Td 6 - Espaces L^p

Exo 1. Étudier l'appartenance à $L^1(\mathbb{R})$ et $L^2(\mathbb{R})$ des fonctions suivantes :

- $f(t) := \dfrac{\sin t}{t}$ $(t \neq 0)$

$$\int_{\mathbb{R}} \left| \frac{\sin t}{t} \right| dt = 2 \int_0^{\infty} \frac{|\sin t|}{t}\, dt \; ; \quad \text{or, } 0 \leq |\sin t| \leq 1$$

$$\Rightarrow \; |\sin t| \geq \sin^2 t = \frac{1-\cos 2t}{2}$$

$$\Rightarrow \int_1^\infty \frac{|\sin t|}{t} \geq \int_1^\infty \frac{\overbrace{1-\cos 2t}^{\geq 0}}{2t}\,dt \underset{\text{CV monotone}}{=} \lim_{A\to\infty} \int_1^A \frac{1-\cos 2t}{2t}\,dt$$

$$\text{Or, } \int_1^A \frac{1-\cos 2t}{2t} = \underbrace{\frac{1}{2}\ln t\Big]_1^A}_{\frac{1}{2}\ln A} - \underbrace{\int_1^A \frac{\cos 2t}{2t}\,dt}_{\left[\frac{\sin 2t}{4t}\right]_1^A - \int_1^A \left(-\frac{\sin 2t}{4t^2}\right)dt} \leftarrow \text{i.p.p.}$$

$$\frac{1}{2}\ln A \xrightarrow[A\to\infty]{} \infty \qquad \text{cf. Th 1}$$

$$\underbrace{\left[\frac{\sin 2t}{4t}\right]_1^A}_{\xrightarrow[A\to\infty]{} -\frac{\sin 2}{4}} \qquad \int_1^A \left(-\frac{\sin 2t}{4t^2}\right)dt \in L^1([1,\infty[) \text{ cf. } 1/t^2 \longrightarrow \int_1^\infty \frac{\sin 2t}{4t^2}\,dt \in \mathbb{R}$$

$$\Rightarrow \int_1^\infty \frac{|\sin t|}{t}\,dt = \infty : \frac{\sin t}{t} \notin L^1(\mathbb{R}).$$

$\frac{\sin t}{t} \in L^2(\mathbb{R})$?

$\infty \overset{?}{>} \int_{\mathbb{R}} \left|\frac{\sin t}{t}\right|^{2} dt$; or, $\int_{\mathbb{R}} \frac{|\sin t|^2}{t^{2}} dt = 2\Big(\overbrace{\int_0^1 \frac{\sin^2 t}{t^2} dt}^{<\infty, \text{ cf } \frac{\sin t}{t} \xrightarrow[t \to 0]{} 1 \neq}$

$+ \int_1^\infty \frac{\sin^2 t}{t^2} dt\Big)$

et $\int_1^\infty \frac{\sin^2 t}{t^2} dt \leq \int_1^\infty \frac{dt}{t^2} < \infty$ (Riemann) : d'où l'appartenance à $L^2(\mathbb{R})$.

Rq : $\frac{\sin t}{t} \notin L^1(\mathbb{R})$, pour autant, $\lim_{A \to \infty} \int_{-A}^{A} \frac{\sin t}{t} dt$ existe (cf. Th 1).

