Information Theory and Coding - Prof. Emere Telatar

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Contents

1 Data compression

Definition 1.1 (Information). *Abstractly, **information** can be thought of as the resolution of uncertainty.*

Given an alphabet $\mathcal{U}$ (e.g. $\mathcal{U} = \{a, ..., z, A, ..., Z, ...\}$), we want to assign binary sequences to elements of $\mathcal{U}$, i.e.

$$\mathcal{C} : \mathcal{U} \to \{0, 1\}^* = \{\emptyset, 0, 1, 00, 01, ...\}$$

For $\mathcal{X}$ a set

$$\mathcal{X}^n \equiv \{(x_0...x_n), x_i \in \mathcal{X}\}$$
$$\mathcal{X}^* \equiv \bigcup_{n \geq 0} \mathcal{X}^n$$

Definition 1.2. *A code $\mathcal{C}$ is called **singular** if*

$$\exists (u, v) \in \mathcal{U}^2, u \neq v \quad s.t. \quad \mathcal{C}(u) = \mathcal{C}(v)$$

Non singular code is defined as opposite

Definition 1.3. *A code $\mathcal{C}$ is called **uniquely decodable** if*

$$\forall u_1, ..., u_n, v_1, ..., v_n \in \mathcal{U}^* \quad s.t. \quad u_1, ..., u_n \neq v_1, ..., v_n$$

we have

$$\mathcal{C}(u_1)...\mathcal{C}(u_n) \neq \mathcal{C}(v_1)...\mathcal{C}(v_n)$$

i.e, $\mathcal{C}$ is non-singular

Definition 1.4. *Suppose $\mathcal{C} : \mathcal{U} \to \{0, 1\}^*$ and $\mathcal{D} : \mathcal{V} \to \{0, 1\}^*$ we can define*

$$\mathcal{C} \times \mathcal{D} : \mathcal{U} \times \mathcal{V} \to \{0, 1\}^* \quad as \quad (\mathcal{C} \times \mathcal{D})(u, v) \to \mathcal{C}(u)\mathcal{D}(v)$$

Definition 1.5. *Given $\mathcal{C} : \mathcal{U} \to \{0, 1\}^*$, define*

$$\mathcal{C}^* : \mathcal{U}^* \to \{0, 1\}^* \quad as \quad \mathcal{C}^*(u_1...u_n) = \mathcal{C}(u_1)...\mathcal{C}(u_n)$$

Definition 1.6. *A code $\mathcal{U} \to \{0, 1\}^*$ is **prefix-free** is for no $u \neq v$ $\mathcal{C}(u)$ is a prefix of $\mathcal{C}(v)$.*

Theorem 1.1. *If $\mathcal{C}$ is prefix-free then $\mathcal{C}$ is uniquely decodable.*

Definition 1.7. *$l(\mathcal{C}(u))$ is the length of the code word $\mathcal{C}(u)$ and $l(\mathcal{C})$ is the expected length of the code:*

$$l(\mathcal{C}) = \sum_u l(\mathcal{C}(u))p(u)$$

Definition 1.8 (Kraft sum). *Given $\mathcal{C} : \mathcal{U} \to \{0, 1\}^*$*

$$kraftsum(\mathcal{C}) = \sum_u 2^{-l(\mathcal{C}(u))}$$

Lemma 1.2. *if $\mathcal{C} : \mathcal{U} \to \{0, 1\}^*$ and $\mathcal{D} : \mathcal{V} \to \{0, 1\}^*$ then*

$$kraftsum(\mathcal{C} \times \mathcal{D}) = kraftsum(\mathcal{C}) \times kraftsum(\mathcal{D})$$

Proof.

$$kraftsum(\mathcal{C} \times \mathcal{D}) = \sum_{u,v} 2^{-(l(\mathcal{C})*l(\mathcal{D}))}$$

$$= \sum_u 2^{-l(\mathcal{C})} \sum_v 2^{-l(\mathcal{D})}$$

$\square$

Corollary 1.2.1. $kraftsum(\mathcal{C}^n) = (kraftsum(\mathcal{C}))^n$

Proposition 1.1. *if $\mathcal{C}$ is non-singular, then*

$$kraftsum(\mathcal{C}) \leq 1 + \max_u l(\mathcal{C}(u))$$

In coding theory, the **Kraft-McMillan inequality** gives a necessary and sifficient condition for the existence of a uniquely decodable code for a given set of codeword lengths.

Wikipedia. *Kraft's inequality limits the lengths of codewords in a prefix code: if one takes an exponential of the length of each valid codeword, the resulting set of values must look like a probability mass function, that is, it must have total measure less than or equal to one. Kraft's inequality can be thought of in terms of a constrained budget to be spent on codewords, with shorter codewords being more expensive.*

Theorem 1.3. *if $\mathcal{C}$ is uniquely decodable, then $kraftsum(\mathcal{C}) \leq 1$*

Proof. $\mathcal{C}$ is uniquely decodable $\equiv \mathcal{C}^*$ is non singular

$$\Rightarrow kraftsum(\mathcal{C}^n) \leq 1 + \max_{u_1,...,u_n} l(\mathcal{C}^n)$$

$$\Rightarrow kraftsum(\mathcal{C})^n \leq 1 + nL, \quad L = \max l(\mathcal{C}(n))$$

A growing exp cannot be bounded by a linear function

$$\Rightarrow kraftsum(\mathcal{C}) \leq 1$$

$\square$

Theorem 1.4. *Suppose $\mathcal{C} : \mathcal{U} \to \mathcal{N}$ is such that $\sum_u 2^{-l(\mathcal{C}(u))} \leq 1$, then, there exists a prefix-free code $\mathcal{C}' : \mathcal{U} \to \{0,1\}$ s.t. $\forall u, l(\mathcal{C}'(u)) = l(\mathcal{C}(u))$*

Proof. Let $\mathcal{U} = \{u_1, ..., u_n\}$ and $\mathcal{C}(u_1) \leq \mathcal{C}(u_2) \leq ... \leq \mathcal{C}(u_k) = \mathcal{C}_{max}$. Consider the complete binary tree up to depth $\mathcal{C}_{max}$ initially all nodes are available to be used as codewords. For $i = 1, 2, ..., n$, place $\mathcal{C}(u_i)$ at an available node at level $\mathcal{C}(u_i)$ remove all descendant of $\mathcal{C}(u_i)$ from the available list.

Corollary 1.4.1. *Suppose $\mathcal{C} : \mathcal{U} \to \{0,1\}^*$ is uniquely decodable, then there exist an $\mathcal{C}' : \mathcal{U} \to \{0,1\}^*$ which is prefix-free and $l(\mathcal{C}'(n)) = l(\mathcal{C}(n))$*

$\square$

Example 1. $\mathcal{U} = \{a, b, c, d\}$, $\mathcal{C} : \{0, 01, 011, 111\}$ and $\mathcal{C}' : \{0, 10, 110, 111\}$
In this case, decoding $\mathcal{C}$ may require delay, while decoding $\mathcal{C}'$ is instanteneous.

2 Alphabet with statistics

Suppose we have an alphabet $\mathcal{U}$, and suppose we have a random variable U taking values in $\mathcal{U}$. We denote by $p(u) = Pr(U = u), u \in \mathcal{U}$ with $p(u) \geq 0$ and $\sum_u p(u) = 1$.

Suppose we have a code $\mathcal{C} : \mathcal{U} \to \{0,1\}^*$. We then have $\mathcal{C}(u)$ a random binary string and $l(\mathcal{C}(u))$ a random integer.

Example 2. $\mathcal{U} = \{a, b, c, d\}$
$p : \{0.5, 0.25, 0.125, 0.125\}$
$\mathcal{C} : \{0, 01, 110, 111\}$

then we have

$$l(\mathcal{C}(u)) = \begin{cases} 1, & p = 0.5 \\ 2, & p = 0.25 \\ 3, & p = 0.125 + 0.125 = 0.25 \end{cases}$$

We can measure how efficient $\mathcal{C}$ represents $\mathcal{U}$ by considering

$$E[l(\mathcal{C}(u))] = \sum_u p(u)l(\mathcal{C}(u)) \quad \text{with} \quad \mathcal{C}(u) = l(\mathcal{C}(u))$$

Theorem 2.1. *if $\mathcal{C}$ is uniquely decodable, then*

$$E[l(\mathcal{C}(u))] \geq \sum_u p(u) \log \frac{1}{p(u)}$$

Proof. let $\mathcal{C}(u) = l(\mathcal{C}(u))$, we know $\sum_u 2^{-\mathcal{C}(u)} \leq 1$ because $\mathcal{C}$ is uniquely decodable. We write $q(u) = 2^{-\mathcal{C}(u)}$ and get

$$E[l(\mathcal{C}(u))] = \sum_u p(u)\mathcal{C}(u) = \sum_u p(u) \log_2 \frac{1}{q(u)}$$

$$\equiv \sum_u p(u) \log \frac{q(u)}{p(u)} \leq 0$$

$$\equiv \sum_u p(u) \ln \frac{q(u)}{p(u)} \leq 0$$

$$\leq \sum_u p(u) \left[\frac{q(u)}{p(u)} - 1 \right] = \underbrace{\sum_u q(u)}_{\leq 1} - \underbrace{\sum_u p(u)}_{=1} \leq 0$$

$\square$

Theorem 2.2. *For any $\mathcal{U}$, there exists a prefix-free code $\mathcal{C}$ s.t.*

$$E[l(\mathcal{C}(u))] < 1 + \sum_{u \in \mathcal{U}} p(u) \log \frac{1}{p(u)}$$

Proof. Given $\mathcal{U}$, let

$$\mathcal{C}(u) = \lceil \log \frac{1}{p(u)} \rceil < 1 + \log \frac{1}{p(u)}$$

$$\Rightarrow \sum_u 2^{-\mathcal{C}(u)} \leq \sum_u p(u) = 1$$

$$\Rightarrow \sum_u p(u)\mathcal{C}(u) < \sum_u p(u) \log \frac{1}{p(u)} + \underbrace{1}_{\sum p(u)}$$

$\square$

Definition 2.1 (Entropy). *Entropy quantifies the amount of uncertainty involved in the value of a random variable or the outcome of a random process.*

Theorem 2.3. *The entropy of a random variable $U \in \mathcal{U}$ is*

$$H(U) = \sum_{u \in \mathcal{U}} p(u) \log \frac{1}{p(u)}$$

with $p(u) = Pr(U = u)$

Wikipedia. *The entropy is a lower bound on the optimal expected length*

$$H(U) \leq E[l(\mathcal{C}(u))]$$

In fact, one can show that there exists a uniquely decodable code such that

$$H(U) \leq E[l(\mathcal{C}(u))] < H(U) + 1$$

Note that $H(U)$ is a fonction of the distribution $\mathcal{C}_u(.)$ of the random variable U, it isn't a function of U.

$$H(U) = E[f(U)] \quad \text{where} \quad f(U) = \log \frac{1}{p(u)}$$

How to design optimal codes (in the sense of minimizing $E[l(\mathcal{C}(u))]$)?
Formally, given a random variable U, find $\mathcal{C}(u) \to \mathcal{N}$ s.t.

$$\sum_{u \in U} 2^{\mathcal{C}(u)} \leq 1 \quad \text{that minimizes} \quad \sum_{u \in U} p(u) l(\mathcal{C}(u))$$

Properties of optimal prefix-free codes

- if $p(u) < p(v)$ then $l(\mathcal{C}(u)) \geq l(\mathcal{C}(v))$

- The two longest codewords have the same length

- There is an optimal code such that the two least probable letters are assigned codewords that differ in the last bit.

Observe that if $\{\mathcal{C}(u_1), ..., \mathcal{C}(u_{k-1}), \mathcal{C}(u_k)\}$ is a prefix-free collection of the property that

$$\begin{aligned} \mathcal{C}(u_{k-1}) &= \alpha 0 \\ \mathcal{C}(u_k) &= \alpha 1 \end{aligned} \quad \text{with} \quad \alpha \in \{0, 1\}^*$$

then $\{\mathcal{C}(u_1), ..., \mathcal{C}(u_{k-2}), \alpha\}$ is also a prefix-free collection.
Also

$$\sum_{u \in \mathcal{U}} p(u) l(\mathcal{C}(u)) = p(u_1) l(\mathcal{C}(u_1)) + ... + p(u_{k-2}) l(\mathcal{C}(u_{k-2})) + [p(u_{k-1}) + p(u_k)](l(\alpha) + 1)$$

$$= (p(u_{k-1}) + p(u_k)) + \sum_{v \in \mathcal{V}} p(v) l(\mathcal{C}'(v))$$

So we have shown that with

$$E[l(\mathcal{C}(U)] = p(u_{k-1}) + p(u_k) + E[l(\mathcal{C}'(V))]$$

if $\mathcal{C}$ is optimal for U, then $\mathcal{C}'$ is optimal for V

3 Entropy and mutual information

Definition 3.1 (Joint entropy)**.** *Suppose U, V are random variables with $p(u, v) = Pr\{U = u, V = v\}$, the joint entropy is*

$$H(UV) = \sum_{u,v} p(u, v) \log \frac{1}{p(u, v)}$$

Theorem 3.1.

$$H(UV) \leq H(U) + H(V)$$

with equality iff U and V are independants.

Proof. We want to show that

$$\sum_{u,v} p(u, v) \log \frac{1}{p(u, v)} \leq \sum_u p(u) \log \frac{1}{p(u)} + \sum_v p(v) \log \frac{1}{p(v)} \iff \sum_{u,v} p(u, v) \log \frac{p(u)p(v)}{p(u, v)} \leq 0$$

We use $\ln z \leq z - 1$ for all z (with equality iff $z = 1$):

$$\sum_{u,v} p(u, v) \log \frac{p(u)p(v)}{p(u, v)} \leq \sum_{u,v} p(u, v) \left[\frac{p(u)p(v)}{p(u, v)} - 1 \right] = \sum_{u,v} p(u)p(v) - \sum_{u,v} p(u, v) = 1 - 1 = 0$$

$\square$

Same definitions of entropy holds for n symbols.

Definition 3.2 (Joint Entropy)**.** *Suppose $U_1, U_2, \ldots, U_n$ are RVs and we are given $p(u_1 \ldots u_n)$, the joint entropy is*

$$H(U_1, \ldots, U_n) = \sum_{u_1 \ldots u_n} p(u_1 \ldots u_n) \log \frac{1}{p(u_1 \ldots u_n)}$$

Theorem 3.2.

$$H(U_1 \ldots U_n) \leq \sum_{i=1}^n H(U_i)$$

with equality iff Us are independants

Corollary 3.2.1. *if $U_1, \ldots, U_n$ are i.i.d. then $H(U_1 \ldots U_n) = nH(U_1)$*

Definition 3.3 (Conditional entropy)**.**

$$H(U|V) = \sum_{u,v} p(u, v) \log \frac{1}{p(u|v)}$$
$$= \sum_v H(U|V = v)Pr\{V = v\}$$

Theorem 3.3.

$$H(UV) = H(U) + H(V|U) = H(V) + H(U|V)$$

Theorem 3.4.

$$H(U) + H(V) \geq H(UV) = H(V) + H(U|V)$$

Definition 3.4 (Mutual information)**.** *Mutual information measures the amount of information that can be obtained about one random variable by observing another.*

$$I(U; V) = I(V; U) = H(U) - H(U|V)$$
$$= H(V) - H(V|U)$$
$$= H(U) + H(V) - H(UV)$$

We can apply the chain rule on the entropy as follow

$$H(U_1 U_2 \ldots U_n) = H(U_1) + H(U_2|U_1) + \cdots + H(U_n|U_1 U_2 \ldots U_{n-1}) = \sum_{i=1}^{n} H(U_i|U^{i-1})$$

Definition 3.5 (Conditional mutual information)**.**

$$
\begin{aligned}
I(U;V|W) &= H(U|W) - H(U|VW) \\
&= H(V|W) - H(V|UW) \\
&= \mathbb{E}_{u,v,w}\left[\log \frac{p(uv|w)}{p(u|w)p(v|w)}\right]
\end{aligned}
$$

Theorem 3.5.

$$I(V;U_1 \ldots U_n) = I(V;U_1) + I(V;U_2|U_1) + \cdots + I(V;U_n|U_1 \ldots U_{n-1})$$

We can apply the chain rule on the mutual information as follows

$$I(U_1, U_2, \ldots; V) = I(U_1;V) + I(U_2;V|U_1) + \ldots$$

Theorem 3.6 (Data processing inequality)**.** *Let* $X \to Y \to Z$ *be a Markov chain, then*

$$I(X;Y) \geq I(X;Z) \quad and \quad I(Y;Z) \geq I(X;Z)$$

We have equality iff $I(X;Y|Z) = 0$ *(i.e.* $X \to Z \to Y$ *form a Markov chain) This important and useful theorem demonstrates that no processing of* Y*, deterministic or random, can increase the information that* Y *contains about* X*.*

Notation 1.
$$U^n \triangleq (U_1 U_2 \ldots U_n)$$

Theorem 3.7.
$$I(U;V|W) \geq 0$$

equality iff conditioned on w*,* u *and* v *are independant, that is iff* $U - V - W$ *is a Markov chain.*

Proof.

$$
\begin{aligned}
I(U;V|W) &= \frac{1}{\ln 2} \sum_{u,v,w} p(uvw) \ln \frac{p(u|w)p(v|w)}{p(uv|w)} \\
&\geq \frac{1}{\ln 2} \sum_{u,v,w} p(uvw) \left[\frac{p(u|w)p(v|w)}{p(uv|w)} - 1\right] \\
&= \frac{1}{\ln 2} \sum_{u,v,w} (p(w)p(u|w)p(v|w) - p(uvw)) \\
&= \frac{1}{\ln 2}(1 - 1) \\
&= 0
\end{aligned}
$$

□

4 Data processing

Theorem 4.1. $U - V - W$ *is a MC* $\iff I(U;W|V) = 0$

Corollary 4.1.1. $I(U;V) \geq I(U;W)$ *and by symetry of MC* $I(W;V) \geq I(U;W)$

Proof.
$$I(U;VW) = I(U;V) + I(U;W|V) = I(U;V)$$
and
$$I(U;VW) = I(U;W) + I(U;V|W) \geq I(U;W)$$

$\square$

Theorem 4.2. *Given U a RV taking values in $\mathcal{U}$ then*
$$0 \leq H(U) \leq \log|\mathcal{U}|$$
and $H(U) = 0$ iff U is constant, $H(U) = \log|\mathcal{U}|$ iff U is $p(u) = 1/|\mathcal{U}|$ for all u.

