# CO 456 Fall 2024 Lecture 8 (Monday September 30)

## 4 Mixed strategies

### 4.8 Two-player zero-sum games

Payoff for P1

| | | P2: 1 | P2: 2 | P2: 3 |
|---|---|---|---|---|
| P1 | 1 | 3 | 5 | −2 |
| | 2 | −5 | 7 | 1 |

Payoff for P2

| | | P2: 1 | P2: 2 | P2: 3 |
|---|---|---|---|---|
| P1 | 1 | −3 | −5 | 2 |
| | 2 | 5 | −7 | −1 |

**Player 1's perspective.**

P2 wants: $\max\{-3x_1^1 + 5x_2^1, -5x_1^1 - 7x_2^1, 2x_1^1 - x_2^1\}$.

P1 wants: $\max\{\min\{3x_1^1 - 5x_2^1, 5x_1^1 + 7x_2^1, -2x_1^1 + x_2^1\}\}$

Let $u_1$ be this maximum. $u_1 \leq$ each term in the min.

$$\begin{aligned} \max\ & u_1 \\ \text{s.t. } & u_1 \leq 3x_1^1 - 5x_2^1 \\ & u_1 \leq 5x_1^1 + 7x_2^1 \\ & u_1 \leq -2x_1^1 + x_2^1 \\ & x_1^1 + x_2^1 = 1 \\ & x_1^1, x_2^1 \geq 0 \end{aligned}$$

ensures $u_1$ is equal to the min

$u_1$ is the utility for P1.

General form:

$$\begin{aligned} \max\ & u_1 \\ \text{s.t. } & u_1 \leq (x^1)^T \cdot \mathrm{col}_j(A) \quad \forall j \in S_2 \\ & \sum_{i \in S_1} x_i^1 = 1 \\ & x^1 \geq 0. \end{aligned}$$

**Player 2's perspective.** Suppose P2 plays $x^2 = (x_1^2, x_2^2, x_3^2)$.

P1 plays their best response $\max\{3x_1^2 + 5x_2^2 - 2x_3^2, -5x_1^2 + 7x_2^2 + x_3^2\}$.

P2 wants $\min\{\max\{\ \ldots\ \}\}$.

$$\begin{aligned} \min\ & u_2 \\ \text{s.t. } & u_2 \geq 3x_1^2 + 5x_2^2 - 2x_3^2 \\ & u_2 \geq -5x_1^2 + 7x_2^2 + x_3^2 \\ & x_1^2 + x_2^2 + x_3^2 = 1 \\ & x_1^2, x_2^2, x_3^2 \geq 0. \end{aligned}$$

Note: $u_2$ is the utility of P1.

P2's utility is $-u_2$.

General form

horizontal vector

$$\begin{aligned} \min\ & u_2 \\ \text{s.t. } & u_2 \geq \mathrm{row}_i(A) \cdot x^2 \quad \forall i \in S_1. \\ & \sum_{j \in S_2} x_j^2 = 1 \\ & x^2 \geq 0 \end{aligned}$$

**Observations.**

The LPs for P1 and P2 are duals (exercise: check).

Both have feasible solutions (any probability distribution along with a small $u_1$, a big $u_2$).

So both have optimal solutions.

By strong duality, both have the same optimal value.

Optimal solution gives NE, optimal value = utility for P1.

A modified simplex solves the LP in poly time.

**Theorem 11.**

Any 2-player zero-sum game with finitely many strategies has a NE, and this can be efficiently computed.

**Optimal solutions to the example.**

P1: $x_1^1 = \frac{6}{11}$, $x_2^1 = \frac{5}{11}$, $u_1 = -\frac{7}{11}$.

P2: $x_1^2 = \frac{3}{11}$, $x_2^2 = 0$, $x_3^2 = \frac{8}{11}$, $u_2 = -\frac{7}{11}$ utility $\frac{7}{11}$.

# 5 Nash's theorem

**Theorem 12.** (Nash's theorem)

Every strategic game with finitely many players and pure strategies has a Nash equilibrium.

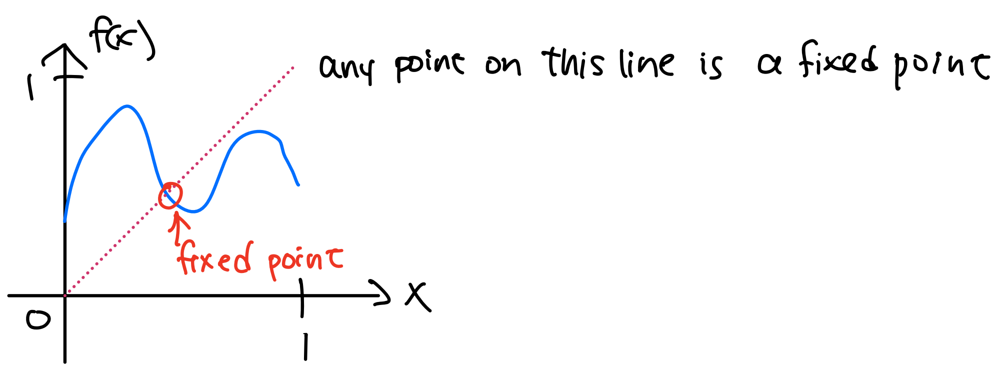
A proof of this uses Brouwer's fixed point theorem.

