About Maxwell Boltzmann Equation.

Claude Bardos

Retired
claude.bardos@gmail.com

The Cauchy Problem in Kinetic Theory. Imperial College September 2015.

Report on an ongoing project with Toan Nguyen and several coworkers: Irene Gamba, Francois Golse, Claudia Negulescu and Rémi Sentis.

September 9, 2015

Figure: Left: Ludwig Boltzmann (1863-1906); right: James Clerk Maxwell(1831-1879).

What is the Maxwell Boltzmann Equation ? I

Two type of particles: Electrons and Ions under the action of electromagnetic forces and inter particles collisions:

$$
\begin{aligned}
&F = \begin{pmatrix} F_+ \\ F_- \end{pmatrix} \; F \mapsto \mathcal{C}(F) = \begin{pmatrix} C^+(F_+, F_-) \\ C^-(F_+, F_-) \end{pmatrix} \\
&= \begin{pmatrix} C^+_{+,+}(F_-+, F_+) + C^+_{+,-}(F_+, F_-) \\ C^-_{-,+}(F_-+, F_+) + C^-_{-,-}(F_-, F_-) \end{pmatrix} \\
&\partial_t F_+ + v \cdot \nabla_x F_+ + \frac{e_+}{m_+}(E + \frac{v \times B}{c}) \nabla_v F_+ = C^+(F_+, F_-) \\
&\partial_t F_- + v \cdot \nabla_x F_- - \frac{e_-}{m_-}(E + \frac{v \times B}{c}) \nabla_v F_- = C^-(F_+, F_-)
\end{aligned}
\tag{1}
$$

What is the Maxwell Boltzmann Equation ? II

Assume time independent given B .
In an open set $\Omega \subset \mathbb{R}^d_x$ Maxwell equations $\Rightarrow$.

$$\nabla \cdot B = 0, \quad B \times \vec{n}_{|\partial\Omega} = 0 \Rightarrow B = \nabla \vec{A}$$

$$\nabla \times E = 0 \Rightarrow \nabla \phi \,, \quad E \cdot \vec{n}_{|\partial\Omega} = 0 \Rightarrow \partial_{\vec{n}} \phi = 0$$

$$-\Delta \phi = e_+ \int_{\mathbf{R}^d_v} F_+(x,v,t)dv - e_- \int_{\mathbf{R}^d_v} F_-(x,v,t)dv$$

$$e_+ \int_\Omega \int_{\mathbf{R}^d_v} F_+(x,t)dx = e_- \int_\Omega \int_{\mathbf{R}^d_v} F_-(x,t)dvdx \,.$$

Assume time independent given B.
In an open set $\Omega \subset \mathbb{R}^d_x$ Maxwell equations $\Rightarrow$.

$$\nabla \cdot B = 0, \quad B \times \vec{n}_{|\partial\Omega} = 0 \Rightarrow B = \nabla \vec{A}$$

$$\nabla \times E = 0 \Rightarrow \nabla\phi\,, \quad E \cdot \vec{n}_{|\partial\Omega} = 0 \Rightarrow \partial_{\vec{n}}\phi = 0$$

$$-\Delta\phi = e_+ \int_{\mathbf{R}^d_v} F_+(x,v,t)dv - e_- \int_{\mathbf{R}^d_v} F_-(x,v,t)dv$$

$$e_+ \int_\Omega \int_{\mathbf{R}^d_v} F_+(x,t)dx = e_- \int_\Omega \int_{\mathbf{R}^d_v} F_-(x,t)dvdx\,.$$

Assume also that the electrons are thermalized then the system which couples the potential and the ions becomes:

$$\partial_t F_+ + v \cdot \nabla_x F_+ + \frac{e_+}{m_+}(E + \frac{v \times B}{c})\nabla_v F_+ = C^+_{+,+}(F_-+, F_+) \tag{2}$$

$$-\Delta\phi = \int_{\mathbf{R}^3_v} F_+(x,v,t)dv - \int_{\mathbf{R}^3_v} F_-(x,v,t)dv$$

What is the Maxwell Boltzmann Equation ? III

The Maxwell Boltzmann equation consists in replacing $\int_{\mathbf{R}^3_v} F_-(x,v,t)dv$ by a convenient function of ϕ (in particular $e^{\beta\phi}$).
Introduce ϵ related to the facts that the media is rarified and that the collision operator (Boltzmann, or Fokker-Planck) is quadratic in term of the density of particles. and take all the other the physical constants equal to 1 . Then in the system for electrons: two almost equivalent approaches

1 ϵ Small enough but fixed and $t \to \infty$.

