CO 456 Fall 2024: Assignment 3 problems

Due: Wednesday October 23 at 11:59pm EDT

Preamble.

- You will receive a Crowdmark link for each problem of the assignment. Submit your solutions on Crowdmark. Your submission must be clear and legible, and in the correct orientation. You need to submit your solutions to the correct problems, otherwise it will not be marked.
- Each of the problems is worth 20 marks.
- You need to justify all of your solutions, unless you are explicitly asked not to.
- You are graded on both your accuracy and your presentation (see below). A correct solution that is poorly presented may not receive full marks. The markers will not spend a lot of time figuring out something you write that is not clear.
- When discussing assignment problems on Piazza, use private posts if you think that you might spoil an answer or major thinking point of a problem.
- Read the assignment policies on the course outline for what is allowed and not allowed in working on the assignment.
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Guidelines to writing solutions. Your goal is to present your solutions so that your reader can easily understand what you are writing and be convinced of your arguments. So the quality of your solutions is very important. Here are some guidelines that you should keep in mind when writing your solutions.

- Write in complete sentences. Separate major steps in the solution into paragraphs.
- When writing a proof, make sure that every statement follows from earlier statements. Cite results or assumptions when you use them. Only use results from the lectures.
- Use proper notation. Mathematics is a precise language, and improper use of notation can easily cause confusion.
- Define a variable before you use it. It can be as simple as "Let $p = 3x$ for some $x \in \mathbb{Z}$" if this is the first time you are using x.
- Do not include irrelevant facts. Only state facts that are necessary in the argument.

Assignment problems.

A3-1. **Multiplayer zero-sum games**

In general, NEs of zero-sum games with more than two players are difficult to compute. But there is a special case where this is easy to compute. Imagine playing multiplayer rock paper scissors (RPS) on a graph, where each vertex is a player, and they are playing against all of their neighbours. Each vertex is only allowed to play one strategy, and each edge represents a game of RPS giving payoffs to players on each end. The utility of a vertex is the sum of the payoffs of all the games that they play with their neighbours.

Formally, let $G = (V, E)$ be a graph with at least one edge. The vertices $V = \{1, \ldots, n\}$ are the players. Each player has the same pure strategy set $T = \{R, P, S\}$. For each edge vw, we define the utility of v and w when playing pure strategies s_v and s_w by $u_v^w(s_v, s_w)$ and $u_w^v(s_w, s_v)$, and this utility is summarized in the following table.

	R	P	S
R	$0, 0$	$-1, 1$	$1, -1$
P	$1, -1$	$0, 0$	$-1, 1$
S	$-1, 1$	$1, -1$	$0, 0$

Given a pure strategy profile s, the utility of a player v is then

$$u_v(s) = \sum_{vw \in E} u_v^w(s_v, s_w).$$

Let x be a mixed strategy profile, where x^v denotes the mixed strategy (x_R^v, x_P^v, x_S^v) played by vertex v. Then the expected utility of player v is

$$u_v(x) = \sum_{vw \in E} \sum_{s_v, s_w \in T} u_v^w(s_v, s_w) x_{s_v}^v x_{s_w}^w.$$

We let Δ be the set of all mixed strategy profiles.

(a) Briefly explain why this is a zero-sum game.

(b) Consider the following linear program, which we call (P).

$$\begin{aligned}
\min \quad & \sum_{v \in V} z_v \\
s.t. \quad & z_v \geq u_v(s, y_{-v}) \quad \forall v \in V, s \in T \\
& y_R^v + y_P^v + y_S^v = 1 \quad \forall v \in V \\
& y \geq \mathbb{0}
\end{aligned}$$

In (P), y is a variable representing any strategy profile, i.e. $y \in \Delta$. Explain what the value of z_i represents when (y, z) is an optimal solution. Then prove that the optimal value is equal to

$$\max_{x \in \Delta} \sum_{v \in V} u_v(x^v, y_{-i})$$

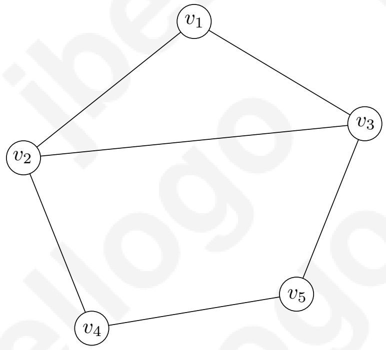
(c) Use part (b) and the fact that this is a zero-sum game to prove that the optimal value of (P) is at least 0.

(d) Nash's theorem implies that there exists a Nash equilibrium y^*. Prove that there exists a vector z such that (y^*, z) is a feasible solution of (P) with objective value 0. What does this say about the optimal value of (P)?

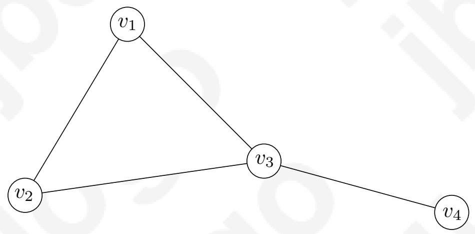
(e) Prove that if (y, z) is an optimal solution to (P), then y is a Nash equilibrium.

