A TOY MODEL FOR THE RELATIVISTIC VLASOV-MAXWELL SYSTEM

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Abstract. [to do]

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Contents

1. Introduction

?⟨sec:intro⟩? Let $f(t,x,v) \geq 0$ denote the one particle distribution in phase space of a monocharged plasma, where $x, v \in \mathbb{R}^3$ and $t \geq 0$. Taking relativistic effects into account, but neglecting collisions among the particles, f satisfies the relativistic Vlasov-Maxwell system:

$$\left.\begin{aligned} \partial_t f + \hat{v}\cdot\nabla_x f + (E + \hat{v}\wedge B)\cdot\nabla_v f = 0 \\ \partial_t E = \nabla\wedge B - 4\pi j, \quad \nabla\cdot E = 4\pi\rho, \\ \partial_t B = -\nabla\wedge E, \quad \nabla\cdot B = 0, \end{aligned}\right\} \tag{RVM}$$ RVM

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where

$$\rho(t,x) = \int_{\mathbb{R}^3} f(t,x,v)\,dv, \quad j(t,x) = \int_{\mathbb{R}^3} \hat{v} f(t,x,v)\,dv \tag{1.1}$$?sources?

are the charge and current density of the plasma, while

$$\hat{v} = \frac{v}{\sqrt{1+|v|^2}} \tag{1.2}$$?phat?

is the relativistic velocity. Additionally, $E(t,x)$ and $B(t,x)$ are the self-consistent electric and magnetic fields generated by the charged particles, and we have chosen units such that the mass and charge of each particle, as well as the speed of light, are normalized to one.

The purpose of this note is to prove a global-in-time existence and uniqueness result for the following toy model of (RVM):

$$\left.\begin{aligned} \partial_t f + \hat{v}\cdot\nabla_x f + \partial_t A \cdot \nabla_v f = 0, \\ \Box A = (\partial_{tt} - \Delta) A = j, \end{aligned}\right\} \tag{Toy}$$ Toy

with initial data $f(0,x,v) = f_0(x,v)$ which is smooth and compactly supported and consistent data for A satisfying $A(0,x) = A_0(x)$ and $\partial_t A(0,x) = A_1(x)$. In this model x, v can be taken to be in $\mathbb{R}^d$ for any $d \geq 1$, with $d = 3$ the case of interest. Our main result, however, considers the simpler case $d = 1$.

1.1. Main result.

⟨thm:main⟩

Theorem 1.1 (Local and global existence). *Suppose that $(f_0, A_0, A_1) \in W^{1,\infty}(\mathbb{R}^2) \times W^{1,\infty}(\mathbb{R}) \times L^\infty(\mathbb{R})$ with compact support, then there exists $T > 0$ such that the Cauchy problem (Toy) has a unique solution*

$$(f, A) \in W^{1,\infty}([0,T]\times\mathbb{R}^2) \times W^{1,\infty}([0,T]\times\mathbb{R}).$$

Furthermore if $(f_0, A_0, A_1) \in \mathcal{C}_c^1(\mathbb{R}^2) \times \mathcal{C}_c^1(\mathbb{R}) \times \mathcal{C}_c(\mathbb{R})$, then the Cauchy problem (Toy) has a unique global solution such that

$$(f, A) \in \mathcal{C}_c^1([0,\infty)\times\mathbb{R}^2) \times \mathcal{C}_c^1([0,\infty)\times\mathbb{R}).$$

1.2. Justification of the toy model. [to do]

1.3. Previous results. [to do. cite [NP14, GP10]]

[to do. cite Glassey-Strauss; Bouchut, Golse, Pallard; Klainerman, Staffilani]

2. Proof of Theorem 1.1

?⟨sec:proof⟩?

2.1. Local existence. We first use a fixed-point argument to prove the local existence result. Due to the assumption of compact support, we can fix $R > 0$ and $M > 0$ such that $f_0 \in \mathcal{C}_c((-R,R)\times(-M,M))$. For a given $T > 0$, we define B_T to be the set of functions $g \in W^{1,\infty}([0,T]\times\mathbb{R}^2)$ that satisfy

(H1) $g(0,x,v) = f_0(x,v)$ and $\|g\|_{L^\infty([0,T]\times\mathbb{R}^2)} \leq \|f_0\|_{W^{1,\infty}(\mathbb{R}^2)}$;
(H2) supp $g \subset [0,T]\times(-R-1,R+1)\times(-M-1,M+1)$;
(H3) $\|g\|_{\mathrm{Lip}} \leq 3\|f_0\|_{W^{1,\infty}(\mathbb{R}^2)}$, where

$$\|g\|_{\mathrm{Lip}} := \sup_{\substack{t\in[0,T] \\ x,v\in\mathbb{R} \\ h=(h_1,h_2)\neq 0}} \frac{|g(t,x+h_1,v+h_2) - g(t,x,v)|}{|h|}$$

(H4) $\|\partial_t g\|_{L^\infty([0,T]\times\mathbb{R}^2)} \leq 3\|f_0\|_{W^{1,\infty}(\mathbb{R}^2)}(2 + \|A_0\|_{L^\infty(\mathbb{R})} + \|A_1\|_{L^\infty(\mathbb{R})})$.

When endowed with the metric $d(g_1, g_2) := \|g_1 - g_2\|_{L^\infty([0,T]\times\mathbb{R}^2)}$, the metric space (B_T, d) is complete. Next, for any given $g \in B_T$, we define A_g to be the solution to the linear wave equation

$$\Box A_g = (\partial_t^2 - \partial_x^2) A_g = \int_{\mathbb{R}} \hat{v} g(t,x,v)dv := j_g(t,x), \tag{2.1} \quad \boxed{\texttt{W}}$$

with initial conditions $A_g(0,x) = A_0(x)$ and $\partial_t A_g(0,x) = A_1(x)$ for any $x \in \mathbb{R}$. With this, we define for any $g \in B_T$ the solution map $\Phi(g) = f$ where $f \in W^{1,\infty}((0,T)\times\mathbb{R}^2)$ is the unique solution of the transport equation

$$\partial_t f + \hat{v}\partial_x f + \partial_t A_g(t,x)\partial_v f = 0, \quad (t,x,v) \in (0,T)\times\mathbb{R}^2, \tag{2.2} \quad \boxed{\texttt{T}}$$

with initial condition $f(0,x,v) = f_0(x,v)$. To complete the fixed-point argument, it suffices to show two properties for T sufficiently small, namely

(1) Φ maps B_T into itself, i.e. for every $g \in B_T$, $f = \Phi(g) \in B_T$;
(2) $\Phi : B_T \to B_T$ is a contraction, i.e. there is $C < 1$ such that

$$\|f_1 - f_2\|_{L^\infty([0,T]\times\mathbb{R}^2)} \le C\|g_1 - g_2\|_{L^\infty([0,T]\times\mathbb{R}^2)}$$

for every $g_1, g_2 \in B_T$.

Step 1: Preliminary estimates. Throughout we will use the fact that the mapping $v \mapsto \hat{v} = v/\sqrt{1+v^2}$ and its derivative are both bounded by one. The solution A_g of (2.1) is given by the formula for the solution of the wave equation, namely

$$A_g(t,x) = (\partial_t Y(t,\cdot) *_x A_0)(t,x) + (Y(t,\cdot) *_x A_1)(t,x) + (Y(\cdot,\cdot) *_{t,x} (j_g \mathbb{1}_{t>0}(t)))(t,x) \tag{2.3} \quad \boxed{\texttt{wave-s1}}$$

where $Y(t,x) = \frac{1}{2}\mathbb{1}_{\{|x|\le t\}}$ is the forward fundamental solution of the one-dimensional wave operator. Note that

$$\partial_x Y(t,x) = \frac{1}{2}\delta_{x=-t} - \frac{1}{2}\delta_{x=t}, \quad \partial_t Y(t,x) = \frac{1}{2}\delta_{x=-t} + \frac{1}{2}\delta_{x=t}.$$

More explicitly, we have

$$A_g(t,x) = \frac{1}{2}[A_0(x+t) + A_0(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} A_1(s)ds + \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} j_g(s,y)dyds. \tag{2.4} \quad \texttt{?\underline{wave-s2}?}$$

Thus, for $t \in [0,T]$ we find

$$\begin{aligned} \|A_g(t)\|_{L^\infty(\mathbb{R})} &\le \|A_0\|_{L^\infty(\mathbb{R})} + t\|A_1\|_{L^\infty(\mathbb{R})} + \frac{1}{2}t^2\|j_g\|_{L^\infty([0,T]\times\mathbb{R})} \\ \|\partial_t A_g(t)\|_{L^\infty(\mathbb{R})} &\le \|A_0'\|_{L^\infty(\mathbb{R})} + \|A_1\|_{L^\infty(\mathbb{R})} + t\|j_g\|_{L^\infty([0,T]\times\mathbb{R})} \\ \|\partial_x\partial_t A_g(t)\|_{L^\infty(\mathbb{R})} &\le \|A_0''\|_{L^\infty(\mathbb{R})} + \|A_1'\|_{L^\infty(\mathbb{R})} + t\|\partial_x j_g\|_{L^\infty([0,T]\times\mathbb{R})}. \end{aligned} \tag{2.5} \quad \boxed{\texttt{A\_g-estimates}}$$