2 Consider the equation with $\alpha > 1$ and $\epsilon \to 0$:

$$\begin{aligned} &\epsilon^\alpha \partial_t F_\epsilon + v \cdot \nabla_x F_\epsilon + (\nabla_x \phi_\epsilon + B \times v)\nabla_v F_\epsilon = \epsilon C(F_\epsilon, F_\epsilon)\,, \\ &-\Delta \phi_\epsilon = I_{\mathrm{ions}}(x) - \int_{\mathbb{R}^3_v} F_\epsilon(x,v,t)dv\,. \end{aligned} \tag{3}$$

Comparison with the Boltzmann equation in a Bounded Domain Ω

$$\partial_t F + v\cdot\nabla_x F = C(F)\,,\quad F(x,v,t) = F(x,v^*,t)\quad \text{on } \partial\Omega\,,$$

$$F(x,v,t)\to m_\infty\left(\frac{\beta_\infty}{2\pi}\right)^{\frac{d}{2}}\exp\left(-\beta_\infty\frac{|v-u_\infty|^2}{2}\right).$$

- With convenient hypothesis on Ω (m_∞,β_∞) are uniquely defined among the stationary solutions by the conservation of mass and energy.
- Convergence and rate of convergence are proven only under extra assumptions or small fluctuation near an absolute maxwellian.

Stationary Solutions for Maxwell Boltzmann with boundary and conserved quantities:

Below will be used the conservation of mass energy H theorem::

$$\partial_t F + v\cdot\nabla_x F + (\nabla\phi + B\times v)\nabla_v F = \epsilon C(F)\,,$$
$$x\in\partial\Omega \Rightarrow F(x,v,t) = F(x,v^*,t)$$
$$-\Delta\phi = I(x) - \int_{\mathbb{R}^d_v} F(x,v,t)dv\,, x\in\partial\Omega \Rightarrow \partial_{\vec{n}}\phi = 0$$

$$\int_{\mathbb{R}^d_v\times\Omega} F(x,v,t)dvdx = \int_\Omega I(x)dx = m_0 \quad \text{Neutrality}$$
$$\int_{\mathbb{R}^d_v\times\Omega} \frac{|v|^2}{2} F(x,v,t)dvdx + \int_\Omega \frac{|\nabla\phi(x)|^2}{2}dx = \mathcal{E}_0$$
$$\frac{d}{dt}\int_{\mathbb{R}^d_v\times\Omega} F\log F dvdx - \int_{\mathbb{R}^d_v\times\Omega} C(F)\log F dvdx = 0$$
$$D(F)=0 \Rightarrow \quad F(v) = \rho\Big(\frac{\beta}{2\pi}\Big)^{\frac{d}{2}} e^{-\beta\frac{|v-u|^2}{2}} = \Big(\frac{\beta}{2\pi}\Big)^{\frac{d}{2}} e^{-\beta\big(\frac{|v-u|^2}{2} - \frac{\log\rho}{\beta}\big)}\,.$$

Theorem

For any given magnetic field B, density of ions $I(x)$ and $\beta > 0$ given the non linear elliptic equation:

$$in \quad \Omega \quad -\Delta\phi + e^{\beta\phi(x)} = I(x) \quad on \quad \partial\Omega \quad \partial_{\vec{n}}\phi = 0 \tag{4}$$

has a unique solution which with

$$F(x,v) = (\frac{\beta}{2\pi})^{\frac{d}{2}} e^{-\beta(\frac{|v|^2}{2} - \phi(x))}$$

provides a solution of the system:

$$v \cdot \nabla_x F + (\nabla_x \phi - v \times B)\nabla_v F - C(F,F) = 0\,, \tag{5a}$$

$$In \quad \Omega \quad -\Delta\phi = I(x) - \int_{\mathbb{R}^3_v} F(x,v,t)dv\,, \tag{5b}$$

$$And\ on\ \partial\Omega \quad F(x,v,t) = F(x,v^*,t)\,, \quad \partial_{\vec{n}}\phi = 0\,. \tag{5c}$$

Proof

Since the mapping $\phi \mapsto \exp(\beta\phi)$ is strictly monotone increasing the existence and uniqueness of a solution of (5b) with Neumann boundary condition is ensured. Then with

$$\int_{\mathbb{R}^d_v} (B \times v)\nabla_v F(v)dv = -\int_{\mathbb{R}^d_v} (B \times v) \cdot vF(v)dv = 0\,,$$

$F(x, v, t)$ is a solution of (4) with density

$$\int_{\mathbb{R}^3_v} F(x, v, t)dv = e^{\beta\phi(x)}\,.$$

Uniqueness of the solution in a simply connected open set of $\Omega \subset \mathbb{R}^d$.

Theorem

In general all stationary solutions are of the form

$$F(x,v) = (\frac{\beta}{2\pi})^{\frac{d}{2}} e^{-\beta(\frac{|v|^2}{2} - \phi(x))}$$

with β uniquely determined by the Energy:

$$\mathcal{E} = \frac{1}{2}\int_{\Omega\times\mathbb{R}^d} |v|^2 F(x,v)dxdv + \frac{1}{2}\int_{\Omega} |\nabla_x \phi(x)|^2 dx\,.$$

Remark

In general the magnetic field does not appear in the Maxwell -Boltzmann relation but will appear in the equation for the Ions.