## 5.1 Brouwer's fixed point theorem

**Theorem 13.** (Brouwer)

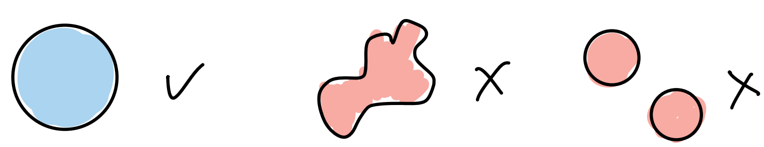
Let $X$ be a convex and compact set in a finite-dimensional Euclidean space. Let $f: X \to X$ be a continuous function. Then there exists $x_0 \in X$ such that $f(x_0) = x_0$ (fixed point).

**Example.** Let $X = [0, 1]$. Consider any continuous function $f : [0, 1] \to [0, 1]$.




**Terminology from the theorem.**

- Euclidean space: essentially $\mathbb{R}^n$ with dot product (define distance & angle)
- Convex: Take any 2 points in the set, the line segment joining them is entirely in the set.



- Compact: closed and bounded.

  Closed: (roughly) any "boundary" points are in the set.

  e.g. $[0,1]$ is closed, $[0,1)$ is not closed.

  Bounded: There exists a constant that bounds the distance between any 2 points. e.g. $[0,1]$ is compact, $\mathbb{R}$ is not.

**Illustration.**

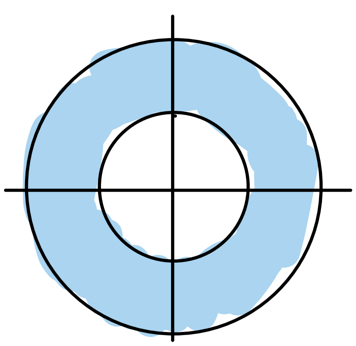

Take this map and put it on the floor. Fixed point theorem $\Rightarrow$ $\exists$ some point on the map that sits directly on top of its actual location. (Assume this part of the Earth is flat.)

**Counterexample when $X$ is not compact.**

$X = (0,1)$. open. $f(x) = x^2$. $f: X \to X$, $x \neq f(x)$.

$X = \mathbb{R}$ unbounded. $f(x) = x+1$ does not have a fixed point.

**Counterexample when $X$ is not convex.**



Not convex

Rotate the region by a bit.

No fixed point.

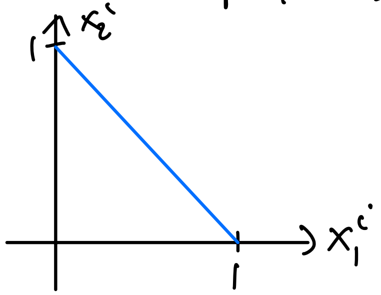
### 5.2 Defining the set $X$

We want to use Brouwer's fixed point theorem when $X$ is the set of all mixed strategy profiles $\Delta$ of a finite strategic game.

For player $i$ with $S_i = \{1, \dots, k\}$, $\Delta^i = \{(x^i_1, \dots, x^i_k) : x^i_j \geq 0,\ x^i_1 + \dots + x^i_k = 1\}$.

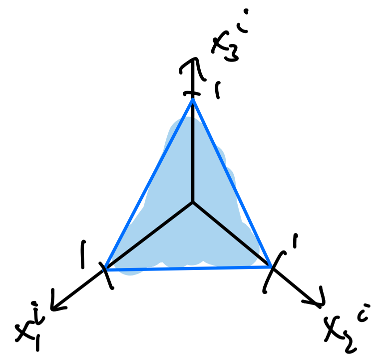
(All strats for player $i$.)

$k=2$




convex & compact

$k=3$




convex & compact

$\Delta^i$ is convex & compact for any $k$.

With $n$ players, $\Delta = \Delta^1 \times \Delta^2 \times \dots \times \Delta^n$. This is also convex and compact. We will use $\Delta$ in applying Brouwer's theorem.

### 5.3 Main idea of applying Brouwer's theorem

**Example.** Rock paper scissors.

| | R | P | S |
|---|---|---|---|
| R | $0,0$ | $-1,1$ | $1,-1$ |
| P | $1,-1$ | $0,0$ | $-1,1$ |
| S | $-1,1$ | $1,-1$ | $0,0$ |