The Spectral Theorem

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1 History

The Spectral Theorem is a general concept, with many different formulations in different settings. We are usually aware of the simple, finite-dimensional case. But the spectral theorem, in its full capacity, is a very powerful tool, used in analysis, PDE, as well as in quantum mechanics.

While the finite case was known before, the inifinite-dimensional case was first brought up by David Hilbert, around 1909-1910, when Banach and Hilbert spaces were formalized. Then and in the subsequent 30-40 years, in conjuction with the rapid developement of quantum mechanics, research into this area had grown. Many of the results brought here are due to mathematicians such as Hilbert himself (who also started working in physics after 1912), John von Neumann and Hermann Weyl, as well as physicists, for whom Hilbert spaces played a central role. Noteworthy are Erwin Schrödinger whose famous equation is studied below, and Werner Heisenberg.

2 Finite-Dimensional Version

Theorem 1. (Finite-Dimensional Version)
Let V be an n-dimensional inner product space (real or complex) with the standard Hermitian inner product. Let A be an $n \times n$ Hermitian matrix. Then there exists an orthonormal basis of V consisting of eigenvectors of A. All eigenvalues are real.

Proof. We prove for the real case.

First, it clearly suffices to prove the existence of one eigenvector v, since we may then look at the orthogonal complement to v in V: A will still be selfadjoint there, and we will be able to apply the proof again.

List all the eigenvalues of A in decreasing order: $\lambda_1, \lambda_2, ..., \lambda_n$. Then for all $v \in V$, we have

$$(Av, v) \;\leq\; \lambda_1(v, v).$$

In fact, we have an equality if and only if $Av = \lambda_1 v$.

Consider the ratio

$$R_A(v) \;=\; \frac{(Av, v)}{(v, v)}$$

which is called the **Rayleigh quotient**.

Now we attempt to find a vector v_1 that maximizes $R_A(v)$ subject to the constraint $(v, v) = 1$. We will only need to show this v_1 is an eigenvector. (At this point we could also prove using Lagrange multipliers)

Clearly, the function

$$v \;\longmapsto\; (Av, v)$$

is continuous (this is where we use finite-dimensionality) and the unit sphere S is compact.

Now, consider the Rayleigh quotient restricted to S. On S, we have $R_A(v) = (Av, v)$. By the compactness of S, and continuity of $R_A(v)$ on S, this function must be bounded, and must attain its maximum. Call this maximum λ_1', and the corresponding vector that attains it, v_1. Then we have

$$(v_1, v_1) \;=\; 1,$$
$$(Av_1, v_1) \;=\; \lambda_1'.$$

We will finish if we show that $Av_1 = \lambda_1' v_1$ (and thus, in fact, $\lambda_1 = \lambda_1'$). To show this, we pick any $u \in V$, and consider the small perturbation $(A(v_1 + \varepsilon u), v_1 + \varepsilon u)$; first we know that

$$(A(v_1 + \varepsilon u), v_1 + \varepsilon u) \ \leq \ \lambda_1'(v_1 + \varepsilon u, v_1 + \varepsilon u).$$

Subtracting the LHS from the RHS, we get

$$\begin{aligned} 0 \ &\leq \ \lambda_1'(v_1 + \varepsilon u, v_1 + \varepsilon u) - (A(v_1 + \varepsilon u), v_1 + \varepsilon u) \\ &= \ \varepsilon\{2\lambda_1'(u, v_1) - (Av_1, u) - (Au, v_1)\} + \varepsilon^2\{\lambda_1'(u, u) - (Au, u)\} \\ &= \ \varepsilon\{2\lambda_1'(v_1, u) - 2(Av_1, u)\} + \varepsilon^2\{\lambda_1'(u, u) - (Au, u)\}, \end{aligned}$$

where the selfadjointness of A was used only in the last transition.

For this expression to be nonnegative for any $\varepsilon \in \mathbb{R}$, the term $2\lambda_1'(v_1, u) - 2(Av_1, u)$ must vanish.