Proof. For the lower bound,
$$H(U) = \sum_u \underbrace{p(u)}_{\geq 0} \underbrace{\log \frac{1}{p(u)}}_{\geq 0} \geq 0$$

For the upper bound,

$$
\begin{aligned}
H(U) - \log|\mathcal{U}| &= \sum_u p(u) \log \frac{1}{p(u)} - \sum_u p(u) \log|\mathcal{U}| \\
&= \frac{1}{\ln 2} \sum_u p(u) \ln \frac{1}{|\mathcal{U}|p(u)} \\
&\leq \frac{1}{\ln 2} \sum_u p(u) \left(\frac{1}{|\mathcal{U}|p(u)} - 1 \right) \\
&= \frac{1}{\ln 2} \left[\sum_u \frac{1}{|\mathcal{U}|} - \sum_u p(u) \right] \\
&= 0
\end{aligned}
$$

$\square$

Theorem 4.3. $I(U;V) = 0 \iff U \perp V$

Definition 4.1 (Entropy rate of a stochastic process)**.**
$$r = \lim_{n \to \infty} \frac{1}{n} H(U^n) \quad \text{if the limit exists}$$

Theorem 4.4. *For stationary stochastic process U^n, the sequences*
$$a_n = \frac{1}{n} H(U^n) \text{ and } b_n = H(U_n|U^{n-1})$$
are positive and non increasing. Then $a = \lim_{n \to \infty} a_n$ and $b = \lim_{n \to \infty} b_n$ exists and $a = b$.

Proof.

$$
\begin{aligned}
b_{n+1} &= H(U_{n+1}|U_1, U_2, \ldots, U_n) \\
&\leq H(U_{n+1}|U_2, \ldots, U_n) \\
&= H(U_n|U_1, U_2, \ldots, U_{n-1}) \\
&= b_n \text{ , because } U_1 \ldots U_n \sim U_2 \ldots U_{n+1} \text{ (Stationarity).}
\end{aligned}
$$

Hence, it is non-increasing.

For the $\{a_n\}$, observe that

$$a_n = \frac{1}{n}H(U^n) = \frac{1}{n}\left[H(U_1) + H(U_2|U_1) + H(U_3|U^2) + \cdots + H(U_n|U^{n-1})\right]$$

$$= \frac{1}{n}\left[b_1 + b_2 + \cdots + b_n\right]$$

and by the "Lemma", whenever $b_n \to b$, $a_n \to b$ □

Lemma 4.5 (Cesaro). *Suppose $b_n \to b$,*

then,

$$a_n = \frac{1}{n}\left[b_1 + b_2 + \cdots + b_n\right] \text{ also converges and to 1.}$$

Proof. Since $b_n \to b$, $\left(\equiv \forall \epsilon > 0 , \exists\, n(\epsilon) \text{ s.t } \forall n > n(\epsilon)\, |b_n - b| < \epsilon\right)$

$\exists B$ s.t. $|b_n| < B$ for all n.

Take $n > n_1(\epsilon) \triangleq \ldots$ then

$$|a_n - b| \leq \frac{|b_1 - b| + |b_2 - b| + |b_3 - b| + \cdots + |b_n - b|}{n}$$

$$\text{so } |a_n - b| \leq \frac{1}{n}\left[\sum_{i=1}^{n_0(\epsilon)} \underbrace{|b_i - b|}_{2B} + \sum_{i=n_0(\epsilon)+1}^{n} \underbrace{|b_i - b|}_{\leq \epsilon}\right] \leq \frac{n_0(\epsilon)2B}{n} + \epsilon < 2\epsilon$$

$$\text{for } n > n_1(\epsilon) \triangleq \max, \{n_0(\epsilon)\frac{1}{\epsilon}n_0(\epsilon)2B\}$$

                                         □

Theorem 4.6. *Given a stationary process with entropy rate r:*

$$r = \lim_{n \to \infty} \frac{1}{n}H(U^n)$$

then

1. for every source coding scheme

$$\mathcal{C}_n : U^n \to \{0, 1\}^*$$

the expected number of bits / letter is given by

$$\frac{1}{n}E[l(\mathcal{C}(U^n))] \geq r$$

2. for any $\epsilon > 0$, there exists a source coding scheme $\mathcal{C}_n : U^n \to \{0, 1\}^*$ s.t.

$$\frac{1}{n}E[l(\mathcal{C}_n(U^n))] < r + \epsilon$$

Proof. 1. we already know

$$\frac{1}{n}E[l(\mathcal{C}_n(U^n))] \geq \frac{1}{n}H(U^n)$$

and the right term is decreasing

2. we also know that for each $n, \exists \mathcal{C}_n$ that is prefix-free s.t.

$$E[l(\mathcal{C}_n(U^n))] < \underbrace{\frac{1}{n}H(U^n)}_{r} + \underbrace{\frac{1}{n}}_{0}$$

we can find n large enough s.t. the right hand side $< r + \epsilon$

$\square$

5   Typicality and typical set

Definition 5.1 (Typicality). *Suppose we have a sequence $U_1, U_2, ...$ of i.i.d. random variables taking values in an alphabet $\mathcal{U}$. Suppose we observe $u_1, u_2..., u_n$. We will call it to be typical-(ϵ, p) if*

$$p(u)(1 - \epsilon) \leq \frac{\# \ of \ times \ u \ appears \ in \ u_1, ..., u_n}{n} \leq p(u)(1 + \epsilon)$$

Theorem 5.1. *u^n is (ϵ, p)-typical then*

$$2^{-nH(u)(1+\epsilon)} \leq Pr(U^n = u^n) \leq 2^{-nH(u)(1-\epsilon)}$$

Proof.

$$Pr(U^n = u^n) = \prod_{i=1}^{n} Pr(U_i = u_i) = \prod_{i=1}^{n} p(u_i) = \prod_{u \in \mathcal{U}} p(u)^{\#_u}$$

with $\#_u$ the number of times u appears in $u_1, ..., u_n$ where

$$n(1 - \epsilon)p(u) \leq \#_u \leq n(1 + \epsilon)p(u)$$

consequently

$$p(u)^{(np(u)(1-\epsilon))} \geq p(u)^{\#_u} \geq p(u)^{np(u)(1+\epsilon)}$$

then

$$(\prod_n p(u)^{p(u)})^{(1-\epsilon)n} \geq Pr(U^n = u^n) \geq (\prod_n p(u)^{p(u)})^{(1+\epsilon)n}$$

but

$$p(u)^{p(u)} = 2^{-p(u)\log \frac{1}{p(u)}} \Rightarrow \prod p(u)^{p(u)} = 2^{-H(u)}$$

$\square$

Definition 5.2 (Typical set).

$$T(n, \epsilon, p) = \{u^n \in \mathcal{U}^n : u^n \ is \ (\epsilon, p)\text{-}typical\}$$

Wikipedia. *Typical sets provide a theoretical means for compressing data, allowing us to represent any sequence X^n using $nH(X)$ bits on average, and, hence, justifying the use of entropy as a measure of information from a source.*

Theorem 5.2.   *1. if $u^n \in T(n, \epsilon, p)$ then*

$$p(u^n) = Pr(U^n = u^n) = 2^{-nH(u)(1 \pm \epsilon)}$$

when U_i i.i.d.

2.

$$\lim_{n \to \infty} Pr(U^n \in T(n, \epsilon, p)) = 1$$

3.

$$|T(n, \epsilon, p)| \leq 2^{n(H(u)(1+\epsilon))}$$

4.

$$|T(n, \epsilon, p)| \geq (1 - \epsilon)2^{nH(u)(1-\epsilon)}$$

Proof.   1. Fix $u \in \mathcal{U}$ let $X_i = 1$ if $U_i = u$ and 0 otherwise

$$\frac{\text{\# of times } u \text{ appears in } U_1...U_n}{n} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

observe that $\{X_i\}$ are i.i.d.

$$X_i = \begin{cases} 1 & \text{w.p. } p(u) \\ 0 & \text{w.p. } 1 - p(u) \end{cases}$$

$$\Rightarrow \mathbb{E}[X_i] = p(u) \quad \text{and} \quad Var[X_i] = p(u) - p(u)^2$$

$$\underbrace{Pr\left\{ |\frac{1}{n} \sum_{i+1}^{n} X_i - p(u)| \geq \epsilon p(u) \right\}}_{u^n \text{ fails the test for letter } u} \leq \frac{Var(\frac{1}{n}\sum X_i)}{(\epsilon p(u))^2} = \frac{(1-p(u))}{\epsilon^2 p(u)}$$

2.

$$Pr\{U^n \notin T(n,\epsilon,p)\} = Pr\left\{ \bigcup_{u \in U} \{u^n \text{ fails the test for u}\} \right\}$$

$$\leq \sum_{u \in U} Pr\{U^n \text{ fails the test for } u\}$$

$$\leq \frac{1}{n} \sum_{u \in U} \frac{(1-p(u))}{p(u)\epsilon^2} \quad \text{which goes to 0 as } n \text{ gets large}$$

3.

$$1 \geq Pr\{U^n \in T(n,\epsilon,p)\} = \sum_{u^n \in T(n,\epsilon,p)} Pr\{U^n = u^n\}$$

$$\geq \sum_{u^n \in T(n,\epsilon,p)} 2^{-n(1+\epsilon)H(u)}$$

$$= 2^{-n(1+\epsilon)H(u)}|T(n,\epsilon,p)|$$

4.

$$1 - \epsilon \leq Pr\{U^n \in T(n,\epsilon,p)\} = \sum_{u^n \in T(n,\epsilon,p)} Pr\{U^n = u^n\}$$

$$\leq \sum_{u^n \in T(n,\epsilon,p)} 2^{nH(u)(1-\epsilon)}$$

$$= 2^{-nH(u)(1-\epsilon)}|T(n,\epsilon,p)|$$

$\square$

Observation 5.1. $Pr\{U^n \in T(n,\epsilon,p)\} \to 1$ *as* $n \to \infty$

Definition 5.3 (Kullback Leibler divergence)**.**

$$D(p||q) = \sum_{u} p(u) \log \frac{p(u)}{q(u)} \geq 0 \text{ with equality iff } p = q$$

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If we compress data in a manner that assumes $q(u)$ is the distribution underlying some data, when, in reality, $p(u)$ is the correct distribution, the Kullback-Leiber divergence is the average number of additional bits per datum necessary for compression. It is also called **relative entropy** and is a measure of how one probability distribution diverges from a second probability distribution.

Lemma 5.3. *if $U_1 \ldots U_n$ are i.i.d. with distribution q and $u_1 \ldots u_n$ is (ϵ, p)-tipycal, then*

$$2^{-n[H(p)+D(p||q)](1+\epsilon)} \leq Pr\left\{U^n = u^n\right\} \leq 2^{-n[H(p)+D(p||q)](1-\epsilon)}$$

Proof. Follows from

$$\left[\prod_u q(u)^{p(u)}\right]^{n(1+\epsilon)} \leq Pr\left\{U^n = u^n\right\} \leq \left[\prod_u q(u)^{p(u)}\right]^{n(1-\epsilon)}$$

$$\prod_u q(u)^{p(u)} = 2^{-\sum p(u)\log\frac{1}{q(u)}}$$

and

$$\sum_u p(u)\log\frac{1}{q(u)} = \underbrace{\sum_u p(u)\log\frac{1}{p(u)}}_{H(p)} + \underbrace{\sum_u p(u)\log\frac{p(u)}{q(u)}}_{D(p||q)}$$

$\square$

Corollary 5.3.1. *if $U_1 \ldots U_n$ are i.i.d. following distribution q, then*

$$2^{-n[(1+\epsilon)D(p||q)+2\epsilon H(p)]} \leq Pr\left\{U^n \in T(n,\epsilon,p)\right\} \leq 2^{-n[(1-\epsilon)D(p||q)-2\epsilon H(p)]}$$

Proof.

$$Pr\left\{U^n \in T(n,\epsilon,p)\right\} = \sum_{u^n \in T(n,\epsilon,p)} Pr\left\{U^n = u^n\right\}$$

We have

$$2^{-n[H(p)+D(p||q)](1+\epsilon)} \leq Pr\left\{U^n = u^n\right\} \leq 2^{-n[H(p)+D(p||q)](1-\epsilon)}$$
$$2^{nH(p)(1-\epsilon)} \leq |T(n,\epsilon,p)| \leq 2^{nH(p)(1+\epsilon)}$$

$\square$

Example 3. $U \in \{0,1\}$, $p = \frac{1}{2}, \frac{1}{2}$, $q = \frac{1}{2} - \delta, \frac{1}{2} + \delta$

$$D(p||q) = \frac{1}{2}\log\frac{1}{1-2\delta} + \frac{1}{2}\log\frac{1}{1+2\delta} = \frac{1}{2}\log\frac{1}{1-4\delta^2} = -\frac{1}{2}\log(1-4\delta^2) \approx \frac{1}{2}4\delta^2 + o(\delta^4)$$

So if we want $2^{-nD(p||q)}$ small, we must pick $n = \Omega(1/\delta^2)$

Example 4. Suppose we are told that U is p distributed and $p(u)$ are powers of 2. We design a prefix-free code $\mathcal{C}$ to minimize $\sum_u p(u)l(\mathcal{C}(u))$. We have been misinformed and $U \sim q$, then:

$$\mathbb{E}\left[l(\mathcal{C}(u))\right] = \sum_u q(u)\log\frac{1}{p(u)}$$
$$= \underbrace{H(q)}_{\text{length for optimal code}} + \underbrace{D(q||p)}_{\text{penalty for misbelief}}$$

6 Universal data compression

Suppose we know that the distribution p of U is either $p_1, p_2 \dots p_k$, can we design a code $\mathcal{C} : U \to \{0,1\}^*$

$$\mathbb{E}\left[l(\mathcal{C}(U))\right] \leq H(U) + \text{small for every } p$$

$$\mathbb{E}\left[\frac{1}{n}l(\mathcal{C}(U))\right] \leq o(n) + \mathbb{E}\left[h_2\left(\frac{K}{n}\right)\right]$$

with $K = \sum_{i=1}^{n} u_i$

We have $\frac{\mathbb{E}[K]}{n} = \theta_1$ and $\mathbb{E}\left[h_2\left(\frac{K}{n}\right)\right] \leq h_2\left(\mathbb{E}\left[\frac{K}{n}\right]\right) = h_2(\theta)$

Design $\mathcal{C}$ Because the probability of a bit string is only dependant of the number of 1s (or 0s), it makes sense to encode two strings with the same numbers of 1 with code words of same lengths. Given $u_1 \dots u_n \in \{0,1\}^n$, first count the number of 1, call it k.

$$\mathcal{C}(u_1 \dots u_n) = \underbrace{\text{describe } k}_{\lceil \log(n+1) \rceil} \underbrace{\text{describe } u_1 \dots u_n}_{\lceil \log \binom{n}{k} \rceil}$$

We now want to evaluate

$$\frac{1}{n}\mathbb{E}\left[l(\mathcal{C}(U))\right]$$

when $U_1 \dots U_n$ are i.i.d with $p_1 = \theta$ and $p_0 = 1 - p_1$

Observation 6.1. *for any* $0 \leq \alpha \leq 1$

$$1 = 1^n = (\alpha + (1-\alpha))^n = \sum_{i=0}^{n} \binom{n}{i} \alpha^i (1-\alpha)^{n-i}$$

$$\geq \binom{n}{k} \alpha^k (1-\alpha)^{n-k}$$

Then for all α

$$\binom{n}{k} \leq \alpha^{-k}(1-\alpha)^{-(n-k)} = 2^{n\left(\frac{k}{n}\log\frac{1}{\alpha} + \left(1-\frac{k}{n}\right)\log\frac{1}{1-\alpha}\right)}$$

We pick $\alpha = \frac{k}{n}$, *and we get*

$$\binom{n}{k} < 2^{nh_2\left(\frac{k}{n}\right)}$$

Using this bound we have

$$\frac{1}{n}l(\mathcal{C}(u_1 \dots u_n)) \leq \frac{2}{n} + \frac{\log(n+1)}{n} + h_2\left(\frac{k}{n}\right)$$

$$\mathbb{E}\left[\frac{1}{n}l(\mathcal{C}(U))\right] \leq o(n) + \mathbb{E}\left[h_2\left(\frac{k}{n}\right)\right]$$

Claim 6.1. *Suppose* U_i *are i.i.d. with* $Pr\{U_1 = 1\} = \theta$. *We have* $\mathbb{E}\left[\frac{k}{n}\right] = \theta$ *and* $\mathbb{E}\left[h_2\left(\frac{k}{n}\right)\right] \leq h_2(\mathbb{E}\left[\frac{k}{n}\right]) = h_2(\theta)$. *So*

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[l(\mathcal{C}(u_1 \dots u_n))\right] \leq h_2(\theta)$$

consequently this scheme is asymptotically optimal.

