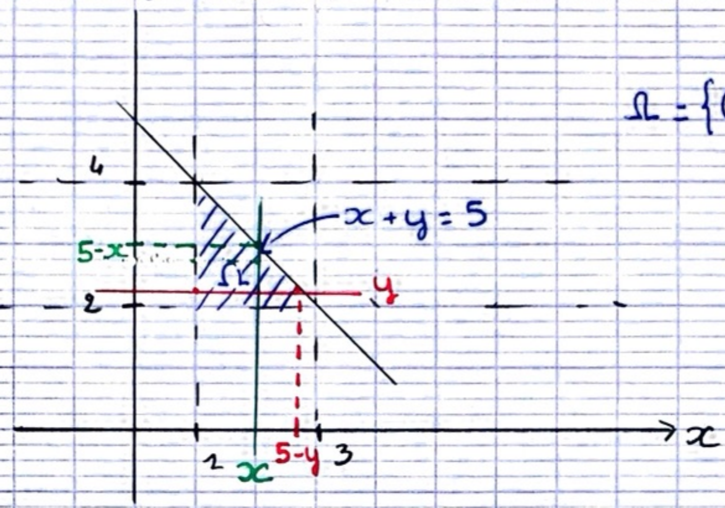
TD 2 : Révisions

Exo 1 :

Soit Ω l'ensemble ci-dessous.



$$\Omega = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{cases} 1 < x < 3 \\ 2 < y \\ x+y < 5 \end{cases} \right\}$$

" $\int_\Omega \frac{dx\,dy}{(x+y)^3}$ " : bien défini (peut être $=\infty$)

car $\frac{1}{(x+y)^3} \geqslant 0$ si $(x,y) \in \Omega$

Par def. de l'intégrale double

$$\int_\Omega \frac{dx\,dy}{(x+y)^3} := \int_2^4 \left(\int_1^{5-y} \frac{dx}{(x+y)^3} \right) dy$$

th. de Tonelli

$$= \int_1^3 \left(\int_2^{5-x} \frac{dy}{(x+y)^3} \right) dx$$

Calculons par exemple,

$$\int_2^4 \left(\int_1^{5-y} \frac{dx}{(x+y)^3} \right) dy = \int_2^4 \underbrace{\left[-\frac{1}{2} (x+y)^{-2} \right]_1^{5-y}}_{\frac{1}{2}\left(\frac{1}{(1+y)^2} - \frac{1}{25}\right)} dy$$

$$= \underbrace{-2 \cdot \frac{1}{2} \cdot \frac{1}{25}}_{-1/25} + \underbrace{\frac{1}{2} \int_2^4 \frac{dy}{(1+y)^2}}_{\frac{1}{2}\left[-\frac{1}{1+y}\right]_2^4}$$

$$= \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{1}{15}$$

$$= -\frac{1}{25} + \frac{1}{15} = \frac{1}{5}\underbrace{\left(-\frac{1}{5} + \frac{1}{3}\right)}_{2/15}$$

$$= \frac{2}{75} > 0$$

TD2 Mi1.

Exo 1. Soit à calculer (pour $a>0$)

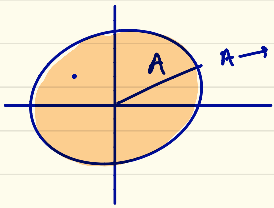
$$I := \int_{(\mathbb{R}_+^*)^2} e^{-a(x^2+y^2)}\, dx\, dy$$

Assez clairement, il suffit de calculer
$\int_{\mathbb{R}^2} \underbrace{e^{-(x^2+y^2)}}_{\geq 0}\, dx\, dy$ pour déterminer I ;

on intègre une fonction mesurable positive (sur un ensemble mesurable) : l'intégrale est bien définie (il se peut que $I = \infty$...)