"semi-CV"

$$\cdot\ g(t) = \frac{1}{\sqrt{t}\,(1+t^2)} \cdot \chi_{\mathbb{R}_+}(t),\quad t \in \mathbb{R}$$

intégrable
cf. Riemann

$$\int_{\mathbb{R}} |g(t)|\,dt \underset{\substack{\uparrow \\ g \in L^1(\mathbb{R})\,?}}{=} \int_0^{\infty} \frac{dt}{\sqrt{t}\,(1+t^2)} : \text{clairement, } g(t) \sim_0 \frac{1}{t^{1/2}}$$

$$g(t) \sim_\infty \frac{1}{t^{5/2}}$$

$$g \notin L^2(\mathbb{R})$$

$$\int_{\mathbb{R}^2} |g(t)|^2\,dt \underset{\substack{\uparrow \\ g \in L^2(\mathbb{R})\,?}}{=} \int_0^{\infty} \frac{dt}{t(1+t^2)^2} = \infty, \text{ cf. } |g(t)|^2 \sim_0 \frac{1}{t} \text{ (+Riemann)}$$

$$\left(\text{cf. } \frac{g^2(t)}{1/t} \xrightarrow[t \to 0]{} 1 \Rightarrow \exists \varepsilon > 0 \text{ tq } t \in \,]0, \varepsilon],\ \frac{g^2(t)}{1/t} \geq \frac{1}{2} \Rightarrow \int_0^{\varepsilon} |g(t)|^2\,dt \geq \int_0^{\varepsilon} \frac{dt}{2t} = \infty\right.$$

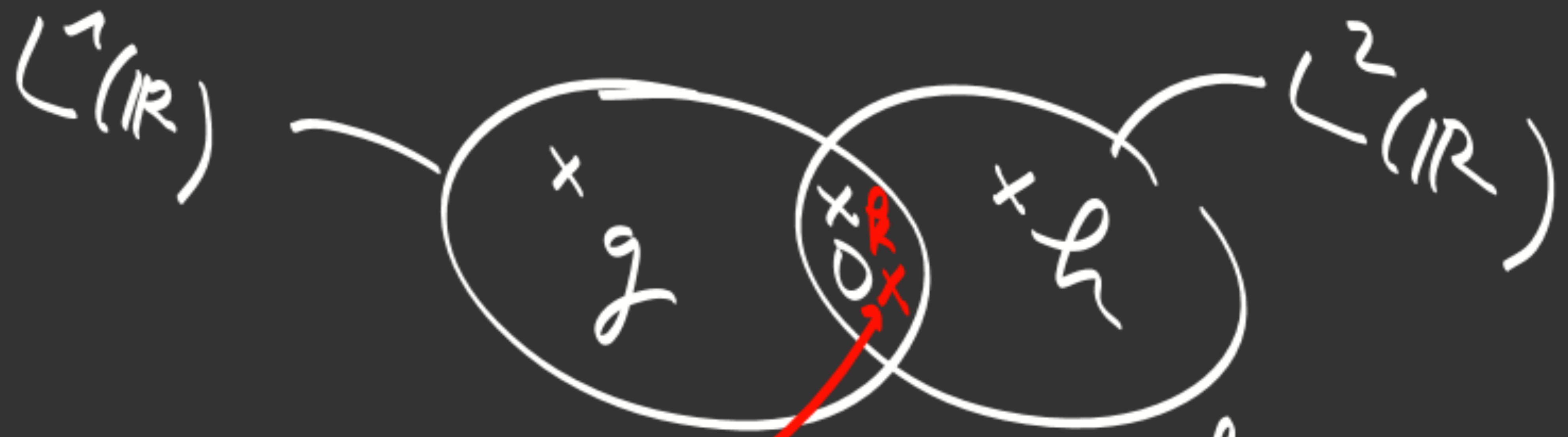
$$\left.\Rightarrow \int_0^{\infty} |g(t)|^2\,dt = \infty.\right)$$

• $h(t) = \dfrac{1}{\sqrt{1+t^2}}, \; t \in \mathbb{R}$;

$$\int_{\mathbb{R}} |h(t)|\,dt = 2\int_0^\infty \frac{dt}{\sqrt{1+t^2}} = \infty, \text{ cf. } h(t) \underset{\infty}{\sim} \frac{1}{t} \text{ (+Riemann)}$$

$$\int_{\mathbb{R}} |h(t)|^2\,dt = 2\int_0^\infty \frac{dt}{1+t^2} < \infty, \text{ cf. } |h(t)|^2 \underset{\infty}{\sim} \frac{1}{t^2} \text{ (+Riemann)}$$

$\Longrightarrow h \notin L^1(\mathbb{R}), \; h \in L^2(\mathbb{R})$



cf. aussi exo 3.

