Imperial College London

Department of Mathematics

Synthesis report

Topics in existence and uniqueness of solutions to Vlasov systems in Kinetic Theory

Author:
Charafeddine MOUZOUNI

Supervisor:
Dr. Jonathan BEN-ARTZI

Abstract

This report presents with details the most known results of existence and uniqueness for Vlasov-Poisson and Vlasov Maxwell systems, in kinetic theory. The first section gives general results about transport equations, which are used to study the linearized equation. The second section contains some properties of Vlasov-Poisson system, and Pfaffelmoser/Schaeffer's proof of existence and uniqueness of a global classical solution for this system. The aim of the last section is to introduce the Vlasov-Maxwell system, and Wollman's local existence and uniqueness theorem. It also provides the details for the proof of Glassey/Strauss's theorem, on global time existence and uniqueness for Vlasov-Maxwell system.

For corrections and suggestions : charafeddine.mouzouni@gmail.com.

Acknowledgments

This project would not have been possible without the support of many people. I would like to express my special thanks of gratitude to my supervisor Mr Jonathan BEN-ARTZI, research fellow at Imperial College London, for his time, his help, his support, his hospitality, genoristy and his advices. I also sincerely thank Mr. Francis Filbet, Professor of Mathematics at the university of Lyon and Mr. Jose Carillo, professor of Mathematics at Imperial College London, who gave me the golden opportunity to do this wonderful project and to be a part of such a rewarding experience. I also thank my teachers at the Ecole Cenrale de Lyon, Prof. Gregory Vial and Prof. Martine Marion, for their support, their time and their advices. Thanks to Imperial College London for providing me everything I needed to complete this project.
And finally, thanks to my soul sister who endured this long process with me, always offering support and love.

Contents

Introduction

Let's take a system of identical point particles, and assume that the total number of particles per unit volume is enough large. In that case we can assume that the distribution of particles is smooth, and the system can be completely described using a smooth distribution function that characterizes the density of particles. Think of a gas or a fluid for example. We all know that a gas is composed of a very large number of molecules. Their number in one mole is in the order of magnitude of 10^{23}. Therefore we usually describe a gas with a smooth distribution function. The general purpose of *kinetic theory* is to describe the dynamics of a system of N identical particles in the limite that N tends to infinity in statistical way. It describes the evolution in time of the distribution of particles in phase space of position and velocity at the mesoscopic scale, which is an intermediate scale between the microscopic and the macroscopic one. In this point of view, the most important thing to know is how many particles do something, and not what each one does.
At the instant t, we define the distribution function $f(t,.,.)$ from the phase space $\Omega_x \times \Omega_v$ in $\mathbb{R}^+$ such that, Ω_x, $\Omega_v \subset \mathbb{R}^d$ and Ω_v and Ω_x are respectively the set of possible velocities and the set of possible positions. The integral of f over any region of phase space gives the proportion of particles which, at that instant of time, have phase space coordinates in that region. Therefore, a basic general requirement is that Ω_x, Ω_v are measurable sets, and for each $t \geq 0$:

$$f(t,.,.) \in L^1_{loc}(\Omega_x \times \Omega_v)$$

In general $\Omega_v = \mathbb{R}^d$. In some problems with boundaries we can take $\Omega_x \subsetneq \mathbb{R}^d$, but we need to precise boundary conditions on $\partial\Omega_x$ which is supposed to be smooth enough. The main boundary conditions studied are :

- Specular reflection : particles bounce off of the boundary

$$\forall x \in \partial\Omega_x,\ \ f(t,x,v) = f(t,x,v - 2(v.n(x))n(x))$$

  with $n(x)$ is the normal on $\partial\Omega_x$ in x.

- Diffusive condition : particles diffuse when they hit the boundary.

$$\forall x \in \partial\Omega_x, n.v < 0,\ \ (\int_{n.u>0} f(t,x,u)du) \times M(v)$$

  with $M(v)$ is some fixed Gaussian. For this condition, f has to be regular enough so that the integral has a sens.

- Periodic boundary conditions : we can require that f is periodic in the space variable "x".

In all this report, we assume that :

$$\Omega_x = \mathbb{R}^d \ ,\ \Omega_v = \mathbb{R}^d$$

To derive an equation satisfied by the distribution function f, we shall start from the Newtonian viewpoint. Let's start with a system of N identical particles and assume that each particle interact with the others through a potential ψ, depending only on the distance between the two bodies, and an external force with some potential ϕ depending only on time and the position of the particles. At the instant $t \geq 0$ we note:

$$Z(t) := (z_i(t))_{1\leq i\leq N} = (x_i(t), v_i(t))_{1\leq i\leq N} = (X(t), V(t))$$

the coordinates of each particle in phase space. Then, with a weight factor equal to 1, the Hamiltonian's formulation of Newtonian mechanics gives for each particle $i \in \{1, .., N\}$:

$$\begin{cases} \dot{x}_i &= \partial_{v_i} H \\ \\ \dot{v}_i &= -\partial_{x_i} H \end{cases} \tag{0.1}$$

with

$$H := \underbrace{\sum_{i=1}^{N} \frac{{v_i}^2}{2}}_{\text{kinetic energy}} + \underbrace{\sum_{i<j} \psi(x_i - x_j)}_{\text{interaction energy}} + \underbrace{\sum_{i=1}^{N} \phi(t, x_i)}_{\text{potential energy}}$$

the corresponding Newton's equations are:

$$\begin{cases} \dot{x}_i &= v_i \\ \dot{v}_i &= -\sum_{i\neq j} \nabla_x \psi(x_i - x_j) - \nabla_x \phi(x_i) \end{cases} \tag{0.2}$$

We assume that all the functions are enough regular so that the derivatives here have a sens[1]. Instead of considering a microscopic description, let's shift now to a statistical viewpoint. Let's consider X and V as a random variables. The probability distribution that defines their simultaneous behavior is called a joint probability distribution. Let $F_N(t, X, V)$ be a joint distribution function of all the particles at the instant t. According to **Liouville's theorem** , this distribution function should be preserved along the particle trajectories :

$$\forall t \geq 0, \quad F_N(t, X(t), V(t)) = F_N(0, X(0), V(0))$$

Hence, writing this condition on the following form:

$$\frac{d}{dt} F_N(t, X(t), V(t)) = 0$$

we get the equation satisfied by F_N, using (0.1) :

$$\partial_t F_N + \sum_{i=1}^{N} (\partial_{v_i} H . \partial_{x_i} F_N - \partial_{x_i} H . \partial_{v_i} F_N) = 0$$

This equation gives a description of all the particles of the system, however we still having a microscopic description of the system. To change the scale of description,

[1]Other assumptions on regularity are made in the next steps, see [11].

we exclude a part of the information by considering the distribution function f with less information:

$$\forall t \geq 0, \quad f(t,x,v) := \int_{(\mathbb{R}^d\times\mathbb{R}^d)^{N-1}} F_N(t,X,V)dx_2dx_3...dx_Ndv_2...dv_N$$

We consider F_N to be symmetric under permutations of particles indexes. Hence, we can choose the first variable in the expression above, without loss of generality. Using the equation satisfied by F_N we can derive an equation satisfied[2] by f :

$$\partial_t f + v.\nabla_x f - \partial_x\phi.\nabla_v f - (N-1)\int \partial_x(\psi(x-y))\nabla_v f^{(2)}(t,x,y,v,w)dydw \quad (0.3)$$

with

$$\forall t \geq 0, \quad f^{(2)}(t,x,y,v,w) := \int_{(\mathbb{R}^d\times\mathbb{R}^d)^{N-2}} F_N(t,X,V)dx_3...dx_Ndv_3...dv_N$$

Now we would like to take the limit when N tends to infinity in the equation (0.3), and we would like to write:

$$f^{(2)}(t,x,y,v,w) = f(t,x,v)f(t,y,w) \quad (0.4)$$

this would be the natural guess if the particle are not too much coupled. To do the passage to the limit rigorously, we need to make some assumptions, and we arrive at deferent models. This is how in general, we derive kinetic equations from physics.

The mean field model is well adapted to electromagnetic and gravitational forces, which are studied in this report. In this point of view, we don't try to describe each binary interaction, but only their collective effect. Mathematically, we let

$$\psi = \frac{1}{N}\bar{\psi}$$

and make the decorelation assumption (0.4). Then, we can obtain the following equation[3]:

$$\partial_t f + v.\nabla_x f - \partial_x\phi.\nabla_v f - (\int \partial_x(\bar{\psi}(x-y))f(y,w)dydw).\nabla_v f(x,v) = 0 \quad (0.5)$$

Hence, in general we have that :

$$\partial_t f + v.\nabla_x f + F.\nabla_v f = 0$$

with F is the superposition of the internal force of binary interactions and external forces. In general, F is a function of t, x and v. This equation is usually called the **Liouville equation** or **Vlasov equation**[4], especially in the case where the force is

[2] The details of this proof are given in [11, Chapter 2].

[3] This is not a proof, this steps only shows in general how we find kinetic equations.

[4] See [16].

depending on the unknown f. In general f is not observable, however we can link f with the following macroscopic quantities that we can measure. For fixed x in $\mathbb{R}^d$:

$$\begin{aligned}
\text{Macroscopic density} \quad \rho &= \int f(t,x,v)dv \\
\text{Mean velocity} \quad \rho u &= \int f(t,x,v)vdv \\
\text{Temperature} \quad d\rho T &= \int f(t,x,v)|v-u|^2dv \\
\text{Local entropy} \quad \rho S &= -\int f(t,x,v)\log(f(t,x,v))dv
\end{aligned}$$

In this report, we are studying the following initial value problem for different models.

$$\begin{cases} \partial_t f + v.\nabla_x f + F.\nabla_v f = 0 \\ f_{|t=0} = f_0 \end{cases} \tag{0.6}$$

The Newton's equations associated to this system are :

$$\begin{cases} \dot{x}(t) &= v(t) \\ \\ \dot{v}(t) &= F(t,x(t),v(t)) \end{cases}$$

In the first section, we are studying the behavior of the system with only an external known force. In the second and third section we are studying two famous Vlasov systems : Vlasov-Poisson and Vlasov-Maxwell.

1 Transport equations

In this section we will give general results about the transport equations and the method of characteristics, which is the key of the study of such equations. The general theory of the transport equations will allow us to give solutions for the system (0.6) in the simple case where the force F is known, and not depending on f. In general, this is not true, in particular for the Vlasov systems studied in this report. The study of this simple case gives a lot of information that will be very useful for the general case, and most of the interesting properties remain satisfied for the general cases. The results given in this section are mostly taken from [8], [11], [3] and [12].

1.1 The free transport equation

At first, let's see what happens in the simple case when the force F is equal to zero. This equation is called the equation of free transportation. In this case the system (0.6) is written on the following form:

$$\begin{cases} \partial_t f + v.\nabla_x f = 0 \\ f_{|t=0} = f_0 \end{cases} \tag{1.1}$$

This is a fundamental kinetic system that describes the behavior of a system of free particles without any collisions or forces. The resolution of the Newton's equations associated to (1.1) gives:

$$\forall t \in \mathbb{R}^+,\ v(t) = v(0) \text{ and } x(t) - x(0) = v(0)t$$

We can notice that if a function f is a solution for (1.1) then f is constant along the trajectories. In fact,

$$\frac{d}{dt} f(t, x(t), v(t)) = \partial_t f + \dot{x}(t).\nabla_x f = \partial_t f + v(t).\nabla_x f$$

Let f be a solution for (1.1); f being constant along the trajectories, it is easy to compute f at time t in terms of f at time 0:

$$\forall (t, x, v) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d,\ f(t, x, v) = f(0, x - vt, v) = f_0(x - vt, v) \tag{1.2}$$

Thus if f_0 is regular enough the previous expression gives a unique solution for (1.1). This is the motivation for the following proposition.

Proposition 1.1 *If $f_0 \in \mathcal{C}^1(\mathbb{R}^d \times \mathbb{R}^d)$, then*

$$(t, x, v) \longmapsto f_0(x - vt, v)$$

is the unique solution for (1.1) in $\mathcal{C}^1(\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d)$.

Proof 1.1 • Uniqueness: *if u is a solution for (1.1), u has to be constant along the trajectories. Thus :*

$$\forall (t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d,\ u(t,x,v) = u(0,x-vt,v) = f_0(x-vt,v)$$

- Existence: *We define for $(t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d,\ f(t,x,v) = f_0(x-vt,v)$. We have that: $f_{|t=0} = f_0$. Since f_0 is $\mathcal{C}^1$, we have that $f \in \mathcal{C}^1(\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d)$. Furthermore, we have that*

$$\partial_t f = -v.\nabla_x f$$

⋆ ⋆ ⋆ ⋆

The physical meaning of this proposition, is that, for a given velocity c, we have a propagation , in the space, of $f_0(.,c)$ which is the initial distribution of particles that has the initial velocity equal to c. Since the particles don't have the same initial velocity, the global evolution of the initial distribution, in phase space, will happens with a deformation[5]. This phenomenon is called **dispersion**.
Let's see now what properties are preserved during the evolution in time of the initial data. Let f be a solution for (1.1) :

- **Positivity :** Using (1.2), if $f_0 \geq 0$, then $f \geq 0$ for all the times.
- **Volume conservation in the phase space :** If we suppose that f_0 has a sufficient decay in x and v so that the integrals converge [6]. Then:

$$\forall t \geq 0,\ \|f(t,.,.)\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)} = \|f_0\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)}$$

Explicitly :

$$\forall t \geq 0,\ \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(t,x,v)dxdv = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f_0(x,v)dxdv$$

In deed, using (1.2),

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(t,x,v)dxdv = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(0,\underbrace{x-vt}_{=u},v)dxdv = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f_0(u,v)dudv$$

More generally we can prove with the same way that , for $p \in [1,\infty]$:

$$\|f(t,.,.)\|_{L^p(\mathbb{R}^d \times \mathbb{R}^d)} = \|f_0\|_{L^p(\mathbb{R}^d \times \mathbb{R}^d)}$$

With the combination of the deformation of the initial data due to the dispersion phenomenon, and the preservation of the volume and the norm $\|.\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d)}$ we intuitively expect the decrease of macroscopic density. The proposition 1.2 gives exactly how the macroscopic density is decreasing.

[5] The figure 1 shows this phenomenon for a specific initial data.

[6] We can assume for example that:

$$\forall m \geq 1,\quad \|x\|^m f_0(x,v) \text{ and } \|v\|^m f_0(x,v) \text{ tend to zero when } \|x\| \text{ and } \|v\| \text{ tend to infinity respectively.}$$

Or simply that f_0 is compactly supported.