En particulier, par CV monotone (cf. CM), on a :

$$\int_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dx\, dy = \lim_{A\to\infty} \int_{B(0,A)} e^{-(x^2+y^2)}\, dx dy$$

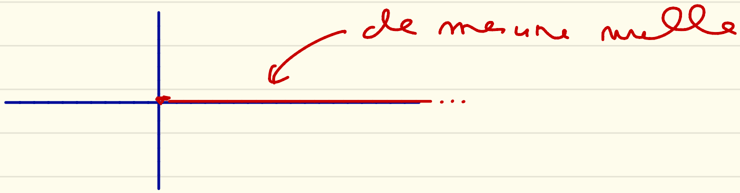


$A \to \infty$ (cf. $e^{-(x^2+y^2)} \cdot \chi_{B(0,A)}(x,y) \xrightarrow[A\to\infty]{} e^{-(x^2+y^2)} \nearrow \ldots$)

Pour calculer $\int_{B(0,A)} e^{-(x^2+y^2)}\,dx\,dy$, on passe en polaires. Ie, on considère le changement de variable ci-dessous :

$$\begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta\end{cases}, \text{ ie } (x,y)=F(\rho,\theta),\quad (\rho,\theta)\in\mathbb{R}_+^*\times\,]0,2\pi[$$

(bijection de l'ens. de déf. sur le plan privé d'une demi-droite :



)

Rappel : Th. de changement de variable (int. de Riemann) :

si $F : A \longrightarrow B$ $\mathcal{C}^1$-difféomorphisme (ie F bij., et F et F^{-1} sont $\mathcal{C}^1$),

$$\int_B f(y).dy \underset{\substack{\uparrow \\ y=F(x) \\ \text{"}dy=\det F'(x).dx\text{"}}}{=} \int_A f(F(x)).|\det F'(x)|dx$$

ex. : en dim 1, $F : [0,1] \longrightarrow [0,1]$, $t \longmapsto 1-t$

$$\int_{[0,1]} (1-s)^2\,ds \underset{\substack{s=F(t) \\ =1-t,\ t\in[0,1]}}{=} \int_{[0,1]} (1-\underbrace{(1-t)}_{t})^2.|\det F'(t)|.dt$$

$F'(t) = -1$, $\quad \det F'(t) = \det [-1] = -1$

$\Rightarrow |\det F'(t)| = +1$

$$\Rightarrow \int_{[0,1]} (1-s)^2 \, ds = \int_{[0,1]} t^2 . 1 . dt$$

$$= 1/3$$

Calcul équivalent à :

$$\int_{\substack{0 \\ \shortparallel \\ F(1)}}^{1=F(0)} (1-s)^2 \, ds \underset{\substack{s = 1-t \\ ds = -dt}}{\overset{\uparrow}{=}} \int_1^0 (1-(1-t))^2 (-dt)$$

$$= \int_0^1 (\overbrace{1-(1-t)}^{t})^2 . dt$$

Ici, $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$(\rho, \theta) \longmapsto (\rho\cos\theta, \rho\sin\theta) = \begin{bmatrix} \rho\cos\theta \\ \rho\sin\theta \end{bmatrix} = \begin{bmatrix} F_1(\rho,\theta) \\ F_2(\rho,\theta) \end{bmatrix}$$

F est dérivable (cf. MIE), et :

$$F'(\rho, \theta) = \underbrace{\begin{bmatrix} \frac{\partial F}{\partial \rho}(\rho,\theta) & \frac{\partial F}{\partial \theta}(\rho,\theta) \end{bmatrix}}_{2} \updownarrow 2$$

$$= \begin{bmatrix} \frac{\partial F_1}{\partial \rho} & \frac{\partial F_1}{\partial \theta} \\ \frac{\partial F_2}{\partial \rho} & \frac{\partial F_2}{\partial \theta} \end{bmatrix} (\rho, \theta) \quad : \text{dérivée ou "mat. jacobienne"}$$

$$= \begin{bmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{bmatrix}$$

$$\Rightarrow \det F'(\rho,\theta) = \rho\cos^2\theta + \rho\sin^2\theta = \rho$$

$$\Rightarrow |\det F'(\rho,\theta)| = \rho > 0$$

Donc, $$\int_{B(0,A)} e^{-(x^2+y^2)}\, dx\, dy$$

$$= \int_{[0,A]\times[0,2\pi]} e^{-\overbrace{((\rho\cos\theta)^2+(\rho\sin\theta)^2)}^{\rho^2}} \cdot \overbrace{|\rho|\, d\rho\, d\theta}^{\rho\, d\rho\, d\theta}$$

$$= \int_{[0,A]\times[0,2\pi]} e^{-\rho^2}\cdot \rho\, d\rho\, d\theta$$

$$\underset{\text{Fubini}}{=} \underbrace{\int_0^A e^{-\rho^2}\rho\, d\rho}_{\left[-\frac{1}{2}e^{-\rho^2}\right]_0^A} \cdot \underbrace{\int_0^{2\pi} d\theta}_{2\pi}$$

$$= \frac{1}{\cancel{2}}(1-e^{-A^2})\cdot \cancel{2}\pi$$

$$= \pi(1-e^{-A^2}) \xrightarrow[A\to\infty]{} \pi : \int_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dx\, dy = \pi.$$

