Linear stability and instability of plasmas

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Abstract

In this note we describe some abstract results of spectral theory for unbounded operators in Hilbert spaces and their application to the system of kinetic equations known as *Vlasov-Poisson* system describing the evolution of an electrostatic plasma. We recall the main definitions of stability and investigate them for some steady states of such a system.

1 Introduction

The main goal of *kinetic theory* is the description of a many-particle system in the limit as the number of particles tends to infinity in a statistical way. There are different levels of description of this sort of systems: the microscopic, the macroscopic and the mesoscopic level. The *microscopic* (or Newtonian) mechanical viewpoint considers every particle as a point in the space and writes down the Newtonian equations for the interactions between each of them. This turns out to be already problematic when the number of particles is equal to 3 (i.e. the 3-body problem) and then, as we are often interested in studying systems such as gases where the order of magnitude of the number of particles is 10^{23}, such an approach is often impossible to use. In the *macroscopic* approach, instead of considering single particles, we can take a cube in the space, small enough in relation to the entire system, but large enough to be considered as a unit. We can then study conservation properties of momenta, matter etc. for such a unit. Such an approach leads to hydrodynamics and Euler and Nevier-Stokes equations. Somewhere in between these two approaches there is the *mesoscopic* viewpoint where we consider molecular interactions, but we do not care about the behaviour of each single particle, but, instead, about the *proportion* of particles. This allows us to use statistical ideas and to merge them with more analytical results.

An interesting application of the mesoscopic viewpoint is the description of the behaviour of particular gases called *plasmas*. A plasma, generally speaking, is a gas of (partially or totally) ionised particles. Among them, many differences exist: the levels of density can be extremely low or extremely high, the pressures can vary considerably, and the proportion of ionised particles can also vary over several orders of magnitude. Nonelastic collisions, recombination processes may be very important as well. Collisionless kinetic theory of gases is generally applied when the density is low and nonelastic processes can be neglected. An interesting and important feature of plasmas, both from the mathematical and the physical point of view, is the presence of Coulomb interactions between charged particles. The main equation in plasma physics describing the evolution of the density of the particles of a plasma is the so called *Vlasov* equation where the oscillations in the plasma are described by the presence of a nonlinear forcing term. Usually such an equation is coupled with a *Poisson* equation for the electric potential where the initial density of the ions is taken into account. The resulting system of equation is the *Vlasov-Poisson* system and in this work we focus on that.

The study of such a system represents a field of great interest and a lot of studies have been developed in the last years on that. From a mathematical point of view, questions as existence, uniqueness and regularity of the solutions have been investigated (see [Vil02], [MP94] and [Pfa92] for general results) as well as important questions regarding the *stability* of the steady states of such a system. Various tools have been developed in order to deal with this last question. Initially, physicists and mathematicians developed parallel methods to analyse stability properties, but in the last thirty years these methods merged together thus giving new sources and tools for such an analysis.

With this spirit in mind, we organize this report as follows: in Section 2 we recall at first some well-known definitions and theorems regarding spectral theory of unbounded operators on Hilbert spaces. We follow mainly the approach given by Kato [Kat95]. We give next some careful definitions of the concept of *stability* mentioned before. In Section 3 we present some of the first tools arisen in a physical framework (see [Pen67] and [Nic83]) that, by exploiting nice and simple results of complex analysis, help us in the investigation of the stability of the Vlasov-Poisson system. The main result in such a contest is the *Penrose criterion* described in detail in this very section. We then move to Section 4, which is written rather informally, where we link some spectral properties of particular operators introduced in order to describe the Vlasov-Poisson system, with some results of instability (see [Lin01]-[BR93a] for references). We conclude our report with Section 5 where, referring mainly to the paper of Guo and Strauss [GS95b], we give a quite detailed proof of *nonlinear* instability exploiting bounds and estimates coming from the mathematical world of spectral theory, linking them with the ones of physics origin via the application of the Penrose criterion.

2 Operators on Hilbert spaces

2.1 Convergence of operators and the Spectral Theorem

We recall some notions about linear and bounded operators, following essentially the approach given by [Kat95, Chapters VI, VIII, X]. Henceforth, we will consider an Hilbert space H defined on $\mathbb{C}$ endowed with a sesquilinear product.

Definition 2.1. *Let $(H, \|\cdot\|)$ a Hilbert space. We denote by $\mathcal{B}(H)$ the space of all the linear and bounded operators on H to H. Such a space is a normed space with norm defined by:*

$$\|T\|_* := \sup_{u\in H, u\neq 0} \frac{\|Tu\|}{\|u\|} = \sup_{u\in H, \|u\|\leq 1} \|Tu\|\,.$$

Endowed with such a norm, $\mathcal{B}(H)$ turns out to be a Banach space.

If the space H is infinite-dimensional, different types of convergence can be defined in $\mathcal{B}(H)$, differently from the finite-dimensional case. We specify these different types of convergence, pointing out the relationships between them.

Definition 2.2 (Convergence in $\mathcal{B}(H)$)**.** *Let $\{T_n\}_{n\geq 1} \subset \mathcal{B}(H)$ and $T \in \mathcal{B}(H)$. Then the sequence $\{T_n\}_n$ is said to be convergent to T*

- ***in norm** (or **uniformly**) if $\|T_n - T\|_* \to 0$ as $n \to \infty$.*
- ***strongly** if for every $u \in H$, $T_n u \to Tu$ or, equivalently, $\|T_n u - Tu\| \to 0$ as $n \to \infty$.*
- ***weakly** if for every $u \in H$, $T_n u$ converges weakly to Tu, that is $(T_n u, g) \to (Tu, g)$ as $n \to \infty$ for every $g \in H^* = H$.*

Convergence in norm implies strong convergence and strong convergence implies weak convergence. In the following, we shall use the notations u-lim, s-lim and w-lim to indicate uniform, strong and weak limit, respectively.

Generally, we deal with operators not everywhere defined in H, but just on a dense subspace of H.

Definition 2.3. *Let T an operator defined on H. T is said to be **unbounded** if its domain $D(T)$ is a dense subspace of H.*

In order to make the following discussion consistent with the forthcoming results in spectral theory we are going to present, we recall the definitions of adjoint and selfadjoint operators. We remark that we are considering Hilbert spaces for the sake of simplicity, but everything can be generalised in the case of Banach spaces.

Definition 2.4. *Given an unbounded operator $T : H \to H$ we say that the operator $T^* : H \to H$ is the **adjoint** of T if $(Tx, f) = (x, T^* f)$ for all $f \in D(T^*)$, $x \in D(T)$ and $D(T^*)$ is defined as the maximal subspace with such a property. T is said to be **selfadjoint** if $T = T^*$, which explicitly means $D(T) = D(T^*)$ and $(Tx, f) = (x, T^* y)$ for all $f \in D(T)$. In the finite-dimensional case T is represented by a Hermitian matrix.*

Before stating the modified version of the Spectral Theorem in this setting, we have to define some objects which, in some sense, simulate the role that in the finite-dimensional case was played by the eigenspaces of a selfadjoint operator.