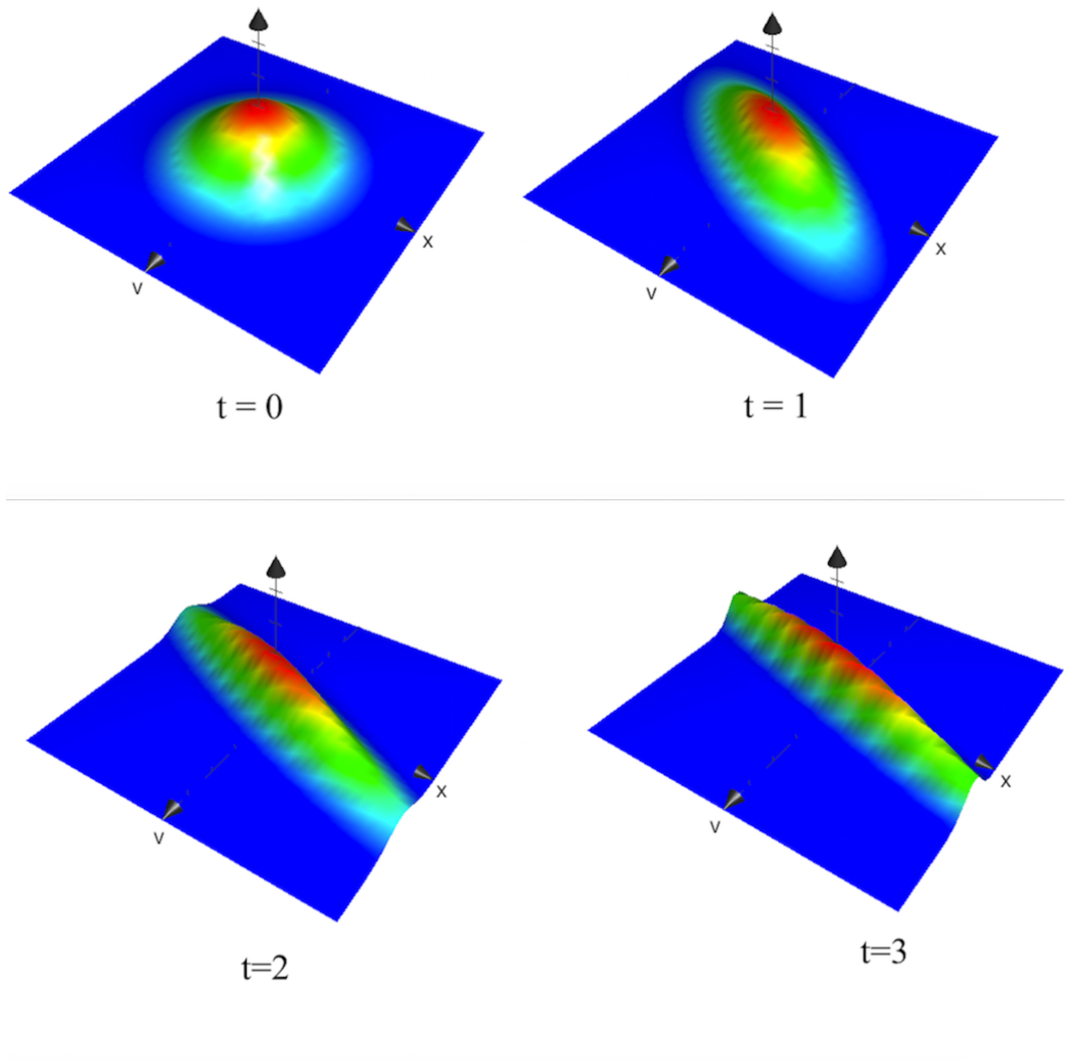

Figure 1: The evolution of the solution of the kinetic transport equation, for the initial data $f_0(x,v) = e^{-(x^2+v^2)}$ for different times t. The figures are realized with `Grapher`.

Proposition 1.2 (Dispersion) *Let f be a solution for (1.1), and let's assume that $f_0 \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d) \cap L^1(\mathbb{R}^d \times \mathbb{R}^d)$. Then, $f \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d) \cap L^1(\mathbb{R}^d \times \mathbb{R}^d)$ and:*

$$\forall t \geq 0 \quad \|f(t,.,.)\|_{L^\infty_x(L^1_v)} \leq |t|^{-d} \|f_0\|_{L^1_x(L^\infty_v)}$$

explicitly,

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} |f(t,x,v)| dv \leq |t|^{-d} \int_{\mathbb{R}^d} \sup_{y \in \mathbb{R}^d} |f_0(u,y)| du$$

Proof 1.2 *For all $(t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d$, we have that :*

$$f(t,x,v) = f_0(x - tv, v)$$

Which gives for all $t \geq 0$, $f(t,.,.) \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d) \cap L^1(\mathbb{R}^d \times \mathbb{R}^d)$. Moreover,

$$\begin{aligned} \forall t \geq 0 \quad \|f(t,.,.)\|_{L^\infty_x(L^1_v)} &= \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} |f_0(x - tv, v)| dv \\ &\leq \sup_{x \in \mathbb{R}^d} |\int_{\mathbb{R}^d} \sup_{y \in \mathbb{R}^d} |f_0(\underbrace{x - tv}_{=u}, y)| dv| \\ &\leq |t|^{-d} \int_{\mathbb{R}^d} \sup_{y \in \mathbb{R}^d} |f_0(u,y)| du = |t|^{-d} \|f_0\|_{L^1_x(L^\infty_v)} \end{aligned}$$

⋆ ⋆ ⋆ ⋆

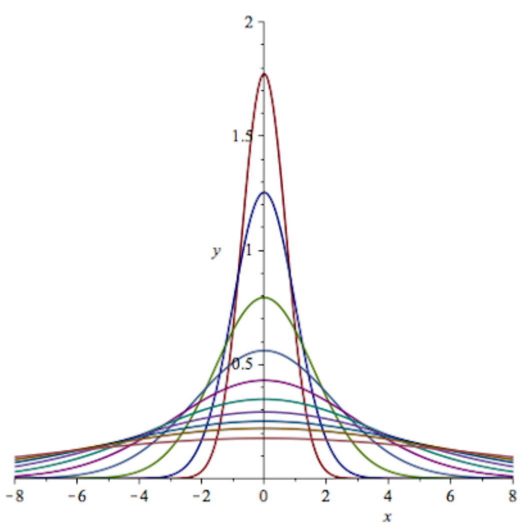

Figure 2: The evolution in time of macroscopic density for the free transport kinetic equation, for the initial data $f_0(x,v) = e^{-(x^2+v^2)}$, and for different times $t \geq 0$. This figure shows that the macroscopic density is spreading in the whole space and vanishes for $t \to \infty$. This figure is realized with `Maple`.

Remark 1.1 *The previous proposition shows that for the kinetic transport equation, the macroscopic density tends to zero uniformly. In fact :*

$$\|\rho(t,.)\|_\infty \leq \frac{1}{t^d}\|f_0\|_{L^1_x(L^\infty_v)}$$

The figure 2 illustrates this phenomenon for:

$$f_0(x,v) = \exp(-x^2 - v^2)$$

1.2 The method of characteristics

An important feature of first order partial differential equations is the method of characteristics, which reduces their study to an ordinary differential system. Let's consider now the general form of the transport equation :

$$\partial_t u + \phi(t,x).\nabla_x u = 0, \qquad t \in]0,T[, \quad x \in \mathbb{R}^d \tag{1.3}$$

with $\phi :]0,T[\times\mathbb{R}^d \longrightarrow \mathbb{R}^d$. The general theory developed here is required for the study of Vlasov's system in the other sections.

Definition 1.1 *A characteristic for the equation (1.3) is a function X from an interval $\mathcal{I}$ of $\mathbb{R}$ in $\mathbb{R}^d$ such that, $X \in \mathcal{C}^1(\mathcal{I})$ and,*

$$\frac{dX}{ds}(s) = \phi(s, X(s))$$

Notice that in this definition, X is a solution for an ordinary differential equation, which is not necessarily linear. With a given initial data, the existence and uniqueness of solutions depend on the regularity of the vector field ϕ. However, in the non linear case, we don't know if the solution is local or global in time. The existence (Cauchy-Peano's theorem) and uniqueness (Cauchy-Lipschitz / Picard-Lindeof theorem & Cauchy-Kovalevskaya theorem (for analytic solutions)) theorems provide only a local result. The following example is a good illustration for the case where the solution blows up in finite time.

Example 1.1 *The unique solution for:*

$$\begin{cases} \frac{dy}{ds}(s) = y(s)^2 \\ y(0) = y_0 > 0 \end{cases}$$

is: $y : t \longmapsto \frac{y_0}{1-y_0 t}$. This solution is obviously not defined for all the times $t \in \mathbb{R}$.

The criteria $(H2)$ in theorem 1.1 is added in order to avoid the finite time blow-up. The theorem below, gives the principal results about the characteristics of (1.3).

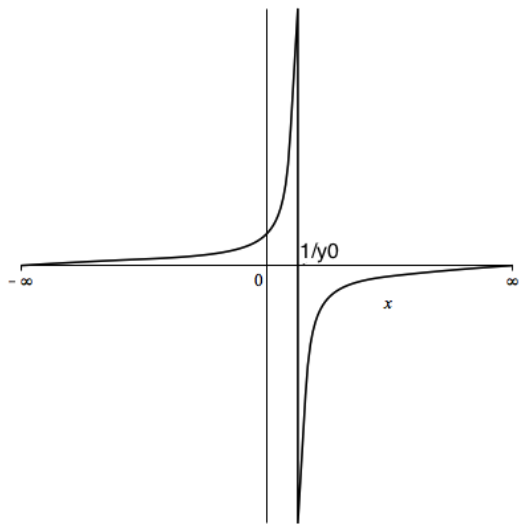

Figure 3: The plot of the function $y : t \longmapsto \frac{y_0}{1-y_0 t}$

Theorem **1.1 (The characteristic flow)** *We assume that ϕ satisfies the two following conditions:*
(H1) : $\phi \in \mathcal{C}([0,T]\times\mathbb{R}^d)$ *and* $\partial_x\phi$ *exists and belongs to* $\mathcal{C}([0,T]\times\mathbb{R}^d)$.
(H2) : $\exists C>0,\quad |\phi(t,x)| \le C(1+|x|)$ *for all* $(t,x)\in[0,T]\times\mathbb{R}^d$.
Then we have the following facts:

1. *For all $(t,x)\in[0,T]\times\mathbb{R}^d$ it exists a unique characteristic defined on $[0,T]$ taking the value x at the instant t. $X(.,t,x)$ denotes this characteristic.*

2. *For all $t_1,t_2,t_3\in[0,T]$ and $x\in\mathbb{R}^d$,*

$$X(t_1,t_2,X(t_2,t_3,x)) = X(t_1,t_3,x)$$

3. *For $(s,t)\in[0,T]\times[0,T]$, $X(s,t,.)$ is a $\mathcal{C}^1$ diffeomorphism from $\mathbb{R}^d$ in $\mathbb{R}^d$, and the inverse function is $X(t,s,.)$.*

4. *The application $(s,t,x)\longmapsto X(s,t,x)$ from $[0,T]\times[0,T]\times\mathbb{R}^d$ to $\mathbb{R}^d$ belongs to $\mathcal{C}^1([0,T]\times[0,T]\times\mathbb{R}^d)$.*

5. *$\partial_x\partial_s X$ and $\partial_s\partial_x X$ exist, are equals and belong to $\mathcal{C}([0,T]\times[0,T]\times\mathbb{R}^d)$.*

Proof for the Theorem **1.1** *1. Let $(t,x)\in[0,T]\times\mathbb{R}^d$.*
By the **Cauchy-Lipschitz theorem**[7]*, the assumption* (H1) *gives the existence and uniqueness of a maximale solution (γ,J) for the Cauchy problem :*

$$\begin{cases} \frac{dy}{ds}(s) = \phi(s,y(s)) \\ y(t) = x \end{cases} \tag{1.4}$$

[7] Theorem B.1

γ is the characteristic $X(.,t,x)$.

Furthermore, (H2) *gives that $J = [0,T]$. In fact, we have that:*

$$\forall s \in [0,T], \quad \gamma(s) = x + \int_t^s \phi(r,\gamma(r))dr$$

thus,

$$\begin{aligned}\forall s \in [0,T], \quad |\gamma(s)| &\leq |x| + |\int_t^s |\phi(r,\gamma(r))|dr| \\ &\leq |x| + CT + C|\int_t^s |\gamma(r)|dr|\end{aligned}$$

using **Gronwall's lemma**[8],

$$\begin{aligned}\forall s \in [0,T], \quad |\gamma(s)| &\leq (|x| + CT)e^{C|s-t|} \\ &\leq (|x| + CT)e^{CT} \qquad (1.5)\end{aligned}$$

If $J \neq [0,T]$, since (γ, J) the maximal solution for (1.4) we necessarily have that :

$$|\gamma(s)| \longmapsto +\infty \quad when\ s \longmapsto inf(J)^+ \ or\ s \longmapsto sup(J)^-$$

which is impossible since we have (1.5). Hence, $J = [0,T]$.

2. *Let $(t_2,t_3) \in [0,T]$ and $x \in \mathbb{R}^d$.*
   $X(.,t_2,X(t_2,t_3,x))$ and $X(.,t_3,x)$ are both solutions for the following Cauchy problem:
   $$\begin{cases} \frac{dy}{ds}(s) = \phi(s,y(s)) \\ y(t_2) = X(t_2,t_3,x) \end{cases}$$
   using uniqueness of the solution, we get:
   $$\forall t_1 \in [0,T], \quad X(t_1,t_2,X(t_2,t_3,x)) = X(t_1,t_3,x)$$

3. *Using the* **derivation theorem relative to the initial conditions**[9], *$\partial_x X$ exists and belongs to $\mathcal{C}([0,T] \times [0,T] \times \mathbb{R}^d)$. In particular, for $(t,s) \in [0,T] \times [0,T]$, $X(s,t,.)$ and $X(t,s,.)$ are $\mathcal{C}^1$. Moreover, using the previous identity, we get:*
   $$\forall (s,t,x) \in [0,T]\times[0,T]\times\mathbb{R}^d\ , \quad [X(s,t,.)\circ X(t,s,.)](x) = X(s,t,X(t,s,x)) = x$$
   Hence, for $(t,s) \in [0,T] \times [0,T]$, $X(s,t,.)$ is a $\mathcal{C}^1$ diffeomorphism and the inverse function is $X(t,s,.)$.

4. • *$X(.,t,x)$ is the unique solution for (1.4). Hence, $\partial_s X$ exists and belongs to $\mathcal{C}([0,T] \times [0,T] \times \mathbb{R}^d)$.*

[8]Proposition B.1 (Appendix)
[9]Theorem B.2 (Appendix)

- *By the* **derivation theorem relative to the initial conditions**, $\partial_x X$ *exists and belongs to* $\mathcal{C}([0,T]\times[0,T]\times\mathbb{R}^d)$.
- *Let's prove now the existence and continuity for the t-derivative. Let* $(s,x)\in[0,T]\times\mathbb{R}^d$ *and*

$$\chi : (t,y)\in[0,T]\times\mathbb{R}^d \longmapsto X(t,s,y)-x$$

A consequence of the regularity proved for X above, is that:

$$\chi\in\mathcal{C}^1([0,T]\times\mathbb{R}^d)$$

Moreover,

$$\forall (t,y)\in[0,T]\times\mathbb{R}^d, \quad det(\partial_y\chi(t,y)) = det(\partial_x X(t,s,y))\neq 0$$

Notice that, $t\longmapsto X(s,t,x)$ *is the unique function y such that :*

$$\forall t\in[0,T], \quad \chi(t,y(t))=0$$

Hence, using **implicit function theorem**[10], *we get that:*

$$t\longmapsto X(s,t,x)$$

is continuous. Moreover, for $t\in[0,T]$ *:*

$$\begin{aligned}
\partial_t X(s,t,x) &= -\partial_y\chi(t,X(s,t,x))^{-1}.\partial_t\chi(t,X(s,t,x))\\
&= -\partial_x X(t,s,X(s,t,x))^{-1}.\partial_t X(t,s,X(s,t,x))\\
&= -\partial_x X(t,s,X(s,t,x))^{-1}.\phi(t,X(t,s,X(s,t,x))\\
&= -\underbrace{\partial_x X(t,s,X(s,t,x))^{-1}.\phi(t,x)}_{\in\mathcal{C}([0,T]\times[0,T]\times\mathbb{R}^d)}
\end{aligned}$$

Hence, $X\in\mathcal{C}^1([0,T]\times[0,T]\times\mathbb{R}^d)$.

5. *Let* $(s,t,x)\in[0,T]\times[0,T]\times\mathbb{R}^d$ *and* $j\in\{1,..,d\}$. *Notice that:*

$$\partial_{x_j}X(t,t,x)=e_j$$

Furthermore, since X is a solution for (1.4), we have that,

$$\partial_{x_j}\partial_s X(s,t,x)=\partial_x\phi(t,X(s,t,x))\partial_{x_j}X(s,t,x)$$

Since $X(s,t,.)$ *is* $\mathcal{C}^1$, $\partial_x\partial_s X$ *exists and belongs to* $\mathcal{C}([0,T]\times[0,T]\times\mathbb{R}^d)$. *Moreover, the* **derivation theorem relative to the initial conditions** *gives that,* $\partial_{x_j}X(,t,x)$ *is the unique solution for the following linear system:*

$$\begin{cases}\partial_s y(s)=\partial_x\phi(t,X(s,t,x))y(s)\\ y(t)=e_j\end{cases}$$

[10] Theorem B.4 (Appendix).

hence, $\partial_s\partial_{x_j}X$ satisfies:

$$\left\{\begin{array}{l}\partial_s\partial_{x_j}X(s,t,x)=\partial_x\phi(t,X(s,t,x)).\partial_{x_j}X(s,t,x)\\ \partial_{x_j}X(t,t,x)=e_j\end{array}\right.$$

which gives by uniqueness:

$$\partial_s\partial_{x_j}X=\partial_{x_j}\partial_sX$$

this completes the proof.