But, letting $u = \lambda_1' v_1 - Av_1$, we attain that $\lambda_1' v_1 = Av_1$, so that λ_1' is an eigenvalue with eigenvector v_1. By maximality, $\lambda_1 = \lambda_1'$, and we are done. $\qquad\square$

3 Definitions and Basic Examples

All operators under consideration are linear!

Notation 2. *For an operator $T: X \longrightarrow Y$ between two Banach spaces, we denote by $D(T)$ and $R(T)$ the domain and range of T respectively.*

Definition 3. *An operator $T: X \longrightarrow Y$ between two Banach spaces is said to be **bounded** if $D(T) = X$, and if there exists $M \in \mathbb{R}$ such that $\|Tx\| \leq M\|x\|$ for all $x \in X$.*
   The space of all bounded operators on X to Y is denoted by $\mathfrak{B}(X, Y)$.

Definition 4. *An operator $T: X \longrightarrow Y$ between two Banach spaces is said to be **closed** if for any sequence $x_n \in D(T)$ such that $x_n \to x$ and $Tx_n \to y$, then $x \in D(T)$ and $y = Tx$.*
   The space of all closed operators on X to Y is denoted by $\mathfrak{C}(X, Y)$.

Remark 5. $\mathfrak{B}(X, Y) \subseteq \mathfrak{C}(X, Y)$.

Remark 6. The definition of a closed operator is equivalent to saying that its graph $G(T)$ is a closed subspace of $X \times Y$.

Definition 7. *An operator $T \in \mathfrak{B}(X, Y)$ is **compact** if the image $\{Tu_n\}$ of any bounded sequence $\{u_n\} \subseteq X$ contains a Cauchy subsequence.*

Definition 8. *Given a densely defined operator $T: X \longrightarrow Y$ we say that $S: Y^* \longrightarrow X^*$ is its **adjoint** if*

$$(f, Tx) \ = \ (Sf, x)$$

for all $f \in D(S)$, $x \in D(T)$, and $D(S)$ is the maximal subspace with this property. I.e., we have the following commutative diagram:

$$\begin{array}{ccc} x \in X & \xrightarrow{\;T\;} & Y \\ \downarrow & \xleftarrow{\;S\;} & \downarrow f \\ \mathbb{C} & & \mathbb{C} \end{array}$$

   The following definitions, and most of this exposition, will deal with a special class of Banach spaces: *Hilbert* spaces. The operators henceforth will be *closed, densely defined and unbounded*, unless otherwise stated.

Remark 9. By Riesz's theorem, a Hilbert space $\mathcal{H}$ is identified with its dual $\mathcal{H}^*$. In particular, for every operator A on $\mathcal{H}$ to itself, the operator A^* also acts on $\mathcal{H}$ to itself.

Definition 10. *A* **selfadjoint** *operator A on a Hilbert space $\mathcal{H}$ is an operator such that $A = A^*$. In particular, $D(A) = D(A^*)$, and*

$$(Ax, y) = (x, Ay)$$

for all $x, y \in D(A)$.

In the finite-dimentional case, A is represented by a Hermitian matrix T ($T = T^*$).

Remark 11. A **symmetric** operator, is such that $A^* \supseteq A$ (A^* is an extension of A). For a historical comment on this matter, see page 414 of [6] Lax.

Definition 12. *Let $T: X \longrightarrow X$ be an operator on a Banach space. If $T - \zeta$ (for $\zeta \in \mathbb{C}$) is invertible, its range all of X, and*

$$(T - \zeta)^{-1} \in \mathfrak{B}(X)$$

then ζ is said to belong to the **resolvent set** *of T, $\boldsymbol{P}(T)$.*

The complementary set $\Sigma(T) = \mathbb{C} \backslash \boldsymbol{P}(T)$ is called the **spectrum** *of T.*

Remark 13. The condition $\text{Range}(T - \zeta) = X$ and T is closed implies already (by the closed graph theorem) that $(T - \zeta)^{-1} \in \mathfrak{B}(X)$.

