Causal Model Fig 3.8 (f)

Jacob Colvin

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1 Causal Diagram

For DAG, see figure 1(a). X is the *do* variable, Y is the response, Z are other measured variables, W are other known but unobserved variables.

1.1 Simulating Data

$$
\begin{aligned}
p(w_1) &= N(w_1|\mu_{w_1}, \sigma^2_{w_1}) && (1)\\
p(w_2) &= N(w_2|\mu_{w_2}, \sigma^2_{w_2}) && (2)\\
p(x|w_1) &= N(\ x|\mu_x + \alpha_1 w_1, \sigma^2_x) && (3)\\
p(z_1|x, w_2) &= N(\ z_1|\mu_{z_1} + \alpha_2 x + \alpha_3 w_2, \sigma^2_{z_1}) && (4)\\
p(z_2|z_1, w_1) &= N(\ z_2|\mu_{z_2} + \alpha_4 z_1 + \alpha_5 w_1, \sigma^2_{z_2}) && (5)\\
p(y|x, z_1, z_2, w_2) &= N(\ y|\mu_y + \alpha_6 x + \alpha_7 z_1 + \alpha_8 z_2 + \alpha_9 w_2, \sigma^2_y) && (6)
\end{aligned}
$$

2 Derivations

2.1 Axioms and Definitions: Do Calculus

The following videos are a good introduction to do calculus from a more mathematical perspective without being too math/theory heavy. Next, key axioms and definitions are restated and will referenced to explain derivations.

Lectures on Causality: Jonas Peters (Broad Institute) (2017 MIT)
Part I: https://www.youtube.com/watch?v=zvrcyqcN9Wo
Part II: https://www.youtube.com/watch?v=bHOGP5o3Vu0
Part III: https://www.youtube.com/watch?v=Jp4UcgpVA2I
Part IV: https://www.youtube.com/watch?v=ytnr_2dyyMU

2.1.1 Conditional Independence

$(X \perp\!\!\!\perp Y|S)_G$

X is independent of Y given the set S. This means that for fixed values of s, there is no correlation (linear or otherwise) between X and Y, given the graph G.

2.1.2 D-separation

A set of nodes S blocks a path p on the DAG G if either (i) p contains at least one arrow-emitting node that is in S, or (ii) p contains at least one collision node that is outside S ($= \bar{S}$) and all such collision nodes have no descendants in S. If S blocks all paths from X to Y (S d-separates X and Y), then X and Y are independent given S, written as $(X \perp\!\!\!\perp Y|S)_G$.

Example of (i): X ... $S \rightarrow$ ... Y

Example of (ii): X ... $\rightarrow \bar{S} \leftarrow$ ... Y

2.1.3 Sub-graphs used in derivations

G is the causal model, and sub graphs (ie arrows are removed) are needed to justify different do-calculus operations.

$G_{\overline{X}}$ is the graph G with all inputs to X removed.
$G_{\underline{X}}$ is the graph G with all outputs from X removed.
$Z(W)$ is the set of nodes in Z that are not parents of nodes in W (ie no directed paths from $Z(W) \to W$).

2.1.4 Rule 1: Ignoring Observations

$p(y|do\{x\}, z, w) = p(y|do\{x\}, w)$ if $(Y \perp\!\!\!\perp Z|X, W)G_{\overline{X}}$.

2.1.5 Rule 2: Action/Observation Exchange

$p(y|do\{x\}, z, w) = p(y|do\{x, z\}, w)$ if $(Y \perp\!\!\!\perp Z|X, W)G_{\overline{X}\underline{Z}}$

2.1.6 Rule 3: Ignoring Actions

$p(y|do\{x\}, z, w) = p(y|z, w)$ if $(Y \perp\!\!\!\perp Z|X, W)G_{\overline{X}\overline{Z(W)}}$.

2.2 Other

For reference, other identities used.

