In their 2013 paper called *price competition and quality differentiation with multiproduct firms*, Yi-Ling Cheng, and Shin-Kun Peng inform the literature on methods of modelling competition when firms may provide a line-up of quality differentiated products. The two authors deliver an examination of the sub-game Nash equilibrium (SPNE) result from models of Cournot quantity competition, and a derivation of possible product quality line-up decisions made by each firm. The scope of the research is development of analysis that takes into consideration vertical product quality-differentiation, as often the literature only considers a single quality of products for analytical convenience.

Product differentiation depends on consumers' attention to one or more key benefits of a product or brand that make it a better choice compared to similar products or brands. In reality, the elements of product quality differentiation are derived from how well a product satisfies customer needs, serves its purpose and meets industry standards. Vertical quality differentiation is when firms offer one or several products of differing levels of quality for the same market. In essence this is when each firm participating in the market for a single item offers a unique lineup of homogenous products with heterogenous levels of customer satisfaction derived from quality. In this regard, each product can be objectively ranked from highest to lowest quality, and a consumer would be able to say one product is better than the other. In particular, the authors are concerned with discovering when and why vertical product differentiation occurs in a market.

The authors approach the problem by a method of defining the equilibrium patterns that arise in a quality-then-price framework of competition. To do this, the authors generalize a standard model of endogenous vertical differentiation in the footsteps of Tirole (1988) and Motta (1993). The authors maintain initial assumptions about the market structure determined in the setting of market size as well as the fixed costs of quality improvement and production line expansion.

The precise framework that the authors use begins with two firms $R = A, B$ with identical cost of quality improvement. Firm $R$ produces $n_R$ quality differentiated products $R_s = R_1, R_2, \ldots, R_{n_R}$ with qualities $q_{R_s} = q_{R_1}, q_{R_2}, \ldots, q_{R_{n_R}}$ in the vertical quality array. $q_{R_s} > q_{R_s} + 1 \quad for \quad s = 1,2, n_{R\text{-}1}$ meaning the level of quality is increasing. Different qualities are sold at different prices $p_{R_s} = p_{R_1}, p_{R_2}, \ldots, p_{R_{n_R}}$.

Each consumer purchases 1 unit from firm A or firm B and consumers are considered to be normally distributed over the taste interval $[\underline{\theta}, \bar{\theta}]$ with density normalized to 1, and $\underline{\theta} = 0$ indicating some consumers do not buy anything. The consumer, indexed by their taste parameter, $\theta_i \epsilon [0, \bar{\theta}]$ maximizes their utility subject to

$$U_\theta(q_{R_s}) = \begin{cases} \theta q_{R_s} - p_{R_s} \\ 0 \end{cases}$$

where $\theta q_{R_s} - p_{R_s}$ is the utility that the consumer receives if they buy a product with quality $q_{R_s}$ for the price $p_{R_s}$from firm $R$, or 0 if they choose not to buy. The utility level of a consumer is therefore derived by their taste. A higher taste parameter $\theta$ indicates a higher willingness to pay for quality. A person with higher taste parameter needs a better quality product to attain the same level of utility as someone with a lower taste parameter.

If there are 3 products in the market, they can be ordered by quality, for example it can be said that if the market has a product line-up such that $q_{A_1} > q_{A_2} > q_{B_1}$ , this would indicate firm A offers the 2 highest quality products, and firm B offers the lowest quality product. The price associated with $q_{A_1}$ is $p_{A_1}$ and price levels would follow the same rank order as the qualities. Each unique quality and price could be simplified as $\widetilde{q_1}$ and $\widetilde{p_1}$.

For the marginal consumer with taste parameter $\theta_i$ where $\theta_i\widetilde{q_1}\text{-}\widetilde{p_1} = \theta_i\widetilde{q_{1+1}}\text{-}\widetilde{p_{1+1}}$, they are indifferent with product quality $\widetilde{q_1}$ and $\widetilde{q_{1+1}}$. Similarly, the consumer with $\theta_{\mathrm{N}}\widetilde{q_{\mathrm{N}}}\text{-}\widetilde{p_{\mathrm{N}}} = 0$ is indifferent between buying the product of lowest quality and not buying anything. It follows that any $\theta_i > \theta_{\mathrm{N}}$ prefers $\widetilde{q_{\mathrm{N}}}$ the lowest quality over nothing.