Proof. To prove the claim we need to show that if $\beta_1 \ldots \beta_k$ are in $[0,1]$ and $q_1 \ldots q_k$ are non negative numbers that sum to 1 then

$$\sum_{i=1}^{k} q_i h_2(\beta_i) \leq h_2\left(\sum_{i=1}^{k} q_i \beta_i\right)$$

Let U and V be random variables with $U \in \{0,1\}$ and $V \in \{1, \ldots, k\}$ with

$$Pr\{V = i\} = q_i$$
$$Pr\{U = 1|V = i\} = \beta_i$$
$$Pr\{U = 0|V = i\} = 1 - \beta_i$$

Then,

$$Pr\{U = 1\} = \sum_i q_i \beta_i$$

$$H(U) = h_2\left(\sum_i q_i \beta_i\right)$$

$$H(U|V) = \sum_i q_i h_2(\beta_i)$$

And we already know that $H(U) \geq H(U|V)$ $\qquad\square$

Suppose we have an infinite string $u_1 u_2 ..., u_i \in \mathcal{U}$, and

$$u_1 u_2 ... = v_1 v_2 ... \text{ with } v_i \in \mathcal{U}^*, v_i \neq v_j \text{ when } i \neq j$$

for any k we have

$$\lim_{m \to \infty} \frac{length(v_1...v_m)}{m} \geq k \Rightarrow \lim_{m \to \infty} \frac{length(v_1...v_m)}{m} = \infty$$

Definition 6.1. *Given an infinite string $u_1 u_2 ...$ and a machine M, let*

$$\rho_M(u_1 u_2 ...) = \overline{\lim_{n \to \infty}} \frac{length \ of \ the \ output \ M \ after \ reading \ u_1 u_2 ...}{n}$$

also given $s > 0$, define

- *The compressibility of $\mathcal{U}^*$ be s-state machines*

$$\rho_s(u_1 u_2 ...) = \min_M \rho_M(u_1 u_2 ...)$$

  with M an s'-state machine with $s' \leq s$

- *Compressibility of $\mathcal{U}^*$ by finite state machines*

$$\rho_{FSM}(u_1 u_2 ...) = \lim_{s \to \infty} \rho_s(u_1 u_2 ...)$$

Definition 6.2. *Suppose $u_1 u_2 ...$ an infinite sequence, define $m(n)$ as the largest m for which $u_1...u_n = v_1...v_m$ with distinct $v_1...v_m$*

Example 5.

$$u = aaaaaaaaaa, \quad \underbrace{\emptyset}_{v_1} \underbrace{a}_{v_2} \underbrace{aa}_{v_3} \underbrace{aaa}_{v_4} \underbrace{aaaa}_{v_5} \quad \Rightarrow m(10) = 5$$

So far we know that

$$\frac{\text{length of the output of any s-state IL machine when it reads } u_1u_2...}{n} \geq \frac{m(n)\log\frac{m(n)}{8s^2}}{n}$$

with

$$\frac{m(n)\log\frac{m(n)}{8s^2}}{n} = \frac{m(n)\log(m(n))}{n} - \frac{m(n)\log(8s^2)}{length(v_1...v_m)}$$

hence if M is a s-state machine

$$\rho_M(u_1u_2...) \geq \varlimsup_{n\to\infty} \frac{m(n)\log(m(n))}{n} \quad \text{then} \quad \rho_{FSM}(u_1u_2...) \geq \varlimsup_{n\to\infty} \frac{m(n)\log m(n)}{n}$$

7   Lemple-Ziv data compression method

Given some alphabet $\mathcal{U}$ to both encoder and decoder, they also agree an order on $\mathcal{U}$:

1. Start with a dictionary $\mathcal{D} = \mathcal{U}$

2. To each word $w \in \mathcal{D}$, assign a $\lceil \log |\mathcal{D}| \rceil$-bit binary description in the dictionary order

3. Parse the first word w in $u_1 u_2 ...$ in the dictionary, output its binary description

4. replace w in $\mathcal{D}$ by $\{wu, \forall u \in \mathcal{U}\}$.

5. Go to 2.

Example 6. Define an alphabet $\mathcal{U} = \{a, b, c\}$ with $a \leq b \leq c$ and an input message

$$u = bbacac$$

- Create the dictionary $\mathcal{D} = \{a, b, c\}$ and its corresponding binary description $\mathcal{D}_{bin} = \{00, 01, 10\}$

- The first word in the message is $'b'$, output its binary description

$$output = 01$$

- Update the dictionary:

$$\mathcal{D} = \{a, ba, bb, bc, c\} \quad \mathcal{D}_{bin} = \{000, 001, 010, 011, 100\}$$

- Parse the next word $'ba'$ and output its binary description

$$output = 01001$$

- Update the dictionary

$$\mathcal{D} = \{a, baa, bab, bac, bb, bc, c\} \quad \mathcal{D}_{bin} = \{000, 001, ...\}$$

- Continue until the end of the input data...

The decoder can proceed in a similar way to iteritavely update the dictionary while decoding the message.

7.1   Analysis of LZ

Observe that LZ parses the string $u_1 u_2 ...$ into $v_1 v_2 ...$ with $v_i \in \mathcal{U}^*$ or $v_i \in \mathcal{D}_i$ where $\mathcal{D}_i$ is the dictionary at step i.

When going from iteration $i \rightarrow i + 1$, v_i is removed from $\mathcal{D}$, consequently v_1, v_2, v_3 are distinct.

The length of the output of LZ after reading $u_1 ... u_m$ is given by

$$\text{LZ output's length} = \lceil \log |\mathcal{U}| \rceil + \lceil \log(2|\mathcal{U}| - 1) \rceil + \lceil \log(3|\mathcal{U}| - 2) \rceil + ... + \lceil \log(m|\mathcal{U}| - m + 1) \rceil$$

we observe that

$$\text{LZ output's length} < m(\log(m|\mathcal{U}|) + 1) = m \log(2m|\mathcal{U}|)$$

Also we have

$$\# \text{ bits / letter} < \frac{m \log(2m|\mathcal{U}|)}{length(u_1...u_m)}$$

$$= \frac{m \log(m)}{length(u_1...u_m)} + \frac{m \log(2|\mathcal{U}|)}{length(u_1...u_m)}$$

therefore

$$\rho_{LZ}(u_1 u_2 ...) = \lim_{m \to \infty} \frac{\# \text{ bits}}{\text{letter}} \leq \lim_{m \to \infty} \frac{m \log(m)}{length(u_1...u_m)} \leq \lim_{n \to \infty} \frac{m(n) \log(m(n))}{n} \leq \rho_{FSM}(u_1 u_2 ...)$$

So we have proved the following theorem:

Theorem 7.1. *for every $u_1 u_2 ...$*

$$\rho_{LZ}(u_1 u_2 ...) \leq \rho_{FSM}(u_1 u_2 ...)$$

Corollary 7.1.1. *if $u_1 u_2 ...$ is stationary*

$$\rho_{LZ}(u_1 u_2 ...) = \ \textit{entropy rate of } u_1 u_2 ...$$

8 Transmission of data

Interesting in the case of unreliable transmission media.

Definition 8.1 (Communication channel). *A communication channel W is a device with an input alphabet $\mathcal{X}$ and an output alphabet $\mathcal{Y}$. Its behabvior is described by*

$$W_i(y_i|x^i, y^{i-1}) = Pr\left\{Y_i = y_i|X^i = x^i, Y^{i-1} = y^{i-1}\right\}$$

Definition 8.2 (Memoryless channel). *a channel W is said to be memoryless if*

$$W_i(y_i|x^i, y^{i-1}) = W(y_i|x_i)$$

Definition 8.3 (Stationary channel). *a channel W is said to be stationary if*

$$W_i(y|x) = W(y|x)$$

Example 7 (Binary erasure channel - BEC). $\mathcal{X} = \{0, 1\}$ and $\mathcal{Y} = \{0, 1, ?\}$, then

$$W(0|0) = 1 - p$$
$$W(?|0) = p$$
$$W(1|0) = 0$$

and same for $x_i = 1$.

Example 8 (Binary symetric channel - BSC).

$$W(0|0) = 1 - p = W(1|1)$$
$$W(1|0) = p = W(0|1)$$

The input $X_1, X_2 \ldots X_n$ to a channel might have memory

$$Pr\left\{X^n = x^n\right\} = p(x_1)p(x_2|x_1)\ldots p(x_i|x^{i-1})\ldots p(x_n|x^{n-1})$$

$$Pr\left\{X^n = x^n, Y^n = y^n\right\} = p(x_1)W_1(y_1|x_1)p(x_2|x_1, y_1)W(y_2|x_1, x_2, y_1)\ldots$$
$$= \prod_i p(x_i| \underbrace{x^{i-1}}_{\text{feedback}} \underbrace{y^{i-1}}_{\text{memory}})W_i(y_i|x^i y^{i-1})$$

Lemma 8.1. *if there is no feedback and the channel is memoryless and stationary, then*

$$Pr\left\{Y^n = y^n|X^n = x^n\right\} = \prod_{i=1}^n W(y_i|x_i)$$

Proof.

$$Pr\left\{Y^n = y^n, X^n = x^n\right\} = \prod_{i=1}^n p(x_i|x^{i-1}y^{i-1})W_i(y_i|x^i y^{i-1})$$
$$= \prod_{i=1}^n p(x_i|x^{i-1})W(y_i|x_i)$$
$$= \prod_{i=1}^n W(y_i|x_i)Pr\left\{X^n = x^n\right\}$$

$\square$

Example 9. Suppose W is BSC(1/2) but we have feedback, defined by $X_1 = 0$ and $X_i = Y_{i-1}$.

$$Pr\left\{Y^2 = 00 | X^2 = 01\right\} = 0$$

$$W(0|0)W(0|1) = \frac{1}{4}$$

Lemma 8.2. *if W is stationary memoryless and there is no feedback, then*

$$H(Y^n|X^n) = \sum_{i=1}^{n} H(Y_i|X_i)$$

Proof.

$$H(Y^n|X^n) = \mathbb{E}\left[\log \frac{1}{Pr\left\{Y^n|X^n\right\}}\right] = \mathbb{E}\left[\log \prod_{i=1}^{n} \frac{1}{Pr\left\{Y_i|X_i\right\}}\right] = \sum_{i=1}^{n}\mathbb{E}\left[\log \frac{1}{Pr\left\{Y_i|X_i\right\}}\right] = \sum_{i=1}^{n} H(Y_i|X_i)$$

$\square$

For a memoryless stationary channel $W(Y|X)$ we can compute, for any distribition $p(x)$, $p(x,y) = p(x)W(y|x)$ and $I(X;Y)$, we can also compute

$$C(W) = \max_{p(x)} I(X;Y)$$

Lemma 8.3. *for a stationary memoryless W without feedback, we have*

$$I(X^n;Y^n) \leq nC(W)$$

Proof.

$$I(X^n;Y^n) = H(Y^n) - H(Y^n|X^n)$$
$$= H(Y^n) - \sum_i H(Y_i|X_i)$$
$$\leq \sum_i H(Y_i) - \sum_i H(Y_i|X_i)$$
$$= \sum_i I(X_i;Y_i)$$

Note that the joint distribution $Pr\left\{X_i, Y_i\right\}$ is of the form $p(x)W(y|x)$, then $I(X_i;Y_i) \leq C(W)$ $\square$

Notation 2. *for simplicity*

$$p * q = (1-q)p + q(1-p)$$

Example 10. Let W be a BSC(p), $Pr\left\{X = 0\right\} = 1 - q$ and $Pr\left\{X = 1\right\} = q$. Then

$$Pr\left\{Y = 0\right\} = (1-q)(1-p) + qp$$
$$Pr\left\{Y = 1\right\} = (1-q)p + q(1-p)$$

$$H(Y|X = 0) = p\log\frac{1}{p} + (1-p)\log\frac{1}{1-p}$$
$$H(Y|X = 1) = p\log\frac{1}{p} + (1-p)\log\frac{1}{1-p}$$
$$H(Y|X) = p\log\frac{1}{p} + (1-p)\log\frac{1}{1-p}$$

$$I(X;Y) = H(Y) - H(Y|X)$$
$$= (p*q)\log\frac{1}{p*q} + (1-(p*q))\log\frac{1}{1-(p*q)} - \left[p\log\frac{1}{p} + (1-p)\log\frac{1}{1-p}\right]$$

We maximize $I(X;Y)$ for $q = 1/2$

$$C(W) = \log 2 - h_2(p)$$

Example 11. Let W be BEC(p) and $Pr\{X = 1\} = q$

$$H(X) = h_2(q)$$
$$H(X|Y = 0) = 0$$
$$H(X|Y = 1) = 0$$
$$H(X|Y =?) = h_2(q)$$

$$I(X;Y) = h_2(q) - ph_2(q) = (1-p)h_2(q)$$
$$C(W) = (1-p)\log 2$$

8.1 Fano's inequality

Suppose U and V take values in the same alphabet $\mathcal{U}$, then

$$H(U|V) \le p_e \log(|\mathcal{U}| - 1) + h_2(p_e)$$

with

$$p_e = Pr\{U \ne V\} \quad \text{and} \quad h_2(p) = p\log\frac{1}{p} + (1-p)\log\frac{1}{(1-p)}$$

Proof. Define

$$Z = \begin{cases} 1 & U \ne V \\ 0 & U = V \end{cases}, \quad H(Z) = h_2(p_e)$$

$$H(UZ|V) = H(U|V) + H(Z|UV)$$
$$= H(Z|V) + H(U|VZ)$$
$$\le H(Z) + H(U|VZ)$$

but

$$H(U|VZ) = \underbrace{H(U|V, Z = 0)}_{0} Pr\{Z = 0\} + \underbrace{H(U|V, Z = 1)}_{\le\log(|\mathcal{U}|-1)}\underbrace{Pr\{Z = 1\}}_{p_e}$$

$\square$

So if $H(U|V) > \lambda \Rightarrow \exists f(\lambda) > 0, p_e > f(\lambda)$

Corollary 8.3.1. *Suppose U^L, V^L are random sequences with common alphabet $\mathcal{U}$, define :*

$$p_{e,i} = Pr\{U_i \ne V_i\}, \quad \bar{p}_e = \frac{1}{L}\sum_{i=1}^{L} p_{e,i}$$

then

$$\frac{1}{L}H(U^L|V^L) \le h_2(\bar{p}_e) + \bar{p}_e \log(|\mathcal{U}| - 1)$$

Proof.

$$\frac{1}{L}H(U^L|V^L) = \frac{1}{L}\sum_{i=1}^{L}H(U_i|U^{i-1}V^L)$$

$$\leq \frac{1}{L}\sum_{i=1}^{L}H(U_i|V_i)$$

$$\leq \frac{1}{L}\sum_{i=1}^{L}(p_{e,i}\log(|\mathcal{U}|-1) + h_2(p_{e,i}))$$

$$= \bar{p}_e\log(|\mathcal{U}|-1) + \frac{1}{L}\sum_{i=1}^{L}h_2(p_{e,i})$$

$$\leq \bar{p}_e\log(|\mathcal{U}|-1) + h_2(\frac{1}{L}\sum_{i=1}^{L}p_{e,i})$$

$$= \bar{p}_e\log(|\mathcal{U}|-1) + h_2(\bar{p}_e)$$

$\square$

Theorem 8.4. *"Bad news" theorem, converse to the coding theorem*

- *Suppose we have a stationary source $U_1 U_2...$ with entropy rate H and produces a letter every τ_s seconds.*

- *Suppose also that we have a channel W that accepts input $X_1 X_2...$ once every τ_c seconds.*

- *Suppose also*

$$\frac{H}{\tau_s} > \frac{C(W)}{\tau_c}$$

then there is a $\lambda > 0$ such that $\bar{p}_e > \lambda$

Definition 8.4. *stable suppose the encoder works by taking blocks of L letters*

$$(U_1...U_L)(U_{L+1}...U_{2L})...$$

and outputs

$$(X_1...X_n)(X_{n+1}...U_{2n})...$$

then the encoder is stable if

$$L\tau_s \geq n\tau_c$$

Proof. Recall that for a stationary source $\frac{H(U_1...U_L)}{L}$ tends to H so

$$H(U_1...U_L) \geq LH$$

We also have

$$I(U^L;V^L) \leq nC(W)$$

therefore, since $\frac{n}{L} \leq \frac{\tau_s}{\tau_c}$

$$H(U^L|V^L) = \frac{1}{L}(H(U^L) - I(U^L;V^L)) \geq H - \frac{n}{L}C(W)$$

$$\geq H - \frac{\tau_s}{\tau_c}C(W)$$

$$= \tau_s(\frac{H}{\tau_s} - \frac{C(W)}{\tau_c})$$

The right hand side is

$$\epsilon(\tau_c, \tau_s, H, C) > 0$$

so for every stable encoder, decoder, we have

$$\bar{p}_e \log(|\mathcal{U}| - 1) + h_2(\bar{p}_e) > \epsilon(\tau_s, \tau_c, H, C)$$

then

$$\bar{p}_e \geq \epsilon(\tau_s, \tau_c, H, C, |\mathcal{U}|)$$

$\square$

Example 12. Suppose $\mathcal{U} = \{0, 1\}$ and $U_1 U_2 ...$ is a Markov process with

$$U_1 = \left\{ \begin{array}{ll} 0 & \text{with } p = 0.5 \\ 1 & \text{with } p = 0.5 \end{array} \right. , \quad p(U_{n+1}|U_n) = \left\{ \begin{array}{ll} 1 - p & u_{n+1} = u_n \\ p & u_{n+1} \neq u_n \end{array} \right. ,$$

$$\begin{aligned} H &= \lim_{n \to \infty} H(U_n | U^{n-1}) \\ &= \lim_{n \to \infty} H(U_n | U_{n_1}) \\ &= H(U_2 | U_1) = h_2(p) \end{aligned}$$

suppose $W = BEC(q), C(W) = (1 - q) \log 2$ and $\tau_s = \tau_c = 1$

$$h_2(\bar{p}_e) \geq h_2(p) - (1 - q) \log(2) \Rightarrow \bar{p}_e \geq \lambda$$

What we want to do next is to show a matching "Good news" theorem:

We could show that if $\frac{H}{\tau_s} \leq \frac{C(W)}{\tau_c}$ then for any $\lambda > 0$, we can find a stable encoder and decoder such that $p_e < \lambda$. Instead, we will show stronger results:

1. **Separation theorem** The encoder can be designed in a modular way:

   - A **source encoder** which encoder message words in bits. The design of this encoder is strongly dependent of the type of the input.

   - A **channel encoder** which encoder the bits to maximize the performance with a specific channel.

2. We will show that

$$Pr\left\{U^L \neq V^L\right\} < \lambda$$

using the fact that

$$(U_i \neq V_i) \Rightarrow (U^L \neq V^L) \quad \text{so} \quad p_{e,i} \leq Pr\left\{U^L \neq V^L\right\} \Rightarrow \bar{p}_e \leq Pr\left\{U^L \neq V^L\right\}$$

We will now show that good channel encoders and channel decoders exist

Definition 8.5. *Given a channel W with input alphabet $\mathcal{X}$, a block encoder is a function*

$$Enc : \{1, ..., M\} \to \mathcal{X}^n$$

with n the block length.
$Enc(1), ..., Enc(M)$ are each called codewords and M is equal to the number of codewords.
The rate of the code can be defined by

$$R = \frac{\log M}{n}$$

Definition 8.6. *Given a channel W with ouput alphabet Y, a block decoder is a function*

$$Dec : \mathcal{Y}^n \to \{?, 1, ..., M\}$$

Definition 8.7.

$$p_{error}(m) = Pr\{\hat{m} \neq m | m\}$$

$$\bar{p}_{error}(m) = \frac{1}{M} \sum_{m=1}^{M} p_{error}(m)$$

$$\hat{p}_{error}(m) = \max_m p_{error}(m)$$

8.2 Computational consideration for $C(W)$

We have an optimization problem

$$\max_{p_X} f(p_X) \quad \text{where} \quad f(p_X) = I(X;Y)$$

See section C for further information on convex optimization.