$p(y) = \int p(x)p(y)\,\mathrm{d}x = p(y)\int p(x)\,\mathrm{d}x$
$p(x, y) = p(y|x)p(x)$
$p(y|x) = p(x, y)/p(x) = g(x)f(x, y) \propto f(x, y)$
$N(x|\mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} exp\left(-\frac{1}{2}(\frac{x-\mu}{\sigma})^2\right)$

2.3 Causal Sub Graphs

2.4 $p(y|do\{x\})$

$$p(y|do\{x\}) \tag{7}$$

$$= \iint p(y|do\{x\}, z_1, z_2)p(z_2|do\{x\}, z_1)p(z_1|do\{x\})\,\mathrm{d}z_1\,\mathrm{d}z_2 \qquad \text{(stats)} \tag{8}$$

$$= \iint p(y|x, z_1, z_2)p(z_2|do\{x\}, z_1)p(z_1|do\{x\})\,\mathrm{d}z_1\,\mathrm{d}z_2 \qquad \text{given\#2} \quad (Y \perp\!\!\!\perp X|Z_1, Z_2)_{G_{\underline{X}}} \tag{9}$$

$$= \iint p(y|x, z_1, z_2)p(z_2|do\{x\}, z_1)p(z_1|x)\,\mathrm{d}z_1\,\mathrm{d}z_2 \qquad \text{given\#2} \quad (Z_1 \perp\!\!\!\perp X)_{G_{\underline{X}}} \tag{10}$$

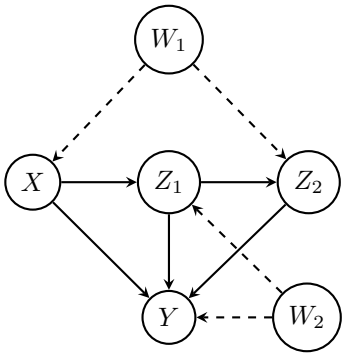
$$= \iint p(y|x, z_1, z_2)p(z_2|do\{x, z_1\})p(z_1|x)\,\mathrm{d}z_1\,\mathrm{d}z_2 \qquad \text{given\#2} \quad (Z_2 \perp\!\!\!\perp Z_1|X)_{G_{\overline{X}\underline{Z}_1}} \tag{11}$$

$$= \iint p(y|x, z_1, z_2)p(z_2|do\{z_1\})p(z_1|x)\,\mathrm{d}z_1\,\mathrm{d}z_2 \qquad \text{given\#3} \quad (Z_2 \perp\!\!\!\perp X|Z_1)_{G_{\overline{Z}_1\overline{X}}} \tag{12}$$

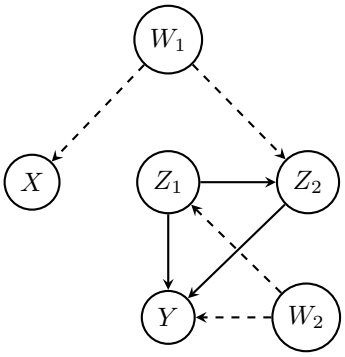
$$= \iint p(y|x, z_1, z_2)\int p(z_2|do\{z_1\}, x)p(x|do\{z_1\})\,\mathrm{d}x\,p(z_1|x)\,\mathrm{d}z_1\,\mathrm{d}z_2 \qquad \text{(stats)} \tag{13}$$

$$= \iint p(y|x, z_1, z_2)\int p(z_2|do\{z_1\}, x)p(x)\,\mathrm{d}x\,p(z_1|x)\,\mathrm{d}z_1\,\mathrm{d}z_2 \qquad \text{given\#3} \quad (X \perp\!\!\!\perp Z_1)_{G_{\overline{Z}_1}} \tag{14}$$

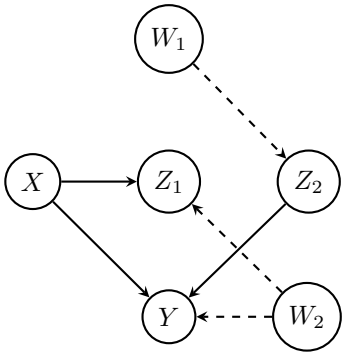
$$= \iint p(y|x, z_1, z_2)\int p(z_2|x, z_1)p(x)\,\mathrm{d}x\,p(z_1|x)\,\mathrm{d}z_1\,\mathrm{d}z_2 \qquad \text{given\#2} \quad (Z_2 \perp\!\!\!\perp Z_1|X)_{G_{\underline{Z}_1}} \tag{15}$$



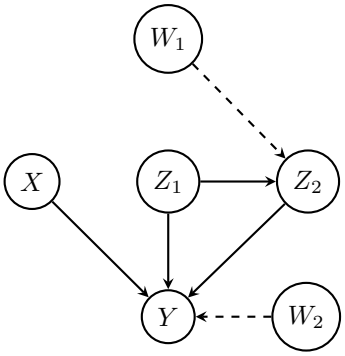
(a) *Figure 3.8 (f) in "Causality" 2E by Pearl.*



