Existence and Blow-up for some Kinetic and Hyperbolic Models of Chemotaxis

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Chemotaxis

- biased motion of cells in the direction of a chemical gradient
- examples: immune response of the body, embryonal morphogenesis, wound healing, metastasis
- tendency of cells to aggregate is counterbalanced by the tendency to diffuse
- motion consists of series of runs and tumbles
- mean time for tumbling is much shorter than the mean time for running; model it as a velocity jump process

The macroscopic point of view: the Keller-Segel model

$$
\begin{aligned}
&\partial_t n = \Delta n - \chi \nabla \cdot (n \nabla S) \\
&-\Delta S = n \\
&(or\ \ S - \Delta S = n \\
&or \quad \epsilon \partial_t S = D \Delta S - \alpha S + \beta n)
\end{aligned}
$$

$S(t,x)$ concentration of the chemoattractant

$n(t,x)$ cell density

mass $M = \int n(t,x)dx$ is conserved

Global existence or blow-up ?

Some results for the Keller-Segel model

- in $d = 1$ dimension we have global existence [Osaki-Yagi 2001] [Hillen-Potapov 2004]
- in $d = 2$ dimensions we have [Jäger-Luckhaus 1992]
  - global existence if the mass is small
  - blow-up in finite time if the mass is large

Blow-up:

$$\frac{d}{dt}\int_{\mathbb{R}^2} |x|^2 n(t,x)dx = 4M\left(1 - \frac{\chi}{8\pi}M\right) < 0 \text{ if } M > \frac{8\pi}{\chi}$$

therefore the solution can't be global.

Global existence:

$$\frac{d}{dt}\int n^p dx = -c\int \left|\nabla n^{p/2}\right|^2 dx + c\int n^{p+1} dx$$

Using a Gagliardo-Nirenberg inequality

$$\int n^{p+1} dx \leq cM \int \left|\nabla n^{p/2}\right|^2 dx$$

we get control of $\int n^p dx$, if M is small.

This method doesn't give the optimal M.

Global existence for $M < 8\pi/\chi$ [Blanchet-Dolbeault-Perthame 2004, 2006]

$$\mathcal{E}(t) = \int_{\mathbb{R}^2} n \, log \, n \, dx + \frac{\chi}{4\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} n(t,x)\, n(t,y)\, log|x-y|\, dxdy$$

$$\frac{d\mathcal{E}}{dt} = -\int_{\mathbb{R}^2} n \left|\nabla logn - \chi \nabla S\right|^2 dx$$

log-HLS inequality with optimal constant [Beckner 1993].

$$\frac{M}{2} \int_{\mathbb{R}^2} n \log \, n \, dx + \int_{\mathbb{R}^2 \times \mathbb{R}^2} n(x) n(y) \log|x-y| \, dxdy \geq C(M)$$

Combine to get control of $\int n \log n dx$ if $M < 8\pi/\chi$.
$M = 8\pi/\chi$: aggregation in infinite time [Blanchet, Carrillo, Masmoudi 2008]

Higher Dimensions

If $d \geq 3$ then critical space is $L^{d/2}$

small $\|n_0\|_{L^{d/2}} \implies$ global existence

$$\int |x|^2 n_0(x)dx \leq C \left(\int n_0(x)dx \right)^{\frac{d}{d-2}} \implies \text{blow-up}$$

[Corrias-Perthame-Zaag 2004] [Corrias-Perthame 2006]

The microscopic point of view: Othmer-Dunbar-Alt kinetic model

$$\partial_t f + v \cdot \nabla_x f = \int_V T[S](t,x,v,v')f(t,x,v')dv'$$
$$- \int_V T[S](t,x,v',v)f(t,x,v)dv',$$

$$-\Delta S = \rho := \int_V f(t,x,v)dv$$

(or $S - \Delta S = \rho \quad or \; ...$)

$f(t,x,v) \geq 0$ is the cell density

$S(t,x)$ is the density of the chemoattractant

$T[S] \geq 0$ is the turning Kernel

V = velocity space = unit ball

mass conserved, but no energy

Why are kinetic models important?

