On computational issues of Vlasov-Maxwell and Vlasov-Poisson-Landau

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The Cauchy problem in kinetic Theory
Recent Developments in Collisionless Models

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Why Vlasov-Maxwell-Landau for plasma modeling?

"Gases" of charged particles: *Fokker-Planck-Landau* (FPL) equation. Collisional plasma
e.g, nano-devices, controlled fusion (sustainable energy), plasma sheath problems, etc.

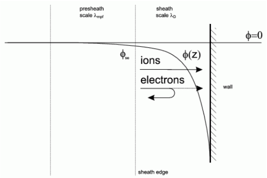


Figure: The plasma sheath

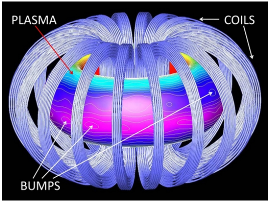


Figure: The fusion plasma [ORNL visualization]

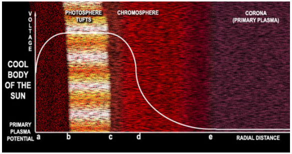


Figure: The Solar Plasma Sheath (W. Thornhill (after W. Allis & R. Juergens), The Electric Universe.)

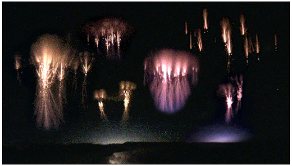


Figure: Various forms of "Sprites" captured from jet planes (occurs at altitude 50-90 km)[TV show "NOVA At the Edge of Space" on PBS, 20 Nov,

Outline

- The Fokker-Planck-Landau (FPL) and Vlasov Poisson/Maxwell (VP/VM) systems
  - Time Splitting
  - A RKDG Method for the Pure Transport Part (Vlasov Poisson / Maxwell)
  - A Conservative Spectral Method for the Collision Part
  - The Linking Process for Vlasov-Poisson Landau
  - Numerical Results & Applications
- Looking ahead

The Fokker-Planck-Landau Equation (FPL) for binary interactions

The strong form

$$\partial_t f_\alpha + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_\alpha + F(t,\mathbf{x}) \cdot \nabla_{\mathbf{v}} f_\alpha = \frac{1}{\varepsilon} \sum_\beta Q_{\alpha,\beta}(f_\alpha, f_\beta), \qquad \mathbf{v} \in \mathbb{R}^{d_{\mathbf{v}}}, \mathbf{x} \in \Omega_{\mathbf{x}} \subseteq \mathbb{R}^{d_{\mathbf{x}}},$$

for $f_\alpha(\mathbf{x},\mathbf{v},t) \geq 0$ the **pdf** of electrons/ions at position $\mathbf{x}$, velocity $\mathbf{v}$ and time t, with collision frequency $\frac{1}{\varepsilon}$ and $Q_{\alpha,\beta}$ modeling $\alpha - \beta$ type collisions (e.g. electron-electron, ion-ion, electron-ion, etc.)
Advection is given by the Lorentzian force $F(t,\mathbf{x}) = \mathbf{E} + \mathbf{v} \times \mathbf{B}$, for $\mathbf{E}$ electric field and $\mathbf{B}$ magnetic potential

$$Q_{\alpha,\beta}(f_\alpha, f_\beta) = \nabla_{\mathbf{v}} \cdot \int_{\mathbb{R}^3} \mathbf{S}(\mathbf{v} - \mathbf{v}_*)(f_\beta(\mathbf{v}_*)\nabla_{\mathbf{v}} f_\alpha(\mathbf{v}) - f_\alpha(\mathbf{v})\nabla_{\mathbf{v}_*} f_\beta(\mathbf{v}_*))d\mathbf{v}_*,$$

with the $d \times d$ non-negative and symmetric projection matrix

$$\mathbf{S}(u) = L|u|^{\gamma+2}(\mathbf{I} - \frac{u \otimes u}{|u|^2}), \qquad \gamma \in [-d, 1],$$

where $\mathbf{I}$ is the $d \times d$ identity matrix. Coulomb potential $\gamma = -d$

- FPL models long-range interactions between charged particles and is the binary grazing collision limit of Botlzmann collision operator;
- Conservation laws; Decay of entropy; Maxwellian equilibria;
- Primary importance in collisional plasmas.

The Vlasov- Poisson- Maxwell system (VMS) for collisionless plasma

Model for a dimensionless single specie: Electron time-space scaling

- Time: Inverse of the plasma frequency ω_p^{-1} ,
- Space Length: by Debye length λ_D,
- Lorentz force: Electric and magnetic fields by $-mc\omega_p/e$

$$
\begin{aligned}
&\partial_t f + \mathbf{v}\cdot\nabla_{\mathbf{x}} f + (\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{v}} f = 0 \ , \qquad \mathbf{v}\in\mathbb{R}^{d_{\mathbf{v}}}, \mathbf{x}\in\Omega_{\mathbf{x}}\subseteq\mathbb{R}^{d_{\mathbf{x}}} \\
&\frac{\partial \mathbf{E}}{\partial t} = \nabla_{\mathbf{x}}\times\mathbf{B} - \mathbf{J}, \qquad \frac{\partial \mathbf{B}}{\partial t} = -\nabla_{\mathbf{x}}\times\mathbf{E} \ , \\
&\nabla_{\mathbf{x}}\cdot\mathbf{E} = \rho - \rho_i, \qquad \nabla_{\mathbf{x}}\cdot\mathbf{B} = 0 \ ,
\end{aligned}
$$

with

$$
\rho(\mathbf{x},t) = \int_{\Omega_{\mathbf{v}}} f(\mathbf{x},\mathbf{v},t)d\mathbf{v}, \qquad \mathbf{J}(\mathbf{x},t) = \int_{\Omega_{\mathbf{v}}} f(\mathbf{x},\mathbf{v},t)\mathbf{v}d\mathbf{v} \ ,
$$

$\rho(\mathbf{x},t)$ is the electron charge density, and $\mathbf{J}(\mathbf{x},t)$ is the current density. The charge density of background ions is denoted by ρ_i.

In the absence of magnetic forces, i.e. $\mathbf{B} = 0$, the system is know as Vlasov-Poisson

$$
\begin{aligned}
&\partial_t f + \mathbf{v}\cdot\nabla_{\mathbf{x}} f + \mathbf{E}\cdot\nabla_{\mathbf{v}} f = 0 \ , \qquad \mathbf{v}\in\mathbb{R}^{d_{\mathbf{v}}}, \mathbf{x}\in\Omega_{\mathbf{x}}\subseteq\mathbb{R}^{d_{\mathbf{x}}} \\
&\nabla_{\mathbf{x}}\cdot\mathbf{E} = \rho - \rho_i,
\end{aligned}
$$

Goal: Solve the coupled Vlasov-Poisson/Maxwell-Landau system for electron-ion species that preserve conserved quantities (total mass, momentum, energy, etc.)

Time Splitting

The strategy consists in a **time splitting** of collisioness and collision times: Denote $f_n(x,v) = f(t_n, x, v)$. In a time interval $[t_n, t_{n+1}]$, solution evolves in two substeps

(1) The Vlasov - Poisson/Maxwell (Collisionless) Problem

$$\partial_t g(\mathbf{x},\mathbf{v},t) + \mathbf{v}\cdot \bigtriangledown_{\mathbf{x}} g(\mathbf{x},\mathbf{v},t) + F(t,\mathbf{x})\cdot \nabla_{\mathbf{v}} g = 0\,,$$

$$\text{Poisson}(\mathbf{E}) \qquad \text{or} \qquad \text{Maxwell}(\mathbf{E},\mathbf{B})$$

$$g(0,\mathbf{x},\mathbf{v}) = f_n(\mathbf{x},\mathbf{v})\,,$$

and

(2) The Homogenous FPL (Collisional) Problem

$$\partial_t \tilde{f}(x,v,t) = \frac{1}{\varepsilon} Q(\tilde{f},\tilde{f})\,,$$
$$\tilde{f}(0,x,v) = g(\Delta t, x, v)\,.$$

The Homogeneous FPL

Type $\alpha = \beta$. Drop the subscript and collision frequency. Weak form of $Q(f,f)$ is writen as the double mixing convolution

$$\int Q(f,f)\varphi(v)dv = \int\int f(v)f(v-u)\left(2[\nabla_v \cdot \mathbf{S}(u)] \cdot \nabla_v \varphi(v) + \mathbf{S}(u) \colon \nabla_v^2 \varphi(v)\right) dvdu$$

Taking the Fourier multiplier $\varphi(v) = (2\pi)^{-d/2} e^{-i\xi \cdot v}$,

$$\widehat{Q}(\xi) = \int_{R^3} \left(\widehat{f}(\xi-\omega)\widehat{f}(\omega)\omega^T \widehat{\mathbf{S}}(\omega)\omega - (\xi-\omega)^T \widehat{\mathbf{S}}(\omega)(\xi-\omega)\widehat{f}(\xi-\omega)\widehat{f}(\omega)\right) d\omega$$

$$\widehat{Q}(f,f)(\xi) = \int \widehat{f}(\xi-\omega)\widehat{f}(\omega)\widehat{G}(\xi,\omega)d\omega$$

where

$$\widehat{G}(\xi,u) = |u|^{-3}\left(i4u\cdot\xi - |u|^2|\xi^{\perp}|^2\right) \qquad \text{with } \xi^{\perp} = \xi - (\frac{\xi \cdot u}{|u|})\frac{u}{|u|}.$$

Remark.