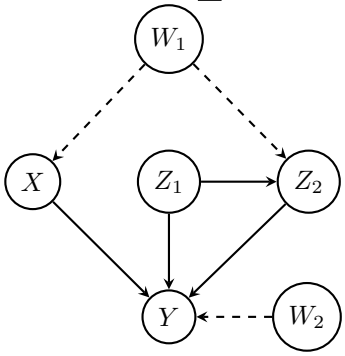
(b) $G_{\underline{X}}$



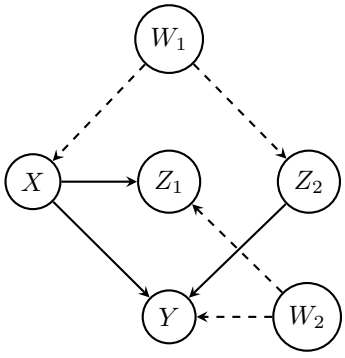
(c) $G_{\overline{X}\underline{Z_1}}$



(d) $G_{\overline{Z_1 X}}$



(e) $G_{\overline{Z_1}}$



(f) $G_{\underline{Z_1}}$

Figure 1: Causal Model DAGs. Figure (a) is the full model, figures (b) through (f) are used in the derivation of $p(y|do\{x\})$

2.5 $p(z_1|x)$

$$
\begin{aligned}
p(z_1|x) &= \iint p(z_1|x,w_2)p(x|w_1)p(w_2)p(w_1)\,\mathrm{d}w_2\,\mathrm{d}w_1 && (16)\\
&= \int p(z_1|x,w_2)p(w_2)\,\mathrm{d}w_2 \int p(x|w_1)p(w_1)\,\mathrm{d}w_1 && (17)\\
&= \int p(z_1|x,w_2)p(w_2)\,\mathrm{d}w_2 && (18)\\
&= \int N(z_1|\mu_{z_1}+\alpha_2 x+\alpha_3 w_2,\sigma^2_{z_1})N(w_2|\mu_{w_2},\sigma^2_{w_2})\,\mathrm{d}w_2 && (19)\\
&\quad \left(\mu_{z_1}+\alpha_2 x+\alpha_3 w_2 \Rightarrow w, \mu_{z_1}+\alpha_2 x+\alpha_3\mu_{w_2}\Rightarrow\mu, \alpha_3^2\sigma^2_{w_2}\Rightarrow\sigma^2_w\right)\\
&= \int N(z_1|w,\sigma^2_{z_1})N(w|\mu,\sigma^2_w)\,\mathrm{d}w_2 && (20)\\
&= N(z|\mu,\sigma_z^2+\sigma_w^2) && (21)\\
&\quad \left(\mu_{z_1}+\alpha_2 x+\alpha_3\mu_{w_2}\Leftarrow\mu, \alpha_3^2\sigma^2_{w_2}\Leftarrow\sigma^2_w\right)\\
&= N(z_1|\mu_{z_1}+\alpha_2 x+\alpha_3\mu_{w_2},\sigma^2_{z_1}+\alpha_3^2\sigma^2_{w_2}) && (22)
\end{aligned}
$$

2.6 $p(z_2|z_1, x)$

$$p(z_2|z_1, x) \tag{23}$$

$$\propto \iint p(z_2|z_1, w_1)p(z_1|w_2, x)p(x|w_1)p(w_2)p(w_1)\,\mathrm{d}w_1\,\mathrm{d}w_2 \tag{24}$$

$$\propto \int p(z_2|z_1, w_1)p(x|w_1)p(w_1)\,\mathrm{d}w_1 \int p(z_1|w_2, x)p(w_2)\,\mathrm{d}w_2 \tag{25}$$

$$\propto \int p(z_2|z_1, w_1)p(x|w_1)p(w_1)\,\mathrm{d}w_1 \tag{26}$$

$$\propto \int N(z_2|\mu_{z_2} + \alpha_4 z_1 + \alpha_5 w_1, \sigma_{z_2}^2)N(x|\mu_x + \alpha_1 w_1, \sigma_x^2)N(w_1|\mu_{w_1}, \sigma_{w_1}^2)\,\mathrm{d}w_1 \tag{27}$$

$$(\mu_{z_2} + \alpha_4 z_1 \Rightarrow \mu_{z_2})$$

$$\propto \int exp\left(-\frac{1}{2}\left(\left(\frac{z_2 - \mu_{z_2} - \alpha_5 w_1}{\sigma_{z_2}}\right)^2 + \left(\frac{x - \mu_x - \alpha_1 w_1}{\sigma_x}\right)^2 + \left(\frac{w_1 - \mu_{w_1}}{\sigma_{w_1}}\right)^2\right)\right)\mathrm{d}w_1 \tag{28}$$

