AVERAGING ALONG DEGENERATE FLOWS ON THE ANNULUS

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ABSTRACT. This paper focuses on the study of flows on the annulus that do not possess a spectral gap. Estimates for the so-called density of states are obtained for small values of the spectrum. Those estimates lead to rates of decay for the averaging dynamic of the flows, using a recent result of the authors on uniform bounds in the ergodic theorem.

CONTENTS

1. INTRODUCTION

In this paper we study self-adjoint operators of the form

$$A = -i\varphi(m)\frac{\partial}{\partial\theta} \qquad \text{acting in } L^2(\mathcal{A}) \tag{1.1}$$

where $\mathcal{A} = [0,1]_m \times \mathbb{S}^1_\theta$ is an annulus and where $\varphi \geq 0$. Specifically, we are interested in obtaining a rate at which the time averages

$$P^T f := \frac{1}{2T}\int_{-T}^{T} e^{itA} f \, \mathrm{d}t \tag{1.2}$$

converge to the spatial average

$$P f := \int_{\mathbb{S}^1} f(m,\theta) \, \mathrm{d}\theta \tag{1.3}$$

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as $T \to +\infty$ in the degenerate case where $\inf_{m \in [0,1]} \varphi(m) = 0$. In this case there cannot be a uniform rate on $L^2(\mathcal{A})$ and our main task is to identify a subspace $\mathcal{X} \subset L^2(\mathcal{A})$ on which a uniform rate *does* hold.

1.1. **Main result.** Our main result is:

Theorem 1.1. *If $\varphi(m) = Cm^\alpha$, $\alpha > 0$ near $m = 0$ (and otherwise φ is bounded away from 0) then $\|P^T - P\|_{\mathcal{H}_0^{s,\gamma} \to (\mathcal{H}_0^{s,\gamma})^*} \lesssim T^{-\frac{s}{s+\alpha}}$ where $s > 1/2$ and $\gamma \geq 0$ satisfy $\gamma + s/\alpha > 1/2$, and where the space $\mathcal{H}_0^{s,\gamma} \subset L^2(\mathcal{A})$ is defined in (4.8) and $(\mathcal{H}_0^{s,\gamma})^*$ is its dual.[1]*

Strategy of the proof. The strong convergence (i.e without a rate) $P^T \to P$ follows immediately from von Neumann's ergodic theorem [vN32]. Von Neumann's idea was to use the spectral theorem to write $A = \int_{\mathbb{R}} \lambda \, dE(\lambda)$, where $\{E(\lambda)\}_{\lambda \in \mathbb{R}}$ is the resolution of the identity of A. This leads to

$$(P^T - P)f = \frac{1}{2T} \int_{-T}^{T} e^{itA} f \, dt - Pf = \frac{1}{2T} \int_{-T}^{T} \int_{\mathbb{R}} e^{it\lambda} \, dE(\lambda) f \, dt - Pf$$

$$= \frac{1}{2T} \int_{-T}^{T} \int_{\mathbb{R} \setminus \{0\}} e^{it\lambda} \, dE(\lambda) f \, dt = \int_{\mathbb{R} \setminus \{0\}} \frac{\sin T\lambda}{T\lambda} \, dE(\lambda) f.$$

This last expression tends to 0 as $T \to +\infty$. In [BAM19] we showed that a rate may be extracted if there exists a subspace $\mathcal{X} \subset L^2(\mathcal{A})$ and some $r > 0$ such that the Stieltjes integral above can be written as a Lebesgue integral via an estimate of the density of states of the form

$$\left| \frac{d}{d\lambda} (E(\lambda)f, g)_{L^2(\mathcal{A})} \right| \leq \psi(\lambda) \|f\|_{\mathcal{X}} \|g\|_{\mathcal{X}}, \qquad \forall f, g \in \mathcal{X}, \, \forall \lambda \in (-r, r) \setminus \{0\}, \qquad (1.4)$$

where $\psi \in L^1(-r, r)$ is strictly positive a.e. on $(-r, r)$. Therefore, to prove Theorem 1.1 the main task is to obtain an estimate of the form (1.4), which involves identifying an appropriate subspace $\mathcal{X}$. This is achieved thanks to the observation that the operator A is unitarily equivalent to the multiplication operator $\varphi\xi$ via a Fourier transform in θ.

To obtain an estimate of the density of states as in (1.4), the first step is to understand the structure of the spectrum. It is evident that A is fibered in m, composed of the one-dimensional operators

$$A(m) = -\frac{2\pi i}{T(m)} \frac{d}{d\theta} \qquad \forall m \in [0, 1], \qquad (1.5)$$

acting on the circle $\mathbb{S}^1$, where we have made the substitution

$$\varphi(m) = \frac{2\pi}{T(m)}$$

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[1] Roughly speaking, $\mathcal{H}_0^{s,\gamma}$ consists of functions that have γ derivatives in θ, whose $\dot{H}^\gamma$ norm is $(s - \frac{1}{2})$–Hölder continuous in m and are constant along the fiber $m = 0$.

with $T(m) > 0$ being the period of the flow along the m-fiber. This point of view is common in the context of the Euler equations, see [Cox14] for instance. The main difficulty is now evident: the spectrum of each fiber $A(m)$ is discrete, while the spectrum of A may have discrete, absolutely continuous and singular continuous parts.

1.2. **Organization of the paper.** In Section 2 we discuss in detail various properties of self-adjoint and self-adjoint *fibered* operators, and in particular their spectrum. In Section 3 we present a toy model which mimics some of the behavior of the operator A in order to gain a better intuition. Then, in Section 4 we turn our attention to the flow in an annulus, first proving a bound on the density of states (Theorem 1.1) and then obtaining a rate for the associated ergodic theorem (Theorem 4.8).