En particulier, comme

$$\int_{\mathbb{R}^2} \underbrace{e^{-(x^2+y^2)}}_{\geq 0}\, dx\, dy$$

$$e^{-x^2} \cdot e^{-y^2}$$

$$\underset{\substack{\uparrow \\ \text{Fubini } (\geq 0,\ \text{Tonelli})}}{=} \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} e^{-(x^2+y^2)}\, dx \Big)\, dy$$

$$\Big(= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} e^{-(x^2+y^2)}\, dy \Big)\, dx \Big)$$

$$= \int_{\mathbb{R}} e^{-y^2} \cdot \Big(\overbrace{\int_{\mathbb{R}} e^{-x^2}\, dx}^{\not\ni y} \Big) \cdot dy$$

$$= \Big(\int_{\mathbb{R}} e^{-x^2}\, dx \Big) \cdot \int_{\mathbb{R}} e^{-y^2}\, dy$$

$$= \Big(\int_{\mathbb{R}} e^{-x^2}\, dx \Big)^2$$

$$\Rightarrow \int_{\mathbb{R}} e^{-x^2}\, dx = \sqrt{\pi}$$

(CV monotone...)

$$\Rightarrow \int_{\mathbb{R}} e^{-\frac{x^2}{2}}\, dx = \lim_{A \to \infty} \int_{-A}^{A} \underbrace{e^{-\frac{x^2}{2}}}_{\uparrow\ e^{-(x/\sqrt{2})^2} \cdot \frac{dx}{\sqrt{2}} \cdot \sqrt{2}}\, dx$$

$$\sqrt{2\pi} = \sqrt{2} \cdot \overbrace{\int_{\mathbb{R}} e^{-u^2} \cdot du}^{\sqrt{\pi}} = \sqrt{2} \cdot \int_{-A/\sqrt{2}}^{A/\sqrt{2}} e^{-u^2}\, du, \quad u = \frac{x}{\sqrt{2}}$$

On en déduit de même la valeur de

$$\int_{\mathbb{R}_+^{*}} e^{-a(x^2+y^2)}\,dx\,dy$$

$$\underset{\text{Fubini}}{=} \int_0^{\infty} e^{-ax^2}\,dx \cdot \int_0^{\infty} e^{-ay^2}\,dy$$

$$= \left(\int_0^{\infty} e^{-ax^2}\,dx\right)^2$$

Et $\displaystyle\int_0^{\infty} e^{-ax^2}\,dx \underset{\text{parité}}{=} \frac{1}{2}\int_{\mathbb{R}} e^{-ax^2}\,dx$

avec $\displaystyle\int_{\mathbb{R}} e^{-ax^2}\,dx \underset{\substack{u = x\sqrt{a} \\ du = dx.\sqrt{a}}}{=} \int_{\mathbb{R}} e^{-(x\sqrt{a})^2} \cdot \frac{x\sqrt{a}}{\sqrt{a}}$

$$= \frac{1}{\sqrt{a}} \underbrace{\int_{\mathbb{R}} e^{-u^2} \cdot du}_{\sqrt{\pi}} = \sqrt{\frac{\pi}{a}}$$

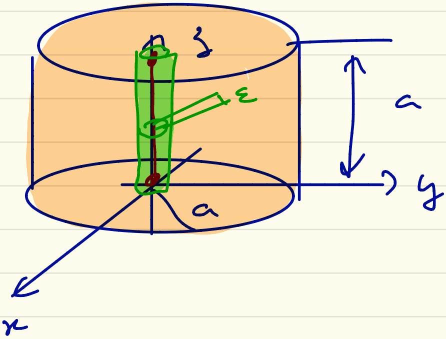
$$\Rightarrow \int_{(\mathbb{R}_+^{*})^2} e^{-a(x^2+y^2)}\,dx\,dy = \left(\frac{1}{2}\sqrt{\frac{\pi}{a}}\right)^2$$

$$= \frac{\pi}{4a}.$$

Exo 3. Soit à intégrer

$$\int_\Omega \frac{z}{\sqrt{x^2+y^2}}\,dx\,dy\,dz$$

où Ω :



