CO 456 Fall 2024 Lecture 8 (Wednesday October 2)

5 Nash's theorem

Theorem 12. (Nash's theorem) Every strategic game with finitely many players and pure strategies has a Nash equilibrium.

Theorem 13. (Brouwer) Let X be a convex and compact set in a finite-dimensional Euclidean space. Let $f : X \to X$ be a continuous function. Then there exists $x_0 \in X$ such that $f(x_0) = x_0$.

Selection of X. We choose $X = \Delta$, the set of all mixed strategy profiles.

5.3 Main idea of applying Brouwer's theorem

Want a function $f: \Delta \to \Delta$ that maps one strategy profile to another;

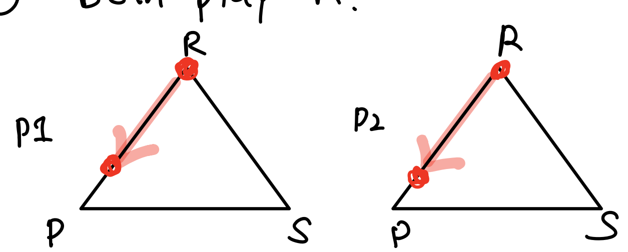
Each player is trying to improve their utility against the remaining players' profiles. Stay the same if they cannot improve.

A fixed point x^* satisfies $f(x^*) = x^*$. No player can improve by switching strategies, then x^* is a NE.

Example. Rock paper scissors.

	R	P	S
R	$0, 0$	$-1, 1$	$1, -1$
P	$1, -1$	$0, 0$	$-1, 1$
S	$-1, 1$	$1, -1$	$0, 0$

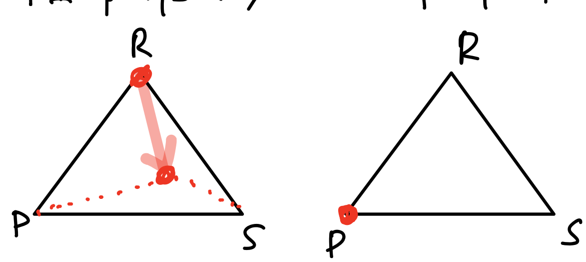
① Both play R.



Both can improve by switching to P.

Our function will move both towards P.

② P1 plays R, P2 plays P



P2 has max utility, does not move.

P1 gains 1 for P, gains 2 for S

P1 moves towards both, but closer to S than P.

5.4 Defining the function f

Define Φ: measuring improvement.

Given strategy profile $x \in \Delta$, a player i, a pure strategy $s \in S_i$,

define $\Phi_s^i(x) = \max\{\underbrace{0}_{\text{if utility does not increase}},\ \underbrace{u_i(s, x^{-i}) - u_i(x)}_{\text{how much utility increases}}\}$.

Note: Φ is continuous. (Check.)

Define f: moving towards improvement.

For player i and strategy $s \in S_i$ where $\Phi_s^i(x) > 0$, we want to increase the probability assigned to s. Say we replace x_s^i by $x_s^i + \Phi_s^i(x)$.

Normalize this by dividing the sum of all the new probabilities:

$$\sum_{s' \in S_i} \left(x_{s'}^i + \Phi_{s'}^i(x) \right) = 1 + \sum_{s' \in S_i} \Phi_{s'}^i(x).$$

Define $f: \Delta \to \Delta$ by $f(x) = \bar{x}$ where for each player i and strat $s \in S_i$,

$$\bar{x}_s^i = \frac{x_s^i + \Phi_s^i(x)}{1 + \sum_{s' \in S_i} \Phi_{s'}^i(x)}.$$

This is continuous

Example. Rock paper scissors.

P1 plays rock, P2 plays paper. $x = ((1,0,0), (0,1,0))$.

For P2, $\Phi_R^2(x) = \Phi_P^2(x) = \Phi_S^2(x) = 0$. So $\bar{x}^2 = x^2$.

For P1, $\Phi_R^1(x) = 0$, $\Phi_P^1(x) = 1$, $\Phi_S^2(x) = 2$.

$$\bar{x}_R^1 = \frac{1+0}{4} = \frac{1}{4}, \quad \bar{x}_P^1 = \frac{0+1}{4} = \frac{1}{4}, \quad \bar{x}_S^1 = \frac{0+2}{4} = \frac{1}{2}$$

So $\bar{x}^1 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$.

Given $x \in \Delta$, consider Φ, f as defined above. Since f is continuous and Δ is convex and compact, we can apply Brouwer's fixed point theorem to conclude that there exists $\hat{x} \in \Delta$ where $f(\hat{x}) = \hat{x}$.

[Prove $\hat{x}$ is a NE by showing $\hat{x}^i \in B_i(\hat{x}^{-i})$]

For player i, we claim that there exists a pure strategy $s \in S_i$ such that $\hat{x}^i_s > 0$ and $u_i(s, \hat{x}^{-i}) \leq u_i(\hat{x})$:

$$u_i(\hat{x}) = \sum_{s \in S_i} \hat{x}^i_s \, u_i(s, \hat{x}^{-i})$$

$$= \sum_{\{s \in S_i : \hat{x}^i_s > 0\}} \hat{x}^i_s \, u_i(s, \hat{x}^{-i})$$

If $u_i(s, \hat{x}^{-i}) > u_i(\hat{x})$ for all such s, then this sum $> u_i(\hat{x})$, contradiction.

Then $\Phi^i_s(\hat{x}) = 0$. Since $\hat{x}$ is a fixed point,

$$\hat{x}^i_s = (f(\hat{x}))^i_s = \frac{\hat{x}^i_s}{1 + \sum_{s' \in S_i} \Phi^i_{s'}(\hat{x})}.$$

Since $\hat{x}^i_s > 0$, this equality holds if and only if $\sum_{s' \in S_i} \Phi^i_{s'}(\hat{x}) = 0$. Since Φ is non-negative,

$\Phi^i_{s'}(\hat{x}) = 0$ for all $s' \in S_i$. So $u_i(s', \hat{x}^{-i}) \leq u_i(\hat{x})$ for all $s' \in S_i$.

Playing $\hat{x}^i$ gives the highest utility against $\hat{x}^{-i}$, so $\hat{x}^i \in B_i(\hat{x}^{-i})$.

This holds for all players, so $\hat{x}$ is a NE. $\square$

Note: This proof does not construct a NE, since proofs of Brouwer's theorem are not constructive.