Stability of Minkowski Space for the Massless Einstein–Vlasov System

Martin Taylor

DPMMS, Cambridge

m.taylor@maths.cam.ac.uk

Recent Progress in Collisionless Models
Imperial College London
9th September 2015

General Relativity and the Einstein Equations

General relativity is Einstein's theory of gravity, governed by the *Einstein equations*:

$$Ric(g)_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R(g) = T_{\mu\nu}.$$

Unknown is a Lorentzian manifold (M, g), called a spacetime, together with fields describing the matter.

Here $Ric(g)_{\mu\nu}$ denotes the Ricci curvature, $R(g)$ the scalar curvature, $T_{\mu\nu}$ the stress–energy–momentum tensor of the matter present.

Terms on the left describe curvature of the spacetime, terms on the right describe the matter present.

Admit a trivial solution $g = -dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$ when $T = 0$, called *Minkowski space*.

The Vlasov Model

A particle at $x \in M$ with velocity $p \in T_x M$ will follow the unique geodesic $\gamma_{x,p}$ such that

$$\gamma_{x,p}(0) = x, \qquad \dot{\gamma}_{x,p}(0) = p.$$

Encoded in a density function: $f(x,p)$ denotes the density of particles at x with velocity p.

Vlasov equation says f conserved along trajectories of the geodesic flow:

$$f(x,p) = f(\exp_s(x,p)),$$

where $\exp_s : T\mathcal{M} \to T\mathcal{M}$ is the *exponential map*,

$$\exp_s(x,p) = (\gamma_{x,p}(s), \dot{\gamma}_{x,p}(s)).$$

Equivalently,

$$X(f) = 0,$$

where $X \in \Gamma(TT\mathcal{M})$ is the *geodesic spray*.

The Energy Momentum Tensor

Assume all particles have equal mass, i.e. f is supported on the mass shell,

$$P = \bigcup_{x \in \mathcal{M}} P_x, \qquad P_x = \{p \in T_x\mathcal{M} : g(p,p) = -m^2\}.$$

Energy momentum tensor takes the form,

$$T^{\alpha\beta}(x) = \int_{P_x} f p^\alpha p^\beta d\mu_{P_x},$$

with p^1, p^2, p^3, p^4 coordinates on $T_x\mathcal{M}$ conjugate to frame e_1, e_2, e_3, e_4 on $\mathcal{M}$ and $d\mu_{P_x}$ volume form on P_x.

Distinguish two cases:

- Massive matter: $m > 0$;
- Massless matter: $m = 0$.

Stability of Minkowski Space with Massless Collisionless Matter

The Einstein–Vlasov system has a well posed Cauchy problem and domain of dependence property (Choquet-Bruhat 1971).

This tells us the dynamic viewpoint is the correct one and allows us to pose the question of stability of Minkowski space:

> *Given an initial data set which is close to that of Minkowski space, what properties does the resulting spacetime have?*

Theorem (M.T. 2015)

Given regular initial data for the massless Einstein–Vlasov system suitably "close" to that of Minkowski space with f_0 compactly supported, the resulting spacetime is geodesically complete and approaches Minkowski space in every direction. Moreover, the spacetime possesses a complete future null infinity whose causal past is $\mathcal{M}$.

Previous Work on Stability of Minkowski Space

Without symmetry:

- Christodoulou–Klainerman (1993): Vacuum;
- Zipser (2000): Maxwell;
- Klainerman–Nicolo (2003): Vacuum;
- Lindblad–Rodnianski (2004): Vacuum, Scalar Field;
- Bieri (2007): Vacuum;
- Loizelet (2008): Maxwell;
- Speck (2010): "Large class of nonlinear electromagnetic equations".

Einstein–Vlasov in spherical symmetry:

- Rein–Rendall (1992): $m > 0$;
- Dafermos (2006): $m = 0$.

Which Gauge to Use?