We can predict the behavior of the population if we know the behavior of individual cells.

On many occasions only kinetic models can describe accurately phenomena such as traveling waves in E-coli.

$$T = \lambda(v')K(v, v')$$

tumbling rate:

$$\lambda(v') = \frac{1}{2}\psi_N\left(\frac{\partial_t N}{N} + v' \cdot \frac{\nabla_x N}{N}\right) + \frac{1}{2}\psi_S\left(\frac{\partial_t S}{S} + v' \cdot \frac{\nabla_x S}{S}\right)$$

where N is a nutrient, S is the the chemoattractant.
$K(v, v')$ accounts for the persistence of the trajectories.

(PNAS 2011, joint work with J. Saragosti, V.Calvez, B.Perthame, A.Buguin and P. Silberzan)

Dispersion estimates

[Bardos-Degond 1985], [Castella-Perthame 1996]

$$\partial_t f + v \cdot \nabla_x f = 0 \quad , \quad f(0,x,v) = f_0(x,v)$$

$$\|f(t)\|_{L^p_x L^q_v} \leq \frac{1}{|t|^{d\left(\frac{1}{q}-\frac{1}{p}\right)}} \|f_0\|_{L^q_x L^p_v}$$

$$1 \leq q \leq p \leq \infty$$

Solution is $f(t,x,v) = f_0(x - tv, v)$.

$$\|f_0(x-tv,v)\|_{L^p_x L^q_v} \leq \frac{1}{|t|^{d\left(\frac{1}{q}-\frac{1}{p}\right)}} \|f_0\|_{L^q_x L^p_v}$$

Strichartz estimates for the kinetic transport equation

[Castella-Perthame 1996 , Keel-Tao 1998]

$$\partial_t f + v \cdot \nabla_x f = g \ , \ \ f(0,x,v) = f_0(x,v)$$

$$\|f\|_{L^r_t L^p_x L^q_v} \lesssim \|f_0\|_{L^a_{x,v}} + \|g\|_{L^{r'}_t L^q_x L^p_v}$$

$$\begin{aligned} &1 \le q \le p \le \infty \\ &\frac{2}{r} = d\left(\frac{1}{q} - \frac{1}{p}\right) < 1 \\ &a = \frac{2pq}{p+q} \le 2 \end{aligned}$$

Global existence results for the O-D-A kinetic model

- $d = 3$ [Chalub, Markowich, Perthame, Schmeiser, 2004]

If

$$T[S](t,x,v,v') \lesssim S(t,x+\epsilon v) + S(t,x-\epsilon v')$$

then we have global existence of weak solutions.

- $d = 2$ [Hwang, Kang, Stevens 2005]

If

$$\begin{aligned} T[S](t,x,v,v') \lesssim S(t,x+\epsilon v) + |\nabla S(t,x+\epsilon v)| \\ + S(t,x-\epsilon v') + |\nabla S(t,x-\epsilon v')| \end{aligned}$$

then we have global existence of weak solutions.

Using dispersion and Strichartz estimates we can show:

Theorem

$d = 3$, $\|f_0\|_{L^{3/2}_{x,v}}$ small and

$$T[S](t,x,v,v') \lesssim S(t,x+v) + |\nabla S(t,x+v)| \\ + S(t,x-v') + |\nabla S(t,x-v')|$$

$\Longrightarrow$ *global existence of weak solutions.*

Theorem

$d = 3$, large data and

$T[S](t,x,v,v') \lesssim S(t,x+v) + S(t,x-v') + |\nabla S(t,x+v)|$.

