Hybrid kinetic-fluid models for magnetized plasmas

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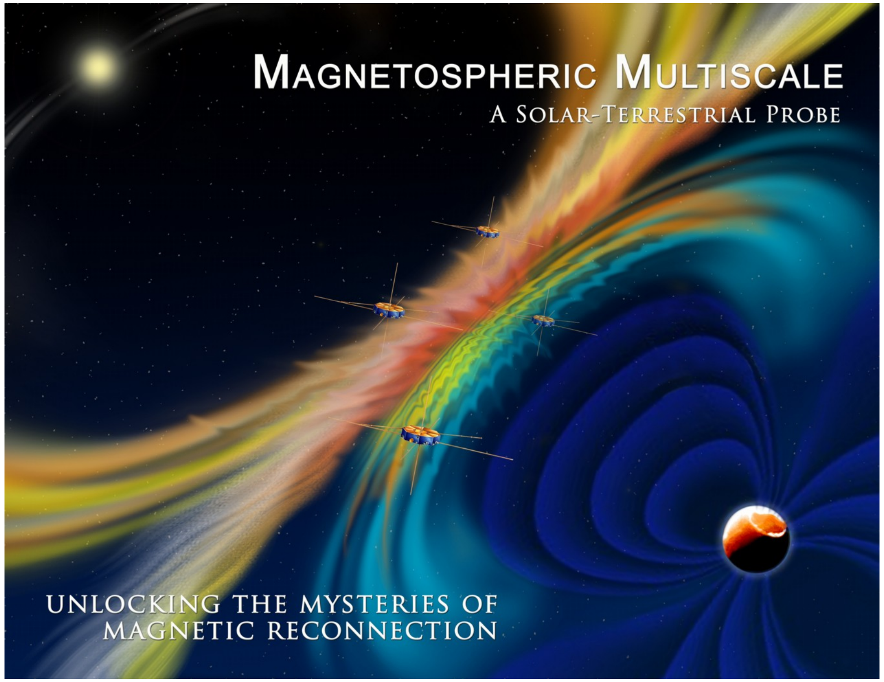
1 Hybrid kinetic-fluid models: why and how

- Most of plasma fusion simulations are based on MHD-like fluid models
- These models are invalidated by presence of an energetic component
- Small-scale processes may control large-scale phenomenology



Energetic Solar wind interacts with magnetosphere

- *Kinetic effects* need to be considered along with fluid macro-scales

- **Hybrid philosophy: a fluid interacts with a hot particle gas**

- Many linear hybrid models exist – here, we focus on *nonlinear models*.

- *Kinetic effects* need to be considered along with fluid macro-scales
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Multi-physics approach!

$\rightarrow$ MHD fluid models need to be coupled to kinetic-like equations

Several coupling options are available, which need special care...

2 Different hybrid models [Park *et al.*(1992)]

$$\rho_b \frac{d\mathbf{v}_b}{dt} + \frac{\partial \rho_h \mathbf{v}_{h,\parallel}}{\partial t} = -\nabla p_b - \nabla\cdot\mathsf{P}_h + \mathbf{J}\times\mathbf{B}, \quad (1)$$

where, $\partial(\rho_h \mathbf{v}_{h,\perp})/\partial t$ is neglected compared to the perpendicular momentum change of the bulk plasma, and $\mathsf{P}_h \equiv \int \mathbf{v}\mathbf{v} f_h \, d^3v$ without the usual velocity shift. For the hot particles alone, we have

$$\frac{\partial \rho_h \mathbf{v}_h}{\partial t} = -\nabla\cdot\mathsf{P}_h + \mathbf{J}_h \times \mathbf{B} + q_h \mathbf{E}_\perp. \quad (2)$$

By subtracting the parallel component of Eq. (2) from Eq. (1), we obtain the pressure coupling equation:

$$\boxed{\rho_b \frac{d\mathbf{v}_b}{dt} = -\nabla p_b - (\nabla\cdot\mathsf{P}_h)_\perp + \mathbf{J}\times\mathbf{B}.} \quad (3)$$

Alternatively, by subtracting all components of Eq. (2) from Eq. (1), we obtain the current coupling equation:

$$\boxed{\rho_b \frac{d\mathbf{v}_b}{dt} = -\nabla p_b + (\nabla\times\mathbf{B} - \mathbf{J}_h)\times\mathbf{B} + q_h \mathbf{v}_b \times \mathbf{B}.} \quad (4)$$

Two couplings are possible: pressure coupling vs. current coupling.

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Two couplings are possible: pressure coupling vs. current coupling.

Let's derive them. . .