Let us consider in H an uncountable family $\mathcal{M} := \{M_\lambda\}_{\lambda\in\mathbb{R}}$ of closed subspaces of H depending on the real parameter λ. We require $\mathcal{M}$ to satisfy the following properties:

- the family $\mathcal{M}$ is *nondecreasing* with respect to λ, that is $M_{\lambda_1} \subset M_{\lambda_2}$ if $\lambda_1 < \lambda_2$;
- the infinite intersection $\bigcap_{\lambda\in\mathbb{R}} M_\lambda$ is $\emptyset$;
- the infinite union $\bigcup_{\lambda\in\mathbb{R}} M_\lambda$ is a dense set in H;
- the family is *right continuous* in the sense that for every fixed λ the equality
$$M_{\lambda^+} := \bigcap_{\lambda' > \lambda : M_\lambda \subset M_{\lambda'}} M_{\lambda'} = M_\lambda$$
holds true.

Because of the assumptions of closedness of the subspaces M_λ, we can also consider for every fixed λ the corresponding orthogonal projection P_λ onto the subspace M_λ. We can then collect all such projections and construct the family $\mathcal{P} := \{P_\lambda\}_{\lambda\in\mathbb{R}}$ of operators.

Definition 2.5 (Spectral family)**.** *The family $\mathcal{P}$ is called a* spectral family *if the following properties are satisfied:*

- *$\mathcal{P}$ is nondecreasing, that is for every $\lambda_1 < \lambda_2$ there holds $P_{\lambda_1} \le P_{\lambda_2}$;*
- *s-$\lim P_\lambda = 0$ as $\lambda \to -\infty$;*
- *s-$\lim P_\lambda = Id$ as $\lambda \to +\infty$;*
- *Defining the projection $P_{\lambda+}$ on $M_{\lambda+}$ as $P_{\lambda+} = s\text{-}\lim_{\varepsilon\to 0} P_{\lambda+\varepsilon}$ we require $\mathcal{P}$ to be right-continuous, meaning that $P_{\lambda+} = P_\lambda$ for every $\lambda \in \mathbb{R}$.*

It is possible now to define for every semiclosed interval $I = (\lambda_1, \lambda_2]$ of the real line the quantity:

$$P(I) := P_{\lambda_2} - P_{\lambda_1}$$

which can be seen as the projection on the subspace $M_{\lambda_2} \cap M_{\lambda_1}^{\perp}$. Using the previous definitions, we aim now to define a measure on all the Borel sets of the real line related to the spectral family $\mathcal{P}$. At first we can define it just on the family of all the union of a finite number of intervals (open, closed and semiclosed) of $\mathbb{R}$. Each element of such a family can be expressed as the union of disjoint sets of the form of I or singletons $\{\lambda\}$. We have already defined the quantity $P(I)$ for the former case and we can use a similar procedure to define the measure we want in correspondence of the singletons (we refer the reader to [Kat95, Chapter VI.2] where such a construction is done rigorously), thus obtaining a measure on the family above which can be extended to the class of all Borel sets using standard measure-theory constructions. Moreover, if for every fixed $u \in H$ we consider the function $f(\lambda) := (P_\lambda u, u)$, thanks to the properties of the family $\mathcal{P}$ given in Definition 2.5 it is easy to show that f is nonnegative, nondecreasing and it tends to 0 as $\lambda \to -\infty$ and to $\|u\|^2$ as $\lambda \to +\infty$. Moreover, it turns out that the function $(P_\lambda u, v)$ is of bounded variation.

We can then define an important object that allows us to state the spectral theorem for unbounded operators. We define an operator A as:

$$A := \int_{\mathbb{R}} \lambda dP_\lambda. \tag{1}$$

The domain of A, $D(A)$ is the set of all the $u \in H$ such that:

$$\int_{\mathbb{R}} \lambda^2 df(\lambda) < \infty$$

where the integral is intended to be a *Stieltjes integral.* For $u \in D(A)$ and every $v \in H$ we have then:

$$(Au, v) = \int_{\mathbb{R}} \lambda d(P_\lambda u, v).$$

The condition on the domain of A ensures that the integral above is convergent and it is possible to show that the operator (1) is symmetric and, mainly, selfadjoint, but the proof of these facts is quite technical and we refer the reader to Kato [Kat95, Chapter VI] for details.

Now, we have been able to define the selfadjoint operator (1) starting from a spectral family, but we would like to reverse the construction. This is ensured thanks to the following statement of the Spectral Theorem.

Theorem 2.6 (Spectral Theorem)**.** *Every selfadjoint operator A admits an expression* (1) *by means of a spectral family $\{P_\lambda\}_{\lambda\in\mathbb{R}}$ which is uniquely determined by A.*

We do not present here the proof of the theorem, but we point out that the key point in it is defining a suitable spectral family which satisfies the properties listed in Definition 2.5 and then checking that the selfadjoint operator defined by such a family coincides exactly with the given A. This is done by exploiting the *polar decomposition* of the operator $(A - \lambda Id)$ (see [Kat95, Section VI.3]).

2.2 Classification of spectra

Let us now consider a selfadjoint operator A in a Hilbert space H. We want now to study the spectrum of such an operator which, in the general case we are investigating, is not made up *just* by the eigenvalues of A, but consists instead of different sets. We are following again the approach given in [Kat95], but we just mention that different authors use slightly different definitions sometimes.

Definition 2.7. *Let $A : H \to H$ be a selfadjoint operator on a Hilbert space H and let A_λ be defined as $A_\lambda := A - \lambda Id$. Then the spectrum of A, $\sigma(A)$, can be subdivided into the following distinct types of spectra:*

- *the **point spectrum** $\sigma_p(A)$, consisting of all the eigenvalues of A, that is*

$$\begin{aligned}\sigma_p(A) &= \{\lambda \in \mathbb{R} : \exists u \in H, u \neq 0, Au = \lambda u\} \\ &= \{\lambda \in \mathbb{R} : A_\lambda \textit{ is not injective } \};\end{aligned}$$

- *the **continuous spectrum** $\sigma_c(A)$, consisting of all $\lambda \in \mathbb{R}$ such that A_λ is invertible and its range is dense in H;*

- *the **residual spectrum** $\sigma_r(A)$, consisting of all $\lambda \in \mathbb{R}$ such that A_λ is invertible and its range is not dense H.*

2.3 Stability of dynamical systems

We now want to give careful definitions of a very important concept both in mathematics and physics, that is the one of *stability.* In this work we want to focus our analysis on the results regarding the Vlasov-Poisson system of equations (see Sections 1 and 3), but such a concept has been widely studied in various other different fields. Heuristically, we can think about stability as the property that a dynamical system possesses whenever typically small perturbations of an orbit of the system do not effect too much the behaviour of the orbit itself. In other words we have a *stable* behaviour when, once perturbed slightly an orbit of the system, the perturbed orbit stays *close* to the original one. According to the different setups we are working with, different notions of stability arise.