For proof recall the "strong " Korn Inequality

For any smooth vector $u : \Omega \mapsto \mathbb{R}^d$ with $u \cdot n = 0$ one has

$$\| \frac{\nabla u + \nabla u^t}{2} \|_{L^2(\Omega)} \geq \overline{K}(\Omega) \| \nabla u \|^2_{L^2(\Omega)} \tag{6}$$

is generally valid.

It holds for the torus and in dimension 2 and 3 if Ω has no axis of symmetry.

For proof recall the "strong " Korn Inequality

For any smooth vector $u : \Omega \mapsto \mathbb{R}^d$ with $u \cdot n = 0$ one has

$$\|\frac{\nabla u + \nabla u^t}{2}\|_{L^2(\Omega)} \geq \overline{K}(\Omega)\|\nabla u\|^2_{L^2(\Omega)} \tag{6}$$

is generally valid.

It holds for the torus and in dimension 2 and 3 if Ω has no axis of symmetry.

Cf. Proposition 13. of Desvillettes and Villani. Invent. Math. 159 (2005) and Desvillettes and Villani ESAIM: Control, Optimisation and Calculus of Variations June 2002, Vol. 8.

Proof I

Multiplication of the equation

$$v\cdot\nabla_x F + (\nabla\phi + B\times v)\nabla_v F - C(F,F) = 0$$

by $\log F$ and integrating over $\Omega\times\mathbb{R}^d_v$ shows that F is a Maxwellan

$$F(x,v) = \rho(x)(\frac{\beta(x)}{2\pi})^{\frac{d}{2}} e^{-\beta(x)(\frac{|v-u(x)|^2}{2})}$$

$$\Rightarrow C(w) = 0,\ \text{and}\ v\cdot\nabla_x F - (E + v\times B)\nabla_v F = 0$$

$$\nabla_x F = F((\frac{\nabla_x\rho(x)}{\rho(x)} - \frac{d}{2}\frac{\nabla_x\beta}{\beta}) + \beta(x)\nabla_x u\cdot(v-u(x))$$

$$-\frac{|v-u(x)|^2}{2}\frac{\nabla_x\beta}{\beta}.$$

$$\nabla_v F(x,v,t) = -\beta(v-u))F$$

$$\Rightarrow (B\times v)\nabla_v F = -\beta B\times v\cdot(v-u)F = -\beta(B\times u)(v-u)F$$

Proof II Identification of the Terms of Order $3, 2, 1, 0$ in $(v-u)$.

Order 3 $(v-u)|v-u|^2 \quad \Rightarrow \nabla_x \beta = 0$,
Order 2 $(v-u)\nabla_x u \cdot (v-u) \quad \Rightarrow \frac{\nabla_x u + \nabla_x u^t}{2} = 0$ With Korn inequality
$\Rightarrow \nabla u = 0$,
Order 1 $(v-u) \Rightarrow \quad \Rightarrow (\frac{\nabla_x \rho(x,)}{\rho(x)} - \beta(\nabla\phi + B \wedge u)) = 0$
Order 0 $u \cdot \nabla_x \log \rho(x) = 0$.
With $\nabla\beta = 0$ and $u = 0$ (u constant and tangent to the boundary) only remain the equations:

$$\nabla \log(\rho(x)) = \beta(\phi(x) \Rightarrow F(x,v) = (\frac{\beta}{2\pi})^{\frac{d}{2}} e^{-\beta(\frac{|v|^2}{2} - \phi(x))}$$

with ϕ changed into $\phi + (\log \mathrm{constante})/\beta$ and solution of

$$-\Delta\phi + e^{\beta\phi} = I(x)\,, \quad \partial_{\vec{n}}\phi = 0 \quad \text{on } \partial\Omega\,.$$

Proof III Existence and Uniqueness with constraints

With $\mathcal{E}_0$ given introduce the sets : $K_{\mathcal{E}_0}$ (resp $\partial K_{\mathcal{E}_0}$) of $(x,v) \mapsto F(x,v)$ with the constraints:

$$\mathcal{E}(F) = \frac{1}{2}\Big(\int_{\Omega\times\mathbb{R}^d_v} |v|^2 F(x,v)dxdv + \int_\Omega |\nabla\phi|^2 dx\Big)$$

$$\leq \text{(resp. } =)\mathcal{E}_0\,,$$

$$\int_\Omega \int_{\mathbb{R}^d_v} F(x,v)dxdv = \int_\Omega I(x)dx\,.$$

$$\int_{\Omega\times\mathbb{R}^d_v} F(x,v)\log F(x,v)dxdv < \infty \text{ on } \partial\Omega \quad F(x,v) = F(x,v^*)\,.$$

$$\text{In } \Omega \quad -\Delta\phi + \int_{\mathbb{R}^d_v} F(x,v)dv = I(x)\,, \text{on } \partial\Omega \quad \partial_{\vec{n}}\phi = 0\,.$$

For the L^1 topology this set is closed convex and non empty.