Two serious options, each with their own merits:

- Something in the spirit of Christodoulou–Klainerman e.g. double null gauge;
- or harmonic gauge à la Lindblad–Rodnianski.

Choose double null gauge:

- There are issues related to this gauge which need to be overcome, with a view to further problems;
- Certain geometric aspects of the proof are more appealing: can show matter is confined to a certain region and appeal to the vacuum result for remaining regions.

The Double Null Gauge

Two *optical* functions u, v whose level hypersurfaces are outgoing and incoming null (characteristic) cones respectively.

Instead of estimating metric, estimate

- Γ – Ricci coefficients;
- ψ – Weyl curvature components;
- $\mathcal{T}$ – energy momentum tensor components;

with respect to double null frame,

$$e_3 \text{ – incoming null direction,} \qquad e_4 \text{ – outgoing null direction,}$$

$$e_A \text{ – spherical directions, } A = 1, 2.$$

Note that, of course, two spherical frames are required.

Overview of Proof

The entire content of the proof is in obtaining global a priori estimates for $\Gamma, \psi, \mathcal{T}$.

Step 0: Appeal to vacuum result to reduce to semi-global existence problem.

Step 1: Global estimates for Γ, ψ.

Step 2: Global estimates for derivatives of f.

Step 3: Behaviour of null geodesics.

Steps 2 and 3 immediately give estimates for $\mathcal{T}$.

Main new difficulties in step 2, so will only briefly discuss the others. Will follow from introducing Sasaki metric on P and estimating Jacobi fields on P.

Null Structure and Bianchi Equations

Content of Einstein equations captured through:

- Null Bianchi equations for Weyl curvature components:

$$\nabla\!\!\!/_3\psi = \mathcal{D}\!\!\!/\psi + \Gamma \cdot \psi + \nabla\mathcal{T}, \qquad \nabla\!\!\!/_4\psi = \mathcal{D}\!\!\!/\psi + \Gamma \cdot \psi + \nabla\mathcal{T}.$$

- Null structure equations for Ricci coefficients

$$\nabla\!\!\!/_3\Gamma = \Gamma \cdot \Gamma + \psi + \mathcal{T}, \qquad \nabla\!\!\!/_4\Gamma = \Gamma \cdot \Gamma + \psi + \mathcal{T}.$$

Weyl curvature components estimated through *energy estimates* for Bianchi equations.

Ricci coefficients estimated through *transport estimates* for null structure equations.

Note it is important that $\mathcal{T}$ estimated *at one degree of differentiability greater than* ψ.

Global Decay Estimates

Due to strongly coupled nature of the system, have to prove boundedness and decay of the geometric quantities simultaneously.

Each $\Gamma, \psi, \mathcal{T}$ has its own decay rate. Proceed by *guessing* correct decay hierarchy then proving guesses are correct.

Then see that error terms in above equations have correct decay for obtaining *global* estimates, i.e. there is some special structure in above equations which is heavily exploited.

Estimates are L^2 based, so to treat nonlinear terms have to also estimate derivatives and use Sobolev inequalities. This is done by introducing a set of differential operators

$$\mathfrak{D} = \{v\nabla\!\!\!\!/, \nabla\!\!\!\!/_3, v\nabla\!\!\!\!/_4\},$$

and showing structure in equations is still present after commutation with elements of $\mathfrak{D}$.

Estimating Derivatives of f

Most obvious way to estimate f is to commute Vlasov equation:

$$X(f) = 0.$$

Note that,

$$X = p^\mu e_\mu - p^\alpha p^\beta \Gamma^\mu_{\alpha\beta} \partial_{p^\mu},$$

so commutation errors will involve derivatives of $\Gamma^\mu_{\alpha\beta}$. Should live at level of differentiability of curvature, so looks promising. There are issues however.