2. SPECTRAL ANALYSIS AND FIBERED OPERATORS

In this section we recall some properties of self-adjoint operators and of self-adjoint fibered operators, and prove some results which are not readily available in the standard literature. Our discussion remains as general as possible, working with abstract self-adjoint operators in abstract Hilbert spaces. Our flow operator defined above is a special example and does not appear in this section.

2.1. **Resolution of the identity.** Since the spectral theorem plays an essential role in our proof, it is worthwhile recalling the definition of the resolution of the identity of a self-adjoint operator.

Let $\mathcal{Y}$ be some Hilbert space, and let H be a self-adjoint operator in it. Its associated spectral family $\{E(\lambda)\}_{\lambda \in \mathbb{R}}$ is a family of projection operators in $\mathcal{Y}$ with the property that, for each $\lambda \in \mathbb{R}$, the subspace $\mathcal{Y}^\lambda = E(\lambda)\mathcal{Y}$ is the largest closed subspace such that

(1) $\mathcal{Y}^\lambda$ *reduces* H, namely, $HE(\lambda)g = E(\lambda)Hg$ for every $g \in D(H)$. In particular, if $g \in D(H)$ then also $E(\lambda)g \in D(H)$.

(2) $(Hu, u)_{\mathcal{Y}} \leq \lambda(u, u)_{\mathcal{Y}}$ for every $u \in \mathcal{Y}^\lambda \cap D(H)$.

Given any $f, g \in \mathcal{Y}$ the spectral family defines a complex function of bounded variation on the real line, given by

$$\mathbb{R} \ni \lambda \mapsto (E(\lambda)f, g)_{\mathcal{Y}}. \tag{2.1}$$

It is well-known that such a function gives rise to a complex measure (depending on f, g) called the *spectral measure*. Recall the following useful fact:

Proposition 2.1 ([Kat95, X-§1.2, Theorem 1.5]). *Let $U \subset \mathbb{R}$ be open. The set of $f, g \in \mathcal{Y}$ for which the spectral measure is absolutely continuous in U with respect to the*

Lebesgue measure forms a closed subspace $\mathcal{AC}_U \subset \mathcal{Y}$. This subspace is referred to as the absolutely continuous subspace of H on U.

2.2. The density of states.

Let $\mathcal{AC}_U \subset \mathcal{Y}$ be the absolutely continuous subspace of H on U and let $\lambda_0 \in U$. If there exists a subspace $\mathcal{X} \subset \mathcal{AC}_U$ equipped with a stronger norm such that the bilinear form $\frac{\mathrm{d}}{\mathrm{d}\lambda}\big|_{\lambda=\lambda_0} (E(\lambda)\cdot,\cdot)_\mathcal{Y} : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ is bounded then it induces a bounded operator $B(\lambda_0) : \mathcal{X} \to \mathcal{X}^*$ defined via

$$\langle B(\lambda_0)f, g \rangle_{(\mathcal{X}^*,\mathcal{X})} = \frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=\lambda_0} (E(\lambda)f, g)_\mathcal{Y}, \quad f, g \in \mathcal{X}, \tag{2.2}$$

where $(\cdot,\cdot)_\mathcal{Y}$ is the inner product in $\mathcal{Y}$, $\langle\cdot,\cdot\rangle_{(\mathcal{X}^*,\mathcal{X})}$ is the $(\mathcal{X}^*, \mathcal{X})$ dual-space pairing, and $\mathcal{X}^*$ is the dual of $\mathcal{X}$ with respect to the inner-product on $\mathcal{Y}$.

Definition 2.2. *We refer to both the bilinear form $\frac{\mathrm{d}}{\mathrm{d}\lambda}\big|_{\lambda=\lambda_0}(E(\lambda)\cdot,\cdot)_\mathcal{Y}$ and the operator $B(\lambda_0)$ as the* density of states *of the operator H at λ_0.*

In physics, the density of states at λ_0 represents the number possible states a system can attain at the energy level λ_0. Another approach for obtaining the density of states is via the *limiting absorption principle*.

2.3. Fibered operators.

We follow the notation of [RS78, p. 283]. Let $\mathcal{H}'$ be a Hilbert space and let $(M, \mathrm{d}\mu)$ be a measure space. Let $A(\cdot) : M \to \mathcal{L}^{s.a.}(\mathcal{H}')$ be a measurable function taking values in the space of self-adjoint operators (not necessarily bounded) on $\mathcal{H}'$ (with appropriate domains). Let $\mathcal{H} = \int_M^\oplus \mathcal{H}'$ and let $A = \int_M^\oplus A(m)\mathrm{d}\mu(m)$. For an element $f \in \mathcal{H}$, we denote its fibers as $f_m \in \mathcal{H}'$ so that

$$f = \int_M^\oplus f_m \mathrm{d}\mu(m). \tag{2.3}$$

We denote the resolution of the identity of A by $\{E(\lambda)\}_{\lambda\in\mathbb{R}}$ and of $A(m)$ by $\{E_m(\lambda)\}_{\lambda\in\mathbb{R}}$.