- $G(\xi,u)$ is calculated by a grazing limit for the nonlinear Boltzmann collision operator for a large class of angular cross-sections [I.M.G, J. Haack, JCP'14)] with convergence rates depending on a sufficient concentration at the angular singularity.
- $\widehat{G}(\xi,\omega)$ is precomputed in phase space; can be exactly evaluated;
- It can be rewritten as a convolution and computed at $O(N \log N)$, with FFTs.

Collision Conservation Routines

Similar to the one introduced in conservative DG solver, define
Conservation Routine: Find $\mathbf{Q}_c$, the minimizer of the problem

$$\min \|\mathbf{Q}_c - \mathbf{Q}\|_{l^2}^2$$
$$\text{s.t.} \quad \mathbf{C}\mathbf{Q}_c = \mathbf{0}$$

where the $(d+2) \times M$ dimensional constraint matrix writes

$$\mathbf{C}_{:,j} = \begin{pmatrix} \omega_j \\ v\omega_j \\ |v|^2\omega_j \end{pmatrix}$$

where ω_j is the j-th integration weight of the quadrature rule (say, Trapezoidal rule).
Approximation estimates are similar to those of isoperimetric inequalities.

$$\mathbf{Q}_c = [\mathbb{I}d - \mathbf{C}^T(\mathbf{C}\mathbf{C}^T)^{-1}\mathbf{C}]\mathbf{Q}$$

Time Discretization

- Diffusive FPL operator requires stability condition–empirical;
- Fourth-order explicit Runge-Kutta scheme;
- Conservation routine is called at every intermediate step.

$$\widehat{\mathbf{F}_n} = \text{FFT}(\mathbf{F}_n), \quad \widehat{\mathbf{K}_n^1} = \text{Compute}\left(\widehat{\mathbf{Q}}(\widehat{\mathbf{F}_n}, \widehat{\mathbf{F}_n})\right), \quad \mathbf{K}_n^1 = \text{IFFT}\left(\widehat{\mathbf{K}_n^1}\right), \quad \mathbf{K}_n^1 = \text{Conserve}(\mathbf{K}_n^1)$$

$$\widetilde{\mathbf{F}}_n = \mathbf{F}_n + \Delta t K_n^1;$$

$$\widehat{\widetilde{\mathbf{F}}_n} = \text{FFT}(\widetilde{\mathbf{F}}_n), \quad \widehat{\mathbf{K}_n^2} = \text{Compute}\left(\widehat{\mathbf{Q}}(\widehat{\widetilde{\mathbf{F}}_n}, \widehat{\widetilde{\mathbf{F}}_n})\right), \quad \mathbf{K}_n^2 = \text{IFFT}\left(\widehat{\mathbf{K}_n^2}\right), \quad \mathbf{K}_n^2 = \text{Conserve}(\mathbf{K}_n^2)$$

$$\widetilde{\mathbf{F}}_n = \mathbf{F}_n + \frac{\Delta t}{2} K_n^1 + \frac{\Delta t}{2} K_n^2;$$

$$\widehat{\widetilde{\mathbf{F}}_n} = \text{FFT}(\widetilde{\mathbf{F}_n}), \quad \widehat{\mathbf{K}_n^3} = \text{Compute}\left(\widehat{\mathbf{Q}}(\widehat{\widetilde{\mathbf{F}}_n}, \widehat{\widetilde{\mathbf{F}}_n})\right), \quad \mathbf{K}_n^3 = \text{IFFT}\left(\widehat{\mathbf{K}_n^3}\right), \quad \mathbf{K}_n^3 = \text{Conserve}(\mathbf{K}_n^3)$$

$$\widetilde{\mathbf{F}}_n = \mathbf{F}_n + \frac{\Delta t}{2} K_n^1 + \frac{\Delta t}{2} K_n^3;$$

$$\widehat{\widetilde{\mathbf{F}}_n} = \text{FFT}(\widetilde{\mathbf{F}_n}), \quad \widehat{\mathbf{K}_n^4} = \text{Compute}\left(\widehat{\mathbf{Q}}(\widehat{\widetilde{\mathbf{F}}_n}, \widehat{\widetilde{\mathbf{F}}_n})\right), \quad \mathbf{K}_n^4 = \text{IFFT}\left(\widehat{\mathbf{K}_n^4}\right), \quad \mathbf{K}_n^4 = \text{Conserve}(\mathbf{K}_n^4)$$

$$\mathbf{F}_{n+1} = \mathbf{F}_n + \frac{1}{6}(3\mathbf{K}_n^1 + \mathbf{K}_n^2 + \mathbf{K}_n^3 + \mathbf{K}_n^4).$$

Numerical Results and Applications

Application: Multi-component Plasmas. The dimensionless system of equation for electro-neutral hydrogen plasma [Bobylev]

$$
\begin{aligned}
\frac{\partial f_e}{\partial t} &= \frac{1}{2}\left[Q^{(1)}_{FPL}(f_e, f_e) + Q^{(\theta)}_{FPL}(f_e, f_i)\right] \\
\frac{\partial f_i}{\partial t} &= \frac{\theta^2}{2}\left[Q^{(1)}_{FPL}(f_i, f_i) + Q^{(1/\theta)}_{FPL}(f_i, f_e)\right]
\end{aligned}
$$

where $\theta < 1$ is the dimensionless mass ratio of electrons to ions. with

$$
Q^{(\theta)}_{FPL}(f, g) = \nabla_v \cdot \int \mathbf{S}(v - v_*)(f(v_*)\nabla_v g(v) - \theta f(v)\nabla_{v_*} g(v_*))dv_*
$$

$$
\widehat{Q^{(\theta)}_{FPL}}(\widehat{f}, \widehat{g})(\xi) = \int \widehat{f}(\xi - w)\widehat{g}(w)\left[(1+\theta)\xi^T\widehat{\mathbf{S}}(w)w - \xi^T\widehat{\mathbf{S}}(w)\xi\right]dw
$$

Multi-component Plasmas

Define the (dimensionless) time-dependent temperatures for electrons and ions

$$T_e(t) = \frac{1}{3}\int f_e(v,t)|v|^2 dv, \quad T_i(t) = \frac{1}{3\theta}\int f_i(v,t)|v|^2 dv$$

$\Longrightarrow$

$$\begin{cases} (\theta \bar{T} + (1-\theta) T_e)^{\frac{3}{2}} \dfrac{dT_e}{dt} = \dfrac{4}{3\sqrt{2\pi}}(T_e - T_i)\theta \\ T_e(t) + T_i(t) = \bar{T} \end{cases}$$

and the temperature difference follows

$$\frac{d(T_i - T_e)}{dt} = -\theta \frac{8}{3\sqrt{2\pi}} \frac{T_i - T_e}{(\theta \bar{T} + (1-\theta) T_e)^{\frac{3}{2}}}$$

which implies

$$|T_i - T_e| \to 0, \quad \text{as } t \to \infty$$

Multi-component Plasmas, cont.

$T_e \approx T_i \approx \frac{\bar{T}}{2}$, as $t \to \infty$. The difference of temperatures decays "almost" exponentially

$$|T_i(t) - T_e(t)| \approx |T_{i,0} - T_{e,0}| \exp\left(-\frac{16}{3\sqrt{\pi}} \frac{\theta}{\left((1+\theta)\bar{T}\right)^{3/2}} t\right)$$

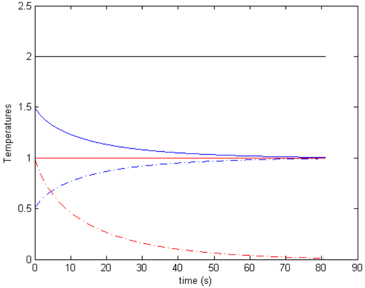


Figure: The relaxation of temperatures for the 2-plasma system: solid blue line: temperatures of ions; dash-dot blue: temperatures of electrons; top solid black: the total temperature; bottom dash-dot red: temperature difference

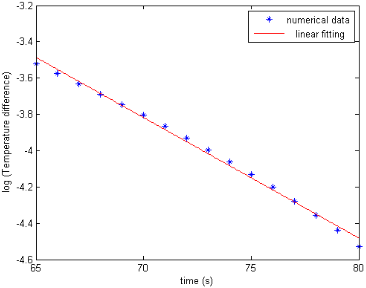


Figure: The logarithm of temperature difference for large time and its linear fitting; slope of linear fitting -0.066343 V.S. prediction -0.061.

RKDG Solver for Vlasov-Poisson Equations

The VP (Collisionless) Problem Coupled with Poisson

$$\begin{aligned}
\partial_t g(x,v,t) + v \cdot \bigtriangledown_x g(x,v,t) - \mathbf{E}(t,x) \cdot \nabla_v g = 0\,, \\
\mathbf{E}(t,x) = -\nabla_x \Phi(t,x)\,, \\
-\Delta_x \Phi(t,x) = 1 - \int_{R^3} g(t,x,v)dv\,, \\
\Phi(t,x) = \Phi_B(t,x) \quad x \in \partial\Omega_x\,, \\
g(0,x,v) = f_n(x,v)\,,
\end{aligned}$$

where the initial transport state is f_n is the current solution of the homogeneous Landau equation.

- Filamentation in phase space and steep gradient in v; Landau damping; BGK formation, etc.