Claim 8.1. *f is a concave function*

We want to compute

$$\frac{\partial I(X;Y)}{\partial p(x)}$$

We have

$$I(X;Y) = \sum_{x,y} p(x)W(y|x) \log \frac{W(y|x)}{p_Y(y)}$$

$$p_Y(y) = \sum_{x} p(x)W(y|x)$$

$$\begin{aligned}
\frac{\partial I}{\partial p(x_0)} &= \sum_{x,y} \frac{\partial}{\partial p(x_0)} \left\{ p(x)W(y|x) \log \frac{W(y|x)}{p_Y(y)} \right\} \\
&= \sum_{x,y} \left\{ I_{x=x_0} W(y|x) \log \frac{W(y|x)}{P_Y(y)} - p(x)W(y|x)\frac{W(y|x_0)}{p_Y(y)} \log e \right\} \\
&= \sum_{y} W(y|x_0) \log \frac{W(y|x_0)}{p_Y(y)} - \sum_{y} p_Y(y)\frac{W(y|x_0)}{p_Y(y)} \log e \\
&= \sum_{y} W(y|x_0) \log \frac{W(y|x)}{P_Y(y)} - \log e
\end{aligned}$$

Theorem 8.5. *p_X maximizes $I(X;Y)$ iff there exists λ such that for all x*

$$\sum_{y} W(y|x) \log \frac{W(y|x)}{P_Y(y)} \leq \lambda$$

with equality when $p_X(x) \neq 0$. Furthermore $\lambda = C(W)$.

Proof. We only need to prove the furthermore part. Observe that for all x

$$p_X(x) \sum_y W(y|x) \log \frac{W(y|x)}{P_Y(y)} = p_X(x)\lambda$$

and then

$$\sum_x p_X(x) \sum_y W(y|x) \log \frac{W(y|x)}{P_Y(y)} = \sum_x p_X(x)\lambda$$

$\square$

Example 13 (Z channel). W is a normal binary channel that maps a 1 input to a 0 output with probability ϵ. Applying theorem 8.5 with $x = 0$ and $x = 1$:

$$W(0|0) \log \frac{W(0|0)}{p_Y(0)} = W(0|1) \log \frac{W(0|1)}{p_Y(1)} + W(1|1) \log \frac{W(1|1)}{p_Y(1)}$$

$$\Longleftrightarrow \log \frac{1}{p_Y(y)} = \epsilon \log \frac{\epsilon}{p_Y(0)} + (1 - \epsilon) \log \frac{1 - \epsilon}{p_Y(1)} = h_2(\epsilon) + \epsilon \log \frac{1}{p_Y(0)} + (1 - \epsilon) \log \frac{1}{p_Y(1)}$$

$$\Longleftrightarrow \log \frac{p_Y(1)}{p_Y(0)} = -\frac{h_2(\epsilon)}{1 - \epsilon} \triangleq -\alpha$$

$$\Longrightarrow p_Y(1) = \frac{2^{-\alpha}}{1 + 2^{-\alpha}} \text{ and } p_Y(0) = \frac{1}{1 + 2^{-\alpha}}$$

$$C(W) = \log(1 + 2^{-\alpha})$$

Lemma 8.6. *For any circle with red segments of cumulative length strictly less than $1/4$, there exists a square whose all corners are on the circle but not on the red segments.*

Proof. By random construction. Place the first corner of the square uniformly at random on the circle (also makes the 3 other uniform).

$$Pr\{\text{1st corner lands on red}\} < \frac{1}{4}$$

$$Pr\{i\text{th corner lands on red}\} < \frac{1}{4}$$

$$Pr\left\{\bigcup_{i=1} i\text{th corner lands on red}\right\} < 1$$

$$Pr\{\text{none of the corners land on red}\} > 0$$

$\square$

Theorem 8.7 (Channel coding - good news). *Given a channel W (discrete, memoryless, stationary), a rate $R < C(W)$ and $\epsilon > 0$, there exists a n large enough and encoding/decoding functions $Enc : \{1 \ldots M\} \to \mathcal{X}^n$ with $M \geq 2^{nR}$ and $Dec : \mathcal{Y}^n \to \{1 \ldots M\}$ such that for all $m \in \{1 \ldots M\}$*

$$Pr\{Dec(Y^n) \neq m | X^n = Enc(m)\} < \epsilon$$

In other words we can communicate reliably at rate greater or equal to R on channel W.

Proof. Given W and $R < C(W)$, fix a p_X such that $I(X; Y) > R$. Pick $\delta > 0$, n large enough (to be determined later) and set $M' = \lceil 2 \cdot 2^{nR} \rceil$. Define the encoding function

$$Enc(1) = X(1)_1 \ldots X(1)_n$$

$$\ldots = \ldots$$

$$Enc(M') = X(M')_1 \ldots X(M')_n$$

choosing $\{X(m)_i : 1 \leq i \leq n, 1 \leq m \leq M'\}$ i.i.d. $\sim p_X$.

For the decoder fix

$$T(n, \delta, p_{XY}) = \left\{ (x^n, y^n) : (1 - \delta)p_{XY}(x, y) \leq \frac{\#\{(x_i, y_i) = (x, y)\}}{n} \leq (1 + \delta)p_{XY}(x, y) \right\}$$

$Dec(y^n)$: check for each m if $(Enc(m), y^n) \in T(n, \delta, p_{XY})$, if there is only a single m for which the pair is in the typical set then $Dec(y^n) = m$ otherwise (if there is none or more than one) $Dec(y^n) = 0$.

We now compute the probability of error $p_{e,m} \triangleq Pr\{Dec(Y^n) \neq m | X^n = Enc(m)\}$. $p_{e,m}$ depends on the choice of $Enc(1) \dots Enc(M)$ and since $Enc(1) \dots Enc(M)$ are randomly chosen, $p_{e,m}$ is a random variable. Supposing m is sent, an error will happen if and only if $(Enc(m), y^n) \notin T$ or for some $m' \neq m : (Enc(m'), y^n) \in T$

$$\begin{aligned}
\mathbb{E}[p_{e,m}] &= E_{Enc}[E_y[I\{\text{error has happened} \mid m \text{ is sent}\}]] \\
&= E_{Enc}[I\{(Enc(m), y) \notin T, \exists m' \neq m(Enc(m', Y^n) \in T\} | m \text{ is sent}] \\
&\leq E[I\{(Enc(m), Y^n) \notin T\}] + \sum_{m' \neq m} I\{(Enc(m'), Y^n) \in T\} | m \text{ is sent}] \\
&= Pr\{(Enc(m), Y^n) \notin T | m \text{ is sent}\} + \sum_{m' \neq m} Pr\{(Enc(m'), Y^n) \in T | m \text{ is sent}\}
\end{aligned}$$

We have

$$\begin{aligned}
Pr\{Enc(m) = x_1 \dots x_n, Y^n = y_1 \dots y_n | m \text{ is sent}\} &= p_x(x_1)p_x(x_2) \dots p_x(x_n)W(y_1|x_1)W(y_2|x_2) \dots W(y_n|x_n) \\
&= p_X(x_1)p_X(x_2) \dots p_X(x_n)p_Y(y_1)p_Y(y_2) \dots p_Y(y_n)
\end{aligned}$$

and as n gets large

$$Pr\{(Enc(m), Y^n) \notin T(p_{XY}, n, \delta)\} = Pr\{\text{iid sequence} \sim p_{XY} \notin T(p_{XY}, n, \delta)\} \to 0$$

because $(Enc(m), Y^n)$ is iid $\sim p_{XY}$. Recall from typicallity that if U^n is iid p_U, then

$$\lim_{n \to \infty} Pr\{U^n \notin T(n, p_U, \delta)\} = 0$$

and if U^n is in reality iid $\sim q_U$

$$Pr\{U^n \in T(n, p, \delta)\} \leq 2^{-n[D(p||q)-o(\delta)]}$$

Then,

$$\begin{aligned}
Pr\{(Enc(m), Enc(m'), Y^n) = (x^n, (x')^n, y^n)\} &= p_X(x^n)p_X((x')^n)W(y^n|x^n) \\
Pr\{(Enc(m), y^n) = (x^n, y^n)\} &= p_X(x^n)W(y^n|x^n) \\
Pr\{(Enc(m'), y^n) = ((x')^n, y^n)\} &= p_X((x')^n)\underbrace{\sum_{x^n} p(x^n)W(y^n|x^n)}_{p_Y(y)}
\end{aligned}$$

$$\iff (Enc(m'), y') \text{ is iid} \sim q_{XY} = p_X p_Y$$

$$\Rightarrow Pr\{(Enc(m'), y^n) \in T(p_{XY}, n, \delta)\} \leq 2^{-n[D(p||q)-o(\delta)]}$$

Also

$$D(p||q) = \sum_{xy} p_{XY}(x, y) \log \frac{p_{XY}(x, y)}{p_X(x)p_Y(y)} = I(X; Y)$$

Remember $M' = \lceil 2 \cdot 2^{nR} \rceil \leq 2 \cdot 2^{nR} + 1$ then $M' - 1 \leq 2 \cdot 2^{nR}$

$$\mathbb{E}\left[p_{e,m}\right] \le o_n(1) + (M'-1)2^{-n[I(X;Y)-o(\delta)]} \le o_n(1) + 2 \cdot 2^{-n(I(X;Y)-R-o(\delta))}$$

We choose δ small enough to have a negative exponent. Then it will go to 0 as n gets large. So we have shown that for n large enough we can make for every m:

$$\mathbb{E}\left[p_{e,m}\right] < \frac{\epsilon}{2}$$

$$\Rightarrow \mathbb{E}\left[\sum_{m=1}^{M'} p_{e,m}\right] \le \frac{M'}{2}\epsilon$$

$$\Rightarrow \exists \text{an encoder such that } \sum_{m=1}^{M'} p_{e,m} \le \frac{M'}{2}\epsilon$$

How many terms in the sumation can be greater or equal to ϵ ? At most $M'/2$, so remaining must be stricly smaller than ϵ but

$$M' - \frac{1}{2}M' = \frac{1}{2}\lceil 2 \cdot 2^{nR}\rceil \ge \frac{1}{2}2 \cdot 2^{nR} = 2^{nR}$$

We throw away the one smaller than ϵ and we have a code with rate greater than R for

$$\max_m p_{e,m} < \epsilon$$

$\square$

Example 14. Suppose $\mathcal{X} = \{a, b, c\}$, $C(W) = 1.3$ and $R = 1.25$, then $Enc(1\dots 32) \to \mathcal{X}^4$ is a valid encoding function for this channel, while $Enc(1\dots 32) \to \mathcal{X}^5$ would not allow reliable transmission.

Example 15. Suppose we want to design a code with $n = 1000$, $R = \frac{1}{2}$. The encoding table will have 1000×2^{500} elements, more than 10^{153} elements. "C'est impossible M'sieur !"

To illustrate the proof technique that we used to prove the coding theorem, we take an example.

Example 16. Assume W is a BEC channel (probability p to have an erasure symbol ?).

$$C(W) = 1 - p = \max_{p_x} I(X;Y)$$

achieved when $p_X(0) = p_X(1) = \frac{1}{2}$. Our coding theorem says that when $R < 1 - p$, $\epsilon > 0$ we can find a code of rate R with error probability $< \epsilon$. In the proof of the theorem 8.7, we generate a $n \times M$ coding matrix with n large and $M = 2^{nR}$ according to p_X defining $C(W)$.

To send a nR-bit message $m \in \{1 \dots M\}$, we send the mth row of the table over the chanel. When we receive $y = (y_1 \dots y_n)$, we compare y to each row of the table and check the tipicality. In our case

$$\frac{1}{n}\{\# \text{ of } (0,0)\} \approx \frac{1-p}{2} \qquad \frac{1}{n}\{\# \text{ of } (0,1)\} = 0 \qquad \frac{1}{n}\{\# \text{ of } (0,?)\} \approx \frac{p}{2}$$

$$\frac{1}{n}\{\# \text{ of } (1,0)\} = 0 \qquad \frac{1}{n}\{\# \text{ of } (1,1)\} \approx \frac{1-p}{2} \qquad \frac{1}{n}\{\# \text{ of } (1,?)\} \approx \frac{p}{2}$$

If there is exactly one row (i.e. $\hat{m}$) return $\hat{m}$, otherwise return 0.

- The correct codeword will pass the test with high probability, thanks to law of large numbers,

- What about an incorrect codeword ?

Recall the definition of typicality (definition 5.1) and suppose

$$y = \underbrace{0 \ldots 0}_{n\frac{1-p}{2}} \underbrace{1 \ldots 1}_{n\frac{1-p}{2}} \underbrace{? \ldots ?}_{np}$$

$m' = x_1 x_2 \ldots x_n$ will be typical only if it is of the type

$$\underbrace{0 \ldots 0}_{n\frac{1-p}{2}} \underbrace{1 \ldots 1}_{n\frac{1-p}{2}} \underbrace{? \ldots ?}_{np}$$

$$Pr\left\{ \begin{pmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{pmatrix} \text{ is typical} \right\} \leq \left(\frac{1}{2}\right)^{n(1-p)} = 2^{-n(1-p)}$$

Then, using that an upperbound to the number of incorrect codewords is 2^{nR},

$$Pr\left\{error\right\} < 2^{-n(1-p)} 2^{nR} + Pr\left\{\text{correct } w \text{ fails the test}\right\}$$

and because $R < 1 - p$

$$\lim_{n \to \infty} Pr\left\{error\right\} = 0$$

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9 Differential entropy

Definition 9.1 (Differential entropy). *Let X be a real valued random variable with probability density function $f(x)$ such that*

$$Pr\{x \leq X \leq x + \delta\} \approx \delta f(x)$$

The differential entropy of X is

$$h(X) \triangleq \int f(x) \log \frac{1}{f(x)} dx$$

Example 17. Uniform random variable in $[0, a]$ then

$$h(A) = \log a = \begin{cases} < 0 & \text{if } a < 1 \\ 0 & \text{if } a = 1 \\ > 0 & \text{if } a > 1 \end{cases}$$

Lemma 9.1. *Suppose $Y = X + a$, a is a constante then $h(Y) = h(X)$*

Proof. We have $f_Y(y) = f_X(y - a)$, then

$$h(Y) = \int f_X(y - a) \log \frac{1}{f_X(y - a)} dy = \int f_X(x) \log \frac{1}{f_X(x)} dx = h(X)$$

$\square$

Lemma 9.2. *Suppose $Y = aX$, then $h(Y) = h(X) + \log|a|$*

Proof. Suppose $a > 0$,

$$f_Y(y) = Pr\{y \leq Y \leq y + \delta\} = Pr\left\{\frac{y}{a} \leq X < \frac{y}{a} + \frac{\delta}{a}\right\} \approx \frac{1}{a} f_X\left(\frac{y}{a}\right)$$

$$\log \frac{1}{f_Y(y)} = \log a + \log \frac{1}{f_X\left(\frac{y}{a}\right)}$$

$$h(Y) = \int f_Y(y) \log \frac{1}{f_Y(y)} dy = \log a + \int f_X\left(\frac{y}{a}\right) \left(\log \frac{1}{f_X\left(\frac{y}{a}\right)}\right) \frac{1}{a} dy = \log a + \underbrace{\int f_X(x) \log \frac{1}{f_X(x)} dx}_{h(X)}$$

$\square$

Example 18. Suppose Y is a gaussian with mean μ and variance σ^2 then $Y = \sigma X + \mu$ where X is $N(0, 1)$

$$h(Y) = h(\sigma X) = \log \sigma + h(X)$$

$$h(X) = \int \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \left[\log \sqrt{2\pi} + \frac{1}{2} x^2 \log e\right] dx \overset{(a)}{=} \frac{1}{2} \log 2\pi + \frac{1}{2} \log e = \frac{1}{2} \log 2\pi e$$

Where the second term of (a) follows from $\mathbb{E}[X^2] = 1$.

Lemma 9.3. *Suppose X is a real value random variable with differentiable entropy $h(X)$. Consider a $\delta > 0$ and X_δ, the quantization of X in interval of width δ*

$$X_\delta = \delta \left\lfloor \frac{X}{\delta} \right\rfloor = n\delta \text{ if } n\delta \leq X \leq (n + 1)\delta$$

then

$$\lim_{\delta \to 0} H(X_\delta) + \log \delta = h(X)$$

Proof.

$$H(X_\delta) = \sum_n Pr\{X_\delta = n\delta\} \log \frac{1}{Pr\{X_\delta = n\delta\}}$$

$$\approx \sum_n \delta f_X(n\delta) \log \frac{1}{\delta f_X(n\delta)}$$

$$= \log \frac{1}{\delta} + \sum_n \left(f_X(n\delta) \log \frac{1}{f_X(n\delta)} \right) \delta$$

$$\stackrel{(a)}{=} \log \frac{1}{\delta} + \int f(x) \log \frac{1}{f(x)} dx$$

We recognize a Riemann sum for equality (a). □

Suppose $X_1...X_n$ are $\mathbb{R}$-valued RV's ($X^n \in \mathbb{R}^n$), we define

$$h(X^n) = h(X_1...X_n) = \underbrace{\int\int\int}_{\mathbb{R}^n} f_{X^n}(x_1...x_n) \log \frac{1}{f_{X^n}(x_1...x_n)} dx_1...dx_n$$

$$h(X|Y) = \int\int f_{XY}(x,y) \log \frac{1}{f_{XY}(X|Y)} dxdy = \mathbb{E}\left[\log \frac{1}{f_X Y(X|Y)}\right]$$

Theorem 9.4.

$$h(X^n) = \sum_{i=1}^n h(X_i|X^{i-1})$$

Proof.

$$f_{X^n}(x_1...x_n) = f_{x_1}(x_1) f_{x_2|x_1}(x_2|x_1)...f_{x_n|x^{n-1}}(x_n|x^{n-1})$$

take log's, take expectation □

Definition 9.2. *Given two densities $f(x), g(x)$, let*

$$D(f||g) = \int f(x) \log \frac{f(x)}{g(x)} dx$$

Lemma 9.5.

$$D(f||g) \geq 0$$

with equality iff $f = g$

Proof. use $\ln(z) \leq z - 1$ to show that $-D(f||g) \leq 0$ □

Definition 9.3. *For X, Y $\mathbb{R}$-valued RV's, define*

$$I(x;Y) = \int f_{XY}(x,y) \log \frac{f_{XY}(x,y)}{f_X(x)f_Y(y)} dy$$

$$= D(f_{XY}||f_X(x)f_Y(y))$$

$$= h(X) + h(Y) - h(XY)$$

$$= h(X) - h(X|Y)$$

$$= h(Y) - h(Y|X)$$

Proposition 9.1.

$$I(X;Y) \geq 0 \quad (= 0 \text{ iff } X \perp Y)$$

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Proof.

$$I(X;Y) = D(f_{XY}||f_x f_y)$$

□

Equivalently: $h(X|Y) \leq h(x)$, with equallity iff X and Y independent.