• $k(t) = e^{-t^2}, \; t \in \mathbb{R}$

$k \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ cf. $t^2 k(t) \xrightarrow[t \to \pm\infty]{} 0$ donc :

si $|t| \geq A$, $t^2 e^{-t^2} \leq 1$ ie $e^{-t^2} \leq \dfrac{1}{t^2}$, donc

$$\int_0^\infty |e^{-t^2}|\,dt = \underbrace{\int_0^A e^{-t^2}dt}_{\in \mathbb{R}} + \underbrace{\int_A^\infty e^{-t^2}dt}_{\leq \int_A^\infty \frac{dt}{t^2} < \infty \text{ (Riemann)}} < \infty : k \in L^1(\mathbb{R})$$

↑ cf. parité

Idem : $t^2 |e^{-t^2}|^2 = t^2 e^{-2t^2} \xrightarrow[t \to \pm\infty]{} 0$, donc $k \in L^2(\mathbb{R})$.

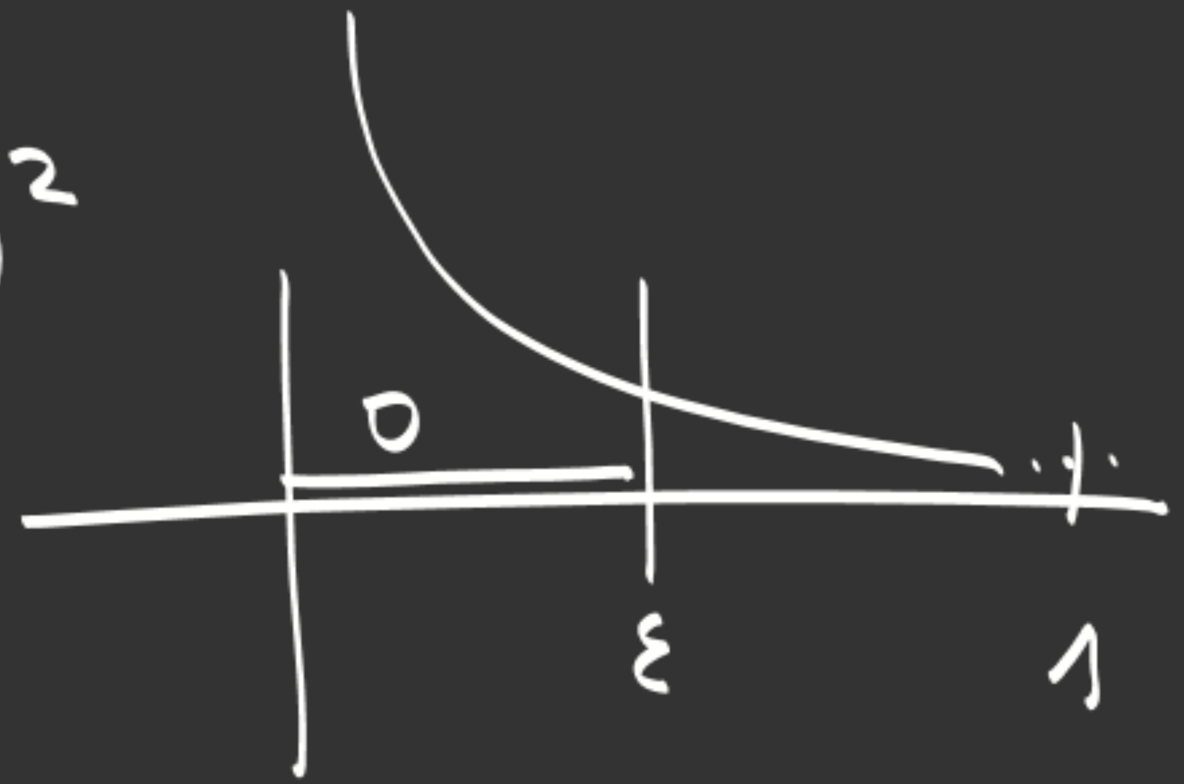
<u>Rq</u> : $k \in L^p(\mathbb{R})$, $p \in [1, \infty]$ ← cf. $|e^{-t^2}| \leq 1$