Taking $g \in B_T$, we have by (H1), (H2) and (H3)

$$\begin{aligned} \|j_g\|_{L^\infty([0,T]\times\mathbb{R})} &= \left\| \int_{\mathbb{R}} \hat{v} g(\cdot,\cdot,v)dv \right\|_{L^\infty([0,T]\times\mathbb{R})} \le 2(M+1)\|f_0\|_{W^{1,\infty}(\mathbb{R}^2)} \\ \|\partial_x j_g\|_{L^\infty([0,T]\times\mathbb{R})} &= \left\| \int_{\mathbb{R}} \hat{v}\partial_x g(\cdot,\cdot,v)dv \right\|_{L^\infty([0,T]\times\mathbb{R})} \le 6(M+1)\|f_0(x,v)\|_{W^{1,\infty}(\mathbb{R}^2)}. \end{aligned}$$

Hence, taking T sufficiently small we obtain the estimates

$$
\begin{aligned}
\|A_g\|_{L^\infty([0,T]\times\mathbb{R})} &\le \|A_0\|_{L^\infty(\mathbb{R})} + \|A_1\|_{L^\infty(\mathbb{R})} + 1 \\
\|\partial_t A_g\|_{L^\infty([0,T]\times\mathbb{R})} &\le \|A_0'\|_{L^\infty(\mathbb{R})} + \|A_1\|_{L^\infty(\mathbb{R})} + 1 \\
\|\partial_x\partial_t A_g\|_{L^\infty([0,T]\times\mathbb{R})} &\le \|A_0''\|_{L^\infty(\mathbb{R})} + \|A_1'\|_{L^\infty(\mathbb{R})} + 1.
\end{aligned}
\tag{2.6}
$$

Step 2: Establishing Φ maps into B_T. Denote by $(X(s;t,x,v), V(s;t,x,v))$ the characteristic curves of (2.2). They satisfy the system of ODEs

$$
\begin{aligned}
&\frac{dX}{ds}(s;t,x,v) = \hat{V}(s;t,x,v) = \frac{V(s;t,x,v)}{(1+V^2(s;t,x,v))^{1/2}}, \\
&\frac{dV}{ds}(s;t,x,v) = (\partial_t A_g)(s, X(s;t,x,v)),
\end{aligned}
$$

with the initial conditions $X(t;t,x,v) = x$ and $V(t;t,x,v) = v$. As is well-known, the solution of (2.2) can then be expressed as

$$
f(t,x,v) = f_0(X(0;t,x,v), V(0;t,x,v)).
$$

From this, we immediately find that $f = \Phi[g]$ satisfies $f(0,x,v) = f_0(x,v)$ and

$$
\|f\|_{L^\infty([0,T])\times\mathbb{R}^2)} = \|f_0(X(0;\cdot,\cdot,\cdot), V(0;\cdot,\cdot,\cdot))\|_{L^\infty([0,T])\times\mathbb{R}^2)} = \|f_0\|_{L^\infty(\mathbb{R}^2)}
$$

so that (H1) is satisfied. Additionally, note that for $T > 0$ sufficiently small, then may use (2.6) to conclude

$$
|x| = |X(t;t,x,v)| \le |X(0;t,x,v)| + \int_0^t |\hat{V}(s;t,x,v)|ds \le |X(0;t,x,v)| + 1
$$

and

$$
|v| = |V(t;t,x,v)| \le |V(0;t,x,v)| + \int_0^t |(\partial_t A_g)(s, X(s;t,x,v))|ds \le |V(0;t,x,v)| + 1.
$$

for $t \in [0,T]$. Now, for $(x,v) \in \operatorname{supp}(f)$, then $0 \ne f(t,x,v) = f_0(X(0;t,x,v), V(0;t,x,v))$, thus $|X(0;t,x,v)| \le R, |V(0;t,x,v)| \le M$. Therefore, $|x| \le R+1$ and $|v| \le M+1$ and (H2) is satisfied.

Next, we verify (H3). By the definition of the Lipschitz norm, we have

$$
\begin{aligned}
\|f\|_{\mathrm{Lip}} &= \sup_{\substack{t\in[0,T] \\ (x,v)\ne(y,p)\in\mathbb{R}^2}} \frac{|f(t,x,v) - f(t,y,p)|}{|(x-y, v-p)|} \\
&= \sup_{(x,v)\ne(y,p)\in\mathbb{R}^2} \frac{|f_0(X(0,t,x,v), V(0,t,x,v)) - f_0(X(0,t,y,p), V(0,t,y,p))|}{|(x-y,v-p)|} \\
&\le \|f_0\|_{W^{1,\infty}(\mathbb{R}^2)}(\|X\|_{\mathrm{Lip}} + \|V\|_{\mathrm{Lip}})
\end{aligned}
$$

where the Lipschitz norm of the characteristics are defined by

$$
\|X\|_{\mathrm{Lip}} := \sup_{\substack{s,t\in[0,T] \\ x,v\in\mathbb{R} \\ h=(h_1,h_2)\ne 0}} \frac{|X(s;t,x+h_1,v+h_2) - X(s;t,x,v)|}{|h|}
$$

and analogously for $\|V\|_{\mathrm{Lip}}$. Since $X(\tau;t,x,v) = x + \int_t^\tau \hat{V}(s;t,x,v)ds$ and $V(\tau;t,x,v) = v + \int_t^\tau (\partial_t A_g)(s, X(s;t,x,v))ds$, we have the following bounds on the Lipschitz norms

$$
\begin{aligned}
&\|X\|_{\mathrm{Lip}} \le 1 + T\|V\|_{\mathrm{Lip}}, \\
&\|V\|_{\mathrm{Lip}} \le 1 + T\|\partial_x\partial_t A_g\|_{L^\infty([0,T]\times\mathbb{R})}\|X\|_{\mathrm{Lip}}.
\end{aligned}
$$

Therefore, we have

$$\|X\|_{\mathrm{Lip}} + \|V\|_{\mathrm{Lip}} \leq 2 + \frac{1}{3}\left(\|V\|_{\mathrm{Lip}} + \|X\|_{\mathrm{Lip}}\right), \tag{2.7}$$ {?}

for T sufficiently small, which implies $\|X\|_{\mathrm{Lip}} + \|V\|_{\mathrm{Lip}} \leq 3$. Inserting this into the estimate on $\|f\|_{\mathrm{Lip}}$, we conclude

$$\|f\|_{\mathrm{Lip}} \leq 3\|f_0\|_{W^{1,\infty}(\mathbb{R}^2)} \tag{2.8}$$ eq:f-lip-norm

and (H3) is satisfied.

Finally, we verify (H4). Computing the time derivative of f, we find

$$\begin{aligned}\|\partial_t f\|_{L^\infty([0,T]\times\mathbb{R}^2)} &= \|\partial_t[f_0(X(0;t,x,v),V(0;t,x,v))]\|_{L^\infty([0,T]\times\mathbb{R}^2)} \\ &\leq \|f_0\|_{W^{1,\infty}(\mathbb{R}^2)}\left(|\partial_t X(0;\cdot,\cdot,\cdot)|_{L^\infty([0,T]\times\mathbb{R}^2)} + |\partial_t V(0;\cdot,\cdot,\cdot)|_{L^\infty([0,T]\times\mathbb{R}^2)}\right).\end{aligned}$$

To bound the two terms on the right hand side, we first estimate

$$\begin{aligned}|\partial_t X(\tau;t,x,v)| &= \left|\partial_t\left(x + \int_t^\tau \hat{V}(s;t,x,v)ds\right)\right| \\ &= \left|-\hat{v} - \int_\tau^t \partial_v(\hat{V}(s;t,x,v))\partial_t V(s;t,x,v)ds\right| \\ &\leq 1 + \int_0^t |\partial_t V(s;t,x,v)|ds\end{aligned}$$

and

$$\begin{aligned}|\partial_t V(\tau;t,x,v)| &= \left|\partial_t\left(v + \int_t^\tau (\partial_t A_g)(s,X(s;t,x,v))ds\right)\right| \\ &= \left|-\partial_t A_g(t,x) - \int_\tau^t (\partial_x\partial_t A_g)(s,X(s;t,x,v))\partial_t X(s;t,x,v)ds\right| \\ &\leq \|\partial_t A_g\|_{L^\infty([0,T]\times\mathbb{R})} + \|\partial_x\partial_t A_g\|_{L^\infty([0,T]\times\mathbb{R})}\int_0^t |\partial_t X(s;t,x,v)|ds.\end{aligned}$$

Therefore, using (2.6) we obtain

$$\begin{aligned}\sup_{\tau\in[0,t]}\left(|\partial_t X(\tau,t,x,v)| + |\partial_t V(\tau,t,x,v)|\right) &\leq 1 + \|\partial_t A_g\|_{L^\infty([0,T]\times\mathbb{R})} \\ &\quad + (1 + \|\partial_x\partial_t A_g\|_{L^\infty([0,T]\times\mathbb{R})})\int_0^t (|\partial_t V(s;t,x,v)| + |\partial_t X(s;t,x,v)|)ds \\ &\leq 2 + \|A_0'\|_{L^\infty(\mathbb{R})} + \|A_1\|_{L^\infty(\mathbb{R})} \\ &\quad + (2 + \|A_0''\|_{L^\infty(\mathbb{R})} + \|A_1'\|_{L^\infty(\mathbb{R})})\int_0^t \sup_{\tau\in[0,s]}\left(|\partial_t X(\tau;t,x,v)| + |\partial_t V(\tau;t,x,v)|\right)ds.\end{aligned}$$