Estimating Derivatives of f

Most obvious way to estimate f is to commute Vlasov equation:

$$X(f) = 0.$$

Note that,

$$X = p^\mu e_\mu - p^\alpha p^\beta \Gamma^\mu_{\alpha\beta} \partial_{p^\mu},$$

so commutation errors will involve derivatives of $\Gamma^\mu_{\alpha\beta}$. Should live at level of differentiability of curvature, so looks promising. There are issues however.

Instead differentiate expression,

$$f(x, p) = f(\exp_s(x, p)),$$

and use the chain rule to get,

$$Vf(x, p) = df|_{(x,p)} V = df|_{\exp_s(x,p)} \cdot d\exp_s|_{(x,p)} V = J(s) f(\exp_s(x, p)),$$

where,

$$J(s) := d\exp_s|_{(x,p)} V.$$

Estimating Derivatives of f

So,

$$Vf(x,p) = J(s)f(\exp_s(x,p)).$$

Take $s < 0$ so that $\exp_s(x,p)$ lies on mass shell over the initial hypersurface.

It remains to estimate the components of J with respect to a suitable frame for P.

Estimating Derivatives of f

So,

$$Vf(x,p) = J(s)f(\exp_s(x,p)).$$

Take $s < 0$ so that $\exp_s(x,p)$ lies on mass shell over the initial hypersurface.

It remains to estimate the components of J with respect to a suitable frame for P.

When P is given the induced *Sasaki metric*, J is a Jacobi field along $s \mapsto \exp_s(x,p)$,

$$\hat{\nabla}_X \hat{\nabla}_X J = \hat{R}(X,J)X.$$

Choice of vectors V are important. Will return to this.

Horizontal and Vertical Lifts

Sasaki metric defined using two geometrically defined maps

$$\mathrm{Hor}_{(x,p)}, \mathrm{Ver}_{(x,p)} : T_x\mathcal{M} \to T_{(x,p)}T\mathcal{M}.$$

In components, if $V = V^\mu e_\mu$,

$$\mathrm{Ver}_{(x,p)}(V) = V^\mu \partial_{p^\mu}, \qquad \mathrm{Hor}_{(x,p)}(V) = V^\mu e_\mu - V^\alpha p^\beta \Gamma^\mu_{\alpha\beta} \partial_{p^\mu}.$$

Horizontal and Vertical Lifts

Sasaki metric defined using two geometrically defined maps

$$\mathrm{Hor}_{(x,p)}, \mathrm{Ver}_{(x,p)} : T_x\mathcal{M} \to T_{(x,p)}T\mathcal{M}.$$

In components, if $V = V^\mu e_\mu$,

$$\mathrm{Ver}_{(x,p)}(V) = V^\mu \partial_{p^\mu}, \qquad \mathrm{Hor}_{(x,p)}(V) = V^\mu e_\mu - V^\alpha p^\beta \Gamma^\mu_{\alpha\beta} \partial_{p^\mu}.$$

Can split $T_{(x,p)}T\mathcal{M}$ into direct sum of vertical and horizontal subspaces,

$$T_{(x,p)}T\mathcal{M} = \mathcal{V}_{(x,p)} \oplus \mathcal{H}_{(x,p)},$$

$$\mathcal{V}_{(x,p)} = \{\mathrm{Ver}_{(x,p)}(V) \mid V \in T_x\mathcal{M}\}, \qquad \mathcal{H}_{(x,p)} = \{\mathrm{Hor}_{(x,p)}(V) \mid V \in T_x\mathcal{M}\}.$$