Poisson Solver

The basic 1-D Poisson solver with periodic boundary conditions: For $x \in [0, L_x]$, with periodic boundary condition $\Phi(0) = \Phi(L_x)$ (slab geometry example), we use the exact integral representation of the approximating potential Φ_h:

For a given $g^n(x, \mathbf{v})$ set $\rho_h^n(x) = \int_{\Omega_\mathbf{v}} g^n(x, \mathbf{v}) d\mathbf{v}$, $\Omega_\mathbf{v} = [-L_\mathbf{v}, L_\mathbf{v}]^3$, and compute the potential Φ_h and electric field E_h by the exact integration (up to a quadrature rule):

$$\Phi_h^n = \int_0^x \int_0^s \rho_h^n(z, t) dz dx - \frac{x^2}{2} - C_E x \,,$$

where $C_E = -\frac{L_x}{2} + \frac{1}{L_x} \int_0^{L_x} \int_0^s \rho_h^n(z, t) dz ds$, and

$$E_h^n = -\frac{d}{dx} \Phi^n = C_E + x - \int_0^x \rho_h^n(z, t) dz \,.$$

We shall see later that this representation gives a pretty accurate total energy conservation as its error vanishes with the order of the approximation.

Remark: See [R.Heath, I.M.G, P. Morrison and C. Michler] for full numerical NIPG Poisson solver or [B. Ayuso, J. Carrillo and C.-W. Shu] for LDG Poisson solver.

From R. Heath, IMG, P.J. Morrison and C.Michler, JCP'12

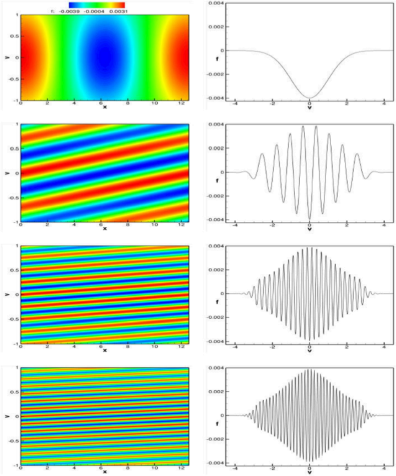


Fig. 3. (Linear Landau damping with Maxwell equilibrium) Contour plots (*left*) and cross-sectional plots (*right*), $x = 2\pi$, for δf at $t = 0$, $t = 25$, $t = 50$, $t = 75$

Linear Landau Damping
For Maxwellian equilibrium:
Formation of filamentation

$$f_0(x,v) = f(x,v,0) = f_{eq} + \delta f(x,v,0).$$

$$\delta f(x,v,0) \;=\; A\cos(kx)\, f_{eq}(v)$$

Solved in

$$\delta f(0,v,t) \;=\; \delta f(L,v,t)\,,$$

$$\Phi(0,t) \;=\; \Phi(L,t)\,,$$

for $(x,v,t) \in [0,L] \times [-V_c, V_c] \times (0,T)$,

Nonlinear Two-Stream Instability: perturbations about $f_{TS}(v) = \frac{1}{\sqrt{2\pi}} v^2 e^{-v^2/2}$

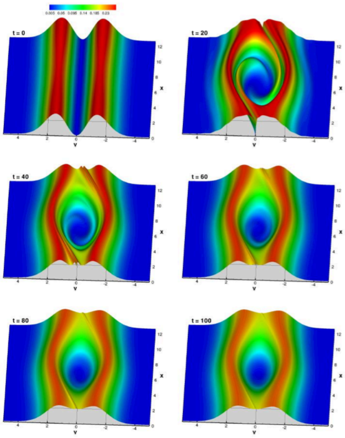


Fig. 10. (Nonlinear two-stream instability) 3D plots of the solution f at $t = 0$, $t = 20$, $t = 40$, $t = 60$, $t = 80$ and $t = 100$.

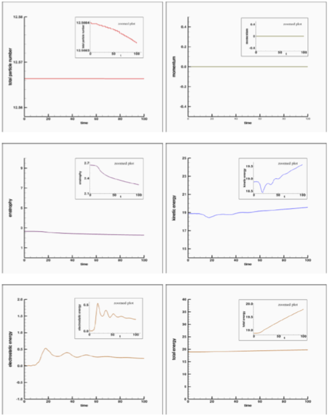
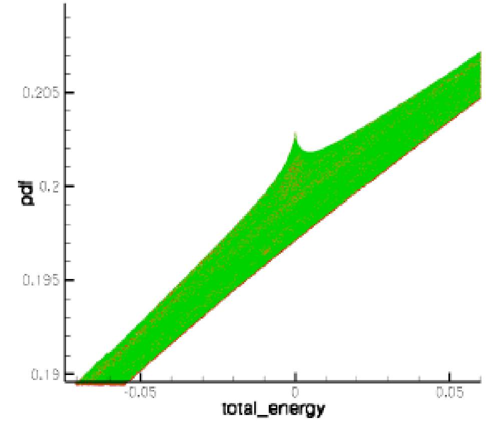
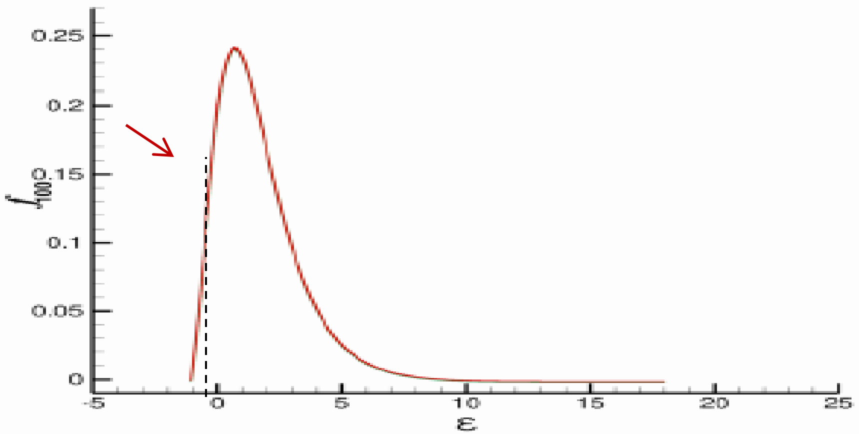


Fig. 14. (Nonlinear two-stream instability) Temporal evolution of the total particle number N *(top left)*, momentum P *(top right)*, enstrophy *(middle left)*, and kinetic energy *(middle right)*, electrostatic energy *(bottom left)*, and the total energy H *(bottom right)*. See Section 2 for definitions of these quantities.

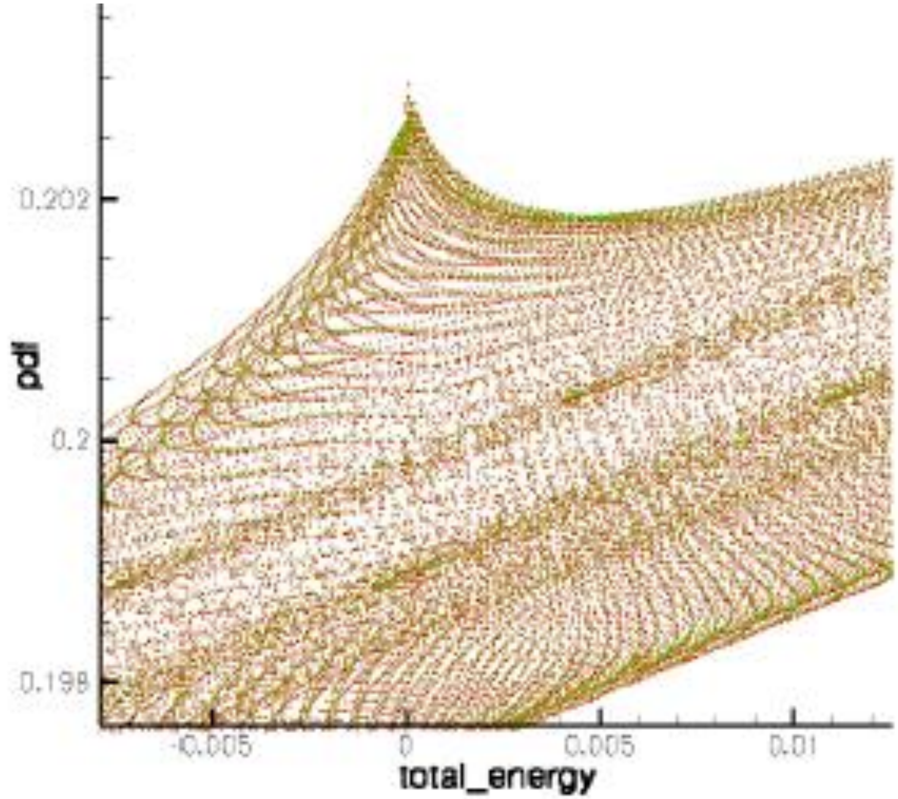
Nonlinear Two-Stream Instability: BGK type mode