$$\propto \int exp\left(-\frac{1}{2}\left(\frac{\alpha_5^2}{\sigma_{z_2}^2}w_1^2 - 2\frac{\alpha_5 z_2 - \alpha_5\mu_{z_2}}{\sigma_{z_2}^2}w_1 + \frac{(z_2 - \mu_{z_2})^2}{\sigma_{z_2}^2} + \frac{\alpha_1^2}{\sigma_x^2}w_1^2 - 2\frac{\alpha_1 x - \alpha_1\mu_x}{\sigma_x^2}w_1 + \frac{1}{\sigma_{w_1}^2}w_1^2 - 2\frac{\mu_{w_1}}{\sigma_{w_1}^2}w_1\right)\right)\mathrm{d}w_1 \tag{29}$$

$$\propto \int exp\left(-\frac{1}{2}\left(\left(\frac{\alpha_5^2}{\sigma_{z_2}^2} + \frac{\alpha_1^2}{\sigma_x^2} + \frac{1}{\sigma_{w_1}^2}\right)w_1^2 - 2\left(\frac{\alpha_5 z_2 - \alpha_5\mu_{z_2}}{\sigma_{z_2}^2} + \frac{\alpha_1 x - \alpha_1\mu_x}{\sigma_x^2} + \frac{\mu_{w_1}}{\sigma_{w_1}^2}\right)w_1 + \frac{(z_2 - \mu_{z_2})^2}{\sigma_{z_2}^2}\right)\right)\mathrm{d}w_1 \tag{30}$$

$$\propto \int exp\left(-\frac{1}{2}\left(aw^2 - 2bw + c\right)\right)\mathrm{d}w \tag{31}$$

$$\propto \int exp\left(-\frac{1}{2}\left(\left(\frac{w - b/a}{a^{-1/2}}\right)^2 + c - \frac{b^2}{a}\right)\right)\mathrm{d}w \tag{32}$$

$$\propto \int \frac{1}{a^{-1/2}} exp\left(-\frac{1}{2}\left(\frac{w - b/a}{a^{-1/2}}\right)^2\right)\mathrm{d}w \tag{33}$$

$$\propto exp\left(-\frac{1}{2}\left(c - \frac{b^2}{a}\right)\right) \tag{34}$$

$$\propto exp\left(-\frac{1}{2}\left(\frac{(z_2 - \mu_{z_2})^2}{\sigma_{z_2}^2} - \left(\frac{\alpha_5^2}{\sigma_{z_2}^2} + \frac{\alpha_1^2}{\sigma_x^2} + \frac{1}{\sigma_{w_1}^2}\right)^{-1}\left(\frac{\alpha_5 z_2 - \alpha_5\mu_{z_2}}{\sigma_{z_2}^2} + \frac{\alpha_1 x - \alpha_1\mu_x}{\sigma_x^2} + \frac{\mu_{w_1}}{\sigma_{w_1}^2}\right)^2\right)\right) \tag{35}$$

$$\propto exp\left(-\frac{1}{2}\left(\frac{1}{\sigma_{z_2}^2}z_2^2 - 2\left(\frac{\mu_{z_2}}{\sigma_{z_2}^2}\right)z_2 - \left(\frac{\alpha_5^2}{\sigma_{z_2}^2} + \frac{\alpha_1^2}{\sigma_x^2} + \frac{1}{\sigma_{w_1}^2}\right)^{-1}\left(\frac{\alpha_5}{\sigma_{z_2}^2}z_2 + \left(\frac{\alpha_1 x - \alpha_1\mu_x}{\sigma_x^2} + \frac{\mu_{w_1}}{\sigma_{w_1}^2} - \frac{\alpha_5\mu_{z_2}}{\sigma_{z_2}^2}\right)\right)^2\right)\right) \tag{36}$$

$$\propto exp\left(-\frac{1}{2}\left(\frac{1}{\sigma_{z_2}^2}z_2^2 - 2\left(\frac{\mu_{z_2}}{\sigma_{z_2}^2}\right)z_2 - \left(\frac{\alpha_5^2}{\sigma_{z_2}^2} + \frac{\alpha_1^2}{\sigma_x^2} + \frac{1}{\sigma_{w_1}^2}\right)^{-1}\left(\frac{\alpha_5^2}{(\sigma_{z_2}^2)^2}z_2^2 + 2\left(\frac{\alpha_1 x - \alpha_1\mu_x}{\sigma_x^2} + \frac{\mu_{w_1}}{\sigma_{w_1}^2} - \frac{\alpha_5\mu_{z_2}}{\sigma_{z_2}^2}\right)\frac{\alpha_5}{\sigma_{z_2}^2}z_2\right)\right)\right) \tag{37}$$