Exo 2.

$$f(t) := \frac{1}{t(1+|\ln t|)^2}, \quad t > 0$$

2.1. Mq $f \in L^1([0,1])$.

$$\int_\varepsilon^1 \frac{dt}{t(1+|\ln t|)^2}$$



$$\int_0^1 \underbrace{\frac{dt}{t(1+|\ln t|)^2}}_{\geq 0} = \lim_{\varepsilon \to 0} \int_0^1 \frac{dt}{t(1+|\ln t|)^2} \cdot \chi_{[\varepsilon,1]}$$

$\uparrow$ cv monotone (cf. $\varepsilon = \frac{1}{n} \ldots$)

On, $$\int_\varepsilon^1 \frac{1}{t} \cdot \frac{dt}{(1-\ln t)^2} =$$

cf. $|\ln t| = -\ln t \geq 0$

on pose $u = 1 - \ln t$ $\quad dt = -t\,du$

$du = -\frac{dt}{t}$

$$\Leftrightarrow \int_{1-\ln\varepsilon}^1 -\frac{du}{u^2} = \left[\frac{1}{u}\right]_{1-\ln\varepsilon}^1$$

$$= \left[\frac{1}{u}\right]_{1-\ln\varepsilon}^{1}$$

$$= 1 - \frac{1}{1-\ln\varepsilon} \xrightarrow[\varepsilon \to 0^{>}]{} 1 < \infty : f \in L^1(]0,1]).$$

<u>Rq.</u> : intégrales de Bertrand :

i) $\displaystyle\int_0^{1/e} \frac{dt}{|t^\alpha \ln^\beta t|}$:
- si $\alpha < 1$, $< \infty$
- si $\alpha = 1$ et $\beta > 1$, $< \infty$
- si $\alpha > 1$, $= \infty$

Situation symétrique en l'∞ :

$\displaystyle\int_e^\infty \frac{dt}{t^\alpha \ln^\beta t}$:
- si $\alpha > 1$, $< \infty$
- si $\alpha = 1$, $\beta > 1$, $< \infty$
- si $\alpha < 1$, $= \infty$.

2.2. Mq $f \notin L^p([0,1])$ si $p \in]1,\infty]$:

$$\int_0^1 \frac{dt}{t^p(1+|\ln t|)^{2p}} = \infty \quad (\text{cf. int. Bertrand})$$

si $p \in]1,\infty[$

et si $p=\infty$, $f \notin L^\infty([0,1])$: en effet,

$$f \notin L^\infty([0,1])$$

$$\Leftrightarrow \underset{\text{négation}}{\neg} \Big(\underbrace{(\exists c \geq 0) : \mu_L(\{t \in [0,1] \mid |f(t)| > c\}) = 0}_{f \in L^\infty([0,1])} \Big)$$

$$\Leftrightarrow (\forall c \geq 0) : \mu_L(\{t \in [0,1] \mid |f(t)| > c\}) > 0$$

Or, $t(1+|\ln t|)^2 \sim t|\ln t|^2 \underset{t \to 0,\ >}{\longrightarrow} 0_+ \Rightarrow \dfrac{1}{t(1+|\ln t|)^2} \underset{t \to 0,\ >}{\longrightarrow} \infty$,

donc, soit $c > 0$, $\exists\, \varepsilon > 0$ si $t \in]0, \varepsilon]$, $|f(t)| > c$