Let us specify the framework where we want to work. We will consider the following autonomous ordinary differential equations, defined for all $t \in \mathbb{R}^+$ and functions $\mathbf{u} : \mathbb{R}^+ \to \mathbb{R}^n$ and $\mathbf{G} : \mathbb{R}^n \to \mathbb{R}^n$:

$$\frac{d}{dt}\mathbf{u}(t) = \mathbf{G}(\mathbf{u}(t)) \tag{2}$$

endowing it with the initial condition $\mathbf{u}(0)$.

Definition 2.8. *We say that the solution $\boldsymbol{u}(t)$ of the system* (2) *is an **equilibrium** or, equivalently, a **steady state** if $\frac{d}{dt}\boldsymbol{u}(t) = \boldsymbol{G}(\boldsymbol{u}(t)) = 0$.*

Definition 2.9 (Lyapunov stability)**.** *An equilibrium $\bar{\mathbf{u}}$ of the system* (2) *is said to be **Lyapunov-stable** if for every neighborhood $U \subset \mathbb{R}^n$ of $\bar{\mathbf{u}}$, there is a neighborhood $V \subset U$ of $\bar{\mathbf{u}}$ such that every solution $\boldsymbol{u}(t)$ with initial condition $\boldsymbol{u}(0) \in V$ remains in U for every $t > 0$. If $\bar{\mathbf{u}}$ is not Lyapunov-stable, we say that it is **unstable**.*

We remark that in the definition above we are not requiring the solution $\mathbf{u}(t)$ to approach the equilibrium $\bar{\mathbf{u}}$ as t tends to infinity. This stronger property is typically called *asymptotic stability*, but it goes beyond the purposes of this work. We point out that in the following we shall refer to the notion of Lyapunov-stability given in Definition 2.9 as, simply, stability.

As we shall see below, a typical procedure often exploited to analyse the behaviour of the solutions to (2) is the so-called *linearisation* technique. Suppose we know that an equilibrium $\bar{\mathbf{u}}$ of the system exists, we now consider the function $\tilde{\mathbf{u}}(t) = \bar{\mathbf{u}} + \varepsilon\zeta(t)$, where the function ζ is a perturbation of $\bar{\mathbf{u}}$ and $\varepsilon > 0$ is a small parameter. Substituting such an expression in (2) we consider now a first-order approximation: we neglect then all the terms with ε^2 and we use the properties of the equilibrium, thus obtaining a new, simpler ODE for ζ called the *linearised* equation around μ.

We point out now the essential difference between *linear* and *nonlinear* stability.

Definition 2.10. *Let $\bar{\mathbf{u}}$ an equilibrium of the system* (2)*. We say that $\bar{\mathbf{u}}$ is:*

- ***linearly stable** if it is Lyapunov-stable for the linearised equation of* (2)*;*
- ***nonlinearly stable** if it is Lyapunov-stable for the original equation.*

Typically, stability properties are analysed from a qualitative point of view, by studying eigenvalues and eigenvectors of the Jacobian matrix $D\mathbf{G}(\bar{\mathbf{u}})$ appearing in the process of linearisation. Informally, we are interested in understanding whether the size of solutions grows, stays constant, or shrinks as t tends to infinity. This can be usually answered just by examining the eigenvalues of $D\mathbf{G}(\bar{\mathbf{u}})$, which seems to justify partially the notions given in Section 2.

3 The Vlasov-Poisson equations in plasma physics

The *Vlasov-Poisson* system of equations is typically used to describe the evolution of the particles of a plasma. The unknown distribution function $f(x, v, t)$

is a probability density function defined usually on the space $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+$ and can be thought of as the number of particles at any given time $t \geq 0$ in a small region of the six-dimensional phase space of a single plasma divided by the volume of the small region itself. The system takes usually the following form:

$$\begin{cases} \partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) + F(t,x) \cdot \nabla_v f(t,x,v) = 0, \\ F(t,x) = -\nabla_x U(t,x), \\ \Delta U(t,x) = \rho_0 \pm \rho(t,x), \\ \rho(t,x) = \displaystyle\int_{\mathbb{R}^3} f(t,x,v) dv. \end{cases} \tag{3}$$

where ρ is the local density of the particles, U their Coulomb potential and ρ_0 is the constant representing the positively charged background density. The system of equations with $\rho_0 = 0$ and a plus in the equation for ΔU where U is the gravitation Newtonian potential, represents the set of equations arising in the description of the distribution of stars in galaxies. We refer the reader to [GL08] and [BMR95] for further details.

In the following we will just focus on the case with $\rho_0 \neq 0$ and a minus in (3). More generally, one can couple the *Vlasov* equation (the first one in the system above) with the Maxwell equations, thus obtaining the *Vlasov-Maxwell* system of equations where the effects of a magnetic field are considered as well (see [Ben11] and [LS07]).

Important questions of existence, uniqueness and (partial) regularity of the solution to (3) have been solved at the end of the eighties, we refer the reader to [Pfa92] for further details.

3.1 Steady states

Recalling Definition 2.8, we are now interested in finding equilibria f to (3), that is, functions for which $\partial_t f = 0$. In Sections 4 and 5 we will analyse the stability of such equilibria under small perturbations. To this purpose, in a very heuristic and informal way, we interpret the left hand side of the Vlasov equation in (3) as the total time derivative of f along a particle orbit. Namely, let $X(t)$ and $V(t)$ the functions that give the position x and the velocity v in the phase-space at time t, respectively. Then, the total time derivative of any quantity measured along the particle's orbit in phase space is:

$$\begin{aligned} \frac{D}{Dt} &= \partial_t + \frac{dX(t)}{dt} \cdot \nabla_x + \frac{dV(t)}{dt} \cdot \nabla_v \\ &= \partial_t + \frac{dx}{dt} \,|_{\text{orbit}} \cdot \nabla_x + \frac{dv}{dt} \,|_{\text{orbit}} \cdot \nabla_v \\ &= \partial_t + v \cdot \nabla_x + F(t,x) \cdot \nabla_v \end{aligned} \tag{4}$$

where we have considered the particle having mass equal to 1 and F then is the Coulomb force. Thanks to (4) the Vlasov equation can be rewritten in the following way:

$$\frac{D}{Dt} f(x,v,t) = 0.$$

Now, suppose we can construct f such that it depends on some functions $C_i(x,v,t), i \geq 1$ which are constants of motion along the orbit of a particle.

Then, by the above expression and the chain rule:

$$\frac{D}{Dt}f(C_1(x,v,t),C_2(x,v,t),\cdots) = \sum_i \frac{\partial f}{\partial C_i}\frac{D}{Dt}C_i = 0$$

and so the Vlasov equation is satisfied. In other words, we are able to generate new solutions of the Vlasov equation once we know that such a function is a function only of the constants of the motion of the individual particle orbits. Such a result is typically known as Jeans Theorem (see [Den90] for further details).

Let us look now at the equilibria. We distinguish the two cases in which the Coulomb force F is 0 and when it is not.