Lemma 9.6. *Given X, Y, $\mathbb{R}$-valued with joint pdf f_{XY}, for $\delta > 0$, define X_δ, Y_δ as $X_\delta = \delta \lfloor \frac{X}{\delta} \rfloor, Y_\delta = \delta \lfloor \frac{Y}{\delta} \rfloor$, then*

$$\underbrace{I(X_\delta;Y_\delta)}_{discrete\ I} \to I(X;Y)\ as\ \delta \to 0$$

Proof. observe

$$Pr\{X_\delta = n\delta\} = Pr\{X \in [n\delta, (n+1)\delta]\} \equiv \delta f_X(n\delta)$$
$$Pr\{Y_\delta = m\delta\} = Pr\{Y \in [m\delta, (m+1)\delta]\} \equiv \delta f_Y(m\delta)$$
$$Pr\{X_\delta = n\delta, Y_\delta = m\delta\} = Pr\{X \in [n\delta, (n+1)\delta], Y \in [m\delta, (m+1)\delta]\} \equiv \delta^2 f_{XY}(n\delta, m\delta)$$

$$I(X_\delta;Y_\delta) = \sum_{n,m} Pr\{X_\delta = n\delta, Y_\delta = m\delta\} \log \frac{Pr\{X_\delta = n\delta, Y_\delta = m\delta\}}{Pr\{X_\delta = n\delta\} Pr\{Y_\delta = m\delta\}}$$

$$\equiv \sum_{n,m} \delta^2 f_{XY}(n\delta, m\delta) \log \frac{\delta^2 f_{XY}(n\delta, m\delta)}{\delta f_X(n\delta) \delta f_Y(m\delta)}$$

$$\equiv \sum_{n,m} \delta^2 f_{XY}(n\delta, m\delta) \log \frac{f_{XY}(n\delta, m\delta)}{f_X(n\delta) f_Y(m\delta)}$$

$$= \text{Riemann sum for}\quad \int \int f_{XY}(x,y) \log \frac{f_{XY}(x,y)}{f_X(x) f_Y(y)} dx dy = I(X;Y)$$

□

In general, define for $X^n \in \mathbb{R}^n, Y^m \in \mathbb{R}^m, Z^k \in \mathbb{R}^k$

$$I(X^n;Y^m|Z^k) = \underbrace{\int ... \int}_{n+m+k} f_{X^n Y^m Z^k}(x^n, y^m, z^k) \log \frac{f_{XY|Z}(x^n y^m | z^k)}{f_{X|Z}(x^n|z^k) f_{Y|Z}(Y^m|z^k)}$$

we then have

Theorem 9.7. *Chain Rule for I*

$$I(X^n;Y) = \sum_{i=1}^{n} I(X_i;Y|X^{i-1})$$

Proof. Same proof as in the discrete case □

Example 19. X^n is a Gaussian Random Variable with $\mathbb{E}\,X^N = \bar{\mu}$ and variance matrix K, $K_{i,j} = \mathbb{E}\,(X_i - \mu_i)(X_j - \mu_j)$

$$h(X^n) = \underbrace{h(X^n - \bar{\mu})}_{\text{Gaussian with zero-mean with covariance}K}$$

consequently we may assume that $\bar{\mu} = \bar{0}$, recall that the joint pdf of a zero-mean gaussian is given by

$$f(\bar{x}) = \underbrace{\frac{1}{det(2\pi K)^{1/2}}}_{(2\pi)^{n/2}(det(K)^{1/2})} e^{0.5(X^T K^{-1} X)}$$

$$\log \frac{1}{f(\bar{x})} = \frac{1}{2}\log(det(2\pi K)) + \frac{1}{2}x^T K^{-1}x(\log(e))$$

$$h(X) = \mathbb{E}\left[\frac{1}{2}\log(det(2\pi K)) + \frac{\log(e)}{2}x^T K^{-1}x\right]$$

$$= \frac{1}{2}\log(det(2\pi K)) + \frac{\log(e)}{2}\underbrace{\underbrace{\mathbb{E}\left[x^T K^{-1}x\right]}_{tr(K^{-1}K)=tr(I_n)=n}}_{\frac{1}{2}\log(e^n)}$$

$$= \frac{1}{2}\log(det(2\pi e K))$$

Side knowledge:

$$\mathbb{E}\left[X^T A X\right] = \mathbb{E}\left[\sum_{i,j} X_i A_{ij} X_j\right] = \sum_{i,j} A_{ij}\mathbb{E}\left[X_i X_j\right] = \sum_{i,j} A_{ij}K_{ij} = \sum_i\left(\sum_j A_{ij}K_{ij}\right) = tr(AK)$$

Theorem 9.8. *Suppose $X \in \mathbb{R}^n$ is a random vector with*

$$\mathbb{E}\left[X_i X_j\right] = k_{ij}$$

Then

$$h(X) \le \frac{1}{2}\log(det(2\pi e K))$$

(Gaussians have maximum entropy among Random vectors with a given 2nd moment)

Proof. Let f be the density of X^n, let g be the gaussian density

$$g(x) = \frac{1}{det(2\pi K)^{1/2}}e^{-\frac{1}{2}(x^T K^{-1}x)}$$

observe that $\log\frac{1}{g(x)} = \frac{1}{2}\log(det(2\pi K) + \frac{1}{2}\log(2)x^T K^{-1}x$ so

$$\int f(\bar{x})\log\frac{1}{g(x)}d\bar{x} = \frac{1}{2}\log(det(2\pi K)) + \frac{\log(e)}{2}\underbrace{\mathbb{E}\left[x^T K^{-1}x\right]}_{n} = \int g(x)\log\frac{1}{g(x)}dx$$

how

$$0 \ge \int f(x)\log\frac{g(x)}{f(x)}dx = -\{\frac{1}{2}\log(det(2\pi e K))\} + h(x)$$

$\square$

Another example of maximum entropy

Suppose we know that $X \in [a,b]$ with probability 1. then $h(X) \le \log(b-a)$ equality iff X is uniform on $[a,b]$.

Proof. Let

$$g(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & \text{else} \end{cases}$$

$$\int f(x)\log\frac{1}{g(x)}dx = \log(b-a) = \int g(x)\log\frac{1}{g(x)}dx$$

$$0 \ge \int f(x)\log\frac{g(x)}{f(x)}dx = -\log(b-a) + h(X)$$

$\square$

Definition 9.4 (Additive Gaussian noise channel). *An additive Gaussian noise channel is a non-discrete communication channel from $\mathcal{X}$ to $\mathcal{Y}$ such that $Y = X + Z$ where Z is $N(0, \sigma^2)$.*

Example 20. A simple encoder for this channel would be to distribute m points in $\mathbb{R}$ such that $m_{i+1} - m_i = 100\sigma$. The decoder pick the point m closest to y. Errrors appear with low probability because the space between point is much larger than the noise. We need a very large range of values (e.g. electrical tension) which is not feasible in practice.

9.1 Channel with cost constraints

To avoid physically unrealisable scenarios, we need to impose some kind of cost constraint in our encoder. Suppose now that we are given $b : \mathcal{X} \to \mathbb{R}$ ($b(x) = x^2$ for the previous example) that associates a cost $b(x)$ to each input symbol. When an encoder $Enc : \{1 \ldots M\} \to \mathcal{X}^n$ is given, in adition to rate $R = \frac{1}{n}\log M$, we will also define

$$b(Enc(m)) = \frac{1}{n}\sum_{i=1}^{n} b(Enc(m)_i)$$

and we let

$$cost(Enc) = b(Enc(m))$$

Definition 9.5 (Channel capacity with power). *Given a channel $W(y|x)$ with input $\mathcal{X}$, output $\mathcal{Y}$, $b : \mathcal{X} \to \mathbb{R}$ and $\beta \in \mathbb{R}$ with $\beta > \inf_x b(x)$ we define*

$$C(W, \beta) = \sup_{x, \mathbb{E}[b(x) \leq \beta]} I(X; Y)$$

Theorem 9.9 (Bad news). *Suppose a channel $\mathcal{U}^L \to \mathcal{X}^n \xrightarrow{W} \mathcal{Y}^n \to V^L$ with $\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\left[b(X_i)\right] \leq \beta$ and $\bar{p}_e = \frac{1}{L}\sum_{i=1}^{L} Pr\{U_i \neq V_i\}$. Then*

$$h(\bar{p}_e) + \bar{p}_e \log(|\mathcal{U}| - 1) \geq H - \frac{L}{n}C(W, \beta)$$

with $H = \lim_{m \to \infty} \frac{H(U^m)}{m}$

Proof. We know that

$$
\begin{aligned}
h(\bar{p}_e) - \bar{p}_e \log(|\mathcal{U}| - 1) &\geq \frac{1}{L}H(U^L|V^L) \\
&= \frac{1}{L}[H(U^L) - I(U^L; V^L)] \\
&\overset{(a)}{\geq} \frac{1}{L}[H(U^L) - I(X^n; Y^n)] \\
&\overset{(b)}{\geq} H - \frac{1}{L}I(X^n; Y^n) \\
&\overset{(c)}{\geq} H - \frac{n}{L}\frac{1}{n}\sum_{i=1}^{N} I(X_i; Y_i) \\
&\geq H - \frac{n}{L}\frac{1}{n}\sum_{i=1}^{N} C(W, \beta_i) \text{ with } \beta_i = \mathbb{E}\left[b(X_i)\right] \\
&\overset{(d)}{\geq} H - \frac{n}{L}C\left(W, \frac{1}{n}\sum_{i=1}^{N} \beta_i\right) \\
&\geq H - \frac{n}{L}C(W, \beta)
\end{aligned}
$$

where we use (a) data processing, (b) stationary sources, (c) memoryless and (d) we use the claim 9.1. $\qquad\square$

Claim 9.1. $\beta \to C(W, \beta)$ *is a non-decreasing concave function.*

Proof. Non decreasing is clear because for $\beta_1 < \beta_2$, any p_X admissible for $C(W, \beta_1)$ is also admissible for $C(W, \beta_2)$.

For concavity, given β_1, β_2 and $\epsilon > 0$, find p_{X_1} and p_{X_2} such that

$$I(X;Y)|_{p_{X_1}} \geq C(W, \beta_1) - \epsilon \text{ with } \mathbb{E}\left[b(X)\right]|_{p_{X_1}} \leq \beta_1$$
$$I(X;Y)|_{p_{X_2}} \geq C(W, \beta_2) - \epsilon \text{ with } \mathbb{E}\left[b(X)\right]|_{p_{X_2}} \leq \beta_2$$

For $0 \leq \lambda \leq 1$ we define $p_X(x) = \lambda p_{X_1}(x) + (1 - \lambda)p_{X_2}(x)$, then

$$\mathbb{E}\left[b(X)\right]|_{p_X} = \lambda \mathbb{E}\left[b(X)\right]|_{p_{X_1}} + (1 - \lambda)\mathbb{E}\left[b(X)\right]|_{p_{X_2}} \leq \lambda\beta_1 + (1 - \lambda)\beta_2$$

Using that $I(\cdot; \cdot)$ is concave, we get

$$C(W, \lambda\beta_1 + (1-\lambda)\beta_2) \geq I(X;Y)|_{p_X} \geq \lambda I(X;Y)|_{p_{X_1}} + (1-\lambda)I(X;Y)|_{p_{X_2}} \geq \lambda C(W, \beta_1) + (1-\lambda)C(W, \beta_2) - \epsilon$$

Since $\epsilon > 0$ is arbitrary, we have shown

$$C(W, \lambda\beta_1 + (1 - \lambda)\beta_2) \geq \lambda C(W, \beta_1) + (1 - \lambda)C(W, \beta_2)$$

$\square$

Theorem 9.10 (Good news). *Given a channel $W(y|x)$ with $x \in \mathcal{X}$, $y \in \mathcal{Y}$, $b : \mathcal{X} \to \mathbb{R}$, $\beta \geq \max_x b(x)$, $\epsilon \geq 0$ and $R < C(W, \beta)$ then there is a $Enc : \{1 \ldots M\} \to \mathcal{X}^n$ and $Dec : \mathcal{Y}^n \to \{0 \ldots M\}$ such that*

$$\frac{1}{n}\log M \geq R$$
$$\forall m \; : \; p_{e,m} = Pr\left\{Dec(Y^n) \neq m | m \text{ is sent}\right\} < \epsilon$$
$$cost(Enc) = \max_m \frac{1}{n}\sum_{i=1}^n b(Enc(m)_i) < \beta + \epsilon$$

Proof. Verbatim as the proof of the coding theorem without costs, namely choose a p_X such that $\mathbb{E}\left[b(X)\right] \leq \beta$ and $I(X;Y) \geq R$. Construct $Enc(\cdot)$ randomly, use the tipycality decoder. Eliminate half the codewords to end up with $Enc(1) \ldots Enc(M)$ with the property that

$$\forall m \; : \; Pr\left\{Dec(Y^n) \neq m | m \text{ is sent}\right\} < \epsilon \tag{1}$$

Recall that the decoder decodes m only if $(Enc(m), y^n) \in T(p_{X,Y}, n, \delta)$ in particular $Enc(m) \in T(p_X, n, \delta)$. Then eq. (1) implies that for each m, $Enc(m) \in T(p_X, n, \delta)$ and

$$\frac{1}{n}\sum_{i=1}^n b(Enc(m)_i) = \frac{1}{n}\sum_{x \in \mathcal{X}} b(x)\{\# \text{ of } i \text{ such that } Enc(m)_i = x\}$$

$$cost(Enc) \leq \underbrace{\mathbb{E}\left[b(x)\right]}_{\leq \beta} + \delta\underbrace{\mathbb{E}\left[|b(X)|\right]}_{< \epsilon}$$

$\square$

Example 21. Assume an additive Gaussian noise channel W, $Y = X + Z$ with $Z \sim N(0, \sigma^2)$, and $b(x) = x^2$.

$$C(W, \beta) = \max_{X, \mathbb{E}[X^2] \leq \beta} I(X; Y)$$

$$= \max_{p_X, \mathbb{E}[X^2] \leq \beta} h(Y) - h(Y|X)$$

$$= \max_{p_X, \mathbb{E}[X^2] \leq \beta} h(Y) - h(Y - X|X)$$

$$= \max_{p_X, \mathbb{E}[X^2] \leq \beta} h(Y) - h(Z|X)$$

$$= \max_{p_X, \mathbb{E}[X^2] \leq \beta} h(Y) - h(Z)$$

$$= \left(\max_{p_X, \mathbb{E}[X^2] \leq \beta} h(X + Z) \right) - h(Z)$$

$$= \left(\max_{p_X, \mathbb{E}[X^2] \leq \beta} h(X + Z) \right) - \frac{1}{2} \log(2\pi e \sigma^2)$$

$$\overset{(a)}{\leq} \frac{1}{2} \log \left(\frac{2\pi e(\beta + \sigma^2)}{2\pi e \sigma^2} \right)$$

$$= \frac{1}{2} \log \left(1 + \frac{\beta}{\sigma^2} \right)$$

For (a), note that

$$\mathbb{E}\left[(X + Z)^2\right] = \mathbb{E}\left[X^2\right] + \sigma^2 \leq \beta + \sigma^2 \rightarrow h(X + Z) \leq \frac{1}{2} \log(2\pi e(\beta + \sigma^2))$$

On the other hand, for $X \sim N(0, \beta)$, we have $\mathbb{E}\left[X^2\right] \leq \beta$. $X + Z$ is $N(0, \beta + \sigma^2)$ so

$$h(X + Z) = \frac{1}{2} \log 2\pi e (\beta + \sigma^2)$$

$$\Rightarrow \max_{p_X, \mathbb{E}[X^2] \leq \beta} h(X + Z) \geq \frac{1}{2} \log \frac{\beta + \sigma^2}{\sigma^2}$$

Then

$$C(W, \beta) = \frac{1}{2} \log \left(1 + \frac{\beta}{\sigma^2} \right) = \frac{1}{2} \log(1 + \text{signal to noise ratio})$$

This formula is well known in industry and abused for other channels. Do not do that!