3 Starting point: Vlasov-multifluid system

Two fluid species (electrons + fluid ions) interact with energetic ions:

$$\rho_s \frac{\partial \boldsymbol{u}_s}{\partial t} + \rho_s \left(\boldsymbol{u}_s \cdot \nabla \right) \boldsymbol{u}_s = a_s \rho_s \left(\epsilon_0^{-1} \mathbf{D} + \boldsymbol{u}_s \times \mathbf{B} \right) - \nabla \mathsf{p}_s$$

$$\frac{\partial \rho_s}{\partial t} + \nabla \cdot \left(\rho_s \boldsymbol{u}_s \right) = 0$$

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m_h} \cdot \frac{\partial f}{\partial \mathbf{x}} + q_h \left(\epsilon_0^{-1} \mathbf{D} + \frac{\mathbf{p}}{m_h} \times \mathbf{B} \right) \cdot \frac{\partial f}{\partial \mathbf{p}} = 0$$

$$\mu_0 \frac{\partial \mathbf{D}}{\partial t} = \nabla \times \mathbf{B} - \mu_0 \sum_s a_s \rho_s \boldsymbol{u}_s - \mu_0 \, a_h \int \mathbf{p} \, f \, \mathrm{d}^3 \mathbf{p}$$

$$\epsilon_0 \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{D}$$

$$\nabla \cdot \mathbf{D} = \sum_s a_s \rho_s + q_h \int f \, \mathrm{d}^3 \mathbf{p} \,, \qquad \nabla \cdot \mathbf{B} = 0$$

Notice: Vlasov eqn is used here, instead of its drift-kinetic approximation

4 Current-coupling scheme for hybrid MHD

- Take the sum $\rho_i \boldsymbol{u}_i + \rho_e \boldsymbol{u}_e$ and neglect electron inertia. Neutrality $\epsilon_0 \to 0$ and ideal Ohm's law $\mathbf{E} + \boldsymbol{u} \times \mathbf{B} = 0$ (*neglects hot charge*) yield

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$$\rho \frac{\partial \boldsymbol{u}}{\partial t} + \rho\,(\boldsymbol{u} \cdot \nabla)\,\boldsymbol{u} = \left(q_h\,\boldsymbol{u} \int f \,\mathrm{d}^3\mathbf{p} - \, a_h \int \mathbf{p}\, f \,\mathrm{d}^3\mathbf{p} + \frac{1}{\mu_0} \nabla \times \mathbf{B} \right) \times \mathbf{B} - \nabla \mathrm{p}$$

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m_h} \cdot \frac{\partial f}{\partial \mathbf{x}} + q_h \left(\frac{\mathbf{p}}{m_h} - \boldsymbol{u} \right) \times \mathbf{B} \cdot \frac{\partial f}{\partial \mathbf{p}} = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0\,, \qquad \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \mathbf{B})\ .$$

Current-coupling scheme (CCS) used in [Belova et al.(1997), Chen et al.(1999)].

- **Exact invariants** [CT(2010),CT&Holm(2012)]: Magnetic and cross helicity, as well as **total energy** (in terms of fluid momentum $\mathbf{m}=\rho\boldsymbol{u}$)

$$H=\underbrace{\frac{1}{2}\int\frac{|\mathbf{m}|^2}{\rho}\,\mathrm{d}^3\mathbf{x}}_{\textit{Cold kinetic energy}}+\underbrace{\frac{1}{2\mathrm{m}_h}\int f\,|\mathbf{p}|^2\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{p}}_{\textit{Hot kinetic energy}}$$
$$+\underbrace{\int\rho\,\mathcal{U}(\rho)\,\mathrm{d}^3\mathbf{x}}_{\textit{Cold internal energy}}+\underbrace{\frac{1}{2\mu_0}\int|\mathbf{B}|^2\,\mathrm{d}^3\mathbf{x}}_{\textit{Magnetic energy}}\,,$$

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$$

- At this point, one would like to insert the assumptions

$$
\frac{1}{\rho}\int f\,\mathrm{d}^3\mathbf{p} \ll 1\,, \qquad \frac{1}{\rho}\int \mathbf{p}\,f\,\mathrm{d}^3\mathbf{p} \ll 1\,, \qquad T_h \gg T_c
$$

where T_h and T_c are the hot and cold temperatures, respectively.