If $\widetilde{p} = (\widetilde{p_1}, \widetilde{p_2}, \dots, \widetilde{p_{\mathrm{N}}})$ and $\widetilde{\mathrm{q}} = (\widetilde{\mathrm{q}_1}, \widetilde{\mathrm{q}_2}, \dots, \widetilde{\mathrm{q}_{\mathrm{N}}})$ the demand as a function of price and quality can be derived for each quality $\widetilde{q_1}$ as

$$\widetilde{x_1}(\widetilde{p}, \widetilde{\mathrm{q}}) \begin{cases} \bar{\theta}\text{-}\theta_i = \bar{\theta}\text{-}\dfrac{\widetilde{p_1}\text{-}\widetilde{p_2}}{\widetilde{\mathrm{q}_1}\text{-}\widetilde{\mathrm{q}_2}} \text{ if i} = 1 \\ \theta_{i\text{-}1}\text{-}\theta_i = \dfrac{\widetilde{p_{1\text{-}1}}\text{-}\widetilde{p_1}}{\widetilde{\mathrm{q}_{1\text{-}1}}\text{-}\widetilde{\mathrm{q}_1}}\text{-}\dfrac{\widetilde{p_1}\text{-}\widetilde{p_{1+1}}}{\widetilde{\mathrm{q}_1}\text{-}\widetilde{\mathrm{q}_{1+1}}} \text{ if i} = 2, \dots, \text{N-1} \\ \theta_{\text{N-1}}\text{-}\theta_{\mathrm{N}} = \dfrac{\widetilde{p_{\text{N-1}}}\text{-}\widetilde{p_{\mathrm{N}}}}{\widetilde{\mathrm{q}_{\text{N-1}}}\text{-}\widetilde{\mathrm{q}_{\mathrm{N}}}}\text{-}\dfrac{\widetilde{p_{\mathrm{N}}}}{\widetilde{\mathrm{q}_{\mathrm{N}}}} \text{ if i} = \mathrm{N} \end{cases}$$

Since it is that a higher $\theta$ indicates a higher willingness to pay for better quality, the quantity demanded $\widetilde{x_1}$ for each quality i , is a function of the price quality ratio subtracted by the next highest quality option. Therefore, demand if calculated such that the market is covered by finding each of the marginal consumers that are no longer willing to pay the premium for a higher quality product but will derive some utility by paying for the quality below.

The unit costs incurred by the firm for improving quality is $\mathrm{c}\left(q_{R_s}\right)$ and the fixed cost for facility expansion is $f > 0$. Profit for firm R is therefore given as

$$\pi\_R = \sum_{\mathrm{N}=1}^{\mathrm{N}} [\tilde{\mathrm{p}}_{\mathrm{i}} - \mathrm{c}\,(\tilde{\mathrm{q}}_{\mathrm{i}})]\tilde{\mathrm{x}}_{\mathrm{i}} - \mathrm{N}f$$

The authors look at the situation for 2 firms that simultaneously choose the number of products and qualities (product line-up). They consider a maximum of two unique qualities for each firm for a total market line-up of 4 possible quality choices facing the consumer. However, a firm may offer 0, 1 or 2 products with unique qualities allowing for monopolies to form. The firm will only enter the market if their profit is greater than 0 with the selected product line-up.

To determine $\pi$ take an example market line up where firm A offers $(\mathrm{q}_{A_1}, \mathrm{q}_{A_2})$ and firm B offers $(\mathrm{q}_{\mathrm{B}_1})$ such that $q_{A_1} > q_{A_2} > q_{B_1}$ and denote the profit incurred by firm R as $\pi_{\mathrm{R}}(\mathrm{aab})$ for this product line-up. The profit function of firm A and B would then be

$$\pi_{\mathrm{A}}(aab) = [\mathrm{p}_{\mathrm{A}_1} - \mathrm{c}\left(q_{\mathrm{A}_1}\right)]\mathrm{x}_1 + [\mathrm{p}_{\mathrm{A}_2} - \mathrm{c}\left(q_{\mathrm{A}_2}\right)]\mathrm{x}_2 - 2f$$

$$\pi_{\mathrm{B}}(aab) = [\mathrm{p}_{\mathrm{B}_1} - \mathrm{c}\left(q_{\mathrm{B}_1}\right)]\mathrm{x}_1 - f$$

Since the first-stage of their competition they choose the number of products and qualities in their line-up and in the second-stage that simultaneously decide on price, these value are taken as given. Therefore the endogenous value of demand quantity can be derived from the firms profit function according to their quality order $q_{\mathrm{A}_1} > q_{\mathrm{A}_2} > q_{\mathrm{B}_1}$.

$$x_{A_1}(\mathrm{p}, q) = \left(\bar{\theta} - \frac{p_{A_1} - p_{A_2}}{\mathrm{q}_{A_1} - \mathrm{q}_{A_2}}\right)$$

$$x_{A_2}(\mathrm{p}, q) = \left(\frac{p_{A_1} - p_{A_2}}{\mathrm{q}_{A_1} - \mathrm{q}_{A_2}} - \frac{p_{A_2} - p_{\mathrm{B}_1}}{\mathrm{q}_{A_2} - \mathrm{q}_{\mathrm{B}_1}}\right)$$

$$x_{\mathrm{B}_1}(\mathrm{p}, q) = \left(\frac{p_{A_2} - p_{\mathrm{B}_1}}{\mathrm{q}_{A_2} - \mathrm{q}_{\mathrm{B}_1}} - \frac{p_{\mathrm{B}_1}}{\mathrm{q}_{\mathrm{B}_1}}\right)$$

