a commutative ring with unit. Using (35) it is straightforward to check that Leibniz' rule holds, namely

$$\{f, g \cdot h\} = \{f, g\} \cdot h + \{f, h\} \cdot g,$$

for every $f, g, h \in C^\infty(\mathcal{S})$. Therefore $\mathcal{A} = (C^\infty(\mathcal{S}), \{\, ,\, \}_{\mathcal{S}}, \cdot)$ is a Poisson algebra. Consider the functions

$$C_1 = \sum_{1 \le i < j \le 4} S_{ij}^2 - 2h \quad \text{and} \quad C_2 = S_{12}S_{34} - S_{13}S_{24} + S_{14}S_{23}.$$

They are Casimirs for $\mathcal{A}$. In other words, $\{C_1, f\} = \{C_2, f\} = 0$ for every $f \in C^\infty(\mathcal{S})$. (From (35) it is enough to show that $\{C_1, S_{ij}\} = \{C_2, S_{ij}\} = 0$ for $1 \le i < j \le 4$. This is a direct verification using table 2.1.) Let $\mathcal{J}$ be the ideal in $(C^\infty(\mathcal{S}), \cdot)$ which is generated
▷ by C_1 and C_2. Then $\mathcal{J}$ is a Poisson ideal in $\mathcal{A}$, that is, if $f \in \mathcal{J}$, then $\{f, g\} \in \mathcal{J}$ for every $g \in C^\infty(\mathcal{S})$.

(2.11) **Proof**: Since $f \in \mathcal{J}$ there are $f_1, f_2 \in C^\infty(\mathcal{S})$ such that $f = f_1C_1 + f_2C_2$. Now

$$\begin{aligned} \{f, g\} &= \{f_1, g\} \cdot C_1 + f_1 \cdot \{C_1, g\} + \{f_2, g\} \cdot C_2 + f_2 \cdot \{C_2, g\}, \text{ by Leibniz' rule} \\ &= \{f_1, g\} \cdot C_1 + \{f_2, g\} \cdot C_2, \quad \text{because } C_1 \text{ and } C_2 \text{ are Casimirs} \\ &\in \mathcal{J}. \quad \square \end{aligned}$$

Therefore we can define a Poisson bracket $\{\, ,\, \}_{C_h}$ on $C^\infty(\mathcal{S}/\mathcal{J})$ by

$$\{f + \mathcal{J}, g + \mathcal{J}\}_{C_h} = \{f, g\}. \tag{36}$$

In order that we can identify the space $C^\infty(\mathcal{S}/\mathcal{J})$ with the space $C^\infty(C_h)$ of smooth functions on C_h, we need to know that $\mathcal{J}$ is the set of smooth functions vanishing identically on C_h. This is a consequence of the following general

Fact: Suppose that 0 is a regular value of the smooth map

$$F : \mathbf{R}^n \rightarrow \mathbf{R}^k : z \rightarrow \big(F_1(z), \dots, F_k(z)\big).$$

Then $M = F^{-1}(0)$ is a smooth submanifold of $\mathbf{R}^n$ defined by

$$F_1(z) = \cdots = F_k(z) = 0.$$

If $G : \mathbf{R}^n \to \mathbf{R}$ is a smooth function, which vanishes identically on M, then there are smooth functions $g_i : \mathbf{R}^n \to \mathbf{R}$, $1 \le i \le k$, such that $G = \sum_{i=1}^k g_i F_i$.

(2.12) **Proof**: Since M is a smooth submanifold of $\mathbf{R}^n$, there is an atlas $\{U_i, \varphi_i\}_{i\in I}$ of $\mathbf{R}^n$ such that for every $i \in I$ for which $U_i \cap M \neq \varnothing$, we have

$$\varphi_i(U_i) = \widetilde{V}_i \times \widetilde{W}_i \subseteq \mathbf{R}^k \times \mathbf{R}^{n-k}$$

where $\widetilde{V}_i$ contains a ball of radius 1 about 0 in $\mathbf{R}^k$, $\widetilde{W}_i$ is an open subset of $\mathbf{R}^{n-k}$, and

$$\varphi_i(U_i \cap M) = \left\{ (x, y) \in \varphi_i(U_i) \middle| x_1 = \cdots = x_k = 0 \right\}.$$

Such an atlas of $\mathbf{R}^n$ is called a *submanifold atlas* for M. Let $\{\psi^\alpha, V^\alpha\}_{\alpha\in A}$ be a smooth *partition of unity* subordinate to the covering $\{U_i\}_{i\in I}$. For every $\alpha \in A$, there is an $i = i(\alpha) \in I$ such that $V^\alpha \subseteq U_{i(\alpha)}$. Let φ^α be the restriction of $\varphi_{i(\alpha)}$ to V^α multiplied by a suitable positive factor $r_{i(\alpha)}$. The factor $r_{i(\alpha)}$ is chosen so that $(r_{i(\alpha)}\varphi_{i(\alpha)})(V^\alpha) \cap \widetilde{V}_{i(\alpha)}$ contains a ball of radius 1 about 0. Then $\{V^\alpha, \varphi^\alpha\}_{\alpha\in A}$ is a submanifold atlas for M.

Let $G^\alpha = \psi^\alpha G$. Then G^α vanishes identically outside of V^α, since ψ^α does. Suppose that $V^\alpha \cap M = \varnothing$. Then $\sum_j F_j^2 \neq 0$. Let $g_j^\alpha = \frac{G^\alpha F_j}{\sum_j F_j^2}$. Then $g_j^\alpha : V^\alpha \to \mathbf{R}$ is smooth and on V^α we have

$$G^\alpha = \sum_{j=1}^k g_j^\alpha F_j. \tag{37}$$

Now suppose that $V^\alpha \cap M \neq \varnothing$. On

$$\varphi^\alpha(V^\alpha) = \left(\varphi_{i(\alpha)}(V^\alpha) \cap \widetilde{V}_{i(\alpha)}\right) \times \left(\varphi_{i(\alpha)}(V^\alpha) \cap \widetilde{W}_{i(\alpha)}\right) = \widetilde{V}^\alpha \times \widetilde{W}^\alpha,$$

the function $\widetilde{G}^\alpha = G^\alpha \circ (\varphi^\alpha)^{-1}$ is defined and vanishes identically on $\varphi^\alpha(V^\alpha \cap M)$. Therefore for every $(x, y) \in \widetilde{V}^\alpha \times \widetilde{W}^\alpha$,

$$\begin{aligned} \widetilde{G}^\alpha(x, y) &= \widetilde{G}^\alpha(0, y) + \int_0^1 \frac{d}{dt}\widetilde{G}^\alpha(tx, y)\,dt \\ &= 0 + \sum_{j=1}^k \Big(\int_0^1 \frac{\partial \widetilde{G}^\alpha}{\partial x_j}(tx, y)\,dt \Big) x_j = \sum_{j=1}^k \widetilde{g}_j^\alpha(x, y) x_j. \end{aligned}$$

Hence on V^α we have

$$G^\alpha = \sum_{j=1}^k g_j^\alpha F_j, \tag{38}$$

where $g_j^\alpha = \widetilde{g}_j^\alpha \circ \varphi^\alpha : V^\alpha \to \mathbf{R}$ is smooth. Pasting together the local results (37) and (38) using the smooth partition of unity (ψ^α, V^α), we obtain

$$G = \sum_\alpha \psi^\alpha G^\alpha = \sum_\alpha \psi^\alpha \Big(\sum_{j=1}^k g_j^\alpha F_j \Big) = \sum_{j=1}^k g_j F_j,$$

where $g_j = \sum_\alpha \psi^\alpha g_j^\alpha : \mathbf{R}^n \to \mathbf{R}$ is smooth. □

Consequently, we may define the quotient Poisson algebra

$$\mathcal{B} = \mathcal{A}/\mathcal{I} = (C^\infty(C_h), \{\,,\,\}_{C_h}, \cdot).$$

Because

$$\{S_{ij} + \mathcal{I}, S_{\ell k} + \mathcal{I}\}_{C_h} = \{S_{ij}, S_{\ell k}\}_{\mathcal{G}},$$

the matrix of Poisson brackets $(\{S_{ij} + \mathcal{I}, S_{\ell k} + \mathcal{I}\}_{C_h})$ has rank 4. Therefore C_h is a co-
▷ symplectic manifold. In other words, the Poisson bracket $\{\,,\,\}_{C_h}$ is *nondegenerate* and hence defines a symplectic form ω_h on C_h, (see appendix A section 4). Moreover, ω_h satisfies

$$\rho_h^* \omega_h = \widetilde{\Omega}_4 | H^{-1}(h). \tag{39}$$

(2.13) **Proof**: For every $(x, y) \in H^{-1}(h)$ we know that $T_{(x,y)}H^{-1}(h)$ is spanned by the vectors $\{X_{S_{ij}}(x, y)\}_{1 \le i < j \le 4}$. Since $(T^+S^3, \widetilde{\Omega}_4)$ is a cosymplectic submanifold of $(T\mathbf{R}^4, \omega_4)$,

$$\begin{aligned}
\widetilde{\Omega}_4(x, y)\big(X_{S_{ij}}(x, y), X_{S_{\ell k}}(x, y)\big) &= \omega_4\big(X_{S_{ij}}(x, y), X_{S_{\ell k}}(x, y)\big) \\
&= \{S_{ij}, S_{\ell k}\}_{\mathcal{G}}(x, y) = \{S_{ij}, S_{\ell k}\}_{C_h}(\rho_h(x, y)) \\
&= \omega_h(\rho_h(x, y))\big(T_{(x,y)}\rho_h X_{S_{ij}}(x, y), T_{(x,y)}\rho_h X_{S_{\ell k}}(x, y)\big) \\
&= (\rho_h^* \omega_h)(x, y)\big(X_{S_{ij}}(x, y), X_{S_{\ell k}}(x, y)\big),
\end{aligned}$$

we obtain (39). □

We now prove the main result of this section, which describes the geometry of the mapping ρ (27). As a consequence, we know the geometry of the SO(4) momentum mapping $\widetilde{\mathcal{J}}$ (22) of the geodesic vector field X_H on (T^+S^3, Ω_4). Note that $\widetilde{\mathcal{J}}$ is also the momentum map of the Delaunay vector field $X_{\widetilde{\mathcal{H}}}$.

Claim: The mapping

$$\rho : T^+S^3 \subseteq T\mathbf{R}^4 \to C \subseteq \bigwedge\nolimits^2 \mathbf{R}^4 : (x, y) \to x \wedge y$$

is a surjective submersion, each of whose fibers is a unique oriented orbit of the geodesic vector field X_H on $(T^+S^3, \widetilde{\Omega}_4)$.

(2.14) **Proof**: We have already shown that ρ is surjective ((2.6)). To show that each of its fibers is a unique oriented orbit of X_H we argue as follows. Suppose that $S = (S_{ij}) \in C$. Because S is nonzero, $\sum_{1 \le i < j \le 4} S_{ij}^2 = 2h$ for some $h > 0$. Therefore $S \in C_h$. Since the fiber $\rho_h^{-1}(S)$ of ρ_h is a unique oriented orbit of X_H of energy h ((2.8)), so is the fiber $\rho^{-1}(S)$ of ρ because $\rho = \rho_h$ on $H^{-1}(h)$.

To show that ρ is a submersion, first note that by ((2.8)) the map $\rho_h : H^{-1}(h) \to C_h$ is a submersion. Note that $H^{-1}(h)$ and C_h are codimension 1 submanifolds of T^+S^3 and C, respectively. Since a normal direction to $H^{-1}(h)$ at $(x, y) \in T^+S^3$ and a normal direction to C_h at $\rho(x, y) \in C$ is spanned by grad $H(x, y)$ and grad $F(\rho(x, y))$ respectively, (where

$$F(S_{ij}) = \sum_{1 \le i < j \le 4} S_{ij}^2 - 2h = 0$$

defines C_h as a submanifold of C), it suffices to show that

$$\left\langle T_{(x,y)}\rho \operatorname{grad} H(x,y),\ \operatorname{grad} F(\rho(x,y))\right\rangle$$

is nonzero. We compute. Clearly $\operatorname{grad} H(x,y) = (0,y)$. Hence

$$T_{(x,y)}\rho\big(\operatorname{grad} H(x,y)\big) = x \wedge y = \big(S_{ij}(x,y)\big).$$

But $\operatorname{grad} F(\rho(x,y)) = 2\big(S_{ij}(x,y)\big)$. Therefore

$$\left\langle T_{(x,y)}\rho\big(\operatorname{grad} H(x,y)\big), \operatorname{grad} F(\rho(x,y))\right\rangle = 2\sum_{1\le i<j\le 4} S_{ij}^2(x,y) = 4h > 0. \quad \square$$

This claim has some interesting consequences.

Corollary 1: The *space of orbits* of the geodesic vector field with positive energy is the manifold C. The orbit map is $\rho : T^+S^3 \to C$.

(2.15) **Proof**: This follows immediately from the claim and the definition of orbit space, (see appendix B section 2). $\square$

Observe that every smooth integral of the geodesic vector field on T^+S^3 is a smooth function of the integrals S_{ij}. More precisely we prove

Corollary 2: Suppose that $G : T^+S^3 \subseteq T\mathbf{R}^4 \to \mathbf{R}$ is a smooth integral of the geodesic vector field X_H on $(T^+S^3, \widetilde{\Omega}_4)$. Then there is a smooth function $\widehat{G} : C \subseteq \bigwedge^2\mathbf{R}^4 \to \mathbf{R}$ such that $G = \rho^*\widehat{G}$.

(2.16) **Proof**: Since G is a smooth integral of X_H on T^+S^3, it is constant on every orbit of X_H on T^+S^3. Because each fiber of ρ is a unique orbit of X_H on T^+S^3, G descends to a smooth function $\widehat{G}$ on the orbit space C. But $\rho : T^+S^3 \to C$ is the orbit map, so $G = \rho^*\widehat{G}$. $\square$
The SO(4) momentum mapping

3 The Kepler problem

We investigate the bounded motion of a particle in $\mathbf{R}^3$ which is under the influence of a gravitational field of a second particle fixed at the origin. This is *Kepler's problem.*

3.1 The Kepler vector field

In this subsection we define the Kepler Hamiltonian system $(H, T_0\mathbf{R}^3, \omega_3)$. We then show that the Kepler Hamiltonian vector field X_H conserves energy H, angular momentum $\mathbf{J}$, and the eccentricity vector $\mathbf{e}$. On the set Σ_- of positions and momenta where the values of H are negative, the orbits of X_H are bounded, yet the flow of X_H is incomplete.

On the phase space $T_0\mathbf{R}^3 = (\mathbf{R}^3 - \{0\}) \times \mathbf{R}^3$ with coordinates (q, p) and symplectic form $\omega_3 = \sum_{i=1}^3 dq_i \wedge dp_i$, consider the *Kepler Hamiltonian*

$$H : T_0\mathbf{R}^3 \to \mathbf{R} : (q,p) \to \tfrac{1}{2}\langle p, p\rangle - \frac{\mu}{|q|}. \tag{40}$$

Here $\langle\ ,\ \rangle$ is the Euclidean inner product on $\mathbf{R}^3$ and $|q|$ is the length of the vector q. The integral curves of the Hamiltonian vector field X_H on $T_0\mathbf{R}^3$ satisfy the equations

$$\begin{cases} \dot{q} &= p \\ \dot{p} &= -\mu \dfrac{q}{|q|^3}, \end{cases} \tag{41}$$

which describe the motion of a particle of mass 1 about the origin under the influence of an inverse $|q|^2$ force — such as Newtonian gravity. We consider the case where the force is attractive, that is, $\mu > 0$. However, much of the following analysis can be carried out mutatis mutandis for $\mu < 0$. (See exercise 3 how to reduce the two body problem to the Kepler problem (41).)