$\Rightarrow \mu_L(\underbrace{\{t \in [0,1] \mid |f(t)| > c\}}_{\supset]0,\varepsilon]}) = \varepsilon > 0.$

2.3. Mq $f \in L^p([1,\infty[)$ pour $p \in [1,\infty]$:

si $p \in [1,\infty[$, $\displaystyle\int_1^\infty \frac{1}{t^p(1+|\ln t|)^{2p}} < \infty$ (soit $p=1$ et $p=2\ (>1)$, soit $p>1$),

si $p=\infty$, $|f(t)| \leq 1$ $\Rightarrow f \in L^\infty([1,\infty[)$.

Exo 3. Soit $(X, \mathcal{B}, \mu)$ esp. mesuré tq $\boxed{\mu(X) < \infty}$.

Soient $1 \leq p < q \leq \infty$, mq

$$L^p(X, \mathcal{B}, \mu) \supset L^q(X, \mathcal{B}, \mu)$$

(avec inclusion stricte, en général)

$\operatorname{Inf}\{C \geq 0 \mid |f(x)| \leq C \ \mu\text{pp } x \in X\}$

- $\underline{q = \infty}$: soit $f \in L^\infty(X, \mathcal{B}, \mu)$ mq

$$\int_X |f|^p \, d\mu < \infty$$

$|f| \leq \|f\|_\infty$ μpp (exo)

On, $\displaystyle\int_X |f|^p d\mu \leq \int_X \|f\|_\infty^p \, d\mu = \|f\|_\infty^p \mu(X) < \infty \Rightarrow f \in L^p(X, \mathcal{B}, \mu)$.

Rq. : $f(t) = \frac{1}{t^{1/2p}} \Rightarrow |f(t)|^p = \frac{1}{t^{1/2}} \Rightarrow f \in L^p([0,1]), p \in [1,\infty[$

$t \in [0,1]$

$f \notin L^\infty([0,1])$ (cf. $f(t) \underset{t \to 0}{\longrightarrow} \infty$ ...)

($\Rightarrow L^p([0,1]) \supsetneq L^\infty([0,1])$)

• $q < \infty$: soit $f \in L^q(X, \mathcal{B}, \mu)$,

$$\int_X |f|^p \, d\mu = \int_X |f|^p \cdot 1 \, d\mu \quad : \text{appliquer Hölder} \quad |f|^q = \left(|f|^p\right)^{q/p}$$

$|f|^p \in L^{q/p}$

(Rappel : Hölder : $f \in L^p$ et $g \in L^q$ avec $\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow fg \in L^1$ et $\|f \cdot g\|_1 \leq \|f\|_p \cdot \|g\|_q$.)

On a $|f|^p \in L^r$ avec $r := \overbrace{q/p}^{>1}$; $1 \in L^s$ avec s conjugué de r,

i.e. $s = \left(1 - p/q\right)^{-1} = \dfrac{q}{q-p} > 1$ (cf. $\int_X |1|^s \, d\mu = \mu(X) < \infty$)

Hölder $\Rightarrow$ $|f|^p . 1 \in L^1(X, \mathcal{T}, \mu)$ ($\Rightarrow f \in L^p(X, \mathcal{T}, \mu)$) et

$$\| |f|^p . 1 \|_1 = \int_X |f|^p \, d\mu \leqslant \| |f|^p \|_r . \| 1 \|_s$$

$$= \left(\int_X (|f|^{\cancel{p}})^{q/\cancel{p}} d\mu \right)^{p/q} . \left(\int_X |1|^{\frac{q}{q-p}} d\mu \right)^{\frac{q-p}{q}}$$

$\Rightarrow \|f\|_p \leqslant \|f\|_q . (\mu(X))^{\frac{1}{p} - \frac{1}{q}}$. (Et l'inclusion est stricte.)