Distributions of pairs $(v^2/2 - \Phi(x,100)\ ,\ \ f(x,v,100)\ )$



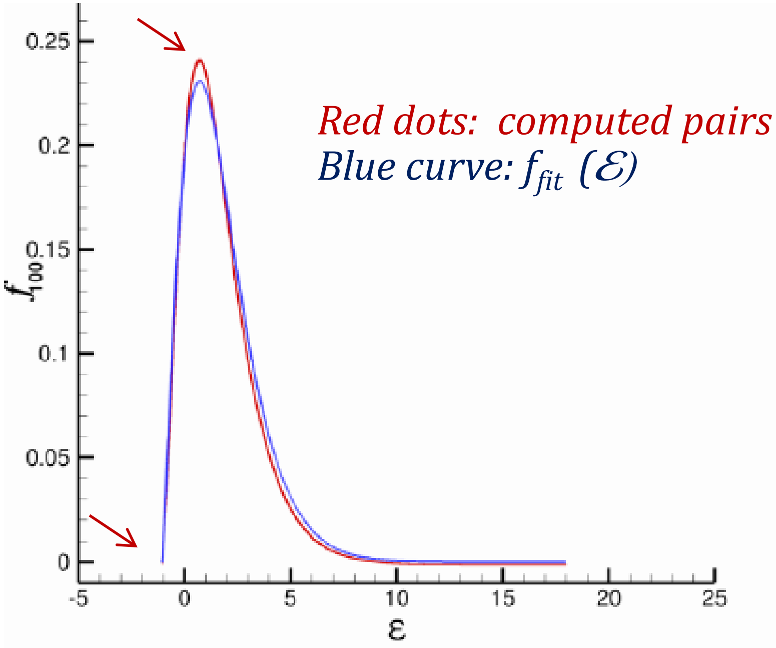


$$\mathcal{E} = v^2/2 - \Phi$$



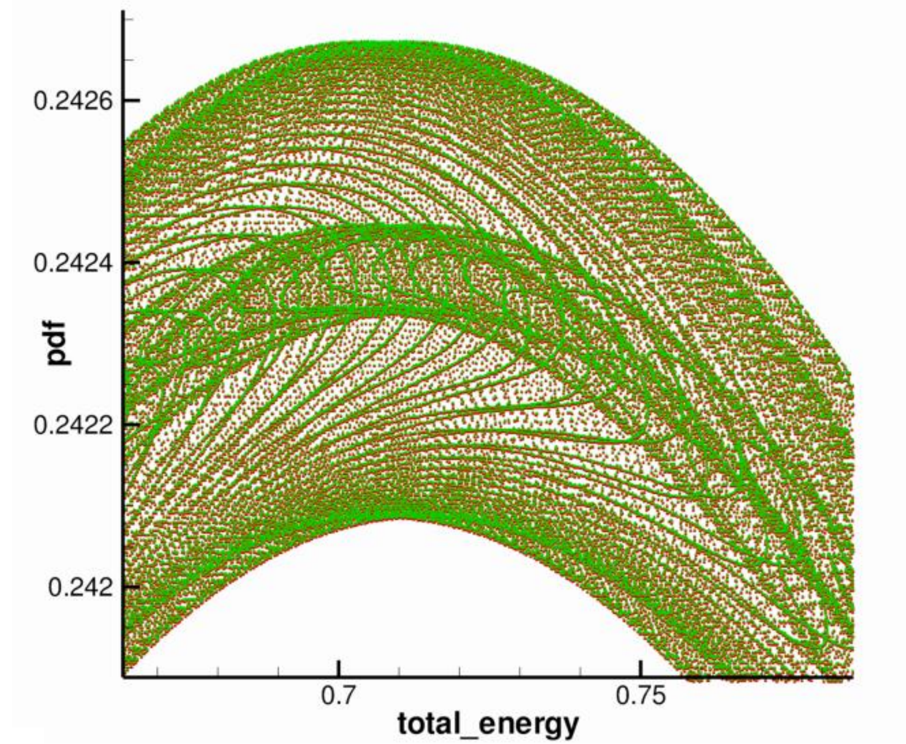
A = 0.05, k = 0.5, L = 4, V_c = 5, and
T = 100,
uniform mesh with (Nx, Nv) = (1800; 1400),

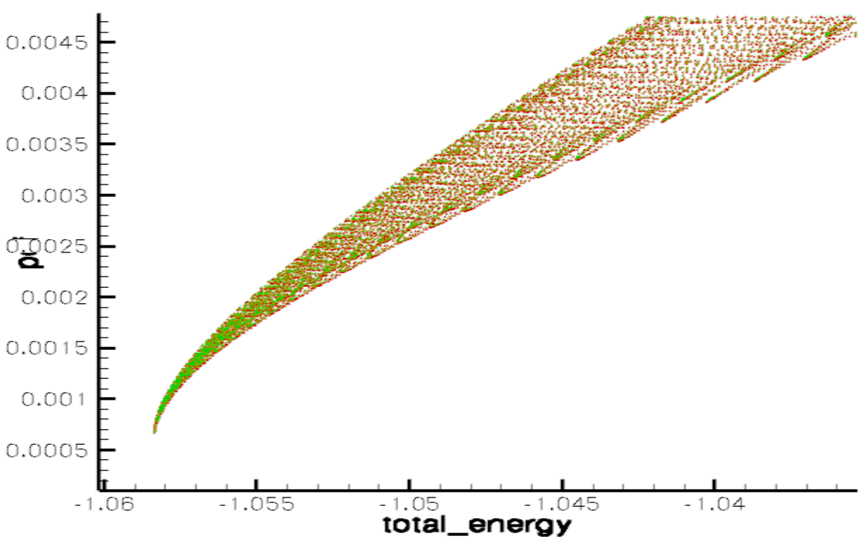
Nonlinear Two-Stream Instability: BGK   type mode

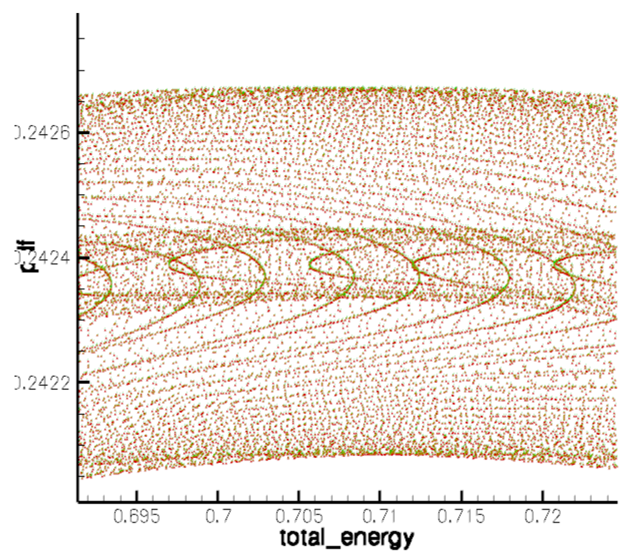


$$f_{\rm fit} = a(\mathcal{E}+\Phi_M)(\mathcal{E}+\mathcal{E}^*)e^{-\beta\mathcal{E}} .$$

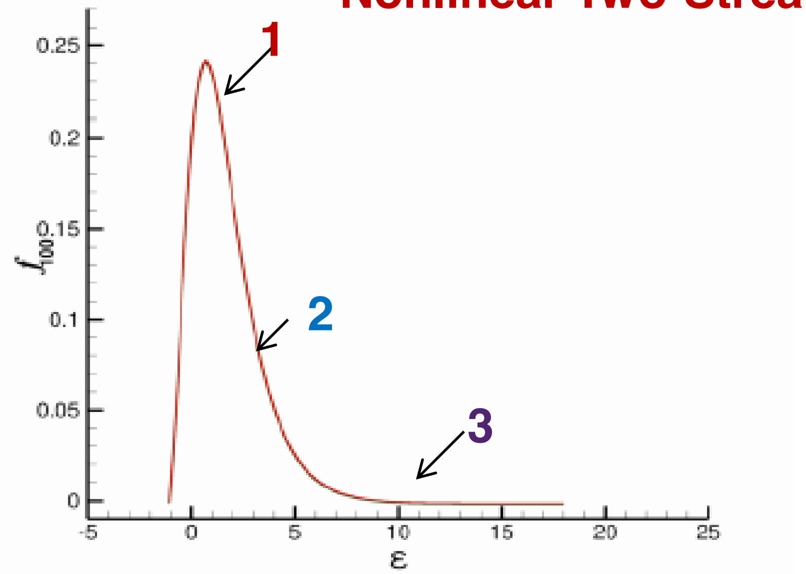
β=1, $\Phi_M = 1.06$ and $\mathcal{E}^* = 1.59.$, *a=0.1148*