The *Kepler vector field* X_H has some obvious integrals: the *total energy*

$$h = \tfrac{1}{2}\langle p, p\rangle - \frac{\mu}{|q|}, \tag{42}$$

which is nothing but the Hamiltonian H, and the *angular momentum*

$$\mathbf{J} = (J_1, J_2, J_3) = q \times p. \tag{43}$$

Here $\times$ is the *vector product* on $\mathbf{R}^3$.

(3.1) **Proof**: A direct way to see that $\mathbf{J}$ is an integral is to compute

$$\begin{aligned} \frac{d\mathbf{J}}{dt} &= \frac{dq}{dt} \times p + q \times \frac{dp}{dt} = p \times p - \frac{\mu}{|q|^3} q \times q, \quad \text{using (41)} \\ &= 0. \end{aligned}$$

A more sophisticated way to see this is to note that the SO(3) action

$$\mathrm{SO}(3) \times \mathbf{R}^3 \to \mathbf{R}^3 : (O, q) \to Oq$$

lifts to a Hamiltonian action

$$\mathrm{SO}(3) \times T_0\mathbf{R}^3 \to T_0\mathbf{R}^3 : \big(O, (q, p)\big) \to (Oq, Op).$$

This action has the momentum mapping

$$\widetilde{J} : T_0\mathbf{R}^3 \to \mathrm{so}(3)^* : (q, p) \to \begin{pmatrix} 0 & J_3 & -J_2 \\ -J_3 & 0 & J_1 \\ J_2 & -J_1 & 0 \end{pmatrix},$$

defined by $\widetilde{J}(q, p)X = \langle p, X(q)\rangle$ where $X \in \mathrm{so}(3)$. Now use the map $k^\flat$ associated to the Killing metric

$$k : \mathrm{so}(3) \times \mathrm{so}(3) \to \mathbf{R} : (X, Y) \to \tfrac{1}{2}\,\mathrm{tr}\, XY^t$$

to identify $\mathrm{so}(3)^*$ with $\mathrm{so}(3)$. This identification boils down to taking transposes. Follow this by the map

$$i : \mathrm{so}(3) \to \mathbf{R}^3 : X = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} \to \mathbf{x} = (x_1, x_2, x_3),$$

which identifies so(3) with $\mathbf{R}^3$, (see chapter 3 section 1). Then $\widetilde{J}$ becomes the usual angular momentum $\mathbf{J} : T_0\mathbf{R}^3 \to \mathbf{R}^3 : (q, p) \to q \times p$. Since the SO(3) action on $(T_0\mathbf{R}^3, \omega_3)$ leaves the Kepler Hamiltonian H (40) invariant, every component of the angular momentum $\mathbf{J}$ is constant on the integral curves of X_H. □

▷ More interestingly, there is another integral of the Kepler vector field. This is the *eccentricity vector*.

$$\mathbf{e} = (e_1, e_2, e_3) = -\frac{q}{|q|} + \frac{1}{\mu}\, p \times (q \times p). \tag{44}$$

(3.2) **Proof**: To see this we calculate

$$\begin{aligned}
\frac{d\mathbf{e}}{dt} &= -\frac{d}{dt}\frac{q}{|q|} + \frac{1}{\mu}\frac{dp}{dt} \times \mathbf{J} = \frac{1}{|q|^3}\langle \frac{dq}{dt}, q\rangle q - \frac{1}{|q|}\frac{dq}{dt} + \frac{1}{\mu}\frac{dp}{dt} \times \mathbf{J} \\
&= \frac{1}{|q|^3}\big(\langle q, p\rangle q - \langle q, q\rangle\, p\big) - \frac{1}{|q|^3}\, q \times \mathbf{J}, \quad \text{using (41)} \\
&= \frac{1}{|q|^3}\big(q \times (q \times p) - q \times (q \times p)\big) = 0. \qquad \square
\end{aligned}$$

We now prove some properties of the flow of the Kepler vector field X_H.

Claim: If the energy h is negative, then the image of every integral curve of the Kepler vector field under the bundle projection $\tau : T_0\mathbf{R}^3 \to \mathbf{R}^3 : (q, p) \to q$ is bounded.

(3.3) **Proof**:
<u>case</u> 1. $\mathbf{J} = 0$. Since $\mathbf{e}$ is an integral of X_H and $\mathbf{J} = 0$, the direction $\mathbf{e} = -\frac{q}{|q|}$ of the motion is constant. Therefore the motion takes place on the line $q(t) = r(t)\,\mathbf{e}$. From conservation of energy we obtain $h + \frac{\mu}{r} = \frac{1}{2}\dot{r}^2 \geq 0$. Therefore $|q(t)| \leq \frac{\mu}{-h}$.

<u>case</u> 2. $\mathbf{J} \neq 0$. Since $J^2 = |q \times p|^2 = |q|^2|p|^2 - \langle q, p\rangle^2$,

$$h = \tfrac{1}{2}\langle p, p\rangle - \frac{\mu}{|q|} = \tfrac{1}{2}\frac{\langle q, p\rangle^2}{|q|^2} + \tfrac{1}{2}\frac{J^2}{|q|^2} - \frac{\mu}{|q|} \geq \tfrac{1}{2}\frac{J^2}{|q|^2} - \frac{\mu}{|q|}.$$

As is easily verified, the function $V_J(|q|) = \frac{1}{2}\frac{J^2}{|q|^2} - \frac{\mu}{|q|}$ has a unique nondegenerate minimum at $|q| = \frac{J^2}{\mu}$ corresponding to the critical value $-\frac{\mu^2}{2J^2}$. Since $\lim_{|q|\to 0^+} V_J(|q|) \nearrow \infty$ and $\lim_{|q|\to\infty} V_J(|q|) \nearrow 0^-$, the function V_J is proper on the set where it has negative values. Therefore $V_J^{-1}([-\frac{\mu^2}{2J^2}, h])$ is compact. Thus the length of $q(t)$ is bounded, when $h < 0$. □

Claim: The flow of the Kepler vector field X_H is *not complete*.

(3.4) **Proof**: Consider a bounded motion with $\mathbf{J} = 0$ and $h < 0$ which starts at $(r(0), \dot{r}(0)) = (\frac{\mu}{-h}, 0)$. The time it takes to reach the origin is

$$T = \int_0^{\frac{\mu}{-h}} \frac{dr}{\sqrt{\frac{2\mu}{r} + 2h}}.$$

This is obtained by separating variables in $\frac{1}{2}\dot{r}^2 = h + \frac{\mu}{r}$ and integrating. Performing the integral gives

$$T = \frac{\pi}{2}\frac{\mu}{(-2h)^{3/2}}, \tag{45}$$

which is finite. □

3.2 The so(4) momentum map

Recall that Σ_- is the open subset of $(T_0\mathbf{R}^3, \omega_3)$ where the energy H is negative. In this subsection we show that on Σ_- the components of the angular momentum $\mathbf{J}$ and the modified eccentricity vector $\widetilde{\mathbf{e}} = -\nu\,\mathbf{e}$ (where $\nu = \mu/\sqrt{-2H}$) form a Lie algebra under Poisson bracket which is isomorphic to so(4). This defines a representation of so(4) on the space of Hamiltonian vector fields on $(\Sigma_-, \widetilde{\omega}_3 = \omega_3|\Sigma_-)$ which has a momentum mapping $\mathcal{J}$. In fact $\mathcal{J}$ is a surjective submersion from Σ_- to

$$C = \left\{(\mathbf{J}, \widetilde{\mathbf{e}}) \in \mathbf{R}^6 \,\middle|\, \langle \mathbf{J} + \widetilde{\mathbf{e}}, \mathbf{J} + \widetilde{\mathbf{e}}\rangle = \langle \mathbf{J} - \widetilde{\mathbf{e}}, \mathbf{J} - \widetilde{\mathbf{e}}\rangle > 0\right\}$$

each of whose nonempty fibers is a unique oriented bounded orbit of X_H.

▷ First we show that on $(\Sigma_-, \widetilde{\omega}_3)$ the components of the angular momentum $\mathbf{J}$ and the modified eccentricity vector $\widetilde{\mathbf{e}}$ satisfy the Poisson bracket relations

$$\{J_i, J_j\} = \sum_k \varepsilon_{ijk}\, J_k, \quad \{J_i, \widetilde{e}_j\} = \sum_k \varepsilon_{ijk}\, \widetilde{e}_k, \quad \text{and} \quad \{\widetilde{e}_i, \widetilde{e}_j\} = \sum_k \varepsilon_{ijk}\, J_k. \tag{46}$$

(3.5) **Proof**: We verify only the third equality in (46). Let $\mathbf{A} = \mu\,\mathbf{e}$. Since $\{q_\ell, p_m\} = \delta_{\ell m}$, $\{q_i, q_j\} = 0$, and $\{p_i, p_j\} = 0$, we have

$$\{J_a, |q|\} = 0, \quad \{q_a, J_b\} = \sum_c \varepsilon_{abc} q_c, \quad \text{and} \quad \{p_a, J_b\} = \sum_c \varepsilon_{abc} p_c.$$

Using bilinearity and the derivation property of Poisson bracket, expand

$$\{A_i, A_j\} = \left\{\sum_{j,k} \varepsilon_{ijk} p_j J_k - \frac{1}{|q|} q_i, \sum_{m,n} \varepsilon_{\ell mn} p_m J_n - \frac{1}{|q|} q_\ell\right\}$$

to obtain

$$\{A_i, A_j\} = -2H \sum_k \varepsilon_{ijk} J_k.$$

It helps to recall the identity

$$\sum_i \varepsilon_{ijk}\varepsilon_{i\ell m} = \delta_{mk}\delta_{\ell j} - \delta_{jm}\delta_{\ell k}. \qquad \square$$

▷ The bracket relations (46) define a Lie algebra which is isomorphic to so(4).

(3.6) **Proof**: For $i = 1, 2, 3$ define

$$\xi_i = \tfrac{1}{2}(J_i + \widetilde{e}_i) \quad \text{and} \quad \eta_i = \tfrac{1}{2}(J_i - \widetilde{e}_i).$$

In terms of ξ_i and η_i the bracket relations (46) become

$$\{\xi_i, \xi_j\} = \sum_k \varepsilon_{ijk}\,\xi_k, \quad \{\eta_i, \eta_j\} = \sum_k \varepsilon_{ijk}\,\eta_k, \quad \text{and} \quad \{\xi_i, \eta_j\} = 0. \tag{47}$$

These relations define the Lie algebra so(3) $\times$ so(3), which is isomorphic to so(4). □

Thus the mappings

$$J_i \to \mathrm{ad}_{J_i} = -X_{J_i} \quad \text{and} \quad \widetilde{e}_i \to \mathrm{ad}_{\widetilde{e}_i} = -X_{\widetilde{e}_i}$$

define a representation of so(4) on the space of Hamiltonian vector fields on $(\Sigma_-, \widetilde{\omega}_3)$. In other words, we have a *Hamiltonian action* of the *Lie algebra* so(4) on $(\Sigma_-, \widetilde{\omega}_3)$. Associated to this Lie algebra action is the mapping

$$\begin{aligned} &\mathcal{J} : \Sigma_- \subseteq T_0\mathbf{R}^3 \to \mathbf{R}^6 : \\ &(q, p) \to (\mathbf{J}, \widetilde{\mathbf{e}}) = \left(q \times p, \nu\Big(\frac{q}{|q|} - \frac{1}{\mu} p \times (q \times p)\Big)\right). \end{aligned} \tag{48}$$

Here we have chosen $\{\epsilon_i\}_{1 \le i \le 6} = \{J_1, J_2, J_3, \widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_2\}$ as a basis for so(4) with Lie bracket $\{\, , \,\}$. Let $\mathcal{J}^{\epsilon_i}$ be the i^{th} component of the mapping $\mathcal{J}$. Then the bracket relations (46) may be written as $\{\mathcal{J}^{\epsilon_i}, \mathcal{J}^{\epsilon_j}\} = \mathcal{J}^{\{\epsilon_i, \epsilon_j\}}$. Therefore we say that the map $\mathcal{J}$ is the *momentum map* of the so(4)-action on $(\Sigma_-, \widetilde{\omega}_3)$.

We now investigate the geometric properties of the mapping $\mathcal{J}$. We begin by noting that the vectors $\mathbf{J}$ and $\widetilde{\mathbf{e}}$ satisfy

$$\begin{cases} \langle \mathbf{J}, \widetilde{\mathbf{e}} \rangle &= 0 \\ \langle \mathbf{J}, \mathbf{J} \rangle + \langle \widetilde{\mathbf{e}}, \widetilde{\mathbf{e}} \rangle &= \nu^2 > 0. \end{cases} \tag{49}$$

The verification of the first equation in (49) is a straightforward. For the second, see (54). These relations define a smooth 4-dimensional manifold C_ν, which is diffeomorphic to $S^2_\nu \times S^2_\nu$ because (49) is equivalent to

$$\langle \mathbf{J} + \widetilde{\mathbf{e}}, \mathbf{J} + \widetilde{\mathbf{e}} \rangle = \langle \mathbf{J} - \widetilde{\mathbf{e}}, \mathbf{J} - \widetilde{\mathbf{e}} \rangle = \nu^2 > 0. \tag{50}$$

Write $\nu = \mu / \sqrt{-2h}$ for some $h < 0$ and consider the map

$$\mathcal{J}_h = \mathcal{J} | H^{-1}(h) : H^{-1}(h) \subseteq \Sigma_- \to C_\nu \subseteq \mathbf{R}^6. \tag{51}$$

Claim: $\mathcal{J}_h$ is a surjective submersion.

(3.7) **Proof**: Let $(q, p) \in H^{-1}(h)$ and let

$$V_{(q,p)} = \operatorname{span}\{X_{J_j}(q, p),\ X_{\widetilde{e}_j}(q, p)\}_{1 \le j \le 3}.$$

Since $\mathbf{J}$ and $\widetilde{\mathbf{e}}$ are integrals of X_H, it follows that $V_{(q,p)} \subseteq \ker dH(q, p)$ which is $T_{(q,p)}H^{-1}(h)$. Therefore

$$\begin{aligned} D\mathcal{J}_h(q, p) | V_{(q,p)} &= \begin{pmatrix} dJ_j(q, p) \\ d\widetilde{e}_j(q, p) \end{pmatrix} V_{(q,p)} \\ &= \begin{pmatrix} (\{J_i, J_j\}(q, p)) & (\{J_i, \widetilde{e}_j\}(q, p)) \\ (\{\widetilde{e}_i, J_j\}(q, p)) & (\{\widetilde{e}_i, \widetilde{e}_j\}(q, p)) \end{pmatrix} = P. \end{aligned}$$

On C_ν (49) the rank of P is 4, because P is conjugate to the matrix

$$\begin{pmatrix} (\{\xi_i, \xi_j\}(q,p)) & 0 \\ 0 & (\{\eta_i, \eta_j\}(q,p)) \end{pmatrix} = \begin{pmatrix} \left(\sum_k \varepsilon_{ijk}\,(J_k + \widetilde{e}_k)(q,p)\right) & 0 \\ 0 & \left(\sum_k \varepsilon_{ijk}\,(J_k - \widetilde{e}_k)(q,p)\right) \end{pmatrix}$$

(see (47) and (50)). Therefore $\mathcal{J}_h$ is a submersion.

