Strichartz estimates for the kinetic transport equation

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7-11 September 2015

Introduction: (Classical) Strichartz estimates

The solution of the kinetic transport equation

$$\partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) = 0, \qquad f(0,x,v) = f^0(x,v)$$

for $(t,x,v) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ satisfies the Strichartz estimates

$$\|f\|_{L^q_t L^p_x L^r_v} \lesssim \|f^0\|_{L^a_{x,v}}, \qquad \text{where}$$

$$\frac{2}{q} = d\left(\frac{1}{r} - \frac{1}{p}\right), \qquad \frac{1}{a} = \frac{1}{2}\left(\frac{1}{r} + \frac{1}{p}\right), \qquad p \geq a \qquad q > a. \tag{1}$$

Castella-Perthame [1996]:

- With the further condition $q > 2 \geq a$.

Keel-Tao [1998]:

- Observed that this condition can be relaxed to $q > a$.
- The homogeneity condition, $p \geq a$ and $q \geq a$ are necessary.
- Conjectured that the Strichartz estimate holds at the endpoint $q = a$.

Endpoint Strichartz estimate: $q = a$

Endpoint conjecture:

$$\|f\|_{L^a_t L_x^{\frac{d}{d-1}a} L_v^{\frac{d}{d+1}a}} \lesssim \|f^0\|_{L^a_{x,v}}$$

Invariance:

$$f^0 \longrightarrow (f^0)^\lambda, \qquad \text{then} \qquad f \longrightarrow f^\lambda$$

If the Strichartz estimate is true for (q, p, r, a) $\Longrightarrow$ Strichartz estimate is ALSO true for $(\lambda q, \lambda p, \lambda r, \lambda a)$.

This invariance allows us to fix/choose one of the Strichartz exponents!

$$\|f\|_{L^a_t L_x^{\frac{d}{d-1}a} L_v^{\frac{d}{d+1}a}} \lesssim \|f^0\|_{L^a_{x,v}}$$

Choosing $\mathbf{a = 2}$ (classical) the endpoint-conjecture rewrites

$$\|f\|_{L^2_t L_x^{\frac{2d}{d-1}} L_v^{\frac{2d}{d+1}}} \lesssim \|f^0\|_{L^2_{x,v}}$$

- $\boxed{d = 1}$ The endpoint estimate

$$\|f\|_{L^2_t L^\infty_x L^1_v} \lesssim \|f^0\|_{L^2_{x,v}} \qquad \text{is FALSE}$$

  - Guo-Peng [2007]. Replacement for the endpoint:

$$\|f\|_{L^2_t \mathrm{BMO}_x L^1_v} \lesssim \|f^0\|_{L^2_{x,v}}$$

  - Ovcharov [2011]: Counterexample based on testing the estimate on the characteristic function of a Besicovich set.

- $\boxed{d > 1}$ Open question.

Aim:

- Prove the failure of the endpoint Strichartz estimate for the kinetic transport equation in any dimension.
- Prove all the non-endpoint Strichartz estimates using multilinear analysis.

Joint work with

Jonathan Bennett (U. Birmingham),
Neal Bez (Saitama University),
Sanghyuk Lee (Seoul National University).

Approach:

- Multilinear analysis.

- Strichartz estimates for the free wave and Schrödinger propagators (and others) have been used extensibly to study the well-posedness of associated nonlinear equations/systems.