⋆ ⋆ ⋆ ⋆

Remark 1.2 *Based on the previous proof, we can notice that if moreover we have ϕ and $\partial_x\phi$ in $\mathcal{C}^k([0,T]\times\mathbb{R}^d)$, then $X\in\mathcal{C}^{k+1}([0,T]\times[0,T]\times\mathbb{R}^d)$.*

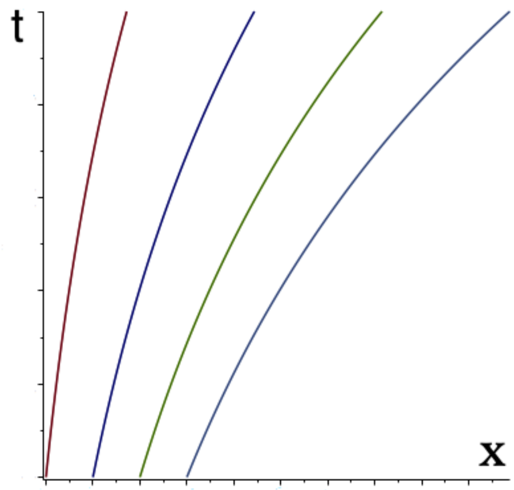

Figure 4: The characteristics $X(.,0,i)$ with $i\in\{1..4\}$ in the case when $\phi(t,x)=x$.

We know that X satisfies, $\partial_sX=\phi(s,X)$. The following result gives another PDE satisfied by X.

Proposition 1.3 *For all $(s,t,x)\in[0,T]\times[0,T]\times\mathbb{R}^d$, we have that :*

$$\partial_tX(s,t,x)+\partial_xX(s,t,x).\phi(t,x)=0$$

Proof 1.3 *In fact,*

$$\forall(s,t,r,x)\in[0,T]\times[0,T]\times[0,T]\times\mathbb{R}^d,\quad X(s,t,X(t,r,x))=X(s,r,x)$$

differentiating relatively to t, we have:

$$\partial_tX(s,t,X(t,r,x))+\partial_xX(s,t,X(t,r,x)).\partial_sX(t,r,x)=0$$

We take $r=t$, then:

$$\partial_tX(s,t,x)+\partial_xX(s,t,x).\partial_sX(t,t,x)=\partial_xX(s,t,x).\phi(t,x)=0$$

⋆ ⋆ ⋆ ⋆

Let's go back now to our first problem (1.3). With the assumptions of the theorem 1.1, we have a result of existence and uniqueness which is a generalization of the proposition 1.1.

<u>Theorem</u> **1.2** *With the assumptions of the theorem 1.1 , and if u_0 is $\mathcal{C}^1$, the problem :*

$$\left\{ \begin{array}{l} \partial_t u + \phi.\nabla_x u = 0 \\ u_{|t=0} = u_0 \end{array} \right. \tag{1.6}$$

has a unique solution in $\mathcal{C}^1([0,T]\times\mathbb{R}^d)$, given by :

$$\forall (t,x) \in [0,T]\times\mathbb{R}^d, \quad u(t,x) = u_0(X(0,t,x))$$

Furthermore, if u_0 is compactly supported, then u is also compactly supported and the support of u depends only on C, T and the support of u_0.

Proof for the Theorem **1.2**

- *Existence: Let $(t,x) \in [0,T]\times\mathbb{R}^d$, and*

  $$u(t,x) = u_0(X(0,t,x))$$

  with the results of the theorem 1.1,

  $$(t,x) \longmapsto X(0,t,x)$$

  is $\mathcal{C}^1$, and since u_0 is $\mathcal{C}^1$, we get:

  $$u \in \mathcal{C}^1([0,T]\times\mathbb{R}^d)$$

  Using the definition of u, we do have that:

  $$u_{|t=0} = u_0$$

  Using the proposition 1.3,

  $$\begin{array}{rcl} \forall (t,x) \in [0,T]\times\mathbb{R}^3, \quad \partial_t u(t,x) & = & \partial_t u_0(X(0,t,x)) \\ & = & \nabla u_0(X(0,t,x))\partial_t X(0,t,x) \\ & = & -\nabla u_0(X(0,t,x))\partial_x X(0,t,x)\phi(t,x) \\ & = & -\nabla_x u(t,x)\phi(t,x) \end{array}$$

  which proves that u is in deed a solution.

- *Uniqueness: Let u be a solution for the problem (1.6) and $(t,x) \in [0,T]\times\mathbb{R}^d$.*

  $$\begin{array}{rcl} \forall s \in [0,T], \quad \dfrac{d}{ds}u(s,X(s,t,x)) & = & \partial_t u(s,X(s,t,x)) + \partial_s X(s,t,x).\nabla_x u(s,X(s,t,x)) \\ & = & \partial_t u(s,X(s,t,x)) + \phi(s,X(s,t,x)).\nabla_x u(s,X(s,t,x)) \\ & = & 0 \end{array}$$

  hence :

  $$u(t,X(t,t,x)) = u(t,x) = u(0,X(0,t,x)) = u_0(X(0,t,x))$$

- *Notice that, for all $t \in [0,T]$,*

$$supp(u(t,.)) = X(t,0,supp(u_0))$$

Let $K := supp(u_0)$. Since the compact sets of $[0,T]\times\mathbb{R}^3$ are the closed bounded sets, we need to prove that:

$$K' = \overline{\{(t,x) \in [0,T] \times \mathbb{R}^d / u(t,x) \neq 0\}}$$

is bounded. Let $(t,x) \in K'$, we have that : $|t| \leq T$. We also have that :

$$K' = \overline{\{(t,x) \in [0,T] \times \mathbb{R}^d / X(0,t,x) \in K\}}$$

hence, $\exists M > 0$, $|X(0,t,x)| = |x + \int_t^0 \phi(s, X(s,t,x))ds| \leq M$.
We get,

$$\begin{aligned} |x| &\leq M + |\int_t^0 |\phi(s,X(s,t,x))|ds| \\ &\leq M + TC + C|\int_t^0 |X(s,t,x)|ds| \\ &\leq M + TC(1 + \|X(.,t,x)\|_\infty) \end{aligned}$$

Moreover,

$$\begin{aligned} \forall s \in [0,T], \quad |X(s,t,x)| &\leq |X(0,t,x)| + |\int_0^s |\phi(r,X(r,t,x))|dr| \\ &\leq M + CT + C|\int_0^s |X(r,t,x)|dr| \end{aligned}$$

hence, using **Gronwall's lemma***,*

$$\begin{aligned} \forall s \in [0,T], \quad |X(s,t,x)| &\leq (M+CT)e^{C|s|} \\ &\leq (M+CT)e^{CT} \end{aligned}$$

which gives $\|X(.,t,x)\|_\infty \leq (M+CT)e^{CT}$. Then, we finally get:

$$|x| \leq M + TC(1 + (M+CT)e^{CT})$$

Which prove that K' is a compact set.

⋆ ⋆ ⋆ ⋆

Generalization

Let's consider now a more generalized form of the problem (1.6), by changing the equation (1.3) with the following equation, which is more general.

$$\partial_t u + \nabla_x(\phi(t,x).u(t,x)) = 0 \quad \text{with } t \in [0,T], \text{ and } x \in \mathbb{R}^d \tag{1.7}$$

With ϕ a function from $]0,T[\times\mathbb{R}^d$ in $\mathbb{R}^d$. We have that :

$$\nabla_x(\phi.u) = \phi.\nabla_x u + u.\nabla_x\phi.$$

We can notice that this equation is the equation (1.3) in the particular case when:

$$\nabla_x\phi = 0$$

For this problem the method of characteristic remains efficient, and gives solutions under the assumptions of the Theorem 1.1 and with additional assumption on ϕ[11]. **A characteristic for the equation (1.7) is defined on the same way as for the equation (1.3)**. *(see Definition 1.1 in this section).*

Proposition 1.4 (Jacobian properties) *Under the assumptions of the Theorem 1.1 and if we define J as the Jacobian of X:*

$$\forall(s,t,x)\in[0,T]\times[0,T]\times\mathbb{R}^d,\quad J(s,t,x) := det(\partial_x X(s,t,x))$$

We have the following results:

- $\partial_s J(s,t,x) = \nabla_x\phi(s,t,X(s,t,x)).J(s,t,x)$
- $J > 0$
- $\forall(s,t_0,x_0)\in[0,T]\times[0,T]\times\mathbb{R}^d$,

$$\frac{J(0,t_0,x_0)}{J(t,t_0,x_0)} = J(0,t,X(t,t_0,x_0)) \tag{1.8}$$

Proof 1.4 *Let $(s,t,x)\in[0,T]\times[0,T]\times\mathbb{R}^d$.*
We start by reminding the following result : if $A\in\mathcal{M}_d(\mathbb{R})$, we have that :

$$\partial det(A).H = Tr(\ {}^tCom(A)H)$$

- *Using the theorem 1.1 and especially the derivatives formulas for X, we get:*

$$\begin{aligned}
\partial_s J(s,t,x) &= Tr(\ {}^tCom(\partial_x X(s,t,x)).\partial_s\partial_x X(s,t,x)) \\
&= Tr(\ {}^tCom(\partial_x X(s,t,x)).\partial_x\phi(s,X(s,t,x))\partial_x X(s,t,x)) \\
&= Tr(\partial_x\phi(s,X(s,t,x)).\partial_x X(s,t,x)\ {}^tCom(\partial_x X(s,t,x))) \\
&= Tr(\partial_x\phi(s,X(s,t,x))).det(\partial_x X(s,t,x)) \\
&= \nabla_x\phi(s,X(s,t,x))J(s,t,x)
\end{aligned}$$

Thus,

$$J(s,t,x) = \exp\left(\int_t^s \nabla_x\phi(u,X(u,t,x))du\right) > 0$$

since $J(t,t,x) = 1$.

[11] We need $\mathcal{C}^1$ regularity for ϕ and $\partial_x\phi$. (see theorem 1.3)

- $$\forall(m,n,r,x)\in[0,T]\times[0,T]\times[0,T]\times\mathbb{R}^d,\quad X(m,n,X(n,r,x))=X(m,r,x)$$

  Differentiating relatively to x and computing the determinant of the expression obtained, we get:

  $$J(m,n,X(n,r,x))J(n,r,x)=J(m,r,x)$$

  by taking :

  $$m=0,\ n=t,\ r=t_0,\ x=x_0$$

  we get the claimed result.

⋆ ⋆ ⋆ ⋆

The following theorem is a generalization of the Theorem 1.2.

<u>THEOREM</u> **1.3** *We assume that ϕ satisfies the two following conditions:*
(H1) : $\phi\in\mathcal{C}^1([0,T]\times\mathbb{R}^d)$ *and $\partial_x\phi$ exists and belongs to* $\mathcal{C}^1([0,T]\times\mathbb{R}^d)$.
(H2) : $\exists C>0,\quad |\phi(t,x)|\leq C(1+|x|)$ *for all* $(t,x)\in[0,T]\times\mathbb{R}^d$.
and $u_0\in\mathcal{C}^1(\mathbb{R}^d)$. Under these assumptions, the following problem :

$$\begin{cases}\partial_t u+\nabla_x(\phi(t,x).u(t,x))=0\\ u_{|t=0}=u_0\end{cases}\tag{1.9}$$

has a unique solution in $\mathcal{C}^1([0,T]\times\mathbb{R}^d)$, given by :

$$\forall(t,x)\in[0,T]\times\mathbb{R}^d,\quad u(t,x)=u_0(X(0,t,x))J(0,t,x)\tag{1.10}$$

Furthermore, if u_0 has a compact support in $\mathbb{R}^d$, u has also a compact support in $[0,T]\times\mathbb{R}^d$ which depends only on C, T and the support of u_0.

PROOF FOR THE THEOREM **1.3**

- *Uniqueness: Let u be a solution for (1.9) and $(t,x)\in[0,T]\times\mathbb{R}^d$. Using the proposition 1.4, we get for $s\in[0,T]$:*

  $$\begin{aligned}\frac{d}{ds}u(s,X(s,t,x)) &= \partial_t u(s,X(s,t,x))+\partial_s X(s,t,x)..\nabla_x u(s,X(s,t,x))\\ &= \partial_t u(s,X(s,t,x))+\phi(s,X(s,t,x)).\nabla_x u(s,X(s,t,x))\\ &= -\nabla_x\phi(s,X(s,t,x)).u(s,X(s,t,x))\\ &= -\frac{\partial_s J(s,t,x)}{J(s,t,x)}.u(s,X(s,t,x))\end{aligned}$$

  thus,

  $$\forall s\in[0,T],\quad \partial_s(J(s,t,x).u(s,X(s,t,x)))=0$$

  hence,

  $$\begin{aligned}J(0,t,x)u_0(X(0,t,x)) &= J(0,t,x)u(0,X(0,t,x))\\ &= J(t,t,x)u(t,X(t,t,x))\\ &= u(t,x)\end{aligned}$$

  which proves the uniqueness.

- *Existence: Let for* $(t,x) \in [0,T] \times \mathbb{R}^d$, $u(t,x) = u_0(X(0,t,x))J(0,t,x)$. *Using theorem 1.1, the following function is* $\mathcal{C}^1$*:*

$$(t,x) \longmapsto X(0,t,x)$$

Moreover $J(0,.,.) = det(\partial_x X(0,.,.)) \in \mathcal{C}^1([0,T] \times \mathbb{R}^d)$. *Then, since* u_0 *is* $\mathcal{C}^1$ *we get:*

$$u \in \mathcal{C}^1([0,T] \times \mathbb{R}^d)$$

Using the proposition 1.3 and (1.8), we get : $\forall (t,t_0,x_0) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d$,

$$\partial_t u(t, X(t,t_0,x_0)) = -\nabla_x (u\phi)(t, X(t,t_0,x_0))$$

thus,

$$\forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d, \quad \partial_t u(t,x) = -\nabla_x (u\phi)(t,x)$$

furthermore,

$$\begin{aligned} \forall x \in \mathbb{R}^d, \quad u(0,x) &= u_0(X(0,0,x))J(0,0,x) \\ &= u_0(x) det(\partial_x X(0,0,x)) \\ &= u_0(x) \end{aligned}$$

- *For the last result (about the compact support) the proof is the same as for the theorem 1.2. In fact, since* $J > 0$, *if* K *and* K' *are respectively the supports of* u_0 *and* u, *we still have:*

$$K' = \overline{\{(t,x) \in [0,T] \times \mathbb{R}^d / X(0,t,x) \in K\}}$$

hence, the proof remains the same.

$\star \quad \star \quad \star \quad \star$

Remark 1.3 *If* $\nabla_x \phi = 0$, *then by the proposition 1.3 we have that:*

$$\forall (s,t,x) \in [0,T] \times [0,T] \times \mathbb{R}^d, \qquad J(s,t,x) = 1$$

In this particular case, we find the result given in the Theorem 1.2.

Application

Let's go back now to the problem (0.6), and let's suppose that F is known and already given. We aim to solve the partial differential system above using the theorems given in this section.

$$\begin{cases} \partial_t f + v.\nabla_x f + F(t,x,v).\nabla_v f = 0 \\ f_{|t=0} = f_0 \end{cases} \tag{1.11}$$

Before solving the equation, let's see what properties are satisfied by a solution for (1.11) if it exists. The following results show that if f is compactly supported in phase space, all the properties satisfied for the free transport equation remain true, even if the force is unspecified. We can even prove a strongest result:

$$\int\int \Psi(f(t,.))dxdv = \int\int \Psi(f_0)dxdv$$

for any enough regular function Ψ.