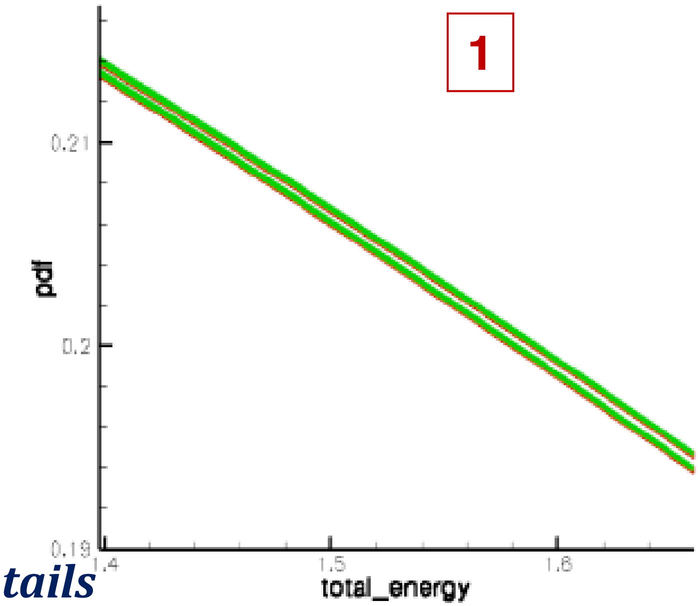




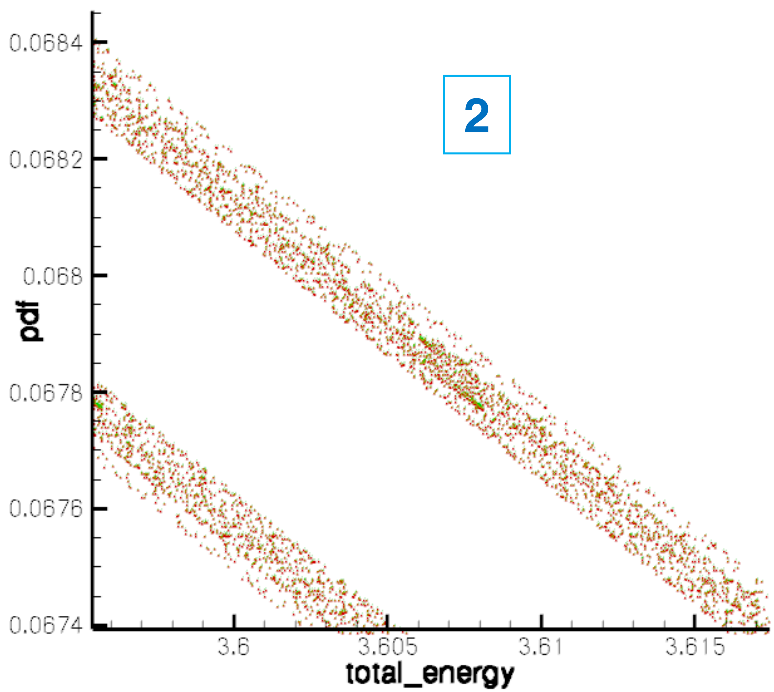


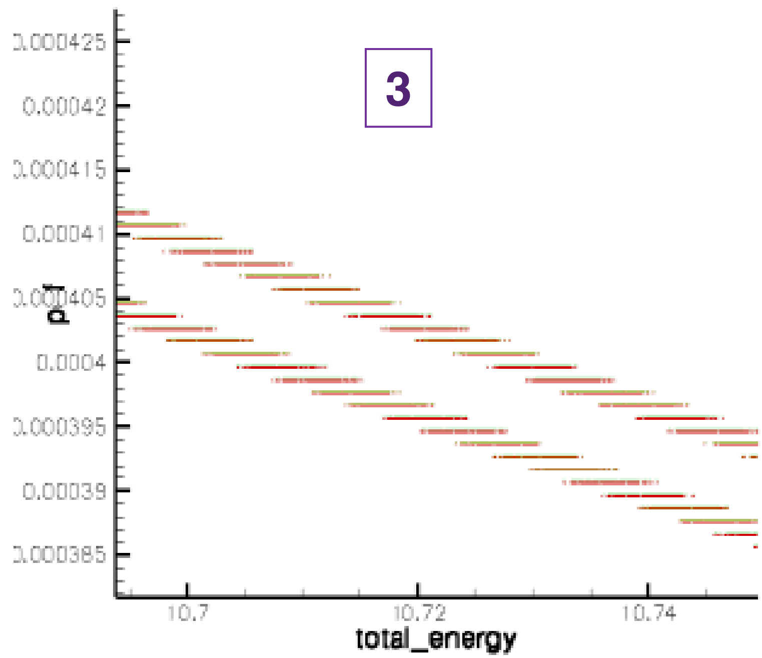
Nonlinear Two-Stream Instability: BGK type mode





Splitting on high energy tails





Semi-discrete DG form

- Computing domain $\Omega = \Omega_x \times \Omega_v = [0, L_x] \times [-L_{\mathbf{v}}, L_{\mathbf{v}}]^3$;
- Triangulation $\mathcal{T}_h = \{\mathcal{E} : \mathcal{E} = I_x \times K_{\mathbf{v}}, \forall I_x \in \mathcal{T}_h^x, \forall K_{\mathbf{v}} \in \mathcal{T}_h^{\mathbf{v}}\}$; with ε_x and $\varepsilon_{\mathbf{v}}$ be set of edges of $\mathcal{T}_h^x$ and $\mathcal{T}_h^{\mathbf{v}}$, respectively.
- Piecewise polynomial approximation $g|_{\mathbf{E}} \in P^{l_x}(I_x) \times P^{l_{\mathbf{v}}}(K_{\mathbf{v}})$.

Semi-discrete DG form

$$\int_{I_i \times K_j} (g_h)_t \varphi_h dx d\mathbf{v} = H_{i,j}(g_h, E_h, \varphi_h)$$

where

$$\begin{aligned}
H_{i,j}(g_h, E_h, \varphi_h) \; &= \\
= \int_{I_i \times K_j} v_1 g_h (\varphi_h)_x dx d\mathbf{v} &- \int_{K_j} (\widehat{v_1 g_h} \varphi_h^-)_{i+\frac{1}{2},\mathbf{v}} d\mathbf{v} + \int_{K_j} (\widehat{v_1 g_h} \varphi_h^+)_{i-\frac{1}{2},\mathbf{v}} d\mathbf{v} \\
- \int_{I_i \times K_j} E_h g_h \partial_{v_1} \varphi_h dx d\mathbf{v} &+ \int_{I_i} \int_{\varepsilon_{\mathbf{v}}} (\widehat{E_h g_h} \varphi_h^-)_{x,j_1+\frac{1}{2}} ds_{\mathbf{v}} dx - \int_{I_i} \int_{\varepsilon_{\mathbf{v}}} (\widehat{E_h g_h} \varphi_h^+)_{x,j_1-\frac{1}{2}} ds_{\mathbf{v}} dx \,.
\end{aligned}$$

with upwind numerical fluxes

$$\widehat{v_1 g_h} = \begin{cases} v_1 g_h^-, & \text{if } v_1 \geq 0 \text{ in } K_j; \\ v_1 g_h^+, & \text{if } v_1 < 0 \text{ in } K_j. \end{cases} \qquad \text{and} \qquad \widehat{E_h g_h} = \begin{cases} E_h g_h^-, & \text{if } \int_{I_i} E_h dx \leq 0; \\ E_h g_h^+, & \text{if } \int_{I_i} E_h dx > 0. \end{cases}$$

The Third Order TVD-RK

The third order total variation diminishing (TVD) Runge-Kutta method

$$
\begin{aligned}
\int_{I_i\times K_j} g_h^{(1)}\varphi_h dx d\mathbf{v} &= \int_{I_i\times K_j} g_h^{n}\varphi_h dx d\mathbf{v} + \Delta t H_{i,j}(g_h^n, E_h^n, \varphi_h)\,, \\
\int_{I_i\times K_j} g_h^{(2)}\varphi_h dx d\mathbf{v} &= \frac{3}{4}\int_{I_i\times K_j} g_h^{n}\varphi_h dx d\mathbf{v} + \frac{1}{4}\int_{I_i\times K_j} g_h^{(1)}\varphi_h dx d\mathbf{v} + \frac{\Delta t}{4} H_{i,j}(g_h^{(1)}, E_h^{(1)}, \varphi_h)\,, \\
\int_{I_i\times K_j} g_h^{n+1}\varphi_h dx d\mathbf{v} &= \frac{1}{3}\int_{I_i\times K_j} g_h^{n}\varphi_h dx d\mathbf{v} + \frac{2}{3}\int_{I_i\times K_j} g_h^{(2)}\varphi_h dx d\mathbf{v} + \frac{2\Delta t}{3} H_{i,j}(g_h^{(2)}, E_h^{(2)}, \varphi_h)
\end{aligned}
$$

Conservations and L^2 Stability

Mass/Charge

$$\frac{d}{dt}\int_{\mathcal{T}_h} g_h dx d\mathbf{v} = \Theta_{h,1}(g_h, E_h)\,,$$

with

$$\Theta_{h,1}(g_h, E_h) = \int_{\mathcal{T}_h^x}\int_{\varepsilon_{\mathbf{v}}^b} (\widehat{E_h g_h})_{x,N_v+\frac{1}{2}}\, ds_{\mathbf{v}}\, dx - \int_{\mathcal{T}_h^x}\int_{\varepsilon_{\mathbf{v}}^b} (\widehat{E_h g_h})_{x,\frac{1}{2}}\, ds_{\mathbf{v}}\, dx\,.$$

Momentum

$$\frac{d}{dt}\int_{\mathcal{T}_h} g_h \mathbf{v} dx d\mathbf{v} = \Theta_{h,2}(g_h, E_h)\,,$$

with

$$\Theta_{h,2}(g_h, E_h) = \int_{\mathcal{T}_h^x}\int_{\varepsilon_{\mathbf{v}}^b} (\widehat{E_h g_h}\mathbf{v})_{x,N_{\mathbf{v}}+\frac{1}{2}}\, ds_{\mathbf{v}}\, dx - \int_{\mathcal{T}_h^x}\int_{\varepsilon_{\mathbf{v}}^b} (\widehat{E_h g_h}\mathbf{v})_{x,\frac{1}{2}}\, ds_{\mathbf{v}}\, dx\,.$$

Both $\Theta_{h,1}(g_h, E_h)$ and $\Theta_{h,2}(g_h, E_h)$ vanished if the computation domain $\Omega = [0, L_x] \times [-L_{\mathbf{v}}, L_{\mathbf{v}}]^3$ is large enough, where the choice of such L_x and $L_{\mathbf{v}}$ depend on the initial state $f_{ini}(x, \mathbf{v})$.