$$\propto exp\left(-\frac{1}{2}\left(\left(\frac{1}{\sigma_{z_2}^2} - \left(\frac{\alpha_5^2}{\sigma_{z_2}^2} + \frac{\alpha_1^2}{\sigma_x^2} + \frac{1}{\sigma_{w_1}^2}\right)^{-1}\left(\frac{\alpha_5^2}{(\sigma_{z_2}^2)^2}\right)\right)z_2^2 - 2\left(\frac{\mu_{z_2}}{\sigma_{z_2}^2} + \frac{\left(\frac{\alpha_1 x - \alpha_1\mu_x}{\sigma_x^2} + \frac{\mu_{w_1}}{\sigma_{w_1}^2} - \frac{\alpha_5\mu_{z_2}}{\sigma_{z_2}^2}\right)\frac{\alpha_5}{\sigma_{z_2}^2}}{\left(\frac{\alpha_5^2}{\sigma_{z_2}^2} + \frac{\alpha_1^2}{\sigma_x^2} + \frac{1}{\sigma_{w_1}^2}\right)}\right)z_2\right)\right) \tag{38}$$

where $\quad (\mu_{z_2} + \alpha_4 z_1 \Leftarrow \mu_{z_2})$

$$\propto exp\left(-\frac{1}{2}\left(\left(\frac{\sigma_x^2 + \alpha_1^2\sigma_{w_1}^2}{(\sigma_x^2 + \alpha_1^2\sigma_{w_1}^2)\sigma_{z_2}^2 + \alpha_5^2\sigma_{w_1}^2\sigma_x^2}\right)z_2^2 - 2\left(\frac{\mu_{z_2} + \alpha_4 z_1}{\sigma_{z_2}^2} + \frac{\left(\frac{\alpha_1 x - \alpha_1\mu_x}{\sigma_x^2} + \frac{\mu_{w_1}}{\sigma_{w_1}^2} - \frac{\alpha_5\mu_{z_2} + \alpha_5\alpha_4 z_1}{\sigma_{z_2}^2}\right)\frac{\alpha_5}{\sigma_{z_2}^2}}{\left(\frac{\alpha_5^2}{\sigma_{z_2}^2} + \frac{\alpha_1^2}{\sigma_x^2} + \frac{1}{\sigma_{w_1}^2}\right)}\right)z_2\right)\right) \tag{39}$$

$$\propto exp\left(-\frac{1}{2}\left(\frac{\sigma_x^2 + \alpha_1^2\sigma_{w_1}^2}{(\sigma_x^2 + \alpha_1^2\sigma_{w_1}^2)\sigma_{z_2}^2 + \alpha_5^2\sigma_{w_1}^2\sigma_x^2}z_2^2 - 2\frac{(\sigma_x^2 + \alpha_1^2\sigma_{w_1}^2)\mu_{z_2} + (\alpha_5\mu_{w_1} + \alpha_4 z_1)\sigma_x^2 - \alpha_1\alpha_5\sigma_{w_1}^2\mu_x + \alpha_1\alpha_5\sigma_{w_1}^2 x + \alpha_1^2\alpha_4\sigma_{w_1}^2 z_1}{(\sigma_x^2 + \alpha_1^2\sigma_{w_1}^2)\sigma_{z_2}^2 + \alpha_5^2\sigma_{w_1}^2\sigma_x^2}z_2\right)\right)$$

$$= N\left(z_2\middle|\frac{(\sigma_x^2 + \alpha_1^2\sigma_{w_1}^2)\mu_{z_2} + (\alpha_5\mu_{w_1} + \alpha_4 z_1)\sigma_x^2 - \alpha_1\alpha_5\sigma_{w_1}^2\mu_x + \alpha_1\alpha_5\sigma_{w_1}^2 x + \alpha_1^2\alpha_4\sigma_{w_1}^2 z_1}{\sigma_x^2 + \alpha_1^2\sigma_{w_1}^2}, \frac{(\sigma_x^2 + \alpha_1^2\sigma_{w_1}^2)\sigma_{z_2}^2 + \alpha_5^2\sigma_{w_1}^2\sigma_x^2}{\sigma_x^2 + \alpha_1^2\sigma_{w_1}^2}\right) \tag{40}$$