Theorem 14. (Compact Operator Spectral Structure)

Let $T \in \mathfrak{B}(X)$ be a compact operator on a Banach space X. $\Sigma(T)$ is a countable set, with no accumulation point other than zero. Each nonzero $\lambda \in \Sigma(T)$ is an eigenvalue of T with finite multiplicity, and $\overline{\lambda}$ is an eigenvalue of T^* with the same multiplicity.

Theorem 15. (Compact Selfadjoint Operator Spectral Theorem)

Let A be a compact selfadjoint operator on a Hilbert space $\mathcal{H}$. Then there exists an orthonormal basis of $\mathcal{H}$ consisting of eigenvectors of A. All eigenvalues are real, and all nonzero eigenvalues are of finite multiplicity.

Corollary 16.

Let $\lambda_1, \lambda_2, ...$ be the sequence of nonzero eigenvalues, and let P_j be the projection on the finite dimensional eigenspace N_j associated with λ_j. Then, for every $u \in \mathcal{H}$, we can represent

$$Au = \sum_{j=1}^{\infty} \lambda_j P_j u,$$

or

$$A = \sum_{j=1}^{\infty} \lambda_j P_j.$$

Example 17. Consider the space $\mathcal{H} = L^2[0, \pi]$, and the operator $A = -\frac{d^2}{dx^2}$. Notice that A is *not* compact (not even bounded)!

1. *The domain $D(A)$:*

   a) A necessary condition for φ to be in $D(A)$ is that $\varphi'' \in \mathcal{H} \subseteq L^1[0, \pi]$. And so, $\varphi \in C^1[0, \pi]$. In particular, $D(A) \subseteq C^1[0, \pi]$.

b) If we restrict further by demanding $\varphi(0) = \varphi(\pi) = 0$, we see (by integration by parts) that A is symmetric:

$$(A\varphi, \psi) \;=\; (\varphi, A\psi).$$

2. *Eigenfunctions of A:*

We are looking for $\varphi \in D(A)$ such that $A\varphi = \lambda\varphi$ for some $\lambda \in \mathbb{R}$ (λ must be real due to symmetry of A). We know that all solutions are given by

$$\varphi_n(x) \;=\; \sin(nx),$$

and the corresponding eigenvalues are given by $\lambda_n = n^2$.

3. *Completeness of the eigenfunctions:*

From the theorem on Fourier series, we know that the eigenfunctions φ_n are a complete orthonormal basis in $\mathcal{H}$.

4. *Conclusion:*

The spectrum, in this case, is $\Sigma = \{1, 4, 9, ..., n^2, ...\}$, where " $=$ " and not " $\supseteq$ " is due to the full version of the spectral theorem, presented later.

Example 18. Consider the space $\mathcal{H} = L^2[0, l]$, and the operator $A = -\frac{d^2}{dx^2}$.

1. *The domain $D(A)$:*

a) A necessary condition for φ to be in $D(A)$ is that $\varphi'' \in \mathcal{H} \subseteq L^1[0, l]$. And so, $\varphi \in C^1[0, l]$. In particular, $D(A) \subseteq C^1[0, l]$.

b) If we restrict further by demanding $\varphi(0) = \varphi(l) = 0$, we see (by integration by parts) that A is symmetric:

$$(A\varphi, \psi) \;=\; (\varphi, A\psi).$$

2. *Eigenfunctions of A:*

We are looking for $\varphi \in D(A)$ such that $A\varphi = \lambda\varphi$ for some $\lambda \in \mathbb{R}$ (λ must be real due to symmetry of A). We know that all solutions are given by

$$\varphi_n(x) \;=\; \sin\left(\frac{n\pi}{l}x\right),$$

and the corresponding eigenvalues are given by $\lambda_n = \left(\frac{n\pi}{l}\right)^2$.

3. *Completeness of the eigenfunctions:*

From the theorem on Fourier series, we know that the eigenfunctions φ_n are a complete orthonormal basis in $\mathcal{H}$.

4. *Conclusion:*

The spectrum, in this case, is $\Sigma = \left\{\left(\frac{\pi}{l}\right)^2, \left(\frac{2\pi}{l}\right)^2, \left(\frac{3\pi}{l}\right)^2, ..., \left(\frac{n\pi}{l}\right)^2, ...\right\}$.