To show that $\mathcal{J}_h$ is surjective, let $(\mathbf{J}, \widetilde{\mathbf{e}}) \in C_\nu$. Then $e = |\mathbf{e}| = \frac{1}{\nu}|\widetilde{\mathbf{e}}| \in [0,1]$ because $\nu^2 = \langle \mathbf{J}, \mathbf{J}\rangle + \langle \widetilde{\mathbf{e}}, \widetilde{\mathbf{e}}\rangle \geq \nu^2 \langle \mathbf{e}, \mathbf{e}\rangle$. Choose

$$(q,p) = \begin{cases} \left(-\frac{\nu}{\mu}\frac{(1-e)}{e}\widetilde{\mathbf{e}},\ -\frac{\mu}{\nu^3}\frac{1}{e(1-e)}\mathbf{J}\times\widetilde{\mathbf{e}}\right), & \text{when } e \in (0,1) \text{ and } \mathbf{J} \neq 0 \\ \left(-\frac{\mu^2}{\nu^2}\,p\times\mathbf{J},\ p\right), & \text{when } e = 0 \text{ and } \mathbf{J}\neq 0. \text{ Here } \langle p, \mathbf{J}\rangle = 0,\ |p| = \frac{\mu}{\nu} \\ \left(-\frac{\nu}{\mu}\widetilde{\mathbf{e}},\ \frac{\nu}{\mu^2}\widetilde{\mathbf{e}}\right), & \text{when } \mathbf{J} = 0. \end{cases}$$

A straightforward calculation shows that $(q,p) \in H^{-1}(h)$ and $\mathcal{J}_h(q,p) = (\mathbf{J}, \widetilde{\mathbf{e}})$. □

Corollary: For every $c \in C_\nu$ the fiber $\mathcal{J}_h^{-1}(c)$ is a union of bounded Keplerian orbits.

(3.8) **Proof**: From the fact that $\mathcal{J}_h$ is a submersion, it follows that

$$\dim\ker D\mathcal{J}_h(q,p) = \dim T_{(q,p)}H^{-1}(h) - \dim \operatorname{im} D\mathcal{J}_h(q,p) = 5 - 4 = 1.$$

But $X_H(q,p) \in \ker D\mathcal{J}_h(q,p)$. Hence for every $c \in C_\nu$

$$T_{(q,p)}H^{-1}(h) = \ker D\mathcal{J}_h(q,p) = \operatorname{span}\{X_H(q,p)\}.$$

Therefore $\mathcal{J}_h^{-1}(c)$ is a union of bounded Keplerian orbits. □

The following claim is a substantial sharpening of the above corollary.

Claim: For every $c \in C_\nu$ the fiber $\mathcal{J}_h^{-1}(c)$ lies over

1. an oriented ellipse, when $c \notin C_\nu \cap \{\mathbf{J} = 0\}$;
2. a line which is the union of two half open line segments

$$\left\{\left(\frac{\sigma}{\nu}\widetilde{\mathbf{e}},\ \pm\frac{1}{\nu}\left(\sqrt{2h + \frac{2\mu}{\sigma}}\right)\widetilde{\mathbf{e}}\right) \in T_0\mathbf{R}^3 \,\middle|\, \sigma \in \left(0, \frac{\mu}{-h}\right]\right\},$$

   that join smoothly at $\left(\frac{\mu}{-\nu h}\widetilde{\mathbf{e}}, 0\right)$, when $c \in C_\nu \cap \{\mathbf{J} = 0\}$.

(3.9) **Proof**:
case 1. $c = (\mathbf{J}, \widetilde{\mathbf{e}}) \in C_\nu - C_\nu \cap \{\mathbf{J} = 0\}$. Let $h = -\mu^2/2\nu^2$. We have to show that the data $h < 0$, $\mathbf{J} \neq 0$, and $\mathbf{e} = -\frac{1}{\nu}\widetilde{\mathbf{e}}$ determine a unique oriented ellipse which is traced out by the projection $t \to q(t)$ of an integral curve $t \to (q(t), p(t))$ of X_H. Because $\mathbf{J} \neq 0$ and $\langle q(t), \mathbf{J}\rangle = \langle p(t), \mathbf{J}\rangle = 0$, the curves $t \to q(t)$ and $t \to p(t)$ lie in a plane $\Pi \subseteq \mathbf{R}^3$ which is perpendicular to $\mathbf{J}$. Since $\langle \mathbf{J}, \mathbf{e}\rangle = \mathbf{0}$, the eccentricity vector $\mathbf{e}$ also lies in Π. Therefore

we may write $\langle q, \mathbf{e}\rangle = |q| e \cos f$, where f is the *true anomaly*. From the definition of the eccentricity vector $\mathbf{e}$ (44) it follows that $\langle q, \mathbf{e}\rangle = -|q| + \frac{1}{\mu} J^2$. Therefore

$$|q| e \cos f = -|q| + \frac{1}{\mu} J^2. \tag{52}$$

Suppose that $e = 0$. Then (52) becomes $|q| = \frac{1}{\mu} J^2$, which defines a circle $\mathscr{C}$ in Π with center at the origin. Since

$$0 = \frac{d|q(t)|^2}{dt} = \langle q, \frac{dq}{dt}\rangle = \langle q, p\rangle,$$

the tangent vector $p(t)$ to $\mathscr{C}$ at $q(t)$ is perpendicular to $q(t)$. Because $\{q, p, p \times q\}$ is a positively oriented basis of $\mathbf{R}^3$, $\{q, p\}$ is a positively oriented basis for Π. Hence the circle traced out by $t \to q(t)$ is positively oriented.

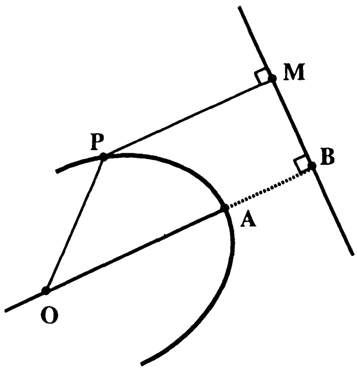

Figure 3.1. Ellipse in the plane Π.

Suppose that $e \neq 0$. Then equation (52) may be written as

$$e\left(\frac{J^2}{\mu e} - |q| \cos f\right) = |q|. \tag{53}$$

Equation (53) describes the locus of points P in the plane Π for which the ratio of the distance $\overline{OP}$ to the origin to the distance $\overline{PM}$ to the line MB is a constant e, (see figure 3.1). Thus the locus is a conic section. To see which conic it is, we calculate the size of e as follows.

$$\begin{aligned} e^2 &= |\mathbf{e}|^2 = 1 - \frac{2}{\mu|q|} |q \times p|^2 + \frac{1}{\mu^2} |p \times (q \times p)|^2, \quad \text{using (44)} \\ &= 1 - \frac{2}{\mu|q|} J^2 + \frac{1}{\mu^2}(|p|^2 J^2 - \langle p, \mathbf{J}\rangle^2), \quad \text{using } \mathbf{J} = q \times p \\ &= 1 + \frac{2}{\mu^2} J^2 h, \quad \text{since } \langle p, \mathbf{J}\rangle = 0 \text{ and } h = \tfrac{1}{2}\langle p, p\rangle - \tfrac{\mu}{|q|}. \end{aligned} \tag{54}$$

Since $h < 0$, it follows that $e \in [0, 1)$. Therefore the locus

$$|q| = \frac{J^2}{\mu} \frac{1}{1 + e \cos f} \tag{55}$$

is an ellipse in Π with *eccentricity* e and major semiaxis lying along $\mathbf{e}$ (which is directed from O to the periapse A) of length

$$a = \frac{J^2}{\mu}\frac{1}{1-e^2} = \frac{\mu}{-2h}.$$

$\triangleright$ When traced out by $t \to q(t)$, this ellipse is oriented in the direction of increasing true anomaly f.

(3.10) **Proof**: From the fact that $\{q, p\}$ is a positively oriented basis of the plane Π, we obtain

$$\begin{aligned} J &= |q \times p| = \text{area of the positively oriented parallogram spanned by } \{q,p\}. \\ &= \det\begin{pmatrix} \langle q, e^{-1}\mathbf{e}\rangle & \langle q, (Je)^{-1}\mathbf{J}\times\mathbf{e}\rangle \\ \langle p, e^{-1}\mathbf{e}\rangle & \langle p, (Je)^{-1}\mathbf{J}\times\mathbf{e}\rangle \end{pmatrix}, \\ &\quad \text{since } \{e^{-1}\mathbf{e}, (Je)^{-1}\mathbf{J}\times\mathbf{e}\} \text{ is a positively oriented orthonormal basis of } \Pi \\ &= |q|^2 \frac{df}{dt}. \end{aligned} \tag{56}$$

Equation (56) follows by first differentiating

$$\langle q, e^{-1}\mathbf{e}\rangle = |q|\cos f \quad \text{and} \quad \langle q, (Je)^{-1}\mathbf{J}\times\mathbf{e}\rangle = |q|\sin f$$

along an integral curve of X_H and then using the fact that $p = \frac{dq}{dt}$ and $\dot{\mathbf{e}} = \dot{\mathbf{J}} = 0$ to obtain

$$\langle p, e^{-1}\mathbf{e}\rangle = \frac{d|q|}{dt}\cos f - |q|\sin f\,\frac{df}{dt}$$

and

$$\langle p, (Je)^{-1}\mathbf{J}\times\mathbf{e}\rangle = \frac{d|q|}{dt}\sin f + |q|\cos f\,\frac{df}{dt}.$$

From (56) we see that $\frac{df}{dt} > 0$. □

case 2. $c = (\mathbf{J}, \widetilde{\mathbf{e}}) \in C_\nu \cap \{\mathbf{J} = 0\}$. Since $\mathbf{J} = 0$, the modified eccentricity vector $\widetilde{\mathbf{e}} = \nu\frac{q}{|q|}$. Because $\widetilde{\mathbf{e}}$ is constant along integral curves $t \to (q(t), p(t))$ of X_H and $h < 0$, the image of $t \to q(t)$ lies along $\widetilde{\mathbf{e}}$ and is the half open line segment

$$\left\{ \frac{\sigma}{\nu}\widetilde{\mathbf{e}} \in \Pi \,\middle|\, \sigma \in \left(0, \frac{\mu}{-h}\right] \right\}.$$

From $\mathbf{J} = 0$ it follows that $p = \lambda\widetilde{\mathbf{e}}$ for some $\lambda \in \mathbf{R}$. In order that $(\frac{\sigma}{\nu}\widetilde{\mathbf{e}}, p) \in H^{-1}(h)$, where $h = -\mu^2/2\nu^2$, we must have $\lambda^2 = \langle p, p\rangle = 2h + 2\mu/\sigma$. Therefore $\mathcal{J}_h^{-1}(c)$ is the line which is the union of the two half open line segments

$$\left\{ \left(\frac{\sigma}{\nu}\widetilde{\mathbf{e}}, \pm\frac{1}{\nu}\left(\sqrt{2h + \frac{2\mu}{\sigma}}\right)\widetilde{\mathbf{e}}\right) \in T_0\mathbf{R}^3 \,\middle|\, \sigma \in \left(0, \frac{\mu}{-h}\right] \right\}$$

which join smoothly at $(\frac{\mu}{-h\nu}\widetilde{\mathbf{e}}, 0)$. □

It is not hard to show that on $\mathcal{J}_h^{-1}\big(C_\nu - (\{\mathbf{J}=0\} \cap C_\nu)\big)$ the mapping $\mathcal{J}_h$ is proper, whereas on $\mathcal{J}_h^{-1}(\{\mathbf{J}=0\} \cap C_\nu)$ it is not.

We now turn to examining the so(4) momentum mapping $\mathcal{J}$ (48). Let C be the submanifold of $\mathbf{R}^3 \times \mathbf{R}^3 - \{(0,0)\}$ defined by

$$\langle \mathbf{J}, \widetilde{\mathbf{e}} \rangle = 0. \tag{57}$$

Claim: The map

$$\begin{aligned}
&\mathcal{J} : \Sigma_- \subseteq T_0\mathbf{R}^3 \to C \subseteq \mathbf{R}^6 : \\
&(q,p) \to \Big(q \times p, \nu\Big(\tfrac{q}{|q|} - \tfrac{1}{\mu}\, p \times (q \times p)\Big)\Big) = (\mathbf{J}, \widetilde{\mathbf{e}})
\end{aligned}$$

is a surjective submersion each of whose fibers is a unique bounded orbit of the Kepler vector field X_H.

(3.11) **Proof**: First we show that $\mathcal{J}$ is surjective. Supppose that $c = (\mathbf{J}, \widetilde{\mathbf{e}}) \in C$. Then

$$\|\mathbf{J} + \widetilde{\mathbf{e}}\|^2 = \|\mathbf{J} - \widetilde{\mathbf{e}}\|^2 = \nu^2$$

for some $\nu > 0$. Hence $(\mathbf{J}, \widetilde{\mathbf{e}}) \in C_\nu$. Let $h = -\mu^2/2\nu^2$. From ((3.7)) it follows that $\mathcal{J}_h^{-1}(c)$ is nonempty. Hence $\mathcal{J}^{-1}(c)$ is nonempty. Because $\mathcal{J}_h^{-1}(c)$ is a unique oriented bounded orbit of the Kepler vector field, $\mathcal{J}^{-1}(c)$ is as well.

Since C is a 5-dimensional smooth manifold, $\mathcal{J}$ is a submersion if for every $(q,p) \in \Sigma_-$ the rank of $D\mathcal{J}(q,p)$ is 5. Actually it suffices to show that for every $(q,p) \in H^{-1}(h)$ the vector $D\mathcal{J}(q,p)\,\mathrm{grad}H(q,p)$ is normal to C_ν at $\mathcal{J}(q,p)$, because

1. by ((3.7)), $D\mathcal{J}(q,p)T_{(q,p)}H^{-1}(h) = T_{\mathcal{J}(q,p)}C_\nu$;
2. a normal space to $H^{-1}(h)$ in Σ_- at (q,p) is spanned by $\mathrm{grad}H(q,p)$;
3. as a submanifold of C the manifold C_ν is defined by

$$F(\mathbf{J}, \widetilde{\mathbf{e}}) = \langle \mathbf{J}, \mathbf{J} \rangle + \langle \widetilde{\mathbf{e}}, \widetilde{\mathbf{e}} \rangle - \nu^2 = 0, \tag{58}$$

where $\nu = \mu/\sqrt{-2H}$.

