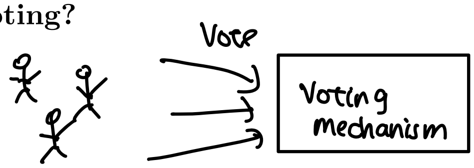
8 Designing voting mechanisms

8.1 Background

What is voting?



result. Represents the will of the people.

This is easy when there are 2 candidates. Decide on the Majority.

More complicated with ≥ 3 candidates.

Voting gone wrong: Condorcet's paradox

3 candidates a, b, c.

3 voters: voter 1 $a >_1 b >_1 c$

Voter 2 $b >_2 c >_2 a$

Voter 3 $c >_3 a >_3 b$

Majority prefers a over b, b over c, c over a. $\Rightarrow a > b > c > a$.

Not possible.

This could lead to strategic voting: a voter might not vote according to their preferences.

Example: a, b front runners, more prefer a than b. A group of voters prefer c but also rather b win than a. So they vote for b instead.

Problem.

Truthful voting: the usual voting mechanism does not incentive voters to vote truthfully, does not represent will of the people.

Spoiler: With ≥ 3 candidates, there are no good voting mechanism.

8.2 Social welfare and social choice

Set up. Set of candidates A, voters $\{1, \ldots, n\}$. Let L be the set of all total orders on A. (A relation $<$ on A that is antisymmetric and transitive.)

Antisymmetric: If $a \neq b$, then $a > b$ or $a < b$, but not both.

Transitive: $a < b$ and $b < c \Rightarrow a < c$.

Each voter i has a total order of preferences $<_i \in L$. ($a <_i b$ means voter i prefers b over a)

A collection of all voter preferences $(<_1, \ldots, <_n) \in L^n$. (Input)

Two types of outcomes...

Definition: social welfare function, social choice function.

A social welfare function $f: L^n \to L$, output another total order "society's preference".

A social choice function $f: L^n \to A$, output one candidate "society's choice".

8.3 Properties of a social welfare function

Let $F: L^n \to L$, $(<_1, \dots, <_n) \in L^n$. Let $< = F(<_1, \dots, <_n)$.

Good property 1. Unanimity.

If every voter prefers a over b, then the society prefers a over b.

(If $b <_i a$ $\forall i \in N$, then $b < a$.)

Good property 2. Independence of irrelevant alternatives (IIA).

The society's preference between two candidates should only dependent on the individual voters' preferences of these two candidates, not on others.

Suppose $(<_1, \dots, <_n), (<_1', \dots, <_n') \in L^n$, $F(<_1, \dots, <_n) = <$, $F(<_1', \dots, <_n') = <'$.

Then for any $a, b \in A$, $a <_i b \Leftrightarrow a <_i' b$ $\forall i \in N$ implies $a < b \Leftrightarrow a <' b$.

(If all voters don't change their relative preference of a, b, then the society doesn't, either.)

Bad property. Dictatorship.

The society's preferences completely agrees with one particular voter's preferences in all cases.

$\exists i \in N$, for all $(<_1, \dots, <_n) \in L^n$, $F(<_1, \dots, <_n) = <_i$

Is there a good voting mechanism?

When $|A| \geq 3$, no.

8.4 Arrow's impossibility theorem

Theorem 17. (Arrow's impossibility theorem)

Every social welfare function over a set of at least 3 candidates that satisfies unanimity and IIA is a dictatorship.

Sketch proof. Suppose $F : L^n \to L$ satisfies unanimity and IIA.

Part 1: We prove that for a given candidate b, if every voter ranks b at the top or bottom of their list, then F also ranks b at the top or bottom.

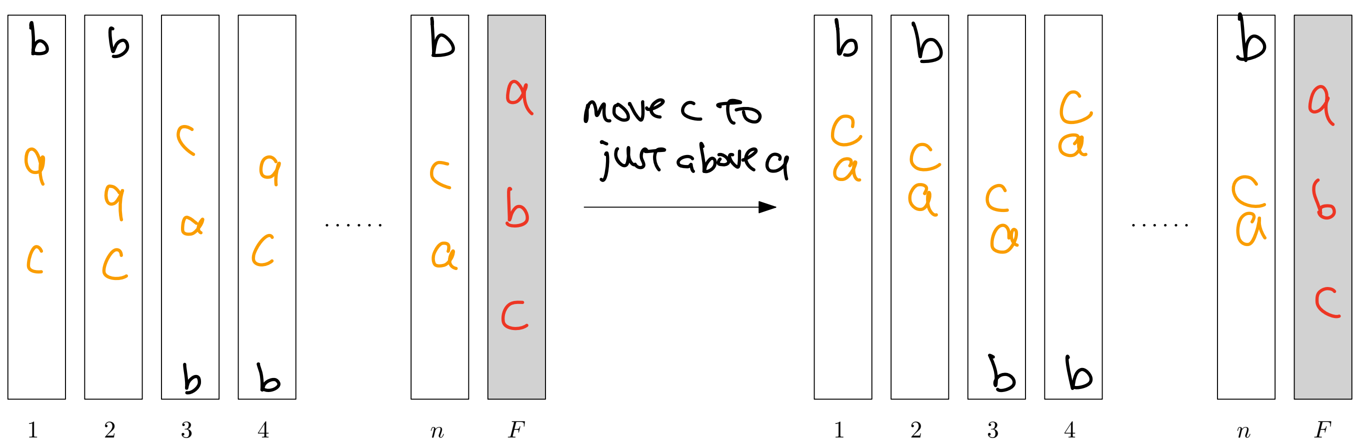
Suppose otherwise, and there exist $a, c \in A$ such that $c < b < a$ (they exist since there are ≥ 3 candidates).