For $Y \in T_{(x,p)}T\mathcal{M}$, define Y^v, Y^h so that,

$$Y = \mathrm{Ver}_{(x,p)}(Y^v) + \mathrm{Hor}_{(x,p)}(Y^h).$$

Curvature of the Mass Shell

Recall,

$$\hat{\nabla}_X \hat{\nabla}_X J = \hat{R}(X, J)X.$$

Can compute,

$$\begin{aligned}\hat{R}(X, J)X = \mathrm{Hor}_{(\gamma,\dot{\gamma})}\left(R(\dot{\gamma}, J^h)\dot{\gamma} + \frac{3}{4}R(\dot{\gamma}, R(\dot{\gamma}, J^h)\dot{\gamma})\dot{\gamma} + \frac{1}{2}(\nabla_{\dot{\gamma}} R)(\dot{\gamma}, J^v)\dot{\gamma}\right) \\ + \frac{1}{2}{}^T\mathrm{Ver}_{(\gamma,\dot{\gamma})}\left((\nabla_{\dot{\gamma}} R)(\dot{\gamma}, J^h)\dot{\gamma} + \frac{1}{2}R(\dot{\gamma}, R(\dot{\gamma}, J^v)\dot{\gamma})\dot{\gamma}\right),\end{aligned}$$

where $(\gamma(s), \dot{\gamma}(s)) = \exp_s(x, p)$.

Note the derivatives are in the "correct" direction.

Can then compute R in terms of $\psi, \mathcal{T} \ldots$

$$
\begin{aligned}
&R(\dot\gamma, J)\dot\gamma = \\
&\quad -\frac{1}{2}\Big\{ J^4\Big[\dot\gamma^3\dot\gamma^3(4\rho + 2\slashed{T}_{34}) + 2\dot\gamma^3\dot\gamma^A(\beta_A + \slashed{T}_{4A}) + \dot\gamma^A\dot\gamma^B(\alpha_{AB} + \frac{1}{2}\slashed{g}_{AB}\slashed{T}_{44})\Big] \\
&\quad + J^3\Big[-2\dot\gamma^3\dot\gamma^4(2\rho + \slashed{T}_{34}) + \dot\gamma^3\dot\gamma^A(2\underline{\beta}_A - \slashed{T}_{3A}) - \dot\gamma^4\dot\gamma^A(2\beta_A + \slashed{T}_{4A}) \\
&\quad - \dot\gamma^A\dot\gamma^B(\rho\slashed{g}_{AB} + \sigma\slashed{\epsilon}_{AB} - \frac{1}{2}\slashed{g}_{AB}\slashed{T}_{34} + \slashed{T}_{AB})\Big] \\
&\quad + \frac{1}{r}J^C\Big[-\dot\gamma^3\dot\gamma^4(2\beta_C + \slashed{T}_{4C}) + \dot\gamma^3\dot\gamma^3(-2\underline{\beta}_C + \slashed{T}_{3C}) \\
&\quad + \dot\gamma^3\dot\gamma^A(\rho\slashed{g}_{AC} + 3\sigma\slashed{\epsilon}_{AC} + \slashed{T}_{AC} - \frac{1}{2}\slashed{g}_{AC}\slashed{T}_{34}) - \dot\gamma^4\dot\gamma^A(\alpha_{AC} + \frac{1}{2}\slashed{g}_{AC}\slashed{T}_{44}) \\
&\quad + \dot\gamma^A\dot\gamma^B(-\slashed{g}_{AB}\beta_C + \slashed{g}_{AC}\beta_B + \frac{1}{2}\slashed{g}_{AB}\slashed{T}_{4C} - \frac{1}{2}\slashed{g}_{AC}\slashed{T}_{4B})\Big]\Big\} e_3
\end{aligned}
$$