Invoking Grönwall's inequality now yields

$$\sup_{\tau\in[0,t]}\left(|\partial_t X(\tau;t,x,v)| + |\partial_t V(\tau;t,x,v)|\right) \leq \left(2 + \|A_0'\|_{L^\infty(\mathbb{R})} + \|A_1\|_{L^\infty(\mathbb{R})}\right)e^{t(2+\|A_0''\|_{L^\infty(\mathbb{R})}+\|A_1'\|_{L^\infty(\mathbb{R})})}.$$

Using this estimate we ultimately have

$$|\partial_t X(0;t,x,v)| + |\partial_t V(0;t,x,v)| \leq 3\left(2 + \|A_0'\|_{L^\infty(\mathbb{R})} + \|A_1\|_{L^\infty(\mathbb{R})}\right)$$

for all $t\in[0,T]$, $x,v\in\mathbb{R}$ and $T>0$ sufficiently small. Taking the supremum and combining this with the estimate of $\|\partial_t f\|_{L^\infty([0,T]\times\mathbb{R}^2)}$ yields

$$\|\partial_t f\|_{L^\infty([0,T]\times\mathbb{R}^2)} \leq 3\|f_0\|_{W^{1,\infty}(\mathbb{R}^2)}(2 + \|A_0\|_{L^\infty(\mathbb{R})} + \|A_1\|_{L^\infty(\mathbb{R})})$$

and (H4) is satisfied.

Step 3: Φ is a contraction. Let $g, \tilde{g} \in B_T$ and $f = \Phi(g)$, $\tilde{f} = \Phi(\tilde{g})$. Then, subtracting the respective Vlasov equations yields

$$\partial_t(f - \tilde{f}) + \hat{v}\partial_x(f - \tilde{f}) + \partial_t A_g(t,x)\partial_v(f - \tilde{f}) + \big(\partial_t A_g(t,x) - \partial_t A_{\tilde{g}}(t,x)\big)\partial_v \tilde{f} = 0$$

with $(f - \tilde{f})(0,x,v) = 0$. Consequently

$$\begin{aligned}&(f - \tilde{f})(t, X(t;0,x,v), V(t;0,x,v))\\ &\qquad = \int_0^t \big(\partial_t A_g - \partial_t A_{\tilde{g}}\big)(s, X(s;0,x,v)) \cdot (\partial_v \tilde{f})(s, X(s;0,x,v), V(s;0,x,v))ds\end{aligned} \tag{2.9}$$

eq:f-ftilde

where the characteristics are given by the ODEs

$$\left.\begin{aligned}&\frac{dX}{ds}(s;0,x,v) = \hat{V}(V(s;0,x,v)) = \frac{V(s;0,x,v)}{(1+V^2(s;0,x,v))^{1/2}},\\ &\frac{dV}{ds}(s;0,x,v) = (\partial_t A_g)(s, X(s;0,x,v)),\\ &X(t;t,x,v) = x, \quad V(t;t,x,v) = v.\end{aligned}\right\}$$

Using (2.5) and (2.8), we have the following estimates for the right hand side of (2.9):

$$\begin{aligned}&\|\partial_v \tilde{f}\|_{L^\infty([0,T]\times\mathbb{R}^2)} \le 3\|f_0\|_{W^{1,\infty}(\mathbb{R}^2)},\\ &\|\partial_t A_g - \partial_t A_{\tilde{g}}\|_{L^\infty([0,T]\times\mathbb{R}^2)} \le 2T\|j_g - j_{\tilde{g}}\|_{L^\infty([0,T]\times\mathbb{R}^2)}.\end{aligned}$$

Therefore, we find

$$\|f - \tilde{f}\|_{L^\infty([0,T]\times\mathbb{R}^2)} \le 3\|f_0\|_{W^{1,\infty}(\mathbb{R}^2)} T^2 \|j_g - j_{\tilde{g}}\|_{L^\infty([0,T]\times\mathbb{R}^2)},$$

which implies, provided T is sufficiently small,

$$\|f - \tilde{f}\|_{L^\infty([0,T]\times\mathbb{R}^2)} \le \frac{1}{2}\|g - \tilde{g}\|_{L^\infty([0,T]\times\mathbb{R}^2)}.$$

Thus, we obtain a unique local solution to the Cauchy problem (Toy) on $[0,T]$ for T sufficiently small. Furthermore, we can extend the life span of the solution as long as derivatives remain finite, that is

$$\|\partial_x f(t,x,v)\|_{L^\infty_{x,v}(\mathbb{R}^2)} + \|\partial_v f(t,x,v)\|_{L^\infty_{x,v}(\mathbb{R}^2)} < \infty, \quad t \in [0,T].$$

2.2. **Global existence.** We assume that the maximal life span is $[0, T^*)$ for some $T^* > 0$, and shall prove that $T^* = +\infty$, hence the solution is global. We need to show

$$(f, \partial_t A) \in W^{1,\infty}([0,T^*) \times \mathbb{R}^2) \times W^{1,\infty}([0,T^*) \times \mathbb{R}).$$

Step 1: Bounds on f and $\partial_t A$. From the proof of local existence, we know that $f \in B_T$ for any $T \in [0, T^*)$. Thus by (H2), the support of $f(t,x,v)$ in the (x,v) variables remains in $(-R-1, R+1) \times (-M-1, M+1)$ for any fixed time $t \in [0,T^*)$. In particular, the v support of f is uniformly bounded for any fixed time $t \in [0,T^*)$ but may tend to $+\infty$ as $t \to T^*$. We therefore define

$$M(t) := \sup\{|v| : \exists\, x \text{ such that } f(t,x,v) \neq 0\}$$

and prove the following result.

?⟨prop:M-bound⟩? **Proposition 2.1.** *Let $T \in (0, T^*)$ be given. Then there exists $C > 0$ independent of T such that*

$$\|\partial_t A\|_{L^\infty([0,T]\times\mathbb{R})} \le C. \tag{2.10}$$

eq:A-bound

Consequently, we have

$$M(t) \le C \text{ for all } t \in [0, T^*). \tag{2.11}$$?eq:M-bound?

Proof. Define

$$B(t,x) = \partial_t A(t,x) - \partial_x A(t,x)$$
$$C(t,x) = \partial_t A(t,x) + \partial_x A(t,x).$$

Then since $\Box A = j$, we have

$$(\partial_t + \partial_x) B(t,x) = j(t,x),$$
$$(\partial_t - \partial_x) C(t,x) = j(t,x).$$

Therefore, for any $h > 0$, we find

$$\partial_\tau [B(\tau, x - (t+h-\tau))] = [(\partial_t + \partial_x)B](\tau, x - (t+h-\tau)) = j(\tau, x - (t+h-\tau))$$
$$\partial_\tau [C(\tau, x + (t+h-\tau))] = [(\partial_t - \partial_x)C](\tau, x + (t+h-\tau)) = j(\tau, x + (t+h-\tau)).$$

Integrating with respect to $\tau \in [t, t+h]$, we obtain

$$B(t+h,x) = B(t, x-h) + \int_t^{t+h} j(\tau, x-(t+h-\tau)) d\tau$$
$$C(t+h,x) = C(t, x+h) + \int_t^{t+h} j(\tau, x+(t+h-\tau)) d\tau.$$

Taking $t = 0$ and replacing h by t, we can represent B and C:

$$B(t,x) = B(0, x-t) + \int_0^t j(\tau, x-(t-\tau)) d\tau,$$
$$C(t,x) = C(0, x+t) + \int_0^t j(\tau, x+(t-\tau)) d\tau.$$

Recalling the definitions of B and C, we obtain the following representation of $\partial_t A$:

$$\partial_t A(t,x) = \frac{1}{2}(B(t,x) + C(t,x)) \tag{2.12}$$ paA

$$= \frac{1}{2}\left(A_0'(x+t) - A_0'(x-t) + A_1(x+t) + A_1(x-t)\right) \tag{2.13}$$ {?}

$$+ \frac{1}{2}\int_0^t \left(j(\tau, x-(t-\tau)) + j(\tau, x+(t-\tau))\right) d\tau. \tag{2.14}$$ {?}

Now we prove (2.10). By (2.12) and the condition on the initial data, we have

$$\|\partial_t A\|_{L^\infty([0,T]\times\mathbb{R})} \le C\left(1 + \sup_{x, t \le T}\left|\int_0^t \left(j(\tau, x-(t-\tau)) + j(\tau, x+(t-\tau))\right) d\tau\right|\right) \tag{2.15}$$ {?}

$$\le C\left(1 + \sum_{\pm} \sup_{x,t\le T}\left|\int_0^t j(\tau, x \pm (t-\tau)) d\tau\right|\right). \tag{2.16}$$ {?}