Since the normal space to C_ν at $\mathcal{J}(q,p) = (\mathbf{J}, \widetilde{\mathbf{e}}) \in C$ is spanned by $\mathrm{grad}F(\mathbf{J}, \widetilde{\mathbf{e}}) = 2(\mathbf{J}, \widetilde{\mathbf{e}})$, it suffices to check that

$$\Big\langle D\mathcal{J}(q,p)\,\mathrm{grad}H(q,p),\ \mathrm{grad}F(\mathcal{J}(q,p))\Big\rangle$$

is nonzero. The following calculation does this.

$$\begin{aligned}
0 \;&\neq\; \Big\langle \mathrm{grad}H(q,p),\ \mathrm{grad}H(q,p)\Big\rangle \;=\; DH(q,p)\,\mathrm{grad}H(q,p) \\
&=\; D\left(-\frac{\mu^2}{2}\left(\langle \mathbf{J}, \mathbf{J}\rangle + \langle \widetilde{\mathbf{e}}, \widetilde{\mathbf{e}}\rangle\right)^{-1}\right)(q,p)\,\mathrm{grad}H(q,p), \\
&\qquad \text{using } H = -\tfrac{\mu^2}{2\nu^2} \text{ and (58)}
\end{aligned}$$

$$= \frac{\mu^2}{2}\big(\langle \mathbf{J},\mathbf{J}\rangle + \langle \widetilde{\mathbf{e}},\widetilde{\mathbf{e}}\rangle\big)^{-2}\Big(\big\langle \mathbf{J}, D\mathbf{J}(q,p)\,\mathrm{grad}H(q,p)\big\rangle + \big\langle \widetilde{\mathbf{e}}, D\mathbf{J}(q,p)\,\mathrm{grad}H(q,p)\big\rangle\Big)$$

$$= \frac{2H(q,p)^2}{\mu^2}\Big\langle D\mathcal{J}(q,p)\,\mathrm{grad}H(q,p), \mathrm{grad}F(\mathcal{J}(q,p))\Big\rangle. \qquad \square$$

The above result has several useful consequences.

Corollary 1. The smooth manifold C (57) is the *space of orbits* of negative energy of the Kepler vector field X_H and the momentum map $\mathcal{J} : \Sigma_- \to C$ is the orbit map.

(3.12) **Proof**: The corollary follows from ((3.11)) and the definition of orbit space. $\square$

The next corollary says that every smooth integral of the Kepler vector field on Σ_- is a smooth function of the components of angular momentum $\mathbf{J}$ and the modified eccentricity vector $\widetilde{\mathbf{e}}$. More precisely,

Corollary 2. Suppose that $G : \Sigma_- \subseteq T_0\mathbf{R}^3 \to \mathbf{R}$ is a smooth integral of the Kepler vector field X_H. Then there is a smooth function $\widehat{G} : C \subseteq \mathbf{R}^6 \to \mathbf{R}$ such that $G = \mathcal{J}^*\widehat{G}$.

(3.13) **Proof**: Since G is an integral of X_H on Σ_-, it is constant on each bounded orbit of X_H and hence is constant on the fibers of the momentum map $\mathcal{J}$. Because C is smooth and is the space of orbits of X_H on Σ_- with orbit mapping $\mathcal{J}$, G descends to a smooth function $\widehat{G} : C \subseteq \mathbf{R}^6 \to \mathbf{R}$. In other words, $G = \mathcal{J}^*\widehat{G}$. $\square$

3.3 Kepler's equation

So far we have only used the constants of motion to describe the orbits of the Kepler vector field X_H of negative energy. This means that we cannot tell where on the orbit the particle is at a given time.

In order to give a time parametrization of a bounded Keplerian orbit, we define a new time scale, the *eccentric anomaly* s, by

$$\frac{ds}{dt} = \frac{\sqrt{-2h}}{|q|}. \tag{59}$$

Before finding a differential equation for $|q(s)|$, we use the integrals of energy and angular momentum to find a differential equation for $|q(t)|$. Multiplying the energy integral

$$h = \tfrac{1}{2}\langle p,p\rangle - \frac{\mu}{|q|}$$

by $2|q|^2$ gives

$$|q|^2|p|^2 = 2\mu\,|q| + 2h\,|q|^2.$$

But

$$|q|^2|p|^2 = |q \times p|^2 + \langle q,p\rangle^2 = J^2 + \langle p,q\rangle^2.$$

In other words,

$$|q|^2\left(\frac{d|q|}{dt}\right)^2 + J^2 = 2\mu\,|q| + 2h\,|q|^2. \tag{60}$$

Using (59) to change to the time variable s and dividing by $-2h$ gives

$$\left(\frac{d|q|}{ds}\right)^2 + a^2(1-e^2) = 2a\,|q| - |q|^2, \tag{61}$$

since

$$a = \mu/(-2h) = \frac{J^2}{\mu}\frac{1}{1-e^2}.$$

Instead of separating variables and immediately integrating (61) we first change variables by $ea\,\rho = a - |q|$. Then (61) simplifies to

$$\left(\frac{d\rho}{ds}\right)^2 + \rho^2 = 1. \tag{62}$$

Since $|q(0)| = a(1-e)$, from the definition of ρ we obtain $\rho(0) = 1$. Therefore

$$|q(s)| = a - ae\cos s. \tag{63}$$

To find the relation between the eccentric anomaly time scale s and the physical time scale t, we substitute (63) into (59) and integrate to obtain

$$\sqrt{-2h}(t-\tau) = \sqrt{-2h}\int_\tau^t dt = \int_0^s (a - ae\cos s)ds = as - ae\sin s. \tag{64}$$

Here τ is a time related to the time of periapse passage. Its precise definition is given below. Dividing (64) by a and using $a = \mu/(-2h) = \nu^2/\mu$ gives *Kepler's equation*

$$s - e\sin s = \frac{\mu^2}{\nu^3}(t-\tau) = n\,\ell, \tag{65}$$

where ℓ is the *mean anomaly* and $n = \mu^2/\nu^3$ is the *mean motion*. Note that

$$\begin{aligned}\langle q, p\rangle &= \langle q, \frac{dq}{dt}\rangle = |q|\frac{d|q|}{dt} = |q|\frac{d|q|}{ds}\frac{ds}{dt}\\ &= \sqrt{-2h}\,ae\,\sin s, \quad \text{using (59) and (63)}\\ &= \nu\,e\sin s.\end{aligned} \tag{66}$$

When $t = \tau$ from Kepler's equation it follows that $s = 0$. Let τ' be the physical time corresponding to $s = 2\pi$ in (65). Then $\tau - \tau'$ is the period of elliptical motion, which according to Kepler's equation, is

$$2\pi/n = 2\pi\nu^3/\mu^2 = 2\pi a^{3/2}/\mu^{1/2}. \tag{67}$$

This is Kepler's third law of motion.

During elliptical motion the particle goes through the periapse periodically. Therefore the time τ in (65) is *not* uniquely determined by the initial condition $(q(0), p(0))$ which defines the integral curve of X_H. We will define τ as follows. In the interval $[-\pi, \pi]$ there are precisely *two* values $\varepsilon\widehat{s}_0$ (with $\varepsilon^2 = 1$) which satisfy $|q(0)| = a(1 - e\cos\varepsilon\widehat{s}_0)$. To fix the choice of ε note that from (66) we have $\varepsilon = \langle q(0), p(0)\rangle/(\nu e \sin\widehat{s}_0)$, unless $\widehat{s}_0 = 0$ in which case ε is irrelevant. Set $s_0 = \varepsilon\widehat{s}_0$ and let $\tau = -\frac{1}{n}(s_0 - e\sin s_0)$. In words, we define τ as follows. If at $t = 0$ the particle is in the upper half of the ellipse, then τ is the first time *before* $t = 0$ when the particle passed through the periapse; otherwise it is the first time *on* or *after* $t = 0$ when the particle passes through the periapse.

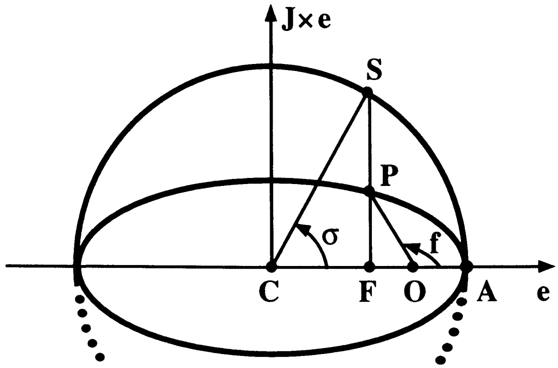

Figure 3.2. The eccentric anomaly.

To describe the geometric meaning of the eccentric anomaly s, consider the figure 3.2. Let O be the center of attraction, A the periapse and C the center of the ellipse of eccentricity e. The arrow on the ellipse indicates the direction of motion and P is the position of the particle on the ellipse with true anomaly f. Construct a line $\mathscr{L}$ through P which is perpendicular to the line CA. Project P parallel along $\mathscr{L}$ to the point S on the circle $\mathscr{C}$ with center C and radius equal to the distance $\overline{CA}$.

Claim: The eccentric anomaly s is the angle $\angle SCA$.

(3.14) **Proof**: Let $\sigma = \angle SCA$. From figure 3.2 we obtain $\overline{CS} = a$ and $\overline{CO} = ae$. Since $\overline{CF} = \overline{CO} + \overline{OF}$, we find that $a\cos\sigma = ae + |q|\cos f$. As the orbit is elliptical, $|q| = \frac{a(1-e^2)}{1+e\cos f}$. This may be rewritten as

$$|q| = a - e(ae + |q|\cos f) = a - ae\cos\sigma.$$

But

$$|q| = a - ae\cos s.$$

Hence $\cos s = \cos\sigma$. Since $s = 0$ when $\sigma = 0$, we obtain $\sigma = s$. □

As the point S traces out the circle $\mathscr{C}$ uniformly with speed n, the point P on the ellipse traces out the projection of an integral curve of the Kepler vector field in configuration space.

3.4 Regularization of the Kepler vector field

The incompleteness of the flow of the Kepler vector field may be removed by embedding it into a complete flow. This process is called regularization. On the subset of phase space where the Kepler Hamiltonian is negative, one can perform regularization in such a way that the embedding not only is symplectic and the resulting vector field is Hamiltonian but also that it linearizes and integrates the so(4)-action. That there is such a large hidden symmetry in the Kepler problem is quite remarkable because this symmetry does *not* arise from a lift of a symmetry on configuration space. We will *regularize all negative energy* Keplerian orbits at once using the *Ligon-Schaaf map* LS. We show that LS is the only symplectic mapping from the negative energy set $(\Sigma_- = \{(q,p) \in T_0\mathbf{R}^3 \,|\, H(q,p) < 0\}, \omega_3|\Sigma_-)$ to $(T^+S^3, \widetilde{\Omega}_4)$ which has the following properties:

1. It intertwines the Kepler and Delaunay vector fields, X_H and $X_{\widetilde{\mathcal{H}}}$, respectively;
2. It intertwines their so(4) momentum mappings $\mathcal{J}$ and $\widetilde{\mathcal{J}}$, respectively;
3. It maps Σ_- onto $T^+(S^3 - \mathrm{np})$, where $\mathrm{np} = (0,0,0,1)$.

We will show that the Ligon–Schaaf map is a diffeomorphism because it maps a one period of a parmetrized integral curve of the Kepler vector field of negative energy onto one period of a parmetrized integral curve of the Delaunay vector field on $T^+(S^3 - \mathrm{np})$.

We begin our search for the Ligon–Schaaf map by noting that the image of the so(4) momentum mapping $\mathcal{J}$ (48) of the Kepler vector field X_H on Σ_- is the *same* as the image of the so(4) momentum mapping $\widetilde{\mathcal{J}}$ (22) of the Delaunay vector field $X_{\widetilde{H}}$ on T^+S^3. Thus we suspect that the momentum maps are somehow related.

Claim: The smooth map

$$\Phi : \Sigma_- \subseteq T_0\mathbf{R}^3 \to T^+S^3 : (q,p) \to (x,y)$$

intertwines the so(4) momentum mappings $\mathcal{J}$ and $\widetilde{\mathcal{J}}$, that is, $\Phi^*\widetilde{\mathcal{J}} = \mathcal{J}$, if and only if

$$(x,y) = \Phi(q,p) = (A\sin\varphi + B\cos\varphi, -\nu A\cos\varphi + \nu B\sin\varphi), \tag{68}$$

where

$$\begin{aligned} A &= (\widetilde{A}, A_4) = \Big(\frac{q}{|q|} - \frac{1}{\mu}\langle q,p\rangle p, \frac{1}{\nu}\langle q,p\rangle\Big) && (69)\\ B &= (\widetilde{B}, B_4) = \Big(\frac{1}{\nu}|q|p, \frac{1}{\mu}\langle p,p\rangle|q| - 1\Big), && (70)\end{aligned}$$

$\nu = \mu/\sqrt{-2H}$, and $\varphi = \varphi(q,p)$ is an arbitrary smooth real valued function on Σ_-.

(3.15) **Proof**: Suppose that Φ intertwines the momentum maps. Write

$$x = (\widetilde{x}, x_4),\; y = (\widetilde{y}, y_4) \in \mathbf{R}^3 \times \mathbf{R} = \mathbf{R}^4.$$

From the definitions of the momentum mappings $\mathcal{J}$ and $\widetilde{\mathcal{J}}$ it is straightforward to check that $\Phi^*\widetilde{\mathcal{J}} = \mathcal{J}$ is equivalent to

$$\begin{aligned} \widetilde{x}\times\widetilde{y} &= q\times p && (71)\\ x_4\widetilde{y} - y_4\widetilde{x} &= \nu\Big(\frac{q}{|q|} - \frac{1}{\mu}p\times(q\times p)\Big) = Mq + Np, && (72)\end{aligned}$$

where

$$M = \nu\left(\frac{1}{|q|} - \frac{1}{\mu}\langle p, p\rangle\right) \quad \text{and} \quad N = \frac{\nu}{\mu}\langle q, p\rangle. \tag{73}$$

Suppose that $q \times p \neq 0$. From (71) it follows that $\widetilde{x}$ and $\widetilde{y}$ are orthogonal to $q \times p$. Since q and p are linearly independent, we obtain

$$\begin{cases} \widetilde{x} &= a\, q + b\, p \\ \widetilde{y} &= c\, q + d\, p \end{cases} \tag{74}$$

Because $q \times p = \widetilde{x} \times \widetilde{y} = (ad - bc)(q \times p)$, we find that

$$ad - bc = 1. \tag{75}$$

Substituting (74) into (72) and using the linear independence of q and p gives a set of linear equations for x_4 and y_4. Using (75) these equations may be solved to give

$$\begin{cases} x_4 &= a\, N - b\, M \\ y_4 &= c\, N - d\, M. \end{cases} \tag{76}$$

Since $(x, y) \in T^+S^3$,

$$\begin{cases} 1 &= \langle \widetilde{x}, \widetilde{x}\rangle + x_4^2 \\ 0 &= \langle \widetilde{x}, \widetilde{y}\rangle + x_4 y_4. \end{cases} \tag{77}$$

Substituting the expressions for $\widetilde{x}$ and $\widetilde{y}$ (74) and the expressions for x_4 and y_4 (76) into (77) gives

$$1 = a^2\langle q, q\rangle + \left(\frac{\nu}{\mu}\langle q, p\rangle\, a + \frac{\nu}{|q|}\, b\right)^2 \tag{78}$$

$$0 = ac\left(\langle q, q\rangle + \frac{\nu^2}{\mu^2}\langle q, p\rangle^2\right) + (ad + bc)\left(\frac{\nu^2}{\mu|q|}\langle q, p\rangle\right) + bd\,\frac{\nu^2}{\langle q, q\rangle}. \tag{79}$$

Here we have used the identities

$$\begin{cases} \langle q, q\rangle + N^2 &= \langle q, q\rangle + \frac{\nu^2}{\mu^2}\langle q, p\rangle^2 \\ \langle q, p\rangle - MN &= \frac{\nu^2}{\mu|q|}\langle q, p\rangle \\ \langle p, p\rangle + M^2 &= \dfrac{\nu^2}{\langle q, q\rangle}, \end{cases}$$

which follow from the definition of M and N (73) and the identity

$$\frac{1}{\mu}\langle p, p\rangle - \frac{1}{|q|} = \frac{1}{|q|} - \frac{\mu}{\nu^2},$$

which is a consequence of the definition of ν. Multiplying (78) by c and (79) by $-a$ and adding the resulting equations gives

$$c = -a\left(\frac{\nu^2}{\mu|q|}\langle q, p\rangle\right) - b\,\frac{\nu^2}{\langle q, q\rangle}, \tag{80}$$

after using (75). Similarly

$$d = a\left(\langle q, q\rangle + \frac{\nu^2}{\mu^2}\langle q, p\rangle^2\right) + b\left(\frac{\nu^2}{\mu|q|}\langle q, p\rangle\right). \tag{81}$$

All solutions of (78) are parametrized by

$$\begin{cases} a &= \dfrac{1}{|q|}\sin\varphi \\ b &= \dfrac{|q|}{\nu}\cos\varphi - \frac{1}{\mu}\langle q, p\rangle\sin\varphi, \end{cases} \tag{82}$$

where $\varphi = \varphi(q, p)$ is an arbitrary smooth function on Σ_-. Substituting (82) into (80) and (81) gives

$$\begin{cases} c &= -\dfrac{\nu}{|q|}\cos\varphi \\ d &= |q|\sin\varphi + \frac{\nu}{\mu}\langle q, p\rangle\cos\varphi. \end{cases} \tag{83}$$

Substituting (82) and (83) into (74) and (76) shows that the mapping Φ has the form (68) with the vectors A and B given by (69) and (70).