Conservations and L^2 Stability

Variation of total Energy

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathcal{T}_h} g_h|\mathbf{v}|^2 dx d\mathbf{v} + \frac{1}{2}\int_{\mathcal{T}_h^x} |E_h|^2 dx\right) = \Theta_{h,3}(g_h, E_h) = \Theta_{h,3}(g_h - g, \Phi_h - \mathbf{P}\Phi_h)\,,$$

with

$$\Theta_{h,3}(g_h, E_h) = \int_{\mathcal{T}_h} (\Phi_h)_x g_h v_1 dx d\mathbf{v} - \int_{\mathcal{T}_h} \Phi_h (g_h)_t dx d\mathbf{v}\,,$$

Numerical solution decays enstrophy

$$\frac{d}{dt}\int_{\mathcal{T}_h} g_h^2 dx d\mathbf{v} = \Theta_{h,4}(g_h, E_h) \leq 0\,,$$

with

$$\Theta_{h,4}(g_h, E_h) = -\frac{1}{2}\int_{\mathcal{T}_h^{\mathbf{v}}}\int_{\varepsilon_x} |\mathbf{v}_1|[g_h]_x^2 ds_x d\mathbf{v} - \frac{1}{2}\int_{\mathcal{T}_h^x}\int_{\varepsilon_{\mathbf{v}}} |E_h|[g_h]_{v_1}^2 ds_{\mathbf{v}} dx\,.$$

Error estimates by R. Heath, PhD thesis 2007, Heath, I.M.G, P.J.Morrison & C. Mishler JCP'11, Y.Cheng, I.M.G & P.J.Morrison,'12

Linking Two Grids

Homogeneous Landau $\rightarrow$ Fourier series solution $\rightarrow$ L^2 projection onto DG grid $\rightarrow$ VP process

RECONSTRUCT

$$Q_N(f,f)(\mathbf{v}) = \frac{(2\pi)^{3/2}}{(2L)^3} \sum_{|k|\le N} \widehat{\mathbf{Q}}(\xi_k) e^{i\xi_k\cdot \mathbf{v}}\,,$$

We want, for the collision invariants $\phi(\mathbf{v}) \in span\{1, \mathbf{v}, |\mathbf{v}|^2\}$

$$\int_{\Omega_{\mathbf{v}}} Q_N(f,f)(\mathbf{v})\phi(\mathbf{v})d\mathbf{v} = 0$$

Find $\widehat{\mathbf{Q}} = [\widehat{\mathbf{Q}}_R^T, \widehat{\mathbf{Q}}_I^T]^T \in \mathbb{R}^{2M}$

$$\min \quad \|\widehat{\mathbf{Q}}_o - \widehat{\mathbf{Q}}\|_2^2 \qquad \text{s.t} \quad \mathbf{C}\widehat{\mathbf{Q}} = \mathbf{0}\,,$$

with $\mathbf{C} = [\mathbf{C}_{\mathcal{R}}, -\mathbf{C}_{\mathcal{I}}] \in \mathbb{R}^{(d+2)\times 2M}$ for $M = N^3$ being the total number of discretizations in $\mathbf{v}$-space.

CONSERVE

$$\widehat{\mathbf{Q}_c} = \left[\mathbf{I} - \mathbf{C}^T\left(\mathbf{C}\mathbf{C}^T\right)^{-1}\mathbf{C}\right]\widehat{\mathbf{Q}}_o\,,$$

Linking Two Grids

That means: the minimization problem to find the vector $\widehat{\mathbf{Q}} = [\widehat{\mathbf{Q}}_R^T, \widehat{\mathbf{Q}}_I^T]^T \in \mathbb{R}^{2M}$ satisfying the constraint $\mathbf{C}\widehat{\mathbf{Q}} = \mathbf{C}_{\mathcal{R}}\widehat{\mathbf{Q}}_{\mathcal{R}} - \mathbf{C}_{\mathcal{I}}\widehat{\mathbf{Q}}_{\mathcal{I}} = \mathbf{0}$.
That is, the constraint matrices $\mathbf{C}_{\mathcal{R}}, \mathbf{C}_{\mathcal{I}} \in \mathbb{R}^5 \times M$ must satisfy

$$\mathbf{C}_{\mathcal{R}}(l,k) + i\mathbf{C}_{\mathcal{I}}(l,k) = \frac{1}{(2L)^3}\int_{\Omega_v} e^{i\xi_k \cdot v}\phi_l(v)d\mathbf{v}\,,$$

which are calculated by taking $\phi_l(\mathbf{v}) = 0, v_1, v_2, v_3, |\mathbf{v}|^2$ with $l = 0, 1, 2, 3, 4$, respectively, by

$$\mathbf{C}_{\mathcal{R}}(0,k) = \prod_{i=1}^{3}\mathrm{sinc}(L\xi_{k_i}), \qquad \mathbf{C}_{\mathcal{I}}(0,k) = 0$$

$$\mathbf{C}_{\mathcal{R}}(l,k) = 0, \qquad \mathbf{C}_{\mathcal{I}}(l,k) = \begin{cases} \frac{\mathrm{sinc}(L\xi_{k_l}) - \cos(L\xi_{k_l})}{\xi_{k_l}} \prod_{i\neq l}^{3}\mathrm{sinc}(L\xi_{k_i}) & \xi_{k_l} \neq 0; \\ 0 & \xi_{k_l} = 0 \end{cases}, \qquad l = 1, 2, 3$$

$$\mathbf{C}_{\mathcal{I}}(4,k) = 0\,, \quad \mathbf{C}_{\mathcal{R}}(4,k) = \sum_{l=1}^{3}\left(\prod_{i\neq l}^{3}\mathrm{sinc}(L\xi_i)\right)\cdot\begin{cases} L^2\mathrm{sinc}(L\xi_l) - 2\frac{\mathrm{sinc}(L\xi_l) - \cos(L\xi_l)}{\xi_l^2} & \xi_l \neq 0; \\ \frac{L^2}{3} & \xi_l = 0 \end{cases}$$

Parallelization

Hybrid OpenMP and MPI

- RKDG: natural due to the locality of basis functions. Once all the nodes can access to the information from previous time step, the evolution of each grid point is done independently without communications across computing nodes;
- Spectral: a single grid point only "sees" the particles at the same spatial grid point, through the collision term. Restrict all information needed for the current time step on the same computing node, and only distribute spatial grid points across the community.

nodes	cores	wall clock time (s)
1	12	1228.18
2	24	637.522
4	48	307.125
8	96	154.385
16	192	80.6144
32	384	41.314

Table: The wall clock time for one single time step of a typical linear Landau damping problem (run on cluster Lonestar-TACC).

Electron plasma waves

$$\frac{\partial}{\partial t} f_e + v \cdot \nabla_x f_e + E(t,x) \cdot \nabla_v f_e = \frac{1}{\varepsilon}\big(Q_{e,e}(f_e, f_e) + Q_{e,i}(f_e)\big)\,.$$

with

$$Q_{e,i}(f_e) = \rho_i \nabla_v \cdot (\mathbf{S}(v - \bar{v}_i)\nabla_v f_e(v))\,,$$

Linear Landau Damping

Perturb the global equilibria $M(v) = (2\pi)^{-\frac{3}{2}} \exp(-\frac{|v|^2}{2})$ by a wave

$$f_0(x,v) = (1 + A\cos(kx))M(v) \qquad (x,v) \in [0, 2\pi/k] \times \mathbb{R}^3$$

k is the wave number. To study linear damping, choose small amplitude $A = 10^{-5}$ to restrict the problem under linear regimes.

The classical Landau theory tells that the square root of the electrostatic energy

$$\frac{1}{2}\int_0^{L_x} |E_h(x)|^2 dx$$

is expected to decay exponentially with frequency ω.

Analytical estimates on damping rate (F.F.Chen; C.J. McKinstrie et al))

$$\lambda = \lambda_l + \lambda_c\,,$$

where, with collision frequency defined $\nu = 1/\varepsilon$,

$$\lambda_c = -\frac{\nu}{3}\sqrt{\frac{2}{\pi}}\,, \quad \lambda_l = \begin{cases} -\sqrt{\frac{\pi}{8}}\frac{1}{k^3}\exp(-\frac{1}{2k^2} - \frac{3}{2}) & \text{for “large” } k \\ -\sqrt{\frac{\pi}{8}}\left(\frac{1}{k^3} - 6k\right)\exp(-\frac{1}{2k^2} - \frac{3}{2} - 3k^2 - 12k^4) & \text{for “small” } k \end{cases}$$

Linear Landau Damping

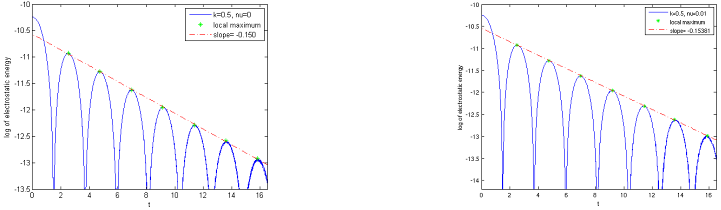


Figure: $k = 0.5$: $\varepsilon = \infty$ (left), $\varepsilon = 100$ (right). Prediction: -0.151(left), -0.154(right).