9.2 K independant gaussian noice channel

Let's study the channel W such that $\vec{X} \rightarrow \vec{Y} = \vec{X} + \vec{Z}$ with

$$\vec{Z} \sim N(\vec{0}, diag(\vec{\sigma^2}))$$

$$b(\vec{x}) = \sum_{i=1}^{k} X_i^2$$

$$C(X^k \rightarrow Y^k = X^k + Z^k, \beta) = \max_{p_{X^k}, \mathbb{E}[||X^k||] \leq \beta} I(X^k, Y^k)$$

$$\text{with } I(X^k, Y^k) = h(Y^k) - h(Y^k|X^k) \leq \sum_{i=1}^{k} \underbrace{(h(Y_i) - h(Z_i))}_{I(X_i, X_i + Z_i)}$$

$$\text{with } h(Y^k|X^k) = h(Y^k - X^k|X^k) = h(Z^k) = \sum_{i=1}^{k} h(Z_i)$$

with equality if X_i's are independent

$$C(X^k \to Y^k = X^k + Z^k, \beta) \leq \sum_{i=1}^{k} \frac{1}{2} \log\left(1 + \frac{\beta_i}{\sigma_i^2}\right) \quad \text{with } \beta_i = \mathbb{E}\left[X_i^2\right]$$

with equality when $X_i \sim N(0, \beta_i)$

$$C(X^k \to Y^k = X^k + Z^k, \beta) = \max_{\beta_1,\ldots,\beta_k \geq 0, \sum \beta_i = \beta} \sum_{i=1}^{k} \frac{1}{2} \log\left(1 + \frac{\beta_i}{\sigma_i^2}\right)$$

$$C(\beta) = \max_{f_1,\ldots,f_k \geq 0, \sum f_i = 1} \sum_{i=1}^{k} \frac{1}{2} \log\left(1 + \frac{f_i \beta_i}{\sigma_i^2}\right)$$

thus, we are maximazing a concave function on the simplex, so the optimal f_i satisfies: *for some λ*

$$\frac{d}{df_j} \sum_{i+1}^{k} \frac{1}{2} \log(1 + f_i \frac{\beta}{\sigma_i^2}) \leq \lambda \text{ for all } j$$

$$= \lambda \text{ when } f_j > 0$$

$$\frac{1}{2} \frac{\frac{\beta}{\sigma_j^2}}{1 + f_j \frac{\beta}{\sigma_j^2}} = \frac{\beta}{2} - \frac{1}{\sigma_j^2 + \beta_j}$$

Using the change of variable $\mu = 1/\lambda$, the optimal $\{\beta_j\}$ satisfies

$$\beta_j + \sigma_j^2 \geq \mu \text{ for all } j$$

$$= \mu \text{ for } j \text{ s.t } \beta_j > 0$$

$$\equiv \beta_j = \mu - \sigma_j^2$$

Notation 3.

$$a^+ \equiv \begin{cases} a, & \text{if } a > 0 \\ 0, & \text{else} \end{cases}$$

So, in terms of μ, we have

$$\beta = \sum_i (\mu - \sigma_i^2)^+$$

$$C(\beta) = \sum_i \underbrace{\log(1 + \frac{(\mu - \sigma_i^2)^+}{\sigma_i^2})}_{(\log \frac{\mu}{\sigma_i^2})^+} = \sum_i \left(\log \frac{\mu}{\sigma_i^2}\right)^+$$

"Water filling" solution pour water of volume β to the basin with altitude map given by σ_i^2's

10 Elementary Coding Theorem

Wikipedia. *The object of coding is to introduce redundancy so that even if some of the information is lost or corrupted, it will still be possible to recover the message at the receiver.*

Restrict ourselves to Binary Symmetric and Binary Erasure Channel. Suppose

$$Enc : \{1, ..., M\} \to \{0,1\}^n, n = 1000, rate = \frac{1}{2} \Rightarrow n = 2^{500}$$

Even the encoding table taken $1000 \cdot 2^{500}$ bits of memory.
We need structure in the Enc, let us try linear structures.

$$Enc(b_1, ..., b_k) = matrix(n, k) \times \vec{b}$$

with $k = \log(M)$

Example 22. Consider $\vec{X} = \{X_1, ..., X_7\}$ s.t.

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ . \\ . \\ . \\ X_7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\equiv \begin{matrix} X_1 = X_4 + X_6 + X_7 \\ X_2 = X_4 + X_5 + X_7 \\ X_3 = X_5 + X_6 + X_7 \end{matrix}$$

$$\equiv \begin{bmatrix} X_1 \\ . \\ . \\ . \\ X_7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_4 \\ X_5 \\ X_6 \\ X_7 \end{bmatrix}$$

$$\equiv Enc\left(\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \right)$$

What is the rate of this code?

$$\frac{4}{7} = \frac{\#bits\ of\ input\ to\ Enc}{\#bits\ of\ output\ to\ Enc}$$

Is there a codeword of $\underbrace{weight}_{\#of\ 1's} = 1$? No because no column of the matrix is $\{0,0,0\}$

Is there a codeword of weight $= 2$? No because no two columns of H are equal.

Definition 10.1. *Given two vectors $X, X' \in \mathbb{F}_2^n$, the Hamming distance is*

$$d_H(X, X') = \sum_{i=1}^n \mathbb{I}\{x_i \neq x_i'\}$$

Definition 10.2. *Given $x \in \mathbb{F}_2^k$, its Hamming weight is*

$$w_H(X) = d_H(X, 0)$$

Remarks:

$$d_H(X, X') = w_H(X + X')$$

because $X_i + X_i' = 0 \Leftrightarrow X_i = X_i'$

d_H is a metrix

1. $d_H(X, X') \geq 0$

2. $d_H(X, X') = d_H(X', X)$

3. $d_H(X, Z) \leq d_H(X, Y) + d_H(Y, Z)$

For the example

$$H \begin{bmatrix} X_1 \\ . \\ . \\ . \\ X_7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

suppose X is a codeword ($HX = 0$) and X' is a codeword ($HX' = 0$), then $X + X'$ is a codeword $H(X + X') = HX + HX' = 0$ (codewords from a linear space)

We saw that there are no codewords of weight 1 or 2.

So, if X and X' are codewords

$$w_H(X + X') = d_H(X, X') \neq 1, 2$$

$$\Rightarrow \min_{X, X', X \neq X'} d_H(X, X') \geq 3$$

and in fact $= 3$.

For each codeword X, consider $\underbrace{B}_{\text{ball}}(\underbrace{X}_{\text{center}}, \underbrace{1}_{\text{radius}}) = \{Y \in \mathbb{F}_2^7 : d_H(X, Y) \leq 1\}$

$$\Rightarrow \text{ if } X \neq X', \text{ then } B(X, 1) \cup B(X', 1) = \emptyset$$

Also, $|B(X, 1)| = \underbrace{1}_{X} + \underbrace{7}_{\text{all } Y\text{'s at } d_H = 1}$

There are 16 coderwords:

$$|\bigcup_{X \in \text{codewords}} B(X, 1)| = 16 * 8 = 128$$

So we conclude that

$$\bigcup_{X \in \text{codewords}} B(X, 1) = \mathbb{F}_2^7$$

meaning that we have perfect cover of $\mathbb{F}_2^7$ with disjoint spheres.

For this reason the code described by it is called a perfect code.

The code we are discussing is called

$$(\underbrace{7}_{\text{length}}, 4, \underbrace{3}_{\text{minimal distance}}) - \text{Hamming code}$$

Let's extend the Hamming code to more general parameters $(2^m - 1, 2^m - 1 - m, 3)$. Then H is a matrix $\mathbb{F}_2^{m \times 2^m - 1}$. $\{B(x, 1) : x \text{ is a codeword}\}$ form a disjoint collection and

$$|B(x, 1)| = 1 + (2^m - 1) = 2^m$$
$$|\bigcup B(x, 1)| = 2^{2^m - 1 - m} \cdot 2^m = 2^{2^n - 1} = |\mathbb{F}_2^{2^m - 1}|$$

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Theorem 10.1 (Sphere packing bound). *Suppose we have a code in $\mathbb{F}_2^n$ with M codewords and the Hamming distance between codewords is*

$$d = \min_{x \neq x'} d_H(x, x')$$

then

$$2^m \geq M \sum_{i=0}^{\lfloor \frac{d-1}{2} \rfloor} \binom{n}{i}$$

Proof. The balls of radius $\lfloor \frac{d-1}{2} \rfloor$ around each codeword are distinct (because any two words are more than d appart).

$$2^n = |\mathbb{F}_2^n| \geq \left| \bigcup_x B\left(x, \left\lfloor \frac{d-1}{2} \right\rfloor\right) \right| = \sum_x \left| \bigcup_x B\left(x, \left\lfloor \frac{d-1}{2} \right\rfloor\right) \right| = M \sum_{i=0}^{\lfloor \frac{d-1}{2} \rfloor} \binom{n}{i}$$

$\square$

Theorem 10.2 (Gilbert bound). *Given n, d there is a code in $\mathbb{F}_2^n$ with min distance ≥ 2 and numbers of codewords M with*

$$M \sum_{i=0}^{d-1} \binom{n}{i} \geq 2^n$$

Proof.

$\quad S \leftarrow F_2^n;$
$\quad \textbf{while } S \neq \emptyset \textbf{ do}$
$\quad\quad \big|\quad$ pick $x \in S$ as a codeword;
$\quad\quad \big|\quad S \leftarrow S \setminus B(x, d-1);$
$\quad \textbf{end}$

$\square$

Observation 10.1. *Suppose we are in $\mathbb{F}_q^n$, we can define the Hamming distance*

$$d_H(x, x') = \mathbb{I}_{x_i \neq x_i'}$$

and

$$|B(x, r)| = \sum_{i=0}^{r} \binom{n}{i} (q-1)^i$$

Theorem 10.3 (Singleton bound). *Suppose we have a code in $\mathbb{F}_2^n$ with $M > 2^k$ codewords. Then the code has minimum distance $\leq n - k$ (i.e. there are $x \neq x$ codewords with $d_H(x, x' \leq n - k)$).*

Proof. Split words of length n in two parts (of length $n - k$ and k). There are 2^k possible right halves but $> 2^k$ codewords. By pigeon hole principle, there exists two codewords $x \neq x'$ with the same right half thus $d_H(x, x') \leq n - k$ (left half part is fully changed). $\square$

10.1 Linear code

Definition 10.3. *A collection $\mathcal{C} \subseteq \mathbb{F}_2^n$ of code words form a linear code if $\mathcal{C}$ is a vector space over $\mathbb{F}_2$ (i.e. $x, x' \in \mathcal{C} \rightarrow x + x' \in \mathcal{C}$).*

Observation 10.2. *The previous definition implies that $0 \in \mathcal{C}$, $-x \in \mathcal{C}$ and $ax + a'x' \in \mathcal{C} \forall a, a' \in \mathbb{F}_2$*

Definition 10.4. *For any collection $\mathcal{C}$, define*

$$d_{\min}(\mathcal{C}) = \min_{x, x' \in \mathcal{C}, x \neq x'} d_H(x, x')$$

Lemma 10.4. *if $\mathcal{C}$ is a linear code, then*

$$d_{\min}(\mathcal{C}) = w_{\min}(\mathcal{C}) \triangleq \min_{x \in \mathcal{C}, x \neq 0} w_H(x)$$

Proof. Let $x, x' \in \mathcal{C}$ have

$$d_H(x, x') = d_{\min} \Rightarrow w_H(x + x') = d_{\min} \Rightarrow w_{\min} \leq d_{\min}$$

Also let $x \in \mathcal{C}$ with $w_H(x) = w_{\min}$ then

$$d_H(0, x) = w_{\min} \Rightarrow d_{\min}(\mathcal{C}) \leq w_{\min}(\mathcal{C})$$

$\square$

Fact 10.1. *if $\mathcal{C}$ is a linear code in $\mathbb{F}_2^n$, then there is a matrix $G \in \mathbb{F}_2^{n \times k}$ (called the generator matrix) so that $\mathcal{C} = \{G_u : u \in \mathbb{F}_2^k\}$ in particular $|\mathcal{C}| = 2^k$. One can even pick G to be of the form*

$$G = \begin{bmatrix} I_k \\ A_{n-k \times k} \end{bmatrix}$$

and its row permutations.

Consequence 10.1. *if $\mathcal{C}$ is a linear code in $\mathbb{F}_2^n$, then there is a matrix $H \in \mathbb{F}_2^{n \times n}$ such that $\mathcal{C} = \{x \in \mathbb{F}_2^n : Hx = 0\}$. One can even choose H or some column permutation of*

$$H = \begin{bmatrix} I_r | A_{r \times n - r} \end{bmatrix}$$

so that $|\mathcal{C}| = 2^{n-r} = 2^k$

Proof.

$$\text{If } G = \begin{bmatrix} I_k \\ A \end{bmatrix} \text{ then } \mathcal{C} = \left\{ \begin{bmatrix} u \\ A_u \end{bmatrix} : u \in \mathbb{F}_2^k \right\} \text{ i.e. } x \in \mathcal{C} \iff \begin{bmatrix} 1 \\ \vdots \\ x_k \\ x_{k+1} \\ \vdots \\ x_n \end{bmatrix} \text{ has } \begin{bmatrix} x_{k+1} \\ \vdots \\ x_n \end{bmatrix} = A \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}$$

$$\iff I_{k-n} \begin{bmatrix} x_{k+1} \\ \vdots \\ x_n \end{bmatrix} = A \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} \iff \underbrace{\begin{bmatrix} A | -I_{k-n} \end{bmatrix}}_{H} \begin{bmatrix} x_1 \\ x_k \\ x_{k+1} \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\square$

Theorem 10.5 (Gilbert-Varshanov). *Given n, d there is a linear code in $\mathbb{F}_2^n$ with $\geq 2^{n-r}$ codewords, provided that*

$$\sum_{i=0}^{d-2} \binom{n-1}{i} < 2^n$$

Suppose a code has a minimum distance d, and we use this cod over the BSC. At the decoder, given the received y^n, decode it to the "nearest" codeword (in d_H). Then, if the channel flips $< \frac{d}{2}$ input symbols, we will decode correctly.

Relationship between the parity check H and d_min, # of codewords

Recall $C = \{x : Hx = 0\}$, e.g. $(7, 4, 3)$−Hamming code: 3 equations, also called the parity-check equations. The matrix H is sometime called the parity-check matrix. Recall also that $d_{min} = w_{min}$.

Suppose H has the property that any k columns are linearly independent. Can there be a non-zero codeword with weight $\equiv k$? No

For such an x, Hx is a linear combination with weights x of the columns of H. Such a combination $\neq 0$

Thus $w_min > k$

Suppose there exists l columns of H which are linearly dependent $\Rightarrow \exists x \neq 0$ with weight $w(x) \leq l$ such that $Hx = 0$. Thus $w_{min} \leq l$

Theorem 10.6. *The code $C = \{x : Hx = 0\}$ has minimum distance d if and only if any collection of $d - 1$ columns of H are linearly independent, and some d columns are linearly independent.*

Theorem 10.7. *Suppose the matrix $H \in \mathbb{F}_2^{m \times n}$ has rank r. Then $C = \{x : Hx = 0\}$ has 2^{n-r} codewords.*

10.1.1 Revisit the singleton bound for general alphabet

Recall $d_H(x, x') = \sum_{i=1}^{n} \mathbb{I}(x_i \neq x_i')$ with $x, x' \in \mathcal{X}^n$

Singleton bound: Suppose $C \subset \mathcal{X}^n, |C| > |X|^k$ then

$$d_{min}(c) \leq n - k$$

Proof. same as in the binary case. □

Definition 10.5. *Reed-Solomon Codes are a family of codes for which the Singleton bound is tight. (they will have $|X|^{k+1}$ codewords and $d_{min} = n - k + 1$)*

Construction of R-S code: we are given a set $\mathbb{F}$, a field with $+, \cdot$ operator defined. (typically $\mathbb{F} = \{0, 1\}^l, l = 8, 16$ is usual), let $q = |\mathbb{F}|$.

We will construct a binary code $C \subset \mathbb{F}^n$ with $n \leq q$, with $|C| = q^k, d_{min}(C) = n - k + 1$. To do so, we will pick $\alpha_1, ..., \alpha_n \in \mathbb{F}$ all distinct. The code then will be the following:

for each $(u_0, ..., u_{k-1}) \in \mathbb{F}^k$ we will define $u(D) = u_0 + u_1 D + u_2 D^2 + ... + u_{k-1} D^{k-1}$, and construct $x(u) \in \mathbb{F}^n$ with

$$x(u) = (u(\alpha_1), ..., u(\alpha_n))$$

and $\mathcal{C} = \{x(u) : u \in \mathbb{F}^k\}$

Observation: Suppose $u \in \mathbb{F}^k, \tilde{u} \in \mathbb{F}^k, u \neq \tilde{u}$

$$x(u) - x(\tilde{u}) = u(\alpha_1) - \tilde{u}(\alpha_1), ..., u(\alpha_n) - \tilde{u}(\alpha_n))$$
$$v = u - \tilde{u} \Rightarrow (v(\alpha), ..., v(\alpha_n))$$

Where $v(D) = v_0 + v_1 D + ... + v_{k-1} D^{k-1}$ a polynomial of degree $\leq k - 1$, with at most $k - 1$ roots. v has atmost $k - 1$ entries. Therefor it has at least $n - k + 1$ non-zero entries. Thus

$$x(u) \neq x(\tilde{u}) \Rightarrow |C| = q^k$$

and $d_{min} \geq n - k + 1$ (actually equal because singleton bound says $d_{min} \leq n - k + 1$)

Furthermore, if $x \in C, \tilde{x} \in C$, if $a, \tilde{a} \in \mathbb{F}$, then $ax + \tilde{a}\tilde{x} \in C$, because $x = x(u), \tilde{x} = x(\tilde{u})$ then $ax + \tilde{a}\tilde{x} = x(au + \tilde{a}\tilde{x})$, so C is a linear code.

Example 23. $\mathbb{F} = \{0, 1, 2\}, n = 3, \alpha_1 = 0, \alpha_2 = 1, \alpha_3 = 2$ with $mod3$ addition and multiplication

u_0	u_1	$u(D)$	||	$x(u)$	
0	0	0	0	0	0
0	1	D	0	1	2
0	2	$2D$	0	2	1
1	0	1	1	1	1
1	1	$1+D$	1	2	0
1	2	$1+2D$	1	0	2
2	0	2	2	2	2
2	1	$2+D$	2	0	1
2	2	$2+2D$	2	1	0

In general, if we are given an X with $|X| = p^l$ with p prime, we identify X with $\mathbb{F}_p^l$
$\rightarrow$ polynomials over $\mathbb{F}_q$ of degree $< l-1$ with addition modulo p and multiplication modulo a suitably chosen polynomial of degree l.