Lemma 2.3. *The resolutions of the identity also satisfy the natural decomposition*

$$E(\lambda) = \int_M^\oplus E_m(\lambda)\mathrm{d}\mu(m).$$

Proof. By standard functional calculus, we apply the characteristic function $\mathbb{1}_{(-\infty,\lambda_0]}$ to

$$\int_\mathbb{R} \lambda \mathrm{d}E(\lambda) = A = \int_M^\oplus A(m)\mathrm{d}\mu(m) = \int_M^\oplus \int_\mathbb{R} \lambda \mathrm{d}E_m(\lambda)\mathrm{d}\mu(m)$$

to obtain the assertion of the lemma (at the point λ_0). $\qquad\square$

It is well-known that since all $A(m)$ are self-adjoint, so is A, with spectrum $\sigma(A)$ characterised as follows:

$$\lambda \in \sigma(A) \qquad \Leftrightarrow \qquad \forall \varepsilon > 0, \quad \mu\left(\{m : \sigma(A(m)) \cap (\lambda - \varepsilon, \lambda + \varepsilon) \neq \emptyset\}\right) > 0. \tag{2.4}$$

An immediate consequence of this is:

$$\sigma(A) \subset \overline{\bigcup_m \sigma(A(m))}. \tag{2.5}$$

Let us mention some other important consequences. First, the following characterisation of eigenvalues:

$$\mu\left(\{m \ : \ \lambda \text{ is an eigenvalue of } A(m)\}\right) > 0 \qquad \Rightarrow \qquad \lambda \text{ is an eigenvalue of } A. \tag{2.6}$$

Hence, if there exist real numbers E_j such that $\cup_m \sigma(A(m)) = \{E_1, \ldots, E_k\}$ with $\mu(\{m \ : \ \sigma(A(m)) = E_j\}) > 0$ for all j, then $\sigma(A) = \{E_1, \ldots, E_k\}$. If, on the other hand, for all $m \in M$ the spectrum of $A(m)$ is purely absolutely continuous then so is the spectrum of A.

Proposition 2.4. *Assume that the measure $\mathrm{d}\mu$ is the Borel measure associated to some given topology and that M is compact. Assume that $\Sigma : M \to \mathrm{cl}(\mathbb{R}) = \{\text{closed subsets in } \mathbb{R}\}$ given by $m \mapsto \sigma(A(m)) \subset \mathbb{R}$ is continuous (we take the Hausdorff distance on $\mathrm{cl}(\mathbb{R})$). Then $\sigma(A) = \cup_m \sigma(A(m))$.*

Proof. Since M is compact and Σ is continuous, we have that $\cup_m \sigma(A(m)) = \overline{\cup_m \sigma(A(m))}$. Hence, considering (2.5) we only need to prove that $\sigma(A) \supset \cup_m \sigma(A(m))$. Suppose that $\lambda \in \cup_m \sigma(A(m)) = \cup_m \Sigma(m)$. In particular there exists some $m_0 \in M$ such that $\lambda \in \Sigma(m_0)$. By the continuity of Σ, for each $\varepsilon > 0$ there exists an open neighborhood U of m_0 such that $\Sigma(m) \cap (\lambda - \varepsilon, \lambda + \varepsilon) \neq \emptyset$ for all $m \in U$, which, by (2.4), implies that $\lambda \in \sigma(A)$ (note that U has positive measure). $\square$

3. Toy example

In most practical applications the space M of fibers is one-dimensional, usually being either $\mathbb{R}$ or an interval within $\mathbb{R}$ (in the case of a flow on the annulus $M = [0,1]$ for instance). In order to gain a better understanding we start with a toy example which exhibits many of the complexities of our problem.

Let $M = [0,1]$ with the usual Borel σ-algebra and suppose that for all $m \in [0,1]$ the spectrum of $A(m)$ in the energy band $I := (a,b) \subset \mathbb{R}$ is given by a single eigenvalue $\mathcal{E}(m)$, with multiplicity 1. Suppose that $\mathcal{E} : [0,1] \to \mathbb{R}$ is measurable (it actually has to be due to the definition of fibered operators). As before, we let $\mathcal{H} = \int_{[0,1]}^{\oplus} \mathcal{H}'$ and $A = \int_{[0,1]}^{\oplus} A(m) \mathrm{d}\mu(m)$ (here in fact $\mathrm{d}\mu(m) = \mathrm{d}m$). We know that

- $\sigma(A) \cap I = \mathrm{ess\,ran}(\mathcal{E})$ and if $\mathcal{E}$ is continuous then $\mathrm{ess\,ran}$ may be replaced by ran, see also Proposition 2.4.

- If $\mathcal{E}$ is constant on some open set U and equal to $\overline{\mathcal{E}}$ there, then $\overline{\mathcal{E}}$ is an eigenvalue of A.

These two statements combined indicate that one can easily construct examples with eigenvalues embedded in the essential spectrum (or at its boundary). Hence even in this simple example, the absolute continuity of the spectrum of A depends on the nature of the function $\mathcal{E}$. Let us assume that there are no embedded eigenvalues:

Assumption A1. $\mathcal{E} \in C^1([0, 1])$ and is not constant on sets of positive measure.

Definition 3.1. *Let $\lambda \in \mathrm{ran}(\mathcal{E})$ and denote $M_\lambda := \mathcal{E}^{-1}(\lambda) \subset [0, 1]$. We say that λ is a regular point if for any $m \in M_\lambda$, $\mathcal{E}'(m) \neq 0$. We also define the set*

$$\sigma_{\mathrm{reg},I}(A) := \{\lambda \in I \; : \; \lambda \in \mathrm{ran}(\mathcal{E}) \text{ is a regular point}\}.$$

Lemma 3.2. $\sigma_{\mathrm{reg},I}(A)$ *is an open subset of I.*

Proof. The claim is almost trivial, as the condition of being a regular point is an open condition. We only note that any point in the set $I \cap \partial(\mathrm{ran}(\mathcal{E}))$ cannot be a regular point. $\square$

The main computation in the proof of the following theorem will appear in the proof of our main theorem in the next section, and therefore it is useful to understand it in this simplified toy model.