By the definition of j, it suffices to show

$$\left|\int_0^t j(\tau, x \pm (t-\tau))\, d\tau\right| \le \sup_{x,t\le T} \int_0^t \int_{\mathbb{R}} \frac{|v|}{\sqrt{1+|v|^2}} f(\tau, x\pm(t-\tau)) dv d\tau \le C \tag{2.17}$$ {?}

uniformly for $x \in \mathbb{R}$ and $t \in [0,T]$. To this end, we first establish the energy identity

$$\partial_t e - \partial_x m = 0 \tag{2.18}$$ en-id

where

$$e(t,x) = \frac{1}{2}(|\partial_t A|^2 + |\partial_x A|^2)(t,x) + \int_{\mathbb{R}} \sqrt{1+|v|^2} f(t,x,v)dv \tag{2.19}$$ {?}

$$m(t,x) = \partial_x A \partial_t A - \int_{\mathbb{R}} \frac{v}{\sqrt{1+|v|^2}} f(t,x,v)dv. \tag{2.20}$$ {?}

Indeed, using (Toy) we have

$$\partial_t e(t,x) = \partial_t \left(\frac{1}{2}(|\partial_t A|^2 + |\partial_x A|^2)(t,x) + \int_{\mathbb{R}} \sqrt{1+|v|^2} f(t,x,v)dv \right) \tag{2.21}$$ {?}

$$= \partial_t A \left(\partial_t^2 A + \int_{\mathbb{R}} \partial_v(\sqrt{1+v^2}) f(t,x,v)dv \right) + \partial_t \partial_x A \partial_x A - \partial_x \int_{\mathbb{R}} v f(t,x,v)dv \tag{2.22}$$ {?}

$$= \partial_t A \partial_x^2 A + \partial_t \partial_x A \partial_x A - \partial_x \int_{\mathbb{R}} v f(t,x,v)dv \tag{2.23}$$ {?}

$$= \partial_x \left(\partial_t A \partial_x A - \int_{\mathbb{R}} v f(t,x,v)dv \right) = \partial_x m(t,x). \tag{2.24}$$ {?}

Define the backward characteristic cone with vertex at (t,x) by

$$\Gamma(t,x) := \{(\tau,y) : 0 \le \tau \le t, |x-y| \le t-\tau\}, \tag{2.25}$$ {?}

and its surface

$$\partial\Gamma(t,x) = S_1 \cup S_2 \cup S_3 \tag{2.26}$$ {?}

where

$$S_1 = \{(\tau,y) : \tau = 0, x-t \le y \le x+t\}, \tag{2.27}$$ {?}

$$S_2 = \{(\tau,y) : 0 \le \tau \le t, y = x-(t-\tau)\}, \tag{2.28}$$ {?}

$$S_3 = \{(\tau,y) : 0 \le \tau \le t, y = x+(t-\tau)\}. \tag{2.29}$$ {?}

Let $\mathbf{n}$ be the normal vector of the surface and let $d\sigma$ be the measure on the surface. Integrating (2.18) over $\Gamma(t,x)$, we use Green's theorem to find

$$0 = \iint_{\Gamma(t,x)} (\partial_t e - \partial_x m)(\tau,y)dyd\tau = \iint_{\partial\Gamma(t,x)} (-m,e)\cdot \mathbf{n} d\sigma \tag{2.30}$$ {?}

$$= \iint_{S_1} (-m,e)\cdot(0,-1)d\sigma + \frac{1}{\sqrt{2}} \iint_{S_2} (-m,e)\cdot(-1,1)d\sigma + \frac{1}{\sqrt{2}} \iint_{S_3} (-m,e)\cdot(1,1)d\sigma \tag{2.31}$$ {?}

$$= -\int_{x-t}^{x+t} e(0,y)dy + \int_0^t (e+m)(\tau, x-(t-\tau))d\tau + \int_0^t (e-m)(\tau, x+(t-\tau))d\tau. \tag{2.32}$$ {?}

Therefore, we obtain

$$\int_0^t (e+m)(\tau, x-(t-\tau))d\tau + \int_0^t (e-m)(\tau, x+(t-\tau))d\tau = \int_{x-t}^{x+t} e(0,y)dy \tag{2.33}$$ {?}

Note that the right side is bounded above by the total energy $\int e(0,y)\, dy$ uniformly for all x and t. Recalling the definitions of e and m, we have

$$(e \pm m)(\tau, x \mp (t-\tau)) = \frac{1}{2}(\partial_t A \pm \partial_x A)^2 + \int_{\mathbb{R}} \left(\sqrt{1+v^2} \mp \frac{v}{\sqrt{1+v^2}} \right) f(\tau, x \mp (t-\tau), v)dv. \tag{2.34}$$ {?}

On the one hand, we see

$$\sqrt{1+v^2} \mp \frac{v}{\sqrt{1+v^2}} = \frac{1+v^2 \mp v}{\sqrt{1+v^2}} = \frac{\frac{3}{4}v^2 + (1 \mp \frac{v}{2})^2}{\sqrt{1+v^2}} \geq \frac{3}{4}\frac{v^2}{\sqrt{1+v^2}} \tag{2.35} \{?\}$$

On the other hand, we see

$$\sqrt{1+v^2} \mp \frac{v}{\sqrt{1+v^2}} = \frac{1+v^2 \mp v}{\sqrt{1+v^2}} = \frac{\frac{3}{4} + (\frac{1}{2} \mp v)^2}{\sqrt{1+v^2}} \geq \frac{3}{4}\frac{1}{\sqrt{1+v^2}} \tag{2.36} \{?\}$$

Therefore, since $f(t,x,v) \geq 0$, we finally have

$$\begin{aligned} C &\geq \int_0^t \int_{\mathbb{R}} \Big(\sqrt{1+v^2} \mp \frac{v}{\sqrt{1+v^2}}\Big) f(\tau, x \mp (t-\tau), v) dv d\tau \\ &\geq \frac{3}{8}\int_0^t \int_{\mathbb{R}} \frac{1+v^2}{\sqrt{1+v^2}} f(\tau, x \mp (t-\tau), v) dv d\tau \\ &\geq \frac{3}{8}\int_0^t \int_{\mathbb{R}} \frac{|v|}{\sqrt{1+v^2}} f(\tau, x \mp (t-\tau), v) dv d\tau. \end{aligned}$$

This quantity then controls j so that

$$\sup_{x\in\mathbb{R}} \left| \int_0^t j(\tau, x \pm (t-\tau)) d\tau \right| \leq C,$$

which completes the proof of (2.10). □

Step 2: Bounds on the derivatives of f and $\partial_t A$. We first make the observation, based on the previous part, that on the support of $f(t)$ the following bound holds:

$$|\hat{v} - 1| = \left| \frac{v}{\sqrt{1+|v|^2}} - 1 \right| \geq \frac{1}{M(t)^2}.$$

The transport equation for f in (Toy) takes the following form for the derivatives of f:

$$(\partial_t + \hat{v}\partial_x + \partial_t A(t,x)\partial_v) \begin{pmatrix} \partial_x f \\ \partial_v f \end{pmatrix} = -\begin{pmatrix} 0 & \partial_t \partial_x A(t,x) \\ (1+|v|^2)^{-3/2} & 0 \end{pmatrix} \begin{pmatrix} \partial_x f \\ \partial_v f \end{pmatrix}.$$

We therefore need to bound $\partial_t \partial_x A$. Recall the expression (2.3) for A. Assuming, without loss of generality, that the initial data for the field is trivial $A_0 = A_1 = 0$, that expression reduces to

$$A_f(t,x) = (Y(\cdot,\cdot) *_{t,x} (j_f \mathbb{1}_{t>0})(t,x),$$

where Y is the forward fundamental solution of the wave operator in one-dimension. Therefore [I don't think that what follows is correct]

$$\begin{aligned} \partial_x \partial_t A_f(t,x) &= (\partial_x \partial_t Y(\cdot,\cdot) *_{t,x} (j_f \mathbb{1}_{t>0})(t,x) \\ &= (\partial_t Y(\cdot,\cdot) *_{t,x} (j_{\partial_x f} \mathbb{1}_{t>0})(t,x) \\ &= (\partial_t Y(\cdot,\cdot) *_{t,x} \int_{v\in\mathbb{R}} \partial_t f dv \mathbb{1}_{t>0})(t,x) \quad \partial_t f + \hat{v}\partial_x f + \partial_t A \partial_v f = 0 \\ &= (\partial_{tt} Y(\cdot,\cdot) *_{t,x} (\int_{v\in\mathbb{R}} f dv \mathbb{1}_{t>0} + f_0 \delta_{t=0}))(t,x) \quad (\partial_{tt} - \partial_{xx})Y = \delta_{(t,x)=(0,0)} \\ &= \int_{v\in\mathbb{R}} f(t,x,v) dv + (\partial_{xx} Y(\cdot,\cdot) *_{t,x} (\int_{v\in\mathbb{R}} f dv \mathbb{1}_{t>0} + f_0 \delta_{t=0}))(t,x) \\ &= \int_{v\in\mathbb{R}} f(t,x,v) dv + (\partial_{xx} Y(\cdot,\cdot) *_{t,x} (\int_{v\in\mathbb{R}} f dv \mathbb{1}_{t>0})(t,x) + (Y(\cdot,\cdot) *_{t,x} \int_{v\in\mathbb{R}} \partial_x^2 f_0 \delta_{t=0} dv))(t,x) \end{aligned} \tag{2.37} \{?\}$$