5 Pressure-coupling MHD scheme (PCS)

- Dynamics of total momentum $\mathbf{M} = \rho\boldsymbol{u} + \int \mathbf{p}\, f\, \mathrm{d}^3\mathbf{p}$ yields

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) + \frac{\partial}{\partial t}\int \mathbf{p}\, f\, \mathrm{d}^3\mathbf{p} \;=\; -\nabla\cdot\mathbb{P} - \nabla\mathsf{p} + \frac{1}{\mu_0}\operatorname{curl}\mathbf{B}\times\mathbf{B}\,.$$

  where $m_h\mathbb{P} = \int \mathbf{pp}\, f\, \mathrm{d}^3\mathbf{p}$ is the kinetic stress tensor (absolute pressure)

- In the literature, the PCS is obtained from above by assuming

$$\frac{\partial}{\partial t}\int \mathbf{p}\, f\, \mathrm{d}^3\mathbf{p} \simeq 0\,,$$

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. . . *we shall use geometry instead!*

Rest of the talk

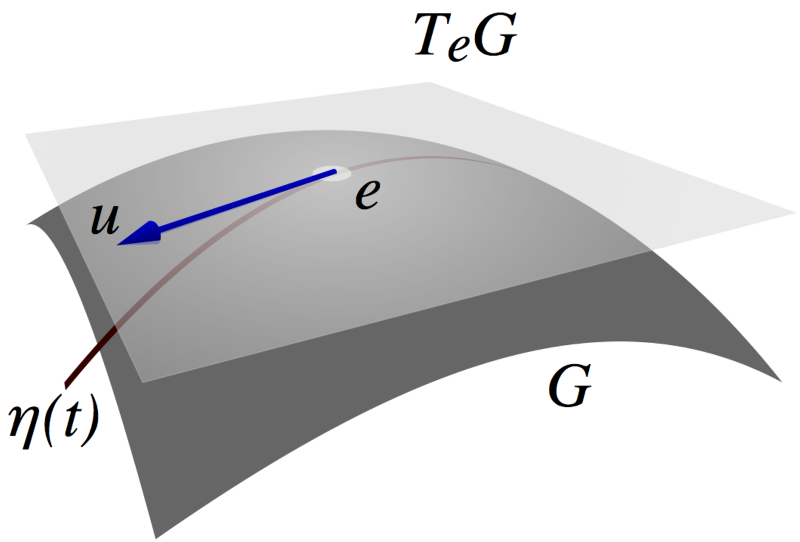
- Hamiltonian (geometric) approaches in plasma physics

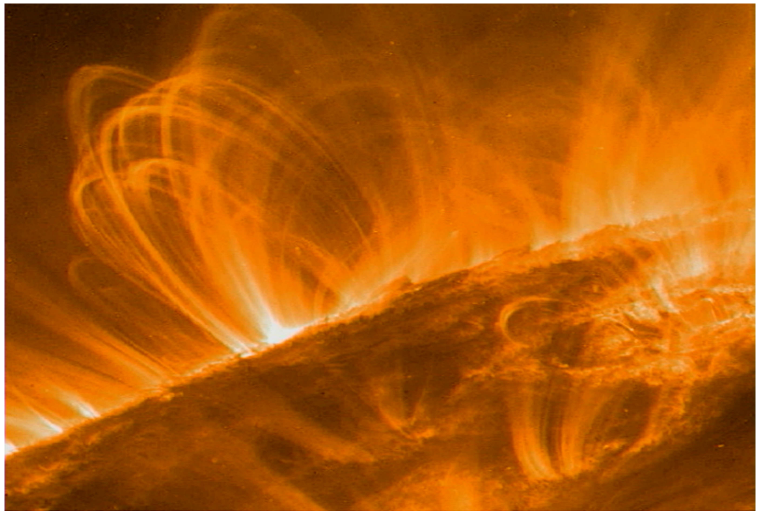
- Application to hybrid MHD models: results on PCS