Since A and B are defined and continuous on all of Σ_-, the map Φ (68) is defined on *all* of Σ_- and not just on the open dense subset where $q \times p \neq 0$. A straightforward calculation shows that a map Φ of the form (68) intertwines the momentum mappings $\mathcal{J}$ and $\widetilde{\mathcal{J}}$. □

Corollary: Φ intertwines the Kepler and Delaunay Hamiltonians, that is, $\Phi^*\widetilde{\mathcal{H}} = H$ on Σ_-.

(3.16) **Proof**: We compute.

$$\begin{aligned} \widetilde{\mathcal{H}}(q, p) &= -\tfrac{1}{2}\frac{\mu^2}{\langle y, y\rangle} = -\tfrac{1}{2}\frac{\mu^2}{\nu^2\,\| - A\cos\varphi + B\sin\varphi\|^2} = -\tfrac{1}{2}\frac{\mu^2}{\nu^2}, \\ &\qquad \text{since } \langle A, A\rangle = \langle B, B\rangle = 1 \text{ and } \langle A, B\rangle = 0, \text{ (see below)} \\ &= H(q, p), \quad \text{by definition of } \nu. \end{aligned}$$

Now

$$\begin{aligned} \langle A, A\rangle &= 1 - \frac{2}{\mu|q|}\langle q, p\rangle^2 + \frac{1}{\mu^2}\langle q, p\rangle^2\langle p, p\rangle + \frac{1}{\nu^2}\langle q, p\rangle^2 \\ &= 1 + \frac{2}{\mu^2}\langle q, p\rangle^2\left(\tfrac{1}{2}\langle p, p\rangle - \frac{\mu}{|q|}\right) + \frac{1}{\nu^2}\langle q, p\rangle^2 \\ &= 1, \quad \text{since } \nu^2 = \mu^2/(-2H), \end{aligned}$$

$$\begin{aligned} \langle B, B\rangle &= \frac{|q|^2}{\nu^2}\langle p, p\rangle \frac{|q|^2}{\mu^2}\langle p, p\rangle^2 - \frac{2|q|}{\mu}\langle p, p\rangle + 1 \\ &= \frac{2|q|^2}{\nu^2}\langle p, p\rangle\left(\tfrac{1}{2}\langle p, p\rangle - \frac{\mu}{|q|} + \tfrac{1}{2}\frac{\mu^2}{\nu^2}\right) + 1 \\ &= 1, \quad \text{since } H = -\mu^2/2\nu^2, \end{aligned}$$

and

$$\langle A, B\rangle = \frac{1}{\nu}|q|\left\langle p, \frac{q}{|q|} - \frac{1}{\mu}\langle q, p\rangle p\right\rangle + \frac{1}{\nu}\langle q, p\rangle\left(\frac{1}{\mu}\langle p, p\rangle|q| - 1\right) = 0. \qquad \square$$

From now on we will use the notation Φ_φ for the map Φ (68) when we wish to emphasize the dependence of Φ on the parameter φ.

We now show that the map Φ is determined up to the flow Ψ_t (16) of the Delaunay vector field $X_{\widetilde{\mathcal{H}}}$ (13). More precisely we prove

Claim: Let Φ_φ and $\Phi_{\varphi'}$ be maps of the form (68) with free parameter φ and φ', respectively. Then for every $(q, p) \in \Sigma_-$

$$\Phi_{\varphi'}(q, p) = \Psi_{\frac{\nu^3}{\mu^2}\left(\varphi(q,p)-\varphi'(q,p)\right)}(\Phi_\varphi(q, p)) \tag{84}$$

(3.17) **Proof**: To verify (84) we write

$$\Phi_\varphi = \begin{pmatrix} \cos\varphi & \sin\varphi \\ \nu\sin\varphi & -\nu\cos\varphi \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} \text{ and } \Phi_{\varphi'} = \begin{pmatrix} \cos\varphi' & \sin\varphi' \\ \nu\sin\varphi' & -\nu\cos\varphi' \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix},$$

where $\varphi = \varphi(q, p)$, $\varphi' = \varphi'(q, p)$, and A and B are given by (69) and (70). Because the product $\begin{pmatrix} \cos(\varphi-\varphi') & \frac{1}{\nu}\sin(\varphi-\varphi') \\ -\nu\sin(\varphi-\varphi') & \cos(\varphi-\varphi') \end{pmatrix}$ and Φ_φ is equal to $\Phi_{\varphi'}$, we obtain (84) using the definition of Ψ_t. $\square$

Claim: The map Φ_φ (68) intertwines the Kepler and Delaunay vector fields, that is,

$$T\Phi_\varphi \circ X_H = X_{\widetilde{\mathcal{H}}} \circ \Phi_\varphi,$$

if and only if

$$\varphi = \varphi(q, p) = \frac{1}{\nu}\langle q, p\rangle - F(q, p), \tag{85}$$

where F is a smooth integral of X_H.

(3.18) **Proof**: Let $t \to (q(t), p(t))$ be an integral curve of the Kepler vector field X_H with negative energy. Then Φ_φ intertwines the Kepler and Delaunay vector fields if and only if $t \to (x(t), y(t)) = \Phi_\varphi(q(t), p(t))$ is an integral curve of the Delaunay vector field $X_{\widetilde{\mathcal{H}}}$. In other words, $t \to (x(t), y(t))$ satisfies

$$\frac{dx}{dt} = \frac{\mu^2}{\nu^4}y \tag{86}$$

$$\frac{dy}{dt} = -\frac{\mu^2}{\nu^2}x. \tag{87}$$

From (86) and (87) it follows that $L_{X_{\widetilde{\mathcal{H}}}}\langle y, y\rangle = 0$; while from

$$-\mu^2/\nu^2 = H(q, p) = \mathcal{H}(x, y) = -\mu^2/\langle y, y\rangle$$

it follows that $\langle y, y\rangle = \nu^2$. Differentiating

$$(x(t), y(t)) = (A\sin\varphi + B\cos\varphi, -\nu A\cos\varphi + \nu B\sin\varphi)$$

with respect to t and using

$$L_{X_H} A = \frac{\mu}{\nu |q|} B \quad \text{and} \quad L_{X_H} B = -\frac{\mu}{\nu |q|} A, \tag{88}$$

(which are proved in ((3.19)) below), gives

$$\begin{aligned} \frac{dx}{dt} &= \frac{dA}{dt} \sin\varphi + A\cos\varphi \frac{d\varphi}{dt} + \frac{dB}{dt}\cos\varphi - B\sin\varphi\frac{d\varphi}{dt} \\ &= \Big(\frac{\mu}{\nu|q|} - \frac{d\varphi}{dt}\Big)(B\sin\varphi - A\cos\varphi) = \frac{1}{\nu}\Big(\frac{\mu}{\nu|q|} - \frac{d\varphi}{dt}\Big)y \end{aligned} \tag{89}$$

and

$$\begin{aligned} \frac{dy}{dt} &= -\nu\frac{dA}{dt}\cos\varphi + \nu A\sin\varphi\frac{d\varphi}{dt} + \nu\frac{dB}{dt}\sin\varphi + \nu B\cos\varphi\frac{d\varphi}{dt} \\ &= -\nu\Big(\frac{\mu}{\nu|q|} - \frac{d\varphi}{dt}\Big)(A\sin\varphi + B\cos\varphi) = -\nu\Big(\frac{\mu}{\nu|q|} - \frac{d\varphi}{dt}\Big)x. \end{aligned} \tag{90}$$

Therefore $t \to (x(t), y(t))$ is an integral curve of $X_{\widetilde{\mathcal{H}}}$ if and only if

$$L_{X_H}\varphi - \frac{\mu}{\nu|q|} = -\frac{\mu^2}{\nu^3}. \tag{91}$$

The following calculation shows that $\varphi_{LS} = \frac{1}{\nu}\langle q, p\rangle$ satisfies (91):

$$\frac{d\varphi_{LS}}{dt} - \frac{\mu}{\nu|q|} = \frac{1}{\nu}\langle\frac{dq}{dt}, p\rangle + \frac{1}{\nu}\langle q, \frac{dp}{dt}\rangle - \frac{\mu}{\nu|q|} = \frac{2}{\nu}\Big(\tfrac{1}{2}\langle p, p\rangle - \frac{\mu}{|q|}\Big) = -\frac{\mu^2}{\nu^3},$$

since $-\frac{\mu^2}{2\nu^2} = \frac{1}{2}\langle p, p\rangle - \frac{\mu}{|q|}$. If φ is another smooth solution of (91), we see that

$$L_{X_H}(\varphi - \varphi_{LS}) = 0.$$

In other words, $\varphi(q, p) - \frac{1}{\nu}\langle q, p\rangle = -F(q, p)$ is a smooth integral of X_H on Σ_-. □

▷ The following argument proves (88).

(3.19) **Proof**: Let $t \to (q(t), p(t))$ be an integral curve of the Kepler vector field X_H. Let $A = A(q(t), p(t))$ and $B = B(q(t), p(t))$ where $A = (\widetilde{A}, A_4)$ and $B = (\widetilde{B}, B_4)$ are given by (69) and (70) respectively. To verify the first equation in (88) we compute.

$$\begin{aligned} \frac{d\widetilde{A}}{dt} &= -\frac{1}{|q|^3}\langle q, \frac{dq}{dt}\rangle q + \frac{1}{|q|}\frac{dq}{dt} - \frac{1}{\mu}\langle\frac{dq}{dt}, p\rangle p - \frac{1}{\mu}\langle q, \frac{dp}{dt}\rangle p - \frac{1}{\mu}\langle q, q\rangle\frac{dp}{dt} \\ &= -\mu\langle p, p\rangle p + \frac{2}{|q|}p \\ &= \frac{\mu}{\nu|q|}|q|p, \quad \text{since } -\tfrac{1}{2}\tfrac{\mu^2}{\nu^2} = \tfrac{1}{2}\langle p, p\rangle - \tfrac{\mu}{|q|} \\ &= \frac{\mu}{\nu|q|}\widetilde{B} \end{aligned}$$

and

$$\frac{dA_4}{dt} = \frac{1}{\nu}\Big[\langle\frac{dq}{dt}, p\rangle + \langle q, \frac{dp}{dt}\rangle\Big] = \frac{\mu}{\nu|q|}\Big(\frac{|q|}{\mu}\langle p, p\rangle - 1\Big) = \frac{\mu}{\nu|q|}B_4.$$

To verify the second equation in (88) we compute

$$\begin{aligned}\frac{d\widetilde{B}}{dt} &= \frac{1}{\nu}\Big[\frac{1}{|q|}\langle q, \frac{dq}{dt}\rangle p + |q|\frac{dp}{dt}\Big], \quad \text{since } L_{X_H}\nu = 0.\\ &= \frac{\mu}{\nu|q|}\Big(\langle q, p\rangle p - \frac{1}{|q|}q\Big) = -\frac{\mu}{\nu|q|}\widetilde{A}\end{aligned}$$

and

$$\begin{aligned}\frac{dB_4}{dt} &= \frac{1}{\mu}\Big[\frac{1}{|q|}\langle q, \frac{dq}{dt}\rangle + 2|q|\langle p, \frac{dp}{dt}\rangle\Big]\\ &= \frac{2\langle q, p\rangle}{\mu|q|}\Big(\tfrac{1}{2}\langle p, p\rangle - \frac{\mu}{|q|}\Big) = -\frac{\mu}{\nu|q|}\frac{1}{\nu}\langle q, p\rangle = -\frac{\mu}{\nu|q|}A_4. \quad \square\end{aligned}$$

If we choose the parameter φ in the mapping Φ_φ to be $\varphi_{LS} = \frac{1}{\nu}\langle q, p\rangle$, we call the resulting mapping LS the *Ligon-Schaaf map*.

Corollary: For every $(q, p) \in \Sigma_-$ we may write

$$\Phi_\varphi(q, p) = \Psi_{\frac{\nu^3}{\mu^2}F(q,p)}(LS(q, p)). \tag{92}$$

(3.20) **Proof**: Using ((3.17)) and $\varphi = \frac{1}{\nu}\langle q, p\rangle - F(q, p)$, we obtain (92). $\square$

Because $\frac{\nu^3}{\mu^2}F(q, p)$ is a smooth integral of the Kepler vector field X_H by ((3.13)) there is a smooth function $\check{F}$ on $C = \mathcal{J}(\Sigma_-) \subseteq \mathbf{R}^6$ such that $\frac{\nu^3}{\mu^2}F = \mathcal{J}^*\check{F}$. By ((3.15)), $(LS)^*\widetilde{\mathcal{J}} = \mathcal{J}$. Thus there is a smooth function $\widetilde{F} = \widetilde{\mathcal{J}}^*\check{F}$ on T^+S^3 such that $\frac{\nu^3}{\mu^2}F = (LS)^*\widetilde{F}$. Therefore we may rewrite (92) as

$$\Phi_\varphi = (LS)^*\widetilde{\Psi}, \tag{93}$$

where $\widetilde{\Psi}$ is the time one map of the flow of the rescaled Delaunay vector field $\widetilde{F}\cdot X_{\widetilde{H}}$. In other words, $\widetilde{\Psi}(x, y) = \Psi_{\widetilde{F}(x,y)}(x, y)$ for every $(x, y) \in \operatorname{im} LS(\Sigma_-)$. Note that the function $\widetilde{F}$ is a smooth integral of $X_{\widetilde{\mathcal{H}}}$. To see this we compute

$$\begin{aligned}L_{X_{\widetilde{\mathcal{H}}}}\widetilde{F}\big(LS(q, p)\big) &= d\widetilde{F}\big(LS(q, p)\big)X_{\widetilde{\mathcal{H}}}\big(LS(q, p)\big)\\ &= d\widetilde{F}\big(LS(q, p)\big)T_{(q,p)}LS\,X_H(q, p)\\ &\qquad \text{since } LS \text{ intertwines } X_H \text{ and } X_{\widetilde{\mathcal{H}}}\\ &= d\Big(\frac{\nu^3}{\mu^2}F(q, p)\Big)X_H(q, p),\\ &\qquad \text{since } \nu^3/\mu^2 = (LS)^*\widetilde{F}\\ &= 0, \qquad \text{since } \nu \text{ and } F \text{ are integrals of } X_H.\end{aligned}$$

We now prove

Claim: If $\Phi_\varphi : (\Sigma_-, \widetilde{\omega}_3) \to (T^+S^3, \widetilde{\Omega}_4)$ is symplectic, in addition to intertwining the momentum mappings $\mathcal{J}$ and $\widetilde{\mathcal{J}}$ and the Kepler and Delaunay vector fields, then the smooth integral $\widetilde{F}$ of the Delaunay vector field $X_{\widetilde{\mathcal{H}}}$ is a smooth function of $\widetilde{\mathcal{H}}$.

