CO 456 Fall 2024: Assignment 2 problems

Due: Friday October 4 at 11:59pm EDT

Preamble.

- You will receive a Crowdmark link for each problem of the assignment. Submit your solutions on Crowdmark. Your submission must be clear and legible, and in the correct orientation. You need to submit your solutions to the correct problems, otherwise it will not be marked.
- Each of the problems is worth 20 marks.
- You need to justify all of your solutions, unless you are explicitly asked not to.
- You are graded on both your accuracy and your presentation (see below). A correct solution that is poorly presented may not receive full marks. The markers will not spend a lot of time figuring out something you write that is not clear.
- When discussing assignment problems on Piazza, use private posts if you think that you might spoil an answer or major thinking point of a problem.
- Read the assignment policies on the course outline for what is allowed and not allowed in working on the assignment.
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Guidelines to writing solutions. Your goal is to present your solutions so that your reader can easily understand what you are writing and be convinced of your arguments. So the quality of your solutions is very important. Here are some guidelines that you should keep in mind when writing your solutions.

- Write in complete sentences. Separate major steps in the solution into paragraphs.
- When writing a proof, make sure that every statement follows from earlier statements. Cite results or assumptions when you use them. Only use results from the lectures.
- Use proper notation. Mathematics is a precise language, and improper use of notation can easily cause confusion.
- Define a variable before you use it. It can be as simple as "Let $p = 3x$ for some $x \in \mathbb{Z}$" if this is the first time you are using x.
- Do not include irrelevant facts. Only state facts that are necessary in the argument.

Assignment problems.

A2-1. **Second-price auction**

(a) In the proof of Theorem 8 in class, we made the assumption that the bids can be any real number. Suppose we change the scenario so that only bids that are integers can be accepted, and all valuations are integers. Prove that for buyers 1 and n, bidding their valuation is not a weakly dominating strategy.

(b) We now suppose that all buyers have distinct non-zero valuations, and all bids must be non-negative. Determine (with proof) all possible pure Nash equilibria where all but one buyer has a zero bid.

A2-2. **Calculating mixed equilibria**

(a) For the two-player strategic game with the following payoff table, derive the best response functions for both players, and draw them on the cartesian plane. Then determine all Nash equilibria in this game (can be pure or mixed).

		P2: C	P2: D
P1	A	$10, 5$	$7, 1$
P1	B	$4, 2$	$7, 5$

(b) For the two-player strategic game with the following payoff table, determine whether or not there is a Nash equilibrium where the support for player 1 is $\{A, C\}$. If so, then derive (with proof) all possible Nash equilibria in this case. If not, then prove that no such Nash equilibria exist.

		P2: D	P2: E	P2: F
P1	A	$8, 3$	$7, 1$	$3, 10$
P1	B	$3, 9$	$4, 5$	$8, 1$
P1	C	$7, 8$	$5, 11$	$9, 5$

A2-3. **Another game on graphs**

Imagine a number of towns within a certain region, connected by a road network. Each town can decide whether or not to build a mall. However, building a mall would only be profitable if none of the neighbouring towns build a mall themselves, since neighbouring malls need to share customers.

We now formulate this as a game theory problem: We are given a connected graph G with at least 2 vertices. The set of players is the set of vertices of G. Each vertex v has two pure strategies: $\{M, N\}$ (mall or no mall). If v plays M, then its utility is 1 if all of its neighbours play N; otherwise its utility is -1. If v plays N, then its utility is 0 regardless of what its neighbours play.

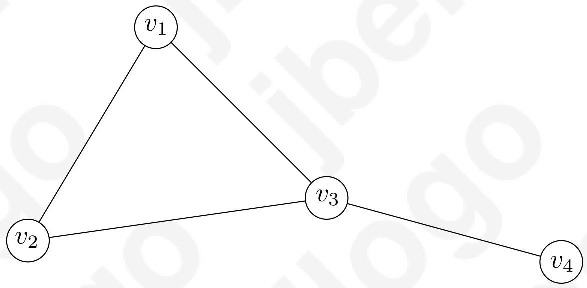
(a) Prove that for any graph G, this game has a pure Nash equilibrium.

(b) Describe all possible pure Nash equilibria when G is a cycle (length at least 3).

(c) We now look for other Nash equilibria. Suppose all vertices play M with probability p. Determine the best response function for a vertex v.

(d) Use part (c) to determine a general class of graphs for which there is a Nash equilibrium when all vertices play M with the same probability.

(e) Suppose we now allow vertices to play different probabilities. Determine all Nash equilibria (pure or mixed) of the following graph. Assume that vertex v_i plays M with probability p_i for each $i = 1, 2, 3, 4$.



A2-4. **Lectures 4-8 review questions**

(a) For the pure strategic game with the following payoff table, determine which strategies are weakly dominating, and which strategies are weakly dominated. (No justifications required.)

	D	E	F	G
A	$1,5$	$3,3$	$5,3$	$7,4$
B	$4,2$	$5,1$	$6,1$	$7,2$
C	$2,3$	$3,4$	$4,4$	$5,3$

(b) Consider the closed-bid first-price auction model. Suppose there are 4 buyers with valuations $v = (3, 1, 4, 1)$. Prove that the bids $b = (2, 1, 3, 3)$ is a Nash equilibrium.

(c) Recall that a set S is *convex* if for any $x, y \in S$, $\lambda x + (1 - \lambda)y \in S$ for all $\lambda \in [0, 1]$. Briefly explain why the best response function for a player i against a fixed $x^{-i} \in \Delta^{-i}$ is a convex set.

(d) In the voting game, prove that there is no pure Nash equilibrium when the two candidates have different number of supporters.

(e) Consider a 2-player zero-sum game with the following payoff table for player 1.

	E	F
A	3	-5
B	1	9
C	-4	-2
D	1	-6

Write down explicitly the linear programs that Player 1 and Player 2 need to solve in order to find the best response to each other's mixed strategies. (Write out the numerical values.) Then solve for an optimal solution to the LPs to determine a Nash equilibrium, along with the expected utilities for both players. (You should use computational tools to solve this.)