7 Finding a DSIC payment rule (Myerson's Lemma)

7.1 Setting up Myerson's Lemma

Myerson's Lemma characterizes auctions that have DSIC payment rule, and gives a formulation for the payment rule.

Definition: single-parameter environment. (Setting up the type of auction)

In a single-parameter environment, we are given

- a set of players N
- a private valuation $v_i \geq 0$ for each $i \in N$ for one unit of goods.
  (only one valuation, "single parameter")
- a set $X \subseteq \mathbb{R}^N$ of vectors $(x_1, \ldots, x_n)$ that describe feasible allocations, i.e. x_i is the amount of goods given to player i.

Examples.

Single-item auction: X is all vectors of $\mathbb{R}^n$ with one 1 and $n-1$ 0's.

Sponsored search auction: k slots with CTR $\alpha_1, \ldots, \alpha_k$.

X is all vectors of $\mathbb{R}^n$ where for each slot j, at most one $i \in N$ has $x_i = \alpha_j$, 0 otherwise.

Definition: direct revelation mechanism, allocation rule, payment rule. (Setting up how to run an auction)

Let $B \subseteq \mathbb{R}^N$ be the set of all possible player bids. Given a single-parameter environment, a direct revelation mechanism ...

- collect bids $b \in B$ from all players.
- choose a feasible allocation $x(b) \in X$ based on the bids
- choose a payment $p(b) \in \mathbb{R}^N$ where player i pays $p_i(b)$.

We call $x : B \to X$ an allocation rule, and $p : B \to \mathbb{R}^N$ a payment rule.

The utility for player i is $u_i(b) = v_i x_i(b) - p_i(b)$.

Assumption.

As part of DSIC, assume $0 \leq p_i(b) \leq v_i x_i(b)$. (Incentive compatible.)

$(b_i = v_i)$

Example.

Second-price auction: Given bids b,

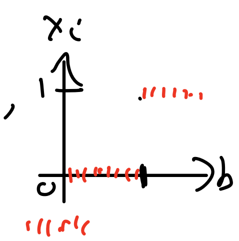
$$x_i(b) = \begin{cases} 1 & b_i \text{ is max in } b \text{ (tie} \Rightarrow \text{lower index)} \\ 0 & \text{else} \end{cases} . \qquad p_i(b) = \begin{cases} \max_{j \neq i} b_j & b_i \text{ is max in } b \text{ (tie} \Rightarrow \text{lower index)} \\ 0 & \text{else} \end{cases} .$$

Two terms related to an allocation rule. $x: B \to X$

- Implementable: An allocation rule x is implementable if there exists a payment rule $p: B \to \mathbb{R}^N$ such that (x, p) is DSIC.
- Monotone: An allocation rule $x: B \to X$ is monotone if for all $i \in N$ and for all $b_{-i} \in B_{-i}$, $x_i(z, b_{-i}) \geq x_i(y, b_{-i})$ whenever $z \geq y$.

  "the higher you bid, the more you get"

Second price auction: Fixing b_{-i}, player i gets 0 when b_i is not max, gets 1 otherwise. This is monotone.



7.2 Myerson's Lemma

Theorem 15. (Myerson's Lemma)

For a single-parameter environment, an allocation rule $x: B \to X$ is implementable if and only if it is monotone. Moreover, given a monotone allocation rule $x: B \to X$, there exists a unique payment rule $p: B \to \mathbb{R}^N$ such that (x, p) is DSIC and $p_i(b) = 0$ whenever $b_i = 0$.

We first prove the forward direction.

Lemma 16. If an allocation rule $x: B \to X$ is implementable, then x is monotone.

Proof: Suppose x is implementable. Then there exists a payment rule $p: B \to \mathbb{R}^N$ such that (x, p) is DSIC. Fix a player $i \in N$ and bids $b_{-i} \in B_{-i}$. We use $x(y)$, $p(y)$ to represent $x_i(y, b_{-i})$, $p_i(y, b_{-i})$.

[Want to prove: If $y \geq z$, then $x(y) \geq x(z)$.]

Key observation: The rules (x, p) are defined independently of the player valuations. They are DSIC for any valuations.

- (x,p) is DSIC if player i's valuation is y. So utility of playing $y \geq$ utility of playing z. (dominant strat)

$$y \cdot x(y) - p(y) \geq y \cdot x(z) - p(z).$$

- (x,p) is DSIC if player i's valuation is z.

$$z \cdot x(z) - p(z) \geq z \cdot x(y) - p(y).$$

Rearrange inequalities to get

$$z(x(y) - x(z)) \leq p(y) - p(z) \leq y(x(y) - x(z))$$ ← "payment sandwich"

Since $y \geq z$, $x(y) - x(z) \geq 0$. So $x(y) \geq x(z)$. $\square$

Special case for the converse. We will prove that the converse is true when the allocation rule x is piecewise constant.

Approach to the proof.

Monotone piecewise constant x.