Theorem 3.3. *Let $\lambda \in \sigma_{\mathrm{reg},I}(A)$. The density of states of A at λ_0 is estimated as*

$$\left| \frac{\mathrm{d}}{\mathrm{d}\lambda}(E(\lambda)f, g)_{\mathcal{H}} \right| \leq \left(\sum_{m \in M_\lambda} \frac{1}{|\mathcal{E}'(m)|} \right) \|f\|_{\mathcal{H}} \|g\|_{\mathcal{H}}. \tag{3.1}$$

Proof. Since $M = [0, 1]$ is compact, $\mathcal{E} \in C^1$, and λ_0 is a regular point, M_λ is a finite set and we denote its elements $M_\lambda = \{m_1, \ldots, m_k\}$. Recall that the spectrum of $A(m)$ in $I = (a, b)$ is made up of a single eigenvalue $\mathcal{E}(m)$ for all $m \in M$. Hence if $\lambda \in I$ and $\mathcal{E}(m) \leq \lambda$ then the projection operator $E_m(\lambda)$ may be represented as $P_m(\mathcal{E}(m)) + E_m(a)$ where $P_m(\mathcal{E}(m))$ is the projection onto the eigenspace in $\mathcal{H}'$ (the mth "copy") corresponding to the eigenvalue $\mathcal{E}(m)$. If $\lambda \in I$ and $\mathcal{E}(m) > \lambda$ then the projection operator $E_m(\lambda)$ is equal to the projection operator $E_m(a)$. Letting $f, g \in \mathcal{H}$, we have

$$\begin{aligned}
(E(\lambda)f, g)_{\mathcal{H}} &= \int_{[0,1]} (E_m(\lambda)f_m, g_m)_{\mathcal{H}'} \, \mathrm{d}m \\
&= \int_{\{m \,:\, \mathcal{E}(m) \leq \lambda\}} (P_m(\mathcal{E}(m))f_m, g_m)_{\mathcal{H}'} \, \mathrm{d}m + \int_{[0,1]} (E_m(a)f_m, g_m)_{\mathcal{H}'} \, \mathrm{d}m.
\end{aligned}$$

Differentiating in λ the second term on the right hand side is eliminated, and one is left with

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(E(\lambda)f,g)_{\mathcal{H}} = \lim_{h\downarrow 0} \left[\frac{1}{h} \int_{\{m \,:\, \lambda < \mathcal{E}(m) \le \lambda+h\}} (P_m(\mathcal{E}(m))f_m, g_m)_{\mathcal{H}'} \, \mathrm{d}m \right]$$

$$= \sum_{i,\ \mathcal{E}'(m_i)>0} \lim_{h\downarrow 0} \left[\frac{1}{h} \int_{(m_i, m_i + \frac{h}{\mathcal{E}'(m_i)}]} (P_m(\mathcal{E}(m))f_m, g_m)_{\mathcal{H}'} \, \mathrm{d}m \right]$$

$$+ \sum_{i,\ \mathcal{E}'(m_i)<0} \lim_{h\downarrow 0} \left[\frac{1}{h} \int_{(m_i + \frac{h}{\mathcal{E}'(m_i)}, m_i]} (P_m(\mathcal{E}(m))f_m, g_m)_{\mathcal{H}'} \, \mathrm{d}m \right].$$

Making the change of variables $\eta_i = \frac{h}{|\mathcal{E}'(m_i)|}$ we have

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(E(\lambda)f,g)_{\mathcal{H}} \le \sum_{i,\ \mathcal{E}'(m_i)>0} \frac{1}{\mathcal{E}'(m_i)} \lim_{\eta_i\downarrow 0} \left[\frac{1}{\eta_i} \int_{(m_i, m_i+\eta_i]} (P_m(\mathcal{E}(m))f_m, g_m)_{\mathcal{H}'} \, \mathrm{d}m \right]$$

$$+ \sum_{i,\ \mathcal{E}'(m_i)<0} \frac{1}{|\mathcal{E}'(m_i)|} \lim_{\eta_i\downarrow 0} \left[\frac{1}{\eta_i} \int_{(m_i-\eta_i, m_i]} (P_m(\mathcal{E}(m))f_m, g_m)_{\mathcal{H}'} \, \mathrm{d}m \right]$$

$$= \sum_{i=1}^{k} \frac{1}{|\mathcal{E}'(m_i)|} \left(P_{m_i}(\mathcal{E}(m_i))f_{m_i}, g_{m_i} \right)_{\mathcal{H}'}. \tag{3.2}$$

Note that one must be careful if one of the m_i is 0 (resp. 1) and $\mathcal{E}'(0) < 0$ (resp. $\mathcal{E}'(1) > 0$) as then the above argument requires a slight adjustment. However the same conclusion holds. A simple use of the Cauchy-Schwartz inequality and the properties of projection operators leads to the desired estimate (3.1). $\qquad\square$

4. Flow in an annulus

4.1. **Description and assumptions.** We are now ready to study our main object of interest, a flow in an annulus. We consider a steady flow in an annulus $\mathcal{A} = [0,1] \times \mathbb{S}^1$. The variables in $[0,1]$ and $\mathbb{S}^1$ shall be denoted m and θ, respectively. For each $m \in (0,1)$ corresponds the self-adjoint operator

$$A(m) = -\frac{2\pi i}{T(m)} \frac{\mathrm{d}}{\mathrm{d}\theta} : H^1(\mathbb{S}^1) \subset L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1) \tag{4.1}$$

generating a flow along the circle $\mathbb{S}^1$ with period $T(m) \in (0,\infty)$. Its spectrum is given by