Proof IV

On the set $K_{\mathcal{E}_0}$ the convex function

$$\mathcal{H}(F) = \int_{\Omega\times\mathbb{R}^d_v} F(x,v)\log F(x,v)dxdv$$

has only one minimum say $\overline{F}$ characterized with not saturated constrain is $(\mathcal{E}(\overline{F}) < \mathcal{E}_0)$ or with saturated constraint $(\mathcal{E}(\overline{F}) = \mathcal{E}_0)$ by.

$$\begin{aligned}\int_{\Omega\times\mathbb{R}^d_v} \nabla\overline{F}(x,v)\delta F)(x,v)dxdv = \\ \int_{\Omega\times\mathbb{R}^d_v}(1+\log\overline{F}(x,v))\delta F(x,v)dxdv = 0\end{aligned} \tag{8a}$$

$$\begin{aligned}&\int_{\Omega\times\mathbb{R}^d_v} \nabla\overline{F}(x,v)\delta F(x,v)dxdv = \\ &\int_{\Omega\times\mathbb{R}^d_v}(1+\log\overline{F}(x,v)(x,v))\delta F(x,v)dxdv \\ &= \beta\int_{\Omega\times\mathbb{R}^d_v}\nabla\mathcal{E}(\overline{F}(x,v))\delta Fdxdv + \mu\int_{\Omega\times\mathbb{R}^d_v}\delta Fdxdv\end{aligned} \tag{8b}$$

Proof V Existence

With $\log \overline{F} = -1 + \mu$ the case $\beta \neq 0$ is excluded. For the saturated case with $\beta \neq 0$ one has:

$$\begin{aligned}\int_{\Omega\times\mathbb{R}^d_v} \nabla\overline{F}(x,v)\delta F(x,v)dxdv = \\ -\beta\Big(\int_{\Omega\times\mathbb{R}^d_v} \frac{|v|^2}{2}\delta F(x,v) + \int_\Omega \nabla_x\phi\nabla\delta\phi dx\Big)\end{aligned} \tag{9}$$

with $$-\Delta\phi = I(x) - \int_{\mathbb{R}^d_v} \overline{F}(x,v)dv \Rightarrow -\Delta\delta\phi = -\int_{\mathbb{R}^d_v}\delta F(x,v)dv \tag{10a}$$

$$\begin{aligned}&\int_{\Omega\times\mathbb{R}^d_v}(1+\log\overline{F}(x,v))\delta F(x,v)dxdv = \\ &-\beta\Big(\int_\Omega\Big(\int_{\mathbb{R}^d_v}\frac{|v|^2}{2}\delta F(x,v) - \overline{\phi}(x)\Big(\int_{\mathbb{R}^d_v}\delta F(x,v)dv\Big)dx\Big) \\ &+\mu\int_{\Omega\times\mathbb{R}^d}\delta F(x,v)dxdv\,.\end{aligned} \tag{10b}$$

Proof of Existence, End

And therefore with $\overline{\phi}$ changed in $\overline{\phi} - 1 + \frac{\log \mu}{\beta}$:

$$\overline{F}(x,v) = e^{-\overline{\beta}(\frac{|v|^2}{2} - \overline{\phi}(x))}$$
$$\int_{\Omega \times \mathbb{R}^d_v} F(x,v)dxdv = \int_\Omega I(x)dx$$
$$-\Delta\phi + e^{\beta\phi} = I(x)\,.$$

This proves the existence of a stationary solution for prescribed density of ions and prescribed energy.

Proof of Uniqueness: Under constraints Stationary solutions coincides with Entropy minimizers

For any solution of the form

$$F(x,v) = \left(\frac{\beta}{2\pi}\right)^{\frac{d}{2}} e^{-\beta(\frac{|v|^2}{2} - \phi(x))}$$

(with given density and energy) one has for energy and entropy:

$$\begin{aligned}
(\nabla\mathcal{E}(F),\delta F) &= \int_\Omega\left(\left(\int_{\mathbb{R}^d_v}\frac{|v|^2}{2}\delta F dv\right) + \nabla_x\phi\nabla_x\delta\phi\right)dx \\
&= \int_{\Omega\times\mathbb{R}^d_v}\left(\frac{|v|^2}{2} - \phi(x)\right)\delta F(x,v)dxdv.
\end{aligned} \tag{11a}$$

$$\begin{aligned}
(\nabla\mathcal{H}(F),\delta F) &= \int_{\Omega\times\mathbb{R}^d_v}(1+\log F(x,v)\delta F(x,v)dxdv \\
&= -\beta\int_{\Omega\times\mathbb{R}^d_v}\left(\frac{|v|^2}{2} - \phi\right)\delta F(x,v)dxdv.
\end{aligned} \tag{11b}$$