(3.21) **Proof**: Because $\widetilde{\Psi}$ is the time one map of the rescaled Delaunay vector field $\widetilde{F} \cdot X_{\widetilde{\mathcal{H}}}$, using ((3.18)) we obtain $\widetilde{\Psi}^*(\widetilde{\Omega}_4) = \widetilde{\Omega}_4 + d\widetilde{F} \wedge d\widetilde{\mathcal{H}}$. From (93) it follows that

$$\Phi_\varphi^*(\widetilde{\Omega}_4) = (LS)^*(\widetilde{\Psi}^*(\widetilde{\Omega}_4)) = (LS)^*(\Omega_4) + (LS)^*(d\widetilde{F} \wedge d\widetilde{\mathcal{H}}). \tag{94}$$

By hypothesis, Φ_φ is symplectic for every choice of smooth φ. Therefore LS is symplectic. Hence (94) becomes $\widetilde{\Omega}_4 = \widetilde{\Omega}_4 + (LS)^*(d\widetilde{F} \wedge d\widetilde{\mathcal{H}})$, that is, $(LS)^*(d\widetilde{F} \wedge d\widetilde{\mathcal{H}}) = 0$. Because LS is symplectic, $T_{(q,p)}LS$ is injective for every $(q, p) \in \Sigma_-$. Hence $d\widetilde{F} \wedge d\widetilde{\mathcal{H}} = 0$. This implies that the 1-form $\widetilde{F} d\widetilde{\mathcal{H}}$ is locally exact, and since T^+S^3 is simply connected, we may conclude that $\widetilde{F} d\widetilde{\mathcal{H}}$ is globally exact. Since $\widetilde{F}$ and $\widetilde{H}$ are both SO(4)-invariant, and SO(4) acts transitively on energy surfaces, we are reduced to understanding

$$\widetilde{F} d\widetilde{\mathcal{H}} = d\widetilde{K} \tag{95}$$

for some smooth function $\widetilde{K}$. Everything in (95) may be thought of as a smooth function of $|p|$, since the orbit space $T^+S^3/\operatorname{SO}(4)$ is $\{|p| > 0\}$. Since $\frac{d\widetilde{\mathcal{H}}}{d|p|} \neq 0$, we may write (95) as $\widetilde{F} = \frac{d\widetilde{K}}{d\widetilde{\mathcal{H}}}$. Consequently, $\widetilde{F}$ is a function of $\widetilde{\mathcal{H}}$, that is,

$$\widetilde{F} = \widetilde{\mathcal{G}} \circ \widetilde{\mathcal{H}}, \tag{96}$$

where $\widetilde{\mathcal{G}} = \frac{d\widetilde{K}}{d\widetilde{\mathcal{H}}}$, on *all* of T^+S^3. □

We now prove the following characterization of the Ligon–Schaaf map.

Theorem: Suppose that $\Phi_\varphi(\Sigma_-) = T^+S^3_{\mathrm{np}}$, in addition to

$$\Phi_\varphi^* \widetilde{\mathcal{J}} = \mathcal{J}, \quad T\Phi_\varphi \circ X_H = X_{\widetilde{\mathcal{H}}} \circ \Phi_\varphi, \quad \text{and} \quad \Phi_\varphi^* \widetilde{\Omega}_4 = \widetilde{\omega}_3.$$

Then $\Phi_\varphi = LS$.

(3.22) **Proof**: We now show that these hypotheses are not vacuous. From the facts that $\Phi_\varphi(\Sigma_-) = T^+S^3_{\mathrm{np}} = LS(\Sigma_-)$ (see ((3.23)) below) and $\Phi_\varphi = \widetilde{\Psi} \circ LS$ (93) it follows that $\widetilde{\Psi}$ maps $T^+S^3_{\mathrm{np}}$ onto itself. Hence $\widetilde{\Psi}$ maps $NP = T^+S^3 - T^+S^3_{\mathrm{np}}$ diffeomorphically onto itself. Note that NP is the fiber of the bundle

$$\tau : T^+S^3 \subseteq \mathbf{R}^4 \to S^3 \subseteq \mathbf{R}^4 : (x, y) \to x$$

over np $= (0, 0, 0, 1)$. For some $\nu > 0$ suppose that $g = \widetilde{\mathcal{G}}(-\mu^2/2\nu^2)$ is nonzero. Here $\widetilde{\mathcal{G}}$ is the smooth function constructed in the proof of ((3.21)). Then the flow Ψ_{gt} has no fixed points on $\widetilde{\mathcal{H}}^{-1}(-\mu^2/2\nu^2)$ because g is nonzero and the Delaunay vector field has no equilibrium points on $\widetilde{\mathcal{H}}^{-1}(-\mu^2/2\nu^2)$. This contradicts the fact that $\widetilde{\Psi}$ maps the set

$NP \cap \widetilde{\mathcal{H}}^{-1}(-\mu^2/2\nu^2)$ into itself. Therefore $g = 0$. Hence $\widetilde{\mathcal{G}}$ is identically zero. From (96) we see that the function $\widetilde{F}$ is identically zero. By definition $\widetilde{\Psi} = \mathrm{id}$. From (93) it follows that $\Phi_\varphi = LS$. □

▷ To prove the converse to the above theorem we need only show that

1. $LS(\Sigma_-) = T^+S^3_{\mathrm{np}}$
2. $(LS)^*(\widetilde{\Omega}_4) = \widetilde{\omega}_3$,

because by definition LS intertwines the momentum maps $\mathcal{J}$ and $\widetilde{\mathcal{J}}$ and by ((3.18)) it also intertwines the vector fields X_H and $X_{\widetilde{\mathcal{H}}}$.

(3.23) **Proof**:
1). Suppose that $(x, y) = LS(q, p)$ for some $(q, p) \in \Sigma_-$ and that $x = \mathrm{np}$. Then

$$1 = x_4 = \frac{1}{\nu}\langle q, p\rangle \sin\varphi + \left(\frac{1}{\mu}\langle p, p\rangle |q| - 1\right)\cos\varphi. \tag{97}$$

Since $(x, y) \in T^+S^3$, it follows that $\langle x, y\rangle = 0$. Therefore

$$0 = \frac{1}{\nu}\, y_4 = -\frac{1}{\nu}\langle q, p\rangle \cos\varphi + \left(\frac{1}{\mu}\langle p, p\rangle |q| - 1\right)\sin\varphi. \tag{98}$$

Multiplying (97) by $\sin\varphi$ and adding the result to $-\cos\varphi$ times (98) gives

$$\sin\varphi = \frac{1}{\nu}\langle q, p\rangle = \varphi, \tag{99}$$

by definition of LS. Since $1 = \langle A, A\rangle$, we find that $\varphi \in [0, 1]$ using (69) and (99). Hence $\varphi = 0$ is the only solution of (99). Therefore (97) becomes $1 = \frac{1}{\mu}|q|\langle p, p\rangle - 1$, which is equivalent to $\frac{1}{2}\langle p, p\rangle - \frac{\mu}{|q|} = 0$. In other words, $(q, p) \notin \Sigma_-$. But this contradicts the hypothesis. Therefore $LS(\Sigma_-) \subseteq T^+S^3_{\mathrm{np}}$. Actually equality holds. To see this look at the proof of ((3.15)). We see that when $\mathbf{J} \neq 0$, the Keplerian ellipse with data h, $\mathbf{J}$, $\widetilde{\mathbf{e}}$ is mapped by LS to an orbit of the Delaunay vector field which is a great circle which does not pass through NP. When $\mathbf{J} = 0$ a linear Keplerian orbit corresponds to a great circle which passes through np when $q = 0$, (see ((3.24))). □

2). This follows by taking the exterior derivative of

$$(LS)^*\langle y, dx\rangle = -\langle q, dp\rangle - d\langle q, p\rangle, \tag{100}$$

which we establish below, (see (104)). First we factor the map LS into the composition of the mapping

$$\begin{aligned} S : \Sigma_- \subseteq T_0\mathbf{R}^3 \to \mathbf{R} \times T_1S^3 : (q, p) \to (\varphi, A, B) = \\ = \left(\nu^{-1}\langle q, p\rangle, \left(\tfrac{q}{|q|} - \mu^{-1}\langle q, p\rangle\, p, \nu^{-1}\langle q, p\rangle\right), \left(\nu^{-1}|q|\, p, \mu^{-1}|q|\,\langle p, p\rangle - 1\right)\right) \end{aligned}$$

followed by the mapping

$$\begin{aligned} L : \mathbf{R} \times T_1S^3 \to T^+S^3 : \\ (\varphi, A, B) \to (x, y) = \big((\sin\varphi)A + (\cos\varphi)B, -\nu(\cos\varphi)A + \nu(\sin\varphi)B\big). \end{aligned}$$

Below we show that

$$L^*\langle y, dx\rangle = -\nu(d\varphi + \langle A, dB\rangle) \tag{101}$$

and

$$\begin{aligned} S^*(\nu\langle A, dB\rangle) &= -\nu\langle q, p\rangle d(\nu^{-1}) + \langle q, dp\rangle & (102)\\ S^*(\nu\, d\varphi) &= \nu\, d\Big(\nu^{-1}\langle q, p\rangle\Big). & (103)\end{aligned}$$

Consequently,

$$\begin{aligned}(LS)^*\langle y, dx\rangle &= S^*(L^*(\langle y, dx\rangle)) = -S^*(\nu d\varphi + \nu\langle A, dB\rangle), \quad \text{by (101)}\\ &= -S^*(\nu d\varphi) + S^*(\nu\langle A, dB\rangle)\\ &= -\nu d\big(\nu^{-1}\langle q, p\rangle\big) + \nu\langle q, p\rangle\, d(\nu^{-1}) + \langle q, dp\rangle, \quad \text{by (102) and (103)}\\ &= -\langle q, dp\rangle - d\langle q, p\rangle. \end{aligned} \tag{104}$$

This proves (100).

We need only verify (101)–(103). Equation (103) follows immediately because $\varphi = \frac{1}{\nu}\langle q, p\rangle$. Equation (101) follows from the definition of L and

$$\langle A, dA\rangle = \langle B, dB\rangle = \langle A, dB\rangle + \langle B, dA\rangle = 0,$$

which is an immediate consequence of $\langle A, A\rangle = \langle B, B\rangle = 1$ and $\langle A, B\rangle = 0$. It remains to prove (102). From the definition of the mapping S we find that

$$S^*(\langle A, dB\rangle) =$$
$$\Big[\mu^{-1}\nu^{-2}\langle q, p\rangle|q|\Big(\langle p, p\rangle - \tfrac{\mu}{|q|}\Big)d\nu + \mu^{-1}\nu^{-1}\langle q, p\rangle|q|\Big(\langle p, dp\rangle + \tfrac{\mu}{|q|^3}\langle q, dq\rangle\Big)\Big] + \nu^{-1}\langle q, dp\rangle.$$

Differentiating $\nu = \mu(-2H)^{-1/2}$ gives $d\nu = \frac{\nu}{(-2H)^{3/2}}\, dH$. Also

$$dH = \langle p, dp\rangle + \frac{\mu}{|q|^3}\langle q, dq\rangle.$$

Hence the terms in the square brackets in the right hand side of the equation for $S^*(\langle A, dB\rangle)$ can be written as

$$\mu^{-1}\nu^{-1}\langle q, p\rangle\left[\frac{\mu - |q|\langle p, p\rangle + 2H|q|}{2H}\right]dH = \nu^{-2}\langle q, p\rangle d\nu,$$

because $\langle p, p\rangle = 2(H + \frac{\mu}{|q|})$ and $d\nu = \frac{\nu}{(-2H)^{3/2}}\, dH$. Therefore

$$S^*(\langle A, dB\rangle) = \nu^{-2}\langle q, p\rangle d\nu + \nu^{-1}\langle q, dp\rangle = -\langle q, p\rangle d(\nu^{-1}) + \nu^{-1}\langle q, dp\rangle.$$

This proves (102) and shows that LS is symplectic. □

▷ We now show that the Ligon–Schaaf map is a diffeomorphism.

(3.24) **Proof**: For every $c \in C$, the fibers $\mathcal{J}^{-1}(c)$ and $\widetilde{\mathcal{J}}^{-1}(c)$ are unique oriented orbits of the Kepler and Delaunay vector fields respectively. Since $(LS)^*\widetilde{\mathcal{J}} = \mathcal{J}$, LS maps a bounded

Kepler orbit onto an orbit of the Delaunay vector field. Thus we need only show that on orbits, LS is a diffeomorphism. Hence it suffices to show that LS preserves the periods of corresponding orbits. When $\mathbf{J} \neq 0$ and $H < 0$, the period of an elliptical Keplerian orbit is $2\pi\nu^3/\mu^2$. Since the period of the corresponding orbit of the Delaunay vector field is also $2\pi\nu^3/\mu^2$, LS preserves periods. When $\mathbf{J} = 0$ and $H < 0$, the Keplerian orbit projects to a linear orbit traversed twice. Hence the period of such a Keplerian orbit is *twice* the time it takes to reach the origin starting with momentum 0. Using (45) we see that again the period is given by $2\pi\nu^3/\mu^2$. The corresponding Delaunay orbit is periodic of period $2\pi\nu^3/\mu^2$. Therefore LS is a diffeomorphism. □

We now put the preceding arguments into perspective. Geometrically we think of T^+S^3 as a regularized model for the negative energy subset Σ_- of the phase space of the Kepler problem. On T^+S^3 the SO(4) symmetry $\widehat{\Phi}$ (21) is *globally* defined and its orbits

$$T_{\mu/\sqrt{-2h}}S^3 = \left\{ (x, y) \in T^+S^3 \middle| \langle y, y\rangle = -\frac{\mu^2}{2h} \right\}$$

are the regularization of the energy surface $H^{-1}(h) \subseteq \Sigma_-$ of the Kepler problem. Let

$$\mathscr{C} = \left\{ (x, y) \in T^+S^3 \middle| x_4 = 1 \right\}.$$

From ((3.24)) we see that $LS^{-1}(\mathscr{C})$ is the collision set

$$\left\{ (q, p) \in T\mathbf{R}^3 \middle| q = 0 \right\}$$

of the Kepler problem. Since $T_{\mu/\sqrt{-2h}}S^3 \cap \mathscr{C}$ is nonempty for every $h < 0$, we conclude that the SO(4) symmetry is *not globally* defined for the Kepler problem. On the other hand the flow of the Delaunay vector field $X_{\widetilde{\mathscr{H}}}$ is globally defined on T^+S^3.

