3.5 Application of weak dominance: Auctions

Theorem 8. In a closed bid second price auction, player $i \in N$ bidding its valuation v_i is a weakly dominating strategy.

Proof of Theorem 8.

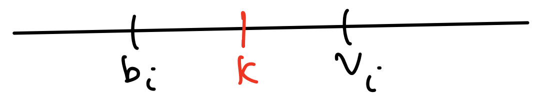
We first show $u_i(v_i, b_{-i}) \geq u_i(b_i, b_{-i})$ for all $b_i \in S_i$, $b_{-i} \in S_{-i}$.

① Suppose v_i is a winning bid against b_{-i}. Let b_j be the second highest bid.
So $u_i(v_i, b_{-i}) = v_i - b_j \geq 0$. Suppose player i changes the bid to b_i.
If $b_i > b_j$ or ($b_i = b_j$ and $i < j$), then b_i is still winning with same utility.
If $b_i < b_j$ or ($b_i = b_j$ and $i > j$), then b_i is losing with utility $0 \leq u_i(v_i, b_{-i})$.

② The case where v_i is a losing bid is left as exercise.

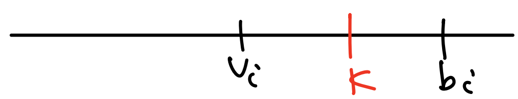
We now show that for all $b_i \neq v_i$, there exists $b_{-i} \in S_{-i}$ such that $u_i(b_i, b_{-i}) < u_i(v_i, b_{-i})$.

① Suppose $b_i < v_i$. Let $k \in \mathbb{R}$ where $b_i < k < v_i$. Set $b_j = k$ $\forall j \neq i$.



Then $u_i(v_i, b_{-i}) = v_i - k > 0$. But $u_i(b_i, b_{-i}) = 0 < u_i(v_i, b_{-i})$.

② Suppose $b_i > v_i$. Let $k \in \mathbb{R}$ where $v_i < k < b_i$. Set $b_j = k$ $\forall j \neq i$.



Then $u_i(v_i, b_{-i}) = 0$. But $u_i(b_i, b_{-i}) = v_i - k < 0 = u_i(v_i, b_{-i})$. $\square$

Note: The way we play this game does not depend on knowing other players' valuations. Easy to play.

4 Mixed strategies

4.1 Matching pennies game

Two players each has a penny. They simultaneously show heads or tails. If they match, then player 1 gains the penny from player 2. If they do not match, then player 2 gets the penny from player 1.

		P2: H	P2: T
P1	H	$1,-1$	$-1,1$
	T	$-1,1$	$1,-1$

No "NE" here. Playing H or T all the time is very bad.

Best way is to play randomly.

Example: P1 plays H $\frac{2}{3}$ of the time, T $\frac{1}{3}$ of the time. [P2 would play T all the time.]

If both P1, P2 plays $\frac{1}{2}$ on each, then no player is incentivized to switch. $\Rightarrow$ A NE.

4.2 Set up for mixed strategies

Definitions: mixed strategy, pure strategy, mixed strategy profile.

Same set up as before: each player i has <u>pure strategies</u> S_i.

A <u>mixed strategy</u> for player i is a vector $x^i \in \mathbb{R}^{S_i}$ such that $x^i \geq 0$ and $\sum_{s \in S_i} x^i_s = 1$. (Probability distribution over all pure strategies.)

The set of all mixed strategies for player i is denoted Δ^i.

A <u>mixed strategy profile</u> is a vector $x = (x^1, \ldots, x^n)$ where $x^i \in \Delta^i$.

The set of all mixed strategy profiles is $\Delta = \Delta^1 \times \cdots \times \Delta^n$.

The mixed strategy profile with player i removed is $x^{-i} \in \Delta^{-i}$.

Example.

In matching pennies, a possible profile is $x = (\underbrace{(\frac{2}{3}, \frac{1}{3})}_{\text{P1 plays } \frac{2}{3} H,\ \frac{1}{3} T}, \underbrace{(0, 1)}_{\text{P2 plays } T})$

Why mixed strategies?

1. With repeated games, good to introduce unpredictability.

2. Model each player as representing a population.

4.3 Expected utility

Example. Matching pennies.

		P2	
		H	T
P1	H	$1,-1$	$-1,1$
	T	$-1,1$	$1,-1$

$x = ((\frac{1}{3}, \frac{2}{3}), (\frac{3}{4}, \frac{1}{4}))$.

P1's utility: $\frac{1}{3}$ of utility for playing H, $\frac{2}{3}$ of utility for T.

Playing H as pure strat: $\frac{3}{4}$ of the time, P2 plays H, utility 1. $\frac{1}{4}$ utility -1. Expected utility is $\frac{3}{4}\cdot 1 + \frac{1}{4}\cdot(-1) = \frac{1}{2}$.

Playing T as pure strat: $-\frac{1}{2}$.

Expected utility for P1: $\frac{1}{3}\cdot\frac{1}{2} + \frac{2}{3}\cdot(-\frac{1}{2}) = \frac{-1}{6}$.

Definitions: expected utility of a pure / mixed strategy.

Given a strategy profile $x = (x^1, \dots, x^n) \in \Delta$, the <u>expected utility of a pure strategy</u> $s_i \in S_i$ for player i is

$$u_i(s_i, x^{-i}) = \underbrace{\sum_{s_{-i}\in S_{-i}}}_{\text{over all pure strat of other players}} \underbrace{u_i(s_i, s_{-i})}_{\text{utility of playing } s_i \text{ against these pure strats}} \underbrace{\prod_{j\neq i} x^j_{s_j}}_{\text{prob that } s_{-i} \text{ happens}}.$$

The <u>expected utility</u> of player i is $u_i(x) = \sum_{s_i \in S_i} x^i_{s_i} u_i(s_i, x^{-i})$.

Example. Suppose 3 players each make a choice between A and B. A \$1 prize is split among players who pick the majority choice.
Suppose $x^1 = (p, 1-p), x^2 = (\frac{1}{2}, \frac{1}{2}), x^3 = (\frac{2}{5}, \frac{3}{5})$.

Expected utility for P1 for playing A:

① $u_1(A, A, A) = \frac{1}{3}$ Prob this happens: $\frac{1}{2}\cdot\frac{2}{5} = \frac{1}{5}$

② $u_1(A, A, B) = \frac{1}{2}$ $\frac{1}{2}\cdot\frac{3}{5} = \frac{3}{10}$

③ $u_1(A, B, A) = \frac{1}{2}$ $\frac{1}{2}\cdot\frac{2}{5} = \frac{1}{5}$

④ $u_1(A, B, B) = 0$

$u_1(A, x^{-1}) = \frac{1}{5}\cdot\frac{1}{3} + \frac{3}{10}\cdot\frac{1}{2} + \frac{1}{5}\cdot\frac{1}{2} + 0 = \frac{19}{60}$.

Check: $u_1(B, x^{-1}) = \frac{7}{20}$.

Expected utility for P1 is $u(x) = p\cdot\frac{19}{60} + (1-p)\frac{7}{20} = \frac{7}{20} - \frac{1}{30}p$.

Maximized at $p = 0$.