$\Longrightarrow$ *global existence of weak solutions.*

Sketch of the proof of the first Theorem

By Strichartz:

$$\|f\|_{L^r_t L^p_x L^q_v} \lesssim \|f_0\|_{L^a_{x,v}} + \|RHS\|_{L^{r'}_t L^q_x L^p_v}$$

$$\begin{aligned} RHS(t,x,v) &= \int |\nabla S(t,x-v')|\, f(t,x,v')dv' + \cdots \\ &\leq \|\nabla S(t,x-v')\|_{L^{q'}_{v'}} \|f(t,x,v')\|_{L^q_{v'}} + \cdots \end{aligned}$$

Take the $L^q_x L^p_v$ norm of both sides:

$$\|RHS(t,x,v)\|_{L^q_x L^p_v} \lesssim \|\nabla S(t,x-v')\|_{L^A_x L^{q'}_{v'}} \|f(t,x,v')\|_{L^p_x L^q_{v'}}$$

$$
\begin{aligned}
\|\nabla S(t,x-v')\|_{L^A_x L^{q'}_{v'}} &\le \|\nabla S(t,x-v')\|_{L^{q'}_{v'} L^A_x} \\
&\le \|\nabla S(t,\cdot)\|_{L^A_x} \\
&\le \|\rho(t,\cdot)\|_{L^p_x} \\
&\le \|f(t,x,v)\|_{L^p_x L^q_v}
\end{aligned}
$$

therefore

$$
\|RHS(t,x,v)\|_{L^p_x L^q_v} \lesssim \|f(t,x,v')\|^2_{L^p_x L^q_{v'}}
$$

Take $L^{r'}_t$

$$
\|RHS(t,x,v)\|_{L^{r'}_t L^p_x L^q_v} \lesssim \left\| \|f(t,x,v')\|^2_{L^p_x L^q_{v'}} \right\|_{L^{r'}_t} = \|f\|^2_{L^r_t L^p_x L^q_v}
$$

Sketch of the proof of the second Theorem

$$
\begin{aligned}
& f(t, x, v) \\
& = \int_0^t \int |\nabla S(t-s, x-sv+v)|\, f(t-s, x-sv, v') dv' ds + \cdots \\
& = \int_0^t |\nabla S(t-s, x-(s-1)v)|\, \rho(t-s, x-(s-1)v-v) ds + \cdots
\end{aligned}
$$

Apply the dispersion estimate with

$$
1 \leq q \leq p \leq \frac{3}{2} \quad , \quad \alpha := 3\left(\frac{1}{q} - \frac{1}{p}\right) < 1
$$

$$\|f\|_{L^p_x L^q_v} \leq$$

$$\int_0^t \|\nabla S(t-s, x-(s-1)v)\rho(t-s, x-(s-1)v-v)\|_{L^p_x L^q_v} \, ds + \cdots$$

$$\leq \int_0^t \frac{1}{|s-1|^\alpha} \|\nabla S(t-s, x)\rho(t-s, x-v)\|_{L^q_x L^p_v} \, ds + \cdots$$

$$= \int_0^t \frac{1}{|s-1|^\alpha} \|\nabla S(t-s, \cdot)\|_{L^q} \|\rho(t-s, \cdot)\|_{L^p} \, ds + \cdots$$

$$\leq M \int_0^t \frac{1}{|s-1|^\alpha} \|f(t-s, x, v\cdot)\|_{L^p_x L^q_v} \, ds + \cdots$$

Turning Kernels without the $x + v$, $x - v'$ structure

$d = 3$, we have global existence in each of the following cases:

- $T[S](t, x, v, v') \lesssim 1 + \|S(t, \cdot)\|_{L^\infty}^{1-\epsilon}$
- $T[S](t, x, v, v') \lesssim 1 + \|S(t, \cdot)\|_{L^\infty}$ and $\|f_0\|_{L^{3/2}_{x,v}}$ is small
- $T[S](t, x, v, v') \lesssim 1 + \|S(t, \cdot)\|_{L^p}^{\alpha}$ where $0 < \alpha < \frac{p}{p-3}$
- $T[S](t, x, v, v') \leq 1 + \|S(t, \cdot)\|_{L^\infty}$ still open

