CO 456: Myerson's Lemma and Knapsack Auctions

(Lecture Notes)*

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Contents

1 Myerson's Lemma

1.1 Definition(s) of the problem

1.1.1 Goal

Definition 1.1. The seller's problem is to design an auction game which has a Nash equilibrium giving him the highest possible expected utility.

*These lecture notes draw from a combination of Prof. Martin Pei's insights on the subject and Prof. Roger Myerson's foundational work

Definition 1.2 (More precisely.)**.** The seller would like to find an auction procedure which can give him the highest expected revenue among different kinds of auctions (e.g., sealed-bid auctions, dutch auctions, or progressive auctions)

1.1.2 Assumptions

1. We study auctions as non-cooperative games with imperfect information.
2. Single-parameter environment
   - One seller who has a single object to sell.
   - The seller faces $n \in \mathbb{N}$ number of bidders.
3. Bidders are allowed to revise their bids
4. The seller has no private information about the object
5. The bidders' value estimates may be unkown to the other bidders or the seller

Question: Why?

Reflect on this with respect to 3. The distinction elaborated next spoils the answer.

1.1.3 Preference and Quality Uncertainity

The bidder's personal preferences might be unknown to the other agents. For example, the item being auction is a philosophy book, we would not enjoy reading it equally, *which what we would refer to as preference uncertanity.*

On the other hand, if the bidder have acquired special information about the intrinsic quality of the object (e.g., it is an ancient philosophical work written by David Hume), *which what we would refer to as quality uncertainity*, they are encouraged to revise their bids.

Question: How does the distinction impact the bidding strategies?

Assuming *only* preference uncertainty should not incentivize the bidders to revise their bids (unless I am jealous, why would I change my valuation of a painting if you enjoy looking at it more than I do?), however, it should be the case if a quality trait is revealed about the item being auctioned.

1.2 Notation

There are n bidders or potential buyers

$$N := \{1, \cdots, n\},$$

we will use i and j to refer to random typical bidders $i \in N$ and $j \in N$, respectively. For each bidder $i \in N$, there is a value estimate

$$t_i \in [a_i, b_i] \subseteq \mathbb{R},$$

where a_i and b_i are the lowerst and highest possible values bidder i is willing to bid *(t_0 is the seller's valuation, which we assumed to be known to every one in 4—this could be thought of as the reserve price of the auction).*

Since we allowed the bidders to revise their bids, we define revision functions

$$e_i : [a_i, b_i] \to \mathbb{R}.$$

An example of how e_i change things would be bidder i learning that t_j was bidder j's valuation. Then, bidder i revises their valuation to $t_i + e_j(t_j)$. As such, we define the private valuation of bidder i as follows

$$v_i(t) = t_i + \sum_{j \in N, j \neq i} e_j(t_j).$$

Intuition

The intuition of the latter sum is that bidder i is revising their valuation according to what they have learned about each other bidder j's valuation.

Question: Assuming *only* preference uncertaintines, what would be the values of e_i for all $i \in N$?

Zero! Why?

Recalling from previous lectures, B is the set of all possible combinations of the bidders' value estimate, which can be defined as follows

$$B = [a_1, b_1] \times \cdots \times [a_n, b_n]$$

The same set, excluding bidder i value estimate, would be denoted B_{-i}. Also, we let

$$f_i : [a_i, b_i] \to \mathbb{R}_+$$

denote the probability denisty for i's value estimates. *Assuming that the bidders have no influence over each other (independence)*, the joint denisty function on the bids B for a vector of value estimates $t = (t_1, \cdots, t_n)$ would be

$$f(t) = \Pi_{j \in N} f_j(t_j),$$

i.e., there is no conditional probability multiplied by each f_j that accounts for bidder i choosing some value t_i.

Having presented the notation, we put the seller's problem in a more technical defintion.

Definition 1.3 (Seller's Problem—More technical)**.** Given the density functions f_i's, revision functions e_i's, and valuations v_i's, the seller's problem is to select an auction direct revelation mechanism to maximize their own expected utility.