- Linear and Lyapunov stability

Geometric mechanics for fluid and kinetic models

Geometry and symmetry in fluids and plasmas

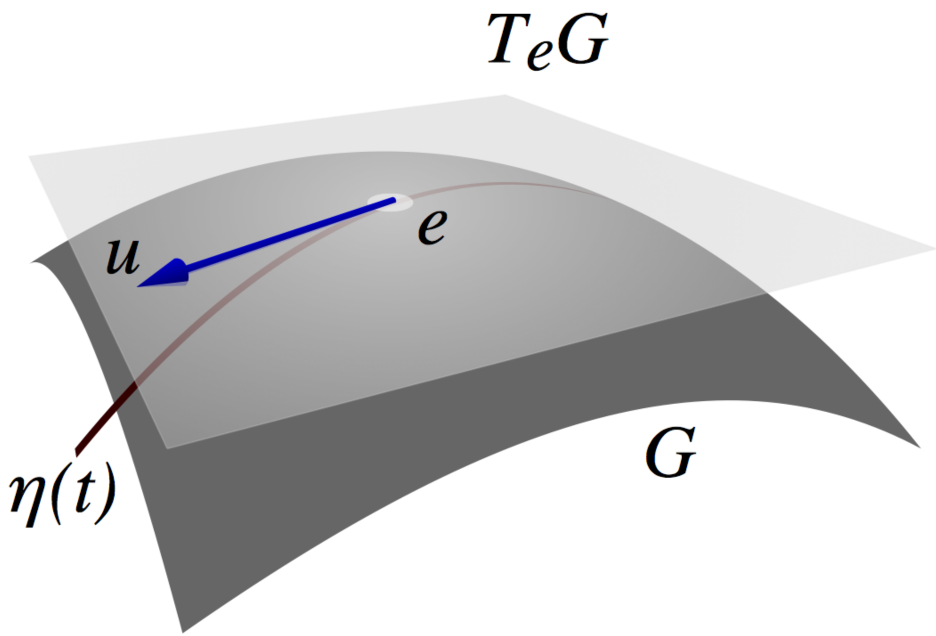




Lagrangian and Eulerian variables are related by the relabeling symmetry, which produces an *intrinsic geometric description* [Arnold (1966)] capturing essential features such as *circulation laws* and dynamical invariants.

Ex. Incompressible ideal fluids move along geodesics on $G = \mathrm{Diff}_{\mathrm{vol}}(M)$

Geometric approach possesses variational and Hamiltonian formulations!

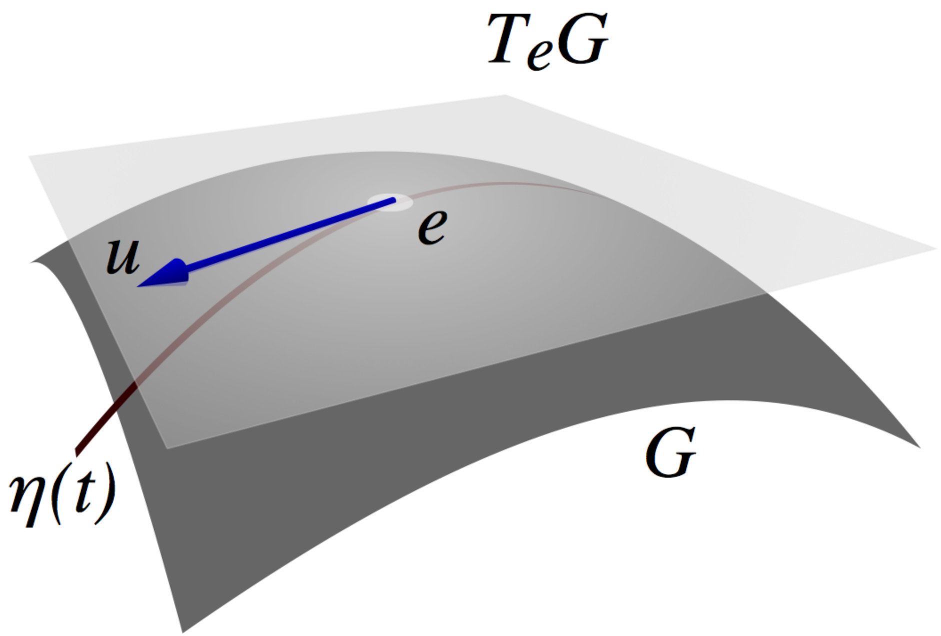


Lagrangian fluid dynamics of $\boldsymbol{\eta}(\mathbf{a},t)$ on the Lie group G possesses the

canonical Poisson bracket: $$\{F,G\} = \int \left(\frac{\delta F}{\delta \boldsymbol{\eta}} \cdot \frac{\delta G}{\delta \boldsymbol{\psi}} - \frac{\delta F}{\delta \boldsymbol{\psi}} \cdot \frac{\delta G}{\delta \boldsymbol{\eta}} \right) \mathrm{d}^3\mathbf{a}\,,$$

Eulerian dynamics on the (dual) tangent space at identity possesses the

Lie-Poisson bracket : $$\{F,G\}(\boldsymbol{\sigma}) = \left\langle \boldsymbol{\sigma}, \left[\frac{\delta F}{\delta \boldsymbol{\sigma}}, \frac{\delta G}{\delta \boldsymbol{\sigma}} \right] \right\rangle$$



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Fluids: *(η,ψ) are Lagrangian coordinates, while $\boldsymbol{\sigma}$ = fluid momentum* $\mathbf{m}$.
Vlasov: *(η,ψ) are Lagrangian coordinates, while σ = distribution function f.*

Symmetry is everywhere in mechanics

- Rotational symmetry for vectors (*rigid body motion*):

$$[\mathbf{g}, \mathbf{k}] = \mathbf{g} \times \mathbf{k} \;\to\; \{F, G\} = \boldsymbol{\mu} \cdot \frac{\mathrm{d}F}{\mathrm{d}\boldsymbol{\mu}} \times \frac{\mathrm{d}G}{\mathrm{d}\boldsymbol{\mu}}$$