$d = 2$, we have global existence in each of the following cases:

- $T[S](t,x,v,v') \lesssim 1 + \|S(t,\cdot)\|_{L^\infty}^{\alpha} + \|\nabla S(t,\cdot)\|_{L^\infty}^{1-\epsilon} \quad , \quad \alpha > 0,\ 0 < \epsilon \leq 1$

- $T[S](t,x,v,v') \lesssim 1 + \exp\left(\|S(t,\cdot)\|_{L^\infty}^{\beta}\right)$ and $0 < \beta < 1$

- $\beta = 1$ and M small ($M < \pi$)

Sketch of the proof of ...

$$d = 3, \; T[S](t,x,v,v') \lesssim 1 + \|S(t,\cdot)\|_{L^\infty}^{\alpha}$$

$$f(t,x,v) = \int_0^t \|S(t-s,\cdot)\|_{L^\infty}^{\alpha} \rho(t-s, x-sv) ds + \cdots$$

$$\|f(t,x,v)\|_{L_x^p L_v^q} \lesssim \int_0^t \|S(t-s,\cdot)\|_{L^\infty}^{\alpha} \|\rho(t-s,\cdot)\|_{L^q} \frac{ds}{s^{3(1/q-1/p)}}$$

$$\|S(t,\cdot)\|_{L^\infty} \leq C(M) \|\rho(t,\cdot)\|_{L^p}^{p'/3}$$

$$\|\rho(t,\cdot)\|_{L^q} \leq C(M) \|\rho(t,\cdot)\|_{L^p}^{p'/q'}$$

$$\|f(t,x,v)\|_{L^p_x L^q_v} \le \int_0^t \|\rho(t-s,\cdot)\|_{L^p}^{\alpha\frac{p'}{3}+\frac{p'}{q'}} \frac{ds}{s^{3(1/q-1/p)}} + \cdots$$

If $\alpha = 1$ then $3(1/q - 1/p) < 1 \iff \alpha\frac{p'}{3} + \frac{p'}{q'} > 1$, it doesn't work.

If $\alpha < 1$ we can find p, q such that

$$3(1/q - 1/p) < 1$$

and

$$\alpha\frac{p'}{3} + \frac{p'}{q'} = 1$$

Blow-up for the kinetic model

Let $d = 2$ and

$$T[S](t,x,v,v') := (v \cdot \nabla S(t,x))_+$$

or

$$T[S](t,x,v,v') = |\nabla S(t,x)| + v \cdot \nabla S(t,x)$$

(cells always choose 'good' directions)

In the spherically symmetric case we can show:

- If the mass is small we have global existence
- If the mass is large we have blow-up in finite time
- optimal mass?

Sketch of the proof of blow-up

$$\frac{d}{dt}\frac{1}{2}\iint |x|^2 f(t,x,v)dxdv = \iint (x\cdot v) f(t,x,v)dxdv$$

$$\begin{aligned}\frac{d^2}{dt^2}\frac{1}{2}\iint |x|^2 f(t,x,v)dxdv = & \\ & \iint |v|^2 f(t,x,v)dxdv \\ & + \ c\iint (x\cdot v)(v\cdot\nabla S(t,x))_+ \rho(t,x)dxdv \\ & - c\iint (x\cdot v)\left|\nabla S(t,x)\right| f(t,x,v)dxdv\end{aligned}$$

$$\iint |v|^2 f(t,x,v)\,dxdv \lesssim M$$

$$\iint (x\cdot v)(v\cdot\nabla S(t,x))_+\rho(t,x)\,dxdv = c\int (x\cdot\nabla S)\,\rho dx = -cM^2$$

$$\iint (x\cdot v)\,|\nabla S(t,x)|\,f(t,x,v)\,dxdv = \frac{dP}{dt},\quad \text{where}\quad P\geq 0$$

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\iint |x|^2 f(t,x,v)\,dxdv &\leq c + (cM - cM^2)t + c(P(0)-P(t))\\ &\leq c + (cM - cM^2)t + P(0)\end{aligned}$$