(f) [Optional, for up to 2 bonus marks.] Find a Nash equilibrium when G is each of the following graphs by solving (P) as applied to the graph. You may use computational tools. Show all of your work.

G_1 is a cycle of length 5 (can this be generalized?). G_2 and G_3 are shown below.



G_2



G_3

[Note 1: This shows that a Nash equilibrium can be computed by solving a linear program that is polynomial in size with respect to the problem, meaning it can be computed efficiently.]

[Note 2: Part (d) can be proved using duality of (P) instead.]

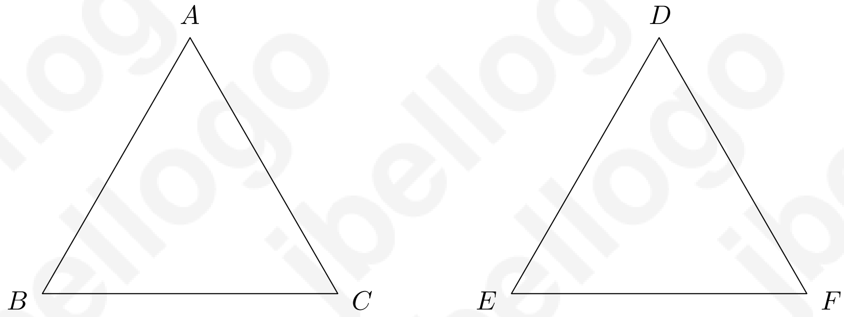
A3-2. **Nash's theorem**

(a) Prove that the set $\Delta = \Delta^1 \times \cdots \times \Delta^n$ is convex and bounded.

(b) Consider the 2-player game from assignment 2 with the following payoff table.

		P2		
		D	E	F
	A	$8, 3$	$7, 1$	$3, 10$
P1	B	$3, 9$	$4, 5$	$8, 1$
	C	$7, 8$	$5, 11$	$9, 5$

Consider the strategy profile $x = ((1,0,0),(1,0,0))$ where $x^1 = (x^1_A, x^1_B, x^1_C)$ and $x^2 = (x^2_D, x^2_E, x^2_F)$. Using the formula for the function f from class (section 5.4), calculate $f(x)$ and $f(f(x))$. Then plot (approximately) $x, f(x), f(f(x))$ as paths on the following diagram for Δ^1 and Δ^2. In addition, plot the actual NE of $((2/5, 0, 3/5), (0, 3/4, 1/4))$.



Δ^1 Δ^2

(c) A strategic game with n players N is *symmetric* if all players have the same strategy set $S = \{1, \ldots, k\}$, and the utility of a strategy profile for any player is dependent only on the number of other players choosing each strategy. To be more precise, for a strategy profile, the utility function for player i is

$$u_i(s) = g(s_i, n_1(s_{-i}), \ldots, n_k(s_{-i}))$$

for some function g, and $n_j(s_{-i})$ is the number of other players who played strategy j. For example, the guessing $\frac{2}{3}$ average game is symmetric.

A mixed Nash equilibrium $x = (x^1, x^2, \ldots, x^n)$ is *symmetric* if $x^1 = x^2 = \cdots = x^n$.

Prove that in every symmetric game, there exists a mixed symmetric Nash equilibrium.

Hint: You may use the following set, which you can assume to be convex and compact.

$$Y = \{(x^1, \ldots, x^n) \in \Delta \ : \ x^1 = \cdots = x^n\}.$$

(If there are parts of the proof that are identical to the proof of Nash's theorem, you just need to state what the parts are without reproving them.)

A3-3. **Ideal auctions**

(a) Consider the one-item auction where the item is given to the player with the second highest bid, and they pay the price of the highest bid. Describe all the ways in which this is not an ideal auction.

(b) For the sponsored search auction, consider the natural extension of the second price auction payment rule: the advertiser who wins the j-th slot will pay the $j+1$-st highest bid multiplied by α_j, the click through rate of the slot they received.

The following example has 3 slots, with $\alpha_1 = 0.03, \alpha_2 = 0.015, \alpha_1 = 0.01$, and 5 advertisers with $v_1 = 10, v_2 = 8, v_3 = 5, v_4 = 2, v_5 = 1$. Determine the allocation that is welfare-maximizing, the payment of each advertiser using the rule stated above if each advertiser bids their valuation, and then show that it is possible that bidding one's valuation is not a dominant strategy.

[Note: Google has allegedly used a payment rule similar to this for their sponsored search auction for a long time initially.]

(c) Consider an auction where there are M units of an item to be given out, and each player i has a valuation of v_i for each unit. In addition, each player has a cap c_i on the maximum number of units they are willing to receive.

i. Write down the set X of all possible feasible allocations.

ii. Determine an allocation rule $x : B \to X$ that is welfare-maximizing. Can this be efficiently implemented?

(d) Prove or disprove: In a single-parameter environment, an allocation rule that is welfare-maximizing is monotone.