We now apply the division lemma (Lemma A.1) with $a(v) = \hat{v} = v(1+v^2)^{-1/2}$ and obtain that

$$\begin{aligned}
\partial_x\partial_t A_f(t,x) &= \int_{v\in\mathbb{R}} f(t,x,v)dv + (\partial_{xx}Y(\cdot,\cdot) *_{t,x} (\int_{v\in\mathbb{R}} f dv \mathbb{1}_{t>0}))(t,x) + (Y(\cdot,\cdot) *_{t,x} \int_{v\in\mathbb{R}} \partial_x^2 f_0 \delta_{t=0} dv))(t,x \\
&= \int_{v\in\mathbb{R}} (1 + \frac{1}{\hat{v}^2 - 1}) f(t,x,v)dv + \int_{v\in\mathbb{R}} \left(\frac{x}{\hat{v}x - t}\partial_x Y\right) *_{t,x} (\partial_t + \hat{v}\partial_x)(f\mathbb{1}_{t>0}))(t,x,v)dv \\
&+ (Y(\cdot,\cdot) *_{t,x} \int_{v\in\mathbb{R}} \partial_x^2 f_0 \delta_{t=0} dv))(t,x) \\
&= \int_{v\in\mathbb{R}} (2+v^2) f(t,x,v)dv + \int_{v\in\mathbb{R}} \left(\frac{x}{\hat{v}x - t}\partial_x Y\right) *_{t,x} ((\partial_t A \partial_v f)\mathbb{1}_{t>0} + f_0\delta_{t=0}))(t,x,v)dv \\
&+ (Y(\cdot,\cdot) *_{t,x} \int_{v\in\mathbb{R}} \partial_x^2 f_0 \delta_{t=0} dv))(t,x) \\
&= \int_{v\in\mathbb{R}} (2+v^2) f(t,x,v)dv + \int_{v\in\mathbb{R}} \left(\frac{-x^2\hat{v}'}{(\hat{v}x - t)^2}\partial_x Y\right) *_{t,x} (\partial_t A f)\mathbb{1}_{t>0})(t,x,v)dv \\
&+ \int_{v\in\mathbb{R}} \left(\frac{x}{\hat{v}x - t}\partial_x Y\right) *_{t,x} (f_0\delta_{t=0}))(t,x,v)dv + (Y(\cdot,\cdot) *_{t,x} \int_{v\in\mathbb{R}} \partial_x^2 f_0 \delta_{t=0} dv))(t,x)
\end{aligned} \tag{2.38}$$

{?}

By $v \in [-M-1, M+1]$, then we have [to complete]

Appendix A. Division lemma

⟨lem:division⟩ **Lemma A.1 (Division lemma).** *Let $Y(t,x) = \frac{1}{2}\mathbb{1}_{\{|x|\le t\}}$ be the forward fundamental solution of the 1d wave operator. Let $a(v) \in \mathbb{R} \setminus \{\pm 1\}$. Then*

$$\partial_x^2 Y = (\partial_t + a(v)\partial_x)\left(\frac{x}{a(v)x - t}\partial_x Y\right) + \frac{1}{a(v)^2 - 1}\delta_{(t,x)=(0,0)}. \tag{A.1}$$ {?}

Proof. Let $L = x\partial_t + t\partial_x$ and recall the commutator $[\Box, L] = 0$ vanishes. Then we have

$$\Box LY = L\Box Y = L\delta_{(t,x)=(0,0)} = 0. \tag{A.2}$$ {?}

By using the uniqueness of the solution of wave equation, wee see

$$LY = 0 \Leftrightarrow x\partial_t Y = -t\partial_x Y. \tag{A.3}$$?LY0?

Furthermore we have

$$\begin{aligned}
0 &= \frac{tv}{t-xv}\partial_x Y - \frac{tv}{t-xv}\partial_x Y \\
&= \frac{tv}{t-xv}\partial_x Y + \frac{xv}{t-xv}\partial_t Y \\
&= \frac{t}{t-xv}v\partial_x Y + (\frac{xv}{t-xv} + 1)\partial_t Y - \partial_t Y \\
&= \frac{t}{t-xv}v\partial_x Y + \frac{t}{t-xv}\partial_t Y - \partial_t Y
\end{aligned} \tag{A.4}$$ {?}

Therefore

$$\partial_t Y = \frac{t}{t-xv}v\partial_x Y + \frac{t}{t-xv}\partial_t Y = \frac{t}{t-xv}(v\partial_x Y + \partial_t Y).$$

Similarly we have

$$\begin{aligned}
0 &= \frac{x}{xv-t}\partial_t Y - \frac{x}{xv-t}\partial_t Y \\
&= \frac{x}{xv-t}\partial_t Y + \frac{t}{xv-t}\partial_x Y \\
&= \frac{x}{xv-t}\partial_t Y + (\frac{t}{xv-t} + 1)\partial_x Y - \partial_x Y \\
&= \frac{x}{xv-t}\partial_t Y + \frac{xv}{xv-t}\partial_x Y - \partial_x Y
\end{aligned} \tag{A.5}$$ {?}

Therefore

$$\partial_x Y = \frac{x}{xv-t}\partial_t Y + \frac{xv}{xv-t}\partial_x Y = \frac{x}{xv-t}(v\partial_x Y + \partial_t Y).$$

Now we compute

$$\begin{aligned}
\partial_x^2 Y &= \partial_x\big(\frac{x}{xv-t}\big)(v\partial_x Y + \partial_t Y) + \frac{x}{xv-t}(v\partial_x + \partial_t)\partial_x Y \\
&= \partial_x\big(\frac{x}{xv-t}\big)(v\partial_x + \partial_t)Y + (v\partial_x + \partial_t)\big(\frac{x}{xv-t}\partial_x Y\big) - (v\partial_x + \partial_t)\big(\frac{x}{xv-t}\big)\partial_x Y \\
&= (v\partial_x + \partial_t)\big(\frac{x}{xv-t}\partial_x Y\big) + \partial_x\big(\frac{x}{xv-t}\big)\partial_t Y - \partial_t\big(\frac{x}{xv-t}\big)\partial_x Y \\
&= (v\partial_x + \partial_t)\big(\frac{x}{xv-t}\partial_x Y\big) + \Big(\frac{-t}{(xv-t)^2}\partial_t Y - \frac{x}{(xv-t)^2}\partial_x Y\Big) \quad (x\partial_t Y = -t\partial_x Y) \\
&= (v\partial_x + \partial_t)\big(\frac{x}{xv-t}\partial_x Y\big) + \frac{1}{2}\Big(\frac{-t}{(xv-t)^2}(\delta_{x=-t} + \delta_{x=t}) - \frac{x}{(xv-t)^2}(\delta_{x=-t} - \delta_{x=t})\Big) \\
&= (v\partial_x + \partial_t)\big(\frac{x}{xv-t}\partial_x Y\big)
\end{aligned} \tag{A.6}$$ {?}

□

Appendix B. OLD section, Fourier methods, to remove?

Instead of the Vlasov-Maxwell system, as an internal goal, we consider a toy model proposed by Francois Golse

$$\begin{cases} \partial_t f + \hat{v}\cdot\nabla_x f + \partial_t A\cdot\nabla_v f = 0 \\ \Box A = (\partial_{tt}-\Delta)A = \int \hat{v} f(t,x,v)dv = j, \qquad \hat{v} = \frac{v}{\sqrt{1+|v|^2}} \end{cases} \tag{B.1} \{?\}$$

with some smooth and compactly supported initial data $f(0,x,v) = f_0(x,v)$, and consistent data for A: $A(0,x) = A_0(x)$ and $\partial_t A(0,x) = A_1(x)$

This toy model preserves many of the dispersive aspects of the Vlasov-Maxwell system but avoids many of the other difficulties (such as loss of derivatives) associated with Maxwell's equations.

We explicitly write

$$\begin{aligned} A(t,x) &= Data + \int_0^t \frac{\sin{(t-\sigma)\sqrt{-\Delta}}}{\sqrt{-\Delta}} F(\sigma,x)d\sigma, \quad F(\sigma,x) = j(\sigma,x) = \int_{\mathbb{R}^n} \hat{v} f(\sigma,x,v)dv \\ &= Data + \int_0^t \int_{\xi\in\mathbb{R}^n} \frac{e^{i(t-\sigma)|\xi|} - e^{-i(t-\sigma)|\xi|}}{i|\xi|} e^{ix\cdot\xi}\hat{F}(\sigma,\xi)d\sigma d\xi \\ &= Data + (A_+ - A_-)(t,x) \end{aligned} \tag{B.2} \{?\}$$

where

$$A_\pm(t,x) = \int_0^t \int_{\mathbb{R}^n} \frac{e^{\pm i(t-\sigma)|\xi| + ix\cdot\xi}}{i|\xi|}\hat{F}(\sigma,\xi)d\sigma d\xi.$$

We use the integration by parts to obtain that

$$\int_0^t e^{\pm i\sigma|\xi|}\hat{F}(\sigma,\xi)d\sigma = \frac{e^{\pm i\sigma|\xi|}}{\pm i|\xi|}\hat{F}(\sigma,\xi)\Big|_{\sigma=0}^{\sigma=t} - \int_0^t \frac{e^{\pm i\sigma|\xi|}}{\pm i|\xi|}(\partial_\sigma \hat{F})(\sigma,\xi)d\sigma. \tag{B.3} \{?\}$$