$$\sigma(A(m)) = \frac{2\pi}{T(m)}\mathbb{Z}. \tag{4.2}$$

The endpoints 0 and 1 correspond to the boundary of the annulus, where the flow may degenerate: i.e. it is possible for $T(m) \to +\infty$ as $m \to 0$ or 1. Assuming that $T \in C((0,1); (0,\infty))$ we can characterize the spectrum of $A = \int_{[0,1]}^{\oplus} A(m) \, \mathrm{d}m$ as

$$\sigma(A) = \left(\bigcup_{-k\in\mathbb{N}} \left[\frac{2\pi k}{T_{\min}}, \frac{2\pi k}{T_{\max}} \right] \right) \cup \left(\bigcup_{k\in\mathbb{N}\cup\{0\}} \left[\frac{2\pi k}{T_{\max}}, \frac{2\pi k}{T_{\min}} \right] \right) \tag{4.3}$$

where $T_{\min} := \inf_{(0,1)} T(m)$ and $T_{\max} := \sup_{(0,1)} T(m)$. For more details on how to derive this expression see for instance [Cox14]. We are particularly interested in the effect of *slow*

flow lines, i.e. when $T_{\max} = +\infty$ and $\sigma(A) = \mathbb{R}$ (there is no spectral gap). We therefore make the following assumption:

Assumption A2. The flow degenerates at $m = 0$: there exists $\alpha > 0$ such that $T(m)$ is of the form

$$T(m) = m^{-\alpha} \qquad \text{as } m \downarrow 0.$$

Without loss of generality, we assume that T is continuous at $m = 1$ and attains some finite value $T(1)$.

Contributions to the spectrum at energy level $\lambda > 0$[2] will come from all $m_{\lambda,k} \in (0, 1)$ such that $\lambda = \frac{2\pi k}{m_{\lambda,k}^{-\alpha}}$ where $k \in \mathbb{N}$. That is,

$$m_{\lambda,k} = \left(\frac{\lambda}{2\pi k} \right)^{1/\alpha}, \qquad k \in \mathbb{N}, \tag{4.4}$$

and conversely, each fiber $m \in (0, 1)$ will contribute to the discrete energy levels

$$\lambda_{m,k} = 2\pi m^{\alpha} k, \qquad k \in \mathbb{N}. \tag{4.5}$$

4.2. **Functional setting.** Before stating our main theorem, we define the functional setting. In the θ variable, periodicity means we consider the usual homogeneous Sobolev space defined through Fourier series by

$$\dot{H}^{\gamma} := \left\{ f \in L^2(\mathbb{S}^1) \, : \, \|f\|_{\dot{H}^{\gamma}}^2 := \sum_{k \in \mathbb{Z}} |k|^{2\gamma} |\widehat{f}(k)|^2 < +\infty \right\}$$

where $\gamma \geq 0$.

Concerning the regularity with respect to the m variable in the open interval $(0, 1)$, we use the following fractional Sobolev setting. We consider functions $f \in L^2(\mathcal{A})$ as fibered in m, as in (2.3):

$$f = \int_{[0,1]}^{\oplus} f_m \, \mathrm{d}m \qquad \text{with} \qquad f_m \in \dot{H}^{\gamma}.$$

We define then, for $s \in (0, 1)$ and $\gamma \geq 0$, the space

$$\mathcal{H}^{s,\gamma} := \left\{ f \in L^2\left((0,1); \dot{H}_{\theta}^{\gamma} \right) \, : \, \frac{\|f_m - f_{m'}\|_{\dot{H}^{\gamma}}}{|m - m'|^{1/2+s}} \in L^2\left((0,1) \times (0,1) \right) \right\}.$$

The norm is defined by

$$\|f\|_{s,\gamma}^2 = \iint_{(0,1) \times (0,1)} \frac{\|f_m - f_{m'}\|_{\dot{H}^{\gamma}}^2}{|m - m'|^{1+2s}} \, \mathrm{d}m \, \mathrm{d}m'. \tag{4.6}$$

One important result is the following Sobolev embedding-type theorem:

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[2]Here we choose to consider positive energy levels $\lambda > 0$ and consequently $k \in \mathbb{N}$. We could have chosen $\lambda < 0$ and then $-k \in \mathbb{N}$.

Proposition 4.1. *Let $s > 1/2$. Then:*

$$\|f_m - f_{m'}\|_{\dot{H}^\gamma} \leq C(s)|m - m'|^{s-1/2}, \qquad \forall m, m' \in (0,1), \forall f \in \mathcal{H}^{s,\gamma} \tag{4.7}$$

where $C(s) > 0$ is independent of m and f.

We refer to the paper of Simon [Sim90] and in particular Corollary 26 therein. We may now define the space in which we will work:

Definition 4.2. *Let $s > 1/2, \gamma \geq 0$. We define $\mathcal{H}_0^{s,\gamma}$ to be the set of functions on the annulus that are constant along $\{m = 0\}$, have s derivatives in the m variable, and γ derivatives in the θ variable:*

$$\mathcal{H}_0^{s,\gamma} = \left\{ f \in L^2(\mathcal{A}) : f \in \mathcal{H}^{s,\gamma} \quad and \quad \|f_0\|_{\dot{H}^\gamma} = 0 \right\}. \tag{4.8}$$

We associate to this subspace of $\mathcal{H}^{s,\gamma}$ the norm $\|f\|_{s,\gamma}$ defined by (4.6).