Proposition 1.5 *We assume that:*

$$\nabla_v F = 0$$

Consider f a $\mathcal{C}^1$ solution on $[0,T]$ for the system (1.11) with initial data f_0 such that:

$$\exists r > 0,\ \forall t \in [0,T],\ support(f(t,.,.)) \subset B(0,r) \tag{1.12}$$

Then we have the following facts:

1. *if $f_0 \geq 0$ then $f(t.,.) \geq 0$ for all $t \in [0,T]$.*
2. $\forall p \in [1,\infty[$, $\forall t \in [0,T]$,
$$\|f(t,.,.)\|_{L^p(\mathbb{R}^d\times\mathbb{R}^d)} = \|f_0\|_{L^p(\mathbb{R}^d\times\mathbb{R}^d)}$$
3. $\forall t \in [0,T]$,
$$\|f(t,.,.)\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)} \leq \|f_0\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)} \tag{1.13}$$
4. $\forall \Psi \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$, *such that* $\Psi(0) = 0$,
$$\forall t \in [0,T], \quad \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Psi(f(t,x,v))dxdv = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Psi(f_0(x,v))dxdv$$

Proof 1.5

- *Let's start by proving (4). Let $\Psi \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$, such that $\Psi(0) = 0$. Then, $\Psi(f)$ is a $\mathcal{C}^1$ solution on $[0,T]$ with the initial data $\Psi(f_0)$. In fact, $\Psi(f)$ is $\mathcal{C}^1$ and :*

$$\begin{cases} \partial_t \Psi(f) + v.\nabla_x \Psi(f) + F(t,x,v).\nabla_v \Psi(f) = \Psi'(f)(\partial_t f + v.\nabla_x f + F(t,x,v).\nabla_v f) = 0 \\ \Psi(f)_{|t=0} = \Psi(f_0) \end{cases} \tag{1.14}$$

Furthermore, the integrals converge, and the function defined below is $\mathcal{C}^1$.

$$t \in [0,T] \longmapsto \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Psi(f(t,x,v))dxdv$$

In fact :

  – *By (1.12) and having $\Psi(0) = 0$,*

$$\forall t \in [0,T], \quad \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Psi(f(t,x,v))dxdv = \int_{B(0,r)} \Psi(f(t,x,v))dxdv < \infty$$

– *To prove the regularity claimed for the integral, we use* **the integral derivtion theorem**[12]. *For $t \in [0,T]$, we have the following domination:*

$$\forall (x,v) \in B(0,r), \quad |\partial_t f(t,x,v)\Psi'(f(t,x,v))| \leq \sup_{|u| \leq \|f\|_\infty} |\Psi'(u)| \sup_{u \in [0,T]\times B(0,r)} |\partial_t f(u)|$$

we recall that:

$$\|f\|_\infty = \sup_{u \in [0,T]\times B(0,r)} |f(u)|$$

Thus, the theorem gives :

$$\frac{d}{dt}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Psi(f(t,x,v))dxdv = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \partial_t \Psi(f(t,x,v))dxdv$$

Using **Green's formula**[13], *and* [14] *(1.14) :*

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Psi(f(t,x,v))dxdv &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \partial_t \Psi(f(t,x,v))dxdv \\ &= -\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \nabla_x(v.\Psi(f)) - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \nabla_v(F.\Psi(f)) \\ &= -\int_{\partial B(0,r)} v.\Psi(f)NdS - \int_{\partial B(0,r)} F.\Psi(f)NdS = 0 \end{aligned}$$

Which proves (4).

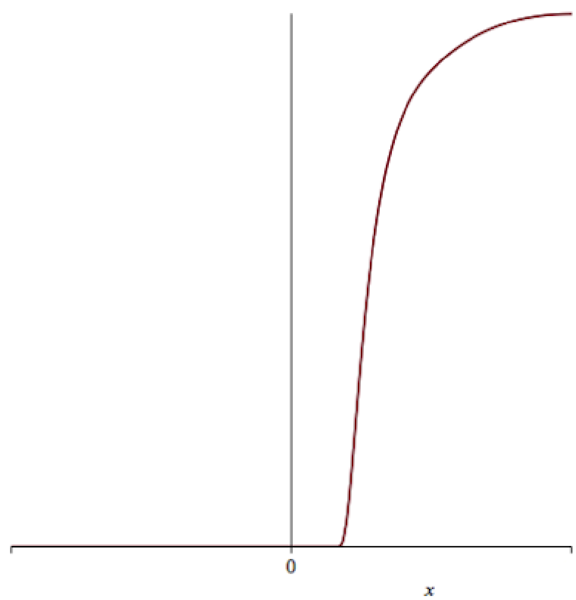

Figure 5: The plot of the function $x \longmapsto \exp(-\frac{1}{(x-1)^2})\chi_{]1,+\infty[}(x)$.

[12] Theorem B.8 (Appendix).
[13] Proposition B.11 (Appendix).
[14] Note that we have used $\nabla_v F = 0$ to move F inside and outside of the v-derivative term.

- *Let's prove now that, if the initial data is positive, then f remains positive for all times. In fact, we take the function Ψ equal to the function below:*

$$\Psi : x \longmapsto \begin{cases} e^{-\frac{1}{x^2}} & x \in]-\infty, 0[\\ 0 & x \in [0, +\infty[\end{cases}$$

We have that $\Psi(f_0) = 0$. Then, using the previous result, we get:

$$\forall t \in [0,T], \quad \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Psi(f(t,x,v))dxdv = 0$$

Since Ψ is a positive and continuous function, we have that: $\Psi(f) = 0$. Which gives: $f \geq 0$.

- *Let $p \in [1, \infty[$. Taking : $\Psi : y \longmapsto y^p$. We get (2).*

- *To prove (3), we will use the same technique used in (1). In deed, let*

$$\Psi : x \longmapsto \begin{cases} \exp(-\frac{1}{(x-\alpha)^2}) & x \in]\alpha, +\infty[\\ 0 & x \in]-\infty, \alpha] \end{cases}$$

with $\alpha = \|f_0\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d)}$, we get $\Psi(f_0) = 0$. Hence, using the previous results we have that: $\Psi(f) = 0$. Which finally gives (1.13).

⋆ ⋆ ⋆ ⋆

Under a few conditions on the force F, the theorem 1.2 can be used to prove the existence and uniqueness of a solution for (1.11). To use the previous theorem, we start by writing the system on the following form :

$$\begin{cases} \partial_t f + (v, F(t,x,v)).\nabla_{(x,v)} f = 0 \\ f_{|t=0} = f_0 \end{cases} \tag{1.15}$$

This system is now analogue to the problem (1.6).

<u>Theorem</u> **1.4** *If $f_0 \in \mathcal{C}^1(\mathbb{R}^d \times \mathbb{R}^d)$ and F satisfies the two following conditions:*

1. *$F \in \mathcal{C}([0,T] \times \mathbb{R}^d \times \mathbb{R}^d)$ and $\partial_x F$ exists and belongs to $\mathcal{C}([0,T] \times \mathbb{R}^d \times \mathbb{R}^d)$.*

2. *$\exists C > 0, \quad |F(t,x,v)| \leq C(1 + |x| + |v|)$ for all $(t,x,v) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d$.*

Then,

$$(t,x,v) \longmapsto f_0(X(0,t,x,v), V(0,t,x,v))$$

is the unique solution for (1.11) in $\mathcal{C}^1([0,T] \times \mathbb{R}^d \times \mathbb{R}^d)$, with $(X(.,t,x,v), V(.,t,x,v))$ is the unique characteristic taking the value (x,v) at the instant t.
Furthermore, if f_0 has a compact support in $\mathbb{R}^d \times \mathbb{R}^d$, f has also a compact support in $[0,T] \times \mathbb{R}^d \times \mathbb{R}^d$ which depends only on C, T and the support of f_0.

PROOF FOR THE THEOREM **1.4** *The proof is a direct application of the theorem 1.2. Let's check the assumptions of the theorem 1.2 :*

- *Let:* $\phi(t,(x,v)) = (v, F(t,x,v))$. *With the assumptions of the theorem,*

  $\phi \in \mathcal{C}([0,T] \times \mathbb{R}^d \times \mathbb{R}^d)$ *and* $\partial_{(x,v)}\phi$ *exists and belongs to* $\mathcal{C}([0,T] \times \mathbb{R}^d \times \mathbb{R}^d)$.

- *Furthermore,* $\forall (t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d$,

$$\begin{aligned} |\phi(t,(x,v))| &= \sqrt{|F(t,x,v)|^2 + |v|^2} \\ &\leq |F(t,x,v)| + |v| \\ &\leq C'(1 + |x| + |v|) \end{aligned}$$

  with $C' > 1$ *a constant.*

$$(x,v) \rightarrow |x| + |v|$$

  define a norm on $\mathbb{R}^d \times \mathbb{R}^d$. *Since all the norms are equivalent in* $\mathbb{R}^d \times \mathbb{R}^d$*:*

$$\exists \eta > 0,\ \forall (x,v) \in \mathbb{R}^d \times \mathbb{R}^d,\ |x| + |v| \leq \eta\sqrt{|x|^2 + |v|^2}$$

  Then, we get the assumption (H2) of the theorem 1.2 :

$$\forall (t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d, \quad |\phi(t,(x,v))| \leq \max(\eta, C')(1 + |(x,v)|)$$

  The system (1.11) can be written in following form:

$$\begin{cases} \partial_t f + \phi(t,z).\nabla_z f = 0 \\ f_{|t=0} = f_0 \end{cases}$$

  with $z := (x,v) \in \mathbb{R}^{2d}$.

- *In this case, if* $(t,x,v) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d$ *and* $z = (x,v)$ *we note* $Z(.,t,z) := (X(.,t,z), V(.,t,z))$, *the unique characteristic solution for:*

$$\begin{cases} \frac{dy}{ds}(s) = \phi(s, y(s)) \\ y(t) = (x,v) \end{cases}$$

⋆ ⋆ ⋆ ⋆

Remark 1.4 *Using the Remark 1.1, if we have moreover* F *and* $\partial_x F$ *in* $\mathcal{C}^k([0,T] \times \mathbb{R}^d)$, *then* $f \in \mathcal{C}^{k+1}([0,T] \times \mathbb{R}^d \times \mathbb{R}^d)$.

Remark 1.5 *It's much easy to get the results of the proposition 1.5 when we have the explicit expression of the solutions. We can even get a strongest result. In fact, we have that:*

$$\forall t \in [0,T], \quad \|f(t.,.)\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d)} = \|f_0\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d)}$$

In fact, for $t \in [0,T]$ *:*

$$\forall (x,v) \in \mathbb{R}^d \times \mathbb{R}^d, \quad f_0(x,v) = f(t, X(t,0,x,v), V(t,0,x,v)) \leq \|f(t.,.)\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d)}$$

1.3 Weak solutions for the transport equation

In this subsection, we introduce the notion of weak solutions for the problem (1.9). We are looking for solutions in a functional space, which contains functions not enough regular so that the derivatives in the equation have a sens for differential calculus. Our motivation is to generalize the classical solutions given in the theorem 1.3, for less regular functions.
In general, the notion of weak solutions is very useful for a large class of problems. The most known one, is the break down of $\mathcal{C}^1$ regularity in finite time for the non linear transport equations with shocks formation[15].

Let's start by defining what we mean here by a weak solution for the problem (1.9). Let $\mathcal{C}([0,T], L^1(\mathbb{R}^d))$ be the space of functions $g \in \mathcal{F}([0,T]\times\mathbb{R}^d, \mathbb{R})$ such that $\forall t \in [0,T], g(t,.) \in L^1(\mathbb{R}^d)$ and $\int g(.,s)ds$ is continuous on $[0,T]$.

Definition 1.2 *We consider the problem (1.9) with $u_0 \in L^1_{loc}(\mathbb{R}^d)$.*
We say that $u \in \mathcal{C}([0,T], L^1(\mathbb{R}^d))$ is a weak solution for (1.9) if :

$$\forall \varphi \in \mathcal{C}^1_c([0,T]\times\mathbb{R}^d),$$

$$\int_0^t \int_{\mathbb{R}^d} u(s,x)(\partial_t\varphi(s,x)+\phi(s,x)\nabla_x\varphi(s,x))dxds = \int_{\mathbb{R}^d} u(t,x)\varphi(t,x)dx - \int_{\mathbb{R}^d} u_0(x)\varphi(0,x)dx$$

for all $t \in [0,T]$.

Remark 1.6
- *We will prove later that this definition extend the classical solutions constructed for the problem (1.9) in the theorem 1.3.*
- *Since $\varphi \in \mathcal{C}^1_c([0,T]\times\mathbb{R}^d)$ and $u \in \mathcal{C}([0,T], L^1(\mathbb{R}^d))$, all the integrals given in the definition are well defined.*
- *This definition holds for a largest class of functions. This notion can be defined also on $L^\infty([0,T], L^1(\mathbb{R}^d))$[16]. However, to give sens to the initial data, we need a minimum of regularity on the time variable. The continuity in the time variable is enough.*

The proposition below, shows that for a classical solution in $\mathcal{C}([0,T], L^1(\mathbb{R}^d))$, the two notions: weak and classical solution, are equivalent.

Proposition 1.6 *We consider the problem (1.9) with $u_0 \in \mathcal{C}^1(\mathbb{R}^d)$.*
Let $u \in \mathcal{C}^1([0,T]\times\mathbb{R}^d)\bigcap\mathcal{C}([0,T], L^1(\mathbb{R}^d))$. Then,

$$u \text{ is a classical solution for (1.9) } \Leftrightarrow u \text{ is a weak solution for (1.9).}$$

[15] A few examples of this phenomenon are given in [3, pages 140-142]

[16] The space of functions $g \in \mathcal{F}([0,T]\times\mathbb{R}^d, \mathbb{R})$ such that $\forall t \in [0,T], g(t,.) \in L^1(\mathbb{R}^d)$ and $\int g(.,s)ds \in L^\infty([0,T])$

Proof 1.6 • *First of all, let's do some calculations that will be very useful later.*

Let $\varphi \in \mathcal{C}_c^1([0,T]\times\mathbb{R}^d)$, $t\in[0,T]$ *and* K_φ *the support of* φ. *Let* Δ *an open ball of* $\mathbb{R}^d$, *such that* $K \subset \Delta$.
We denote :

$$I := \int_0^t \int_{\mathbb{R}^d} \varphi(s,x)(\partial_t u(s,x) + \nabla_x(\phi(s,x)u(s,x)))dxds$$

Performing an **integration by parts** *with respect to* t, *we have that:*

$$\begin{aligned}
I &= \int_0^t \int_\Delta \varphi(s,x)(\partial_t u(s,x) + \nabla_x(\phi(s,x)u(s,x)))dxds \\
&= \int_\Delta \int_0^t \varphi(s,x)\partial_t u(s,x)dsdx + \int_0^t \int_\Delta \varphi(s,x)\nabla_x(\phi(s,x)u(s,x))dxds \\
&= \int_\Delta [\varphi(s,x)u(s,x)]_0^t dx - \int_\Delta \int_0^t u(s,x)\partial_t\varphi(s,x)dsdx \\
&\quad + \int_0^t \int_\Delta \varphi(s,x)\nabla_x(\phi(s,x)u(s,x))dxds \\
&= \int_\Delta [\varphi(s,x)u(s,x)]_0^t dx - \int_0^t \int_\Delta u(s,x)\partial_t\varphi(s,x)dxds \\
&\quad + \int_0^t \int_\Delta \varphi(s,x)\nabla_x(\phi(s,x)u(s,x))dxds
\end{aligned}$$

the integral, interversion here is justified by **Fubini-Lebesgue's theorem**.[17]
In fact,

$$(s,x) \longmapsto \varphi(s,x)\partial_t u(s,x) \quad \text{and} \quad (s,x) \longmapsto u(s,x)\partial_t\varphi(s,x)$$

are integrable on $[0,t]\times\Delta$, *since the functions are bounded and* $[0,t]\times\Delta$ *is a compact set. Furthermore, using* **Green's formula** *we have that:*
$\forall s \in [0,T]$,

$$\begin{aligned}
\int_\Delta \varphi(s,x)\nabla_x(\phi(s,x)u(s,x))dx &= -\int_\Delta u(s,x)\phi(s,x)\nabla_x\varphi(s,x)dx \\
&\quad + \int_{\partial\Delta} u(s,x)\varphi(s,x)\phi(s,x)N(x)dS
\end{aligned}$$

moreover, $\forall s\in[0,T]$, $\forall x \in \partial\Delta$,

$$\varphi(s,x) = 0$$

Hence,

$$\int_\Delta \varphi(s,x)\nabla_x(\phi(s,x)u(s,x))dx = -\int_\Delta u(s,x)\phi(s,x)\nabla_x\varphi(s,x)dx$$

[17] Theorem B.10 (Appendix).