Now: Suppose we let l grow: then the spectrum has more and more points very close to 0. The actual limit $l \longrightarrow \infty$ will have to be solved in another method: while the solution for this exaple and the previous may be done using Fourier *series*, the case $l \longrightarrow \infty$ requires the Fourier *transform*. Heuristically, one might think of the spectrum tending towards the continuum $(0, \infty)$. This spectrum is a new type of spectrum, one that is *not* comprised of eigenvalues, as A has no eigenvalues on $\mathcal{H}' = [0, \infty)$ with the appropriate domain.

Definition 19. *Let $T: X \longrightarrow X$ be an operator on a Banach space X, and let $T_\lambda = \lambda - T$. The spectrum of T, $\Sigma(T)$ may be subdivided into the following distinct types of spectra:*

- *The **point spectrum** of T, $\sigma_p(T)$, consists of eigenvalues of T. $\lambda \in \sigma_p(T)$ iff T_λ is not injective.*

- *The **continuous spectrum** of T, $\sigma_c(T)$, consists of all λ for which T_λ is invertible and $\mathrm{Range}(T_\lambda)$ is dense in X.*

- *The **residual spectrum** of T, $\sigma_r(T)$, consists of all λ for which T_λ is invertible and $\mathrm{Range}(T_\lambda)$ is not dense in X.*

Example 20. Consider the space $\mathcal{H} = L^2(\mathbb{R})$, and the operator $A = -\frac{d^2}{dx^2}$ (we consider the action of this operator on functions in $\mathcal{H}$ formally).

Using the Fourier transform, we may write:

$$
\begin{aligned}
A\varphi &= -\frac{d^2}{dx^2}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{\varphi}(\xi)e^{ix\xi}\,d\xi\right) \\
&= -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{d^2}{dx^2}\left(\hat{\varphi}(\xi)e^{ix\xi}\right)d\xi \\
&= -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(i\xi)^2\hat{\varphi}(\xi)e^{ix\xi}\,d\xi \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\xi^2\hat{\varphi}(\xi)e^{ix\xi}\,d\xi.
\end{aligned} \tag{1}
$$

More abstractly, let $\mathfrak{F}\colon L^2(\mathbb{R}_x) \longrightarrow L^2(\mathbb{R}_\xi)$ be the Fourier transform $(\mathfrak{F}f)(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-ix\xi}$. Then the Fourier inversion theorem says that $\mathfrak{F}$ is unitary (an onto isometry), and its inverse is given by $\left(\mathfrak{F}^{-1}\hat{\varphi}\right)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{\varphi}(\xi)e^{ix\xi}\,d\xi$. The equality (1) can then be written

$$
\left(\mathfrak{F}A\mathfrak{F}^{-1}\right)\hat{\varphi}(\xi) = \xi^2\hat{\varphi}(\xi). \tag{2}
$$

Thus $D(A)$ is mapped by $\mathfrak{F}$ onto the functions $\hat{\varphi}(\xi) \in L^2(\mathbb{R}_\xi)$ such that also $\xi^2\hat{\varphi}(\xi) \in L^2(\mathbb{R}_\xi)$. This actually defines $D(A)$.

So what is the spectrum of A, $\Sigma(A)$?

First, we note that a one-to-one and onto transformation as $\mathfrak{F}$ does not affect the behavior of the operator, and, what is important for us here, does not affect the spectrum.

Thus, we may consider the ξ-space, where A translates into multiplication by ξ^2. To find the spectrum, we must first find the resolvent, $P(A)$. We know from Definition 12 that $\zeta \in \mathbb{C}$ is in the resolvent of A if and only if $A - \zeta$ is onto $\mathcal{H}$, is invertible, and the inverse is bounded.