$$= N\left(z_2\middle|\frac{(\sigma_x^2 + \alpha_1^2\sigma_{w_1}^2)\mu_{z_2} + \alpha_5\mu_{w_1}\sigma_x^2 - \alpha_1\alpha_5\sigma_{w_1}^2\mu_x}{\sigma_x^2 + \alpha_1^2\sigma_{w_1}^2} + \frac{\alpha_1\alpha_5\sigma_{w_1}^2}{\sigma_x^2 + \alpha_1^2\sigma_{w_1}^2}x + \frac{(\alpha_4\sigma_x^2 + \alpha_1^2\alpha_4\sigma_{w_1}^2)}{\sigma_x^2 + \alpha_1^2\sigma_{w_1}^2}z_1, \frac{(\sigma_x^2 + \alpha_1^2\sigma_{w_1}^2)\sigma_{z_2}^2 + \alpha_5^2\sigma_{w_1}^2\sigma_x^2}{\sigma_x^2 + \alpha_1^2\sigma_{w_1}^2}\right) \tag{41}$$

2.7 $p(y|x, z_1, z_2)$

$$p(y|z_1, z_2, x) \tag{42}$$

$$\propto \iint p(y|x, z_1, z_2, w_2)p(z_2|z_1, w_1)p(z_1|x, w_2)p(x|w_1)p(w_2)p(w_1)\,\mathrm{d}w_2\,\mathrm{d}w_1 \tag{43}$$

$$\propto \int p(y|x, z_1, z_2, w_2)p(z_1|x, w_2)p(w_2)\,\mathrm{d}w_2 \int p(z_2|z_1, w_1)p(x|w_1)p(w_1)\,\mathrm{d}w_1 \tag{44}$$

$$\propto \int p(y|x, z_1, z_2, w_2)p(z_1|x, w_2)p(w_2)\,\mathrm{d}w_2 \tag{45}$$

$$\propto \int N(y|\mu_y + \alpha_9 w_2, \sigma_y^2)N(z_1|\mu_{z_1} + \alpha_3 w_2, \sigma_{z_1}^2)N(w_2|\mu_{w_2}, \sigma_{w_2}^2)\,\mathrm{d}w_2 \tag{46}$$

where $\quad (\mu_y + \alpha_6 x + \alpha_7 z_1 + \alpha_8 z_2 \Rightarrow \mu_y, \mu_{z_1} + \alpha_2 x \Rightarrow \mu_{z_1})$

$$\propto \int exp\left(-\frac{1}{2}\left(\left(\frac{y - \mu_y - \alpha_9 w_2}{\sigma_y}\right)^2 + \left(\frac{z_1 - \mu_{z_1} - \alpha_3 w_2}{\sigma_{z_1}}\right)^2 + \left(\frac{w_2 - \mu_{w_2}}{\sigma_{w_2}}\right)^2\right)\right)\mathrm{d}w_2 \tag{47}$$

$$\propto \int exp\left(-\frac{1}{2}\left(\frac{(y - \mu_y)^2 - 2\alpha_9(y - \mu_y)w_2 + \alpha_9^2 w_2^2}{\sigma_y^2} + \frac{(z_1 - \mu_{z_1})^2 - 2\alpha_3(z_1 - \mu_{z_1})w_2 + \alpha_3^2 w_2^2}{\sigma_{z_1}^2} + \frac{w_2^2 - 2\mu_{w_2}w_2 + \mu_{w_2}^2}{\sigma_{w_2}^2}\right)\right)\mathrm{d}w_2 \tag{48}$$

$$\propto \int exp\left(-\frac{1}{2}\left(\left(\frac{\alpha_9^2}{\sigma_y^2} + \frac{\alpha_3^2}{\sigma_{z_1}^2} + \frac{1}{\sigma_{w_2}^2}\right)w_2^2 - 2\left(\frac{\alpha_9(y - \mu_y)}{\sigma_y^2} + \frac{\alpha_3(z_1 - \mu_{z_1})}{\sigma_{z_1}^2} + \frac{\mu_{w_2}}{\sigma_{w_2}^2}\right)w_2 + \left(\frac{(y - \mu_y)^2}{\sigma_y^2} + \frac{(z_1 - \mu_{z_1})^2}{\sigma_{z_1}^2} + \frac{\mu_{w_2}^2}{\sigma_{w_2}^2}\right)\right)\right)\mathrm{d}w_2$$

$$\propto \int exp\left(-\frac{1}{2}\left(aw^2 - 2bw + c\right)\right)\mathrm{d}w \tag{49}$$

