Perturbing a non homogeneous stationary state of the Vlasov equation

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Outlook

General goal: explore the Vlasov phase space.

I have no theorem for what I am going to tell
$\rightarrow$ heuristic asymptotic expansions backed by numerical simulations

I hope it may catch your interest nonetheless!

An astrophysical motivation

Radial Orbit Instability: take a family of spherically symmetric stationary state of the gravitational Vlasov-Poisson equation, depending on a parameter α.
Few low angular momentum stars (large α) $\rightarrow$ stable
Many low angular momentum stars (small α) $\rightarrow$ unstable, real eigenvalue

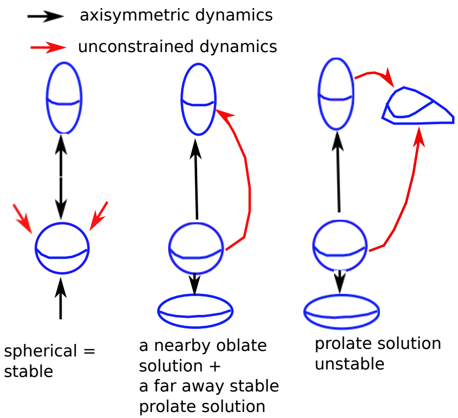
What happens when the instability develops? Supposed to play an important role in determining the shape of some galaxies.

Palmer et al. (1990): detailed numerics and approximate computations. Ex:

$$f(E, L) \propto \frac{1}{J^2 + \alpha^2}$$

An astrophysical motivation, 2

Scenario according to Palmer et al.:



How general is it? Can we quantify this (what does "nearby" means)?

Vlasov equations: dynamics close to a stationary state

General problem: investigate the dynamics close to a stationary state

• Difficulties: even the linearized equation around a stable stationary state is not so easy... (eg: Landau damping)

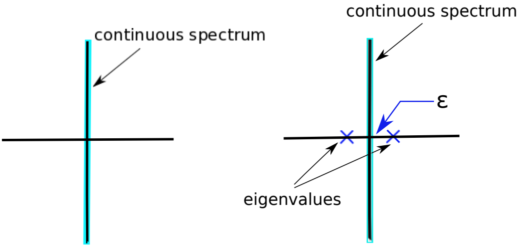
• Close to homogeneous stationary states: tremendous recent mathematical progresses (Mouhot-Villani, Lin-Zeng).

• Close to non homogeneous stationary states (relevant setting in astrophysics!): even more difficult. There are powerful criteria for stablity in the gravitational case (Guo,Rein,Lin,Lemou, Méhats,Raphaël...).

Close to a homogeneous weakly unstable state

- Unstable eigenvector ψ; unstable eigenvalue ε. Nonlinear dynamics?

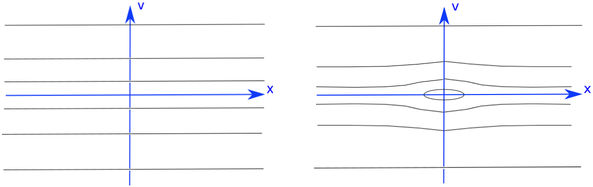
$\rightarrow$ A bifurcation problem; not a standard one.



- An old problem in plasma physics, many contributions (O'Neil, Crawford, Del-Castillo-Negrete……), with an interesting story. Nice recent review by Balmforth, Morrison, Thiffeault (preprint 2013)

No theorem here to my knowledge $\rightarrow$ heuristics and numerics!

Homogeneous weakly unstable state - Phase space



Left: unperturbed homogeneous stationary state
Right: perturbed stationary state $\rightarrow$ resonance phenomenon

• J.D. Crawford: unstable manifold computations $\rightarrow$ reduced 1D model with diverging coefficients.
Correct scaling in ε (not trivial!), but not much more.
• D. Del-Castillo-Negrete $\rightarrow$ an infinite dimensional reduced model (Single Wave Model), which seems a satisfactory answer.
Messages: strong non linear effects, divergences in the unstable manifold expansion; universality of the Single Wave Model.

Close to a NON homogeneous weakly unstable state

Goal: address a similar question close to a NON homogeneous weakly unstable state.

Example: radial orbit instability (NB: real unstable eigenvalue).

• 3D gravitational Vlasov-Poisson: technical difficulties, even at linear level.
→ use simpler 1D models, for which explicit computations can be carried out, and numerics is easy.