10.1.2 Polar codes

Channel 1:

$$y_0 \cap y_1 = \emptyset \quad C(W) = 1$$

Easy to achieve capacity: send unencoded binary data

Polar cosing is a technique to convert n copies of a "mediocre" channel W to n extremal channels, while preserving the total capacity

2×2 building blocks:

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

$$\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

$$\begin{matrix} X_1 = U_1 \oplus U_2 \\ X_2 = U_2 \end{matrix} \quad \Longleftrightarrow \quad \begin{matrix} U_1 = X_1 \oplus X_2 \\ U_2 = X_2 \end{matrix}$$

$$\Rightarrow I(U_1 U_2; Y_1 Y_2) = I(X_1 X_2; Y_1 Y_2)$$

$$U_1 U_2 \text{ i.i.d. } \sim B\left(\frac{1}{2}\right) \Rightarrow \begin{array}{cc|c} U_1 & U_2 & p \\ \hline 0 & 0 & 1/4 \\ 0 & 1 & 1/4 \\ 1 & 0 & 1/4 \\ 1 & 1 & 1/4 \end{array} \Rightarrow \begin{array}{cc|c} X_1 & X_2 & p \\ \hline 0 & 0 & 1/4 \\ 0 & 1 & 1/4 \\ 1 & 0 & 1/4 \\ 1 & 1 & 1/4 \end{array} \Rightarrow X_1, X_2 \text{ i.i.d.} \sim B\left(\frac{1}{2}\right)$$

$$I(X_1 X_2; Y_1 Y_2) \stackrel{(*)}{=} I(X_1; Y_1) + I(X_2; Y_2) = 2I(W)$$
$$I(U_1 U_2; Y_1 Y_2) = I(U_1; Y_1 Y_2) + I(U_2; Y_1 Y_2 | U_1)$$
$$\triangleq I(W^-) + I(W^+)$$

where $(*)$ is because X's are i.i.d.

$$W^- : U_1 \rightarrow Y_1 Y_2, \ W^-(y_1 y_2 | u_1) = \frac{1}{2} Pr\left\{Y_1 Y_2 = y_1 y_2 | X_1 X_2 = U_1 0\right\} + \frac{1}{2} Pr\left\{Y_1 Y_2 = y_1 y_2 | X_1 X_2 = \bar{U}_1 1\right\}$$

$$= \frac{1}{2} W(y_1 | u_1) W(y_2 | 0) + \frac{1}{2} W(y_1 | \bar{u}_1) W(y_2 | 1)$$

$$W^+ : U_2 \to Y_1 Y_2 U_1, \ W^+(y_1 y_2 | u_2) = \frac{1}{2} Pr\{Y_1 Y_2 = y_1 y_2 | X_1 = u_1 + u_2, X_2 = u_2\}$$
$$= \frac{1}{2} W(y_1 | u_1 + u_2) W(y_2 | u_2)$$

Question: Are W^- and W^+ real ?

In the general case, say $U_1 U_2 \ldots U_n$

Oracle aided decoder

$$\hat{U}_1 = \Phi_1(Z)$$
$$\hat{U}_2 = \Phi_2(ZU_1)$$
$$\hat{U}_3 = \Phi_3(ZU_1 U_2)$$
$$\vdots$$
$$\hat{U}_n = \Phi_n(ZU_1 U_2 \ldots U_{n-1})$$

Unaided decoder (with same Φ_i's as the oracle aided decoder)

$$\tilde{U}_1 = \Phi_1(Z)$$
$$\tilde{U}_2 = \Phi_2(Z\tilde{U}_1)$$
$$\tilde{U}_3 = \Phi_3(Z\tilde{U}_1 \tilde{U}_2)$$
$$\vdots$$
$$\tilde{U}_n = \Phi_n(Z\tilde{U}_1 \tilde{U}_2 \ldots \tilde{U}_{n-1})$$

Suppose that

$$\hat{U}^n = U^n \Rightarrow \tilde{U}_1 = \hat{U}_1 = U_1$$
$$\Rightarrow \tilde{U}_2 = \Phi_2(Z\tilde{U}_1 = U_1) = \hat{U}_2 = U_2$$
$$\Rightarrow \tilde{U}_3 = \Phi_3(Z, \tilde{U}_1 \tilde{U}_2 = U_1 U_2) = \tilde{U}_3 = U_3$$
$$\Rightarrow \ldots$$

Check that $\hat{U}^n \neq U^n \Rightarrow \tilde{U}^n \neq U^n$

$$\Rightarrow Pr\{\hat{U}^n = U^n\} = Pr\{\tilde{U}^n = U^n\}$$

If we decide U_1 first ($\equiv$ decode W^- first) and use the decision $\tilde{U}_1$ as if it were true (as if it was the oracle's answer) when decoding W^+. So W^+ is a "real" channel.

So far $2I(W) = I(W^-) + I(W^+)$

Example 24. $W = \text{BEC}(p)$

$$Y_1 = \begin{cases} X_1 & \text{with probability } 1-p \\ ? & \text{with probability } p \end{cases}$$

$$W^- : U_1 \to Y_1 Y_2 = \begin{cases} X_1 X_2 & \text{with probability } wp(1-p)^2 \\ X_1? & \text{with probability } p(1-p) \\ ?X_2 & \text{with probability } p(1-p) \\ ?? & \text{with probability } p^2 \end{cases}$$

$$X_1 = U_1 \oplus U_2 \quad X_2 = U_2$$

$$U_1 \text{ independent of } U_2 \equiv \begin{cases} U_1 & wp(1-p)^2 \\ ? & wp(2p - p^2) \\ ? & wp(2p - p^2) \\ ? & wp(2p - p^2) \end{cases} \equiv \text{BEC}(2p - p^2)$$

$$W^+ : U_2 \to Y_1 Y_2 U_1 = \begin{cases} X_1 X_2 U_1 \\ X + 1? U_1 \\ ? X_2 U_1 \\ ?? U_1 \end{cases} \equiv \begin{cases} U_2 & \text{with probability } (1 - p^2) \\ U_2 & \text{with probability } (1 - p^2) \\ U_2 & \text{with probability } (1 - p^2) \\ ? & \text{with probability } p^2 \end{cases} \equiv \text{BEC}(p^2)$$

W^+ is a much better channel than W. In general

$$I(W^+) = I(U_2; Y_1 Y_2 U_1) \geq O(X_2, Y_2) = I(W) \Rightarrow I(W^-) \leq I(W) \leq I(W^+)$$

How to decoder the output $\mathcal{Y}$ of a binary input channel W? Given y (the output of W) Compute $W(y|0), W(y|1)$, max likelihood decoding rule

$$\varphi(y) = \begin{cases} 0 & \text{if } W(y|0) > W(y|1) \\ 1 & \text{if } W(y|1) < W(y|1) \end{cases}$$

Encoding / decoding complexity

Encoding complexity:

$$\frac{nk}{2} XOR - \frac{1}{2} n \log n$$

Decoding complexity

$$\frac{nk}{2} proc^{-/+}$$

We are now left with the question: does this W result in extremal channel?
Restrict $W = BEC(p)$:

$$W^+ = BEC(p^2) = BEC(p(+))$$
$$W^- = BEC(p(2 - p)) = BEC(p(-))$$
$$p(++) = (p^2)^2$$
$$p(+-) = p^2(2 - p^2)$$
$$p(-+) = (p(2 - p))^2$$
$$etc...$$

Define a $BEC(p)$ to be ϵ-mediocre if $p \ni (\epsilon, 1 - \epsilon)$ define for a $BEC(p)$ its ugliness to be $\sqrt{4p(1 - p)}$
If W is ϵ-mediocre then $ugl(W) \geq \sqrt{4\epsilon(1 - \epsilon)}$
Suppose $W = BEC(p)$

$$ugl(W^+) = \sqrt{4p^2(1 - p^2)} = ugl(W)\sqrt{p(1 + p)}$$
$$ugl(W^-) = \sqrt{4p(2 - p)(1 - p)^2} = ugl(W)\sqrt{(2 - p)(1 + p)} = f(1 - p)$$

$$\frac{1}{2}[ugl(W^+) + ugl(W^-)] = ugl(W)\frac{1}{2}[f(p) + f(1 - p)] \quad \text{with } f(p) = \sqrt{p(1 - p)}$$

$$\leq ugl(W)\frac{1}{2}2f(\frac{1}{2}) = ugl(W)\sqrt{\frac{3}{4}}$$

$$\Rightarrow \frac{1}{2^k} \sum_{s^k \in \{+,-\}^k} ugl(W^{(s^k)}) \leq (\frac{3}{4})^{k/2} \quad \text{tends to 0 as } k \text{ gets large}$$

$$= \frac{1}{2^k} \sum_{s^k \in \{+,-\}^k} \mathbb{I}\{W^k \text{ is } \epsilon\text{-mediocre}\}\sqrt{4\epsilon(1-\epsilon)}$$

$$\Rightarrow \text{fraction of } \epsilon\text{-mediocre channel s.t. stage } k$$

$$\leq \frac{(3/4)^{k/2}}{\sqrt{4\epsilon(1-\epsilon)}} \quad \text{tends to 0 as } k \text{ gets large}$$

So we have shown the following: let $\mu_k(\epsilon)$ be the function of ϵ-mediocre channels s.t. stage k, then

$$\lim_{k \to \infty} \mu_k(\epsilon) = 0$$

$\Rightarrow$ the fraction of ϵ-good channels $\to (1-p)$
$\Rightarrow$ the fraction of ϵ-bad channels $\to p$

This suggest the following coding methods: Given $R < 1 - p = C(BEC(p))$, pick $\epsilon > 0$ and k large enough so that fraction of ϵ-good channels is $> R$. Then

$$Pr\left\{\hat{U}^n \neq U^n\right\} \leq \sum_{i:\epsilon\text{-good}} Pr\left\{\hat{U}_i \neq U_i\right\} \leq nR\epsilon$$

we need to show that the good channels are in fact very good.

For this we need

Definition 10.6. *A channel W^k is called ϵ-pure if for all $\sqrt{k} \leq i \leq k$, W^{si} is not ϵ-mediocre.*
Observation: $\mu_i(\epsilon) \leq \frac{1}{\sqrt{4(\epsilon(k\epsilon))}}(\frac{3}{4})^{1/2}$ fraction of ϵ-mediocre channels at gen u.
$\Rightarrow$ *fraction of ϵ-inpure channels at gen k is*

$$\leq \sum_{i=\sqrt{k}}^{k} \mu_i(\epsilon) \leq \sum_{i=\sqrt{2}}^{\infty} (\frac{3}{4})^i \frac{1}{\sqrt{\epsilon(1-\epsilon)}} \leq const(\epsilon)(\frac{3}{4})^{\sqrt{k}/2}) \to 0$$

$\Rightarrow$ *fraction of ϵ-pure channels $\to 1$ as k gets large.*

Observation: p cannot convert good to bad, bad to good $\Rightarrow$ an ϵ-pure channel comes from a set of ϵ-good ancestor and ϵ-good$_-$

11 Quantization / Rate-Distortion Theory

We have seen (lossless) source coding

$$U^n \to \text{bits} \to \hat{U}^n = U^n$$

Lossless is not possible if U takes on values in $\mathbb{R}$. Then the question is: What is the trade-off between fidelity (how close $\hat{U}^n$ is to U^n) and "rate" (# of bits / source-letter)?

Consider $U_1, U_2, \ldots$ iid $\sim P_u$ on an alphabet $\mathcal{U}$. We also have a reproduction alphabet $\mathcal{V}$ and a function

$$d : \mathcal{U} \times \mathcal{V} \to \mathbb{R} \quad \text{(measures the penalty for reprenting } u \text{ as } v\text{)}$$

Example 25.
$$\mathcal{U} = \mathcal{V} = \{0,1\}, \quad d(u,v) = \mathbb{I}\{u \neq v\}$$

Example 26.
$$\mathcal{U} = \mathcal{V} = \mathbb{R}, \quad d(u,v) = |u-v|^2$$

Example 27.

$$\mathcal{U} = \{0,1\}, \mathcal{V} = \{0,1,?\} \quad \begin{aligned} d(0,0) &= d(1,1) = 0 \\ d(0,1) &= d(1,0) = 10 \\ d(0,?) &= d(?,1) = 1 \end{aligned}$$

Definition 11.1. *Given $\mathcal{U}$ and $\mathcal{V}$, a quantizer consists of two maps: f and ϕ*

$$\begin{aligned} f : \quad & \mathcal{U}^n \to \{1, \ldots, n\} (\log_2 M \text{ bits description of } (U_1, \ldots, U_n)) \\ \phi : \quad & \{1, \ldots, n\} \to \mathcal{V}^n \end{aligned}$$

We will assess the quality of the quantizer by

$$rate = \frac{\log M}{n}$$

$$distortion = \mathbb{E}[d(U^n, \phi(f(u^n)))] \quad where \quad d(u^n, v^n) = \sum_{i=1}^{n} d(u_i, v_i)$$

Theorem 11.1. *Suppose that for a source iid U with statistics P_u and distortion measure d, we have designed a quantizer with rate R, and distortion Λ. Then there is a joint distribution P_{uv} such that*

$$\Lambda = \mathbb{E}[d(U,V)] \quad and \quad R \geq I(U;V)$$

Corollary 11.1.1. *For any quantizer with distortion Λ, we will have rate*

$$R \geq \min_{\mathbb{E}[d(u,v)]=\Lambda} I(U;V)$$

Example 28. U is Gaussian 0-mean, variance σ^2, $\mathcal{V} = \mathbb{R}$ and $d(u,v) = (u-v)^2$.
Suppose (u,v) is such that $\mathbb{E}[(u-v)^2] = \Lambda$, then

$$\begin{aligned} I(U;V) &= h(U) - h(U|V) \\ &= h(U) - h(U-V|V) \\ &\geq h(U) - h(U-V) \\ &\geq h(U) - \frac{1}{2}\log(2\pi e \Lambda) \\ &= \frac{1}{2}\log\frac{\sigma^2}{\Lambda} \end{aligned}$$

$$\Rightarrow I(U;V) \geq (\frac{1}{2}\log\frac{\sigma^2}{\Lambda})^+$$

Indeed

$$\min_{\mathbb{E}[(u,v)^2]=\Lambda} I(U;V) = \frac{1}{2}\log\frac{\sigma^2}{\Lambda} \quad \text{when} \quad \Lambda \leq \sigma^2$$

Recall: $I(U;V)$ as a function of $(P_u, P_{v|u})$ is concave in P_u and convex in $P_{v|u}$

Theorem 11.2. *Given an iid source $U_1, U_2, ...$, given $d : \mathcal{U} \times \mathcal{V} \to \mathbb{R}$, given a P_{uv} with $\mathbb{E}[d(U,V)] = \Lambda$ and $I(U;V) = R$, given an $\epsilon > 0$, there exists a quantizer (f, ϕ) with*

$$rate \leq R + \epsilon \quad and \quad |distortion - \Lambda| < \epsilon$$

Proof. See **chapter 10 section 4** in "Elements of information theory, second edition" $\qquad\square$

Given i.i.d. source $U_1 U_2 \dots$ and a distorsion measure $d : \mathcal{U} \times \mathcal{V} \to \mathbb{R}$ the possible (R rate, D distortion) pairs are

$$R \geq \min_{p_{V|U}:\mathbb{E}[d(U,V)]=D} I(U;V)$$

Definition 11.2 (Rate distorsion)**.**

$$R \geq \min_{p_{V|U}:\mathbb{E}[d(U,V)]\leq D} I(U;V)$$

Example 29. When $U \sim N(0,\sigma^2)$ and $d(u,v) = (u-v)^2$

$$R(D) = \left(\frac{1}{2}\log\frac{\sigma^2}{D}\right)^+$$

$$D(R) = \sigma^2 2^{-2R}, R \geq 0$$

11.1 Multi party communications

Encode two message $m_1 \in \{1, \dots, M_1\}$ and $m_2 \in \{1, \dots, M_2\}$ with two encoders Enc_i. Transmit with one channel and decode with onde decoder Dec that gives $(\hat{m}_1, \hat{m}_2)$.

$$R_i = \frac{1}{n}\log M_i$$

$$p_e = \frac{1}{M_1 M_2} \sum_{m_1, m_2} Pr\{(\hat{m}_1, \hat{m}_2) \neq (m_1, m_2)|m_1, m_2 \text{ is sent}\}$$

Definition 11.3. *Given a channel $W(y|x_1, x_2)$ (memoryless), a rate pair (R_1, R_2) is said to be achievable if $\forall \epsilon > 0$ there is (Enc_1, Enc_2, Dec) such that the rate of Enc_1 is greater that $R_1 - \epsilon$, the rate of Enc_2 is greater that $R_2 - \epsilon$ and $p_e \leq \epsilon$. The set of all achievable rate-pairs (for the given W) is denoted $C(W)$ and called the capacity region.*

Theorem 11.3. *Given $W(y|x_1, x_2)$ for any p_{X_1}, p_{X_2} then the achievable pair rates are*

$$\{(R_1, R_2) : R_1 \leq I(X_1; YX_2), R_2 \leq I(X_2; YX_1), R_1 + R_2 \leq I(X_1 X_2; Y)\}$$

Proof. Suppose we are given W, p_{X_1} and p_{X_2}, $R_1 \leq I(X_1; YX_2)$, $R_2 \leq I(X_2; YX_1)$, $R_1 + R_2 \leq I(X_1 X_2; Y)$. Fix ϵ

Let $M_1 = 2^{n(R_1-\epsilon)}$, $M_2 = 2^{n(R_2-\epsilon)}$, set $p_{X_1 X_2 Y} = p_{X_1}(x_1)p_{X_2}(x_2)W(y|x_1 x_2)$. Create the random encoder with

$$\{X_{ij}(m) : i \in \{1,2\}, 0 \leq j < n, m \in M_i\} \sim p_{X_i}$$

$\qquad\square$

We use the typicality decoder: given Y^n search for $\hat{m}_1, \hat{m}_2$ such that

$$(Enc_1^n(\hat{m}_1), Enc_2^n(\hat{m}_2), Y^n) \in T(n, \delta, p)$$

If only one pair $(\hat{m}_1, \hat{m}_2)$ is found declare it, otherwise declare 0. We are fine concerning the rates. For the probability of error assume (w.l.o.g.) that $m_1 = 1, m_2 = 1$ is sent.

$$\{error\} = \{(X_1^n(1), X_2^n(1), Y^n) \notin T\} \cup \bigcup_{(m_1', m_2') \neq (1,1)} \{(X_1^n(m_1'), X_2^n(m_2'), Y^n) \in T\}$$

$$p_e = Pr\left\{(X_1^n(1), X_2^n(1), Y^n) \notin T\right\} + \sum_{(m_1', m_2') \neq (1,1)} Pr\left\{(X_1^n(m_1'), X_2^n(m_2'), Y^n) \in T\right\}$$

$$(m_1', m_2') \neq (1,1) \Rightarrow \begin{cases} m_1' = 1, m_2' \neq 1, \text{genericity } (1,2), (M_2 - 1) \text{ such terms} \\ m_1' \neq 1, m_2' = 1, \text{genericity } (2,1), (M_1 - 1) \text{ such terms} \\ m_1' \neq 1, m_2' \neq 1, \text{genericity } (2,2), (M_1 - 1)(M_2 - 1) \text{ such terms} \end{cases}$$