Hence $\quad \nabla\mathcal{H}(F) = -\beta\nabla\mathcal{E}(F)\,.$

Assume the existence besides the minimun $\overline{F}$ of an other solution $F \in \partial K_{\overline{\mathcal{E}}}$. Introduce the function: $F_t : [0,1] \mapsto \overline{F} + t(F - \overline{F}) \in K_{\overline{\mathcal{E}}}$.

$$\frac{d\mathcal{E}(F_t)}{dt} = (\nabla \mathcal{E}(F_t), F - \overline{F}) \text{ and } \frac{d\mathcal{H}(F_t)}{dt} = (\nabla \mathcal{H}(F_t), F - \overline{F}) \quad (12)$$

For $F_0 = \overline{F} \in \partial K_{\overline{\mathcal{E}}}$ and $F_1 = F \in \partial K_{\overline{\mathcal{E}}}$ uses the explicit form of the solutions to write:

$$\frac{d\mathcal{E}(F_t)}{dt}_{|t=0} = \int_{\Omega\times\mathbb{R}^d_v} (\frac{|v|^2}{2} - \overline{\phi}(x))(F - \overline{F})dxdv \le 0$$
$$\frac{d\mathcal{H}(F_t)}{dt}_{|t=0} = -\overline{\beta}\int_{\Omega\times\mathbb{R}^d_v} (\frac{|v|^2}{2} - \overline{\phi}(x))(F - \overline{F})dxdv \ge 0 \tag{13a}$$

$$\frac{d\mathcal{E}(F_t)}{dt}_{|t=1} = \int_{\Omega\times\mathbb{R}^d_v} (\frac{|v|^2}{2} - \phi(x))(F - \overline{F})dxdv \ge 0$$
$$\frac{d\mathcal{H}(F_t)}{dt}_{|t=1} = -\beta\int_{\Omega\times\mathbb{R}^d_v} (\frac{|v|^2}{2} - \phi(x))(F - \overline{F})dxdv \ge 0 \tag{13b}$$

The two equations (13a) at the minimum $\overline{F}$ contain no contradiction.
However for the two equations (13b) which result from the convexity of $K_{\overline{\mathcal{E}}}$ and of the entropy $\mathcal{H}$ contain a contradiction and therefore the uniqueness is proven.

Exemple of solutions with $u \neq 0$ The Flat Torus $\Omega = Q(\subset \mathbb{R}^2_{x_1,x_2}) \times (\mathbb{R}_{x_3}/\mathbb{Z})$

Consider the Vlasov equation with specular reflection on $\partial\Omega = \partial Q \times (\mathbb{R}_{x_3}/\mathbb{Z})$ with a given external potential magnetic potential

$$B = \nabla_x \times \vec{A} \quad \text{with} \quad \vec{A} = (0, 0, A(x_1, x_2)) \quad \text{and} \quad I(x) = I(x_1, x_2),$$
$$\partial_t F + v \cdot \nabla_x F + (\nabla\phi + B \times v) \cdot \nabla_v F = C(F)\,,$$
$$-\Delta\phi + \int_{\mathbb{R}^3_v} F(x, v, t)dv = I(x) \ \text{on}\, \Omega \,\text{and}\ \partial_{\vec{n}}\phi = 0 \ \text{on}\ \partial\Omega\,. \tag{14}$$

Beside the total density and the energy the dynamic (14) preserve the x_3 independence of the $F(x, v, t)$ and in this class support an extra invariant: The Axial Momentum $\mathcal{M}$.

Proposition

For the x_3 independent solutions of the dynamic given by (14) one has the following extra conserved quantity:

$$\frac{d}{dt}\int_{\Omega\times\mathbb{R}^3_v}(v_3-A)F(x,v,t)dxdv=0\,. \qquad (15)$$

Proof

$$\int_{\Omega\times\mathbf{R}^3_v}(B\times v)\cdot\nabla_v F v_3 dvdx=-\int_{\Omega\times\mathbf{R}^3_v}(v\cdot\nabla_x A)Fdvdx$$

$$\int_{\Omega\times\mathbb{R}^3_v}(v\cdot\nabla_x F+(\nabla\phi+B\times v)\cdot\nabla_v F)A(x)dxdv$$

$$=-\int_{\Omega\times\mathbf{R}^3_v}(v\cdot\nabla_x A)Fdvdx\,.$$

Theorem

In the above dynamic all the stationary solutions are given by:

$$F(x,v)=(\frac{\beta}{2\pi})^{\frac{d}{2}}e^{-\beta(\frac{|v-\vec{u}|^2}{2}-(\phi(x_1,x_2))-u_3A(x_1,x_2)))}$$

with β *and* $\vec{u}=(0,0,u_3)$ *uniquely determined by the Energy:*

$$\mathcal{E}=\frac{1}{2}\int_{\Omega\times\mathbb{R}^d}|v|^2F(x,v)dxdv+\frac{1}{2}\int_{\Omega}|\nabla_x\phi(x)|^2dx$$

and the Axial Momentum :

$$\int_{\Omega\times\mathbb{R}^3_v}(v_3-A)F(x,v)dxdv=\mathcal{M}_3$$

Proof

Adapted from the previous case .