$$
\begin{aligned}
&- \dot\gamma^A \dot\gamma^B (\rho \not{g}_{AB} + \sigma \not{\epsilon}_{AB} - \frac{1}{2} \not{g}_{AB} \not{T}_{34} + \not{T}_{AB})\Big] \\
&+ J^3 \Big[\dot\gamma^4 \dot\gamma^4 (4\rho + 2\not{T}_{34}) - 2\dot\gamma^4 \dot\gamma^A (2\underline{\beta}_A - \not{T}_{3A}) + \dot\gamma^A \dot\gamma^B (\underline{\alpha}_{AB} + \frac{1}{2} \not{g}_{AB} \not{T}_{33})\Big] \\
&+ \frac{1}{r} J^C \Big[\dot\gamma^3 \dot\gamma^4 (2\underline{\beta}_C - \not{T}_{3C}) + \dot\gamma^4 \dot\gamma^4 (2\beta_C + \not{T}_{4C}) \\
&- \dot\gamma^3 \dot\gamma^A (\underline{\alpha}_{AC} + \frac{1}{2} \not{g}_{AC} \not{T}_{33}) + \dot\gamma^4 \dot\gamma^A (\rho \not{g}_{AC} + 3\sigma \not{\epsilon}_{AC} + \not{T}_{AC} - \frac{1}{2} \not{g}_{AC} \not{T}_{34}) \\
&+ \dot\gamma^A \dot\gamma^B (\not{g}_{AB} \underline{\beta}_C - \not{g}_{AC} \underline{\beta}_B + \frac{1}{2} \not{g}_{AB} \not{T}_{3C} - \frac{1}{2} \not{g}_{AC} \not{T}_{3B})\Big]\Big\} e_4 \\
&+ \Big\{ J^4 \Big[- \dot\gamma^3 \dot\gamma^4 (2\beta^D + \not{T}_4{}^D) + \dot\gamma^3 \dot\gamma^3 (-2\underline{\beta}^D + \not{T}_3{}^D) \\
&+ \dot\gamma^3 \dot\gamma^A (\rho \delta_A^D + 3\sigma \not{\epsilon}_A{}^D + \not{T}_A{}^D - \frac{1}{2} \delta_A^D \not{T}_{34}) - \dot\gamma^4 \dot\gamma^A (\alpha_A{}^D + \frac{1}{2} \delta_A^D \not{T}_{44})
\end{aligned}
$$

$$
\begin{aligned}
&+\dot\gamma^A\dot\gamma^B\Big(-\slashed{g}_{AB}\beta^D+\delta_A^D\beta_B+\frac{1}{2}\slashed{g}_{AB}\slashed{T}_4{}^D-\frac{1}{2}\delta_A^D\slashed{T}_{4B}\Big)\Big]\\
&+J^3\Big[\dot\gamma^3\dot\gamma^4(2\underline{\beta}^D-\slashed{T}_3{}^D)+\dot\gamma^4\dot\gamma^4(2\beta^D+\slashed{T}_4{}^D)-\dot\gamma^3\dot\gamma^A(\underline{\alpha}_A{}^D+\frac{1}{2}\delta_A^D\slashed{T}_{33})\\
&+\dot\gamma^4\dot\gamma^A(\rho\delta_A^D+3\sigma\slashed{\epsilon}_A{}^D+\slashed{T}_A{}^D-\frac{1}{2}\delta_A^D\slashed{T}_{34})\\
&+\dot\gamma^A\dot\gamma^B(\slashed{g}_{AB}\underline{\beta}^D-\delta_A^D\underline{\beta}_B+\frac{1}{2}\slashed{T}_3{}^D-\frac{1}{2}\delta_A^D\slashed{T}_{3B})\Big]\\
&+\frac{1}{r}J^C\Big[\dot\gamma^4\dot\gamma^4(\alpha_C{}^D+\frac{1}{2}\delta_C^D\slashed{T}_{44})+\dot\gamma^3\dot\gamma^3(\underline{\alpha}_C{}^D+\frac{1}{2}\delta_C^D\slashed{T}_{33})\\
&+\dot\gamma^3\dot\gamma^4(-2\rho\delta_C^D-2T_C{}^D+\delta_C^D\slashed{T}_{34})+\dot\gamma^A\dot\gamma^B\Big(-\rho(\slashed{g}_{AB}\delta_C^D-\delta_A^D\slashed{g}_{BC})\\
&+\dot\gamma^4\dot\gamma^A(\slashed{g}_{AC}\beta^D+\delta_A^D\beta_C-2\delta_C^D\beta_A+\delta_C^D\slashed{T}_{4A}-\frac{1}{2}\slashed{g}_{AC}\slashed{T}_4{}^D-\frac{1}{2}\delta_A^D\slashed{T}_{4C})\\
&+\dot\gamma^3\dot\gamma^A(-\slashed{g}_{AC}\underline{\beta}^D-\delta_A^D\underline{\beta}_C+2\delta_C^D\underline{\beta}_A+\delta_C^D\slashed{T}_{3A}-\frac{1}{2}\slashed{g}_{AC}\slashed{T}_3{}^D-\frac{1}{2}\delta_A^D\slashed{T}_{3C})\\
&+\frac{1}{2}(\slashed{g}_{AB}\slashed{T}_C{}^D+\delta_A^D\slashed{T}_{AB}-\slashed{g}_{BC}\slashed{T}_A{}^D-\delta_A^D\slashed{T}_{BC})\Big)\Big]\Big\}e_D.
\end{aligned}
$$