Move c to just above a. Relative ranking of a, b do not change.

By IIA, F still ranks a above b.

Similar for b, c. F ranks b over c.

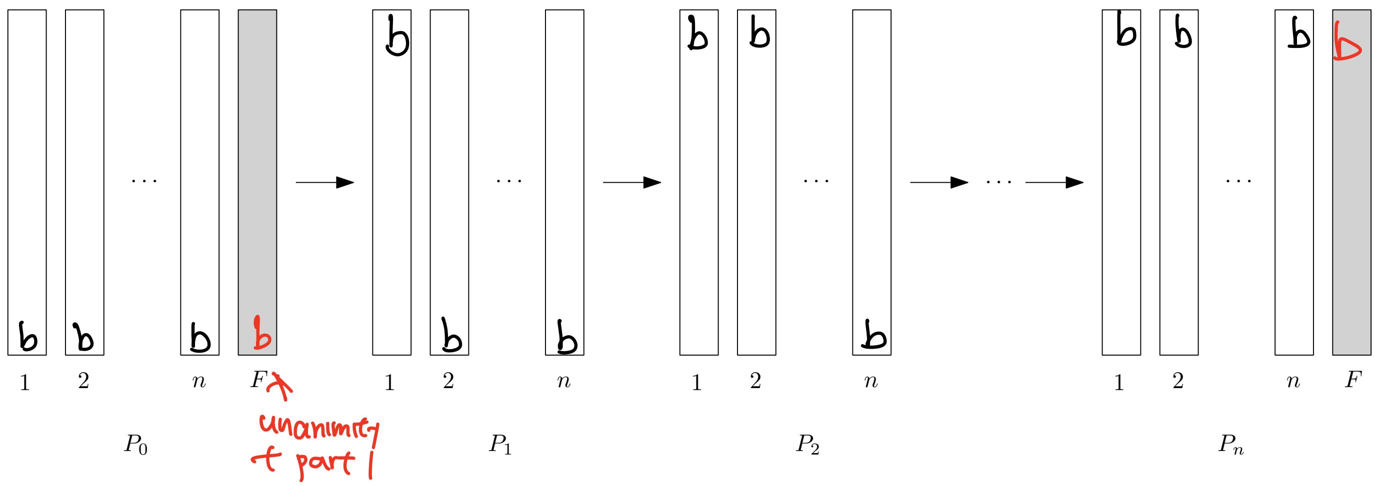
So F prefers a over c, while every voter prefers c over a, contradicting unanimity.



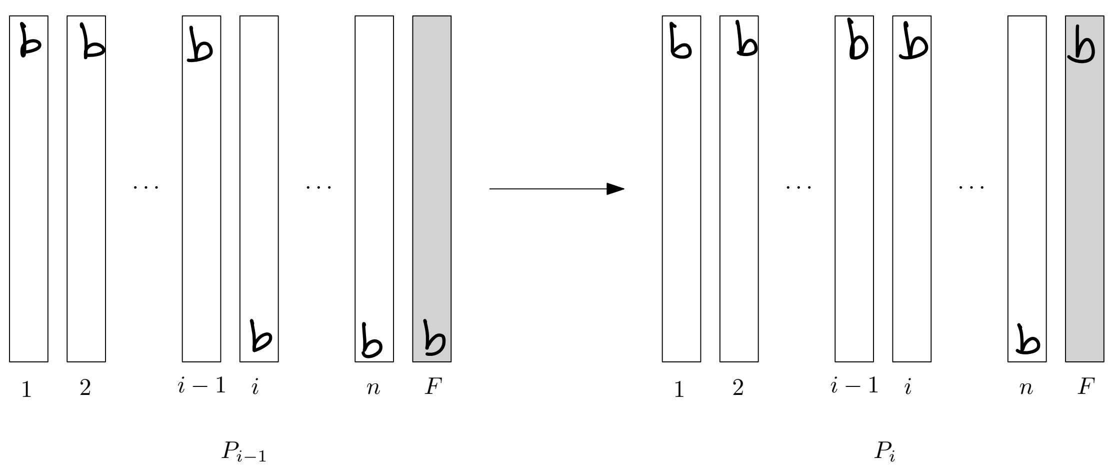
Part 2: Find a potential dictator. Let $P \in L^n$.

Create P_i by raising b to the bottom for voters $0, \ldots, i$

dropping b to the top for voters $i+1, \ldots, n$.



F must switch b from bottom to top at some point. Say $P_{i-1} \Rightarrow P_i$.



Claim: Voter i is the dictator.