3.1.1 Case $F = 0$

In absence of external fields, the kinetic energy and momentum of a particle are constants of the motion and both these quantities depend just on the velocity $v \in \mathbb{R}^3$ of the particle itself. Thus, any function $f = f(v_x, v_y, v_z)$ is a solution of the time-independent Vlasov equation. From a mathematical point of view, this can be seen just writing down the time-independent Vlasov equation with no external forces, which turns out to be simply:

$$v \cdot \nabla_x f = 0.$$

3.1.2 Case $F = F(x) \neq 0$

In the presence of an arbitrary electric field having a given direction, the momenta and the sum of kinetic and the electric potential energy are constants of the motion and then every function of them is an equilibrium distribution function.

3.2 Criteria for stability: the Nyquist method and the Penrose criterion

We want now to analyse the behaviour of the equilibria of the Vlasov-Poisson system (3) which are not depending on space (the so-called *homogeneous* equilibria). In the following we shall describe some powerful ideas and tools that have been exploited for a lot of years and that arose around 1960 thanks to Penrose in [Pen67] and others. These ideas have, substantially, physical origins and they have not been developed from a mathematical point of view till 1980. In Sections 4 and Section 5 we will develop a more mathematical approach, using these same tools, but presenting them in a more rigorous way.

We consider then an homogeneous equilibrium $\mu(v)$ and look for plane wave solutions of the linearised equation. By applying Laplace transform to the perturbation, it is possible to replace the presence of time and space derivatives, finding a complex-valued expression containing just v-derivatives of μ. The expression that one obtains is:

$$f(\omega,k,v) = \frac{eE(\omega)\mu'(v) + f(0,\omega,v)}{-i\omega + ikv}$$

where $E(\omega)$ is the spatially homogeneous part of the perturbed electric field and ω is the frequency variable replacing t after applying Laplace transform. Substituting into Poisson's equation we get:

$$ik\varepsilon(k,\omega)E(\omega) = \frac{-4\pi e}{ik}\int \frac{f(0,k,v)}{v-\omega/k}dv$$

where

$$\varepsilon(k,\omega) = 1 - \frac{4\pi e^2}{k^2}\int \frac{\mu'(u)}{u-\omega/k}du \quad \forall\omega\in\mathbb{C}. \tag{5}$$

is the so-called *dielectric function*. Hence we have:

$$E(\omega) = \frac{4\pi e}{k^2\varepsilon(k,\omega)}\int \frac{f(0,k,v)}{v-\omega/k}dv \tag{6}$$

and, therefore, using inverse Laplace transform we have:

$$E(t) = \int_C \frac{E(\omega)}{2\pi}e^{-i\omega t}d\omega$$

where the integration must be carried over a contour C that passes above all the poles of $E(\omega)$. By representation (6) we are then interested in studying the behaviour of the function $\varepsilon(\omega,k)$ in the complex ω-plane. Such a function contains in fact all the information concerning the linear stability of the equilibrium μ (for further details see [Nic83, Chapter 6]). Let us study the function $\varepsilon(k,\cdot)$, considering k as a fixed real parameter so that we can consider ε as a function of ω only. Consider the new function $(1/\varepsilon)\partial\varepsilon/\partial\omega$. We observe that we have:

$$\frac{1}{2\pi i}\int_{c_\omega}\frac{1}{\varepsilon}\frac{\partial\varepsilon}{\partial\omega}d\omega = N = \#\text{ zeros of }\varepsilon\text{ inside }c_\omega \tag{7}$$

where c_ω is any closed contour in the complex ω-plane and the integration is performed in the counterclockwise direction. We are assuming that $\partial\varepsilon/\partial\omega$ has no poles in the enclosed region and ε has only simple zeros. The first equality in (7) could be obtained simply by Taylor expanding both the function ε and $\partial\varepsilon/\partial\omega$ near any simple zero ω_0 of ε and applying then the residue theorems to get the right hand side of (7). As it known, the region of instability in the ω-complex plane is the upper half-plane of all the $\omega\in\mathbb{C}$ having imaginary part $\omega_i > 0$. Hence, we need to consider a contour that includes *all* of the upper half-plane. In (7) we can change the variables inside the integral, considering now a corresponding contour c_ε in the complex ε-plane. Such a contour is obtained simply by evaluating $\varepsilon(k,\cdot)$ at each point on the countour c_ω in the ω-plane. Thus, we have:

$$\frac{1}{2\pi i}\int_{c_\omega}\frac{1}{\varepsilon}\frac{\partial\varepsilon}{\partial\omega}d\omega = \frac{1}{2\pi i}\int_{c_\varepsilon}\frac{1}{\varepsilon}d\varepsilon = N \tag{8}$$

which says that c_ω must contain N zeros of ε so that in the ε-plane the contour c_ε must circle the origin N times. Figure 1 contains a representation of what we have just described.

Definition 3.1 (Nyquist method for stability)**.** *A system described by a dielectric function $\varepsilon(k,\omega)$ is stable if the curve c_ε, found by mapping the curve c_ω encircling the upper half ω-plane, does not encircle the origin $\varepsilon = 0$. If c_ε encircles the origin one or more times the system is unstable.*

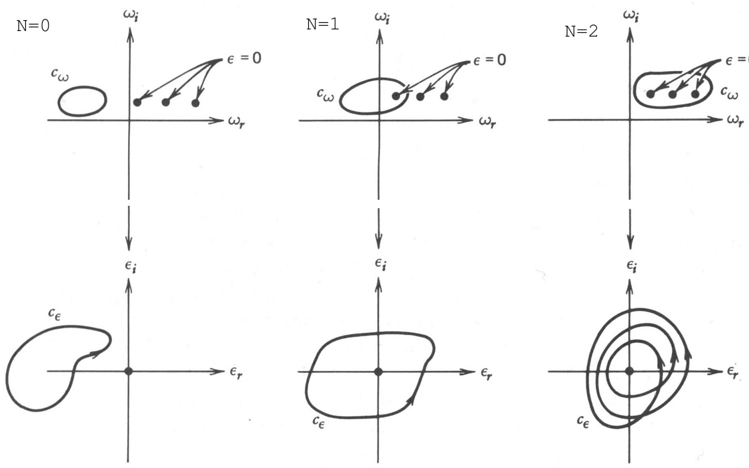


Figure 1: The ω-plane (on the top row) and the ε-plane (on the bottom row) in three different situations (N=0,1,2). The contours are drawn in the counter-clockwise direction (see [Nic83, Fig. 6.19]

Using a particular expression for ε and with really simple arguments which can be nicely and easily understood by exploiting the Nyquist method, it is possible to show the following result which gives us a result of stability in the case in which the equilibrium μ is single-humped (see [Nic83, Section 6.9]):

Theorem 3.2 (Gardner)**.** *Let $\mu(v)$ a single-humped equilibrium of* (3)*. Then, the origin $\varepsilon = 0$ is not encircled by c_ε or, equivalently, μ is stable.*

We want now to focus on the more general case in which the homogeneous steady state μ has a double hump, with a relative minimum between them, as shown in Figure 2.