We state three important properties of this L^2 subspace:

Proposition 4.3. *The set $\mathcal{H}_0^{s,\gamma}$ is a closed L^2 subspace of $\mathcal{H}^{s,\gamma}$.*

Proposition 4.4. *Since $s > 1/2$, the norm $\|f_m\|_{\dot{H}^\gamma}$ decays as $m \downarrow 0$, as there holds*

$$\|f_m\|_{\dot{H}^\gamma} \leq C(s) \|f\|_{s,\gamma} \, m^{s-1/2}, \qquad \forall m \in (0,1), \forall f \in \mathcal{H}_0^{s,\gamma}. \tag{4.9}$$

We state finally a result on the Lipschitz regularity of the Fourier coefficients of functions in $\mathcal{H}^{s,\gamma}$:

Proposition 4.5. *For any $f \in \mathcal{H}_0^{s,\gamma}$, there holds*

$$\left| \widehat{f}_m(k) - \widehat{f}_{m'}(k) \right| \leq C\|f\|_{s,\gamma}|k|^{-\gamma}|m - m'|^{s-1/2}, \qquad \forall m, m' \in (0,1), \forall k \in \mathbb{Z}, \tag{4.10}$$

where the constant $C > 0$ does not depend on f, k nor m.

Proof. We compute

$$\left| \widehat{f}_m(k) - \widehat{f}_{m'}(k) \right|^2 \leq |k|^{-2\gamma} \sum_{k' \in \mathbb{Z}} \left| \widehat{f}_m(k') - \widehat{f}_{m'}(k') \right|^2 \leq |k|^{-2\gamma} \|f_m - f_{m'}\|_{\dot{H}^\gamma}^2.$$

By the Hölder continuity (4.7) of f_m in the $\dot{H}^\gamma$ norm, we obtain (4.10). $\qquad \square$

4.3. **Main result.** We may now state our main result concerning the density of states of the operator A:

Theorem 4.6. *Let A be the operator defined in* (1.1) *satisfying Assumption A2. Further assume that $s > 1/2$ and $\gamma \geq 0$ satisfy the constraint*

$$\gamma + s/\alpha > 1/2. \tag{4.11}$$

Then there exists $r > 0$ such that the density of states of A satisfies

$$\left| \frac{\mathrm{d}}{\mathrm{d}\lambda} (E(2\pi\lambda)f, g)_{L^2(\mathcal{A})} \right| \leq C \left\| f \right\|_{s,\gamma} \left\| g \right\|_{s,\gamma} |\lambda|^{2s/\alpha - 1}, \qquad \forall \lambda \in (-r, r) \backslash \{0\}, \forall f, g \in \mathcal{H}_0^{s,\gamma}, \tag{4.12}$$

where $C > 0$ does not depend on f, g or λ.

Remark 4.7. In the case $\alpha \leq 1$ the constraint (4.11) is satisfied for all $s > 1/2$ with $\gamma = 0$. This means that for $\alpha \leq 1$, the subspace $\mathcal{H}_0^{s,0}$ is sufficient to get the estimate of the density of states.

In the case $\alpha \geq 1$ however, the constraint (4.11) is stronger and working in the subspace $\mathcal{H}_0^{s,0}$ is not sufficient to obtain estimates of the density of states. This is mainly due to the fact that the eigenvalues $\lambda_{m,k} = 2\pi k m^\alpha$ concentrate at 0 faster as $m \to 0$ and $k \to \infty$. To balance that effect, more regularity in the θ is required, as the equation $\lambda = \lambda_{m,k}$ allows to trade control of high frequencies in smallness in m.
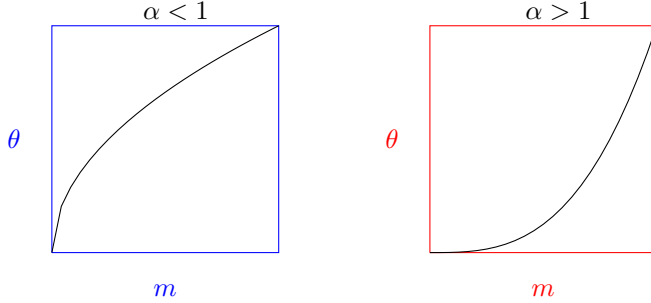
Note also that the power of λ_0 does not depend on γ, the parameter measuring regularity in the θ variable. In particular, there is no threshold $\alpha = 1$ regarding the estimate (4.12) of the density of states, neither regarding the rate of convergence in the ergodic theorem. This is physically pertinent.

Proof of Theorem 4.6. As in the case of the toy model presented in Section 3, we write

$$(E(\lambda)f, g)_{L^2(\mathcal{A})} = \int_{[0,1]} (E_m(\lambda)f_m, g_m)_{L^2(\mathbb{T})} \, \mathrm{d}m$$

$$= \int_{[0,1]} \sum_{\substack{k \in \mathbb{N} \\ \lambda_{m,k} \leq \lambda}} (P_m(\lambda_{m,k})f_m, g_m)_{L^2(\mathbb{T})} \, \mathrm{d}m$$

where $P_m(\lambda_{m,k})$ is the projection on the Fourier coefficient $\widehat{f_m}(k)$ of f_m:

$$\widehat{f_m}(k) = \int_{[0,1]} f_m(\theta)e^{-i2\pi k\theta}\,\mathrm{d}\theta.$$

As before, we compute the density of states starting from the definition of a derivative, i.e. the limit of $\frac{1}{h}\left((E(2\pi\lambda + h)f, g)_{L^2(\mathcal{A})} - (E(2\pi\lambda)f, g)_{L^2(\mathcal{A})}\right)$ as $h \to 0$. We therefore define the energy band

$$\mathcal{B}(\lambda, m, h) = \{k \in \mathbb{Z} : \lambda < km^\alpha \leq \lambda + h\}$$

and have that

$$\frac{1}{h}\left((E(2\pi\lambda_0 + h)f, g) - (E(2\pi\lambda_f, g))\right) = \frac{1}{2\pi h}\int_{[0,1]}\sum_{k\in\mathcal{B}(\lambda,m,h)}(P_m(\lambda_{m,k})f_m, g_m)_{L^2(\mathbb{S}^1)}\,\mathrm{d}m$$

$$= \frac{1}{2\pi h}\int_{[0,1]}\sum_{k\in\mathcal{B}(\lambda,m,h)}\widehat{f_m}(k)\overline{\widehat{g_m}(k)}\,\mathrm{d}m.$$