Combining the two previous results, we get :

$$\begin{aligned} I &= -\int_0^t \int_\Delta u(s,x)(\partial_t\varphi(s,x) + \phi(s,x)\nabla_x\varphi(s,x))dxds + \int_\Delta u(t,x)\varphi(t,x)dx \\ &\quad - \int_\Delta u_0(x)\varphi(0,x)dx \\ &= -\int_0^t \int_{\mathbb{R}^d} u(s,x)(\partial_t\varphi(s,x) + \phi(s,x)\nabla_x\varphi(s,x))dxds + \int_{\mathbb{R}^d} u(t,x)\varphi(t,x)dx \\ &\quad - \int_{\mathbb{R}^d} u_0\varphi(0,.) \end{aligned} \tag{1.16}$$

- $\Longrightarrow$)*Let's assume that u is a classical solution for (1.9).*
  Let $t \in [0,T]$,

  We have that:

  $$\forall \varphi \in \mathcal{C}_c^1([0,T]\times\mathbb{R}^d), \quad \int_0^t \int_{\mathbb{R}^d} \varphi(s,x)(\partial_t u(s,x) + \nabla_x(\phi(s,x)u(s,x)))dxds = 0$$

  Thus, using (1.16)
  $\forall \varphi \in \mathcal{C}_c^1([0,T]\times\mathbb{R}^d)$,

  $$\int_0^t \int_{\mathbb{R}^d} u(s,x)(\partial_t\varphi(s,x)+\phi(s,x)\nabla_x\varphi(s,x))dxds = \int_{\mathbb{R}^d} u(t,x)\varphi(t,x)dx - \int_{\mathbb{R}^d} u_0(x)\varphi(0,x)dx$$

  which gives that u is in deed a weak solution.

- $\Longleftarrow$) *Let's assume that u is a weak solution for (1.9).*
  We have that :

  $$\forall t \in [0,T], \quad \int_0^t \int_{\mathbb{R}^d} u(s,x)(\partial_t\varphi(s,x) + \phi(s,x)\nabla_x\varphi(s,x))dxds = \int_{\mathbb{R}^d} u(t,x)\varphi(t,x)dx - \int_{\mathbb{R}^d} u_0(x)\varphi(0,x)dx$$

  - Taking $t = T$, *we get :*

  $\forall \varphi \in \mathcal{C}_c^1([0,T]\times\mathbb{R}^d)$,

  $$\int_0^T \int_{\mathbb{R}^d} u(s,x)(\partial_t\varphi(s,x)+\phi(s,x)\nabla_x\varphi(s,x))dxds = \int_{\mathbb{R}^d} u(T,x)\varphi(T,x)dx - \int_{\mathbb{R}^d} u_0(x)\varphi(0,x)dx$$

  Hence, $\forall \varphi \in \mathcal{C}_c^1([0,T]\times\mathbb{R}^d)$,

  $$\int_0^T \int_{\mathbb{R}^d} \varphi(s,x)(\partial_t u(s,x) + \nabla_x(\phi(s,x)u(s,x)))dxds = 0$$

  using Theorem B.12 we get:

  $$\forall (t,x) \in]0,T[\times\mathbb{R}^d, \quad \partial_t u(t,x) + \nabla_x(\phi(t,x)u(t,x)) = 0$$

- Furthermore, taking $t = 0$ we get: $\forall \varphi \in \mathcal{C}_c^1([0,T] \times \mathbb{R}^d)$,

$$\begin{aligned}\int_{\mathbb{R}^d} u(0,x)\varphi(0,x)dx - \int_{\mathbb{R}^d} u_0(x)\varphi(0,x)dx &= \int_{\mathbb{R}^d} (u(0,x) - u_0(x))\varphi(0,x)dx \\ &= 0\end{aligned}$$

Thus, $\forall \varphi \in \mathcal{C}_c^1(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} (u(0,x) - u_0(x))\varphi(x)dx = 0$$

hence, using theorem B.12, we deduce that:

$$\forall x \in \mathbb{R}^d, \quad u(0,x) = u_0(x)$$

⋆ ⋆ ⋆ ⋆

The theorem below is an extension of the theorem 1.3 for weak solutions. The proof gives how the weak solutions are constructed from the classical ones.

<u>THEOREM</u> **1.5** *We consider the problem (1.9) and we assume that ϕ satisfies the following conditions:*
(H1) : $\phi \in \mathcal{C}^1([0,T] \times \mathbb{R}^d)$ *and* $\partial_x \phi$ *exists and belongs to* $\mathcal{C}^1([0,T] \times \mathbb{R}^d)$.
(H2) : $\exists C > 0, \quad |\phi(t,x)| \leq C(1 + |x|)$ *for all* $(t,x) \in [0,T] \times \mathbb{R}^d$.
If $u_0 \in L^1(\mathbb{R}^d)$, *the problem (1.9) has a unique weak solution given by (1.10).*

PROOF FOR THE THEOREM **1.5** • *Existence: the density of* $\mathcal{C}_c^\infty(\mathbb{R}^d)$ *in* $L^1(\mathbb{R}^d)$ *gives the existence of a sequence* $(u_0^n)_n$ *such that :*

$$\|u_0^n - u_0\|_{L^1(\mathbb{R}^d)} \overset{n\to\infty}{\longrightarrow} 0$$

If we denote for all $n \in \mathbb{N}$, u^n *the classical solution of (1.9) associated to the initial condition* u_0^n, *then using the Theorem 1.4,* $\forall n \in \mathbb{N}$, <u>u^n *is compactly supported.*</u>
Using **Lebesgue's theorem***:*

$$\forall n \in \mathbb{N}, \quad u^n(.,.) = u_0^n(X(0,.,.))J(0,.,.) \in \mathcal{C}([0,T], L^1(\mathbb{R}^d))$$

In fact,

– *Performing a change of variables we have that:* $\forall n \in \mathbb{N}, \forall t \in [0,T]$,

$$\int_{\mathbb{R}^d} u_0^n(\underbrace{X(0,t,x)}_{=s})J(0,t,x)dx = \int_{\mathbb{R}^d} u_0^n(s)ds < \infty$$

Thus,

$$\forall n \in \mathbb{N}, \forall t \in [0,T], \; u_0^n(X(0,t,.))J(0,t,.) \in L^1(\mathbb{R}^d)$$

- *Moreover, the domination condition is also satisfied. In deed: if K_n is the support of u_n we have :*

$$\forall (t,x) \in [0,T] \times K_n, \quad |u^n(t,x)| \leq \underbrace{\|u^n\|_\infty}_{\textit{integrable on } K_n}$$

Since K_n is a bounded set, all the constants are integrable.

Furthermore, we have uniformly in $t \in [0,T]$:

$$u^n(t,.) \overset{n\to\infty}{\to} u(t,.) = u_0(X(0,t,.))J(0,t,.)$$

in $L^1(\mathbb{R}^d)$. In fact,

- *Let $t \in [0,T]$,*

$$\begin{aligned}&|\int_{\mathbb{R}^d} |u_0^n(X(0,t,x))J(0,t,x)| - |u_0(X(0,t,x))J(0,t,x)|dx| = \\ &|\int_{\mathbb{R}^d} (|u_0^n(X(0,t,x))| - |u_0(X(0,t,x))|)J(0,t,x)dx|\end{aligned}$$

performing a change of variable: $u = X(0,t,x)$ ($X(0,t,.)$ is a $\mathcal{C}^1$ diffeomorphism), we get :

$$\begin{aligned}&|\int_{\mathbb{R}^d} |u_0^n(X(0,t,x))J(0,t,x)| - |u_0(X(0,t,x))J(0,t,x)|dx| = \\ &|\int_{\mathbb{R}^d} |u_0^n(u)| - |u_0(u)|du| \overset{n\to\infty}{\longrightarrow} 0\end{aligned}$$

Since the convergence is uniform in $t \in [0,T]$, we also have that:

$$u \in \mathcal{C}([0,T], L^1(\mathbb{R}^d))$$

Moreover, using the proposition 1.6:
$\forall t \in [0,T],\ \forall n \in \mathbb{N},\ \forall \varphi \in \mathcal{C}_c^1([0,T] \times \mathbb{R}^d)$,

$$\overbrace{\int_0^t \int_{\mathbb{R}^d} u^n(s,x)(\partial_t \varphi(s,x) + \phi(s,x)\nabla_x \varphi(s,x))dxds}^{a_n(t)} = \overbrace{\int_{\mathbb{R}^d} u^n(t,x)\varphi(t,x)dx}^{b_n(t)} - \overbrace{\int_{\mathbb{R}^d} u_0^n \varphi(0,.)}^{c_n}$$

Let,

$$\begin{aligned} a(t) &:= \int_0^t \int_{\mathbb{R}^d} u(s,x)(\partial_t \varphi(s,x) + \phi(s,x)\nabla_x \varphi(s,x))dxds \\ b(t) &:= \int_{\mathbb{R}^d} u(t,x)\varphi(t,x)dx \\ c &:= \int_{\mathbb{R}^d} u_0(x)\varphi(0,x)dx \end{aligned}$$

Let K be the support of $\varphi \in \mathcal{C}_c^1([0,T] \times \mathbb{R}^d)$ and $t \in [0,T]$. Since the convergence is uniform in t, we have that:

–

$$|a_n(t) - a(t)| \leq \underbrace{\|\partial_t\varphi(.,.) + \phi(.,.)\nabla_x\varphi(.,.)\|_{\infty,[0,T]\times K}\int_0^t\int_{\mathbb{R}^d}|u^n(s,x) - u(s,x)|dxds}_{\overset{n\longrightarrow\infty}{\longmapsto}0}$$

–

$$|b_n(t) - b(t)| \leq \underbrace{\|\varphi(.,.)\|_{\infty,[0,T]\times K}\int_{\mathbb{R}^d}|u^n(t,x) - u(t,x)|dx}_{\overset{n\longrightarrow\infty}{\longmapsto}0}$$

–

$$|c_n - c| \leq \underbrace{\|\varphi(0,.)\|_{\infty,K}\int_{\mathbb{R}^d}|u_0^n(x) - u_0(x)|dx}_{\overset{n\longrightarrow\infty}{\longmapsto}0}$$

Taking the limit when n tends to infinity we get:

$$\int_0^t\int_{\mathbb{R}^d}u(s,x)(\partial_t\varphi(s,x)+\phi(s,x)\nabla_x\varphi(s,x))dxds = \int_{\mathbb{R}^d}u(t,x)\varphi(t,x)dx - \int_{\mathbb{R}^d}u_0(x)\varphi(0,x)dx$$

Which proves that u is in deed a weak solution for the problem (1.9).

- *Uniqueness: Let u be a weak solution for the problem (1.9) and let f be a function defined by:*

$$\forall(t,x) \in [0,T]\times\mathbb{R}^d, \quad f(t,x) := u(t,X(t,0,x))J(t,0,x)$$

Let $t \in [0,T]$ and $\varphi \in \mathcal{C}_c^1([0,T]\times\mathbb{R}^d)$. We have that:

$$\int_0^t\int_{\mathbb{R}^d}u(s,y)(\partial_t\varphi(s,y)+\phi(s,y)\nabla_x\varphi(s,y))dyds = \int_{\mathbb{R}^d}u(t,y)\varphi(t,y)dy - \int_{\mathbb{R}^d}u_0(y)\varphi(0,y)dy$$

Performing a the change of variable : $y = X(s,0,x)$ we get :

$$\int_0^t\int_{\mathbb{R}^d}u(s,X(s,0,x))(\partial_s\varphi(s,X(s,0,x))+\phi(s,X(s,0,x))\nabla_x\varphi(s,X(s,0,x)))J(s,0,x)dxds$$

$$= \int_{\mathbb{R}^d}u(t,X(t,0,x))\varphi(t,X(t,0,x))J(t,0,x)dx - \int_{\mathbb{R}^d}u_0(x)\varphi(0,x)J(0,0,x)dx$$

Using the fact that $J(0,0,x) = 1$ given in the proposition 1.4, and remarking that :

$$\frac{\partial}{\partial s}(\varphi(s,X(s,0,x))) = \partial_s\varphi(s,X(s,0,x)) + \phi(s,X(s,0,x))\nabla_x\varphi(s,X(s,0,x))$$

we get:

$$\begin{aligned}\int_0^t \int_{\mathbb{R}^d} f(s,x) \frac{\partial}{\partial s}(\varphi(s, X(s,0,x)))dxds &= \int_{\mathbb{R}^d} f(t,x)\varphi(t, X(t,0,x))dx \\ &- \int_{\mathbb{R}^d} f(0,x)\varphi(0,x)dx \qquad (1.17)\end{aligned}$$

Let $\mathcal{A}$ be the set of the solutions for (1.3) with a compact support. By the Theorem 1.2 we know that $\mathcal{A} \neq \emptyset$.
Let $\varphi \in \mathcal{A}$, then φ is constant along the characteristics :

$$\forall s \in [0,T], \quad \frac{\partial}{\partial s}(\varphi(s, X(s,0,x))) = 0$$

Thus, (1.17) gives:

$$\begin{aligned}\int_{\mathbb{R}^d} f(0,x)\varphi(0,x)dx &= \int_{\mathbb{R}^d} f(t,x)\varphi(t, X(t,0,x))dx \\ &= \int_{\mathbb{R}^d} f(t,x)\varphi(0,x)dx\end{aligned}$$

A consequence of the Theorem 1.2 is :

$\forall \varphi_c \in \mathcal{C}_c^1(\mathbb{R}^d)$, *if we take* $\varphi(.,.) = \varphi_0(X(0,.,.))$, *we have* $\varphi \in \mathcal{A}$.

Using this remark we get:

$$\forall \varphi_0 \in \mathcal{C}_c^1(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} f(t,x)\varphi_0(x)dx = \int_{\mathbb{R}^d} f(0,x)\varphi_0(x)dx$$

Thus,

$$\forall (t,x) \in [0,T] \times \mathbb{R}^d, \quad f(t,x) = f(0,x) = u_0(x)$$

Since $X(0,t,.)$ is a diffeomorphism we get:

$$\forall (t,x) \in [0,T]\times\mathbb{R}^d, \quad f(t, X(0,t,x)) = u(t,x)J(t,0,X(0,t,x)) = u_0(X(0,t,x))$$

hence,

$$\forall (t,x) \in [0,T] \times \mathbb{R}^d, \quad u(t,x) = u_0(X(0,t,x))J(0,t,x)$$

which prove the uniqueness of the solution.

⋆ ⋆ ⋆ ⋆

Application

As for the previous section, we are going to use the previous theorem to prove the existence and uniqueness of weak solutions for (1.11). Here, we assume that:

$$\nabla_v F = 0$$

in order to write the problem (1.11) in the form of the problem (1.9). In deed, under this assumption, $\phi(t,(x,v)) = (v, F(t,x,v))$, and $\nabla_{(x,v)}\phi = 0$ which means that $\nabla_{(x,v)}(\phi f) = \phi \nabla_{(x,v)} f$.

Proposition 1.7 *If $f_0 \in L^1(\mathbb{R}^d \times \mathbb{R}^d)$ and F satisfy the two following conditions:*

1. $F \in \mathcal{C}^1([0,T] \times \mathbb{R}^d \times \mathbb{R}^d)$ *and $\partial_x F$ exists and belongs to $\mathcal{C}^1([0,T] \times \mathbb{R}^d \times \mathbb{R}^d)$.*

2. $\exists C > 0, \quad |F(t,x,v)| \leq C(1 + |x| + |v|)$ *for all $(t,x) \in [0,T] \times \mathbb{R}^d$.*

Then,

$$(t,x,v) \longmapsto f_0(X(0,t,x,v), V(0,t,x,v))$$

is the unique weak solution for (1.11)

Proof 1.7 *It's a direct application of the theorem 1.5 for the problem (1.15). In this case $\phi(t,(x,v)) = (v, F(t,x,v))$, and $\nabla_{(x,v)}\phi = 0$, thus :*

$$\forall (t,x,v) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d, \quad J(0,t,x,v) = 1$$

$\star \quad \star \quad \star \quad \star$

Example

We consider the system (1.11) in the case when $d = 1$ and

$$F(t,x,v) = -Kx$$

with $K > 0$. In this case the system is written on the following form :

$$\begin{cases} \partial_t f + v.\partial_x f + F(t,x,v).\partial_v f = 0 \\ f_{|t=0} = f_0 \end{cases}$$

We are modeling here a system of particles in one dimension space with a restoring force of stiffness K, and we start with a very smooth repartition function:

$$f_0(x,v) := exp(-(x^2 + (v-1)^2))$$

We could have started with a less regular function. In fact, according to the results given in the previous section, the solution still exist and it is also unique. But for simplicity, we start with a very smooth function.