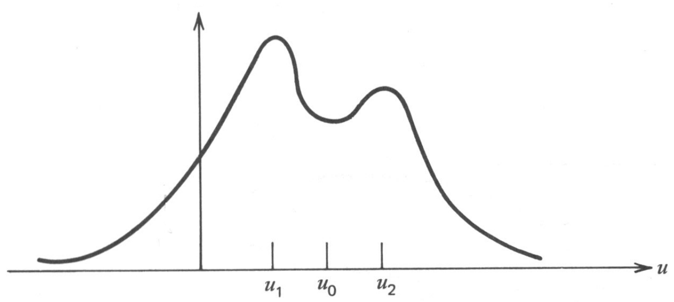


Figure 2: The behaviour of a double-humped equilibrium [Nic83, Fig. 6.25].

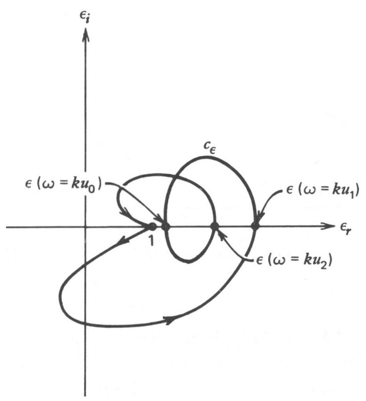
We start the investigation in such a case using the representation of ε given by (5). Let us start by considering the original contour c_ω, containing all the upper ω-half-plane. In order to do that we start from considering the semicircle at infinity. We have that:

$$\varepsilon(k, \omega) \longrightarrow 1 \quad \text{as } |\omega| \to +\infty$$

since the second term in (5) vanishes at infinity. So, in correspondence of such a semicircle-limit we do not have any root of ε. Pictorially, our c_ε is then going to end in 1 and, as it must be a closed orbit, we need that it starts from 1 as well. We want now to understand what could be the behaviour along the real ω-axis. By exploiting a representation similar to (5), but holding just for $\omega \in \mathbb{C}$ with imaginary part equal to 0 it is possible to 'foresee' the behaviour of the contour c_ε as such elements tend to $+\infty$ and $-\infty$, thus finding that:

$$\text{if } \omega \to +\infty \text{ then } \varepsilon_i > 0$$
$$\text{if } \omega \to -\infty \text{ then } \varepsilon_i < 0$$

where, for notational simplicity, we have indicated with ε_i the imaginary part of ε. We know now how the contour is going to be *close to infinity*, but we want to close it taking into account the structure of μ. As a double-humped $\mu(v)$ has three different positions such that $\mu'(v) = 0$, the contour c_ε must encounter the crossings of the real ε-axis three times, namely in correspondence of the values $\varepsilon(\omega_1 = ku_1)$, $\varepsilon(\omega_0 = ku_0)$ and $\varepsilon(\omega_2 = ku_2)$, in this very order. It is possible to show that if u_1 is the absolute maximum of μ, there holds $Re(\varepsilon(\omega_0 = ku_0)) > 1$, which pictorially means that such crossing is on the right of the starting point 1. We are now allowed two more crossings of the ε-real axis. The only two possibilities allowed (i.e. respecting the order) are shown in the following Figure 3. We see that the first one gives stability, while the second



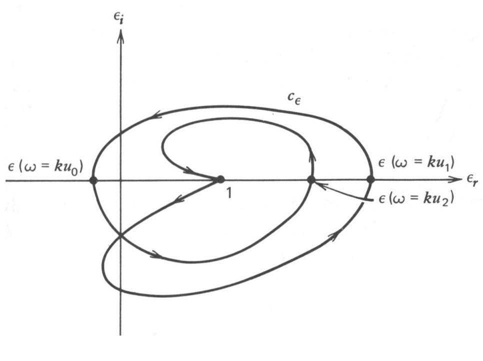


Figure 3: Contour c_ε for a stable and unstable distribution, respectively [Nic83, Fig. 6.26].

one indicates one unstable behaviour. This suggests that a necessary condition for instability is $\varepsilon(\omega_0 = ku_0) < 0$ which, recalling representation (5), means:

$$\varepsilon(k, \omega_0 = ku_0) = 1 - \frac{4\pi e^2}{k^2} \int \frac{\mu'(u)}{u - u_0} du < 0.$$

By subtracting $0 = \mu'(u_0)$ in the numerator, we can then integrate by parts to obtain:

$$\varepsilon(k, \omega_0 = ku_0) = 1 + \frac{4\pi e^2}{k^2} \int \frac{\mu(u_0) - \mu(u)}{(u - u_0)^2} du$$

which, by what we have just seen, must be less than 0 for instability. A sufficient condition for it is:

$$\int \frac{\mu(u_0) - \mu(u)}{(u - u_0)^2} du < 0. \tag{P}$$

In fact, if (P) is true, reestablishing the dependence of ε on the variable k we see that:

$$\begin{aligned} \varepsilon(k, \omega_0 = ku_0) &\to 1 \qquad \text{as } k \to \infty \\ &\to -\infty \text{ as } k \to 0 \end{aligned}$$

which entails the existence of a $\bar{k}$ such that $\varepsilon(\omega_0 = \bar{k}u_0) < 0$ while $\varepsilon(\omega_2 = \bar{k}\omega_2 > 0$, which is exactly the condition for instability we want. Equation (P) is called the *Penrose criterion* for instability ant it is a sufficient and necessary condition for the linear instability of the Vlasov-Poisson equilibrium (see [Pen67] for a complete description). In Section 5 we will link such a condition also with the *non*linear instability of a steady-state.

4 Linear instability of Vlasov-Poisson system

In this section we adopt a more mathematical approach trying to give a flavour of the reasons why and how the abstract results described in Section 2 can be used in order to analyse *linear* stability properties of the system (3). In the following we are not going to be very rigorous, so we refer the reader to [Lin01] and [Ben11] for further and more precise details.

As pointed out in Definition 2.10, the first step in the analysis of the linear stability of the equilibria of (3) is the linearisation of the system. We observe that the only nonlinearity appearing in the Vlasov-Poisson system is the forcing F term in the Vlasov equation. We apply a standard linearisation technique considering then an homogeneous equilibrium $\mu(v)$ and studying the Vlasov equation for the function $\mu(v) + f(t, x, v)$, where f is a perturbation of μ. Recalling the definition of F given by the Poisson equations in (3), we furthermore denote by F^μ and F the quantities related to the equilibrium and to the perturbed quantity, respectively. We obtain then the linearised Vlasov equation:

$$\partial_t f(t, x, v) + v \cdot \nabla_x f(t, x, v) + F^\mu(t, x) \cdot \nabla_v f(t, x, v) = -F(t, x) \cdot \nabla_v \mu(v) \tag{9}$$