So, we consider the operator $\left(\xi^2 - \zeta\right)^{-1}$. Now, since $\xi \in \mathbb{R}$, $\xi^2 \geq 0$. Thus, for any $\zeta \in \mathbb{C}\backslash[0,\infty)$ we have that $\left(\xi^2 - \zeta\right)^{-1}$ is bounded. Otherwise it is not (it is not even defined at $\zeta = \xi^2$).

The above also shows that $\left(\xi^2 - \zeta\right)$ is onto (for $\zeta \in \mathbb{C}\backslash[0,\infty)$), since we may simply write

$$
\left(\xi^2 - \zeta\right)\left(\xi^2 - \zeta\right)^{-1}f = f
$$

for any $f \in D(A)$.

All these imply that $\Sigma(A) = [0,\infty)$.

Another way of solving this problem, is an ODE method. As mentioned above, to find the resolvent, we look for all $\zeta \in \mathbb{C}$ such that $A - \zeta$ is onto $\mathcal{H}$, is invertible, and the inverse is bounded.

Thus we may let $h \in \mathcal{H}$,o write

$$
(A - \zeta)f = -h,
$$

and ask for which ζ we have a solution in $\mathcal{H}$. Such a ζ would be in the resolvent. We have to solve:

$$
f''(t) + \zeta f(t) = h(t).
$$

Here's a sketch of a possible method of solving:

We first consider the homogeneous problem, which we translate into the quadratic equation

$$r^2 + \zeta = 0,$$

which has solutions $r_{1,2} = \pm \sqrt{-\zeta}$.

1. Assume $\zeta \in (-\infty, 0)$: Then we would have two distinct, real solutions $\pm \zeta_1 \in \mathbb{R}$, and the solution to the homogeneous problem would be

$$f_{\text{homogeneous}}(t) = C_1 e^{\zeta_1 t} + C_2 e^{-\zeta_1 t}.$$

2. Assume $\zeta \in \mathbb{C} \backslash \mathbb{R}$: Then $\pm \sqrt{-\zeta}$ would have a nonzero real part. The solution will contain two exponents, as above, as well as sines and cosines.

3. Assume $\zeta = 0$: Then the solution to the homogeneous problem would be

$$f_{\text{homogeneous}}(t) = C_1 + C_2 t.$$

4. Assume $\zeta \in (0, \infty)$: Then $\pm \sqrt{-\zeta} = \pm i \sqrt{\zeta}$ are purely imaginary, and the solution is given by

$$f_{\text{homogeneous}}(t) = C_1 \cos\left(\sqrt{\zeta} t\right) + C_2 \sin\left(\sqrt{\zeta} t\right).$$

Now, assume for simplicity, that h has compact support. Then outside the support of h we will have a solution comprised of the homogeneous solution, and on the support of h, the solution will take into account both the homogeneous, and the nonhomogeneous solutions.

Only in cases (1) and (2) above we will be able to construct a solution that will be globally L^2: for the part to the right of the support of h we choose a sloution that has a factor $e^{-\zeta t}$ with $\zeta > 0$, and for the part to the left of the support we choose the factor $e^{\zeta t}$. In between, on the support of h, the special solution will "unite" these two solutions to the right and to the left.

Thus we have a solution.

In cases (3) and (4) however, such a construction is not possible. We cannot obtain an L^2 solution.

This corresponds to our above finding that $\Sigma = [0, \infty)$.

Example 21. Consider the Schrödinger equation in $\mathbb{R}^3$:

$$i h \frac{\partial \psi}{\partial t} = \left[-\frac{h^2}{2m} \Delta + V(\boldsymbol{r}) \right] \psi,$$

where we assume ψ to be sufficiently "nice".