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- Relabeling symmetry for velocities (*Euler fluid dynamics*):

$$[\mathbf{v}, \mathbf{u}] = (\mathbf{v} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v} \;\rightarrow\; \{F, G\} = \int \boldsymbol{\mu}(\mathbf{x}) \cdot \left[\frac{\delta F}{\delta \boldsymbol{\mu}}, \frac{\delta G}{\delta \boldsymbol{\mu}}\right] \mathrm{d}^3\mathbf{x}$$

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- Unitary symmetry for matrix operators (*quantum dynamics*):

$$[A,B] = AB - BA \;\rightarrow\; \{F,G\} = \hbar\,\mathsf{Tr}\left(i\rho\left[\frac{\delta F}{\delta\rho},\frac{\delta G}{\delta\rho}\right]\right)$$

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- Canonical symmetry for phase-space functions (**Vlasov equation**):

$$[h,k]=\frac{\partial h}{\partial\mathbf{x}}\cdot\frac{\partial k}{\partial\mathbf{p}}-\frac{\partial h}{\partial\mathbf{p}}\cdot\frac{\partial k}{\partial\mathbf{x}}\;\rightarrow\;\{F,G\}=\int f(\mathbf{x},\mathbf{p})\left[\frac{\delta F}{\delta f},\frac{\delta G}{\delta f}\right]\mathsf{d}^3\mathbf{x}\,\mathsf{d}^3\mathbf{p}$$

Magnetohydrodynamics (MHD)

- Fluid plasma model in which the magnetic field $\mathbf{B}$ is 'frozen in':

$$\partial_t(\mathbf{B}\cdot d\mathbf{S})+\pounds_{\boldsymbol{u}}(\mathbf{B}\cdot d\mathbf{S})=0\,,\quad \text{or, equivalently,}\quad \partial_t\mathbf{B}+\nabla\times(\mathbf{B}\times\boldsymbol{u})=0$$

- Fluid equation is

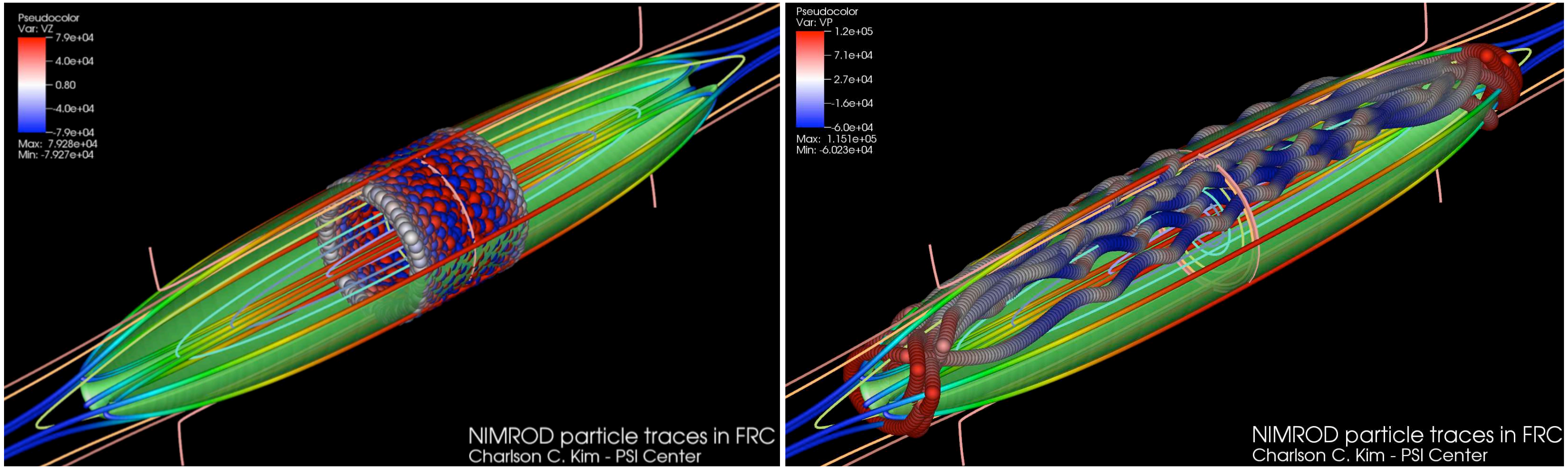
$$\frac{\partial \boldsymbol{u}}{\partial t}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}=-\frac{1}{\rho}\nabla \mathsf{p}-\frac{1}{\mu_0\rho}\mathbf{B}\times\nabla\times\mathbf{B}$$

  where ρ is the transported mass density and p denotes pressure

- $\mathbf{J}=\nabla\times\mathbf{B}$ is an electric current, so $\mathbf{J}\times\mathbf{B}$ arises as a Lorentz force

- Most plasma studies are based on this Hamiltonian (Lie-Poisson) model!

Still, energetic particles require kinetic theory!