In the case of one particle, the system is an harmonic oscillator, and the trajectory of the particle in phase space is an ellipse. The Figure 6 gives the trajectory of an harmonic oscillator in phase space. Let's see what we get with the Kinetic theory description. Let's start by calculating the characteristics :

For all $(s,t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$,

$$(X(s,t,x,v), V(s,t,x,v)) = \exp((s-t)A).\begin{pmatrix} x \\ v \end{pmatrix}$$

with

$$A := \begin{pmatrix} 0 & 1 \\ -K & 0 \end{pmatrix}$$

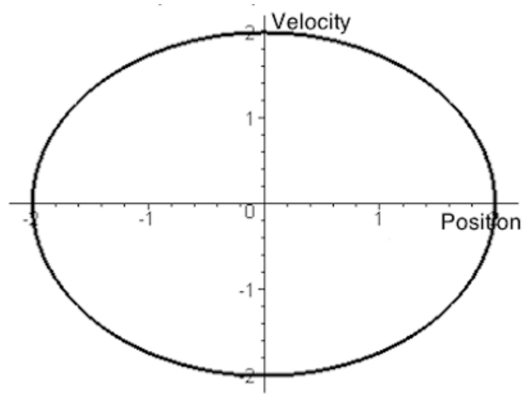

Figure 6: Phase portrait of an harmonic oscillator

using `Maple` we get :

$$\forall (s,t) \in \mathbb{R}^+ \times \mathbb{R}^+, \ \exp((s-t)A) = \begin{pmatrix} \cos(\sqrt{K}(s-t)) & \frac{1}{\sqrt{K}} \sin(\sqrt{K}(s-t)) \\ -\sin(\sqrt{K}(s-t))\sqrt{K} & \cos(\sqrt{K}(s-t)) \end{pmatrix}$$

Using the theorem 1.4 we can get a simulation of the evolution in time of the solution for different initial data f_0. In fact,

$$f(t,x,v) = f_0(\cos(\sqrt{K}t)x - \frac{1}{\sqrt{K}} \sin(\sqrt{K}t)y, \sin(\sqrt{K}t)\sqrt{K}x + \cos(\sqrt{K}t)y)$$

With `Grapher` we plot the solution for different times. The figure 7 gives the evolution in time of the solution for the initial data:

$$f_0(x,v) := exp(-(x^2 + (v-1)^2))$$

The initial distribution rotates around the origin with a certain deformation related to dispersion phenomenon. This general behavior of the solution is expected, since the trajectory of each particle in the phase space is an ellipse. The deformation is due to the fact that the particles do not have the same initial condition, thus the ellipse parameters are specific for each particle. Furthermore, The evolution of the density translates the oscillatory movement of the particles around the origin.

The general behavior of this system translate simply the behavior of each particle. This is related to the fact that, a particle is not acting on the other particles. There's no collisions, and the force doesn't depends on the particle distribution.

For other systems, the evolution of f is not directly related to the behavior of each particle, since the force it self is depending on the distribution function.
In the case of a plasma, the fields created by the distribution of particles acts on the particles themselves. Moreover, since the number of particles considered here is very large, it is expected to have some **collective effects**. To be able to see what we mean here by collective effects, we will give an exemple of one of these phenomenons called **plasma oscillations** or **Langmuir waves**.

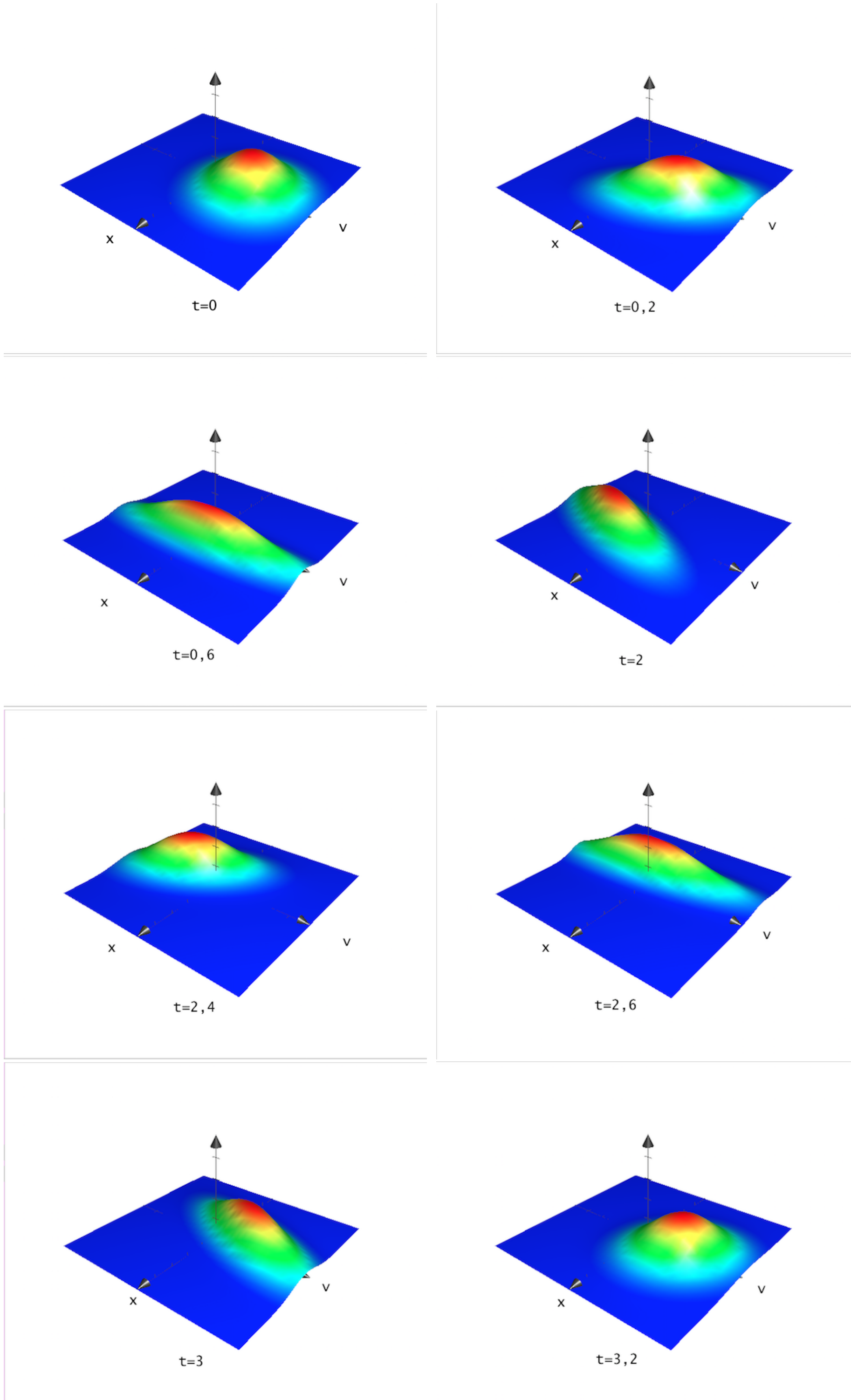

Figure 7: The evolution of the solution in different times for the initial data $f_0(x,v) = exp(-(x^2 + (v-1)^2))$.

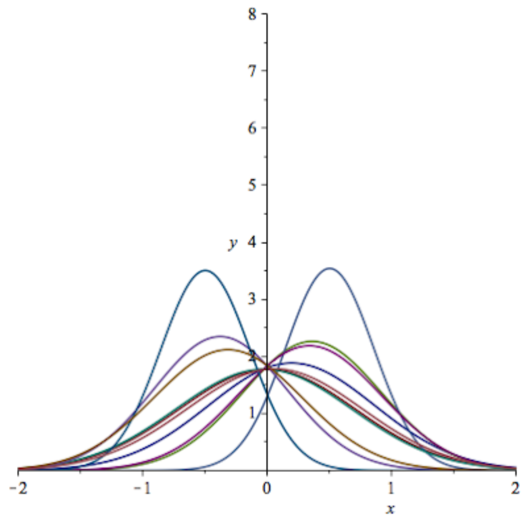

Figure 8: Oscillations of macroscopic density for $K = 4$

We start with a plasma with two charged species $+e$ and $-e$ with the same mass, and we disturb it with an uniform and constant electric field. This creates a displacement of the positive charged particles from x_0 to $x_0 + \zeta(x_0, t)$ and the negative ones from x_0 to $x_0 - \zeta(x_0, t)$. Hence, we get situation illustrated in the figure 9.

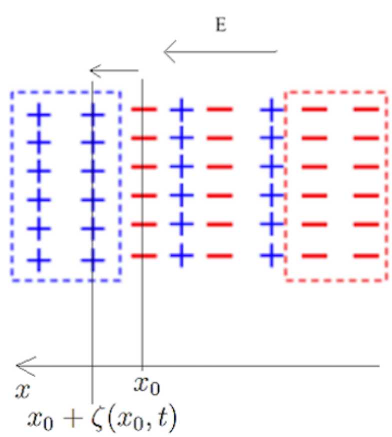

Figure 9: Displacement of charges after the electric perturbation

The inhomogeneity of the distribution of charges creates an electric field that can be calculated using **Gauss theorem** :

$$E = \frac{\rho_0 e \zeta(x_0, t)}{\varepsilon_0}$$

with ρ_0 the particle density in the equilibrium and ε_0 the vacuum permittivity.

Hence, the equation of mouvement is:

$$\frac{\partial^2 \zeta}{\partial t^2} + \frac{\rho_0 e^2}{\varepsilon_0 m_e}\zeta = 0$$

with m_e the mass of the charged particles. Thus, the particles oscillates around x_0, with the following frequency, called **Langmuir frequency**:

$$\omega_p := \sqrt{\frac{\rho_0 e^2}{\varepsilon_0 m_e}}$$

Hence, the behavior of this system is not directly related to what each particle does under the effect of the disturbance. In the following sections we will study, two examples of these non-linear systems : Vlasov-Poisson system, and Vlasov-Maxwell system, which can also describe both the behavior of a plasma.

2 The Vlasov-Poisson system

The Vlasov-Poisson system is a model where the Vlasov equation is coupled with poisson's equation, relative to a system of particles with Newtonian interactions. Vlasov- Poisson's model, is usually used in galactic dynamics and plasma physics. In the case of a plasma with only an electric field,[18] we consider the Coulombian interactions between particles. Then, the Vlasov-Poisson system is the combination of Vlasov equation and the electric poisson's equation. For simplicity, we assume that all particles in the Plasma are identically charged. In this case, the force is repulsive.

In galactic dynamics, we can think of the particles as stars in a galaxy , then we can study the evolution of the galaxy using the Vlasov-Poisson system. In this case the interaction force is the gravitational force which is attractive and the Vlasov-Poisson system is the combination of Vlasov equation and the gravitational Poisson's equation. Vlasov equation have played a great role in galactic dynamics. In fact, a good approximation to the orbit of any star can be obtained using Vlasov equation. The true orbit deviates significantly from the orbit of the model, however, in systems with more than thousand stars, the deviation is small in some time interval [1]. To take into account the two cases we introduce a constant γ with $\gamma = -1$ in the case of electric plasma with identical charges and $\gamma = 1$ in the gravitational case. Physical constants : masses, charges, permittivity and the gravitational constant are all chosen equal to 1. The results given in this section are mostly taken from [4] , [3] and [13].

In this section the force is not depending on v, and we denote for each $(t, x, v) \in \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^3$:

$$F(t, x, v) = F(t, x) = E_\rho(t, x)$$

with E_ρ the electric field in the case of a plasma, and the gravitational force in the gravitational case. The notation E_ρ is used to indicate that in the two cases the forces are depending on the macroscopic density of particles. The potential V_ρ defined in (2.1) is also related to the macroscopic density with Poisson's equation (2.2):

$$\begin{aligned} E_\rho &= -\nabla_x V_\rho && (2.1) \\ \Delta V_\rho &= \gamma \rho && (2.2) \end{aligned}$$

Furthermore, we assume that the potential is zero at infinity, which is physically justified since the force has a weak range :

$$\lim_{|x| \to \infty} V_\rho(t, x) = 0 \tag{2.3}$$

Under some assumptions, we can solve (2.2) assuming (2.3) and get:

$$V_\rho(t, x) = -\frac{\gamma}{4\pi} \int_\Omega \frac{\rho(t, y)}{|x - y|} dy \tag{2.4}$$

[18]The Vlasov-Maxwell model is used for magnetic plasmas, such as tokamak, or the solar wind, for instance.

assuming that the functions are regular enough and using (2.1) we can also get :

$$E_\rho(t,x) = -\frac{\gamma}{4\pi}\int_\Omega \frac{x-y}{|x-y|^3}\rho(t,y)dy \tag{2.5}$$

or:

$$E_\rho = \nabla W * \rho = \nabla W * \int f dv$$

with W is the function defined by $W(x) := -\frac{\gamma}{4\pi}\frac{1}{|x|}$.
Hence,Vlasov-Poisson equation can be written on the following form:

$\forall (t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^3,$

$$\partial_t f + v.\nabla_x f - (\nabla W * \int f dv).\nabla_v f = 0 \tag{2.6}$$

In this section we are interested in the following system, called the Vlasov-Poisson system with the a initial data f_0:

$$\begin{cases} \partial_t f + v.\nabla_x f - \nabla_x V_\rho.\nabla_v f = 0 \\ \rho = \int_{\mathbb{R}^3} f dv \\ \Delta V_\rho = \gamma\rho \\ \lim_{|x|\to\infty} V_\rho(t,x) = 0 \\ f_{|t=0} = f_0 \end{cases} \tag{2.7}$$

Under assumptions on ρ, the following Lemma proves (2.4) and (2.5), and gives some estimates on the field and derivatives of the field.

Lemma 2.1 *Let $t \in \mathbb{R}^+$,*
If $\rho(t,.) \in \mathcal{C}_c^1$ then we have the following results :

1. *$V_\rho(t,.)$ given by (2.4) is the unique solution for :*

$$\begin{cases} \Delta V_\rho(t,.) = \gamma\rho(t,.) \\ \lim_{|x|\to\infty} V_\rho(t,x) = 0 \end{cases} \tag{2.8}$$

   in $C^2(\mathbb{R}^3)$.

2. $\forall x \in \mathbb{R}^3$,

$$\nabla_x V_\rho(t,x) = \gamma\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{x-y}{|x-y|^3}\rho(t,y)dy$$

$$V_\rho(t,x) = \mathcal{O}(\frac{1}{|x|}) \quad \textit{and} \quad \nabla_x V_\rho(t,x) = \mathcal{O}(\frac{1}{|x|^2}) \quad \textit{for} \quad |x| \longrightarrow \infty$$

3. $\forall p \in [1,3[$,

$$\|\nabla_x V_\rho(t,.)\|_\infty \leq c_p \|\rho(t,.)\|_p^{p/3} \|\rho(t,.)\|_\infty^{1-p/3}$$

   where $c_p > 0$ is a constant depending only on p.

4. $\exists c > 0$ *for any* $0 < d \leq R$,

$$\|\partial_x^2 V_\rho(t,.)\|_\infty \leq c[\frac{\|\rho(t,.)\|_1}{R^3} + d\|\nabla_x \rho(t,.)\|_\infty + (1+\ln(R/d))\|\rho(t,.)\|_\infty]$$

Proof 2.1 *Let* K_ρ *be the support of* ρ, *and* $\eta > 0$ *such that* $K_\rho \subset B(0,\eta)$.