Hope: the weakly non linear dynamics may be "universal"

Simplest possible model: cosine potential

- Periodic in space, $x \in [0, 2\pi]$; cosine interaction potential

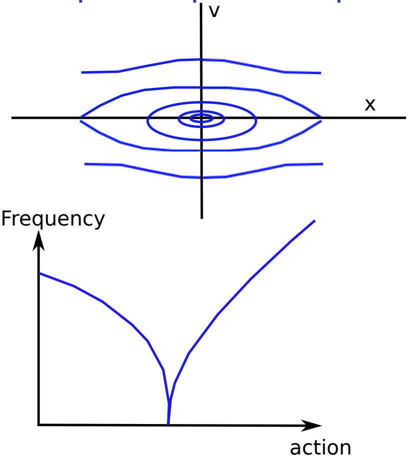
$$
\begin{aligned}
\partial_t f + v\partial_x f - \partial_x \Phi \partial_v f &= 0 \\
\Phi(x) = -C \int f(y,v,t)\cos(x-y)dy &= -CM\cos(x-\varphi)
\end{aligned}
$$

$M =$ "magnetization", the only parameter for the potential.
Characteristics, particles trajectories = simple pendulum dynamics.

- Actually introduced in an astrophysical context! (Lynden-Bell, Pichon).

- Only model I know where linear computations in the NON homogeneous case are completely explicit.
$\rightarrow$ will be helpful

Cosine potential: phase space, frequency



Real eigenvalue (zero frequency) $\rightarrow$ no resonance; non linear effects may be weaker.

Hope: unstable manifold computations less singular than in the homogeneous case; $\rightarrow$ more reliable

Unstable manifold computations,1

Expansion around the reference stationary state $f_0(x, v)$:

$$f(x, v, t) = A(t)\Psi(x, v) + R[A](x, v, t)$$

$A\psi =$linear part; $R[A] =$ equation of the unstable manifold.
Reduced dynamics:

$$\dot{A} = \varepsilon A + C(\varepsilon)A^2 + \dots$$

with $C(\varepsilon) \sim c/\varepsilon$. Explicit expression for c available (complicated).

$\rightarrow$ $1/\varepsilon$ singularities appear; next order computation show they are weaker than in the homogeneous case

Unstable manifold computations,2

$$\dot{A} = \varepsilon A + C(\varepsilon)A^2 + \dots$$

Conclusions:

- There is an attractive (*on the unstable manifold) stationary state with* $A \propto \varepsilon^2$
- *Asymmetry between the two directions on the unstable manifold: one direction goes to a "nearby stationary state", the other one goes far away, out of range for the present theory*
- *All this can be directly checked numerically*

Numerics

• Standard semi-lagrangian method; uses GPU (cf Rocha Filho 2013)

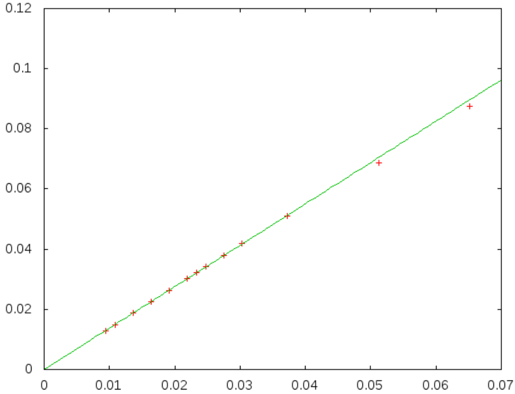
•1 spatial dimension $\rightarrow$ possible to reach good resolution (1024x1024)

• Order of magnitude of the unstable eigenvalue $\varepsilon \simeq 0.05$

How close is the "nearby stationary state"?

Stationary state $f_0 \propto e^2 e^{-\alpha e}$; α_c: instability threshold.

Plot: $M^\infty - M_0$ *vs* $\alpha - \alpha_c$



$\rightarrow$ confirms the scaling $A(t \to \infty) \propto \varepsilon^2$

Back to Radial Orbit Instability

NB: Radial Orbit Instability associated with a real eigenvalue $\rightarrow$ consistent with the present theory

Some of the findings in Palmer et al. 1990 are recovered:

• Existence of a nearby stationary state, attractive at least for a restricted dynamics; we have a prediction for the *distance* of this state from the reference stationary state.

• Otherwise the system goes far away from the original reference stationary state

Conclusions

- Only heuristic computations; more numerical tests are needed
- What about more complicated models? 1D, 3D gravitational. Universality of this scenario?
- Exploring the case of complex eigenvalues. . .