Field Reversed Configuration experiments (FRCs) for nuclear fusion require kinetic descriptions as ordinary fluid approximations do not apply. *Right*: low energy particles colored by poloidal velocity. *Left*: high energy particles colored by axial velocity. Hot particles confine to the outboard region (higher magnetic gradients) and never cross the origin. *(Figure by the Plasma Science and Innovation Center, University of Washington).*

Kinetic theory & electromagnetism: Maxwell-Vlasov

- Vlasov kinetic equation for $f(\mathbf{x}, \mathbf{p}, t)\ldots$

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m} \cdot \frac{\partial f}{\partial \mathbf{x}} + q\left(\mathbf{E} + \frac{\mathbf{p}}{m} \times \mathbf{B}\right) \cdot \frac{\partial f}{\partial \mathbf{p}} = 0$$

- ...coupled to Maxwell's equations

$$\epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{B} - \mu_0 \, \frac{q}{m} \int \mathbf{p} \, f \, \mathrm{d}^3\mathbf{p}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

$$\epsilon_0 \nabla \cdot \mathbf{E} = q \int f \, \mathrm{d}^3\mathbf{p} \,, \qquad \nabla \cdot \mathbf{B} = 0$$

- Again, this is a Lie-Poisson Hamiltonian system!

Let's apply geometric mechanics to formulate our hybrid models!

Current-coupling scheme for hybrid MHD

- CCS derived in [Park & al. (1992)] and used by [Belova & al. (1997)]:

$$
\rho\frac{\partial \boldsymbol{u}}{\partial t} + \rho\,(\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} = \left(q_h\,\boldsymbol{u}\int f\,\mathrm{d}^3\mathbf{p} - \; a_h\int \mathbf{p}\,f\,\mathrm{d}^3\mathbf{p} + \frac{1}{\mu_0}\nabla\times\mathbf{B}\right)\times\mathbf{B} - \rho\nabla\mathsf{p}
$$

$$
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$$

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- Based on quasi-neutrality, massless electrons and ideal Ohm's law

Problem: is this a Hamiltonian system?

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$$H = \frac{1}{2} \int \frac{|\mathbf{m}|^2}{\rho} \, \mathrm{d}^3\mathbf{x} + \frac{1}{2\mathrm{m}_h} \int f \, |\mathbf{p}|^2 \, \mathrm{d}^3\mathbf{x} \, \mathrm{d}^3\mathbf{p} + \int \rho \, \mathcal{U}(\rho) \, \mathrm{d}^3\mathbf{x} + \frac{1}{2\mu_0} \int |\mathbf{B}|^2 \, \mathrm{d}^3\mathbf{x} \,,$$

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$$\frac{\partial f}{\partial t}+\left(\boldsymbol{u}+\frac{\mathbf{p}}{m_h}\right)\cdot\frac{\partial f}{\partial \mathbf{x}}-(\mathbf{p}\cdot\nabla\boldsymbol{u})\cdot\frac{\partial f}{\partial \mathbf{p}}+a_h\,\mathbf{p}\times(\mathbf{B}-\nabla\times\boldsymbol{u})\cdot\frac{\partial f}{\partial \mathbf{p}}=0$$

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\,\boldsymbol{u})=0\,,\qquad\qquad \frac{\partial \mathbf{B}}{\partial t}=\nabla\times(\boldsymbol{u}\times\mathbf{B})\;,$$

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A geometric hybrid model: equations

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- Inertial force terms emerge since hot particle trajectories are now computed in the cold fluid frame.

- We get magnetic and (new) cross helicity invariants:

$$\mathcal{H} = \int \mathbf{A} \cdot \mathbf{B} \, \mathsf{d}^3\mathbf{x} \,, \qquad \Lambda = \int \left(\boldsymbol{u} - m_h \frac{\mathbf{K}}{\rho} \right) \cdot \mathbf{B} \, \mathsf{d}^3\mathbf{x}$$

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- Circulation law [Holm&CT(2011)]

$$\frac{\mathrm{d}}{\mathrm{d}t} \oint_{\gamma_t} \boldsymbol{u} \cdot \mathrm{d}\mathbf{x} = - \oint_{\gamma_t} \frac{1}{\rho} \left(\frac{1}{\mu_0} \mathbf{B} \times \nabla \times \mathbf{B} + m_h \nabla \cdot \mathbb{P} \right) \cdot \mathrm{d}\mathbf{x} \,;$$

where

$$\mathbb{P} = \int \mathbf{p}\mathbf{p} f \, \mathrm{d}^3\mathbf{p}$$

denotes the kinetic stress (pressure) tensor.

What happens to the physics?