1. • *(Existence): Let* $t \in \mathbb{R}^+$,

$$x \to V_\rho(t,x) = -\frac{\gamma}{4\pi}\int_{\mathbb{R}^3} \frac{\rho(t,y)}{|x-y|}dy$$

is in deed a solution for (2.8). In fact:
- $V_\rho(t,.)$ *is well defined:*

$$\begin{aligned}\int_{\mathbb{R}^3} \frac{\rho(t,y)}{|x-y|}dy = \int_{B(0,\eta)} \frac{\rho(t,y)}{|x-y|}dy &\leq \|\rho(t,.)\|_\infty \int_{B(0,\eta)} \frac{1}{|x-y|}dy \\ &\leq \|\rho(t,.)\|_\infty \int_{B(x,\eta)} \frac{1}{|y|}dy \\ &\leq \int_0^{2\pi}\int_0^{\pi}\int_0^{\eta} \frac{1}{\eta} sin\theta\ \eta^2 d\eta d\theta d\phi < \infty\end{aligned}$$

- Moreover, V_ρ *is* $\mathcal{C}^2$ *and* $\Delta V_\rho(t,.) = \gamma\rho(t,.)$.
In fact:

– *Let* $i \in \{1,2,3\}$, $(t,u) \in \mathbb{R}^+ \times \mathbb{R}^3$ *and* e_i *the i-th vector of the canonic base of* $\mathbb{R}^3$.
We have that:

$$\frac{V_\rho(t,u+he_i) - V_\rho(t,u)}{h} = -\frac{\gamma}{4\pi}\int_{\mathbb{R}^3} W(y)[\frac{\rho(t,u+he_i-y) - \rho(t,u-y)}{h}]dy$$

Moreover, ρ *being compactly supported,*

$$[\frac{\rho(t,u+he_i-y) - \rho(t,u-y)}{h}] \to \partial_{u_i}[\rho(t,u-y)] = \partial_{x_i}\rho(t,u-y)$$

[19] *uniformely on* $\mathbb{R}^3$ *as* $h \to 0$;
Hence,

$$\nabla_x V_\rho(t,u) = -\frac{\gamma}{4\pi}\int_{\mathbb{R}^3} W(y)\nabla_x \rho(t,u-y)dy$$

This expression gives also that $V_\rho(t,.) \in \mathcal{C}^1(\mathbb{R}^3)$. *In fact, this property is a consequence of the* **Lebesgue Theorem**. *Since the function inside the integral has the required regularity, we only need to check* the domination hypothesis , *on a compact set* $\mathcal{K} \subset \mathbb{R}^3$:

[19] The derivatives relatively to x denotes the derivatives relatively to the second variable.

For $R > 0$, $u \in \mathbb{R}^3$ and $i \in \{1,2,3\}$ we have:

$$\begin{aligned}\partial_{x_i} V_\rho(t,u) &= -\frac{\gamma}{4\pi}\int_{\mathbb{R}^3} W(y)\partial_{x_i}\rho(t,u-y)dy \\ &= -\frac{\gamma}{4\pi}\Big(\underbrace{\int_{B(0,R)} W(y)\partial_{x_i}\rho(t,u-y)dy}_{I_1} + \underbrace{\int_{\|y\|\geq R} W(y)\partial_{x_i}\rho(t,u-y)dy}_{I_2}\Big)\end{aligned}$$

taking $R = \sup_{u\in\mathcal{K}} |u| + \eta$ we have that[20] *:*

$$I_2 = 0$$

for I_1 the domination is given by:

$$\forall x \in \mathcal{K}, \forall y \in B(0,R), \quad |W(y)\partial_{x_i}\rho(t,x-y)| \leq \|\partial_{x_i}\rho(t,.)\|_\infty \qquad \underbrace{|W(y)|}_{\textit{integrable on } B(0,R)}$$

Since we have only $\mathcal{C}^1$ regularity assumption on ρ, we can't use the same technique as above to get the second order derivative. To avoid this difficulty, we use the following calculations :

Notice that without loss of generality, we can always assume that $0 \in \overset{\circ}{K}_\rho$. Let $\varepsilon > 0$ such that $B(0,\varepsilon) \subset K_\rho$,[21] *and $u \in K_\rho$,*

$$\begin{aligned}\nabla_x V_\rho(t,u) &= -\frac{\gamma}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|y|}\nabla_x\rho(t,u-y)dy \\ &= -\frac{\gamma}{4\pi}\Big(\underbrace{\int_{\mathbb{R}^3 - B(0,\varepsilon)} \frac{1}{|y|}\nabla_x\rho(t,u-y)dy}_{J_\varepsilon} + \underbrace{\int_{B(0,\varepsilon)} \frac{1}{|y|}\nabla_x\rho(t,u-y)dy}_{I_\varepsilon}\Big)\end{aligned}$$

we have that:

$$|I_\varepsilon| \leq \|\nabla_x\rho(t,.)\|_\infty \int_{B(0,\varepsilon)} \frac{1}{|y|}dy \leq \underbrace{\|\nabla_x\rho(t,.)\|_\infty 2\pi\varepsilon^2}_{\overset{\varepsilon\to 0}{\longmapsto} 0}$$

hence,

$$\lim_{\varepsilon\to 0} I_\varepsilon = 0$$

Furthermore, using **Green's formula**, *we have that:*

$$\begin{aligned}J_\varepsilon &= -\int_{B(0,\varepsilon)} \frac{1}{|y|}\nabla_y[\rho(t,u-y)]dy \\ &= -\underbrace{\int_{\partial B(0,\varepsilon)} \frac{1}{|y|}\rho(t,x-y)dS(y)}_{K_\varepsilon} - \int_{\mathbb{R}^3 - B(0,\varepsilon)} \frac{y}{|y|^3}\rho(t,x-y)dy\end{aligned}$$

[20] $|u-y| \geq |y| - |u| \geq \eta$

[21] Such an ε exists since ρ is a continuous function and $\rho \neq 0$ (i.e: $\overset{\circ}{K}_\rho \neq \emptyset$).

moreover:

$$|K_\varepsilon| \leq \underbrace{\|\rho(t,.)\|_\infty 2\pi\varepsilon}_{\overset{\varepsilon\to 0}{\longmapsto} 0}$$

thus,

$$\lim_{\varepsilon\to 0} K_\varepsilon = 0$$

Taking the limit when $\varepsilon \to 0$, and shifting the convolution product, we finally get:

$$\nabla_x V_\rho(t,x) = \frac{\gamma}{4\pi}\int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3}\rho(t,y)dy$$

Having this expression, we can now use the same technique as for the first derivative, since here the derivation doesn't require higher regularity for ρ.

$$\frac{\nabla_x V_\rho(t,u+he_i) - \nabla_x V_\rho(t,u)}{h} = -\frac{\gamma}{4\pi}\int_{\mathbb{R}^3} \nabla W(y)[\frac{\rho(t,u+he_i-y)-\rho(t,u-y)}{h}]dy$$

Taking the limit where h goes to infinity (uniformly), we get :

$$\forall (i,j) \in \{1,2,3\}^2, \quad \partial_{x_i}\partial_{x_j} V_\rho(t,u) = -\frac{\gamma}{4\pi}\int_{\mathbb{R}^3} \partial_{x_i} W(y).\partial_{x_j}\rho(t,u-y)dy$$

which gives:

$$\nabla_x^2 V_\rho(t,u) = \Delta_x V_\rho(t,u) = -\frac{\gamma}{4\pi}\int_{\mathbb{R}^3} \nabla W(y).\nabla_x \rho(t,u-y)dy$$

Since $\forall i \in \{1,2,3\}$, $\partial_{x_i} W(t,.)$ is integrable around the origine, we can use the same steps as for the $\mathcal{C}^1$ case, to prove that $V_\rho(t,.)$ is $\mathcal{C}^2$.

– *Let's prove now that V_ρ is in deed a solution for the equation. We start with the last expression found above. We have that :*

$$\forall x \in \mathbb{R}^3, \quad \Delta_x V_\rho(t,x) = -\frac{\gamma}{4\pi}\int_{\mathbb{R}^3} \nabla W(y).\nabla_x \rho(t,x-y)dy$$

Let $\varepsilon > 0$ and $x \in \mathbb{R}^3$,

$$\begin{aligned}
\Delta_x V_\rho(t,x) &= -\frac{\gamma}{4\pi}\int_{\mathbb{R}^3} \nabla W(y).\nabla_x \rho(t,x-y)dy \\
&= -\frac{\gamma}{4\pi}\int_{B(0,\varepsilon)} \nabla W(y).\nabla_x \rho(t,x-y)dy \\
&\quad -\frac{\gamma}{4\pi}\underbrace{\int_{\mathbb{R}^3 - B(0,\varepsilon)} \nabla W(y).\nabla_x \rho(t,x-y)dy}_{=K_\varepsilon}
\end{aligned}$$

moreover,

$$\begin{aligned}|\int_{B(0,\varepsilon)} \nabla W(y).\nabla_x \rho(t, x-y)dy| &\leq \|\nabla_x \rho(t,.)\|_\infty \int_{B(0,\varepsilon)} \nabla W(y)dy \\ &\leq \underbrace{\|\nabla_x \rho(t,.)\|_\infty \varepsilon}_{\longrightarrow 0}\end{aligned}$$

furthermore, we can prove that: $K_\varepsilon(x) \longrightarrow -4\pi\rho(x)$ *when* ε *tends to 0. [3, pages 24-25].*
Taking the limit when ε *goes to 0, we get finally that:*

$$\Delta_x V_\rho = \gamma\rho$$

- Let (x_n) *a sequence of* $\mathbb{R}^3$ *such that* $|x_n| \to \infty$. *We have that:*

$$\lim_n V_\rho(t, x_n) = 0$$

In fact :

$$\frac{1}{4\pi}\int_{K_\rho} \frac{\rho(t,y)}{|x_n - y|}dy \leq \frac{\|\rho\|_{\infty,K_\rho}}{4\pi} \underbrace{\int_{K_\rho} \frac{1}{|x_n - y|}dy}_{<\infty}$$

using **the dominated convergence theorem**[22],

$$\frac{\|\rho\|_{\infty,K_\rho}}{4\pi}\int_{K_\rho} \frac{1}{|x_n - y|}dy \longrightarrow_{n\to\infty} 0$$

In deed, $\exists N > 0$ *such that* $\forall n \geq N$, $|x_n| \geq 1 + \max_{y\in K_\rho}(|y|)$. *Then we have the domination :*

$$\forall n \geq N, \ \frac{1}{|x_n - y|} \leq 1$$

Conclusion :

$$\lim_{|x|\to\infty} V_\rho(t, x) = 0$$

- *(Uniqueness) Let* $u(t,.)$ *and* $v(t,.)$ *be two solutions for (2.8). Then*

$$w(t,.) := u(t,.) - v(t,.)$$

is a solution for **Laplace's equation***:*

$$\left\{\begin{array}{l}\Delta w(t,.) = 0 \\ \lim_{|x|\to\infty} w(t,x) = 0\end{array}\right. \tag{2.9}$$

Let $\varepsilon > 0$.

$$\begin{aligned}\exists A_\varepsilon > 0, \quad |x| \geq A_\varepsilon \ &\Rightarrow \ |w(t,x)| \leq \varepsilon \\ &\Rightarrow \ \left\{\begin{array}{l} w(t,x) \leq \varepsilon \\ -w(t,x) \leq \varepsilon\end{array}\right.\end{aligned}$$

[22]Theorem B.6 (Appendix).

hence,

$$\begin{cases} \Delta w(t,x) = 0 & x \in B(0, A_\varepsilon) \\ \pm w(t,x) \leq \varepsilon & x \in \partial B(0, A_\varepsilon) \end{cases}$$

using the **strong maximum principale***([3, page 27]) for* w *and* $-w$ *we finally get:*

$$\forall \varepsilon > 0,\ \forall x \in \mathbb{R}^3, \quad |w(t,x)| \leq \varepsilon$$

Which gives $w(t,.) = 0$*, and proves uniqueness.*

2. • *From the first point (above), we have that :*

$$\nabla_x V_\rho(t,x) = \frac{\gamma}{4\pi} \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \rho(t,y) dy$$

• *Let* $x \in \mathbb{R}^3$ *and* $t \geq 0$.

$$|x||V_\rho(t,x)| \leq \frac{\|\rho(t,.)\|_\infty}{4\pi} \int_{K_\rho} \frac{|x|}{|x-y|} dy$$

and

$$|x|^2 |\nabla_x V_\rho(t,x)| \leq \frac{\|\rho(t,.)\|_\infty}{4\pi} \int_{K_\rho} \frac{|x|^2}{|x-y|^2} dy$$

For $|x| \geq 2.\sup_{y \in K_\rho} |y|$*, we have:*

$$\begin{aligned} |x||V_\rho(t,x)| &\leq \frac{\|\rho(t,.)\|_\infty}{4\pi} \int_{K_\rho} \frac{1}{1 - \frac{|y|}{|x|}} dy \\ &\leq \frac{\|\rho(t,.)\|_\infty}{2\pi} \nu(K_\rho) \end{aligned}$$

and

$$\begin{aligned} |x|^2 |\nabla_x V_\rho(t,x)| &\leq \frac{\|\rho(t,.)\|_\infty}{4\pi} \int_{K_\rho} \frac{1}{(1 - \frac{|y|}{|x|})^2} dy \\ &\leq \frac{\|\rho(t,.)\|_\infty}{\pi} \nu(K_\rho) \end{aligned}$$

which proves that,

$$|x| V_\rho(t,x) = \mathcal{O}(1) \quad \textit{and} \quad |x|^2 \nabla_x V_\rho(t,x) = \mathcal{O}(1) \quad \textit{for} \quad |x| \longrightarrow \infty$$

3. • *Let $p \in]1,3[$, $x \in \mathbb{R}^3$, $t \in \mathbb{R}^+$ and $R > 0$:*

$$\begin{aligned}
|\nabla_x V_\rho(t,x)| &\leq \frac{1}{4\pi}[\int_{|x-y|<R} \frac{|\rho(t,y)|}{|x-y|^2}dy + \int_{|x-y|\geq R} \frac{|\rho(t,y)|}{|x-y|^2}dy] \\
&\leq \frac{1}{4\pi}[\|\rho(t,.)\|_{\infty,K_\rho} \int_{|x-y|<R} \frac{1}{|x-y|^2}dy \\
&\quad \underbrace{+ \|\rho(t,.)\|_p(\int_{|x-y|\geq R} \frac{1}{|x-y|^{2q}}dy)^{\frac{1}{q}}]}_{Holder's\ inequality\ with\ \frac{1}{p}+\frac{1}{q}=1} \\
&\leq \frac{1}{4\pi}[\|\rho(t,.)\|_{\infty,K_\rho} \int_0^{2\pi}\int_0^{\pi}\int_0^{R} sin\theta dr d\theta d\phi \\
&\quad + \|\rho(t,.)\|_p \underbrace{(\int_0^{2\pi}\int_0^{\pi}\int_R^{+\infty} r^{2-2q} sin\theta dr d\theta d\phi)^{\frac{1}{q}}]}_{<\infty\ since\ p<3\Rightarrow 2q-2>1} \\
&\leq \|\rho(t,.)\|_\infty R + \|\rho(t,.)\|_p(\frac{R^{3-2q}}{3-2q})^{\frac{1}{q}}
\end{aligned}$$

We optimize this estimate by choosing $R = (c\frac{\|\rho\|_p}{\|\rho\|_\infty})^{p/3}$ with a suitable constant $c > 0$. Then :

$$|\nabla_x V_\rho(t,x)| \leq c_p \|\rho(t,.)\|_p^{p/3} \|\rho(t,.)\|_\infty^{1-p/3}$$

with $c_p > 0$ a constant.

• *If p=1, let $x \in \mathbb{R}^3$, $t \in \mathbb{R}^+$ and $R > 0$:*

$$\begin{aligned}
|\nabla_x V_\rho(t,x)| &\leq \frac{1}{4\pi}[\int_{|x-y|<R} \frac{|\rho(t,y)|}{|x-y|^2}dy + \int_{|x-y|\geq R} \frac{|\rho(t,y)|}{|x-y|^2}dy] \\
&\leq \|\rho(t,.)\|_\infty R + \frac{1}{R^2}\|\rho(t,.)\|_1
\end{aligned}$$

Taking $R = (c\frac{\|\rho\|_1}{\|\rho\|_\infty})^{1/3}$ with a constant $c > 0$, we finally get:

$$|\nabla_x V_\rho(t,x)| \leq c_1 \|\rho(t,.)\|_1^{1/3} \|\rho(t,.)\|_\infty^{2/3}$$

with $c_1 > 0$ a constant.

4. A proof of this fact is given in [13, page 7].

⋆ ⋆ ⋆ ⋆

2.1 LOCAL EXISTENCE FOR THE VLASOV-POISSON SYSTEM

The local existence and uniqueness result is the necessarily starting step for any other investigation. Let's first start by defining what we mean by a classical solution for the Vlasov-Poisson system.