Question: What if there were other mechanisms, which are not direct revelation, that could have brought higher utility for the seller?

Lemma 1.4 (The Revelation Principle)**.** *Given any implementable auction mechanism, there exists an equivalent implementable direct revelation mechanism which gives to the seller and all bidders the same expected utilities as in the given mechanism.*

Definition 1.5 (Implementable)**.** Recall: an allocation rule $x : B \to \mathbb{R}^n$ is implementable if there exists a payment rule $p : B \to \mathbb{R}^n$ such that (x, p) is DSIC.

For a bidder i, the expected utility from an auction mechanism (x, p) is defined as follows

$$u_i(x, p, t_i) = \int_{B_{-1}} (v_i(t)x_i(t) - p_i(t)) f_{-i}(t_{-i}) dt_{-i}, \tag{1}$$

with $dt_{-i} = dt_1 \cdots d_{t_{i-1}} d_{t_{i+1}} dt_n$.

Intuition

In other words, (1) measures $[\int_{B_{-i}} \cdots dt_{-i}]$ how much benefit $[v_i(t)x_i(t) - p_i(t)]$ the bidder gets from their choice $[t_i]$, factoring in the probability that they would win $[x_i(t)]$ and the uncertainty of not knowing others' bids $[f_{-i}(t_{-i})]$.

Similarly, the auctioneer's utility would be

$$u_0(x,p) = \int (v_0(t)(1 - \sum_{j \in N} x_j(t)) + \sum_{j \in N} p_j(t)) f(t) dt,$$

where $dt = dt_1 \cdots dt_n$.

We expound on the properties of an allocation rule being implementable in the following definition.

Definition 1.6 (Implementable-Mathematical Details)**.** Equivalently, an allocation rule $x : B \to \mathbb{R}^n$ is implementable if there exists a payment rule $p : B \to \mathbb{R}^n$ such that for all bidders $i \in N$

- $$u_i(x,p,t_i) \geq 0 \tag{2}$$

  Intuition

  There has to be a nonnegative utility; otherwise, unless you are forcing me, why would I participate?

- $$u_i(x,p,t_i) \geq u_i(x,p,s_i) = \int_{B_{-i}} (v_i(t) x_i(t_{-i}, s_i) - p_i(t_{-i}, s_i)) f_{-i}(t_{-i}) dt_{-i}, \tag{3}$$

  where $(t_{-i}, s_i) = (t_1, \cdots, t_{i-1}, s_i, \cdots, t_n)$.

  Intuition

  That is, assuming a bidder i lies about their bid being s_i instead of t_i, our mechanism should guarantee that their utility then is less had they been honest, i.e., do not lie!

Define the conditional probability that bidder i will get the object from the auction mechanism (x,p) given that their value estimate is t_i as follows

$$Q_i(x,t_i) = \int_{B_{-i}} x_i(t) f_{-i}(t_{-i}) dt_{-i}.$$

1.3 Proof

Lemma 1.7 ([1, Lemma 2])**.** *The auction mechanism (x,p) is implementable if and only if the following conditions hold*

- $$\textit{if } s_i \leq t_i \Rightarrow Q_i(x, s_i) \leq Q_i(x, t_i) \qquad (\forall i \in N)(\forall t_i, s_i \in [a_i, b_i]) \tag{4}$$

- $$u_i(x, p, t_i) = u_i(x, p, a_i) + \int_{a_i}^{t_i} Q_i(x, s_i) ds_i \tag{5}$$

Intuition

The first condition (monotonicity) says that if a bidder i bids more, they are guaranteed a higher chance of winning. As for the second, the condition expresses that a bidder's utility at any bid t_i can be understood as their utility at the lowest possible bid a_i, plus the sum of all marginal gains they would get by increasing their bid from a_i to t_i.