Consider the function

$$\widetilde{J}(x, y) = \sum_{1 \le i < j \le 3} (x_i y_j - x_j y_i)^2 | T^+S^3 = \langle \widetilde{x} \times \widetilde{y}, \widetilde{x} \times \widetilde{y} \rangle.$$

Then $\widetilde{J}$ is an integral of the geodesic flow on T^+S^3 and hence is an integral of the Delaunay vector field $X_{\widetilde{\mathscr{H}}}$.

Claim: $\widetilde{J}^{-1}(0)$ is the set of all integral curves of $X_{\widetilde{\mathscr{H}}}$ which pass through $\mathscr{C}$.

(3.25) **Proof**: As all orbits of $X_{\widetilde{\mathscr{H}}}$ when projected onto S^3 are great circles, each projected orbit intersects the equator $\{x_4 = 0\} \cap S^3$. Hence every orbit of $X_{\widetilde{\mathscr{H}}}$ intersects the equator at some point $(\widetilde{x}, 0, \widetilde{y}, y_4)$. If $\widetilde{y} = 0$ then $\widetilde{J} = 0$. As seen from the formula (16) for the flow of $X_{\widetilde{\mathscr{H}}}$, the $X_{\widetilde{\mathscr{H}}}$ orbit through $(\widetilde{x}, 0, 0, y_4)$ passes through $\{x_4 = 1\}$. Now suppose that $\widetilde{y} \neq 0$. $\widetilde{J} = 0$ if and only if $\widetilde{x} \times \widetilde{y} = 0$, that is, $\widetilde{x}$ and $\widetilde{y}$ are linearly dependent. But this contradicts the fact that $0 = \langle x, y\rangle = \langle \widetilde{x}, \widetilde{y}\rangle$, and both $\widetilde{x}$ and $\widetilde{y}$ are nonzero. Therefore $\widetilde{y} = 0$. □

From ((3.15)) we see that $(LS)^{-1}(\widetilde{J}^{-1}(0)) = \mathscr{J}^{-1}(0) \cap \Sigma_-$. An integral curve γ of $X_{\widetilde{\mathscr{H}}}$ which lies on $\mathscr{J}^{-1}(0)$ is the image under LS of an integral curve of the Kepler vector field

X_H on Σ_- with zero angular momentum. In particular the projection of $(LS)^{-1}(\gamma - \mathrm{np})$ is a ray from the origin. The particle moves on this ray being reflected at the origin. This is the regularized picture of collision in the Kepler problem. This regularization is geometrically faithful as the mapping LS is an open dense embedding of $(H, \Sigma_-, \widetilde{\omega}_3)$ into $(\widetilde{\mathcal{H}}, T^+S^3, \widetilde{\Omega}_4)$ with $(LS)^*\widetilde{\mathcal{H}} = H$.

4 Exercises

1. (The virial group.) For the Kepler problem show that the change of variables $(t, q, p) \to (a^3 t, a^2 q, a^{-1} p)$ defines an action of the multiplicative group $\mathbf{R}^*$ and leaves the equations of motion invariant. Deduce the period energy relation (Kepler's third law) from this symmetry.

2. (sl(2, **R**) and the Kepler problem.) For the Kepler problem with rotationally symmetric Hamiltonian $H = \frac{1}{2} p \cdot p - \frac{1}{|q|}$ let $j = q \times p$, $x = q \cdot q$, $y = p \cdot p$, and $z = q \cdot p$.

   (a). Show that the functions x, y, z Poisson commute with the components of j. Moreover, show that the Poisson brackets of x,y, and z define a representation of sl(2, **R**). Conclude that so(3) and sl(2, **R**) form a dual pair in the Lie algebra of homogeneous quadratic polynomials.

   (b). In xyz-space draw that level sets of $j^2 =$ const. for different values of the constant including zero. These are models for the SO(3) reduced space.

   (c). Draw the intersections of the $h =$ const. surfaces with a given $j =$ const. surface to see the integral curves of the reduced dynamics.

   (d). Show that the level sets $j^2 =$ const. are symplectic leaves for the Poisson manifold $\mathrm{sl}(2, \mathbf{R}) = \mathbf{R}^3$ with coordinates (x, y, z)

3. (Center of mass and the two body problem.) (a). For the two body problem in space show that regular reduction by the translation group can be interpreted as passing to a center of mass frame. Do the reduction of the translation and rotational symmetries in one step by using the Euclidean group $E(3)$.

   (b). Consider the spherical analogue of the planar two body problem. This is the motion of two particles connected by a spring constrained to move on the surface of a 2-sphere. The rotation group SO(3) is an obvious symmetry group of the problem, as compared to the Euclidean group $E(2)$ for the planar problem. Construct all the SO(3) reduced spaces. Show that there is *no notion* of a center of mass *frame*.

   (c)*. Is the spherical two body problem Liouville integrable?

4. (Kepler's equation.) Consider Kepler's equation

$$t = s - e \sin s = f(s). \tag{105}$$

   (a). For $|e| < 1$ show that f is a one to one real analytic diffeomorphism of **R**. Expand $f(s)$ in a power series: $t = \sum_{k=0}^{\infty} a_k s^k$ and invert: $s = f^{-1}(t) =$

$\sum_{k=0}^{\infty} c_k t^k$. Estimate the radius of convergence of the series for f^{-1}. (Hint: see the discussion in Markushevich [145].)

(b). Another way to invert Kepler's equation goes as follows. Suppose we are given a function $t(s)$ in the form $t = F(s) = s - \varepsilon f(s)$, where $f(0) = 0$, $f(s+2\pi) = f(s)$, and $|\varepsilon f'(s)| < 1$. In other words, εf is a "small" periodic perturbation of the identity. Since $|\varepsilon f'(s)| < 1$, we have $\frac{df}{ds} = 1 - \varepsilon f'(s) > 0$. Thus by the inverse function theorem there is a well-defined inverse function $s(t)$. Furthermore, it is of the form $s = s(t) = t + g_\varepsilon(t)$, where $g_\varepsilon(t)$ is 2π-periodic with $g_\varepsilon(0) = 0$. We would like to expand g_ε in a Fourier series: $g_\varepsilon(t) = \frac{1}{2} a_0 + \sum_{k=1}^{\infty}(a_k \cos kt + b_k \sin kt)$, where

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} g_\varepsilon(t)\, dt, \quad a_k = \frac{1}{\pi} \int_0^{2\pi} g_\varepsilon(t) \cos kt\, dt, \text{ and } b_k = \frac{1}{\pi} \int_0^{2\pi} g_\varepsilon(t) \sin kt\, dt.$$

The difficulty with this direct calculation is that we do not know the function $g_\varepsilon(t)$. We circumvent this by expressing $g_\varepsilon(t)$ in terms of s, namely, $g_\varepsilon(t) = s - F(s) = \varepsilon f(s)$, where s and t are related by $t = F(s)$. Now

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_0^{2\pi} g_\varepsilon(t)\, dt \\ &= \frac{1}{\pi} \int_0^{2\pi} g_\varepsilon(t(s)) \frac{dt}{ds}\, ds \\ &= \frac{1}{\pi} \int_0^{2\pi} \varepsilon f(s)\big(1 - \varepsilon f'(s)\big)\, ds, \end{aligned}$$

using the change of variable $f(s)$. A similar formula holds for a_k and b_k. Show that for the case of Kepler's equation, where $f(s) = e \sin s$, we have $a_k = 0$ for all $k \geq 0$. Change variable from t to s and use the fact that

$$J_k(x) = \int_0^{\pi} \cos(ks - x \sin s)\, ds$$

to conclude that $b_k = \frac{2}{k} J_k(ke)$.

(c). Show that $y(x) = J_k(x)$ satisfies

$$x^2 y'' + xy' + (x^2 - k^2)y = 0,$$

which is Bessel's equation.

5. (a). Let $G(2,4)$ be the space of all 2-planes in $\mathbf{R}^4$. Show that the orthogonal group $\mathrm{O}(4)$ acts transitively on $G(2,4)$ and that the subgroup of all elements of $\mathrm{O}(4)$, which leave the 2-plane span$\{e_1, e_2\}$ fixed, is isomorphic to $\mathrm{O}(2) \times \mathrm{O}(2)$. Deduce that $G(2,4)$ is diffeomorphic to the orbit space $\mathrm{O}(4)/(\mathrm{O}(2) \times \mathrm{O}(2))$.

   (b). Let $\widetilde{G}(2,4)$ be the space of all *oriented* 2-planes in $\mathbf{R}^4$. Show that the group $\mathrm{SO}(4)$ acts transitively on $\widetilde{G}(2,4)$ and that the subgroup of all elements of $\mathrm{SO}(4)$, which leave the oriented 2-plane span$\{e_1, e_2\}$ fixed, is isomorphic to $\mathrm{SO}(2) \times \mathrm{SO}(2)$. Deduce that $\widetilde{G}(2,4)$ is diffeomorphic to the orbit space $\mathrm{SO}(4)/(\mathrm{SO}(2) \times \mathrm{SO}(2))$.

   (c). Show that $\widetilde{G}(2,4)$ is diffeomorphic to $S^2 \times S^2$ and hence is simply connected. Deduce that $\widetilde{G}(2,4)$ is a 2-fold covering space of $G(2,4)$.

6. (a). Construct an isomorphism between the Lie algebra so(4) and so(3) $\times$ so(3).

   (b). Show that the corresponding Lie-Poisson algebras are isomorphic.

   (c). Write out Hamilton's equations on the Lie-Poisson algebra corresponding to so(3) $\times$ so(3).

7. (Time rescaling). Consider the Hamiltonian system (H, M, ω) with h a regular value of the Hamiltonian H. Let $f : M \to \mathbf{R}_{>}$ be a positive smooth function and define a new Hamiltonian $K : M \to \mathbf{R}$ by $K(m) = f(m)(H(m) - h)$. Show that $X_K | K^{-1}(0) = f\left(X_H | H^{-1}(h)\right)$. Show that the integral curves of X_K on $K^{-1}(0)$ are an orientation preserving time rescaling of the integral curves of X_H on $H^{-1}(h)$ obtained by defining a new time variable s by $\frac{ds}{dt} = \frac{1}{f}$.

8. (Souriau's linearization and regularization.) In the Kepler problem

$$\begin{aligned} \dot{q} &= p \\ \dot{p} &= -\frac{1}{r^3} q, \quad r = |q|, \end{aligned} \tag{106}$$

   let $H = \frac{1}{2}\langle p, p\rangle - \frac{1}{r}$ be the Hamiltonian and define a new time variable s by

$$s = \langle q, p\rangle - 2ht.$$

   (a). Show that $\frac{ds}{dt} = \frac{1}{r}$. Thus s is the eccentric anomaly.

   (b). Define a 4-vector ξ by $\xi = \begin{pmatrix} t \\ q \end{pmatrix}$. Let $\Xi = \mathrm{col}(\xi, \xi', \xi'', \xi''')$ be a 4×4 matrix, where $'$ is differentiation with respect to s. Show that Ξ satisfies the *linear* differential equation

$$\Xi' = A\,\Xi \tag{107}$$

   where $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2h & 0 & 0 \end{pmatrix}$.

   (c). Solve (107) and thus find $\xi(s)$. Note that because $\xi(s)$ is defined for *all* s and hence for all t by Kepler's equation, it follows that the Kepler problem has been regularized.

9. (Bacry-Györgyi variables and the conformal group.) Using the same notation in the Kepler problem as in exercise 8, set $\alpha = \sqrt{-2h}$, $P = \begin{pmatrix} t''' \\ \alpha^{-1} q''' \end{pmatrix}$ and $Q = \begin{pmatrix} \alpha t'' \\ q'' \end{pmatrix}$. Here we are confining ourselves to the case of bounded motions, namely, $h < 0$.

   (a). Show that $P^t P = Q^t Q = 1$ and $P^t Q = 0$.

   (b). Let ζ be the 6×6 matrix

$$\begin{pmatrix} QP^t - PQ^t & P & Q \\ P^t & 0 & 1 \\ Q^t & -1 & 0 \end{pmatrix}$$

(c). Show that $\zeta^2 = 0$.

(d). Show that the components of ζ satisfy the Poisson bracket relations for the Lie algebra so(4, 2).

(e). Show that the map from the regularized phase space of the negative energy orbits $(q, p, h) \to \zeta$ is a symplectic diffeomorphism if we equip the SO(4, 2)-coadjoint orbit through ζ with the symplectic structure given in appendix 1 section 1 example 2. The tricky part of this is deciding which component of the variety $\zeta^2 = 0, \zeta \neq 0$ in so$(4, 2)^*$ you need to map to.

10. (Levi-Civita regularization). (a). Let $\mathbf{R}_0^2 = \mathbf{R}^2 - \{0\}$. On $T^*\mathbf{R}_0^2 = \mathbf{R}_0^2 \times \mathbf{R}^2$ with coordinates (x, y) and symplectic form $\omega = \sum_{i=1}^{3} dx_i \wedge dy_i$ consider the Kepler Hamiltonian

$$H(x, y) = \tfrac{1}{2}\,(y_1^2 + y_2^2) - \frac{1}{\sqrt{x_1^2 + x_2^2}}. \tag{108}$$

Identify $\mathbf{R}_0^2$ with $\mathbf{C}_0 = \mathbf{C} - \{0\}$ and $T^*\mathbf{R}_0^2$ with $T^*\mathbf{C}_0 = \mathbf{C}_0 \times \mathbf{C}$. Introduce complex coordinates $q = x_1 + i\,x_2$ and $p = y_1 + i\,y_2$ on $T^*\mathbf{C}_0$. Show that $\omega = \mathrm{Re}(dq \wedge d\overline{p})$ and that the Kepler Hamiltonian becomes

$$H(q, p) = \tfrac{1}{2}\,|p|^2 - \frac{1}{|q|}. \tag{109}$$

(b). Using exercise 7 and the time rescaling $\frac{ds}{dt} = \frac{k}{2|q|}$ show that the integral curves of X_H on the level set $H^{-1}(-k^2/2)$ are a time reparametrization of the integral curves of the vector field $X_{\widetilde{K}}$ on the level set $\widetilde{K}^{-1}(0)$ where

$$\widetilde{K}(q, p) = \frac{2\,|q|}{k}\left(\tfrac{1}{2}\,|p|^2 - \frac{1}{|q|} + \frac{k^2}{2}\right) = \frac{1}{k}\,|q|\,|p|^2 + k\,|q| - \frac{2}{k}. \tag{110}$$

(c). Define the Levi-Civita map

$$\mathscr{L} : T^*\mathbf{C}_0 \to T^*\mathbf{C}_0 : (u, v) \to (q, p) = \Big(\frac{1}{2k^2}\,u^2,\ k\,\frac{v}{\overline{u}}\Big). \tag{111}$$

Show that $\mathscr{L}$ has the following properties:

1). $\mathscr{L}$ is a smooth two to one surjective submersion with $\mathscr{L}(-u, -v) = \mathscr{L}(u, v)$.

2). $\mathscr{L}^*(\mathrm{Re}\, q\, d\overline{p}) = \mathrm{Re}\,(u\, d\overline{v})$. Hence $\mathscr{L}$ is symplectic.