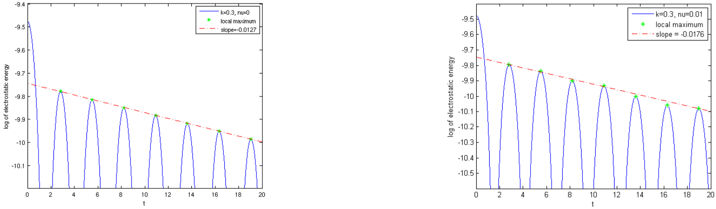


Figure: $k = 0.3$: $\varepsilon = \infty$ (left), $\varepsilon = 100$ (right). Prediction: -0.0132(left), -0.0167(right).

Nonlinear Landau Damping

Perturbation by wave

$$f_0(x, v) = (1 + A\cos(kx))M(v)\,,\,(x, v) \in [0, 2\pi/k] \times \mathbb{R}^3\,,$$

for a relatively large amplitude $A = 0.5$. Here, we choose the Maxwellian

$$M(v) = \frac{1}{2\pi T}\exp(-\frac{|v|^2}{2T})\,,$$

with $T = 0.25$.

Electron trapping: electrons trapped in a potential well

$$F(t, x, v_x) = \int_{\mathbb{R}^2} f(t, x, v_x, v_y, v_z)dv_y dv_z\,.$$

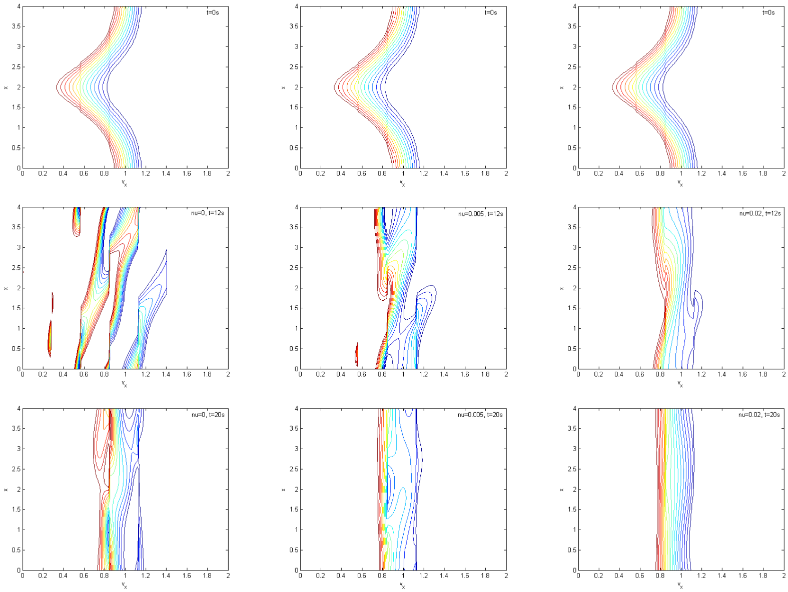


Figure: Evolution of contours of $F(t, x, v_x)$ for $\nu = 0$ (left), $\nu = 0.005$ (middle) and $\nu = 0.02$ (right)

A few comments about Vlasov - Maxwell (Cheng, Li, IMG,& Morrison, SINUM'14)

Scheme formulation: ***Semi discrete formulation***

For any $K = K_x \times K_\xi \in \mathcal{T}_h$, look for $f_h \in \mathcal{G}_h^k$, $\mathsf{E}_h, \mathbf{B}_h \in \mathcal{U}_h^r$, such that for any $g \in \mathcal{G}_h^k$, $\mathbf{U}, \mathbf{V} \in \mathcal{U}_h^r$,

Vlasov

$$\left\{\begin{aligned} &\int_K \partial_t f_h g dx d\xi - \int_K f_h \xi \cdot \nabla_x g dx d\xi - \int_K f_h (\mathsf{E}_h + \xi \times \mathbf{B}_h) \cdot \nabla_\xi g dx d\xi \\ &+ \int_{K_\xi} \int_{\partial K_x} \widehat{f_h \xi \cdot \mathbf{n}_x} g ds_x d\xi + \int_{K_x} \int_{\partial K_\xi} (\widehat{f_h (\mathsf{E}_h + \xi \times \mathbf{B}_h) \cdot \mathbf{n}_\xi}) g ds_\xi dx = 0 \,, \end{aligned}\right.$$

$$\left.\begin{aligned} &\int_{K_x} \partial_t \mathsf{E}_h \cdot \mathbf{U} dx = \int_{K_x} \mathbf{B}_h \cdot \nabla \times \mathbf{U} dx + \int_{\partial K_x} \widehat{\mathbf{n}_x \times \mathbf{B}_h} \cdot \mathbf{U} ds_x - \int_{K_x} \mathbf{J}_h \cdot \mathbf{U} dx \,, \\ &\int_{K_x} \partial_t \mathbf{B}_h \cdot \mathbf{V} dx = - \int_{K_x} \mathsf{E}_h \cdot \nabla \times \mathbf{V} dx - \int_{\partial K_x} \widehat{\mathbf{n}_x \times \mathsf{E}_h} \cdot \mathbf{V} ds_x \,, \end{aligned}\right\}$$

Maxwell

with

$$\mathbf{J}_h(\mathsf{x}, t) = \int_{\mathcal{T}_h^\xi} f_h(\mathsf{x}, \xi, t) \xi d\xi \,.$$

All the hat functions are numerical fluxes, and they are taken to be upwinding **for Vlasov, and**
either upwind, central or alternating flux for Maxwell 's eqs

Theorem (Mass conservation)

The numerical solution $f_h \in \mathcal{G}_h^k$ with $k \geq 0$ satisfies

$$\frac{d}{dt} \int_{\mathcal{T}_h} f_h dx d\xi + \Theta_{h,1}(t) = 0 \,,$$

with $\Theta_{h,1}(t) = \int_{\mathcal{T}_h^x} \int_{\mathcal{E}_\xi^b} f_h \max((\mathsf{E}_h + \xi \times \mathbf{B}_h) \cdot \mathbf{n}_\xi, 0) ds_\xi dx \,.$ *with* $0 \approx f_h |_{\mathcal{E}_\xi^b} \ll \sup_{\mathcal{E}_\xi^b}(|\mathbf{E}_h| + 2V|\mathbf{B}_h|) \,.$

A few comments about Vlasov - Maxwell (Cheng, Li, IMG,& Morrison, SINUM'14)

Theorem (Total energy conservation)

For $k \geq 2$, $r \geq 0$, the numerical solution $f_h \in \mathcal{G}_h^k$, $\mathsf{E}_h, \mathbf{B}_h \in \mathcal{U}_h^r$ with the upwinding numerical fluxes satisfies

$$\frac{d}{dt}\left(\int_{\mathcal{T}_h} f_h|\xi|^2 d\mathsf{x} d\xi + \int_{\mathcal{T}_h^x} (|\mathsf{E}_h|^2 + |\mathbf{B}_h|^2) d\mathsf{x}\right) + \Theta_{h,2}(t) + \Theta_{h,3}(t) = 0$$

with

$$\Theta_{h,2}(t) = \int_{\mathcal{E}_x} \left(|[\mathsf{E}_h]_\tau|^2 + |[\mathbf{B}_h]_\tau|^2\right) ds_x$$

jump of tangential components in (E,B) across interfaces

$$\Theta_{h,3}(t) = \int_{\mathcal{T}_h^x} \int_{\mathcal{E}_\xi^b} f_h|\xi|^2 \max((\mathsf{E}_h + \xi \times \mathbf{B}_h) \cdot \mathbf{n}_\xi, 0) ds_\xi d\mathsf{x}\ .$$

with $\quad 0 \approx f_h\,|_{\mathcal{E}_\xi^b} \ll 4V^2 \sup_{\mathcal{E}_\xi^b}(|\mathbf{E}_h| + 2V|\mathbf{B}_h|)$

Remark:. *The energy conservation holds as long as* $|\xi|^2 \in \mathcal{G}_h^k$.
For k < 2, the energy conservation can be obtained if one $\mathcal{G}_h^k \oplus \{|\xi|^2\}$.

Remark: It is not possible to have a positive preserving limiter (Zhang-Shu) and energy conservation: over –determine systems for the degrees of freedom

A few comments about Vlasov - Maxwell (Cheng, Li, IMG,& Morrison, SINUM'14)

Corollary: Total energy conservation: For different choices of numerical fluxes

(1) *Central flux:* $\widetilde{E_h} = \{E_h\}$ *and* $\widetilde{B_h} = \{B_h\}$, **or**

(2) *Alternating flux:* $\widetilde{E_h} = E_h^+$ *and* $\widetilde{B_h} = B_h^-$, *or* $\widetilde{E_h} = E_h^-$ *and* $\widetilde{B_h} = B_h^+$, for **Maxwell's** eqs.