Now we perform a key assumption: we suppose in fact that there exists a growing mode for the equation (9), that is there is a solution of the form $f(t, x, v) = e^{\lambda t} \tilde{f}(x, v)$ with $\lambda > 0$. In other words, we are making the ansatz of linear instability with real and positive growth parameter λ. For notational simplicity we denote by D the spatial operators appearing on the left hand side of such equation, that is:

$$D := v \cdot \nabla_x + F^\mu \cdot \nabla_v$$

and we observe that our ansatz allows us to substitute the time derivative $\partial_t f$ appearing in (9) with the multiplication by the parameter λ itself. In fact:

$$\partial_t f(t, x, v) = \partial_t e^{\lambda t} \tilde{f}(x, v) = \lambda f(t, x, v)$$

by which we can rewrite the equation (9) as:

$$(\lambda + D) f(t, x, v) = -F(t, x) \cdot \nabla_v \mu(v). \tag{10}$$

The operator $\lambda + D$ is a first order differential operator by which we can then define the trajectories of a particle by the following system of characteristics:

$$\begin{cases} \dot{\mathbf{X}}(s; x, v) = \mathbf{V}(s; x, v) \\ \dot{\mathbf{V}}(s; x, v) = F^{\mu}(\mathbf{X}(s; x, v)) \end{cases}$$

whose solutions $\mathbf{X}(t; x, v)$ and $\mathbf{V}(t; x, v)$ represent the position and the velocity at time t of particles starting at the point (x, v), meaning that the initial condition of the system above is simply $(\mathbf{X}(0; x, v), \mathbf{V}(0; x, v)) = (x, v)$. Integrating equation (10) along the particles trajectories we are able to define some further operators that help us to study the behaviour of solutions of (9) by studying their spectrum. Without going into details as there are a lot of technicalities in the rigorous integration steps that go beyond the purpose of this work, we restrict ourselves to say that we can obtain an integral representation of f in terms of the right hand side of (10) by 'inverting' the first order differential operator on the left hand side integrating along the trajectories given by the system above. Once we obtain such a representation, we can recall the Poisson equations of (3) holding for the potential U of F and find another equation in which an operator A_λ dependent on the parameter λ appears. Such operator A_λ is typically called *dispersion operator*. It turns out that looking for some growing mode as the one we have supposed to exist above is equivalent to finding some non-constant U belonging to the kernel of the operator A_λ which, typically, possesses some interesting spectral properties.

In the more general case not covered by the Penrose criterion in which μ is a periodic (non-homogeneous) BGK wave (see [Lin01] for such a case) we can perform the same argument as above and find that the operator A_λ is of the form:

$$A_\lambda U = \frac{\partial^2}{\partial x^2} U - \int \mu'(e) U(x) dv + \int \mu'(e) dv \int_{-\infty}^{0} \lambda e^{\lambda s} U(\mathbf{X}(s)) ds \tag{11}$$

where e is the local energy given by $e = 1/2 v^2 + \beta(x)$ and β is a periodic solution of a particular differential equation, as defined in [Lin01]. Again, the study of the stability of μ reduces to find some non-constant U such that $A_\lambda U = 0$ (see [Lin01, Lemma 3]). The main spectral property of such an operator is the following ([Lin01, Lemma 4]):

Proposition 4.1. *A_λ defined in* (11) *is selfadjoint and has purely point spectrum.*

We point out once more that we should make our arguments more precise by defining the ambient spaces where we are working, but this is, in some sense, not closely related to the spectral properties involved to show the linear instability result.

Even though a complete study of the operator A_λ is quite difficult (as it depends on the particle orbits), its behaviour in correspondence of the limiting values $\lambda = 0$ and $\lambda = +\infty$ can be analysed quite easily. Such analysis and some good properties entailed by the fact that the spectrum of A_λ is entirely a point spectrum, show that it is possible to find $\lambda_0 > 0$ such that A_{λ_0} has nontrivial kernel, proving, equivalently, that the equilibrium μ is linearly unstable thanks to what we said above ([Lin01, Lemma 8]).

Results of linear instability for non-homogeneous equilibria have been proved in a number of papers with a rigorous and mathematical approach. In [GS95a] and [GS99], for instance, the authors were able to do that by exploiting the Penrose criterion described in Section 3.2. In these works, the authors deal with inhomogeneous periodic BGK waves which are slight perturbations of homogeneous equilibria satisfying Penrose condition for instability, thus proving that the inhomogeneous waves are unstable as well, thus preserving somehow, the "hint" of instability given by (P). Further references regarding these cases are given in [Lin01, Introduction].

For the sake of completeness, we just mention that similar tools have been used in [GS95b] to show a similar result of linear instability in the particular case of a double-humped equilibrium μ for (3), defined in a cube. In order to do that, the authors exploit an extension of the Weyl's theorem about the perturbation of linear operators, exhibiting an unstable eigenfunction of the linearised problem. Such arguments have been moreover extended, for instance, to the case of the more general *Vlasov-Maxwell* system in [Ben11].

5 Nonlinear instability of Vlasov-Poisson system

We devote this section to the description of the key points needed in the proof of the result of nonlinear instability of double-humped equilibria for the system (3) defined in a cube with specular boundary condition. Such a result is shown in [GS95b] where the authors are able to prove it assuming some decay on the equilibrium as well as a Penrose linear instability condition as the one described in 3.2, typically used, as we have seen, to show such a result in the *linearised* case.

We consider a plasma described by the Vlasov-Poisson system (3) in the cube $Q := [\pi, \pi]^3$ with the specular boundary condition on the density and Neumann boundary conditions on the potential. We denote by $\bar{\partial} Q$ the set of points belonging to the boundary ∂Q which are not corners of edges and by n_x any outward normal vector at the boundary point $x \in \bar{\partial} Q$. System (3) takes the form:

$$\begin{cases} \partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) + \nabla_x U(t,x) \cdot \nabla_v f(t,x,v) = 0 \quad \text{in } Q, \\ f(0,x,v) = f_0(x,v) \quad \text{in } Q, \\ f(t,x,v) = f(t,x,v - 2(n_x \cdot v) n_x) \quad \text{on } \bar{\partial} Q, \\ \Delta U(t,x) = \displaystyle\int_{\mathbb{R}^3} f(t,x,v) dv - \rho_0, \\ \dfrac{\partial U}{\partial n} = 0 \quad \text{on } \partial Q, \quad \displaystyle\int_Q U(t,x) dx = 0. \end{cases} \tag{12}$$

Furthermore, we assume that the plasma is initially neutral, meaning that:

$$\int_{Q \times \mathbb{R}^3} f_0(x,v) dx dv - \int_Q \rho_0 dx = 0. \tag{13}$$

We just state the global existence result of a classical solution to (12) ([GS95b, Theorem 1.1]):

Theorem 5.1. *Let $f_0 \in C^1(Q \times \mathbb{R}^3)$ be positive and satisfy* (13)*, $n_x \cdot \nabla_x f_0 = 0$ on $\bar{\partial} Q$ and:*

$$\int_{\mathbb{R}^3} (1 + |v|^2)^{1/2} f_0(x, v) dv < \infty.$$

Moreover, fix $p > 3$ such that:

$$|f_0(x, v)| + |\nabla_x f_0(x, v)| + |\nabla_v f_0(x, v)| \leq C(1 + |v|^2)^{-p/2}.$$

Then, there exists a unique pair solution (f, U) of (12) *such that $f \in C^1$ and $U \in C^2$.*

The proof exploits a simple reflection method by which the difficulties arising from the complicated particle paths can be overcome thanks to the special geometric structure and the boundary conditions in (12).

