2 Nash equilibrium (NE)

2.1 Examples

Prisoner's dilemma.

		P2 Share	P2 Steal
P1	Share	$5, 5$	$0, 10$
	Steal	$10, 0$	$0.1, 0.1$

Let $s^* = $ (steal, steal)

P1: $u_1(s^*) = 0.1$

$u_1(\text{share}, s^*_{-1}) = 0 < u_1(s^*)$.

P1 switch to share   P2 remains steal

P2: $u_2(s^*) = 0.1$ $u_2(\text{share}, s^*_{-2}) = 0 < u_2(s^*)$

So s^* is a Nash equilibrium.

Bach or Stravinsky.

		P2 Bach	P2 Stravinsky
P1	Bach	$2, 1$	$0, 0$
	Stravinsky	$0, 0$	$1, 2$

Both (B, B) and (S, S) are NEs.

Rock paper scissors.

		P2 R	P2 P	P2 S
P1	R	$0, 0$	$-1, 1$	$1, -1$
	P	$1, -1$	$0, 0$	$-1, 1$
	S	$-1, 1$	$1, -1$	$0, 0$

For any strategy profile, one player with utility 0 or -1 can get higher utility by switching to a winning gesture.

No NE ... (sort of a lie, to be resolved later).

2/3 average game. 3 players. Each player simultaneously pick from $\{1, \ldots, 10\}$. A \$1 prize is split among all players closest to 2/3 of the average of the 3 numbers, other players get \$0.

$S = (6, 4, 1)$ average $\frac{11}{3}$. $\frac{2}{3}$ average $\frac{2}{3} \cdot \frac{11}{3} = 2.4\ldots$

$u_3(s) = 1$, $u_1(s) = u_2(s) = 0$.

P1: $u_1(1, s_{-1}) = 0.5 > u_1(s)$.   Not a NE.

$(1, 1, 1)$ is the only NE (needs proof)

2.2 Best response function

Definition: best response function.

Player i's <u>best response function</u> (BRF) for $s_{-i} \in S_{-i}$ is

$$B_i(s_{-i}) = \{ s_i' \in S_i : \underbrace{u_i(s_i', s_{-i})}_{\text{utility of a best response}} \geq \underbrace{u_i(s_i, s_{-i})}_{\text{utility of all possible responses}} \ \forall s_i \in S_i \}.$$

Examples. Prisoner's dilemma.

$B_1(\text{share}) = \{\text{steal}\}$ $\qquad$ $B_1(\text{steal}) = \{\text{steal}\}$.

P2's strat $\qquad$ $u_1(\text{steal}, \text{share}) > u_1(\text{share}, \text{share})$

2/3 average game.

$S_{-1} = (5,5)$ (P2, P3 play 5). $\qquad$ $u_1(x,5,5) = \begin{cases} 0 & x \geq 6 \\ 1/3 & x = 5 \\ 1 & x \leq 4 \end{cases}$

$B_1(5,5) = \{1,2,3,4\}$.

Lemma 1.

A strategy profile $s^* \in S$ is a Nash equilibrium if and only if for each player i, $s_i^* \in B_i(s_{-i}^*)$.

[No incentive for any player to switch, since each player played their best response already.]

Examples.

		P2	
		Share	Steal
P1	Share	5, 5	0, $\boxed{10}$
	Steal	10, 0	0.1, $\boxed{0.1}$

BR given P2 share

best responses to each other, NE.

○ best response for P1 given P2's choices

□ " " P2 " P1's "

		P2		
		D	E	F
P1	A	1, $\boxed{2}$	2, 1	1, 0
	B	2, $\boxed{1}$	0, $\boxed{1}$	0, 0
	C	0, 1	0, 0	1, $\boxed{2}$

NEs: (B, D), (C, F).

best responses to each other.

2.3 Cournot's oligopoly model

Firms $N = \{1, \dots, n\}$ are producing a single type of goods sold on the common market.

- Each firm i decides the number of units of goods q_i to produce.
- Production cost for firm i is $C_i(q_i)$ where C_i is a given increasing function.
- Given a strategy profile $q = (q_1, \dots, q_n)$, a unit of the goods sell for the price of $P(q)$, where P is a given non-increasing function on $\sum_i q_i$.
- The utility of firm i is $u_i(q) = q_i P(q) - C_i(q_i)$.

Szidarovszky and Yakowitz proved that a NE exists under some continuity and differentiability assumptions on P, C_i.

Special case: linear costs and prices

Assume $C_i(q_i) = c q_i \quad \forall i \in N$, fixed c.

$P(q) = \max\{0, \alpha - \sum_j q_j\}$ fixed α. $\quad 0 < c < \alpha$.

Utility is $u_i(q) = q_i P(q) - C_i(q_i) = \begin{cases} q_i(\alpha - \sum_j q_j - c) & \alpha - \sum_j q_j \geq 0 \\ -c q_i & \alpha - \sum_j q_i < 0. \end{cases}$

Find BRF for firm i: If $\alpha - \sum_{j \neq i} q_j \leq 0$, then BR is 0 (price is 0, no need to produce).

When $\alpha - \sum_j q_j - c > 0$, we make a profit. Take q_i out of the sum.

$\alpha - c - q_i - \sum_{j \neq i} q_j > 0$. The utility is $q_i(\alpha - c - q_i - \sum_{j \neq i} q_j)$.

Given fixed q_j, $j \neq i$, we want q_i that maximizes utility (BR).

This is maximized at $q_i = (\alpha - c - \sum_{j \neq i} q_j)/2$. (Using calculus?)

$$B_i(q_{-i}) = \begin{cases} \{(\alpha - c - \sum_{j \neq i} q_j)/2\} & \alpha - c - \sum_{j \neq i} q_j > 0 \\ \{0\} & \text{otherwise.} \end{cases}$$