- Strichartz estimates for the kinetic transport equation have been used to prove global existence of (weak) solutions for kinetic models of chemotaxis with smallness assumption of the initial data.
  [Bournaveas-Calvez-Gutierrez-Perthame, 2008]

Failure of the endpoint

$$\|f\|_{L^a_t L_x^{\frac{d}{d-1}a} L_v^{\frac{d}{d+1}a}} \lesssim \|f^0\|_{L^a_{x,v}}$$

Choosing $a = \frac{d+1}{d}$, the endpoint-conjecture rewrites

$$\|f\|_{L_t^{\frac{d+1}{d}} L_x^{\frac{d+1}{d-1}} L_v^1} \lesssim \|f^0\|_{L_{x,v}^{\frac{d+1}{d}}}$$

Notice that $f(t,x,v) = f^0(x - tv, v)$, and then

$$\|f(t,x,\cdot)\|_{L^1_v} = \underbrace{\int f(t,x,v)\,dv}_{\textbf{Velocity average}} = \int f^0(x - tv, v)\,dv = \rho(f^0)(t,x)$$

The endpoint-conjecture rewrites as the following estimate for $\rho(f^0)$

$$\|\rho(f^0)\|_{L_t^{\frac{d+1}{d}} L_x^{\frac{d+1}{d-1}}} \lesssim \|f^0\|_{L_{x,v}^{\frac{d+1}{d}}}$$

$$\|\rho(f^0)\|_{L_t^{\frac{d+1}{d}} L_x^{\frac{d+1}{d-1}}} \lesssim \|f^0\|_{L_{x,v}^{\frac{d+1}{d}}}$$

$\Updownarrow$ **Duality**

$$\|\rho^*(g)\|_{L_{x,v}^{d+1}} \lesssim \|g\|_{L_t^{d+1} L_x^{\frac{d+1}{2}}}$$

where ρ^* is the adjoint operator given by

$$(\rho^*(g))(x,v) = \int_{\mathbb{R}} g(t, x+tv)\, dt$$

Theorem

The following estimate

$$\|\rho^*(g)\|_{L_{x,v}^{d+1}} \lesssim \|g\|_{L_t^{d+1} L_x^{\frac{d+1}{2}}} \qquad \textit{is FALSE!}$$

Argument: Frank-Lewin-Lieb-Seiringer [2014]

Refined Strichartz estimates for solutions of the free Schrödinger equation associated with systems of orthonormal functions.

The argument follows the authors' approach in the proof of the failure of a conjectured endpoint.

Proof: Recall that

$$(\rho^*(g))(x,v) = \int_{\mathbb{R}} g(t, x + tv)\, dt.$$

Suppose $g \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^d)$ is nonnegative and such that $\widehat{g} \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$ is also nonnegative. Formally,

$$\|\rho^*(g)\|_{\mathbf{L}^{\mathbf{d+1}}_{\mathbf{x,v}}}^{d+1} = \int \prod_{j=1}^{d+1} g(t_j, x + t_j v)\, \mathrm{d}\vec{t}\mathrm{d}x\mathrm{d}v.$$

Denoting $\widehat{g}$ the space-time Fourier transform of g, using Fourier inversion

$$\begin{aligned} g(t_j, x + t_j v) &= c \int e^{i\tau_j t_j} e^{\xi_j \cdot (x + t_j v)} \widehat{g}(\tau_j, \xi_j)\, \mathrm{d}\tau_j\, \mathrm{d}\xi_j \\ &= c \int \underbrace{e^{t_j(\tau_j + \xi_j \cdot v)}}\, \underbrace{e^{i\xi_j \cdot x}}\, \underbrace{\widehat{g}(\tau_j, \xi_j)}\, d\tau_j\, d\xi_j. \end{aligned}$$