We would now like to commute the integration and summation, to obtain the sum of integrals over small subintervals of $[0, 1]$, much like in the computations in (3.2). However, for fixed h the sets $\mathcal{B}(\lambda_0, m, h)$ will contain arbitrarily many elements as $m \to 0$. Moreover, as $m \to 0$ the subintervals are no longer disjoint, resulting in many redundant integrations. Indeed, intervals $(\lambda/m^\alpha, (\lambda + h)/m^\alpha]$ are of size larger than one if $m < h^{1/\alpha}$, and then may contain more than one integer. Our strategy is to split the previous integral into two parts

$$\frac{1}{2\pi h}\int_{[0,1]}\sum_{k\in\mathcal{B}(\lambda,m,h)}\widehat{f_m}(k)\overline{\widehat{g_m}(k)}\,\mathrm{d}m = \mathcal{I}(h) + \mathcal{R}(h)$$

where

$$\mathcal{I}(h) = \frac{1}{2\pi h}\int_{[h^{1/\alpha},1]}\sum_{k\in\mathcal{B}(\lambda,m,h)}\widehat{f_m}(k)\overline{\widehat{g_m}(k)}\,\mathrm{d}m$$

and

$$\mathcal{R}(h) = \frac{1}{2\pi h}\int_{[0,h^{1/\alpha})}\sum_{k\in\mathcal{B}(\lambda,m,h)}\widehat{f_m}(k)\overline{\widehat{g_m}(k)}\,\mathrm{d}m.$$

1. The term $\mathcal{R}(h)$. We show that $\lim_{h\to 0}\mathcal{R}(h) = 0$. Using the inequalities

$$\left|\sum_{k\in\mathcal{B}(\lambda,m,h)}\widehat{f_m}(k)\overline{\widehat{g_m}(k)}\right| \leq \sum_{k\in\mathcal{B}(\lambda,m,h)}\left|\widehat{f_m}(k)\overline{\widehat{g_m}(k)}\right| \leq \sum_{k>\lambda/m^\alpha}\left|\widehat{f_m}(k)\overline{\widehat{g_m}(k)}\right|$$

and taking $f, g \in \mathcal{H}_0^{s,\gamma}$ we have

$$|\mathcal{R}(h)| \leq \frac{1}{2\pi h}\int_{[0,h^{1/\alpha})}\sum_{k>\lambda/m^\alpha}|k|^{-2\gamma}\left(|k|^\gamma\left|\widehat{f_m}(k)\right|\right)\left(|k|^\gamma\left|\overline{\widehat{g_m}(k)}\right|\right)\mathrm{d}m$$

$$\leq \frac{1}{2\pi h}\int_{[0,h^{1/\alpha})}\sup_{k>\lambda/m^\alpha}\left(|k|^{-2\gamma}\right)\|f_m\|_{\dot{H}^\gamma}\|g_m\|_{\dot{H}^\gamma}\,\mathrm{d}m$$

using the Hölder inequality for the last line. We use now inequality (4.9) that gives decay as $m \to 0$ for the norm $\|f_m\|_{\dot{H}^\gamma}$ and compute then

$$
\begin{aligned}
|\mathcal{R}(h)| &\leq \frac{1}{2\pi h} \int_{[0,h^{1/\alpha})} \left(\frac{\lambda}{m^\alpha}\right)^{-2\gamma} \|f_m\|_{\dot{H}^\gamma} \|g_m\|_{\dot{H}^\gamma} \, \mathrm{d}m \\
&\leq C^2 \|f\|_{s,\gamma} \|g\|_{s,\gamma} \frac{\lambda^{-2\gamma}}{2\pi h} \int_{[0,h^{1/\alpha})} m^{2\alpha\gamma + 2(s-1/2)} \, \mathrm{d}m \\
&\leq C' \|f\|_{s,\gamma} \|g\|_{s,\gamma} \frac{\lambda^{-2\gamma}}{2\pi h} h^{2\gamma + 2(s-1/2)/\alpha + 1/\alpha} \\
&\lesssim \|f\|_{s,\gamma} \|g\|_{s,\gamma} \frac{\lambda^{-2\gamma}}{2\pi} h^{2\gamma + 2s/\alpha - 1}.
\end{aligned}
$$

As soon as the constraint (4.11) on s and γ is satisfied there holds

$$
\lim_{h \to 0} \mathcal{R}(h) = 0. \tag{4.13}
$$

2. The term $\mathcal{I}(h)$. As $m \geq h^\alpha$, the energy band $\mathcal{B}(\lambda, m, h)$ contains at most one integer. For $\mathcal{B}(\lambda, m, h)$ to contain one integer, by definition m has to be in an interval of the form $((\lambda/k')^{1/\alpha}, ((\lambda+h)/k')^{1/\alpha}]$ for some $k' \in \mathbb{N}^\star$. As $m \in [h^{1/\alpha}, 1]$, the integer k' has to satisfy the bounds

$$
N(\lambda, h) := \lfloor (\lambda+h)/h \rfloor \geq k' \geq k_0 := \max(1, \lfloor \lambda \rfloor).
$$