We observe that any function $f(t, x, v) = \mu(v)$ is a stationary solution of (12) if μ is non-negative, even in each coordinate v_1, v_2 and v_3 and such that:

$$\int_{\mathbb{R}^3} \mu(v) dv = \rho_0.$$

This is the type of equilibrium we are going to consider in the following. On such a μ we assume now a Penrose-type *linear instability* condition similar to (P). The following condition is the form that the Penrose Criterion takes in the cube Q:

$$\int_{\mathbb{R}^3} \frac{\mu(v) - \mu(0, v_2, v_3)}{v_1^2} dv > 1. \tag{14}$$

Such a condition is compatible with the physically meaningful assumption that μ could be double-humped, i.e. of the form shown in Figure 2. Likewise as in Theorem 5.1, we finally assume a stronger finiteness condition on μ, for a fixed $3 < p < \infty$:

$$|\mu(v)| + |\nabla \mu(v)| + |\nabla^2 \mu(v)| \leq C(1 + |v|^2)^{-p/2}. \tag{15}$$

For such a p we define also:

$$\|f\|_X := \sup_{x,v} [(1 + |v|^2)^{-p/2}(|f| + |\nabla_x f| + |\nabla_v f|)]$$

and we consider as ambient space the space of C^1 functions on $Q \times \mathbb{R}^3$ for which $\|f\|_X < \infty$. We can now state the main nonlinear stability result:

Theorem 5.2. *Let $\mu(v)$ be as described above and satisfying, in particular,* (14) *and* (15)*. Then, $\mu(v)$ is a nonlinearly unstable equilibrium for* (12) *with respect to $\| \cdot \|_X$.*

We explicitly write down what being *nonlinearly unstable* means in order to prove it later on.

Claim 5.3. *There exist initial data $f_0^n(x, v)$ and times $t_n \geq 0$ such that $\|f_0^n - \mu\|_X$ tends to 0, but $\|f^n(t_n) - \mu\|_X$ does* not *tend to 0 as $n \to \infty$.*

In order to prove that, we will need several tools and lemma. The first intermediate step is the forthcoming Lemma 5.4 which proves the existence of a growing mode for the linearisation of (12) around $\mu(v)$. We consider then the function $\mu(v) + g(t,x,v)$ where g is a perturbation. We are thus looking at the following, linearised system:

$$\begin{cases} \partial_t g(t,x,v) + v \cdot \nabla_x g(t,x,v) + \nabla_x V(t,x) \cdot \nabla_v \mu(v) = 0 \quad \text{in } Q, \\ g(t,x,v) = g(t,x,v - 2(n_x \cdot v)n_x) \quad \text{on } \bar{\partial} Q, \\ \Delta V(t,x) = \displaystyle\int_{\mathbb{R}^3} g(t,x,v) dv, \\ \dfrac{\partial V}{\partial n} = 0 \quad \text{on } \partial Q, \quad \displaystyle\int_Q V(t,x) dx = 0. \end{cases} \tag{16}$$

We state and prove now the following Lemma which allows us to assume the existence of a growing mode for (16). We are going to use that in the proof of the main result of nonlinear instability. We emphasize that such a Lemma is exactly the key point where the Penrose condition (14) comes in.

Lemma 5.4. *If $\mu(v)$ satisfies* (14) *and* (15)*, there is a growing mode with positive eigenvalue for the problem* (16)*.*

Proof. Condition (14) can be equivalently expressed in the following way:

$$\int_{\mathbb{R}} \frac{[\partial_{v_1} \int_{\mathbb{R}^2} \mu(v) dv_2 dv_3]}{v_1} dv_1 > 1.$$

Using the fact that:

$$\lim_{\lambda \to +\infty} \int_{\mathbb{R}^3} \frac{v_1 \partial_{v_1} \mu(v)}{v_1^2 + \lambda^2} dv = 0$$

we can link these two conditions together finding $\lambda > 0$ such :

$$\int_{\mathbb{R}^3} \frac{v_1 \partial_{v_1} \mu(v)}{v_1^2 + \lambda^2} dv = 1.$$

For such a λ we define:

$$\begin{aligned} r(t,x,v) &:= e^{\lambda t} \frac{v_1 \partial_{v_1} \mu(v)}{v_1^2 + \lambda^2} \cos x_1 - e^{\lambda t} \frac{\lambda \partial_{v_1} \mu(v)}{v_1^2 + \lambda^2} \sin x_1 \\ V(t,x) &:= -e^{\lambda t} \cos x_1. \end{aligned} \tag{17}$$

Plugging such definitions in the Vlasov equation of the linearised problem (16) we have that:

$$\begin{aligned} &\partial_t r + v \cdot \nabla_x r + \nabla_x V \cdot \nabla_v \mu \\ &= e^{\lambda t} \{ \frac{\partial_{v_1} \mu(v)}{v_1^2 + \lambda^2} [\lambda v_1 \cos x_1 - \lambda^2 \sin x_1 - v_1^2 \sin x_1 - \lambda v_1 \cos x_1] \\ &+ \partial_{v_1} \mu(v) \sin x_1 \} = 0 \end{aligned}$$

which shows that r is an eigenvector related to the eigenvalue λ of the operator

$$-v \cdot \nabla_x - \nabla_v \mu \cdot \nabla \Delta^{-1} \int dv$$

with Neumann boundary conditions, that is exactly the operator defining (16). $\square$

Before using such an eigenvalue and relating it to the nonlinear original problem (16), we find useful bounds for it in the $\|\cdot\|_X$ norm defined above, assuming bounds in another, auxiliary norm defined as follows:

$$\|f\|_Z := \|(1+|v|^2)^{\alpha/2} f\|_{L^p(Q\times\mathbb{R}^3)} \tag{18}$$

for the same p as before and for α such that: $3-3/p < \alpha < p-3/p$. We introduce, furthermore, a third auxiliary norm:

$$\|f\|_Y^p := \int_{\mathbb{R}^3} \left\{ [(1+|v|^2)^{\alpha/2}(|f|+|\nabla_x f|+|\nabla_v f|)]^s dx \right\}^{p/s} dv \tag{19}$$

for p and α as above and $s = p^2/(p-1) > 1$.

We now state some of the results that will allow us to prove Theorem 5.2. The first one shows that for each eigenvector of the linearised problem (16) we are guaranteed of some finiteness conditions.

Lemma 5.5. *Let $R(x,v)$ any eigenvalue of the problem* (16) *with a nonzero eigenvalue for which $\|R\|_Z < \infty$. Then $\|R\|_X < \infty$.*

The proof of the lemma above is, essentially, based on some estimates that exploit the periodicity of the boundary condition.

The next lemma we are going to state holds for every solution of the nonlinear problem (12), establishing some relations between the different auxiliary norms introduced above in terms of the time derivative of spatial quantities and the equilibrium μ.