Global existence for small mass

We use a comparison argument.

$$\partial_t f + v \cdot \nabla_x f \leq (v \cdot \nabla S)_+ \rho$$

spherical symmetry implies: $\nabla S = \left(-\frac{1}{r} \int_0^r \lambda \rho(\lambda) d\lambda \right) \frac{x}{r}$

$$\partial_t f + v \cdot \nabla_x f \leq \frac{1}{r} \int_0^r \lambda \rho(\lambda) d\lambda \left(\frac{x \cdot v}{r} \right)_- \rho \leq \frac{M}{2\pi r} \left(\frac{x \cdot v}{r} \right)_- \rho$$

Find comparison function $k(x, v)$ such that

$$f_0(x, v) \leq k(x, v) \ , \ v \cdot \nabla_x k \geq \frac{M}{2\pi r} \left(\frac{x \cdot v}{r} \right)_- \int k dv$$

If M is small then $f(t, x, v) \leq k(x, v)$ for all t.

Hyperbolic models of chemotaxis

model proposed by Dolak and Hillen, 2003

$$
\begin{aligned}
&\partial_t \rho + \nabla \cdot j = 0\\
&\partial_t j + \nabla \rho = \rho \nabla S - j\\
&-\Delta S = \rho
\end{aligned}
$$

[Natalini, Di Russo, 2010] Local existence of classical solutions, Global existence of small solutions, initial data in $H^s(\mathbb{R}^d)$, $s \geq d/2 + 1$

Proof uses old method of Klainerman combined with recent estimates of [Natalini et al] for dissipative equations.

Can we improve?

Nonlinear Wave equation

$$\Box\rho + \rho_t = -\nabla\left(\rho\nabla S\right)$$
$$\rho(0) \in H^s(\mathbb{R}^d),\ \rho_t(0) \in H^{s-1}(\mathbb{R}^d)$$

Scaling shows that critical s is $\frac{d-4}{2}$.

Can we prove local or global well-posedness at the critical level?

Low-regularity LWP is related to GWP.

Example: for some nonlinear wave equations with power nonlinearities $\Box u = F(u) \simeq u^p$ it is possible to prove GWP by proving LWP in H^1 (energy class) and using the fact that the energy is conserved

At the critical regularity, if we can prove LWP then we can usually also prove GWP for small data.

$$\|u\|_X \lesssim D_0 + T^\alpha \|u\|_X^N$$

if $s >$ *critical* then $\alpha > 0$, so assume small T to bootstrap, hence LWP.
if $s =$ *critical* then $\alpha = 0$, so assume small D_0 to bootstrap, hence GWP.
for LWP:

$$\|u\|_{X([0,T]\times\mathbb{R}^d)} \lesssim \|u_{hom}\|_{X([0,T]\times\mathbb{R}^d)} + \|u\|_{X([0,T]\times\mathbb{R}^d)}^N$$

so assume small T to get LWP.

$$\Box u = \nabla \cdot (u \nabla S)\ , \quad -\Delta S = u\ ,\ u(0) \in H^s,\ \partial_t u(0) \in H^{s-1}$$

Energy estimates:

$$\|u(t)\|_{H^s} \lesssim \|u(0)\|_{H^s} + \int_0^T \|\nabla \cdot (u \nabla S)\|_{H^{s-1}}\, dt$$

$$\|\nabla \cdot (u \nabla S)\|_{H^{s-1}} \lesssim \|u \nabla S\|_{H^s} \lesssim \|u\|_{H^s} \|\nabla S\|_{L^\infty} + \cdots$$

$$\|\nabla S\|_{L^\infty} \lesssim \|\nabla S\|_{H^{d/2+\epsilon}} \lesssim \|u\|_{H^{d/2-1+\epsilon}}$$

If $s > d/2 - 1 = \textit{critical} + 1$ we can bootstrap.

