4 Mixed strategies

4.7 Example: Voting game

Downs paradox.

Voting has costs. Low probability that one vote is the deciding vote.

Expectation: People don't vote.

Reality: People do vote.

Model for voter participation.

There are 2 candidates A, B. The number of supporters of A, B are a, b, respectively. WLOG $a \geq b$. If anyone votes, they will incur a cost of c, where $0 < c < 1$. Regardless of participation, each person gets a payoff of 2 if their supporting candidate wins; 1 for a tie, 0 for a loss.

Pure NE when $a = b = 1$.

A = abstain, V = vote.

P2: B

P1: A	A	V
A	1, 1	0, 2−c
V	2−c, 0	1−c, 1−c

NE: both votes. Prisoner's dilemma.

Note: $a = b$, there is a pure NE.

$a \neq b$, there is no pure NE.

One possible scenario for a mixed NE. Suppose $a > b$. Among all A supporters, b of them will vote and $a - b$ of them will abstain. Every B supporter will vote with probability p. $b \geq 2$.

Check: $p = 0, 1$ are not NE. Assume $p \in (0, 1)$.

- Best response for a B supporter.

If a B supporter abstains, B cannot win, utility is 0.

If a B supporter votes, then either B ties $(1-c)$ or B loses $(-c)$

$$\text{Expected utility: } \underbrace{(1-c)}_{\text{utility for a tie}} \cdot \underbrace{p^{b-1}}_{\text{prob everyone else votes}} + \underbrace{(-c)}_{\text{loss}} \underbrace{(1-p^{b-1})}_{\text{Someone abstains}} = p^{b-1} - c$$

Since both pure strats have positive prob, both have maximum utility (by support characterization).

$$\Rightarrow 0 = p^{b-1} - c \Rightarrow p = c^{\frac{1}{b-1}}.$$

- Based on B's best response, should an A supporter change their strategy?

  Currently all are playing pure strats. In order to switch to a diff prob, the utility for switching to the other pure strat > current utility.

  - An A supporter who abstained.

    $$\text{Expected utility: } \underbrace{1}_{\text{tie}}\cdot\underbrace{p^b}_{\text{b voted}} + \underbrace{2}_{\text{win}}\cdot\underbrace{(1-p^b)}_{\substack{\text{not all B}\\ \text{voted}}} = 2-p^b.$$

    Expected utility when they switch to voting: $2-c$ (A wins)

    $$2-c = 2-p^{b-1} < 2-p^b.$$

    Switching does not increase utility. ✓

  - An A supporter who voted.

    $$\text{Expected utility for voting: } (1-c)\,p^b + (2-c)(1-p^b) = 2-p^b-c.$$

    Expected utility when they switch to abstaining:

    $$\underbrace{0\cdot p^b}_{\text{loss}} + 1\cdot\underbrace{p^{b-1}}_{\substack{b-1\\ \text{voted}}}\cdot\underbrace{b}_{\substack{\text{ways}\\ \text{this}\\ \text{happen}}}\cdot\underbrace{(1-p)}_{\text{1 abstain}} + 2\cdot\underbrace{\left(1-p^b-p^{b-1}b(1-p)\right)}_{\text{remaining case.}}$$

    $$= 2-2\cdot p^b - b\,p^{b-1}(1-p)$$

    Exercise: $2-2p^b-bp^{b-1}(1-p) \le 2-p^b-c.$

    Switching does not increase utility. ✓

    $\Rightarrow$ NE when $p = c^{\frac{1}{b-1}}$.

Note: c increases $\Rightarrow$ p increases. weird.

4.8 Two-player zero-sum games

NEs for two-player zero-sum games are easy to compute.

Definition: zero-sum games.

A strategic game is zero-sum if for all strategy profiles $x \in \Delta$,

$\sum_{i \in N} u_i(x) = 0$. (e.g. rock paper scissors)

Payoff matrix for 2-player zero-sum games.

Let $S_1 = \{1, \ldots, m\}$, $S_2 = \{1, \ldots, n\}$. Define payoff matrix $A \in \mathbb{R}^{m \times n}$ for P1. Then P2 get $-A$.

Example.

Payoff for P1

		P2: 1	2	3
P1	1	3	5	−2
	2	−5	7	1

Payoff for P2

		P2: 1	2	3
P1	1	−3	−5	2
	2	5	−7	−1

Min-maxing argument.

To find NE, given a strategy we play, opposing playing will maximize their utility, which minimizes ours. Knowing how they play, how can we maximize our own utility?

Player 1's perspective. Suppose P1 plays x'. Expect P2 to play their best response.

Example: $x' = (x_1', x_2')$. P2: $u_2(1, x') = -3x_1' + 5x_2'$

$u_2(2, x') = -5x_1' - 7x_2'$

$u_2(3, x') = 2x_1' - x_2'$

P2 wants: $\max\{-3x_1' + 5x_2', -5x_1' - 7x_2', 2x_1' - x_2'\}$

P1 want: $\min\{\max\{\ldots \quad \ldots \quad \ldots\}\}$

$\Rightarrow \max\{\min\{3x_1' - 5x_2', 5x_1' + 7x_2', -2x_1' + x_2'\}$

Turn into LP.