...

$$\propto exp\left(-\frac{1}{2}\left(c - \frac{b^2}{a}\right)\right) \tag{50}$$

$$\propto exp\left(-\frac{1}{2}\left(\frac{(y - \mu_y)^2}{\sigma_y^2} + \frac{(z_1 - \mu_{z_1})^2}{\sigma_{z_1}^2} + \frac{\mu_{w_2}^2}{\sigma_{w_2}^2} - \frac{\left(\frac{\alpha_9(y-\mu_y)}{\sigma_y^2} + \frac{\alpha_3(z_1-\mu_{z_1})}{\sigma_{z_1}^2} + \frac{\mu_{w_2}}{\sigma_{w_2}^2}\right)^2}{\frac{\alpha_9^2}{\sigma_y^2} + \frac{\alpha_3^2}{\sigma_{z_1}^2} + \frac{1}{\sigma_{w_2}^2}}\right)\right) \tag{51}$$

$$\propto exp\left(-\frac{1}{2}\left(\frac{1}{\sigma_y^2}y^2 - 2\frac{\mu_y}{\sigma_y^2}y - \frac{\sigma_y^2\sigma_{z_1}^2\sigma_{w_2}^2}{\alpha_9^2\sigma_{z_1}^2\sigma_{w_2}^2 + \alpha_3^2\sigma_y^2\sigma_{w_2}^2 + \sigma_y^2\sigma_{z_1}^2}\left(\frac{\alpha_9}{\sigma_y^2}y - \left(\frac{\alpha_9\mu_y}{\sigma_y^2} - \frac{\alpha_3 z_1 - \alpha_3\mu_{z_1}}{\sigma_{z_1}^2} - \frac{\mu_{w_2}}{\sigma_{w_2}^2}\right)\right)^2\right)\right) \tag{52}$$

$$\propto exp\left(-\frac{1}{2}\left(\frac{1}{\sigma_y^2}y^2 - 2\frac{\mu_y}{\sigma_y^2}y - \frac{\sigma_y^2\sigma_{z_1}^2\sigma_{w_2}^2}{\alpha_9^2\sigma_{z_1}^2\sigma_{w_2}^2 + \alpha_3^2\sigma_y^2\sigma_{w_2}^2 + \sigma_y^2\sigma_{z_1}^2}\left(\frac{\alpha_9^2}{(\sigma_y^2)^2}y^2 - 2\frac{\alpha_9}{\sigma_y^2}\left(\frac{\alpha_9\mu_y}{\sigma_y^2} - \frac{\alpha_3 z_1 - \alpha_3\mu_{z_1}}{\sigma_{z_1}^2} - \frac{\mu_{w_2}}{\sigma_{w_2}^2}\right)y\right)\right)\right) \tag{53}$$

$$\propto exp\Big(-\frac{1}{2}\Big(\left(\frac{1}{\sigma_y^2} - \frac{\alpha_9^2}{(\sigma_y^2)^2}\frac{\sigma_y^2\sigma_{z_1}^2\sigma_{w_2}^2}{\alpha_9^2\sigma_{z_1}^2\sigma_{w_2}^2 + \alpha_3^2\sigma_y^2\sigma_{w_2}^2 + \sigma_y^2\sigma_{z_1}^2}\right)y^2 +$$

$$-2\left(\frac{\mu_y}{\sigma_y^2} - \frac{\alpha_9}{\sigma_y^2}\left(\frac{\alpha_9\mu_y}{\sigma_y^2} - \frac{\alpha_3 z_1 - \alpha_3\mu_{z_1}}{\sigma_{z_1}^2} - \frac{\mu_{w_2}}{\sigma_{w_2}^2}\right)\frac{\sigma_y^2\sigma_{z_1}^2\sigma_{w_2}^2}{\alpha_9^2\sigma_{z_1}^2\sigma_{w_2}^2 + \alpha_3^2\sigma_y^2\sigma_{w_2}^2 + \sigma_y^2\sigma_{z_1}^2}\right)y\Big)\Big) \tag{54}$$