Thus

$$\begin{aligned} \|\rho^*(g)\|_{L^{d+1}_{x,v}}^{d+1} &= \int \prod_{j=1}^{d+1} g(t_j, x + t_j v)\, \mathrm{d}\vec{t}\mathrm{d}x\mathrm{d}v \\ &= c \int \prod_{j=1}^{d+1} \widehat{g}(\tau_j, \xi_j) \left(\prod_{k=1}^{d+1} e^{it_k(\tau_k + v \cdot \xi_k)} \right) e^{ix \cdot \sum_{\ell=1}^{d+1} \xi_\ell}\, \mathrm{d}\vec{\tau}\mathrm{d}\vec{\xi}\mathrm{d}\vec{t}\mathrm{d}x\mathrm{d}v \\ &= c \int \prod_{j=1}^{d+1} \widehat{g}(\tau_j, \xi_j) \prod_{k=1}^{d+1} \delta(\tau_k + v \cdot \xi_k)\, \delta\left(\sum_{\ell=1}^{d+1} \xi_\ell \right) \mathrm{d}\vec{\tau}\mathrm{d}\vec{\xi}\mathrm{d}v \\ &= c \int \prod_{j=1}^{d+1} \widehat{g}(-v \cdot \xi_j, \xi_j)\, \delta\left(\sum_{l=1}^{d+1} \xi_l \right) \mathrm{d}\vec{\xi}\mathrm{d}v \end{aligned}$$

$$\|\rho^*(g)\|_{L^{d+1}_{x,v}}^{d+1} = c \int \prod_{j=1}^{d} \widehat{g}(-v\cdot\xi_j, \xi_j)\ \widehat{g}\left(v\cdot\sum_{k=1}^{d}\xi_k, -\sum_{\ell=1}^{d}\xi_\ell\right) \mathrm{d}v\,\mathrm{d}\vec{\xi}. \qquad (2)$$

On $\mathbb{R}^d$, consider the change of variables: $v \longrightarrow w_j = -\xi_j \cdot v, \quad j = 1, \dots d$

$$\left|\frac{\partial w}{\partial v}\right| = |\det(-\xi_1, \dots, -\xi_d)|$$

$$\|\rho^*(g)\|_{L^{d+1}_{x,v}}^{d+1} = c \int \prod_{j=1}^{d} \widehat{g}(w_j, \xi_j)\ \widehat{g}\left(-\sum_{k=1}^{d} w_k, -\sum_{\ell=1}^{d}\xi_\ell\right) \frac{1}{|\det(-\xi_1, \dots, -\xi_d)|}\,\mathrm{d}w\mathrm{d}\vec{\xi}.$$

- Notice that $\widehat{g}$ is continuous and $\widehat{g}(0,0) = \int g(t,x) > 0$
- Writing $\xi_j = r_j\theta_j, \quad j = 1, \dots d$ (*polar coordinates*)

$$|\det(-\xi_1, \dots, -\xi_d)| = \left(\prod_{j=1}^{d} r_j\right) |\det(\theta_1, \dots, \theta_d)|$$

$$\|\rho^* g\|_{L^{d+1}_{x,v}}^{d+1} \gtrsim \int_{|r|\lesssim 1} \int_{(\mathbb{S}^{d-1})^d} \left(\prod_{j=1}^{d} r_j^{d-2} \right) \frac{1}{|\det(\theta_1, \ldots, \theta_d)|} \,\mathrm{d}\vec{r}\mathrm{d}\vec{\theta}. \qquad (3)$$

- $d = 1$: the radial integral

$$\int_{|r|\lesssim 1} \frac{dr}{r} = \infty$$

- $d \geq 2$: the angular integral

$$\int_{(\mathbb{S}^{d-1})^d} \frac{1}{|\det(\theta_1, \ldots, \theta_d)|} \,\mathrm{d}\vec{\theta} = \infty,$$

Remarks:

- For the free Schrödinger propagator:

  The endpoint Strichartz estimate is TRUE for all $d \geq 3$ [Keel-Tao, 1998]

  This theorem highlights a fundamental difference in the Strichartz estimates for the kinetic transport and the Schrödinger equations.

- The endpoint estimate fails rather **generically**.

- Replacements for the endpoint: Keel-Tao, Ovcharov, Guo-Peng...

