CO 456 Fall 2024 Lecture 15 (Monday November 4)

8 Designing voting mechanisms

8.4 Arrow's impossibility theorem

Theorem 17. (Arrow's impossibility theorem) Every social welfare function over a set of at least 3 candidates that satisfies unanimity and IIA is a dictatorship.

Sketch proof so far. Suppose $F : L^n \to L$ satisfies unanimity and IIA. We have found a potential dictator i, F matches the ranking of b with voter i.

Remains to prove. For $a, c \neq b$, F agrees with voter i on the relative ranking of a and c.
For $d \neq b$, F agrees with voter i on the relative ranking of b and d.

Sketch proof continues.

Part 3: Let a, c be two candidates other than b. (They exist as there are ≥ 3 candidates)
WLOG voter i ranks $a > c$ in original P. Create a new voter preference P^* from P_i by moving a to the top of the list for voter i.

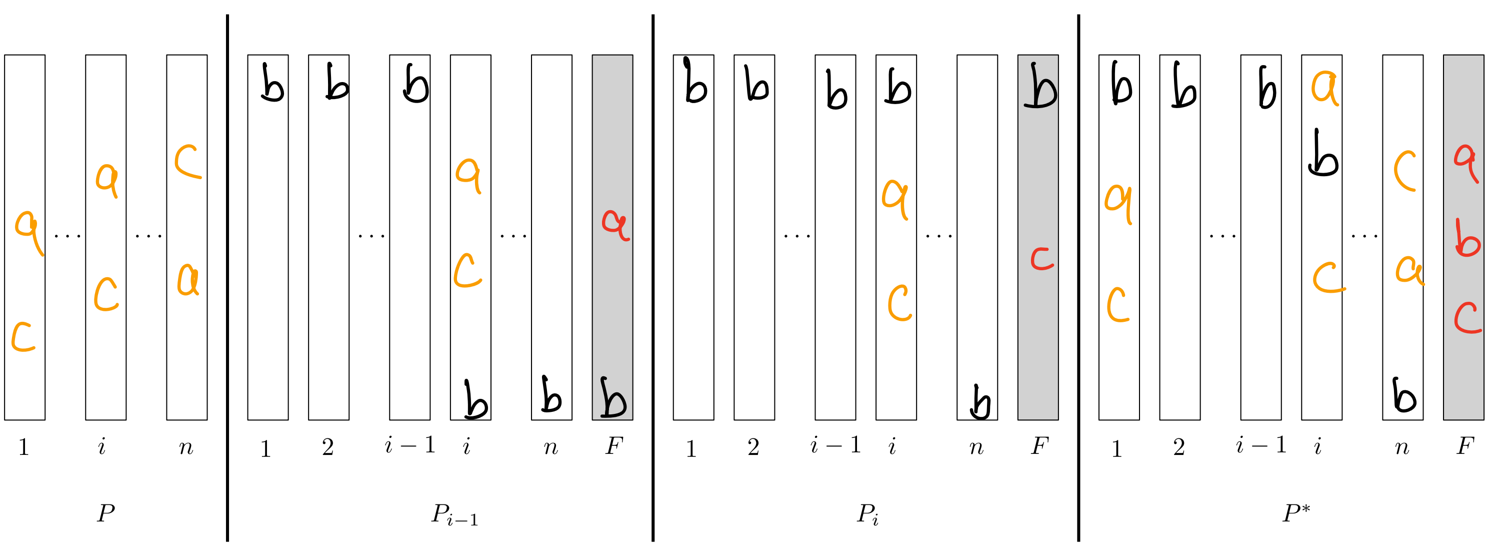
Note: relative rankings of a, c are the same for P, P^*, so same for F.

In P_{i-1} and P^*, relative rankings of a, b are the same. F ranks b at the bottom in P_{i-1}, so we must have $a > b$ by F in P_{i-1} and P^* (IIA).

In P_i and P^*, relative rankings of b, c are the same. F ranks $b > c$ in P_i, so by IIA, F ranks $b > c$ in P^*.

F ranks $a > b > c$ in P^*, by transitivity, $a > c$ in P^*. Also in P.

So F agrees with voter i for a, c.



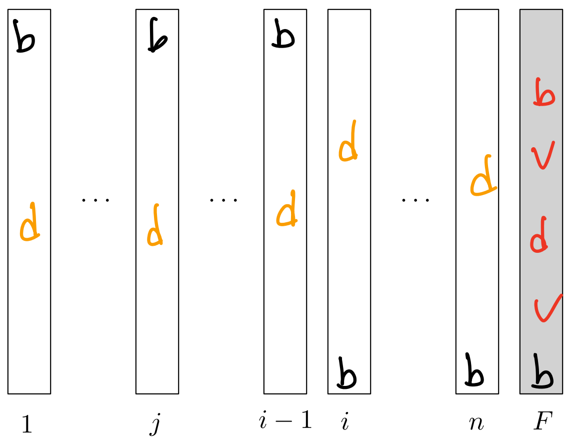
Part 4: Let $d \neq b$ be any candidate. Show F agrees with voter i on b, d.

