4 Mixed strategies

4.4 Mixed equilibria

Definitions: (mixed) Nash equilibrium, best response function.

A mixed strategy profile $\bar{x} \in \Delta$ is a Nash equilibrium if for each $i \in N$, $u_i(\bar{x}) \geq u_i(x^i, \bar{x}^{-i})$ for all $x^i \in \Delta^i$.

Given $\bar{x}^{-i} \in \Delta^{-i}$, the best response function for player i is the set of all mixed strategies for player i with maximum utility against $\bar{x}^{-i}$.

$$B_i(\bar{x}^{-i}) = \{ \bar{x}^i \in \Delta^i : u_i(\bar{x}^i, \bar{x}^{-i}) \geq u_i(x^i, \bar{x}^{-i}) \ \forall x^i \in \Delta^i \}.$$

Proposition 9. A strategy profile $\bar{x} = (\bar{x}^1, \ldots, \bar{x}^n) \in \Delta$ is a Nash equilibrium if and only if $\bar{x}^i \in B_i(\bar{x}^{-i})$ for all $i \in N$.

4.5 Finding mixed NE for 2-player 2-strategy games

Example. Matching pennies. Suppose $x^1 = (p, 1-p)$ and $x^2 = (q, 1-q)$.

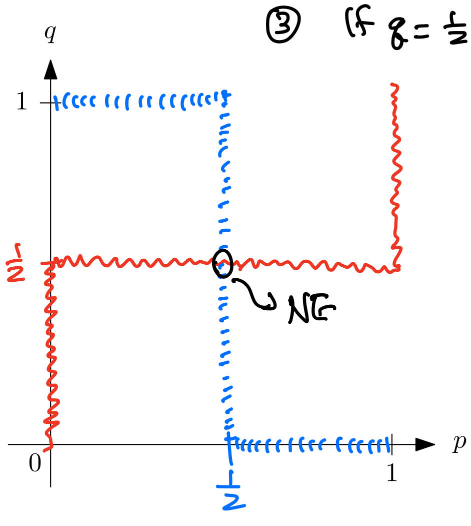
		P2	
		H	T
P1	H	$1, -1$	$-1, 1$
	T	$-1, 1$	$1, -1$

For P1, expected utility for playing H is $1 \cdot q + (-1)(1-q)$ $= 2q - 1$. Playing T: $(-1) \cdot q + 1 \cdot (1-q) = -2q + 1$.

Overall expected utility for P1 is: $p \cdot (2q-1) + (1-p)(-2q+1)$ $= p(-2+4q) + (1-2q)$.

Given q, what is best response for P1? $1-2q$ constant. Maximizing $p(-2+4q)$.

3 cases:
1. If $q < \frac{1}{2}$, then $-2+4q < 0$. Maximized at $p = 0$
2. If $q > \frac{1}{2}$, then $-2+4q > 0$. " " $p = 1$.
3. If $q = \frac{1}{2}$, then $-2+4q = 0$. Any $p \in [0,1]$ maximizes it.



BRF for P1:

$$B_1(x^2) = \begin{cases} \{(0,1)\} & q < \frac{1}{2} \\ \{(p, 1-p) : p \in [0,1]\} & q = \frac{1}{2} \\ \{(1,0)\} & q > \frac{1}{2}. \end{cases}$$

BRF for P2:

$$B_2(x^1) = \begin{cases} \{(0,1)\} & p > \frac{1}{2} \\ \{(q, 1-q) : q \in [0,1]\} & p = \frac{1}{2} \\ \{(1,0)\} & p < \frac{1}{2}. \end{cases}$$

NE: $p = \frac{1}{2}$, $q = \frac{1}{2}$. (where BRF meet).

Example. Bach or Stravinsky. Suppose $x^1 = (p, 1-p)$ and $x^2 = (q, 1-q)$.

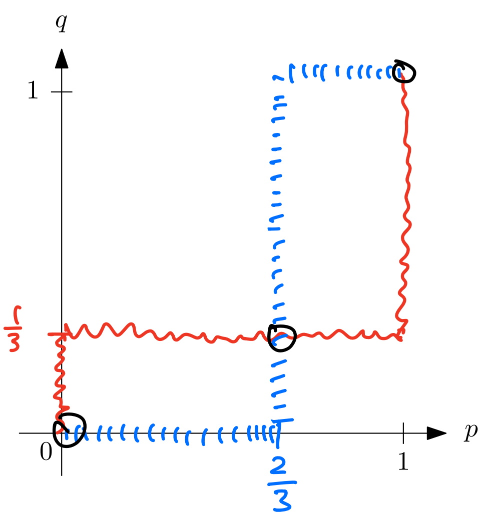
		P2	
		B	S
P1	B	$2, 1$	$0, 0$
	S	$0, 0$	$1, 2$