Question: Does monotonicity hold in an auction where we give the item being auctioned to the bidder with second highest bid?

No. Why?

Proof. We first write an equivalent expression to (3) that will simplify the succeeding steps. Note that for a bidder i, we could rewrite their *lying* utility (i.e., when they change it their bid to s_i) as follows

$$\begin{aligned}
&\int_{B_{-i}} (v_i(t)x_i(t_{-i}, s_i) - p_i(t_{-i}, s_i)) f_{-i}(t_{-i}) dt_{-i} \\
&= \int_{B_{-i}} ((v_i(t_{-i}, s_i) + (t_i - s_i))x_i(t_{-i}, s_i) - p_i(t_{-i}, s_i)) f_{-i}(t_{-i}) dt_{-i} \qquad (6) \\
&= u_i(x, p, s_i) + (t_i - s_i)Q(x, s_i),
\end{aligned}$$

where (6) follows from $v_i(t_{-i}, s_i) = s_i + \sum_{i=1} e_i(t_i) \Rightarrow v_i(t) = v_i(t_{-i}, s_i) + (t_i - s_i)$. Hence, we can rewrite (3) as

$$u_i(x, p, t_i) \geq u_i(x, p, s_i) + (t_i - s_i)Q_i(x, s_i). \tag{7}$$

As such, (x, p) is implementable if and only if (2) and (7) hold. We now show that (2) and (7) imply (4) and (5). Apply (7) twice with the roles of s_i and t_i switched to learn the following

$$(t_i - s_i)Q_i(x, s_i) \leq u_i(x, p, t_i) - u_i(x, p, s_i) \leq (t_i - s_i)Q_i(x, t_i).$$

Then, (4) follows when $s_i \leq t_i$. Since $Q_i(x, s_i)$ is increasing in s_i we obtain

$$\int_{a_i}^{t_i} Q_i(x, s_i) ds_i = u_i(x, p, t_i) - u_i(x, p, a_i),$$

which gives us (5) after rearranging. Conversely, suppose that (4) and (5) hold. Since $Q_i(x, s_i) \geq 0$, we have (2) following from

$$0 \leq u_i(x, p, t_i) = \underbrace{u_i(x, p, a_i)}_{\geq 0} + \underbrace{\int_{a_i}^{t_i} Q_i(x, s_i) ds_i}_{\geq 0}.$$

To show (7) (which is equivalent to (3)), suppose that $s_i \leq t_i$, then from (4) and (5) we have for some bid $r_i \in [a_i, b_i]$

$$\begin{aligned} u_i(x, p, t_i) &= u_i(x, p, s_i) + \int_{s_i}^{t_i} Q_i(x, r_i) dr_i && (8) \\ &\geq u_i(x, p, s_i) + \int_{s_i}^{t_i} Q_i(x, s_i) dr_i && (9) \\ &\geq u_i(x, p, s_i) + (t_i - s_i) Q_i(x, s_i) && (10) \end{aligned}$$

Similarly, if $s_i > t_i$ then the following holds

$$\begin{aligned} u_i(x, p, t_i) &= u_i(x, p, s_i) - \int_{t_i}^{s_i} Q_i(x, r_i) dr_i && (11) \\ &\geq u_i(x, p, s_i) - \int_{t_i}^{s_i} Q_i(x, s_i) dr_i && (12) \\ &\geq u_i(x, p, s_i) + (t_i - s_i) Q_i(x, s_i), && (13) \end{aligned}$$

yielding (7). □

A visual proof of Myerson's lemma is depicted in Figure 1.