Start from a Maxwellian with density temperature and velocity (ρ, β, u)

As above the temperature and the velocity are constant.

Moreover $u = (0, 0, u_3)$ ($2d$ Korn inequality in Q and u tangent to the boundary).

Then the identification of the term of order 0 gives $u_3 \partial_{x_3} \rho = 0$ which implies that ρ is independent of x_3 and with the Laplace equation same is true for ϕ .

From the identification of the terms of order 1 in $(v - u)$ one has observing that:

$$B = \nabla_x \times \vec{A} = (\partial_{x_2} A, -\partial_{x_1} A, 0) \Rightarrow u \times B = u_3(\partial_{x_1} A, \partial_{x_2} A)\,,$$

$$(\frac{\nabla_x \rho(x,)}{\rho(x)} - \beta \nabla_x (\phi - uA)) = 0\,.$$

Therefore any stationary solution must be of the form:

$$F(x, v) = (\frac{\beta}{2\pi})^{\frac{d}{2}} e^{-\beta(\frac{|v - \vec{u}|^2}{2} - (\phi(x)) - u_3 A))} \qquad (16)$$

Existence and uniqueness for given Energy and Axial Momentum.

For $\mathcal{E}_0$ and $\mathcal{M}$ given introduce the sets $K_{\mathcal{E}_0,\mathcal{M}}$ (resp $\partial K_{\mathcal{E}_0,\mathcal{M}}$) of $F(x,v)$ with

$$\mathcal{E}(F) = \frac{1}{2}(\int_{\Omega\times\mathbb{R}^d_v} |v|^2 F(x,v)dxdv + \int_\Omega |\nabla\phi|^2\, dx \leq (\text{resp.} =)\mathcal{E}_0\,,$$

$$\int_{\Omega\times\mathbb{R}^3_v} F(x,v)dxdv = \int_\Omega I(x)dx\,,$$

$$\text{In } \Omega \quad -\Delta\phi + \int_{\mathbb{R}^d_v} F(x,v)dv = I(x)\,, \text{On } \partial\Omega \quad \partial_{\vec{n}}\phi = 0\,,$$

$$\int_{\Omega\times\mathbb{R}^3_v} (v_3 - A)F(x,v)dxdv = \mathcal{M}\,.$$

As above this set is convex closed and non empty.

Moreover as above the entropy $\mathcal{H}(F)$ has one and only one minimum on $K_{\mathcal{E}_0,\mathcal{M}}$ which belongs to $\partial K_{\mathcal{E}_0,\mathcal{M}}$ and which is characterized by the relation :

$$\int_{\Omega\times\mathbb{R}^d_v}(1+\log F)\delta F dxdv$$
$$=\beta(\int_{\Omega\times\mathbb{R}^d_v}(\frac{|v|^2}{2}-\phi)\delta F dxdv)+\gamma\int_{\Omega\times\mathbb{R}^d_v} v_3\delta F dxdv \qquad (17)$$
$$+\mu\int_{\Omega\times\mathbb{R}^d_v}\delta F dxdv\,.$$

This minimum provides the existence of a stationary solution $(\overline{F},\overline{u},\overline{\beta})$. And as above since for any stationary solution one has:

$$\nabla\mathcal{H}(F)=-\beta\nabla\mathcal{E}(F) \qquad (18)$$

the argument of the previous section provides the uniqueness of the solution.

The “Real Torus"cf. T. Nguyen and W. Strauss ARMA 2014.

$$\Omega = \Big\{ x = (x_1, x_2, x_3) \in \mathcal{R}^3 \ : \ \Big(a - \sqrt{x_1^2 + x_2^2}\Big)^2 + x_3^2 < 1 \Big\},$$

With the toroidal coordinates (r, θ, φ)

$$x_1 = (a + r\cos\theta)\cos\varphi, \quad x_2 = (a + r\cos\theta)\sin\varphi, \quad x_3 = r\sin\theta.$$

$$e_r = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, \sin\theta),$$

$$e_\theta = (-\sin\theta\cos\varphi, -\sin\theta\sin\varphi, \cos\theta),$$

$$e_\varphi = (-\sin\varphi, \cos\varphi, 0).$$

$$e_\theta \times e_r = e_\varphi, \qquad e_r \times e_\varphi = e_\theta, \qquad e_\varphi \times e_\theta = e_r.$$

$$u = u_r e_r + u_\theta u_\theta + u_\varphi e_\varphi = [u_r, u_\theta, u_\varphi]$$

Assume that $B = \nabla \times A \quad \vec{A} = (0, 0, A_\varphi)$ and that the density of ion $I(x)$ is also independent of φ