And finally, by inputting the demand functions into the profit function, the profits for firm A and B can be written as:

$$\pi_{\mathrm{A}}(aab) = [\mathrm{p}_{\mathrm{A}_1} - \mathrm{c}\left(q_{\mathrm{A}_1}\right)]\left(\bar{\theta} - \frac{p_{A_1} - p_{A_2}}{\mathrm{q}_{A_1} - \mathrm{q}_{A_2}}\right) + [\mathrm{p}_{\mathrm{A}_2} - \mathrm{c}\left(q_{\mathrm{A}_2}\right)]\left(\frac{p_{A_1} - p_{A_2}}{\mathrm{q}_{A_1} - \mathrm{q}_{A_2}} - \frac{p_{A_2} - p_{\mathrm{B}_1}}{\mathrm{q}_{A_2} - \mathrm{q}_{\mathrm{B}_1}}\right) - 2f$$

$$\pi_{\mathrm{B}}(aab) = [\mathrm{p}_{\mathrm{B}_1} - \mathrm{c}\left(q_{\mathrm{B}_1}\right)]\left(\frac{p_{A_2} - p_{\mathrm{B}_1}}{\mathrm{q}_{A_2} - \mathrm{q}_{\mathrm{B}_1}} - \frac{p_{\mathrm{B}_1}}{\mathrm{q}_{\mathrm{B}_1}}\right) - f$$

To find the equilibrium product qualities that are offered in market equilibrium, a quadratic unit cost is first given to $\mathrm{c}\left(q_{R_s}\right)$, simply as $\alpha {q^2}_{R_s}$ where $\alpha > 0$. To find the market equilibrium parameters of interest for each of the 18 possible product line-ups i.e. *(a)* being a monopoly*, (abab), (bba),* etc., the authors undergo the complex process of taking the first order conditions of the profit functions with respect to quality and price, and then plugged back into the profit function and demand functions to find the sub game perfect Nash equilibrium, which is term for the equilibrium in dynamic games with more than 1 stage.

For example, if one is interested in the market for product line-up *(ab)*, the following first order conditions would need to be taken and set to zero,

$$\frac{\partial \pi_{\mathrm{A}}(ab)}{\partial q_{\mathrm{A}_1}} = 0, \qquad \frac{\partial \pi_{\mathrm{A}}(ab)}{\partial q_{\mathrm{B}_1}} = 0$$

$$\frac{\partial \pi_{\mathrm{A}}(ab)}{\partial \mathrm{p}_{\mathrm{A}_1}} = 0, \qquad \frac{\partial \pi_{\mathrm{A}}(ab)}{\partial \mathrm{p}_{\mathrm{B}_1}} = 0$$

The authors provide a table with the resulting equilibrium values that could be found from the product line-ups that had an equilibrium, in the case of *(ab)*, it was found that $q_{\mathrm{A}_1} = 0.410\frac{\bar{\theta}}{\alpha}$, $q_{\mathrm{B}_1} = 0.199\frac{\bar{\theta}}{\alpha}$, and for prices, $\mathrm{p}_{\mathrm{A}_1} = 0.227\frac{\overline{\theta^2}}{\alpha}$, $\mathrm{p}_{\mathrm{A}_1} = 0.075\frac{\overline{\theta^2}}{\alpha}$. Like the case of *(ab)*, all other product line-ups that resulted in an equilibrium, exhibit the fact that quality and prices are proportional to $\frac{\bar{\theta}}{\alpha}$. This implies the notion that both the equilibrium quality and price decrease with cost of improvement and increase with the taste parameter $\bar{\theta}$, which is a constant proportion of willingness to pay for quality. By testing for ranges of the fixed cost parameter of facility expansion, $f$, the authors found the feasible equilibrium product line-ups that firms will choose if they enter the market at all. The authors finalize their paper by making some conclusions in the form of 2 propositions.

Proposition 1 states that a natural monopolist never sells multiple products. They derive this proof from the fact that the product line-up *(aa)* or *(bb)* cannot be an equilibrium. This implies that if it is sufficiently profitable for a firm to offer two products, it must be profitable for an other firm to enter the market. Also, since in a case where firm A faces two-sided price competition in both the high quality and low markets, as in *(aba)* there is an incentive for firm A to corner the low quality market and deviate to the line-up *(aab).* In fact, the authors show that sandwich *(aba)*, enclosure *(abba)*, and interlacing *(abab)* configurations of product line-ups never emerge in equilibrium, and high-quality firms never offer more products than low-quality firms for this reason.

Proposition 2 states that in the segmentation equilibrium, the quality differentiation of two neighbouring products between firms is greater than that within a firm, and the quality differentiation in the low-quality market is greater than the that in the high-quality market. This proposition is logical because neighbouring products offered by competing firms are bound to face more competition than those offered by a single firm.