Curvature of the Mass Shell

Recall,

$$\hat{\nabla}_X \hat{\nabla}_X J = \hat{R}(X, J)X,$$

with

$$\hat{R}(X, J)X = \mathrm{Hor}_{(\gamma,\dot{\gamma})}\left(R(\dot{\gamma}, J^h)\dot{\gamma} + \frac{3}{4} R(\dot{\gamma}, R(\dot{\gamma}, J^h)\dot{\gamma})\dot{\gamma} + \frac{1}{2}(\nabla_{\dot{\gamma}} R)(\dot{\gamma}, J^v)\dot{\gamma} \right)$$
$$+ \frac{1}{2} {}^T\mathrm{Ver}_{(\gamma,\dot{\gamma})}\left((\nabla_{\dot{\gamma}} R)(\dot{\gamma}, J^h)\dot{\gamma} + \frac{1}{2} R(\dot{\gamma}, R(\dot{\gamma}, J^v)\dot{\gamma})\dot{\gamma} \right).$$

Allowing certain components of J to grow like v, use guess for decay of Γ, ψ and bounds for components of null geodesics to check,

$$\hat{R}(X, J)X = \mathcal{O}(v(s)^{-\frac{5}{2}}),$$

and hence can be integrated twice.

Choice of Vectors

The goal is to estimate,

$$v e_4(f), \qquad e_3(f), \qquad v\left(\frac{1}{v}e_A\right)(f) \text{ for } A = 1, 2.$$

More natural to take vertical and horizontal lifts for V.

Choice of Vectors

The goal is to estimate,

$$ve_4(f), \qquad e_3(f), \qquad v\left(\frac{1}{v}e_A\right)(f) \text{ for } A = 1, 2.$$

More natural to take vertical and horizontal lifts for V.

In uncoupled problem in Minkowski space,

$$J(s) = \mathrm{Par}_{(\gamma,\dot\gamma),s}(V) + s\mathrm{Par}_{(\gamma,\dot\gamma),s}\left(\mathrm{Hor}_{(x,p)}(V^v)\right).$$

Choice of Vectors

The goal is to estimate,

$$ve_4(f), \qquad e_3(f), \qquad v\left(\frac{1}{v}e_A\right)(f) \text{ for } A = 1, 2.$$

More natural to take vertical and horizontal lifts for V.

In uncoupled problem in Minkowski space,

$$J(s) = \mathrm{Par}_{(\gamma,\dot{\gamma}),s}(V) + s\mathrm{Par}_{(\gamma,\dot{\gamma}),s}\left(\mathrm{Hor}_{(x,p)}(V^v)\right).$$

Take $V_{(3)} = \mathrm{Hor}_{(x,p)}(e_3)$. Then $V_{(3)} \sim 1$ and hence $J_{(3)}(s) \sim 1$ for all $s < 0$.

Choice of Vectors

The goal is to estimate,

$$ve_4(f), \qquad e_3(f), \qquad v\left(\frac{1}{v}e_A\right)(f) \text{ for } A = 1, 2.$$

More natural to take vertical and horizontal lifts for V.