Theorem

In the class of ϕ independent solution the angular momentum is preserved:

$$\frac{d}{dt}\int_{\Omega\times\mathbb{R}^3_v}(a + r\cos\theta)(v_\phi - A_\phi)F(x, , v, t)dxdv = 0 \qquad (19)$$

With prescribed totale density and momentum there exists unique a stationary solution given by:

$$\vec{u} = (u_r, u_\theta, u_\varphi) = (0, 0, u_\varphi)$$

$$F(x, v) = (\frac{\beta}{2\pi})^{\frac{d}{2}} e^{-\beta(\frac{|v-\vec{u}|^2}{2} - (\phi(x_1,x_2)) - (a+r\cos\theta)u_\varphi A_\varphi(r,\theta))} \qquad (20)$$

$$-\Delta\phi + e^{\beta\phi} = I(x)$$

Formal Remarks about Convergence

Starting from

$$
\begin{aligned}
&\partial_t F + v \cdot F + (\nabla \phi + B \times v)\nabla_v F = \epsilon_0 C(F,F) \\
&-\Delta \phi = I_{\mathrm{ions}}(x) - \int_{\mathbb{R}^3_v} F(x,v,t)dv \\
&\text{or } \epsilon^\alpha \partial_t F_\epsilon + v \cdot F_\epsilon + (\nabla_x \phi_\epsilon + B \times v)\nabla_v F_\epsilon = \epsilon C(F_\epsilon, F_\epsilon) \\
&-\Delta \phi_\epsilon = I_{\mathrm{ions}}(x) - \int_{\mathbb{R}^3_v} F_\epsilon(x,v,t)dv
\end{aligned}
\tag{21}
$$

one can extract subsequences $F(x,v,t_k)$or $F_{\epsilon_k}(x,v,t)$ which converge weakly to a solution of

$$
v \cdot F + (\nabla_x \phi + B \times v)\nabla_v F = 0 \quad \text{and} \quad C(F,F) = 0 \text{ with } \alpha > 1 \tag{22}
$$

The uniqueness of the limit proven above shows convergence not only of subfamily but of the complete set of solutions.

About full proofs

- It seems that there is no chance to do better that what is already done for the genuine Boltzmann equation (i.e. with $E = B = 0$)
- On the other hand it seems also that all the classical results can be adapted.

Below consider the Vlasov Maxwell Boltzmann equation in Ω a simply connected (which satisfies the strong Korn inequality) with constant magnetic field an initial data $F_0(x,v)$ an external density of ions $I(x)$ and the electric field given bien the relation:

$$\begin{gathered} -\Delta\Phi = I(x) - \int_{\mathbb{R}^d_v} F(x,v,t)dv \quad \partial_n \phi_{|\partial\Omega} = 0 \\ \int_\Omega I(x)dx = \int_{\Omega\times\mathbf{R}^d_v} F(x,v,t)dxdv = \int_{\Omega\times\mathbf{R}^d_v} F_0(x,v)dxdv \end{gathered} \tag{23}$$

Theorem

Let $(\overline{F},\overline{\phi})$ be any stationary solution and F a " weak solution"

$$\overline{F}(x,v)=\left(\frac{\beta}{2\pi}\right)^{3/2}e^{-\beta\left(\frac{|v|^2}{2}-\overline{\phi}(x)\right)}$$
$$In\ \Omega \quad -\Delta\overline{\phi}+\int_{\mathbb{R}^d_v}\overline{F}(x,v)dv=I(x)\,,\ on\ \partial\Omega \quad \partial_{\vec{n}}\overline{\phi}=0 \tag{24}$$

There holds

$$\frac{d}{dt}\mathcal{H}(F|\overline{F})+\frac{\beta}{2}\frac{d}{dt}\int_\Omega|\nabla\phi-\nabla\overline{\phi}|^2+\leq D(F)$$
$$D(F):=\iint_{\Omega\times\mathbb{R}^3}C(F,F)\log F\ dxdv \tag{25}$$
$$\mathcal{H}(f|\overline{f}):=\iint_{\Omega\times\mathbb{R}^3}\left[f\log\left(\frac{f}{\overline{f}}\right)-f+\overline{f}\right](x,v)\ dxdv,\leq 0.$$

In particular

• For any "reasonable weak " solution there is a "weak strong " stability control

$$
\begin{aligned}
&\mathcal{E}(F(0)) - \mathcal{E}(\overline{F}) + \mathcal{H}(F(0)) - \mathcal{H}(\overline{F}) \\
&\geq \mathcal{E}(F(t)) - \mathcal{E}(\overline{F}) + \mathcal{H}(F(t)) - \mathcal{H}(\overline{F}) \\
&\geq \frac{1}{2}\int_\Omega |\nabla\Phi(x,t) - \nabla\overline{\Phi}(x,t)|^2 dx\,.
\end{aligned}
\tag{26}
$$

And the conditional results of convergence of Desvillettes -Villani can be adapted.