L'intégrale de la fonction positive est bien définie (peu importe le cas $x=y=0$ qui correspond à l'ensemble de mesure nulle $\{0,0\}\times[0,A]$) et :

$$\int_\Omega \frac{z}{\sqrt{x^2+y^2}}\,dx\,dy\,dz \underset{\text{CV monotone}}{=} \lim_{\varepsilon\to 0} \int_{\Omega\cap\{x^2+y^2\geq\varepsilon^2\}} \frac{z}{\sqrt{x^2+y^2}}\,dx\,dy\,dz$$

On Fubinise (**) et on passe en polaires :

$$\int_{\Omega\cap\{\rho\geq\varepsilon\}} \frac{z}{\sqrt{x^2+y^2}}\,dx\,dy\,dz \underset{\text{Fubini}}{=} \underbrace{\int_0^a z\,dz}_{a^2/2} \cdot \int_{\{\varepsilon\leq\rho\leq a\}} \frac{dx\,dy}{\sqrt{x^2+y^2}}$$

$$\underset{\text{polaires (*)}}{=} \frac{a^2}{2}\cdot\int_{[\varepsilon,a]\times[0,2\pi]} \frac{\cancel{\rho}\,d\rho\,d\theta}{\cancel{\rho}} = \frac{a^2}{2}\cdot\int_\varepsilon^a d\rho\cdot\int_0^{2\pi} d\theta$$

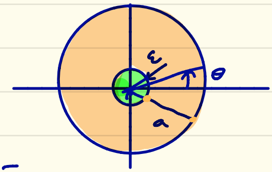
$$= a^2\pi(a-\varepsilon) \xrightarrow[\varepsilon\to 0]{} \pi a^3.$$

$$(*)\quad \int_{\{\varepsilon^2 \le x^2+y^2 \le a^2\}} \frac{dx\,dy}{\sqrt{x^2+y^2}} = \int_{\underbrace{[\varepsilon, a]}_{\rho} \times \underbrace{[0, 2\pi]}_{\theta}} \frac{1}{\cancel{\rho}} \cdot \cancel{\rho}\, d\rho\, d\theta$$

cf. $(x,y) = F(\rho,\theta)$

$\qquad\qquad = (\rho\cos\theta, \rho\sin\theta)$

$|\det F'(\rho,\theta)| = \rho$



$$= \int_\varepsilon^a \left(\int_0^{2\pi} d\theta \right) d\rho$$

$$= \int_\varepsilon^a 2\pi\, d\rho = 2\pi(a-\varepsilon).$$

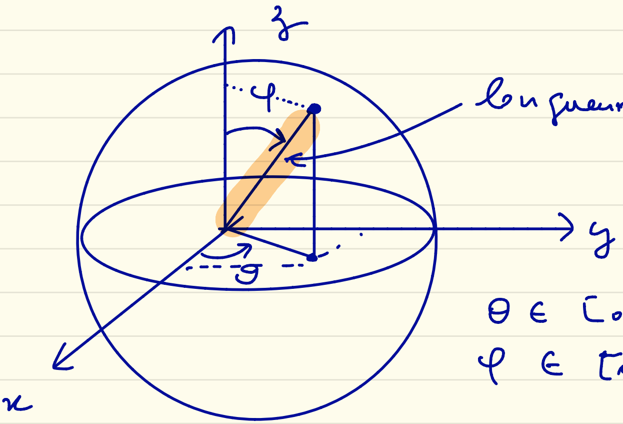
$$(**)\quad \int \frac{z\, dx\,dy\,dz}{\sqrt{x^2+y^2}} = \int z \cdot \frac{1}{\sqrt{x^2+y^2}}\, dx\,dy\,dz$$

$$\underset{\text{Fubini}}{\overset{\uparrow}{=}} \int \underbrace{\left(\int_0^a z\, dz \right)}_{\left[\frac{z^2}{2}\right]_0^a = \frac{a^2}{2}} \cdot \frac{1}{\sqrt{x^2+y^2}}\, dx\,dy$$

$$= \frac{a^2}{2} \int \frac{dx\,dy}{\sqrt{x^2+y^2}}.$$

Exo 4.