We look for steady energy solutions, i.e. functions ψ of the form

$$\psi(\boldsymbol{r}, t) = e^{-\frac{i}{h} E t} \psi(\boldsymbol{r}),$$

with E negative, and potentials of the form $V = -\frac{e^2}{r}$, so that ψ must satisfy

$$-\frac{h^2}{2m} \Delta \psi = \left(E + \frac{e^2}{r} \right) \psi.$$

Rewriting in spherical coordinates, we find that the equation becomes

$$\frac{1}{r} \frac{\partial^2}{\partial r^2}(r\psi) + \frac{1}{r^2} \left\{ \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2 \psi}{\partial \phi^2} \right\} = -\frac{2m}{h}\left(E + \frac{e^2}{r} \right) \psi.$$

Now, further assume that ψ is spherically symmetrical. Then the equation reduces to

$$\frac{1}{r} \frac{\partial^2}{\partial r^2}(r\psi) = -\frac{2m}{h^2}\left(E + \frac{e^2}{r} \right) \psi. \tag{3}$$

By changing coordinates into

$$r = \frac{h^2}{me^2}\rho$$
$$E = \frac{me^4}{2h^2}\varepsilon$$

and multiplying (3) by ρ, we have

$$\frac{d^2(\rho\psi)}{d\rho^2} = -\left(\varepsilon + \frac{2}{\rho}\right)\rho\psi.$$

The rescaling factors h^2/me^2 and $me^4/2h^2$ are called the "Bohr radius" and the "Rydberg" energy unit respectively.

Letting $f = \rho\psi$, we have

$$\frac{d^2 f}{d\rho^2} = -\left(\varepsilon + \frac{2}{\rho}\right)f. \tag{4}$$

The Solution of a Physicist (Feynman, in this case):

To make solving easier, we first write f as the product $f(\rho) = e^{-\alpha\rho}g(\rho)$, with $0 < \alpha \in \mathbb{R}$, so that we need to find g instead of f. Writing this f in (4), one gets

$$\frac{d^2 g}{d\rho^2} - 2\alpha\frac{dg}{d\rho} + \left(\frac{2}{\rho} + \varepsilon + \alpha^2\right)g = 0.$$

Choosing $\alpha^2 = -\varepsilon$ (since $\varepsilon < 0$ this is okay), we get

$$\frac{d^2 g}{d\rho^2} - 2\alpha\frac{dg}{d\rho} + \frac{2}{\rho}g = 0. \tag{5}$$

We solve (5) by means of a power series in ρ: we assume g may be written as

$$g(\rho) = \sum_{k=1}^{\infty} a_k\rho^k,$$

and plug into (5) to get

$$\sum_{k=1}^{\infty} \{(k+1)ka_{k+1} - 2\alpha ka_k + 2a_k\}\rho^{k-1} = 0. \tag{6}$$

To satisfy this, the coefficients must vanish for all k. Thus we get the system of equations

$$(k+1)ka_{k+1} - 2(\alpha k - 1)a_k = 0.$$

These are easy to satisfy: pick any a_1, and for $k > 1$ set

$$a_{k+1} = \frac{2(\alpha k - 1)}{k(k+1)}a_k. \tag{7}$$

Now, consider large values of ρ. For such ρ, high order terms will be more significant. For large values of k, one finds

$$a_{k+1} \approx \frac{2\alpha}{k}a_k$$
$$\approx \frac{(2\alpha)^k}{k!}.$$

So we get that

$$g(\rho) \;\sim\; \sum_{k=1}^{\infty} \frac{(2\alpha)^{k-1}}{(k-1)!}\rho^k$$
$$\sim\; e^{2\alpha\rho}.$$

This implies that

$$f(\rho) \;\sim\; e^{\alpha\rho}.$$

This means that $f \notin L^2[0,\infty)$. The reason this is not good comes originally from physics, where f (or, rather, ψ) represents the probability distribution function for the location of an electron orbitting a proton (this is the hydrogen atom model). A probability distribution function must be L^2, and, in addition, we know that we expect to find the electron in proximity with the proton, which is not the case with the solution that we have just found.

To solve this problem, we notice that for any any integer n, if $\alpha = 1/n$, we get that $a_{n+1} = a_{n+2} = \cdots = 0$. This would imply that $g(\rho)$ has polynomial growth, so that $f(\rho) = e^{-\alpha\rho}g(\rho)$ will tend (rapidly) to 0 as $\rho \to \infty$. In fact, we'll have $f \in L^2[0,\infty)$. The same would obvioulsy apply to the wave function ψ.