combine energy estimates + Strichartz estimates:

$$\Box u = F \ , \ u(0) = u_0 \ , \ u_t(0) = u_1$$

$$\|u\|_{L^q_t L^r_x} \lesssim \|u_0\|_{H^\mu} + \|u_1\|_{H^{\mu-1}} + \|F\|_{L^1_t H^{\mu-1}_x}$$

$$d \geq 2 \ , \ 2 \leq q \leq \infty, \ 2 \leq r < \infty, \mu \geq 0$$

$$\frac{2}{q} \leq (d-1)\left(\frac{1}{2} - \frac{1}{r}\right) \ , \ \frac{1}{q} + \frac{d}{r} = -\mu + \frac{d}{2}$$

$$\begin{aligned} \int_0^T \|u \nabla S\|_{H^s} \, dt &\leq T^{1/2} \left\|u \partial^{-1} u\right\|_{L^2_t H^s_x} \\ &\leq T^{1/2} \|u\|_{L^\infty_t H^s_x} \left\|\partial^{-1} u\right\|_{L^2_t L^\infty_x} + \cdots \end{aligned}$$

$$\begin{aligned} \left\|\partial^{-1} u\right\|_{L^2_t L^\infty_x} &\lesssim \left\|\partial^\epsilon \partial^{-1} u\right\|_{L^2_t L^r_x} \quad (\epsilon > d/r) \\ &\lesssim \left\|\partial^\epsilon \partial^{-1} u(0)\right\|_{H^\mu} + \cdots \end{aligned}$$

$$\epsilon - 1 + \mu = \cdots = \mathit{critical} + \frac{1}{2} + \epsilon'$$

$X^{s,b}$ spaces (Bourgain, Klainerman and Machedon, KPV,...)

$$\|u\|_{X^{s,b}} = \left\| w_-(\tau,\xi)^b w_+(\tau,\xi)^s \ \widetilde{u}(\tau,\xi) \right\|_{L^2_{\tau,\xi}}$$

$$w_\pm(\tau,\xi) = 1 + ||\tau| \pm |\xi||$$

Energy estimates become:

$$\left\| \Box^{-1} F \right\|_{X^{s,b}} \lesssim \|F\|_{X^{s-1,b-1}}$$

Strichartz estimates become embeddings:

$$\|u\|_{L^q_t L^r_x} \lesssim \|u\|_{X^{s,b}}$$

(Almost) all product estimates are now known:

$$\|uv\|_{X^{s_1,b_1}} \lesssim \|u\|_{X^{s_2,b_2}} \|v\|_{X^{s_3,b_3}}$$

[D'Ancona, Foschi and Selberg, 2012]

By 'energy estimate':

$$\|u\|_{X^{s,b}} \lesssim D_0 + \left\|\partial(u\partial^{-1}u)\right\|_{X^{s-1,b-1}} = D_0 + \left\|u\partial^{-1}u\right\|_{X^{s,b-1}}$$

Want

$$\left\|u\partial^{-1}u\right\|_{X^{s,b-1}} \lesssim \|u\|^2_{X^{s,b}}$$

which is the same as

$$\|uv\|_{X^{s,b-1}} \lesssim \|u\|_{X^{s,b}} \|v\|_{X^{s+1,b}}$$

By [D'Ancona, Foschi and Selberg] this works provided that $s > \mathit{critical} + \frac{1}{4}$ (and $b = 3/4$).