  It seems natural to conjecture that ρ^* satisfies the weak-type estimate

$$\|\rho^*(g)\|_{L^{d+1,\infty}_{x,v}} \lesssim \|g\|_{L^{d+1}_t L_x^{\frac{d+1}{2}}}.$$

Multilinear approach to nonendpoint cases

- **Argument:** Multilinear variant of Perthame-Castella.
- Suffices to prove the estimate for $r = 1$:

$$\|\rho(f^0)\|_{L^q_t L^p_x} \lesssim \|f^0\|_{L^a_{x,v}},$$

$$\frac{2}{q} = d\left(1 - \frac{1}{p}\right); \qquad \frac{1}{a} = \frac{1}{2}\left(1 + \frac{1}{p}\right); \qquad q > a, \qquad p \geq a$$

$$\Updownarrow \quad \textbf{Duality}$$

$$\|\rho^*(g)\|_{L^{a'}_{x,v}} \lesssim \|g\|_{L^{q'}_t L_x^{\frac{a'}{2}}},$$

$$\frac{1}{q'} + \frac{d}{a'} = 1,; \qquad a' > (d+1)$$

$$\Updownarrow \quad a' = \sigma(d+1);\ \text{with} \quad \sigma = \frac{a'}{d+1}$$

$$\|\rho^*(g)\|_{L^{\sigma(d+1)}_{x,v}} \lesssim \|g\|_{L^{q_\sigma}_t L_x^{\frac{(d+1)\sigma}{2}}},$$

$$\frac{1}{q_\sigma} + \frac{d}{(d+1)\sigma} = 1, \qquad \sigma > 1.$$

Theorem

The following inequalities are true:

$$\|\rho^*(g)\|_{L^{\sigma(d+1)}_{x,v}} \lesssim \|g\|_{L^{q_\sigma}_t L_x^{\frac{(d+1)\sigma}{2}}}, \qquad (4)$$

$$\frac{1}{q_\sigma} + \frac{d}{(d+1)\sigma} = 1, \qquad \sigma > 1.$$

Proof: Suppose g is nonnegative.

$$\begin{aligned} \|\rho^*(g)\|^{d+1}_{L^{\sigma(d+1)}_{x,v}} &= \left(\int \left(\int \prod_{j=1}^{d+1} g(t_j, x + t_j v) \, \mathrm{d}\vec{t} \right)^{\sigma} \mathrm{d}x \mathrm{d}v \right)^{1/\sigma} \\ &\leq \int \Bigg(\underbrace{\int \prod_{j=1}^{d+1} g^{\sigma}(t_j, x + t_j v) \, \mathrm{d}x \mathrm{d}v}_{} \Bigg)^{1/\sigma} \mathrm{d}\vec{t}. \end{aligned}$$

Now fix $t_1, \dots, t_{d+1}$, $g_j = g^\sigma$ $j = 1, \dots d$, and consider

$$\int \prod_{j=1}^{d+1} g_j(t_j, x + t_j v) \, \mathrm{d}x \mathrm{d}v \quad \text{(Multilinear form)}$$

$$\int \prod_{j=1}^{d+1} g_j(t_j, x + t_j v)\,\mathrm{d}x\mathrm{d}v \quad \text{(Multilinear form)}$$

$$\begin{aligned} &\leq \prod_{k\neq i,j}^{d+1} \|g_k(t_k,\cdot)\|_{L^\infty_x} \int g_i(t_i, \underbrace{x + t_i v}_{y})\, g_j(t_j, \underbrace{x + t_j v}_{z})\,\mathrm{d}x\mathrm{d}v \\ &= \left(\prod_{k\neq i,j}^{d+1} \|g_k(t_k,\cdot)\|_{L^\infty_x}\right) \frac{1}{|t_i - t_j|^d} \|g_i(t_i,\cdot)\|_{L^1_x} \|g_j(t_j,\cdot)\|_{L^1_x} \end{aligned}$$

for each $1 \leq i < j \leq d$.