Lemma 5.6. *If f is a C^1 solution of the nonlinear problem* (12) *with $f(t) \in X$, then:*

$$\begin{aligned}
&\frac{d}{dt}\|\nabla_x f\|_Z^p \le C\|f-\mu\|_Z \|f\|_Y^p,\\
&\frac{d}{dt}\|\nabla_v (f-\mu)\|_Z \le C\|f-\mu\|_Z(1+\|f-\mu\|_Y) + C\|\nabla_x f\|_Z.
\end{aligned}$$

Using the lemma above we are able to find other estimates that give direct bounds on the difference between the solution of the nonlinear problem (12) and the equilibrium in terms of the initial values f_0.

Lemma 5.7. *Let f be a C^1 solution of the nonlinear problem* (12) *and $f \in L^\infty([0,T);X)$. If there exists $\beta > 0$ such that:*

$$\|f(t)-\mu\| \le Ce^{\beta t}\|f_0-\mu\|_Z$$

in $[0,T)$. Then

$$\|\nabla_v(f(t)-\mu)\|_Z \le Be^{\beta t/p}\|f_0-\mu\|_Y^{1/p} \tag{20}$$

for a.e. $t \in [0,T)$ and the constant B depends on μ, β and $\sup_t \|f(t)\|_Y$ only.

We are now able to prove the main Theorem 5.2 by contradiction of our Claim 5.3.

Proof. [of Theorem 5.2]

Suppose Claim 5.3 is false so that μ is nonlinearly stable. For every $\varepsilon > 0$ we can then find a $\delta > 0$ such that $f \in C([0,\infty); X)$ and that, if

$$\|f_0 - \mu\|_X < \delta, \tag{21}$$

there holds:

$$\sup_{t\in[0,\infty)} \|f(t) - \mu\|_X \leq \varepsilon. \tag{22}$$

Condition (21) is satisfied with the choice:

$$f_0(x,v) - \mu(v) = \delta R(x,v)$$

where R is the eigenvector given in Lemma 5.5 with $\|R\|_X = 1$ such that its eigenvalue λ has the largest real part. Now, we write down how the difference $f(t,x,v) - \mu(v)$ looks like in terms of the perturbed strongly continuous semigroup associated to the linearised equation (16):

$$\begin{aligned} &f(t,x,v) - \mu(v) \\ &= \delta R(x,v)e^{\lambda t} - \int_0^t e^{-(A+K)(t-\tau)}(\nabla_x U(\tau,x)\cdot[\nabla_v(f(\tau,x,v) - \mu(v)])d\tau \end{aligned} \tag{23}$$

where $A+K$ is the linearised operator related to (12). The operators A and K are of the form:

$$\begin{aligned} &A(h) = v\cdot\nabla_x\bar{h} \\ &K(g) = \nabla_x\bar{V}\cdot\nabla_v\bar{\mu} \quad \text{where } \Delta\bar{V} = \int \bar{g}dv \end{aligned}$$

and the bar notation indicates the extension of the functions defined at first in $\mathbb{R}^+\times Q\times\mathbb{R}^3$ to the whole $\mathbb{R}^+\times\mathbb{R}^3\times\mathbb{R}^3$ by using simply a reflection method as performed in [GS95b, Theorem 1.1]. In [GS95b, Theorem 2.4] the authors show that A generates a strongly continuous semigroup on the space of periodic functions, whereas K turns out to be a compact operator on the same space. Such considerations allow the authors to use a variant of Weyl's Theorem for perturbed linear operators (see [Kat95, Section VIII]).

Let now Λ be such that $Re\lambda < \Lambda < Re\lambda(1+1/p)$ and T defined as:

$$T := \sup\left\{t' : \left\|f(t) - \mu - \delta Re^{\lambda t}\right\|_Z \leq \frac{1}{2}\delta e^{Re\lambda t}\|R\|_Z \text{ for } 0\leq t\leq t'\right\}. \tag{24}$$

Then T is positive and finite. For every time t in $[0,T]$ we have then:

$$\|f(t)-\mu\|_Z \leq \left\|f(t)-\mu\pm\delta Re^{\lambda t}\right\|_Z \leq \frac{1}{2}\delta e^{Re\lambda t}\|R\|_Z + \delta\left\|Re^{\lambda t}\right\| \leq \frac{3\delta}{2}e^{tRe\lambda}\|R\|_Z.$$

From such an inequality, by using properties of the operators A and K it is possible to give the following bound to $\nabla_x U$:

$$\|\nabla_x U(\cdot,t)\|_\infty \leq C\|f(t)-\mu\|_Z \leq C\delta e^{tRe\lambda}.$$

We are now in the conditions of applying Lemma 5.7 with the choice $\beta = Re\lambda$ to obtain from (23):

$$\left\|f(t) - \mu - \delta R e^{\lambda t}\right\|_Z \leq \int_0^t e^{\Lambda(t-\tau)} \left\|\nabla_x U(x,\tau)\right\|_\infty \left\|\nabla_v (f(\tau,x,v) - \mu(v))\right\|_Z d\tau$$

$$\leq C \int_0^t e^{\Lambda(t-\tau)} \delta e^{\tau Re\lambda} \delta^{1/p} e^{\tau Re\lambda/p} d\tau \leq C\delta^{1+1/p} e^{(Re\lambda)(1+1/p)t}. \tag{25}$$

For every $0 \geq t \geq T$ we then have:

$$\|f(t) - \mu\|_Z \geq \|R\|_Z \,\delta e^{tRe\lambda} - C(\delta e^{tRe\lambda})^{1+1/p}. \tag{26}$$

We solve now the following equation with respect to the t unknown:

$$\delta e^{tRe\lambda} = (\frac{\|R\|_Z}{2C})^p$$

thus finding t_δ such that it is satisfied. We claim now that this t_δ is in $[0, T]$ and leads to our instability result. To see that $t_\delta \leq T$ we just use the definition of T given by (24). In correspondence of T we have in fact:

$$\left\|f(T) - \mu - \delta R e^{\lambda T}\right\|_Z = \frac{1}{2}\delta e^{TRe\lambda} \|R\|_Z \leq C(\delta e^{TRe\lambda})^{1+1/p}$$

where the last inequality comes directly from (25). We observe that this implies:

$$(\frac{\|R\|_Z}{2C})^p \leq \delta e^{TRe\lambda}.$$

With our choice of t_δ we have then that:

$$\delta e^{t_\delta Re\lambda} \leq \delta e^{TRe\lambda}$$

and then, comparing with definition (24), we see that actually $t_\delta \leq T$. We can now use (26) for such a value, as we showed that it belongs to the interval $[0, T]$. We get:

$$\|f(t_\delta) - \mu\|_Z \geq \frac{\|R\|_Z^{p+1}}{2^{p+1} C^p}$$

which contradicts (22) as ε is arbitrarily small. □

More works on stability and instability of plasmas in more general cases have been developed in [BR93b], [BMR95] and [Ben11]. Furthermore, the relationships between them and the phenomenon of the *Landau damping* have been studied in [MV11] where the authors present also the Penrose criterion (P) in a much more mathematical way, giving interesting connections and links with such a phenomenon.

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