GWP for initial data in $B^s_{2,1}$, $s = critical = \frac{n-4}{2}$, $n \geq 6$

Following [Tataru] we define, for $\lambda = 2^k$, $k \in \mathbb{Z}$

$$\widetilde{S_\lambda u}(\tau,\xi) = s_\lambda(\tau,\xi)\widetilde{u}(\tau,\xi), \;\; s_\lambda(\tau,\xi) \;\text{ adapted to }\; |\tau| + |\xi| \simeq \lambda$$

$$\widetilde{P_\lambda u}(\xi) = p_\lambda(\xi)\widetilde{u}(\tau,\xi), \;\; p_\lambda(\xi) \;\text{ adapted to }\; |\xi| \simeq \lambda$$

$$\widetilde{C_\lambda u}(\tau,\xi) = c_\lambda(\tau,\xi)\widetilde{u}(\tau,\xi), \;\; c_\lambda(\tau,\xi) \;\text{ adapted to }\; ||\tau| - |\xi|| \simeq \lambda$$

For fixed spacetime frequency λ let:

$$\|u\|_{X^{1/2}_{\lambda,1}} = \sum_d d^{1/2} \|S_\lambda C_d u\|_{L^2_{t,x}}$$

$$\|u\|_{Y_\lambda} = \lambda^{-1} \|\Box S_\lambda u\|_{L^1_t L^2_x}$$

$$\|u\|_{F_\lambda} = \|S_\lambda u\|_{L^\infty_t L^2_x} + \|u\|_{X^{1/2}_{\lambda,1} + Y_\lambda}$$

Define

$$\|u\|_{F^{s,p}} = \left(\sum_{\lambda} \lambda^{sp} \|u\|_{F_\lambda}^{p} \right)^{1/p}$$

Frequency localized Strichartz:

$$\left\| |\nabla|^{-\sigma} e^{\pm it|\nabla|} P_\lambda f \right\|_{L_t^q L_x^r} \lesssim \lambda^{\gamma-\sigma} \|P_\lambda f\|_{L_x^2}$$

$$\left\| |\nabla|^{-\sigma} e^{\pm it|\nabla|} P_{\leq\lambda} f \right\|_{L_t^q L_x^r} \lesssim \lambda^{\gamma-\sigma} \|P_{\leq\lambda} f\|_{L_x^2}$$

$$n \geq 4, 2 \leq q, r \leq \infty, \ \frac{1}{q} \leq \frac{d-1}{2}\left(\frac{1}{2} - \frac{1}{r}\right)$$

$$\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - \gamma, \ \sigma \leq \gamma, \ (\sigma < \gamma \ if \ r = \infty)$$

F_λ controls $L_t^q L_x^r$:

$$\left\| |\nabla|^{-\sigma} S_\lambda u \right\|_{L_t^q L_x^r} \lesssim \lambda^{\gamma-\sigma} \left\| u \right\|_{F_\lambda}$$

For proving existence we need something like:

$$\left\| \Box^{-1} \partial \left(u \partial^{-1} v \right) \right\|_{F^{s,1}} \lesssim \left\| u \right\|_{F^{s,1}} \left\| v \right\|_{F^{s,1}}$$

Write $u = \sum_{\mu_1} S_{\mu_1} u$, $v = \sum_{\mu_2} S_{\mu_2} v$.
We need to estimate

$$\left\| \Box^{-1} \partial \left(S_{\mu_1} u \partial^{-1} S_{\mu_2} v \right) \right\|_{F_\lambda}$$

Three cases: $\mu_1 << \mu_2$, $\mu_1 \simeq \mu_2$, $\mu_1 >> \mu_2$.

For $\mu_1 << \mu_2$:

$$\begin{aligned}\left\|\Box^{-1}\nabla\left(S_{\mu_1}u|\nabla|^{-1}S_{\mu_2}v\right)\right\|_{Y_\lambda} &\lesssim \left\|S_{\mu_1}u|\nabla|^{-1}S_{\mu_2}v\right\|_{L^1_tL^2_x}\\ &\lesssim \left\|S_{\mu_1}u\right\|_{L^2_tL_x^{\frac{2n}{3}}}\left\||\nabla|^{-1}S_{\mu_2}v\right\|_{L^2_tL_x^{\frac{2n}{2n-3}}}\\ &\lesssim \mu_1^{\frac{n-4}{2}}\left\|u\right\|_{F_{\mu_1}}\left\|v\right\|_{F_{\mu_2}}\end{aligned}$$