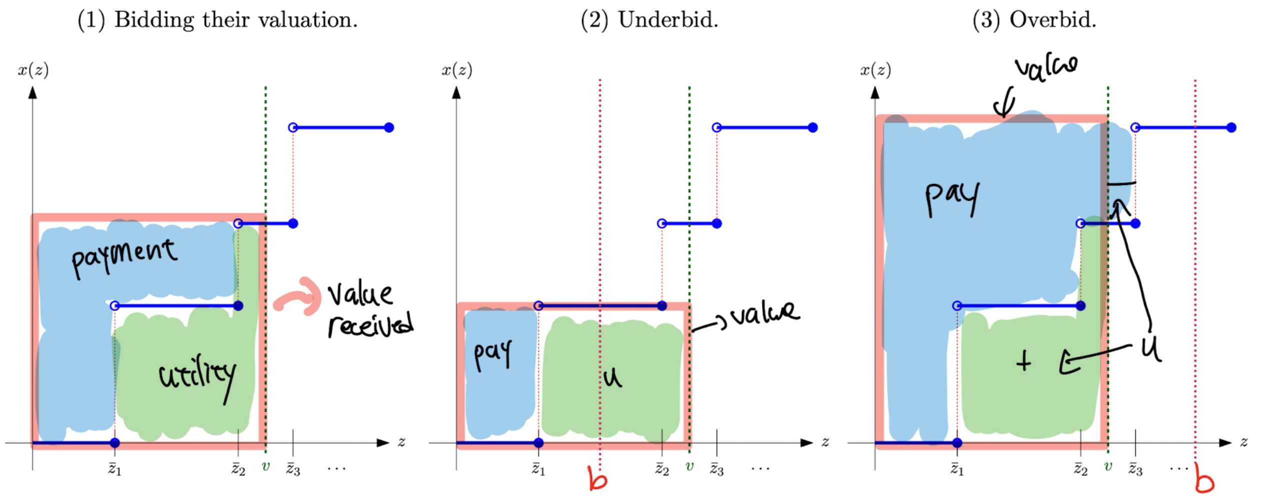

Figure 1: Proof by picture for Myerson's lemma

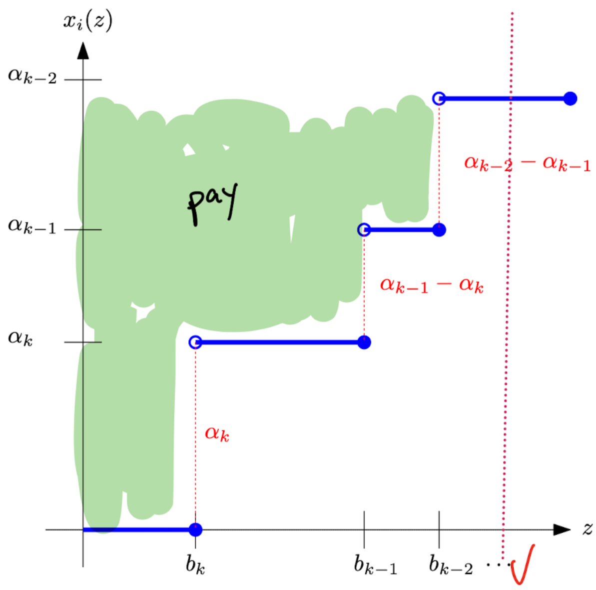

Figure 2: Monotone allocation rule for sponsored search auction

1.4 Sponsored Search Auctions

The sponsored search industry typically runs separate auctions for each separate query, e.g., "car insurance" or "veterinarian near me" have two distinct auctions. However, each is a single-parameter environment.

Let $k <= n$ be the number of slots available for advertisements. The search estimates α_{r_i}, the probability that a user will click on the r-th slot when occupied by bidder i. The qunatity α_{r_i} is called the click through rate (CTR). We assume that $\alpha_1 \geq \cdots \geq \alpha_k$.

Consider a player i and bids $b_{-i} \in B_{-i}$ such that $b_1 \geq b_2 \geq \cdots \geq b_k$. We assume without generality that they are k bids, since one could remove the $n - k$ lowest bids.

Let x be the allocation rule such that slot α_j is assigned to the j-th highest bidder. The allocation is visualizaed in Figure 2.