3). The Hamiltonian

$$K(u, v) = (\mathscr{L}^*\widetilde{K})(u, v) = \tfrac{1}{2}\,|v|^2 + \tfrac{1}{2}\,|u|^2 - \frac{2}{k} \tag{112}$$

is defined on $K^{-1}(0)$ which is a 3-sphere centered at the origin and having radius $2/\sqrt{k}$. Since $K^{-1}(0)$ is compact, all the integral curves of X_K on $K^{-1}(0)$ are defined for *all* time. Thus K is the Levi-Civita regularization of the Kepler Hamiltonian for negative energy orbits. Note that up to an additive constant, K is the harmonic oscillator Hamiltonian.

(d). The Levi-Civita map $\mathcal{L}$ is *not* an equivalence between the Hamiltonian systems $(K, T^*\mathbf{C}_0, \omega)$ and $(\widetilde{K}, T^*\mathbf{C}_0, \omega)$, because it is *not* a diffeomorphism. Show that that vector fields X_K on $K^{-1}(0)$ and $X_{\widetilde{K}}$ on $\widetilde{K}^{-1}(0)$ are $\mathcal{L}$-related, that is, $T\mathcal{L}\circ X_K = X_{\widetilde{K}}\circ\mathcal{L}$. Thus the image of an integral curve of X_K on $K^{-1}(0)$ under the Levi-Civita map $\mathcal{L}$ is an integral curve of $X_{\widetilde{K}}$ on $\widetilde{K}^{-1}(0)$.

(e). On $T^*\mathbf{C}_0$ define a $\mathbf{Z}_2$-action generated by $(u, v) \to (-u, -v)$. Show that this action is free, preserves the symplectic form ω, and preserves the Hamiltonian K. Thus there is an induced Hamiltonian $\mathcal{K}$ on $(T^*\mathbf{C}_0/\mathbf{Z}_2, \omega)$. Since the map $\mathcal{L}$ is invariant under the $\mathbf{Z}_2$-action, it induces an equivalence between the Hamiltonian systems $(\mathcal{K}, T^*\mathbf{C}_0/\mathbf{Z}_2, \omega)$ and $(K, T^*\mathbf{C}_0, \omega)$. Thus the regularized energy surface $H^{-1}(-k^2/2)$ of the Kepler Hamiltonian is $\mathcal{K}^{-1}(0) = S^3_{2/\sqrt{k}}/\mathbf{Z}_2$, which is real projective three space $\mathbf{RP}^3$.

11. (Moser regularization). (a). Let $\mathbf{R}^3_0 = \mathbf{R}^3 - \{0\}$. On $T^*\mathbf{R}^3_0 = \mathbf{R}^3_0 \times \mathbf{R}^3$ with coordinates (x, y) and symplectic form $\omega = \sum_{i=1}^3 dx_i \wedge dy_i$ consider the Kepler Hamiltonian

$$H(x, y) = \tfrac{1}{2}\,|y|^2 - \frac{\mu}{|x|}. \tag{113}$$

The Hamiltonian system $(H, T\mathbf{R}^3_0, \omega)$ describes the two body problem in 3-*space* with one body located at the origin.

(b). Introduce a new time scale s defined by $\frac{ds}{dt} = \frac{k}{|x|}$ and consider the new Hamiltonian

$$\widetilde{K}(x, y) = \frac{|x|}{k}\left(\tfrac{1}{2}\,|y|^2 - \frac{\mu}{|x|} + \frac{k^2}{2}\right) + \frac{\mu}{k} = \frac{1}{2k}\,|x|\,(|y|^2 + k^2). \tag{114}$$

Show that the integral curves of $X_{\widetilde{K}}$ on $\widetilde{K}^{-1}(\mu/k)$ are a time reparametrization of the integral curves of X_H on $H^{-1}(-k^2/2)$. Since the total energy of the Kepler Hamiltonian H is *negative*, we are only considering bounded Keplerian orbits.

(c). Define Moser's mapping by

$$\mathcal{M} : T^+S^3_{\mathrm{np}} \subseteq T\mathbf{R}^4 \to T\mathbf{R}^3_0 : (q, p) \to (x, y)$$

where $S^3_{\mathrm{np}} = S^3 - \{(0, 0, 0, 1)\}$ and for $i = 1, 2, 3$

$$\begin{aligned} x_i &= -\tfrac{1}{k}\,(p_i + q_i p_4 - p_i q_4) \\ y_i &= \frac{k q_i}{1 - q_4}. \end{aligned} \tag{115}$$

Note that the tangent bundle TS^3 of S^3 is defined by

$$q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1 \quad \text{and} \quad q_1 p_1 + q_2 p_2 + q_3 p_3 + q_4 p_4 = 0.$$

1). Show that $\mathcal{M}$ is given by composing the tangent of stereographic projection of S^3 from the north pole $(0, 0, 0, 1)$ to the plane tangent to S^3 at $(0, 0, 0, -1)$ followed by the map $(x, y) \to (-y, x)$ which interchanges positions and velocities.

2). Show that

$$\begin{cases} |x|^2 &= \frac{1}{k^2}\,|p|^2(1-q_4)^2 \\ |y|^2+k^2 &= \dfrac{2k^2}{1-q_4} \\ \langle x, y\rangle &= -p_4. \end{cases} \tag{116}$$

Deduce that Moser's mapping $\mathcal{M}$ is a diffeomorphism with inverse given by

$$\begin{cases} q_i = \dfrac{2k\, y_i}{|y|^2+k^2}, \\ p_i = -\dfrac{1}{2k}\left[(|y|^2+k^2)x_i - 2\langle x, y\rangle\, y_i\right] \end{cases} \qquad i = 1, 2, 3$$

and

$$q_4 = \frac{|y|^2-k^2}{|y|^2+k^2} \qquad\qquad p_4 = -\langle x, y\rangle.$$

3). Show that

$$\mathcal{M}^*\Big(\sum_{i=1}^{3} x_i\, dy_i\Big) = -\left(\sum_{i=1}^{4} p_i\, dq_i\right)\Bigg|_{T^+S^3_{\mathrm{np}}}. \tag{117}$$

Hence Moser's map is symplectic when we restrict the standard symplectic form $\sum_{i=1}^4 dq_i \wedge dp_i$ on $T\mathbf{R}^3$ to $T^+S^3_{\mathrm{np}}$.

4). Show that

$$(\mathcal{M}^*\widetilde{K})(q, p) = |p| = \mathcal{K}(q, p). \tag{118}$$

$\mathcal{K}$ is defined and smooth on all of $T^+S^3 = TS^3 - \{(q, 0)|\, q \in S^3\}$. The flow of $X_{\mathcal{K}}$ is complete, because it is the geodesic flow on S^3. Thus we have regularized the flow of Kepler Hamiltonian on a negative energy level. The regularized negative energy level of the Kepler Hamiltonian is diffeomorphic to the unit tangent sphere bundle on S^3, that is, $S^2 \times S^3$.

12. (Kustaanheimo-Stiefel regularization). Let $x = (x_1, x_2, x_3)$ be a vector in $\mathbf{R}^3_0 = \mathbf{R}^3 - \{0\}$ and let $z = (z_1, z_2) \in \mathbf{C}^2_0 = \mathbf{C}^2 - \{0\} = \mathbf{R}^4 - \{0\}$. Define the 2×2 skew Hermitian matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $\langle z, w\rangle = z_1\overline{w_1} + z_2\overline{w_2}$ be the standard Hermitian inner product on $\mathbf{C}^2$. Show that the mapping

$$\pi : \mathbf{C}^2_0 \to \mathbf{R}^3_0 : z \to \big(\langle z, \sigma_1(z)\rangle, \langle z, \sigma_2(z)\rangle, \langle z, \sigma_3(z)\rangle\big), \tag{119}$$

is the Hopf map.

(a). On $\mathbf{C}_0^2$ define an action

$$\varphi : U(1) \times \mathbf{C}_0^2 \to \mathbf{C}_0^2 : \big(\mathrm{e}^{is}, (z_1, z_2)\big) \to (\mathrm{e}^{is} z_1, \mathrm{e}^{is} z_2).$$

Let $T^*\mathbf{C}_0^2 = (\mathbf{C}^2 - \{0\}) \times \mathbf{C}^2$. Lift φ to a $U(1)$-action

$$\Phi : S^1 \times T^*\mathbf{C}_0^2 \to T^*\mathbf{C}_0^2 : \big(\mathrm{e}^{is}, z, w\big) \to \big(\mathrm{e}^{is} z, \mathrm{e}^{is} w\big).$$

Define a 1-form θ on $T^*\mathbf{C}_0^2$ by $\theta = -2i \,\mathrm{Im}\, \langle w, dz\rangle$. Show that $\Omega = -d\theta$ is a symplectic form on $T^*\mathbf{C}_0^2$ and that Φ is a Hamiltonian action with momentum map

$$\mathcal{J} : T^*\mathbf{C}_0^2 \to \mathbf{R} : (z, w) \to 2\,\mathrm{Re}\, \langle w, z\rangle.$$

Let $\mathcal{J}_0 = \mathcal{J}^{-1}(0) - \{0\}$.

(b). The map π (119) lifts to the *Kustaanheimo-Stiefel* map

$$\mathcal{KS} : T^*\mathbf{C}_0^2 \to T^*\mathbf{R}_0^3 : (z, w) \to (x, y) =$$
$$\Big((\langle z, \sigma_j(z)\rangle), \frac{1}{\langle z, z\rangle} (\mathrm{Re}\, \langle w, \sigma_j(z)\rangle) \Big), \qquad j = 1, 2, 3.$$

The following calculation shows that

$$(\mathcal{KS})^*(\vartheta | \mathcal{J}_0) = \theta | \mathcal{J}_0, \tag{120}$$

where $\vartheta = \langle y, dx\rangle$ is the canonical 1-form on $T^*\mathbf{R}^3$. For every $u, w, z \in \mathbf{C}^2$

$$\sum_{j=1}^{3} \mathrm{Re}\, \langle u, \sigma_j(z)\rangle\, \sigma_j(w) = 2\langle w, z\rangle\, u - \langle u, z\rangle\, w. \tag{121}$$

Interchanging u with z in (121) and subtracting the result from (121) gives

$$\sum_{j=1}^{3} \mathrm{Re}\, \langle u, \sigma_j(z)\rangle\, \sigma_j(w) = \langle w, z\rangle\, u - \langle w, u\rangle\, z - i\, \mathrm{Im}\, \langle u, z\rangle\, w. \tag{122}$$

Taking the inner product of (122) with z and then adding the result to its complex conjugate gives

$$\sum_{j=1}^{3} \mathrm{Re}\, \langle u, \sigma_j(z)\rangle\, \mathrm{Re}\, \langle z, \sigma_j(w)\rangle = \mathrm{Re}\, \langle z, w\rangle\, \mathrm{Re}\, \langle u, z\rangle - \langle z, z\rangle\, \mathrm{Re}\, \langle u, w\rangle. \tag{123}$$

Replacing w in (123) with $-iw$ gives

$$\sum_{j=1}^{3} \mathrm{Re}\, \langle u, \sigma_j(z)\rangle\, \mathrm{Im}\, \langle z, \sigma_j(w)\rangle = \mathrm{Im}\, \langle z, w\rangle\, \mathrm{Re}\, \langle u, z\rangle - \langle z, z\rangle\, \mathrm{Im}\, \langle u, w\rangle. \tag{124}$$

Finally, replacing w by dz and u by w in (124) gives

$$\sum_{j=1}^{3} \mathrm{Re}\, \langle u, \sigma_j(z)\rangle\, \mathrm{Im}\, \langle z, \sigma_j(dz)\rangle = \mathrm{Im}\, \langle z, dz\rangle\, \mathrm{Re}\, \langle z, w\rangle - \langle z, z\rangle\, \mathrm{Im}\, \langle w, dz\rangle. \tag{125}$$

Consequently

$$(\mathcal{KS})^*\vartheta = 2i\left(\frac{\mathrm{Im}\langle z, dz\rangle\,\mathrm{Re}\,\langle z, w\rangle - \langle z, z\rangle\,\mathrm{Im}\,\langle w, dz\rangle}{\langle z, z\rangle}\right). \tag{126}$$

From (126) it follows that $(\mathcal{KS})^*(\vartheta\,|\mathcal{I}_0) = \theta\,|\mathcal{I}_0$.

(c). On $T^*\mathbf{R}^3_0$ with coordinates (x, y) and symplectic form $\omega = \sum_i dx_i \wedge dy_i$, consider the time rescaled Kepler Hamiltonian

$$\widetilde{K}(x, y) = \frac{1}{2k}\,\|x\|(\|y\|^2 + k^2)$$

whose $\frac{\mu}{k}$-level set corresponds to the $-k^2/2$-level set of the Kepler Hamiltonian (see exercise 11). Setting $u = w$ in (123) show that on $\mathcal{I}_0$ $\|(\mathcal{KS})^*y\|^2 = \frac{\langle w,w\rangle}{\langle z,z\rangle}$ and $\|(\mathcal{KS})^*x\|^2 = \langle z, z\rangle$. Therefore on $\mathcal{I}_0$ we obtain the regularized Hamiltonian

$$K = (\mathcal{KS})^*\widetilde{K} = \frac{1}{2k}\big(\langle w, w\rangle + k^2\langle z, z\rangle\big). \tag{127}$$

When $k = 1$ the regularized Hamiltonian is the harmonic oscillator Hamiltonian on $(T^*\mathbf{C}^2, \Omega)$ restricted to the open cone $\mathcal{I}_0$. Show that the regularized Hamiltonian K (127) is invariant under the $U(1)$-action Φ. Since the mapping $\mathcal{KS}$ is *not* a diffeomorphism, the harmonic oscillator vector field X_K is not equivalent to the Kepler vector field $X_{\widetilde{K}}$. Show that they are $\mathcal{KS}$-related on $\mathcal{I}_0$, that is, on $\mathcal{I}_0$ we have $T(\mathcal{KS})\circ X_K = X_{\widetilde{K}}\circ(\mathcal{KS})$. Moreover, show that after dividing out the S^1 action Φ on $\mathcal{I}_0$ we obtain an equivalence of Hamiltonian systems. Show that the orbit space $\mathcal{I}_0/S^1$ is diffeomorphic to T^+S^3, the tangent bundle to S^3 less its zero section.

13. (Ligon-Schaaf map.) Consider the Ligon-Schaaf map

$$LS : \Sigma_- \subseteq T_0\mathbf{R}^3 \to T^+(S^3 - \mathrm{np}) \subseteq T\mathbf{R}^4 : (q, p) \to (x, y),$$

(see (68)) with $\varphi = \frac{1}{\nu}\langle q, p\rangle$. Show that its inverse is given by

$$\begin{aligned} q &= \mu^{-1}\langle y, y\rangle\big((\sin\varphi - \langle y, y\rangle^{-1/2}y_4)\widetilde{x} + \langle y, y\rangle^{-1/2}(x_4 - \cos\varphi)\widetilde{y}\big) \\ p &= \mu\langle y, y\rangle^{-1/2}\Big(\frac{\widetilde{x}\cos\varphi + \langle y, y\rangle^{-1/2}\widetilde{y}\sin\varphi}{1 - x_4\cos\varphi - \langle y, y\rangle^{-1/2}y_4\sin\varphi}\Big), \end{aligned}$$

where $x = (\widetilde{x}, x_4)$, $y = (\widetilde{y}, y_4)$ and φ is a smooth solution of

$$\varphi - x_4\sin\varphi - y_4\langle y, y\rangle^{-1/2}\cos\varphi = 0.$$