Stability results

Dispersion relation for κ-distributions

Linearize around static equilibria with $f_0 = f_0(p^2/2)$ and define $F = \int f_0 \, \mathrm{d}^2\mathbf{p}_\perp$. For longitudinal propagation, one obtains (with $v_A = b/\sqrt{\mu_0}$)

$$\omega^2 - k_z^2 v_A^2 + \omega\left(\alpha\omega \mp \omega_c\right)\left(n_0 + (\omega \mp \omega_c)\int_{-\infty}^{+\infty} \frac{F \, \mathrm{d}p_z}{k_z p_z - \omega \pm \omega_c}\right) = 0.$$

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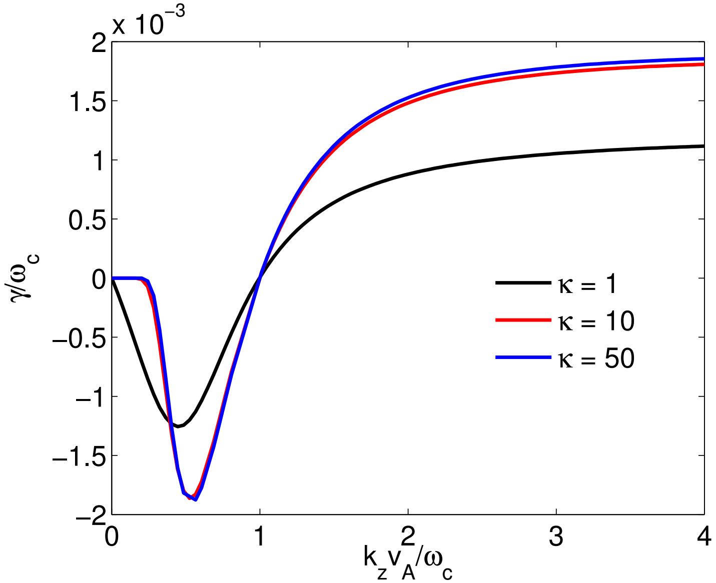
Old hybrid model – growth rate



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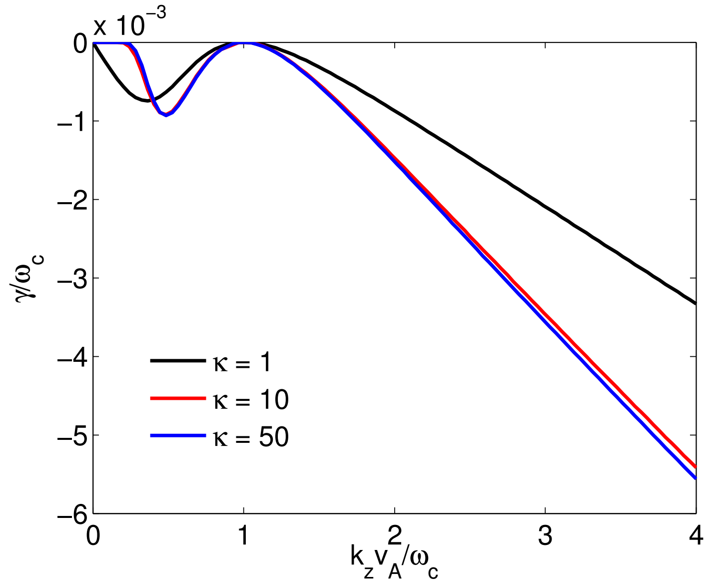

New model ($\alpha = 1$) gives magnetized 'Landau damping'

Spurious instability in the non-Hamiltonian model! ($\alpha = 0$)

[C.T., Tassi, Camporeale & Morrison (2014)]

Lyapunov stability of planar Vlasov-MHD

- let H be the energy of a Hamiltonian system and let C denote the (sum of all) other invariant(s)

- Energy-Casimir method: equilibrium conditions and corresponding Lyapunov stability are found by, respectively,

$$\delta(H + C) = 0\,, \qquad\qquad \delta^2(H + C) > 0$$

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- Let Ψ be any function. Upon denoting by $\omega=\hat{\mathbf{z}}\cdot\nabla\times\boldsymbol{u}$ the fluid vorticity and by $\phi=-\boldsymbol{\Delta}^{-1}\omega$ its stream function, the equilibria are

$$\phi_e=\phi_e(A_e)\,,\qquad\qquad f_e=f_e\Big(\frac{1}{2}\,|\mathbf{p}+\boldsymbol{u}_e|^2-\frac{1}{2}\,|\boldsymbol{u}_e|^2\Big)$$

Hybrid Grad-Shafranov: $-\boldsymbol{\Delta}A_e-(\omega_e-\hat{\mathbf{z}}\cdot\nabla_\perp\times\mathbf{K}_\perp)\,\phi_e'(A_e)+\Psi'(A_e)=0$