Note that $\partial_\sigma \hat{f}(\sigma,\xi,v) + i\hat{v}\cdot\xi\hat{f}(\sigma,\xi,v) + (\widehat{\partial_t A} * \nabla_v \hat{f})(\sigma,\xi,v) = 0$ where $*$ denotes the convolution, hence

$$(\partial_\sigma \hat{f})(\sigma,\xi,v) = -\left(i\hat{v}\cdot\xi\hat{f}(\sigma,\xi,v) + \left(\widehat{\partial_t A} * \nabla_v \hat{f}\right)(\sigma,\xi,v)\right). \tag{B.4} \{?\}$$

Plug this into the above to obtain

$$\begin{aligned} \int_0^t e^{\pm i\sigma|\xi|}\hat{f}(\sigma,\xi,v)d\sigma = \frac{e^{\pm i\sigma|\xi|}}{\pm i|\xi|}\hat{f}(\sigma,\xi,v)\Big|_{\sigma=0}^{\sigma=t} &\pm \int_0^t \frac{e^{\pm i\sigma|\xi|}}{|\xi|}\left(\hat{v}\cdot\xi\hat{f}(\sigma,\xi,v)\right)d\sigma \\ &\mp \int_0^t \frac{e^{\pm i\sigma|\xi|}}{i|\xi|}\left(\widehat{\partial_t A} * \nabla_v \hat{f}\right)d\sigma, \end{aligned} \tag{B.5} \{?\}$$

thus

$$\begin{aligned} &\left(1 \mp \frac{\hat{v}\cdot\xi}{|\xi|}\right)\int_0^t e^{\pm i\sigma|\xi|}\hat{f}(\sigma,\xi,v)d\sigma \\ &= \int_0^t e^{\pm i\sigma|\xi|}\hat{f}(\sigma,\xi,v)d\sigma \mp \int_0^t \frac{e^{\pm i\sigma|\xi|}}{|\xi|}\left(\hat{v}\cdot\xi\hat{f}(\sigma,\xi,v)\right)d\sigma \\ &= \frac{e^{\pm i\sigma|\xi|}}{\pm i|\xi|}\hat{f}(\sigma,\xi,v)\Big|_{\sigma=0}^{\sigma=t} \pm i\int_0^t \frac{e^{\pm i\sigma|\xi|}}{|\xi|}\left(\widehat{\partial_t A} * \nabla_v \hat{f}\right)d\sigma. \end{aligned} \tag{B.6} \{?\}$$

Thus

$$\partial_t A(t,x) = Data + \partial_t A_+(t,x) - \partial_t A_-(t,x) \tag{B.7} \{?\}$$

Now we bound

$$\begin{aligned}\partial_t A_\mp(t,x) &= \int_0^t \int_{\mathbb{R}^n} e^{\mp i(t-\sigma)|\xi|+ix\cdot\xi} \hat{F}(\sigma,\xi)d\sigma d\xi = \int_{\mathbb{R}^n} e^{ix\cdot\xi\mp it|\xi|}\int_0^t e^{\pm i\sigma|\xi|}\hat{F}(\sigma,\xi)d\sigma d\xi \\ &= \int_{\mathbb{R}^n} e^{ix\cdot\xi\mp it|\xi|}\int_0^t e^{\pm i\sigma|\xi|}\int_{v\in\mathbb{R}^n}\hat{v}\hat{f}(\sigma,\xi,v)dvd\sigma d\xi \\ &= \int_{\mathbb{R}^n} e^{ix\cdot\xi\mp it|\xi|}\int_{v\in\mathbb{R}^n}\hat{v}\left(1\mp\frac{\hat{v}\cdot\xi}{|\xi|}\right)^{-1}\left(1\mp\frac{\hat{v}\cdot\xi}{|\xi|}\right)\int_0^t e^{\pm i\sigma|\xi|}\hat{f}(\sigma,\xi,v)d\sigma dv d\xi \\ &= \int_{\mathbb{R}^n} e^{ix\cdot\xi\mp it|\xi|}\int_{v\in\mathbb{R}^n}\hat{v}\left(1\mp\frac{\hat{v}\cdot\xi}{|\xi|}\right)^{-1}\left(\frac{e^{\pm i\sigma|\xi|}}{\pm i|\xi|}\hat{f}(\sigma,\xi,v)\Big|_{\sigma=0}^{\sigma=t}\pm i\int_0^t\frac{e^{\pm i\sigma|\xi|}}{|\xi|}\left(\widehat{\partial_t A}*\nabla_v\hat{f}\right)d\sigma\right)du\end{aligned} \tag{B.8}$$

{?}

Now we need to estimate the following terms:

$$I_\mp = \int_{\mathbb{R}^n} e^{ix\cdot\xi\mp it|\xi|}\int_{v\in\mathbb{R}^n}\hat{v}\left(1\mp\frac{\hat{v}\cdot\xi}{|\xi|}\right)^{-1}\left(\frac{e^{\pm i\sigma|\xi|}}{\pm i|\xi|}\hat{f}(\sigma,\xi,v)\Big|_{\sigma=0}^{\sigma=t}\right)dvd\xi, \tag{B.9}$$

{?}

and

$$II_\mp = \int_{\mathbb{R}^n} e^{ix\cdot\xi\mp it|\xi|}\int_{v\in\mathbb{R}^n}\hat{v}\left(1\mp\frac{\hat{v}\cdot\xi}{|\xi|}\right)^{-1}\left(\int_0^t\frac{e^{\pm i\sigma|\xi|}}{|\xi|}\left(\widehat{\partial_t A}*\nabla_v\hat{f}\right)d\sigma\right)dvd\xi. \tag{B.10}$$

{?}

Let $\theta=\angle(\hat{v},\xi/|\xi|)$, we are worried about the terms when

$$\left|1\mp\frac{\hat{v}\cdot\xi}{|\xi|}\right| \geq 1-|\hat{v}|\cos\theta \geq \begin{cases}(1+|v|^2)^{-1/2}(\sqrt{1+|v|^2}+|v|)^{-1}\to 0\\ 1-\cos\theta\sim\theta^2\to 0\end{cases} \tag{B.11}$$

{?}

Now we consider the following terms:

$$\begin{aligned}I_\mp &= \int_{\mathbb{R}^n}\frac{e^{ix\cdot\xi}}{\pm i|\xi|}\int_{v\in\mathbb{R}^n}\hat{v}\left(1\mp\frac{\hat{v}\cdot\xi}{|\xi|}\right)^{-1}\hat{f}(t,\xi,v)dvd\xi \\ &\quad -\int_{\mathbb{R}^n}\frac{e^{i(x\cdot\xi\mp t|\xi|)}}{\pm i|\xi|}\int_{v\in\mathbb{R}^n}\hat{v}\left(1\mp\frac{\hat{v}\cdot\xi}{|\xi|}\right)^{-1}\hat{f}(0,\xi,v)dvd\xi\end{aligned} \tag{B.12}$$

{?}

and

$$\begin{aligned}II_\mp &= \int_{\mathbb{R}^n} e^{ix\cdot\xi\mp it|\xi|}\int_{v\in\mathbb{R}^n}\hat{v}\left(1\mp\frac{\hat{v}\cdot\xi}{|\xi|}\right)^{-1}\left(\int_0^t\frac{e^{\pm i\sigma|\xi|}}{|\xi|}\left(\widehat{\partial_t A}*\nabla_v\hat{f}\right)d\sigma\right)dvd\xi \\ &= \int_0^t\int_{\mathbb{R}^n}\frac{e^{\mp i(t-\sigma)|\xi|+ix\cdot\xi}}{|\xi|}\int_{v\in\mathbb{R}^n}\hat{v}\left(1\mp\frac{\hat{v}\cdot\xi}{|\xi|}\right)^{-1}\left(\widehat{\partial_t A}*\nabla_v\hat{f}\right)dvd\xi d\sigma \\ &= \int_0^t\int_{\mathbb{R}^n}\frac{e^{\mp i(t-\sigma)|\xi|+ix\cdot\xi}}{|\xi|}\int_{v\in\mathbb{R}^n}\nabla_v\left[\hat{v}\left(1\mp\frac{\hat{v}\cdot\xi}{|\xi|}\right)^{-1}\right]\left(\widehat{\partial_t A}*\hat{f}\right)dvd\xi d\sigma\end{aligned} \tag{B.13}$$

{?}

where we use the integrating by parts and density argument to ignore the boundary term in the last equality. In other word, we assume that $\hat{f}(t,\xi,\cdot) \in C_c^\infty(\mathbb{R}^n)$ has compact support in v but the norm is independent of the compact support. We need to make it be rigorous.