There exists candidate $e \neq b, d$ (≥ 3 candidates).

Run the arguments from parts 2 and 3 on candidate e, and this gives us a voter j such that F agrees with voter j for any candidates that are not e.

We show that $j = i$.

Suppose otherwise. Say $j < i$. Consider P_{i-1}.



P_{i-1}

Voter j ranks $d < b$ in P_{i-1}.

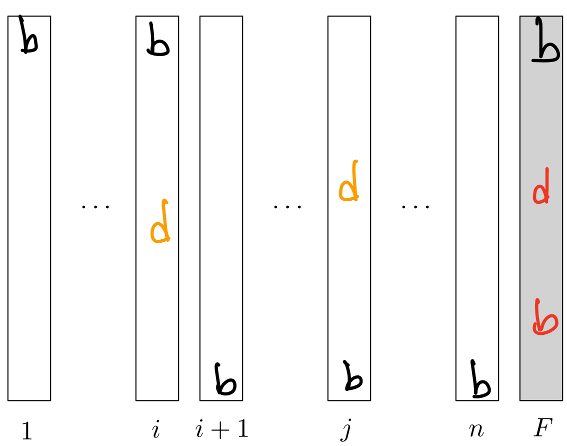
So F agrees with voter j on b, d (not e).

So F ranks $b > d$ in agreeing with voter j.

But F ranks $d > b$ in agreeing with voter i.

Contradiction.

Suppose $j > i$. Consider P_i.



P_i

F agrees with voter i on b, d, so $b > d$.

F agrees with voter j on b, d, so $d > b$.

Contradiction.

$\Rightarrow i = j$.

$\Rightarrow$ F agrees with voter i in all cases. So voter i is a dictator. $\square$

8.5 Gibbarth-Satterthwaite theorem

Do things get better with social choice functions $f : L^n \to A$? No.

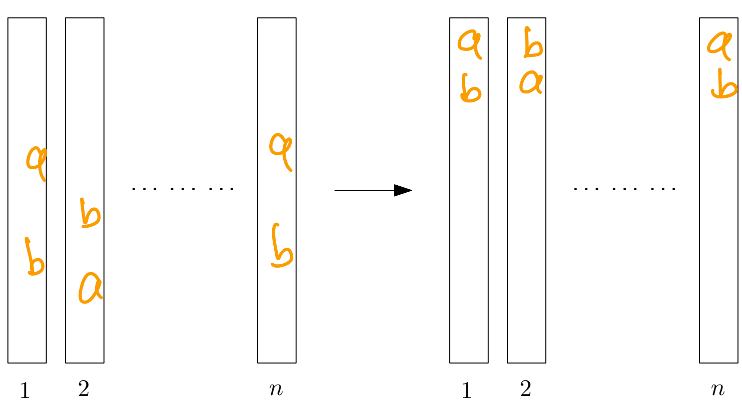
Good property. Strategy-proof: a voter cannot change the society's choice to someone they like better by changing their preferences.

Formally, we say f can be <u>strategically manipulated</u> by voter i if there exist $a, b \in A$, $(<_1, \dots, <_n) \in L^n$ and $<_i' \in L$ such that $a <_i b$ (voter i prefers b), $f(<_1, \dots, <_n) = a$ (society picks a), and $f(<_1, \dots, <_i', \dots, <_n) = b$ (society picks b when voter i changes their preference). We say f is <u>strategy-proof</u> if it cannot be strategically manipulated by any voter.

Bad property. Dictatorship. There exists a voter i such that f will always choose the candidate at the top of voter i's list.

Theorem 18. (Gibbarth-Satterthwaite theorem)
Any strategy-proof social choice function onto a set of at least 3 candidates is a dictatorship.

Very sketch proof.
Assume $f: L^n \to A$ is strategy-proof. Construct a social welfare function $F : L^n \to L$ as follows: Take any $a, b \in A$. Raise them to the top of the rankings. Then f will pick a or b as society's choice (exercise). If f picks a, then F ranks $a > b$. Otherwise, F ranks $b > a$.



Need to show F is antisymmetric (easy) and transitive (hard).
Then show F satisfies unanimity and IIA (not hard).
By Arrow, F is a dictatorship. Since f picks the top choice, f is also a dictatorship.