(3) and upwind fluxes for **Vlasov**

then $\Theta_{h,2}(t) = 0$ so

$$\frac{d}{dt}\left(\int_{\mathcal{T}_h} f_h|\xi|^2 dx d\xi + \int_{\mathcal{T}_h^x} (|E_h|^2 + |B_h|^2) dx\right) + \Theta_{h,3}(t) = 0 ,$$

with

$$\Theta_{h,3}(t) = \int_{\mathcal{T}_h^x}\int_{\mathcal{E}_\xi^b} f_h|\xi|^2 \max((E_h + \xi \times B_h)\cdot n_\xi, 0) ds_\xi dx .$$

with $0 \approx f_h|_{\mathcal{E}_\xi^b} \ll 4V^2 \sup_{\mathcal{E}_\xi^b}(|E_h| + 2V|B_h|)$

Theorem (L^2-stability of f_h)

For $k \geq 0$, the numerical solution $f_h \in \mathcal{G}_h^k$ satisfies

$$\frac{d}{dt}\left(\int_{\mathcal{T}_h} |f_h|^2 dx d\xi\right) + \int_{\mathcal{T}_h^\xi}\int_{\mathcal{E}_x} |\xi\cdot n_x||[f_h]_x|^2 ds_x d\xi + \int_{\mathcal{T}_h^x}\int_{\mathcal{E}_\xi} |(E_h + \xi\times B_h)\cdot n_\xi||[f_h]_\xi|^2 ds_\xi dx = 0$$

A few comments about Vlasov - Maxwell (Cheng, Li, IMG,& Morrison, SINUM'14)

Example 1&1/2 dimension magnetic confinement plasma model

A reduced version of the **VME's** with **one** spatial**,** $\mathbf{x_2}$, and **two** velocity variables, ξ_1 and ξ_2, for the **pdf** $\mathbf{f(x_2; \xi_1; \xi_2; t)}$, under **a 2D electric field** $\mathbf{E = (E_1(x_2; t);\ E_2(x_2; t);\ 0)}$ **and a 1D magnetic field** $\mathbf{B = (0; 0; B_3(x_2; t))}$ →

Reduced VM system

$$\left\{\begin{array}{l} f_t + \xi_2 f_{x_2} + (E_1 + \xi_2 B_3) f_{\xi_1} + (E_2 - \xi_1 B_3) f_{\xi_2} = 0 \\ \dfrac{\partial B_3}{\partial t} = \dfrac{\partial E_1}{\partial x_2}, \quad \dfrac{\partial E_1}{\partial t} = \dfrac{\partial B_3}{\partial x_2} - j_1, \quad \dfrac{\partial E_2}{\partial t} = -j_2\ , \\ j_1 = \displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_2, \xi_1, \xi_2, t)\xi_1\, d\xi_1 d\xi_2, \quad j_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_2, \xi_1, \xi_2, t)\xi_2\, d\xi_1 d\xi_2\ . \end{array}\right.$$

$x=x_2$ in $\Omega_x = [0,L]$, $L=2\pi/k_0$ with periodic b.c. in x-space and

$(\xi_1; \xi_2)$ in $\Omega_\xi = [-1.5; 1.5]^2$ with vanishing b.c. in ξ-space

Initial conditions for a Streaming Weibel (SW) instability: derives its free energy from transverse counter-streaming as opposed to temperature anisotropy (Weibel Inst.)

$$f(x_2, \xi_1, \xi_2, 0) = \frac{1}{\pi\beta} e^{-\xi_2^2/\beta}[\delta e^{-(\xi_1 - v_{0,1})^2/\beta} + (1-\delta)e^{-(\xi_1 + v_{0,2})^2/\beta}],$$

$$E_1(x_2, \xi_1, \xi_2, 0) = E_2(x_2, \xi_1, \xi_2, 0) = 0, \qquad B_3(x_2, \xi_1, \xi_2, 0) = b\sin(k_0 x_2)\ ,$$

with $\beta = 0.01$ $k_0 = 0.2$ and $b=0.001$

A few comments about Vlasov - Maxwell (Cheng, Li, IMG,& Morrison, SINUM'14)

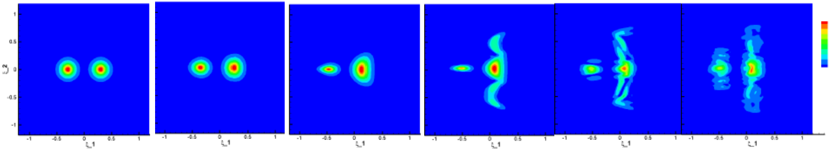
Streaming Weibel (SW) instability: $L=2\pi/k_0$ - two cases: Symmetric and non-Symmetric depending on the choice of d and initial currents $v_{0,1} = v_{0,2}$

Init. Cond:

$$f(x_2,\xi_1,\xi_2,0) = \frac{1}{\pi\beta} e^{-\xi_2^2/\beta}[\delta e^{-(\xi_1 - v_{0,1})^2/\beta} + (1-\delta)e^{-(\xi_1+v_{0,2})^2/\beta}],$$

$$E_1(x_2,\xi_1,\xi_2,0) = E_2(x_2,\xi_1,\xi_2,0) = 0, \qquad B_3(x_2,\xi_1,\xi_2,0) = b\sin(k_0 x_2)\ ,$$

$x_2=0.05\pi$



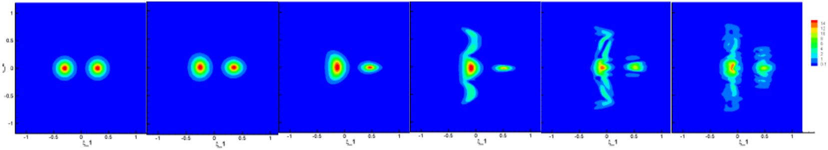
(b) $x_2 = 4.95\pi$, $t = 0$. (c) $x_2 = 0.05\pi$, $t = 55$. (e) $x_2 = 0.05\pi$, $t = 82$. (a) $x_2 = 0.05\pi$, $t = 100$. (c) $x_2 = 0.05\pi$, $t = 125$. (e) $x_2 = 0.05\pi$, $t = 175$.

→ **time line**

Symmetric case: 2D contour plots for the approximate **pdf** f_h for **the SW instability,** with parameter $\delta = 0.5$, $v_{0,1} = v_{0,2} = 0.3$, $k_0 = 0.2$, at selected locations x_2 and times t. The mesh is 100^3 with piecewise quadratic polynomials and upwind flux is used.

$L=10\pi$

$x_2=4.95\pi$



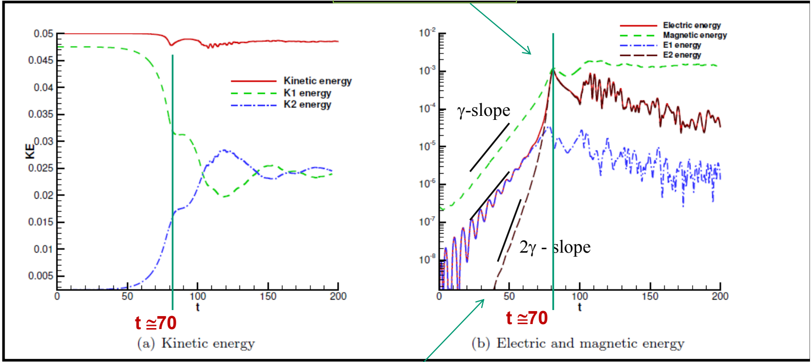
(b) $x_2 = 4.95\pi$, $t = 0$. (d) $x_2 = 4.95\pi$, $t = 55$. (f) $x_2 = 4.95\pi$, $t = 82$. (b) $x_2 = 4.95\pi$, $t = 100$. (d) $x_2 = 4.95\pi$, $t = 125$. (f) $x_2 = 4.95\pi$, $t = 175$.

A few comments about Vlasov - Maxwell (Cheng, Li, IMG,& Morrison, SINUM'14)

SW instability For the symmetric i.s.: Energy conservation analysis (alter. flux in Maxwell)

Total energy $\int_{\mathcal{T}_h} f_h|\xi|^2 d\mathbf{x}d\xi + \int_{\mathcal{T}_h^x} (|\mathbf{E}_h|^2 + |\mathbf{B}_h|^2)d\mathbf{x} = \underbrace{\mathbf{K_1} + \mathbf{K_2}}_{\text{Kinetic energy}} + \underbrace{\mathbf{E_1} + \mathbf{E_2}}_{\text{Electric energy}} + \underbrace{\mathbf{B}}_{\text{Magnetic energy}} = \mathbf{ITE}$

Equipartition of elec & mag. energy



t ≅70

(a) Kinetic energy

t ≅70

(b) Electric and magnetic energy

There is no contribution of E_2 until t=40 and then kicks-in twice as fast to peak at t=70

Summary & Looking ahead

- **Today's summary**
  - Extended the conservative spectral method developed for Boltzmann collisional flows to homogeneous FPL equations; Generalized the RKDG method to study VP in 3D;
  - Developed a conservative solver for the inhomogeneous FPL equations coupled with Poisson equations, tested for periodic boundary conditions. Applications to plasma sheath formation is underway.
- **Looking ahead**
  - Implementation for the coupling of Vlasov Maxwell Landau (VML).
  - Hybrid coupling of VML to MHD - new formulation of Hamiltonian dynamics recently develop by M. Lingam, P.J. Morrison, et al (2014 - 15) may call for a revision to MHD closures and collision forms.
  - Develop a solver for Balescu-Lenard operator: **strongly non-linear/non-local**: They weight function $S(u)$ become $S(u, f(u,t))$ needs to be updated at time steps...

With many thanks to Bob, Jack and Walter,
for so many inspirational writings and
for motivated this wonderful gathering this past week!!

Thank you very much for your attention!

Form preprints/references check at
http://www.ma.utexas.edu/users/gamba/publications-web.htm