Rewriting equation (4) as

$$-\left(\frac{d^2}{d\rho^2} + \frac{2}{\rho} \right)f \;=\; \varepsilon f$$

we have just found that the eigenvalues of this equation are given by $\varepsilon_n = -1/n^2$:

$$\left\{ -1, -\frac{1}{4}, -\frac{1}{9}, ..., -\frac{1}{n^2}, ... \right\}.$$

4 The Spectral Family

4.1 Definitions

Now we define an extremely important notion in analysis, the notion of the *spectral family*:

We consider a Hilbert space $\mathcal{H}$, and a parameter $\lambda \in \mathbb{R}$. Suppose that there exists a nondecreasing family $\{\mathcal{M}(\lambda)\}$ of subspaces of $\mathcal{H}$, such that $\cap_\lambda \mathcal{M}(\lambda) = 0$ and $\overline{\cup_\lambda \mathcal{M}(\lambda)} = \mathcal{H}$.

Denote $\mathcal{M}(\lambda+0) = \cap_{\lambda' > \lambda} \mathcal{M}(\lambda')$ and $\mathcal{M}(\lambda-0) = \overline{\cup_{\lambda' < \lambda} \mathcal{M}(\lambda')}$.

For the family $\{\mathcal{M}(\lambda)\}$ we say that

- $\mathcal{M}(\lambda)$ is **right continuous** at λ if $\mathcal{M}(\lambda) = \mathcal{M}(\lambda+0)$,

- $\mathcal{M}(\lambda)$ is **left continuous** at λ if $\mathcal{M}(\lambda) = \mathcal{M}(\lambda-0)$,

- $\mathcal{M}(\lambda)$ is **continuous** at λ if it is both left and right continuous.

We usually assume right continuity.

Definition 22. *The* **spectral family** *(or resolution of the identity) is a family* $\{E(\lambda)\}$ *of orthogonal projections onto* $\mathcal{M}(\lambda)$ *that satisfies:*

- $E(\lambda)$ *is nondecreasing:* $E(\lambda') \le E(\lambda'')$ *for* $\lambda' < \lambda''$.

- $\underset{\lambda\to-\infty}{s\text{-}lim}\, E(\lambda) = 0$ *and* $\underset{\lambda\to+\infty}{s\text{-}lim}\, E(\lambda) = 1$.

Notation 23. $\mathcal{M} \ominus \mathcal{N} = \mathcal{M} \cap \mathcal{N}^{\perp}$ *is the orthogonal complement of* $\mathcal{N}$ *in* $\mathcal{M}$.

For any semiclosed interval $I = (\lambda', \lambda'']$ we set

$$E(I) \;=\; E(\lambda'') - E(\lambda'),$$

so that $E(I)$ is the projection on the subspace $\mathcal{M}(I) = \mathcal{M}(\lambda'') \ominus \mathcal{M}(\lambda')$.

4.2 The Selfadjoint Operator Associated with a Spectral Family

The following is the converse to our main goal in this presentation: we discuss a selfadjoint operator that *comes from* a spectral family.

We define the operator

$$A \;=\; \int_{-\infty}^{\infty} \lambda \, dE(\lambda). \tag{8}$$

The domain $D(A)$ is the set of all $u \in \mathcal{H}$ such that

$$\int_{-\infty}^{\infty} \lambda^2 \, d(E(\lambda)u, u) \;<\; \infty. \tag{9}$$

Then (Au, v), for any $u \in D(A)$, and $v \in \mathcal{H}$, is given by

$$(Au, v) \;=\; \int_{-\infty}^{\infty} \lambda \, d(E(\lambda)u, v).$$

We may also define a family of operators, in the following way:

$$\phi(A) \;=\; \int_{-\infty}^{\infty} \phi(\lambda) \, dE(\lambda),$$

with $\phi(\lambda)$ being any complex-valued, continuous function. If $\phi(\lambda)$ is bounded on the support Σ of $\{E(\lambda)\}$ then the condition (9) is always fulfilled, so that $D(\phi(A)) = \mathcal{H}$, and $\phi(A)$ is bounded:

$$\|\phi(A)\| \;\leq\; \sup_{\lambda \in \Sigma} |\phi(\lambda)|.$$

Thus $\phi(A) \in \mathfrak{B}(\mathcal{H})$. Furthermore, it is normal (i.e. $AA^* = A^*A$).