$$\begin{aligned}&\|S_\lambda\Box^{-1}\underbrace{\nabla\left(S_{\mu_1}u|\nabla|^{-1}S_{\mu_2}v\right)}_{F}\|_{L^\infty_tL^2_x}\\ &\lesssim \lambda^{-1}\left(\|S_\lambda F\|_{L^1_tL^2_x}+\|P_\lambda F\|_{L^1_tL^2_x}\right)\\ &\lesssim \mu_1^{\frac{n-4}{2}}\left\|u\right\|_{F_{\mu_1}}\left\|v\right\|_{F_{\mu_2}}\end{aligned}$$

relevant only when $\lambda \lesssim \mu_2$.

$$\sum_{\lambda} \lambda^{s_c} \sum_{\mu_2 \gtrsim \lambda} \sum_{\mu_1 \lesssim \mu_2} \mu_1^{s_c} \|u\|_{F_{\mu_1}} \|v\|_{F_{\mu_2}}$$

$$\lesssim \|u\|_{F^{s,1}} \sum_{\mu_2} \|v\|_{F_{\mu_2}} \sum_{\lambda \lesssim \mu_2} \lambda^{s_c}$$

$$\lesssim \|u\|_{F^{s_c,1}} \sum_{\mu_2} \|v\|_{F_{\mu_2}} \mu_2^{s_c}$$

$$\lesssim \|u\|_{F^{s_c,1}} \|v\|_{F^{s_c,1}}$$

If $\mu_1 \simeq \mu_2$ then

$$\begin{aligned}\left\|\Box^{-1}\partial\left(S_{\mu_1}u\partial^{-1}S_{\mu_2}v\right)\right\|_{Y_\lambda} &\lesssim \left\|S_{\mu_1}u\partial^{-1}S_{\mu_2}v\right\|_{L_t^1L_x^2}\\ &\lesssim \left\|S_{\mu_1}u\right\|_{L_t^2L_x^4}\left\|\partial^{-1}S_{\mu_2}v\right\|_{L_t^2L_x^4}\\ &\lesssim \mu_1^{\frac{n-2}{4}}\mu_2^{\frac{n-6}{4}}\left\|u\right\|_{F_{\mu_1}}\left\|v\right\|_{F_{\mu_2}}\\ &= \mu_1^{s_c}\left\|u\right\|_{F_{\mu_1}}\left\|v\right\|_{F_{\mu_2}}\end{aligned}$$

When we sum up we find:

$$\sum_\lambda \lambda^{s_c}\sum_{\mu_1\simeq\mu_2\gtrsim\lambda}\mu_1^{s_c}\left\|u\right\|_{F_{\mu_1}}\left\|v\right\|_{F_{\mu_2}} \lesssim$$
$$\left\|u\right\|_{F^{s_c,1}}\sum_{\mu_2}\sum_{\lambda\lesssim\mu_2}\lambda^{s_c}\left\|v\right\|_{F_{\mu_2}} \lesssim \left\|u\right\|_{F^{s_c,1}}\left\|v\right\|_{F^{s_c,1}}$$

In the remaining case $\mu_1 >> \mu_2$ we need to decompose near/away from the cone $|\tau| = |\xi|$.

Blow-up for $d = 2$, large mass.

Let $I(t) = \frac{1}{2}\int |x|^2 \rho dx$. Then

$$\frac{dI}{dt} = \int x \cdot j$$

$$\begin{aligned}\frac{d^2 I}{dt^2} &= \int x \cdot (-\nabla\rho + \rho\nabla S - j)dx \\ &= cM - c'M^2 - \frac{dI}{dt}\end{aligned}$$

$$cM - c'M^2 < 0 \Rightarrow I(t) < 0 \;\text{ in finite time}$$

Therefore, positive solutions of large mass cannot exist globally.

Open problems

- GWP for $n \leq 5$?
- Spherical symmetry?
- Is there a critical mass in 2 dimensions?
- Other hyperbolic models.