In uncoupled problem in Minkowski space,

$$J(s) = \mathrm{Par}_{(\gamma,\dot{\gamma}),s}(V) + s\mathrm{Par}_{(\gamma,\dot{\gamma}),s}\left(\mathrm{Hor}_{(x,p)}(V^v)\right).$$

Take $V_{(3)} = \mathrm{Hor}_{(x,p)}(e_3)$. Then $V_{(3)} \sim 1$ and hence $J_{(3)}(s) \sim 1$ for all $s < 0$.

Choosing $V_{(A)} = v\mathrm{Hor}_{(x,p)}\left(\frac{1}{v}e_A\right)$ leads to $J_{(A)}(s) \sim v$ for all $s < 0$, so instead take

$$V_{(A)} = v\mathrm{Hor}_{(x,p)}\left(\frac{1}{v}e_A\right) + {p^4}^T\mathrm{Ver}_{(x,p)}\left(\frac{1}{v}e_A\right).$$

Choice of Vectors

In uncoupled problem in Minkowski space,

$$J(s) = \mathrm{Par}_{(\gamma,\dot{\gamma}),s}(V) + s\mathrm{Par}_{(\gamma,\dot{\gamma}),s}\left(\mathrm{Hor}_{(x,p)}(V^v)\right).$$

Similarly, choosing $V_{(4)} = v\mathrm{Hor}_{(x,p)}(e_4)$ leads to $J_{(4)}(s) \sim v$ for all $s < 0$. Instead use the fact that,

$$v\mathrm{Hor}_{(x,p)}(e_4) = \frac{v}{p^4}X - \frac{v^2 p^A}{p^4}\mathrm{Hor}_{(x,p)}\left(\frac{1}{v}e_A\right) - \frac{vp^3}{p^4}\mathrm{Hor}_{(x,p)}(e_3),$$

and

$$\frac{vp^A}{p^4} \sim 1, \qquad \frac{vp^3}{p^4} \sim \frac{1}{v}.$$

Choice of Vectors

In uncoupled problem in Minkowski space,

$$J(s) = \mathrm{Par}_{(\gamma,\dot{\gamma}),s}(V) + s\mathrm{Par}_{(\gamma,\dot{\gamma}),s}\left(\mathrm{Hor}_{(x,p)}(V^v)\right).$$

Similarly, choosing $V_{(4)} = v\mathrm{Hor}_{(x,p)}(e_4)$ leads to $J_{(4)}(s) \sim v$ for all $s < 0$. Instead use the fact that,

$$v\mathrm{Hor}_{(x,p)}(e_4) = \frac{v}{p^4}X - \frac{v^2p^A}{p^4}\mathrm{Hor}_{(x,p)}\left(\frac{1}{v}e_A\right) - \frac{vp^3}{p^4}\mathrm{Hor}_{(x,p)}(e_3),$$

and

$$\frac{vp^A}{p^4} \sim 1, \qquad \frac{vp^3}{p^4} \sim \frac{1}{v}.$$

For vertical parts, $V_{(4+A)} = \frac{p^4}{v}{}^T\mathrm{Ver}_{(x,p)}\left(\frac{1}{v}e_A\right)$ is of size $\frac{p^4}{v}$, so $J_{(4+A)}(s) \sim \frac{sp^4}{v} \leq 1$.

Choice of Vectors

In uncoupled problem in Minkowski space,

$$J(s) = \mathrm{Par}_{(\gamma,\dot{\gamma}),s}(V) + s\mathrm{Par}_{(\gamma,\dot{\gamma}),s}\left(\mathrm{Hor}_{(x,p)}(V^v)\right).$$