Expected utility for P1: $u_1(x) = p(-1+3q) + (1-q)$

$$B_1(x^2) = \begin{cases} \{(0,1)\} & q < \frac{1}{3} \\ \{(p,1-p) : p \in [0,1]\} & q = \frac{1}{3} \\ \{(1,0)\} & q > \frac{1}{3}. \end{cases}$$

Expected utility for P2: $u_2(x) = q(-2+3p) + (2-2p)$

$$B_2(x^1) = \begin{cases} \{(0,1)\} & p < \frac{2}{3} \\ \{(q,1-q) : q \in [0,1]\} & p = \frac{2}{3} \\ \{(1,0)\} & p > \frac{2}{3} \end{cases}$$



3 NEs:

① $p = q = 0$ (pure)

② $p = \frac{2}{3}$, $q = \frac{1}{3}$

③ $p = q = 1$. (pure)

4.6 Support characterization

Suppose $\bar{x}^{-i} \in \Delta^{-i}$ is fixed. Which $x^i \in \Delta^i$ maximizes $u_i(x^i, \bar{x}^{-i})$? Using $x^i = (x_1^i, \dots, x_m^i)$ as variables.

$$\begin{aligned} \max \quad & \sum_{s \in S_i} x_s^i u_i(s, \bar{x}^{-i}) \\ \text{s.t.} \quad & \sum_{s \in S_i} x_s^i = 1 \\ & x^i \geq 0 \end{aligned}$$

Dual: 1 dual var y

$$\begin{aligned} \min \quad & y \\ \text{s.t.} \quad & y \geq u_i(s, \bar{x}^{-i}) \quad \forall s \in S_i \\ & y \text{ free.} \end{aligned}$$

$S_i = \{1, 2, 3\}$

$$\begin{aligned} \max \quad & x_1^i u_i(1, \bar{x}^{-i}) + x_2^i u_i(2, \bar{x}^{-i}) + x_3^i u_i(3, \bar{x}^{-i}) \\ \text{s.t.} \quad & x_1^i + x_2^i + x_3^i = 1 \\ & x_1^i, x_2^i, x_3^i \geq 0. \end{aligned}$$

$$\begin{aligned} \min \quad & y \\ \text{s.t.} \quad & y \geq u_i(1, \bar{x}^{-i}) \\ & y \geq u_i(2, \bar{x}^{-i}) \\ & y \geq u_i(3, \bar{x}^{-i}). \end{aligned}$$

Optimal solutions.

For the dual, the optimal solution is $y = \max\{u_i(s, \bar{x}^{-i}) : s \in S_i\}$.

By Strong Duality Theorem, the primal has the same optimal value.

For P1, the best response has the same utility as playing the pure strategy with the highest expected utility.

Complementary slackness conditions.

Either $x_s^i = 0$ or $y = u_i(s, \bar{x}^{-i})$ for each $s \in S_i$. (In an optimal soln.)

$\Leftrightarrow$ If $x_s^i > 0$, then $y = u_i(s, \bar{x}^{-i})$. Only pure strategies with maximum utility could have positive probabilities.

Definition: support.

For $x^i \in \Delta^i$, the <u>support</u> of x^i is the set of pure strategies with positive probability in x^i.

Theorem 10. (Support characterization)

$x^i \in B_i(\bar{x}^{-i})$ if and only if the support of x^i are pure strategies with maximum utility against $\bar{x}^{-i}$.

Example. Consider a 2-player game with the following payoff table. Suppose $x^2 = (0, \frac{1}{3}, \frac{2}{3})$. What is $B_1(x^2)$?

		P2 D	P2 E	P2 F
P1	A	$2, 1$	$3, 3$	$1, 1$
	B	$3, 1$	$0, 4$	$2, 1$
	C	$3, 4$	$5, 1$	$0, 7$

Utilities for P1 in pure strats:

$u_1(A, x^2) = 0 + \frac{1}{3} \cdot 3 + \frac{2}{3} \cdot 1 = \frac{5}{3}$

$u_1(B, x^2) = \frac{4}{3}$

$u_1(C, x^2) = \frac{5}{3}$

max utility

Best response : $B_1(x^2) = \{(p, 0, 1-p) : p \in [0,1]\}$.

To be a NE, we need p such that x^2 is in the best response of x^1.

Let $x^1 = (p, 0, 1-p)$.

$u_2(D, x^1) = 4 - 3p$, $u_2(E, x^1) = 1 + 2p$, $u_3(F, x^1) = 7 - 6p$.

For $x^2 = (0, \frac{1}{3}, \frac{2}{3})$ to be in the BRF, E, F have highest utility.

$\Rightarrow 1 + 2p = 7 - 6p \geq 4 - 3p$. $\Rightarrow p = \frac{3}{4}$ works.

So $x^1 = (\frac{3}{4}, 0, \frac{1}{4})$, $x^2 = (0, \frac{1}{3}, \frac{2}{3})$ is a NE.