Proposition 24. A *is symmetric.*

Proof. Let $u, v \in D(A)$. Then

$$
\begin{aligned}
(Au, v) \;&=\; \int_{-\infty}^{\infty} \lambda \, d(E(\lambda)u, v) \\
&=\; \int_{-\infty}^{\infty} \lambda \, \overline{d(E(\lambda)v, u)} \\
&=\; \overline{(Av, u)} \\
&=\; (v, Au).
\end{aligned}
$$

$\square$

Proposition 25. A *is selfadjoint.*

Proof. This proof is lengthier, and is ommitted here. $\square$

Returning to Example 17:

Recall that for $\mathcal{H} = L^2[0, \pi]$ and the operator $A = -\frac{d^2}{dx^2}$, we found that the spectrum was the set $\Sigma = \sigma_p = \{1, 4, 9, ..., n^2, ...\}$. Now we can explain this further:

Let N_j be the eigenspace associated to $\lambda_j = j^2$. Let P_j be the projection onto N_j. Let

$$
\begin{aligned}
E(\lambda) \;&=\; \text{Projection onto all } N_j \text{ with } j^2 < \lambda \\
&=\; \sum_{j^2 < \lambda} P_j.
\end{aligned}
$$

Then $\{E(\lambda)\}$ forms a spectral family, and by the construction of A,

$$
A \;=\; \int_{-\infty}^{\infty} \lambda\, dE(\lambda),
$$

so that A is selfadjoint by Proposition 25, and, by Theorem 26 below, A uniquely generates a spectral family. This implies that $\{E(\lambda)\}$ that we defined above is unique, which, in turn, implies that, indeed $\Sigma = \sigma_p$, rather than $\Sigma \supseteq \sigma_p$.

5 Infinite-Dimensional Versions

Theorem 26. (Selfadjoint Operator Version 1)
 Let A be a selfadjoint operator on a Hilbert space $\mathcal{H}$. Then A uniquely generates a spectral family.

Theorem 27. (Selfadjoint Operator Version 2)
 Let A be a selfadjoint operator on a Hilbert space $\mathcal{H}$. Then there exists a measure space (X, Σ, μ) and a real valued, measurable function ϕ on X, and an isometry $U\colon L^2(X, \Sigma, \mu) \longrightarrow \mathcal{H}$ that is onto, such that

$$
\big(U^{-1}AUf\big)(x) \;=\; \phi(x)f(x)
$$

for any $f \in L^2(X, \Sigma, \mu)$ such that also $\phi f \in L^2(X, \Sigma, \mu)$.

Returning to Example 20:
 We now see that the expression in (2) is the same as the expression in Theorem 27. This theorem explains to us how it is possible that we found that an operator is equivalent to a multiplication by some function.

Bibliography

1. Elkies, N., *Spectral Theorem for self-adjoint oprators on a finite-dimensional real inner product space*, `http://www.math.harvard.edu/~elkies/M55a.99/spectral.html`

2. Feynman, R., *The Feynman Lectures on Physics: Quantum Mechanics*, Addison-Wesley, 1965.

3. Halmos, P. R., *Introduction to Hilbert space and the theory of spectral multiplicity*, AMS, 1957.

4. Halmos, P. R., *What does the Spectral Theorem say?*, The American Mathematical Monthly, Vol. 70, No. 3 (March, 1963), pp. 241-247.

5. Kato, T., *Perturbation Theory for Linear Operators*, Springer-Verlag, New York, 1976.

6. Lax, P., *Functional Analysis*, Wiley-Interscience, New York, 2002.

7. Reed, M. and Simon, B., *Methods of Modern Mathematical Physics I: Functional Analysis*, Academic Press, 1972.

8. Yosida, K, *Functional Analysis*, Springer-Verlag, 1980.