Then

$$\begin{aligned} p_e \leq & Pr\left\{(X_1^n(1), X_2^n(1), Y^n) \notin T\right\} + (M_2 - 1)Pr\left\{(X_1^n(1), X_2^n(2), Y^n) \in T\right\} \\ & + (M_1 - 1)Pr\left\{(X_1^n(2), X_2^n(1), Y^n) \in T\right\} \\ & + (M_1 - 1)(M_2 - 1)Pr\left\{(X_1^n(2), X_2^n(2), Y^n) \in T\right\} \end{aligned}$$

Given that $(X_1^n(1), X_2^n(1), Y^n)$ i.i.d. $\sim p_{X_1, X_2, Y}$ then

$$\lim_{n \to \infty} Pr\left\{(X_1^n(1), X_2^n(1), Y^n) \notin T\right\} = 0$$

$(X_1^n(1), X_2^n(2), Y^n)$ i.i.d. $\sim p_{X_1, Y} p_{X_2} \triangleq q_1$ then

$$Pr\left\{(X_1^n(1), X_2^n(2), Y^n) \in T(p)\right\} = 2^{-n(D(p||q_1) + o(\delta))}$$

$$D(p||q_1) = \sum_{x_1, x_2, y} p(x_1, x_2, y) \log \frac{p(x_1, x_2, y)}{p(x1, y)p(x_2)} = I(X_2; Y X_1)$$

Similarly

$$Pr\left\{(X_1^n(2), X_2^n(1), Y^n) \in T(p)\right\} = 2^{-n(I(X_1; Y X_2) + o(\delta))}$$
$$Pr\left\{(X_1^n(2), X_2^n(2), Y^n) \in T(p)\right\} = 2^{-n(I(X_1 X_2; Y) + o(\delta))}$$

so

$$p_e \leq 2^{-n(I(X_2; Y X_1) + o(\delta))} + 2^{-n(I(X_1; Y X_2) + o(\delta))} + 2^{-n(I(X_1 X_2; Y) + o(\delta))} \to 0$$

as $R_2 \leq I(X_2; Y X_1)$, $R_1 \leq I(X_1; Y X_2)$ and $R_1 + R_2 \leq I(X_1 X_2; Y)$

Suppose we are given a "working system", normally

with $\frac{1}{M_1 M_2} Pr\left\{\hat{m}_1 \hat{m}_2 \neq m_1 m_2 | m_1 m_2 \text{ is sent}\right\} < \epsilon$

We want to show that (R_1, R_2) belongs to convex hull $\left(\bigcup_{P_{x_1} P_{x_2}} \{...\}\right)$ To that end, let U_1 be uniformly distributed on $1...M_1$ with $R_1 = \frac{1}{n} H(U_1)$ and let U_2 be uniformly distributed on $1...M_2$ with $R_2 = \frac{1}{n} H(U_2)$ also with

$$\Rightarrow Pr\left\{V_1 V_2 \neq U_1 U_2\right\} < \epsilon \Rightarrow \begin{cases} Pr\left\{V_1 \neq U_1\right\} < \epsilon \\ Pr\left\{V_2 \neq U_2\right\} < \epsilon \end{cases}$$

$$H(U_1|V_1) \leq h_2(\epsilon) + \epsilon \log(M_1 - 1) \leq h_2(\epsilon) + \epsilon_n R_1$$
$$H(U_2|V_2) \leq h_2(\epsilon) + \epsilon \log(M_2 - 1) \leq h_2(\epsilon) + \epsilon_n R_2$$
$$H(U_1 U_2|V_1 V_2) \leq h_2(\epsilon) + \epsilon \log(M_1 M_2 - 1) \leq h_2(\epsilon) + \epsilon_n (R_1 + R_2)$$

$$\rightarrow \frac{1}{n} I(U_1; V_1) = \frac{1}{n} H(U_1) - \frac{1}{n} H(U_1|V_1) \geq (1 - \epsilon) R_1 - \frac{1}{n} h_2(\epsilon)$$
$$\frac{1}{n} I(U_2; V_2) = \frac{1}{n} H(U_2) - \frac{1}{n} H(U_2|V_2) \geq (1 - \epsilon) R_2 - \frac{1}{n} h_2(\epsilon)$$
$$\frac{1}{n} I(U_1 U_2; V_1 V_2) \geq (1 - \epsilon)(R_1 + R_2) - \frac{1}{n} h_2(\epsilon) \leftarrow (\boldsymbol{\Omega})$$

Observe

$$U_1 \rightarrow X_1^n \rightarrow Y^n \rightarrow V_1 \text{ so } I(U_1; V_1) \leq I(X_1^n; Y^n)$$

and

$$U_2 \rightarrow X_2^n \rightarrow Y^n \rightarrow V_2 \text{ so } I(U_2; V_2) \leq I(X_2^n; Y^n)$$

so

$$(U_1 U_2) \rightarrow (X_1^n X_2^n) \rightarrow Y^n \rightarrow (V_1 V_2) \text{ so } I(U_1 U_2; V_1 V_2) \leq I(X_1^n X_2^n; Y^n)$$

and also X_1^n is indep. of X_2^n

$$
\begin{aligned}
I(X_1^n; Y^n) &\leq I(X_1^n; Y^n X_2^n) \\
&= I(X_1^n; Y^n|X_2^n) + \underbrace{I(X_i^n; X_2^n)}_{\rightarrow 0} \\
&= H(Y^n|X_2^n) - H(Y^n|X_1^n X_2^n) \\
&\leq \sum_{i=1}^{n} H(Y_i|X_{2i}) - \sum_{i=1}^{n} H(Y_i|X_{1i} X_{2i}) \\
&= \sum_{i=1}^{n} I(X_{1i}; Y_i|X_{2i}) \\
&= \sum_{i=1}^{n} \underbrace{I(X_{1i}; Y_i X_{2i})}_{a_i} - \underbrace{I(X_{1i}; X_{2i})}_{\rightarrow 0}
\end{aligned}
$$

Similarly

$$I(X_2^n; Y^n) \leq \sum_{i=1}^{n} \underbrace{I(X_{2i}; Y_i X_{1i})}_{b_i}$$

and

$$
\begin{aligned}
I(X_1^n X_2^n; Y^n) &= H(Y^n) - H(Y^n) - H(Y^n|X_1^n X_2^n) \\
&\leq \sum_i H(Y_i) - H(Y_i|X_{1i} X_{2i}) \\
&= \sum_i \underbrace{I(X_{1i} X_{2i}; Y_i)}_{c_i}
\end{aligned}
$$

Therefore we have

$$(\mathbf{\Omega}) \leq \frac{1}{n} \sum_{i=1}^{n} a_i$$

$$(1 - \epsilon)R_2 - \frac{1}{n}h_2(\epsilon) \leq \frac{1}{n} \sum_{i=1}^{n} b_i$$

$$(1 - \epsilon)(R_1 + R_2) - \frac{1}{n}h_2(\epsilon) \leq \frac{1}{n} \sum_{i=1}^{n} c_i$$

$$\Rightarrow (R_1, R_2) - O(\epsilon) \in Pent' = Pent = \frac{1}{n} \sum_{i=1}^{n} Pent_i$$

$$\subseteq ConvexHull(\bigcup_i Pent_i) \subseteq ConvexHull(\bigcup_{P_{X_1}P_{X_2}} \{R_1 \leq I(X_1; YX_2), R_2 \leq I(X_2; YX_1), R_1 + R_2 \leq I(X_1X_2; Y)\})$$

So we have proved that

Theorem 11.4.

$$C(W) \subseteq ConvexHull(\bigcup_{P_{X_1}P_{X_2}} \{...\})$$

So, using what we have shown before, we get the equality

$$C(W) = ConvexHull(\bigcup_{P_{X_1}P_{X_2}} \{...\})$$

Example 30.

$$X_1 \in \{0, 1\}, X_2 \in \{0, 1\}, Y = X_1 + X_2 = \{0, 1, 2\}$$

so

$$R_1 \leq 1, R_2 \leq 1, R_1 + R_2 \leq \frac{3}{2}$$

Appendices

A Markov chains

$U_1 - U_2 - \cdots - U_n$ forms a Markov chain if the joint probability distribution of the RVs is

$$p(a, b, c, d) = p(a)p(b|a)p(c|b)p(d|c)$$

which is equivalent to $(U_1, \ldots, U_{k-1})$ are independant of $(U_{k+1}, \ldots, U_n)$ when conditionned on U_k for any k.

Theorem A.1. *The reverse of a MC is a MC*

B Stochastic processes

A stochastic process is a collection $U_1, U_2 \ldots U_n$ of RVs each taking values in $\mathcal{U}$. It is described by its joint probability

$$p(u^n) = P(U_1 \ldots U_n = u_1 \ldots u_n) = P(U^n = u^n)$$

Definition B.1 (Stationary stochastic process). *A process $U_1, U_2, \ldots$ is called stationary if for every n and k and $u_1 \ldots u_n$, we have*

$$p(u^n) = p(U_1 \ldots U_n = u_1 \ldots u_n) = p(U_{1+k} \ldots U_{n+k} = u_1 \ldots u_n)$$

In other words, the process is time shift invariant.

C Concave/convex functions

A function $f : S \to \mathbb{R}$ is called convex if

$$\forall x, y \in S, 0 \leq \lambda \leq 1, f(\lambda x - (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

where S is a convex set.

Definition C.1. *A set $S \subseteq \mathbb{R}^k$ is called to be convex if*

$$\forall x, y \in S, 0 \leq \lambda \leq 1, \lambda x + (1 - \lambda)y \in S$$

Definition C.2. *f is called concave if $-f$ is convex.*

Definition C.3. *k-simplex*

$$S_k = \{(p_1, ..., p_k) \in \mathbb{R}^k, p \geq 0, \sum_i p_i = 1\}$$

as the k-simplex (a $(k-1)$-dimentional subset of $\mathbb{R}^k$)

Remark: Given S_k a convex set and $p, q \in S_k$, let

$$r = \lambda p + (1 - \lambda)q$$
$$r_i = \lambda p_i + (1 - \lambda)q_i \geq 0$$

$$\sum r_i = \lambda + (1 - \lambda) = 1$$

Example 31. Let $f : S_k \to \mathbb{R}$, with

$$f(p_1, ..., p_k) = \sum^k p_i \log_{\frac{1}{p_i}}$$

claim: f is concave

Proof. Given $p, q \in S_k, 0 \leq \lambda \leq 1$, define (U, V) with $U \in \{0, 1\}$ and $V \in \{1, ..., k\}$

$$P_{UV}(u, v) = \begin{cases} \lambda p_i, & u = 0, v = i \\ (1 - \lambda)q_i, & u = 1, v = i \end{cases}$$

therefore we have

$$Pr\{V = i\} = \lambda p_i + (1 - \lambda)q_i$$
$$H(V) = f(\lambda p + (1 - \lambda)q)$$
$$H(V|U) = \lambda f(p) + (1 - \lambda)f(q)$$

$\square$

Example 32. For $W(Y|X)$ let $f(p_X) = I(X; Y)$ when $p(x, y) = p_X(x)W(Y|X)$ **Claim:** f is concave,

$$I(X; Y) = H(Y) - H(X|Y)$$

and

$$H(Y|X) = \sum_x p_X(x) \sum_y W(Y|X) \log \frac{1}{W(Y|X)}$$

We see that $H(Y|X)$ is a linear function of $p_X(x)$.
$H(Y)$ is a concave function of $p_Y(y)$ with

$$p_Y(y) - \sum_x p_X(x)W(Y|X)$$

$$p_X \underbrace{\longrightarrow}_{\text{linear}} p_Y \underbrace{\longrightarrow}_{\text{concave}} H(Y) \Longrightarrow p_X \underbrace{\longrightarrow}_{\text{concave}} H(Y)$$

How to maximize a function on the simplex?

Theorem C.1. *Karush-Kuhn-Tucker conditions - (KKT)*
Suppose $f : S_k \to \mathbb{R}$, smooth ($\frac{df}{dp_i dp_j}$ exists), then if $p = \{p_1, ..., p_k\}$ maximizes f, then $\exists \lambda$ s.t.

$$\forall i, \frac{df}{dp_i} \leq \lambda$$

with equality $\forall i$ for which $p_i > 0$

Proof. Suppose $(p_1, ..., p_k)$ maximizes f, then suppose that $p_i > 0$. Then we can consider a $p' \in S_k$ as follow:
Pick $j \neq i$ and a small $\epsilon, 0 < \epsilon < p_i$

$$p'_k = \begin{cases} p_i - \epsilon, & k = i \\ p_j + \epsilon, & k = j \\ p_k, & \text{else} \end{cases}$$

$$f(p') = f(p) + \frac{df(p)}{dp_i}(-\epsilon) + \frac{df(p)}{dp_j}(\epsilon) + O(\epsilon^2)$$

$$= f(p) + \epsilon \left[\frac{df}{dp_j} - \frac{df}{dp_i} \right] + O(\epsilon^2)$$

So for every i, j with $p_i > 0$ we have

$$\frac{df}{dp_j} \frac{df}{dp_i}$$

$\Rightarrow$ equality if i and j are such that $p_i > 0, p_j > 0$
$\Rightarrow$ for i's such that $p_i > 0, \frac{df}{dp_i} = \lambda$ and all the indices j have $\frac{df}{dp_j} \leq \lambda$ $\square$

Theorem C.2. *Suppose $f : S_k \to \mathbb{R}$, suppose f is concave and suppose for $p \in S_k$, the KKT condition hold. Then $\forall q \in S_k, f(q) \leq f(p)$*

Proof.

$$f(\epsilon q + (1 - \epsilon)p) \geq (1 - \epsilon)f(p) + \epsilon f(q)$$

$$\frac{f(\epsilon q + (1 - \epsilon)p) - f(p)}{\lambda} \geq f(q) - f(p), \quad \forall 0 < \epsilon \leq 1$$

$$\Rightarrow f(q) - f(p) \leq \lim_{\epsilon \to 0} \frac{f(p + \epsilon(q - p)) - f(p)}{\epsilon}$$

$$f(p + \epsilon(q - p)) = f(p) + \sum \epsilon(q_i - p_i)\frac{df(p)}{dp_i} + O(\epsilon^2)$$

$$\frac{f(p + \epsilon(q - p)) - f(p)}{\epsilon} = \sum_i (q_i - p_i)\frac{df(p)}{dp_i} + O(\epsilon)$$

So

$$\lim_{\epsilon \to 0} \frac{f(p + \epsilon(q - p)) - f(p)}{\epsilon} = \sum_i (q_i - p_i)\frac{df(p)}{dp_i}$$

with

$$(q_i - p_i)\frac{df}{dp_i} = \begin{cases} \lambda(q_i - p_i), & p_i > 0 \\ \underbrace{(q_i - p_i)}_{\geq 0} \underbrace{\frac{df}{dp_i}}_{\leq \lambda}, & p_i = 0 \end{cases} \leq \lambda(q_i - p_i)$$

$$\Rightarrow f(q) - f(p) \leq \lim_{\epsilon \to 0}[...] \leq 0$$

$\square$

Example 33. Supppose $f(p_1, p_2, p_3) = p_1 p_2^2 p_3^3$. We want to maximize it. If it isn't concave, we know that $\log(f(..))$ is concave. A try with KKT:

$$\frac{df}{dp_1} = \frac{1}{p_1}, \frac{df}{dp_2} = \frac{2}{p_2}, \frac{df}{dp_3} = \frac{3}{p_3}$$

setting then all λ yeild

$$(p_1, p_2, p_3) = \lambda(1, 2, 3) = (\frac{1}{6}, \frac{2}{6}, \frac{3}{6})$$

Example 34.

$$f(p_1, p_2, p_3) = (1 + p_1)p_2 p_3$$

maximize f on the simplex by considering

$$\log(f) = \log(1 + p_1) + \log(p_2) + \log(p_3)$$

therefore:

$$\frac{df}{dp_1} = \frac{1}{1 + p_1}, \frac{df}{dp_2} = \frac{1}{p_2}, \frac{df}{dp_3} = \frac{1}{p_3}$$

suggest $p = (0, 0.5, 0.5)$ the $\frac{df}{dp} = (1, 2, 2) \to$ satisfy KKT with $\lambda = 2$

D A story of two pentagons

Consider two pentagons:

$$Pent_1 = \{(R_1, R_2) : 0 \le R_1 \le a_1, 0 \le R_2 \le b_1, 0 \le R_1 + R_2 \le c_1\}$$

$$Pent_2 = \{(R_1, R_2) : 0 \le R_1 \le a_2, 0 \le R_2 \le b_2, 0 \le R_1 + R_2 \le c_2\}$$

Set

$$Pent = \frac{1}{2}Pent_1 + \frac{1}{2}Pent_2 = \{(R_1, R_2) : (R_1, R_2) = \frac{1}{2}(R_1', R_2') + \frac{1}{2}(R_1'', R_2'')\}$$

such that $(R_1', R_2') \in Pent_1$ and $(R_1'', R_2'') \in Pent_2$

And

$$Pent' = \{(R_1, R_2) : R_1 \le \frac{a_1 + a_2}{2}, 0 \le R_2 \le \frac{b_1 + b_2}{2}, 0 \le R_1 + R_2 \le \frac{c_1 + c_2}{2}\}$$

Suppose $(R_1, R_2) \in Pent$, then

$$\{0 \le R_1 \le \frac{a_1 + a_2}{2}, 0 \le R_2 \le \frac{b_1 + b_2}{2}, 0 \le R_1 + R_2 \le \frac{c_1 + c_2}{2}\} \Rightarrow (R_1, R_2) \in Pent'$$

So we have shown that $Pent \subseteq Pent'$
However, if we assume further

$$\{\max(a_1, b_1) \le c_1 \le a_1 + b_1, \max(a_2, b_2) \le c_2 \le a_2 + b_2\}$$

then $Pent' = Pent$

Theorem D.1. *Suppose $(a_i, b_i, c_i) \ge 0, i = 1, ..., n$ with $max(a_i, b_i) \le c_i \le a_i + b_i$ and*

$$Pent_i = \{(R_1, R_2) : R_1 \le a_i, R_2 \le b_i, R_1 + R_2 \le c_i\}$$

$$Pent' = \{(R_1, R_2) : R_1 \le \frac{1}{n}\sum q_i, R_2 \le \frac{1}{n}\sum b_i, R_1 + R_2 \le \frac{1}{n}\sum c_i\}$$

$$Pent = \frac{1}{n}\sum Pent_i = \{\frac{1}{n}\sum(R_{1i}, R_{2i}) : (R_{1i}, R_{2i}) \in Pent_i\}$$

then $Pent = Pent'$