$$
\begin{aligned}
\partial_t A_\mp(t,x) = &\int_{\mathbb{R}^n} \frac{e^{ix\cdot\xi}}{\pm i|\xi|} \int_{v\in\mathbb{R}^n} \hat{v}\left(1 \mp \frac{\hat{v}\cdot\xi}{|\xi|}\right)^{-1} \hat{f}(t,\xi,v)\,dvd\xi \\
&- \int_{\mathbb{R}^n} \frac{e^{i(x\cdot\xi \mp t|\xi|)}}{\pm i|\xi|} \int_{v\in\mathbb{R}^n} \hat{v}\left(1 \mp \frac{\hat{v}\cdot\xi}{|\xi|}\right)^{-1} \hat{f}(0,\xi,v)\,dvd\xi \\
&+ \int_0^t \int_{\mathbb{R}^n} \frac{e^{\mp i(t-\sigma)|\xi| + ix\cdot\xi}}{|\xi|} \int_{v\in\mathbb{R}^n} \nabla_v \left[\hat{v}\left(1 \mp \frac{\hat{v}\cdot\xi}{|\xi|}\right)^{-1}\right] \left(\widehat{\partial_t A} * \hat{f}\right) dvd\xi d\sigma
\end{aligned}
\tag{B.14}
$$
$\boxed{\texttt{sol}}$

Return to our final goal. The trajectory equation

$$
\begin{cases}
\frac{dX}{ds}(s;t,x,p) = \hat{V}(s;t,x,p), \\
\frac{dV}{ds}(s;t,x,p) = (\partial_t A)(s, X(s;t,x,p)),
\end{cases}
\tag{B.15}
$$
{?}

shifting t to 0, it gives

$$
\begin{cases}
\frac{dX}{ds}(s;0,x,p) = \hat{V}(s;0,x,p), \\
\frac{dV}{ds}(s;0,x,p) = (\partial_t A)(s, X(s;0,x,p)).
\end{cases}
\tag{B.16}
$$
{?}

Briefly write $V(s) = V(s;0,x,p)$ and $X(s) = X(s;0,x,p)$, then for the maximal existence interval $[0,T^*)$ our final goal is to control

$$
\begin{aligned}
\sup_{s\in[0,T^*)} V(s) &= \sup_{s\in[0,T^*)} \left(V(0) + \int_0^s (\partial_t A)(\tau, X(\tau;0,x,p))d\tau\right) \\
&= V(0) + \sup_{s\in[0,T^*)} \left(\int_0^s (\partial_t A)(\tau, X(\tau))d\tau\right)
\end{aligned}
\tag{B.17}
$$
{?}

for $s \in [0,T^*]$ and uniformly in x,p. Plug the (B.14) into the above and replace t by τ and x by $X(\tau;0,x,p)$, therefore we main task is to estimate

$$
\begin{aligned}
&\int_0^s (\partial_t A)(\tau, X(\tau))d\tau \\
&= \sum_{\pm} \Big(\int_0^s \int_{\xi\in\mathbb{R}^n} \frac{e^{iX(\tau;0,x,p)\cdot\xi}}{\pm i|\xi|} \int_{v\in\mathbb{R}^n} \hat{v}\left(1 \mp \frac{\hat{v}\cdot\xi}{|\xi|}\right)^{-1} \hat{f}(\tau,\xi,v)\,dvd\xi d\tau \\
&- \int_0^s \int_{\xi\in\mathbb{R}^n} \frac{e^{i(X(\tau;0,x,p)\cdot\xi \mp \tau|\xi|)}}{\pm i|\xi|} \int_{v\in\mathbb{R}^n} \hat{v}\left(1 \mp \frac{\hat{v}\cdot\xi}{|\xi|}\right)^{-1} \hat{f}(0,\xi,v)\,dvd\xi d\tau \\
&+ \int_0^s \int_0^\tau \int_{\xi\in\mathbb{R}^n} \frac{e^{\mp i(\tau-\sigma)|\xi| + iX(\tau;0,x,p)\cdot\xi}}{|\xi|} \int_{v\in\mathbb{R}^n} \nabla_v \left[\hat{v}\left(1 \mp \frac{\hat{v}\cdot\xi}{|\xi|}\right)^{-1}\right] \left(\widehat{\partial_t A} * \hat{f}\right) dvd\xi d\sigma \Big)
\end{aligned}
\tag{B.18}
$$
{?}

for $s \in [0, T^*]$ and uniformly in x, p.

$$
\begin{aligned}
&\nabla_x V(s;0,x,p) = \int_0^s (\partial_t \nabla_x A)(\tau, X(\tau)) \nabla_x X(\tau;0,x,p) d\tau \\
&= \sum_{\pm} \Big(\int_0^s \nabla_x X(\tau;0,x,p) \int_{\xi\in\mathbb{R}^n} \frac{e^{iX(\tau;0,x,p)\cdot\xi}}{\pm i|\xi|} \int_{v\in\mathbb{R}^n} \hat{v} \left(1 \mp \frac{\hat{v}\cdot\xi}{|\xi|}\right)^{-1} \xi \hat{f}(\tau,\xi,v) dv d\xi d\tau \\
&- \int_0^s \nabla_x X(\tau;0,x,p) \int_{\xi\in\mathbb{R}^n} \frac{e^{i(X(\tau;0,x,p)\cdot\xi \mp \tau|\xi|)}}{\pm i|\xi|} \int_{v\in\mathbb{R}^n} \hat{v} \left(1 \mp \frac{\hat{v}\cdot\xi}{|\xi|}\right)^{-1} \xi \hat{f}(0,\xi,v) dv d\xi d\tau \\
&+ \int_0^s \nabla_x X(\tau;0,x,p) \int_0^\tau \int_{\xi\in\mathbb{R}^n} \frac{e^{\mp i(\tau-\sigma)|\xi| + iX(\tau;0,x,p)\cdot\xi}}{|\xi|} \int_{v\in\mathbb{R}^n} \nabla_v \left[\hat{v} \left(1 \mp \frac{\hat{v}\cdot\xi}{|\xi|}\right)^{-1} \xi \right] \left(\widehat{\partial_t A} * \hat{f} \right) dv d\xi d\sigma \Big)
\end{aligned}
\tag{B.19}
$$
{?}

Now we estimate the first term

$$
\begin{aligned}
&\int_0^s \nabla_x X(\tau;0,x,p) \int_{\xi\in\mathbb{R}^n} \frac{e^{iX(\tau;0,x,p)\cdot\xi}}{\pm i|\xi|} \int_{v\in\mathbb{R}^n} \hat{v} \left(1 \mp \frac{\hat{v}\cdot\xi}{|\xi|}\right)^{-1} \xi \hat{f}(\tau,\xi,v) dv d\xi d\tau \\
&= \int_0^s \nabla_x X(\tau;0,x,p) \int_{\xi\in\mathbb{R}^n} e^{iX(\tau;0,x,p)\cdot\xi} \frac{\xi}{\pm i|\xi|} \int_{v\in\mathbb{R}^n} \hat{v} \left(1 \mp \frac{\hat{v}\cdot\xi}{|\xi|}\right)^{-1} \hat{f}(\tau,\xi,v) dv d\xi d\tau
\end{aligned}
\tag{B.20}
$$
{?}

Now we need the following lemma

Lemma B.1. *If $a(v,\xi) \in S^0(v,\xi)$, that is,*

$$
|\partial_\xi^\alpha a(v,\xi)| \le C(1+|\xi|)^{-|\alpha|} \tag{B.21}
$$
{?}

then

$$
\int_{\mathbb{R}^n} \frac{e^{ix\cdot\xi}}{|\xi|} a(v,\xi) \hat{g}(\xi) d\xi \tag{B.22}
$$
{?}

and

$$
\int_{\mathbb{R}^n} \frac{e^{ix\cdot\xi \pm it|\xi|}}{|\xi|} a(v,\xi) \hat{g}(\xi) d\xi \tag{B.23}
$$
{?}

Proof.
Consider $n = 1$, recall $\hat{v} = v/(1+|v|^2)^{1/2}$, we compute

$$
\left(1 - \frac{\hat{v}\cdot\xi}{|\xi|}\right)^{-1} = \begin{cases} \sqrt{1+|v|^2}(\sqrt{1+|v|^2}+v), & \xi > 0, \\ (1+\hat{v})^{-1}, & \xi < 0. \end{cases} \tag{B.24}
$$
{?}

and

$$
\left(1 + \frac{\hat{v}\cdot\xi}{|\xi|}\right)^{-1} = \begin{cases} \sqrt{1+|v|^2}(\sqrt{1+|v|^2}+v), & \xi < 0, \\ (1+\hat{v})^{-1}, & \xi > 0. \end{cases} \tag{B.25}
$$
{?}