Similarly, choosing $V_{(4)} = v\mathrm{Hor}_{(x,p)}(e_4)$ leads to $J_{(4)}(s) \sim v$ for all $s < 0$. Instead use the fact that,

$$v\mathrm{Hor}_{(x,p)}(e_4) = \frac{v}{p^4}X - \frac{v^2 p^A}{p^4}\mathrm{Hor}_{(x,p)}\left(\frac{1}{v}e_A\right) - \frac{vp^3}{p^4}\mathrm{Hor}_{(x,p)}(e_3),$$

and

$$\frac{vp^A}{p^4} \sim 1, \qquad \frac{vp^3}{p^4} \sim \frac{1}{v}.$$

For vertical parts, $V_{(4+A)} = \frac{p^4}{v}{}^T\mathrm{Ver}_{(x,p)}\left(\frac{1}{v}e_A\right)$ is of size $\frac{p^4}{v}$, so $J_{(4+A)}(s) \sim \frac{sp^4}{v} \leq 1$.

Finally, $V_{(4)} = v\mathrm{Hor}_{(x,p)}(e_4) + p^{4\,T}\mathrm{Ver}_{(x,p)}(e_4)$ gives $J_{(4)}(s) \sim v(s)$.

Estimates for f

Use these facts to conclude

$$v|e_4(f)| \leq C, \qquad |e_3(f)| \leq C, \qquad v|\frac{1}{v}e_A(f)| \leq C \text{ for } A = 1, 2.$$

$$p^4|\partial_{\overline{p}^4} f| \leq C, \qquad \frac{p^4}{v^2}|\partial_{\overline{p}^A} f| \leq C \text{ for } A = 1, 2.$$

Note that Jf satisfies,

$$Jf(x, p) = Jf(\exp_s(x, p)).$$

Can therefore repeat to obtain estimates for higher order derivatives.

Error terms will involve higher order derivatives of ψ so have to estimate in appropriate L^2 spaces.

Geometry of Null Geodesics

Decay estimates for $\mathcal{T}$ follow from understanding geometry of null geodesics and estimates for derivatives of f.

If λ is a null geodesic in Minkowski space with,

$$\dot{\lambda}(s) = p^A(s)e_A + p^3(s)e_3 + p^4(s)e_4,$$

can easily see,

$$v(s)^2|p^A(s)| \leq C, \qquad v(s)^2|p^3(s)| \leq C, \qquad |p^4(s)| \leq C.$$

Geometry of Null Geodesics

Decay estimates for $\mathcal{T}$ follow from understanding geometry of null geodesics and estimates for derivatives of f.

If λ is a null geodesic in Minkowski space with,

$$\dot{\lambda}(s) = p^A(s)e_A + p^3(s)e_3 + p^4(s)e_4,$$

can easily see,

$$v(s)^2|p^A(s)| \leq C, \qquad v(s)^2|p^3(s)| \leq C, \qquad |p^4(s)| \leq C.$$

In general can be seen, using pointwise bounds for Γ, through geodesic equations:

$$\dot{p}^\mu(s) + \Gamma^\mu_{\alpha\beta} p^\alpha(s) p^\beta(s) = 0,$$

or, according to guess,

$$\frac{d}{ds}p^4(s) = \mathcal{O}(v(s)^{-2}), \quad \frac{d}{ds}v(s)^2 p^3(s) = \mathcal{O}(v(s)^{-2}), \quad \frac{d}{ds}v(s)^2 p^A(s) = \mathcal{O}(v(s)^{-2}).$$

Zeroth Order Estimates for Energy Momentum Tensor

With these bounds estimates for the energy momentum tensor are straightforward,

$$T^{\alpha\beta}(x) = \int_{P_x} f p^\alpha p^\beta d\mu_{P_x} \leq \varepsilon_0 \int_{P_x} \mathbb{1}_{\mathrm{supp}(f)} p^\alpha p^\beta d\mu_{P_x} \leq \frac{C\varepsilon_0}{v^q},$$

with q depending on α, β, using the fact that $P_x \cap \mathrm{supp}(f)$ is shrinking as $v \to \infty$.

Given bounds for derivatives of f, higher order derivatives of $\mathcal{T}$ are estimated the same way.