$$I := \int_{\{1 \leq \rho \leq 2\}} \frac{dx\,dy\,dz}{\sqrt{x^2+y^2+z^2}}$$



$\theta \in [0, 2\pi]$

$\varphi \in [0, \pi]$

$$\begin{cases} x = \rho \sin\varphi \cdot \cos\theta \\ y = \rho \sin\varphi \cdot \sin\theta \\ z = \rho \cos\varphi \end{cases}$$

$$F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$(\rho, \theta, \varphi) \longmapsto \begin{bmatrix} \rho \sin\varphi \cdot \cos\theta \\ \rho \sin\varphi \cdot \sin\theta \\ \rho \cos\varphi \end{bmatrix}$$

Formule de changement de variables :

$$\int_{\{1 \leq x^2+y^2+z^2 \leq 4\}} \frac{dx\,dy\,dz}{\sqrt{x^2+y^2+z^2}}$$

$$= \int_{\underbrace{[1,2]}_{\rho} \times [0,2\pi] \times [0,\pi]} \frac{|\det F'(\rho,\theta,\varphi)| \cdot d\rho\, d\theta\, d\varphi}{\rho}$$

$F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ est C^∞, et :

$$(\rho, \theta, \varphi) \longmapsto \begin{bmatrix} \rho \sin\varphi \cdot \cos\theta \\ \rho \sin\varphi \cdot \sin\theta \\ \rho \cos\varphi \end{bmatrix} \updownarrow 3$$

$$F'(\rho, \theta, \varphi) = 3 \updownarrow \begin{bmatrix} \frac{\partial F}{\partial \rho}(\rho,\theta,\varphi) & \frac{\partial F}{\partial \theta}(\rho,\theta,\varphi) & \frac{\partial F}{\partial \varphi}(\rho,\theta,\varphi) \end{bmatrix}$$

$$= \begin{bmatrix} \sin\varphi \cdot \cos\theta & -\rho \sin\varphi \cdot \sin\theta & \rho \cos\varphi \cdot \cos\theta \\ \sin\varphi \cdot \sin\theta & \rho \sin\varphi \cdot \cos\theta & \rho \cos\varphi \cdot \sin\theta \\ \cos\varphi & 0 & -\rho \sin\varphi \end{bmatrix}$$

$$\Rightarrow \det F'(\rho, \theta, \varphi) = \cos\varphi \cdot \underbrace{\begin{vmatrix} -\rho \sin\varphi \cdot \sin\theta & \rho \cos\varphi \cdot \cos\theta \\ \rho \sin\varphi \cdot \cos\theta & \rho \cos\varphi \cdot \sin\theta \end{vmatrix}}_{A}$$

$$- \rho \sin\varphi \cdot \underbrace{\begin{vmatrix} \sin\varphi \cdot \cos\theta & -\rho \sin\varphi \cdot \sin\theta \\ \sin\varphi \cdot \sin\theta & \rho \sin\varphi \cdot \cos\theta \end{vmatrix}}_{B}$$

$$A = -\rho^2 \sin\varphi \cdot \cos\varphi \, (\sin^2\theta + \cos^2\theta)$$
$$= -\rho^2 \sin\varphi \cdot \cos\varphi$$

$$B = \rho \sin^2\varphi \, (\cos^2\theta + \sin^2\theta)$$
$$= \rho \sin^2\varphi$$

$$\Rightarrow \det F'(\rho, \theta, \varphi) = -\rho^2 \sin\varphi \cdot \cos^2\varphi - \rho^2 \sin^3\varphi$$
$$= -\rho^2 \sin\varphi \, (\cos^2\varphi + \sin^2\varphi) = -\rho^2 \sin\varphi$$

$$\Rightarrow \left|\det F'(\rho,\theta,\varphi)\right| = \rho^2 \sin\varphi \quad (>0, \text{ si } \varphi \in \,]0,\pi[)$$

Donc,

$$I = \int_{[1,2]\times[0,2\pi],\,[0,\pi]} \frac{\rho^{\cancel{2}}\sin\varphi}{\cancel{\rho}}\, d\rho\, d\theta\, d\varphi$$

$$= \int_1^2 \rho\, d\rho \cdot \int_0^{2\pi} d\theta \cdot \int_0^{\pi} \sin\varphi$$

$$= \underbrace{\left[\frac{\rho^2}{2}\right]_1^2}_{2-\frac{1}{2}} \cdot 2\pi \cdot \underbrace{\left[-\cos\varphi\right]_0^{\pi}}_{2}$$

$$= 6\pi.$$