We write then

$$
\begin{aligned}
\mathcal{I}(h) &= \frac{1}{2\pi h} \int_{[h^{1/\alpha}, 1]} \sum_{k \in \mathcal{B}(\lambda, m, h)} \widehat{f_m}(k) \overline{\widehat{g_m}(k)} \, \mathrm{d}m \\
&= \frac{1}{2\pi h} \sum_{N(\lambda, h) \geq k' \geq k_0} \int_{((\lambda/k')^{1/\alpha}, ((\lambda+h)/k')^{1/\alpha}]} \sum_{k \in \mathcal{B}(\lambda, m, h)} \widehat{f_m}(k) \overline{\widehat{g_m}(k)} \, \mathrm{d}m \\
&= \frac{1}{2\pi h} \sum_{N(\lambda, h) \geq k' \geq k_0} \int_{((\lambda/k')^{1/\alpha}, ((\lambda+h)/k')^{1/\alpha}]} \widehat{f_m}(k') \overline{\widehat{g_m}(k')} \, \mathrm{d}m.
\end{aligned}
$$

Next, we use the same kind of computation as in the proof of Theorem 3.3. For that we use in particular inequality (4.10) in Proposition 4.5 which ensures some uniform regularity for the Fourier coefficients. This leads to

$$
\lim_{h \to 0} \frac{1}{h} \int_{((\lambda/k)^{1/\alpha}, ((\lambda+h)/k)^{1/\alpha}]} \widehat{f_m}(k) \overline{\widehat{g_m}(k)} \, \mathrm{d}m = \frac{1}{\alpha k} \left(\frac{\lambda}{k}\right)^{1/\alpha - 1} \widehat{f}_{(\lambda/k)^{1/\alpha}}(k) \overline{\widehat{g}_{(\lambda/k)^{1/\alpha}}(k)}
$$

which gives

$$
\lim_{h \to 0} \mathcal{I}(h) = \frac{1}{2\pi} \sum_{k \geq k_0} \frac{1}{\alpha k} \left(\frac{\lambda}{k}\right)^{1/\alpha - 1} \widehat{f}_{(\lambda/k)^{1/\alpha}}(k) \overline{\widehat{g}_{(\lambda/k)^{1/\alpha}}(k)}.
$$

by uniform boundedness. To conclude we use inequality (4.10) and the fact that f_0 is trivial in $\dot{H}^\gamma$ to obtain

$$
\left| \widehat{f}_{(\lambda/k)^{1/\alpha}}(k) \right| \leq C \|f\|_{s,\gamma} |k|^{-\gamma} \left| (\lambda/k)^{1/\alpha} \right|^{s-1/2}
$$

with the constant $C > 0$ independent of k and λ. There holds now for $\lim_{h\to 0} \mathcal{I}(h)$ the bound

$$
\begin{aligned}
\left| \lim_{h\to 0} \mathcal{I}(h) \right| &= \frac{1}{\alpha 2\pi} \left| \sum_{k\geq k_0} \frac{1}{k} \left(\frac{\lambda}{k} \right)^{1/\alpha-1} \widehat{f}_{(\lambda/k)^{1/\alpha}}(k) \overline{\widehat{g}_{(\lambda/k)^{1/\alpha}}(k)} \right| \\
&\leq C \, \|f\|_{s,\gamma} \, \|g\|_{s,\gamma} \, \frac{\lambda^{1/\alpha-1}}{2\pi\alpha} \sum_{k\geq k_0} k^{-1/\alpha-2\gamma} \left(\frac{\lambda_0}{k} \right)^{2(s-1/2)/\alpha} \\
&\leq C \, \|f\|_{s,\gamma} \, \|g\|_{s,\gamma} \, \frac{\lambda^{2s/\alpha-1}}{2\pi\alpha} \sum_{k\geq k_0} k^{-(1/\alpha+2\gamma+2(s-1/2)/\alpha)} \\
&\leq C \, \|f\|_{s,\gamma} \, \|g\|_{s,\gamma} \, \frac{\lambda^{2s/\alpha-1}}{2\pi\alpha} \sum_{k\geq 1} k^{-(2\gamma+2s/\alpha)}
\end{aligned}
$$

where we use $k_0 \geq 1$ in the last inequality. The sum on k is then finite if the constraint (4.11) on s and γ is satisfied. This suffices to end the proof. $\qquad\square$

4.4. **Uniform ergodic theorem.** We use now the main result of [BAM19] to immediately deduce a rate of convergence for the associated ergodic theorem:

Theorem 4.8. *The operators P^T and P defined in (1.2) and (1.3) respectively have the uniform rate of convergence*

$$
\|P^T - P\|_{\mathcal{H}_0^{s,\gamma} \to (\mathcal{H}_0^{s,\gamma})^*} \lesssim T^{-\frac{s}{s+\alpha}}.
$$

The proof of the Theorem uses the polynomial bound of the density of states obtained in Theorem 4.6 and Corollary 1.5 in [BAM19]. Note that the projection P writes, thanks to the Fourier coefficients of f, as

$$
Pf = \widehat{f}_m(0).
$$

It is hence natural to work in spaces $\mathcal{H}_0^{s,\gamma}$ rather than in spaces $\mathcal{H}^{s,\gamma}$. We follow up by a few remarks on Theorem 4.8:

Remark 4.9. 1. Note that the rate does not depend on γ. Note also that, even if there is a threshold regarding α less or greater than 1 in the regularity constraint (4.11) (see Remark 4.7), there is no such thing regarding the exponent $-\frac{s}{s+\alpha}$.

2. As $\alpha \downarrow 0$ the exponent tends to 1 which is the rate of convergence in the case of a spectral gap. Intuitively, smaller α brings us closer to the case of a spectral gap in the sense that there are "fewer" slow trajectories.

3. We also observe that $\frac{s}{s+\alpha} \to 1$ as $s \to +\infty$. Informally, this expresses the fact that "infinite regularity" in the m variable gives the rate of convergence of the spectral gap case.

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