CS 383 Fall 2013
Homework # 5 Solutions
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Honor Code:
I affirm that I have adhered to the Honor Code on this assignment.

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Problem 1: Are we truly free?

Construction

Let us create a PDA, M such that:

$$M = (Q, \Sigma, \Gamma, \delta, s, F, \perp)$$

M will have transitions defined as follows:

$$(q, a, A) \xrightarrow[M]{1} (q', \alpha)$$

$$(r, a, \perp) \xrightarrow[M]{1} (r', \perp)$$

Now let us create a second PDA, N that models a regular language:

$$N = (Q', \Sigma, \{\text{II}\}, \delta', s'.F', \text{II})$$

With transition:

$$(q, a, \text{II}) \xrightarrow[N]{1} (\delta'(q, a), \text{II})$$

Finally, let us create a third PDA M' that models $L(M) \cap L(N)$

$$M' = (Q \times Q', \Sigma, \Gamma \times \{\text{II}\}, \Delta, (s, s'), F \times F', (\perp, \text{II}))$$

With transition:

$$((q, r), a, (A, \text{II})) \xrightarrow[M']{1} ((q', r'), (\alpha, \text{II}))$$

As can be seen, all of our transitions are defined for some states q or r, some $a \in \Sigma$ and some $\alpha \in \Gamma$ that gets pushed onto the stack.
Let all of out PDA's define acceptance in terms of reaching some final state.

Lemma 1. *$(q, xy, \alpha) \xrightarrow[M]{*} (q', y, \alpha')$ and $(r, xy, \text{II}) \xrightarrow[N]{*} (r', y, \text{II}) \iff ((q, r), xy, \alpha \times \{\text{II}\}) \xrightarrow[M']{*} ((q', r'), y, \alpha' \times \{\text{II}\})$*

Proof. ($\Rightarrow$) Induction on the length of the derivation.
Base Case n = 0: $(q, xy, \alpha) \xrightarrow[M]{0} (q, xy, \alpha)$ and $(r, xy, \text{II}) \xrightarrow[N]{0} (r, xy, \text{II}) \Rightarrow ((q, r), xy, \alpha \times \{\text{II}\}) \xrightarrow[M']{0} ((q, r), xy, \alpha \times \{\text{II}\})$

Inductive Hypothesis: Assume this is true for n+1 derivations and let $a \in \Sigma$
Inductive Step:

$$(q, axy, \alpha) \xrightarrow[M]{1} (q, xy, \alpha) \xrightarrow[M]{n} (q, y, \alpha')$$

$$(r, axy, \text{II}) \xrightarrow[N]{1} (r, xy, \text{II}) \xrightarrow[N]{n} (r, y, \text{II})$$

$$((q, r), axy, \alpha \times \{\text{II}\}) \xrightarrow[M']{1} ((q, r), xy, \alpha \times \{\text{II}\}) \xrightarrow[M']{n} ((q, r), y, \alpha \times \{\text{II}\})$$

($\Leftarrow$) This direction is trivial as it is symmetrical to the opposite direction. $\square$

Theorem 1.

$$(q, aA) \xrightarrow[M]{1} (q', \alpha)$$

and

$$(r, a, \text{II}) \xrightarrow[N]{1} (\delta'(r', a), \text{II})$$

are accept states

$$\Longleftrightarrow$$

$$((q, r), a, \alpha \times \{\text{II}\}) \xrightarrow[M']{1} ((q', r'), (\alpha, \text{II}))$$

is also an accept state.

Proof. If M and N are in an equivalent state, then by our construction, M' is also in an equivalent state as proven by our lemma. Therefore, if M and N are in accept states, then M' must also be in an accept state. $\square$