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- Define $\mathbb{T}_\perp=-\!\int\! f_e'\,\mathbf{p}_\perp\mathbf{p}_\perp\,\mathrm{d}^3\mathbf{p}$. Then, the stability conditions are

$$|\phi_e'|^2<(1+2\,\mathrm{Tr}\,\mathbb{T}_\perp)^{-1}\,,\qquad\qquad f_e'<0\ \textit{(e.g. Maxwellians)}$$
$$(\omega_e-\hat{\mathbf{z}}\cdot\nabla_\perp\times\mathbf{K}_\perp)\,\phi''-\Psi''-\phi'\Delta\phi'+2|\nabla_\perp\phi'|^2\,\mathrm{Tr}\,\mathbb{T}_\perp<0$$

[Morrison, Tassi & C.T. (2013); C.T., Tassi & Morrison (2014)]

More hybrid directions. . .

- Hybrid models for (conservative) complex fluid dynamics [CT(2012)]: *order parameter correlations*

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- Multiscale models for the mean-fluctuation decomposition in fluid turbulence: *fluctuation dynamics* [Holm & CT (2012)]

- Hybrid models for classical-quantum systems: new semidirect product $\mathcal{H}(\mathbb{R}^{2n})\circledS\,\mathcal{U}(H)$, now between the *Heisenberg and the unitary group* [Bonet-Luz & CT (2015)]

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Geometric Mechanics message: *in multiscale dynamics, the bigger scales 'push' the smaller scales*, which also undergo their own micromotion

THANK YOU!

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Task force in mathematical methods for plasma modeling

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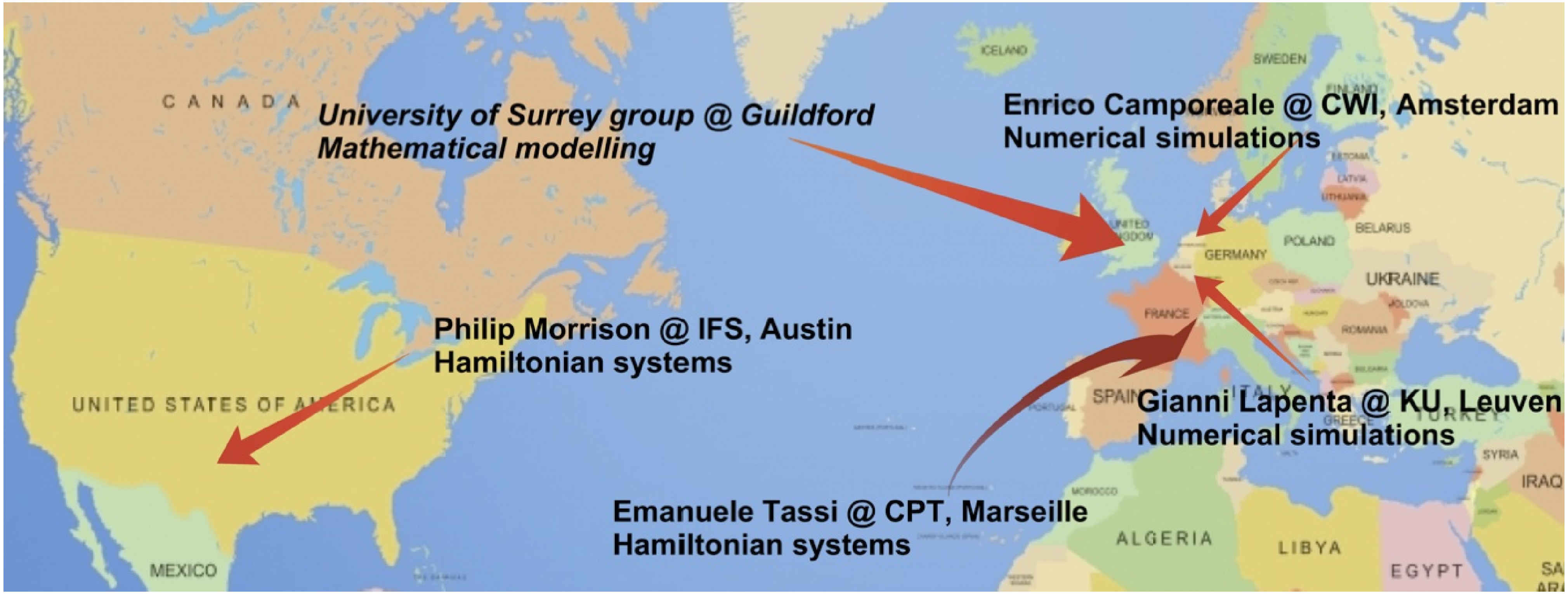

Activity is coordinated by the University of Surrey group (Maths Dept.): Mrs Esther Bonet-Luz, Mr Alexander Close, Dr Paul Skerritt & C.T.