Corollary 1.8. *There is an ideal auction for sponsored search.*

Proof. A direct consequence of Myerson's lemma. □

2 Knapsack Auctions

Suppose we are managing advertising slots for a TV station. We have T seconds to fill and there are N potential advertisers. For each advertise $i \in N$, there is an ad length $a_i \leq T$ and has a value v_i.

The knapsack auction is running an auction to fill the T seconds by $A \subseteq \{a_i, \cdots, a_N\}$ such that $\sum_{i \in N} v_i x_i$ is maximized. This is an NP-hard problem[1]. Consequently, there is no ideal auction!

Question: Which condition to relax from the ideal auction properties?

Well, relaxing the DSIC property would not help. Why? Since it is the other two which are conflicting: welfare maximization and efficiency. Abandoning welfare maximization would discourage potential players from participating. So, perhaps hoping for a polynomial approximation is one of the ways to fix this!

2.1 Greedy Algorithm for Knapsack Auctions

Let $N = \{1, \cdots, n\}$ and $a_i \leq T$ for all $i \in N$ (we can eliminate all bidders with $a_i > T$)

(1) Sort the bidders as follows

$$\frac{v_1}{a_1} \geq \frac{v_2}{a_2} \geq \cdots \geq \frac{v_n}{a_n}$$

(2) Select bidder i according to the sorted list in (1) as long as $a_1 + \cdots + a_i \leq T$.

(3) If $v_1 + \cdots + v_i \geq v_{i+1}$, then $\{1, \cdots, i\}$ are the winning bidders. Otherwise, only $\{i+1\}$ is the winner.

2.2 Approximation Result

Let APX be the value obtained by the greedy algorithm described in the previous section and OPT be hte optimal value of the given problem (which we cannot compute in polynomial time). Then, we prove the following result.

Theorem 2.1. $APX \geq \frac{1}{2} OPT$

[1]We refer the read to Chapter 2: NP and NP completeness of Boaz Barak and Sanjeev Arora's Computational Complexity: A modern approach.

Proof. OPT is the optimal solution of the integer program

$$\max\{v^T x : a^T x \leq T, x_i \in \{0,1\}, i \in N\} \tag{IP}$$

The linear programming relaxation is

$$\max\{v^T x : a^T x \leq T, 0 \leq x_i \leq 1, i \in N\} \tag{LP}$$

Suppose that i is the *largest* index with $a_1 + \cdots + a_i \leq T$. Consider the following (handcrafted) solution x where

$$x_1 = \cdots = x_i = 1, x_{i+1} = \frac{T - t_1 - \cdots - t_i}{t_{i+1}}, x_{i+2} = \cdots = x_n = 0$$

Intuition

Filling the remaining space of the knapsack with a fraction of item $i+1$

Then, since the knapsack is filled with the highest density items, x is optimal. Let v^* be the optimal value of (LP) obtained by x. Since (LP) is a relaxation of (IP), $OPT \leq v^*$. We would like to prove

$$APX \geq \frac{1}{2} v^*.$$

Let $S = \{1, \cdots, i\}$. Then,

$$v^* \leq v_1 + \cdots + v_i + v_{i+1} = \underbrace{v(S)}_{\text{(a little abuse of notation)}} + v_{i+1}.$$

Consider the following two cases

(i) $v(S) \geq v_{i+1}$ Then, the greedy algorithm picks S. So, $APX = v(S)$. Hence,

$$v^* \leq v(S) + v_{i+1} \leq v(S) + v(S) = 2 \cdot APX$$

(ii) $v(S) < v_{i+1}$ Then, the algorithm picks $\{i+1\}$. So, $APX = v_{i+1}$. Hence,

$$v^* \leq v(S) + v_{i+1} \leq v_{i+1} + v_{i+1} = 2 \cdot APX$$

Consequently, $2 \cdot APX \geq v^* \geq OPT$ $\square$

References

[1] Roger B. Myerson. Optimal auction design. *Math. Oper. Res.*, 6(1):58–73, February 1981.