In particular

• For any "reasonable weak " solution there is a "weak strong " stability control

$$\begin{aligned} &\mathcal{E}(F(0)) - \mathcal{E}(\overline{F}) + \mathcal{H}(F(0)) - \mathcal{H}(\overline{F}) \\ &\geq \mathcal{E}(F(t)) - \mathcal{E}(\overline{F}) + \mathcal{H}(F(t)) - \mathcal{H}(\overline{F}) \\ &\geq \frac{1}{2}\int_\Omega |\nabla\Phi(x,t) - \nabla\overline{\Phi}(x,t)|^2 dx\,. \end{aligned} \tag{26}$$

And the conditional results of convergence of Desvillettes -Villani can be adapted.

Here also the main difficulty being the absence of result concerning in such generality the conservation energy which is essential for uniqueness .

About full proofs III

- Consider fluctuations of the initial data $F_0(x,v)$ near a stationary solution (with same mass and same energy) the form:

$$e^{-|v|^2+2\overline{\phi}} = (\frac{1}{\pi})^{\frac{d}{2}} e^{-2(\frac{|v|^2}{2}-\tilde{\phi})}, \quad \overline{\phi} = \tilde{\Phi} + \frac{d}{2}\log\pi$$
$$-\Delta\tilde{\Phi} + e^{2\tilde{\Phi}} = I(x) \tag{27}$$
$$F_0(x,v) = e^{-|v|^2}(1+\delta_0 g_0(x,v))e^{-\frac{|v|^2}{2}})e^{2\overline{\phi}}$$

Then the results of Y. Guo should be adapted. For $g_0(x,v)$ smooth enough and small enough with respect to the collision operator (δ_0 small enough) one has

$$F(x,v,t) = e^{-|v|^2}(1+g(x,v,t)e^{-\frac{|v|^2}{2}})e^{2\overline{\phi}} \tag{28}$$

with $g(x,v,t)$ smooth and convergingg to zero. Hence the convergence.

Final Formal Remarks

The nature of the limit depends on the geometry and on the initial value.

1 For a domain with no symmetry or in the above case (flat torus, real torus) with genuine dependence of the initial data in term of x_3 or φ the bulk velocity of the limit is 0 .

2 It is may not be zero in the torus if the initial data do not depend on (x_3 resp. φ) then the bulk velocity is determined by the conservation of moment.

3 The limit is completely different from what is obtained in the case of Landau Damping. In some sense the influence of the collision operator remains and the initial data is not close to an homogenous distribution.

4 The situation is completely different from the contribution of Aoki, Golse and Kosuge. First the force is not $f \cdot \nabla_v$ but $(B \times v)\nabla_v$ and therefore support any centered Maxwellian and second even in such case the coupling with the Laplace equation generate a non trivial density.

5 The situations of course completely different if $\Omega = \mathbb{R}^3$ because then the equation

$$\partial_t F + v \cdot F + (\nabla\phi + B \times v)\nabla_v F = 0\,, \quad C(F,F) = 0\,.$$

may have time dependent solutions:
Fo instance with $\phi = 0$ the global Maxwellians of Dave Levermore.

$$\mathcal{M} = \frac{\rho(x,t)}{\big(2\pi\theta(t)\big)^{\frac{\mathrm{D}}{2}}} \exp\left(-\frac{|v-u(x,t)|^2}{2\theta(t)}\right),$$

$$\theta(t) = \frac{1}{at^2-2bt+c}, \qquad u(x,t) = \theta(t)\big(axt - bx - Bx\big)$$

$$\rho(x,t) = m\left(\frac{1}{2\pi}\,\theta(t)\right)^{\frac{\mathrm{D}}{2}} \sqrt{\det(Q)}\, \exp\left(-\frac{\theta(t)}{2}\, x^T Qx\right)$$

$$\begin{aligned}
\partial_t\rho + \nabla_x\cdot(\rho u) &= 0\,,\\
\partial_t(\rho u) + \nabla_x\cdot(\rho u\otimes u) + \nabla_x\rho\theta &= 0\,,\\
\partial_t\left(\tfrac{1}{2}\rho|u|^2 + \tfrac{\mathrm{D}}{2}\rho\theta\right) + \nabla_x\cdot\left(\tfrac{1}{2}\rho|u|^2u + \tfrac{\mathrm{D}+2}{2}\rho\theta u\right) &= 0\,.
\end{aligned}$$

Collaborators:

Toan Nguyen, Irene Gamba, Francois Golse, Claudia Negulescu and Rémi Sentis.

Collaborators:

Toan Nguyen, Irene Gamba, Francois Golse, Claudia Negulescu and Rémi Sentis.

Thanks for ATTENTION ;

Thanks for INVITATION.

Collaborators:

Toan Nguyen, Irene Gamba, Francois Golse,
Claudia Negulescu and Rémi Sentis.

Thanks for ATTENTION ;

Thanks for INVITATION.

Many Thanks to Bob , Jack and Walter for Science and Friendship.

Happy Birthday Bob!,