We see that

$$
\begin{aligned}
II_- &= \int_0^t \int_{\mathbb{R}} \frac{e^{-i(t-\sigma)|\xi|+ix\cdot\xi}}{|\xi|} \int_{v\in\mathbb{R}^n} \nabla_v \left[\hat{v}\left(1-\frac{\hat{v}\cdot\xi}{|\xi|}\right)^{-1}\right]\left(\widehat{\partial_t A} * \hat{f}\right) dvd\xi d\sigma \\
&= \int_0^t \int_0^\infty \frac{e^{-i(t-\sigma)|\xi|+ix\cdot\xi}}{|\xi|} \int_{v\in\mathbb{R}^n} \nabla_v \left[\hat{v}\sqrt{1+|v|^2}(\sqrt{1+|v|^2}+v)\right]\left(\widehat{\partial_t A} * \hat{f}\right) dvd\xi d\sigma \\
&+ \int_0^t \int_{-\infty}^0 \frac{e^{-i(t-\sigma)|\xi|+ix\cdot\xi}}{|\xi|} \int_{v\in\mathbb{R}^n} \nabla_v \left[\hat{v}(1+\hat{v})^{-1}\right]\left(\widehat{\partial_t A} * \hat{f}\right) dvd\xi d\sigma \\
&= \int_0^t \int_0^\infty \frac{e^{-i(t-\sigma)|\xi|+ix\cdot\xi}}{|\xi|} \int_{v\in\mathbb{R}^n} \nabla_v \left[\hat{v}\sqrt{1+|v|^2}(\sqrt{1+|v|^2}+v)\right]\left(\widehat{\partial_t A} * \hat{f}\right) dvd\xi d\sigma \\
&+ \int_0^t \int_0^\infty \frac{e^{-i(t-\sigma)|\xi|-ix\cdot\xi}}{|\xi|} \int_{v\in\mathbb{R}^n} \nabla_v \left[\hat{v}(1+\hat{v})^{-1}\right]\left(\widehat{\partial_t A} * \hat{f}\right) (-\xi) dvd\xi d\sigma
\end{aligned} \tag{B.26} \{?\}
$$

and

$$
\begin{aligned}
II_+ &= \int_0^t \int_{\mathbb{R}} \frac{e^{i(t-\sigma)|\xi|+ix\cdot\xi}}{|\xi|} \int_{v\in\mathbb{R}^n} \nabla_v \left[\hat{v}\left(1+\frac{\hat{v}\cdot\xi}{|\xi|}\right)^{-1}\right]\left(\widehat{\partial_t A} * \hat{f}\right) dvd\xi d\sigma \\
&= \int_0^t \int_0^\infty \frac{e^{i(t-\sigma)|\xi|+ix\cdot\xi}}{|\xi|} \int_{v\in\mathbb{R}^n} \nabla_v \left[\hat{v}(1+\hat{v})^{-1}\right]\left(\widehat{\partial_t A} * \hat{f}\right) dvd\xi d\sigma \\
&+ \int_0^t \int_{-\infty}^0 \frac{e^{i(t-\sigma)|\xi|+ix\cdot\xi}}{|\xi|} \int_{v\in\mathbb{R}^n} \nabla_v \left[\hat{v}\sqrt{1+|v|^2}(\sqrt{1+|v|^2}+v)\right]\left(\widehat{\partial_t A} * \hat{f}\right) dvd\xi d\sigma.
\end{aligned} \tag{B.27} \{?\}
$$

Therefore we obtain that

$$
\begin{aligned}
II_+ - II_- &= \int_0^t \int_0^\infty \frac{e^{i(t-\sigma)|\xi|+ix\cdot\xi} - e^{-i(t-\sigma)|\xi|-ix\cdot\xi}}{|\xi|} \int_{v\in\mathbb{R}^n} \nabla_v \left[\hat{v}(1+\hat{v})^{-1}\right]\left(\widehat{\partial_t A} * \hat{f}\right) dvd\xi d\sigma \\
&+ \int_0^t \int_0^\infty \frac{e^{i(t-\sigma)|\xi|-ix\cdot\xi} - e^{-i(t-\sigma)|\xi|+ix\cdot\xi}}{|\xi|} \int_{v\in\mathbb{R}^n} \nabla_v \left[\hat{v}\sqrt{1+|v|^2}(\sqrt{1+|v|^2}+v)\right]\left(\widehat{\partial_t A} * \hat{f}\right) dvd\xi \\
&= \int_0^t \int_0^\infty \frac{2i\sin((t-\sigma+x)\xi)}{\xi} \int_{v\in\mathbb{R}^n} \nabla_v \left[\hat{v}(1+\hat{v})^{-1}\right]\left(\widehat{\partial_t A} * \hat{f}\right) dvd\xi d\sigma \\
&+ \int_0^t \int_0^\infty \frac{2i\sin((t-\sigma-x)\xi)}{\xi} \int_{v\in\mathbb{R}^n} \nabla_v \left[\hat{v}\sqrt{1+|v|^2}(\sqrt{1+|v|^2}+v)\right]\left(\widehat{\partial_t A} * \hat{f}\right) dvd\xi d\sigma.
\end{aligned} \tag{B.28} \{?\}
$$

Now we consider the hardest terms which involves $\sqrt{1+|v|^2}(\sqrt{1+|v|^2}+v)$.
We compute that

$$
\partial_t A = Data + (\partial_t A_+ - \partial_t A_-) \tag{B.29} \{?\}
$$

note that

$$
\int_{-\infty}^\infty \frac{\sin\lambda y}{\lambda} d\lambda = \pi\mathrm{sgn}(y),
$$

$$
\begin{aligned}
&\int_{\xi\in\mathbb{R}} e^{\pm i(t-\sigma)|\xi|+i(x-y)\cdot\xi}|\xi|^{-1}d\xi \\
&= \int_{\xi\in\mathbb{R}} (\cos((t-\sigma)\xi)\sin(x-y)\xi)\,\xi^{-1}d\xi \\
&= \int_{\xi\in\mathbb{R}} \frac{\sin(\xi(t-\sigma+x-y))+\sin(\xi(t-\sigma-x+y))}{\xi} d\xi = \mathrm{sgn}
\end{aligned} \tag{B.30} \{?\}
$$

$$
\begin{aligned}
&\left|\int_0^s (\partial_t A)(\tau, X(\tau))d\tau\right| \\
&= \left|\int_0^s B(\tau, X(\tau))d\tau\right| \\
&= \int_0^s \int_0^\tau \int_{y\in\mathbb{R}^n} \int_{\rho\in[0,\infty)} e^{\pm i(\tau-\sigma)\rho} \frac{J_{\frac{n-2}{2}}(|X(\tau)-y|\rho)}{(|X(\tau)-y|\rho)^{\frac{n-2}{2}}} \rho^{n-1} d\rho F(\sigma,y) d\sigma dy d\tau
\end{aligned}
\tag{B.31} \{?\}
$$

If one can prove that there exist a constant C such that

$$
\left|\int_0^s (\partial_t A)(\tau, X(\tau))d\tau\right| \le C\|F\|_{L^1_x}, \quad \forall s\in[0,T^*]
\tag{B.32} \{?\}
$$

then we have done.

Method I: Since $s\in[0,T^*]$, we have

$$
\begin{aligned}
\left|\int_0^s (\partial_t A)(\tau, X(\tau))d\tau\right| &\le C_{T^*}\left(\int_0^s |(\partial_t A)(\tau, X(\tau))|^2 d\tau\right)^{1/2} \\
&\le C_{T^*}\left(\int_0^s \left(\sup_X |(\partial_t A)(\tau, X(\tau))|\right)^2 d\tau\right)^{1/2} \\
&\le C_{T^*}\|\partial_t A(t,x)\|_{L^2([0,T^*];L^\infty_x)}.
\end{aligned}
\tag{B.33} \{?\}
$$

At this moment, we recall the dispersive estimate for half wave propagator. Let $U(t) = e^{it\sqrt{-\Delta}}$, and define

$$
U_j(t) = \int_{\mathbb{R}^n} e^{ix\cdot\xi}\beta(\frac{|\xi|}{2^j})e^{it|\xi|}d\xi
\tag{B.34} \{?\}
$$

then we have

$$
\|U_j(t)\|_{L^1\to L^\infty} \le C2^{j(n+1)/2}(2^{-j}+|t|)^{-(n-1)/2}.
\tag{B.35} \{?\}
$$

Therefore

$$
\begin{aligned}
B_\pm(t,x) &= \sum_{j\in\mathbb{Z}} B^j_\pm(t,x) \\
&= \sum_{j\in\mathbb{Z}} \int_0^t \int_{\mathbb{R}^n} e^{\pm i(t-\sigma)|\xi| + ix\cdot\xi}\beta(2^{-j}|\xi|)\hat{F}(\sigma,\xi)d\sigma d\xi \\
&= \sum_{j\in\mathbb{Z}} \int_0^t U_j(\pm(t-\sigma))F(\sigma)d\sigma
\end{aligned}
\tag{B.36} \{?\}
$$

By the conservation law, we have

$$
\iint_{\mathbb{R}^3_x\times\mathbb{R}^3_v} e(v)f(t,x,v)dxdv + \frac{1}{2}\int_{\mathbb{R}^3}\left(|\partial_t A|^2 + |\nabla A|^2\right)(t,x)dx = \text{const}
\tag{B.37} \{?\}
$$

where

$$
\nabla_v e = \hat{v} \implies e(v) = (1+|v|^2)^{1/2}.
$$

The best result is to use the conservation quantity to control $V(s,t,x,v)$. Now we analyze the integral

$$
\int_0^{T^*}\int_{\sigma\in[0,s]}\int_{\mathbb{R}^3} e^{\pm i(s-\sigma)|\xi| + iX(s;0,x,v)\cdot\xi}\hat{F}(\sigma,\xi)d\sigma d\xi ds
\tag{B.38} \{?\}
$$

References

Gerard2010 [GP10] Patrick Gerard and Christophe Pallard. A mean-field toy model for resonant transport. *Kinetic and Related Models*, 3(2):299–309, may 2010.

Nguyen2014c [NP14] Charles Nguyen and Stephen Pankavich. A one-dimensional kinetic model of plasma dynamics with a transport field. *Evolution Equations and Control Theory*, 3(4):681–698, oct 2014.

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