$\Downarrow$ **Multilinear Interpolation**

$$\int \prod_{j=1}^{d+1} g_j(t_j, x + t_j v)\,\mathrm{d}x\mathrm{d}v \lesssim \prod_{1\leq i<j\leq d} |t_i - t_j|^{-\frac{2}{d+1}} \prod_{k=1}^{d+1} \|g_k(t_k,\cdot)\|_{L_x^{\frac{d+1}{2}}}.$$

$$\int \prod_{j=1}^{d+1} g_j(t_j, x + t_j v)\, \mathrm{d}x\mathrm{d}v \lesssim \prod_{1\le i<j\le d} |t_i - t_j|^{-\frac{2}{d+1}} \prod_{k=1}^{d+1} \|g_k(t_k,\cdot)\|_{L_x^{\frac{d+1}{2}}}$$

Since $g_j = g^\sigma$ for each $j = 1, \dots d$,

$$\begin{aligned} \|\rho^*(g)\|_{L^{\sigma(d+1)}_{x,v}}^{d+1} &\lesssim \int_{\mathbb{R}^{d+1}} \prod_{1\le i<j\le d} |t_i - t_j|^{-\frac{2}{(d+1)\sigma}} \prod_{k=1}^{d+1} \|g(t_k,\cdot)\|_{L_x^{\frac{(d+1)\sigma}{2}}} \,\mathrm{d}\vec{t} \\ &\lesssim \|g\|^{d+1}_{L_t^{q\sigma} L_x^{\frac{(d+1)\sigma}{2}}} \end{aligned}$$

Multilinear Hardy–Littlewood–Sobolev [Christ, 1985]

Remark:

- Natural question:

  Can one prove the validity of all the non-endpoint Strichartz estimates for the free Schrödinger propagator using this "multilinear argument"?

  Not clear

Some Prespectives

Geometric interpretation

$$(\rho^*(g))(x,v) = \int_{\mathbb{R}} g(t, x+tv)\,dt$$

Notice that $(t, x+tv) = (0,x) + t(1,v)$, so

$(\rho^*(g))(x,v)$ is the integral of g along the line $l(x,v)$ given by

$$l(x,v) = \{(0,x) + t(1,v) \ : t \in \mathbb{R}\}.$$

Thus

$$\rho^*(g)(x,v) = \int g(t, x+tv)\,dt = \int_{l(x,v)} g = X[g] \quad X\text{-ray transform of } g$$

Sharp constants and extremizers

There is an emerging literature on sharp constants and extremisers for Strichartz estimates in a variety of contexts (Schrödinger, Wave, Klein-Gordon...).

Drouot and Flock [2014]: The best constant C for

$$\|\rho^*(g)\|_{L^{d+2}_{x,v}} \leq C\|g\|_{L^{\frac{d+2}{2}}_{t,x}} \qquad (*)$$

is attained (up to symmetries of the inequality) if and only if

$$g(t,x) = \frac{1}{1+t^2+|x|^2}$$

- $(*)$ is the endpoint in the scale of pure norm estimates for ρ^* (or ρ).
- All other exponents are open from this point of view (conjecture of Baernstein&Loss 1997).

Other related problems

Validity of estimates of the type:

$$\|\rho^*(g)\|_{L^q_v L^r_x} \leq \|g\|_{W^{p,\alpha}} \qquad (**)$$

- Drury&Christ (1984): conjecture on the range of exponents (p, q, r, α).
- Partial results (certain $r < \infty$) due to Wolff (98) and Laba-Tao (2001).
- Conjecture $\Leftrightarrow$ $(**)$ for $(p, q, r, \alpha) = (n, n, \infty, \varepsilon)$, $\forall\, \varepsilon > 0$.

$\boxed{r = \infty}$

$$\|\rho^*(g)\|_{L^\infty_x} = \sup_{l \| (1,v)} \int_l g = \text{Variant of the Kakeya maximal function}$$

Conjecture $(**)$ for ρ^* $\Leftrightarrow$ **Kakeya maximal conjecture!**

THANK YOU FOR YOUR ATTENTION!