$$= N\left(y\middle|\frac{(\mu_y + \alpha_9\mu_{w_2})\sigma_{z_1}^2 - \alpha_3\alpha_9\sigma_{w_2}^2\mu_{z_1} + \alpha_3^2\sigma_{w_2}^2\mu_y + z_1\alpha_3\alpha_9\sigma_{w_2}^2}{(\sigma_{z_1}^2 + \alpha_3^2\sigma_{w_2}^2}, \frac{(\sigma_y^2 + \alpha_9^2\sigma_{w_2}^2)\sigma_{z_1}^2 + \alpha_3^2\sigma_{w_2}^2\sigma_y^2}{\sigma_{z_1}^2 + \alpha_3^2\sigma_{w_2}^2}\right) \tag{55}$$

where $\qquad (\mu_y + \alpha_6 x + \alpha_7 z_1 + \alpha_8 z_2 \Leftarrow \mu_y, \mu_{z_1} + \alpha_2 x \Leftarrow \mu_{z_1})$ (56)

$$= N\Big(y\Big|\frac{(\mu_y + \alpha_6 x + \alpha_7 z_1 + \alpha_8 z_2 + \alpha_9\mu_{w_2})\sigma_{z_1}^2 - \alpha_3\alpha_9\sigma_{w_2}^2(\mu_{z_1} + \alpha_2 x) + \alpha_3^2\sigma_{w_2}^2(\mu_y + \alpha_6 x + \alpha_7 z_1 + \alpha_8 z_2) + z_1\alpha_3\alpha_9\sigma_{w_2}^2}{\sigma_{z_1}^2 + \alpha_3^2\sigma_{w_2}^2},$$

$$\frac{(\sigma_y^2 + \alpha_9^2\sigma_{w_2}^2)\sigma_{z_1}^2 + \alpha_3^2\sigma_{w_2}^2\sigma_y^2}{\sigma_{z_1}^2 + \alpha_3^2\sigma_{w_2}^2}\Big) \tag{57}$$

$$= N\Big(y\Big|\frac{(\mu_y + \alpha_9\mu_{w_2})\sigma_{z_1}^2 - \alpha_3\alpha_9\sigma_{w_2}^2\mu_{z_1} + \alpha_3^2\sigma_{w_2}^2\mu_y}{\sigma_{z_1}^2 + \alpha_3^2\sigma_{w_2}^2} + \frac{\alpha_6\sigma_{z_1}^2 - \alpha_3\alpha_9\sigma_{w_2}^2\alpha_2 + \alpha_3^2\sigma_{w_2}^2\alpha_6}{\sigma_{z_1}^2 + \alpha_3^2\sigma_{w_2}^2}x + \frac{\alpha_7\sigma_{z_1}^2 + \alpha_3^2\sigma_{w_2}^2\alpha_7 + \alpha_3\alpha_9\sigma_{w_2}^2}{\sigma_{z_1}^2 + \alpha_3^2\sigma_{w_2}^2}z_1 +$$

$$\frac{\alpha_8\sigma_{z_1}^2 + \alpha_3^2\sigma_{w_2}^2\alpha_8}{\sigma_{z_1}^2 + \alpha_3^2\sigma_{w_2}^2}z_2, \qquad \frac{(\sigma_y^2 + \alpha_9^2\sigma_{w_2}^2)\sigma_{z_1}^2 + \alpha_3^2\sigma_{w_2}^2\sigma_y^2}{\sigma_{z_1}^2 + \alpha_3^2\sigma_{w_2}^2}\Big) \tag{58}$$

3 Simulation

Given a regression model estimated using observational data for $N(z_1|\mu_{z_1} + \alpha_2 x, \sigma^2_{z_1})$, $N(z_2|\mu_{z_2} + \alpha_4 z_1, \sigma^2_{z_2})$, $N(y|\mu_y + \alpha_6 x + \alpha_7 z_1 + \alpha_8 z_2, \sigma^2_y)$, and $N(x|\bar{x}, s^2_x)$.

$$p(y|do\{x\}) = \iint p(y|x, z_1, z_2) \int p(z_2|x, z_1) p(x) \, \mathrm{d}x \, p(z_1|x) \, \mathrm{d}z_1 \, \mathrm{d}z_2 \tag{59}$$

Create an empty data table with n rows;
Sample z_1^i for i in $1...n$ from $p(z_1|x)$ where $x = do\{x\}$;
Sample x^i for i in $1...n$ from the observed population $p(x)$;
Sample z_2^i for i in $1...n$ from $p(z_2|x^i, z_1^i)$;
Sample y^i for i in $1...n$ from $p(y|x^i, z_1^i, z_2^i)$;

Algorithm 1: Generating Samples y^i From $p(y|do\{x\})$