Definition 2.1 *A function* $f : I \times \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty[$ *is a classical solution on* I [23] *for the Vlasov-Poisson system with* $f(0, ., .) = f_0$ *if the following holds:*

1. $f \in \mathcal{C}^1(I \times \mathbb{R}^3 \times \mathbb{R}^3)$
2. *If* ρ *is the macroscopic density associated to* f, $\rho \in \mathcal{C}^1(I \times \mathbb{R}^3)$, *and* $V_\rho \in \mathcal{C}^1(I \times \mathbb{R}^3)$. *Furthermore,* V_ρ *is twice continuously differentiable with respect to* x.
3. *For any compact subinterval* $J \subset I$, $\partial_x V_\rho$ *is bounded on* $J \times \mathbb{R}^3$.
4. f, ρ *and* V_ρ *satisfy the Vlasov-Poisson system (2.7).*

The following existence result does not only provides a local result of existence and uniqueness, but also says in which way the solution can stop to exist at a finite time.

<u>Theorem</u> **2.1 (Local existence and uniqueness for VPS)** *If* $f_0 \in \mathcal{C}_c^1(\mathbb{R}^3 \times \mathbb{R}^3)$ *and* $f_0 \geq 0$. *The Vlasov-Poisson system (2.7) has a unique solution on some time interval* $[0, T[$, *and for all* $t \in [0, T[$, f *is compactly supported and positive. Furthermore, for a maximal* $T > 0$, *if :*

$$\sup_{t \in [0,T[} \sup\{|v| / \ \exists x \in \mathbb{R}^3, (x, v) \in supp(f(t, ., .))\} < \infty$$

or

$$\sup\{\rho(t, x) / \ t \in [0, T[, x \in \mathbb{R}^3\} < \infty$$

then $T = \infty$.

Proof for the Theorem **2.1** *For a later use let :* $P > 0$ *and* $Q > 0$ *such that:*

$$f_0(x, v) = 0 \ \textit{if} \ |x| \geq P \ \textit{or} \ |v| \geq Q$$

The proof follows different steps. The main idea of the proof is to use an iterative scheme to avoid the nonlinearity, then it will be possible to use the properties of the linear problem given in section 1. The estimations on the force and the gradient of the force given in lemma 2.1, will allow us to control the size of the support and prove the convergence of the scheme. We define the following iterative scheme :

$$\begin{cases} \forall (t, z) \in \mathbb{R}^+ \times \mathbb{R}^6, \ f_0(t, z) := f_0(z) \\ \\ \forall n \in \mathbb{N}, \ \rho_n := \int f_n dv \ \textit{and} \ W_n := V_{\rho_n} \\ \\ \forall n \in \mathbb{N}, \ Z_n \ \textit{are the solutions for the charactertics system :} \ \dot{x} = v, \quad \dot{v} = -\nabla_x W_n(s, x)). \\ \\ \forall n \in \mathbb{N}, \forall (t, z) \in \mathbb{R}^+ \times \mathbb{R}^6, \ f_{n+1}(t, z) := f_0(Z_n(0, t, z)) = f_0(X_n(0, t, z), V_n(0, t, z)) \end{cases}$$

[23] I is an interval of $\mathbb{R}^+$ that contains 0.

- *(step 0)[Observations] We aim to prove that this scheme converges in enough strong sens so that the limit be a classical solution in the sens of the previous definition. Furthermore, to prove the criteria of prolongation, we also need to have an information on the evolution of the support. That's why we introduce the following quantities, which are very important for the proof :*

$$\begin{cases} \forall t \geq 0, \; Q_0(t) := Q \\ \forall n \in \mathbb{N}, \; Q_{n+1}(t) := \sup\{|V_n(s,0,z)| / z \in supp(f_0), 0 \leq s \leq t\} \end{cases}$$

We assume here that $(f_n), (\rho_n), (Z_n)$ and (W_n) are well defined [24]*, we have that:*

$$\forall n \in \mathbb{N}, \forall t \geq 0, \quad f_n(t,x,v) = 0 \text{ if } |v| \geq Q_n(t) \text{ or } |x| \geq P + \int_0^t Q_n(s)ds$$

and

$$\begin{aligned} \forall n \in \mathbb{N}, \forall t \geq 0, \quad \|\rho_n(t,.)\|_\infty &\leq \|f_0\|_\infty \int_{|v| \leq Q_n(t)} dv \\ &\leq \frac{4\pi}{3} \|f_0\|_\infty Q_n^3(t) \end{aligned}$$

Moreover, using the estimates on the forces given in lemma 2.1, we get :

$$\|\nabla_x W_n(t,.)\|_\infty \leq C(f_0) Q_n^2(t)$$

with $C(f_0) = c_1 \|f_0\|_\infty^{2/3} \|f_0\|_1^{1/3}$ and c_1 denotes the constant of the lemma 2.1. Using the characteristic system, for $z = (x,v) \in supp(f_0)$ and $n \in \mathbb{N}$, we have that :

$$\begin{aligned} |V_n(t,0,z)| &= |v| + \int_0^t |\nabla_x W_n(s, X(s,0,z))| ds \\ &\leq Q + \int_0^t \|\nabla_x W_n(s,.)\|_\infty ds \\ &\leq Q + C(f_0) \int_0^t Q_n^2(s) ds \end{aligned}$$

Hence,

$$\forall n \in \mathbb{N}, \forall t \geq 0, \quad Q_n(t) \leq Q + C(f_0) \int_0^t Q_n^2(s) ds$$

This relation characterizes the evolution of the support for the different iterations. If we assume now that we are able to take the limit of n in this relation, the support of the limit function should satisfy a similar relation. And that leads to consider the maximal solution for:

$$y(t) = Q + C(f_0) \int_0^t y^2$$

[24] In this stage of the proof we can't affirm that all the iterations exist. In fact, even for the linear problem we must have conditions to insure existence and uniqueness of the characteristics (see Theorem 1.4).

- *(step 1) Let* $(Q_\infty, [0,\delta[)$ *the maximal solution* [25] *for:*

$$y(t) = Q + C(f_0)\int_0^t y^2 \quad \textit{with} \quad \delta > 0$$

We have that,

$$\forall n \in \mathbb{N}, \forall t \geq 0, \quad Q_n(t) \leq Q_\infty(t)$$

In fact, let $t \in [0,\delta[$, *the assertion being obvious for* $n = 0$, *we suppose that the assertion holds for* $n \in \mathbb{N}$. *Using the previous estimates, we have that :*

$$\begin{aligned} \forall s \in [0,t], \quad |Z_{n+1}(s,0,v)| &\leq |v| + \int_0^s |\nabla_x W_n(u,.)|_\infty du \\ &\leq Q + C(f_0)\int_0^s Q_n^2(u)du \\ &\leq Q + C(f_0)\int_0^s Q_\infty^2(u)du = Q_\infty(s) \\ &\leq Q_\infty(t) \end{aligned}$$

Hence,

$$Q_{n+1}(t) \leq Q_\infty(t)$$

- *(step 2) Let* $0 < T < \delta$, *we have that:*

$$\forall n \in \mathbb{N}, \forall t \in [0,T], \quad \|\nabla_x W_n(t,.)\|_\infty \leq C(f_0)Q_\infty(T)$$

Hence, using theorem (1.4), and for all $n \in \mathbb{N}$, *the characteristics* Z_n *exist on* $[0,T]$. *Furthermore, for all* $n \in \mathbb{N}$ *and* $t \in [0,T]$, $\rho_n(t,.)$ *is continuously differentiable* [26] *and compactly supported since,*

$$supp(\rho_n(t,.)) \subset B(0, P + TQ_\infty(T))$$

Thus, using lemma 2.1 and proposition 1.5, the sequences are well defined and enjoy of the following properties :

  - $\forall n \in \mathbb{N}$, $f_n \in \mathcal{C}^1([0,\delta[\times\mathbb{R}^6)$ *and* $\forall t \in [0,\delta[$, $\|f_n(t,.)\|_\infty = \|f_0\|_\infty$ *and* $\|f_n(t,.)\|_1 = \|f_0\|_1$
  - $\forall n \in \mathbb{N}$, $\nabla_x W_n \in \mathcal{C}^1([0,\delta[\times\mathbb{R}^6)$.

- *(step 3) In order to get estimations on the derivatives of the potential, we need some boundaries on the second derivative of the potential, that we will use in the next step to prove the convergence. The constant* $C > 0$ *may takes different values here, but it does not depend on* t *or* n. *Let* $0 < T < \delta$, *we aim to prove*

[25] We recall that :

$$Q_\infty : t \to \frac{Q}{1 - QC(f_0)t} \quad \text{and} \quad \delta = \frac{1}{QC(f_0)}$$

[26] This is a consequence of the integral derivation theorem (Theorem B.8).

that there exist a constant $C > 0$ depending on the initial data and T such that:

$$\forall n \in \mathbb{N}, \forall t \in [0,T], \quad \|\nabla_x \rho_n(t,.)\|_\infty + \|\partial_x^2 W_n(t,.)\|_\infty \leq C$$

Let $t \in [0,T]$, $n \in \mathbb{N}$ and $(x,v) \in \mathbb{R}^6$. First, we have that :

$$\begin{aligned}
\|\nabla_x \rho_{n+1}(t,x)\| &\leq \int_{|v|\leq Q_\infty(t)} |\partial_x Z_n(t,x,v) \nabla_x f_0(Z_n(t,x,v))| dv \\
&\leq \|\partial_x Z_n(t,.)\|_\infty \|\nabla_x f_0\|_\infty \frac{4\pi}{3} Q_\infty^3(T) \\
&\leq C\|\partial_x Z_n(t,.)\|_\infty
\end{aligned} \tag{2.10}$$

moreover,

$$\forall s \in [0,T], \ |\partial_x \dot{X}_n(s,t,z)| \leq |\partial_x V_n(s,t,z)|, \ |\partial_x \dot{V}_n(s,t,z)| \leq \|\partial_x^2 W_n(s,.)\|_\infty |\partial_x X_n(s,t,z)|$$

By the integration of the previous inequalities, adding the results and observing that $\partial_x X_n(t,t,z) = I_3$ and $\partial_x V_n(t,t,z) = 0$, we have that:

$\forall 0 \leq s \leq T,$

$$|\partial_x X_n(s,t,z)| + |\partial_x V_n(s,t,z)| \leq 1 + \int_s^t (1 + \|\partial_x^2 W_n(u)\|_\infty)(|\partial_x X_n(u,t,z)| + |\partial_x V_n(u,t,z)|) du$$

Using **Gronwall's lemma** [27] *we get :*

$$|\partial_x X_n(s,t,z)| + |\partial_x V_n(s,t,z)| \leq exp(\int_0^t (1 + \|\partial_x^2 W_n(u,.)\|_\infty) du)$$

Hence, using the estimate (2.10) on $\nabla_x \rho_n$, we have that:

$$\|\nabla_x \rho_{n+1}(t,.)\|_\infty \leq C \exp(\int_0^t \|\partial_x^2 W_n(u,.)\|_\infty du) \tag{2.11}$$

Using the estimate on the force derivative given in lemma 2.1, (choosing for example $R = d$ with the notation of the lemma), we finally get:

$$\|\partial_x^2 W_{n+1}(t,.)\|_\infty \leq C(1 + \int_0^t \|\partial_x^2 W_n(u,.)\|_\infty du)$$

By induction :

$$\|\partial_x^2 W_n(t,.)\|_\infty \leq Ce^{CT}$$

Using, (2.11) we get that $\|\nabla_x \rho_n(t,.)\|_\infty$ is also bounded.

- *(step 4) The constant $C > 0$ may takes different values here, but it does not depend on t, z or n. Let $0 < T < \delta$, (f_n) converges uniformly on $[0,T]$.*

[27] We use here the generalized form of Gronwall's lemma: Proposition B.2 (Appendix).

In fact, using **Mean value theorem**[28] *, we have that :*

$\forall n \in \mathbb{N}^*, \forall t \in [0,T], \forall z \in \mathbb{R}^6,$

$$|f_{n+1}(t,z) - f_n(t,z)| \leq C|Z_n(0,t,z) - Z_{n-1}(0,t,z)| \tag{2.12}$$

Let $t \in [0,T]$, $z \in \mathbb{R}^6$ *and* $s \in [0,t]$. *We have that :*

$$|X_n(s,t,z) - X_{n-1}(s,t,z)| \leq \int_s^t |V_n(u,t,z) - V_{n-1}(u,t,z)|du$$

moreover,

$$\begin{aligned} |V_n(s,t,z) - V_{n-1}(s,t,z)| &\leq \int_s^t |\nabla_x W_n(u, X_n(u,t,z)) - \nabla_x W_{n-1}(u, X_n(u,t,z))|du \\ &\leq \int_s^t |\nabla_x W_n(u, X_n(u,t,z)) - \nabla_x W_n(u, X_{n-1}(u,t,z))|du \\ &\quad + \int_s^t \|\nabla_x W_n(u,.) - \nabla_x W_{n-1}(u,.)\|_\infty du \end{aligned} \tag{2.13}$$

Using the fact that $\partial_x^2 W_n(t,.)$ *is bounded (from the previous step), and using* **Mean value theorem***, we get:*

$$\begin{aligned} |V_n(s,t,z) - V_{n-1}(s,t,z)| &\leq \int_s^t C|X_n(u,t,z)) - X_{n-1}(u,t,z))|du \\ &\quad + \int_s^t \|\nabla_x W_n(u,.) - \nabla_x W_{n-1}(u,.)\|_\infty du \end{aligned}$$

adding these two estimates, we have that:

$$|Z_n(s,t,z) - Z_{n-1}(s,t,z)| \leq C \int_s^t |Z_n(u,t,z)) - Z_{n-1}(u,t,z))|du + \int_0^t \|\nabla_x W_n(u) - \nabla_x W_{n-1}(u)\|$$

Hence, using **Gronwall's lemma** *, we get :*

$$|Z_n(s,t,z) - Z_{n-1}(s,t,z)| \leq (\int_0^t \|\nabla_x W_n(u,.) - \nabla_x W_{n-1}(u,.)\|_\infty du) \exp(CT)$$

Hence,

$$\begin{aligned} |Z_n(s,t,z) - Z_{n-1}(s,t,z)| &\leq C \int_0^t \|\nabla_x W_n(u,.) - \nabla_x W_{n-1}(u,.)\|_\infty du \\ &\leq C \int_0^t \|\rho_n(u,.) - \rho_{n-1}(u,.)\|_\infty du \\ &\leq C \int_0^t \|f_n(u,.) - f_{n-1}(u,.)\|_\infty du \end{aligned}$$

[28]Theorem B.5

Using (2.12) we get:

$$\forall t \in [0,T], \forall n \in \mathbb{N}, \quad \|f_{n+1}(t,.) - f_n(t,.)\|_\infty \leq C \int_0^t \|f_n(u,.) - f_{n-1}(u,.)\|_\infty du$$

which implies by induction:

$$\forall t \in [0,T], \forall n \in \mathbb{N}, \ \|f_{n+1}(t,.) - f_n(t,.)\|_\infty \leq \frac{C'.C^n t^n}{n!} \leq \frac{C^{n+1}}{n!}$$

thus,

$$\forall n \in \mathbb{N}, \quad \|f_{n+1} - f_n\|_\infty \leq \frac{C'.C^n t^n}{n!} \leq \frac{C^{n+1}}{n!}$$

The series $\sum f_{n+1} - f_n$ is then absolutely convergent in the space of bounded and continuous function on $[0,T] \times \mathbb{R}^6$ with the norme $\|.\|_\infty$. Then by completeness of this space [29]*, (f_n) converges uniformly on $[0,T] \times \mathbb{R}^6$ to some function $f \in \mathcal{C}([0,T] \times \mathbb{R}^6)$. And the function f satisfies the following properties:*

$$\forall t \in [0,T], \ f(t,x,v) = 0 \ \textit{if} \ \ |v| \geq Q_\infty(t) \ \textit{or} \ \ |x| \geq P + \int_0^t Q_\infty(s) ds$$

Moreover,

$$\rho_n \to \rho_f = \rho \quad \textit{and} \quad W_n \to V_\rho$$

uniformly on $[0,T] \times \mathbb{R}^3$. In fact,

– $\forall (t,x) \in [0,T] \times \mathbb{R}^3$,

$$\begin{aligned} |\rho_n(t,x) - \rho_f(t,x)| &\leq \int_{|v| \leq Q_\infty} |f_n(t,x,v) - f(t,x,v)| dv \\ &\leq \underbrace{\frac{4\pi Q_\infty^3(T)}{3} \|f_n - f\|_\infty}_{\overset{n\to\infty}{\longrightarrow} 0} \end{aligned}$$

– *Let $R > 0$,* $\forall (t,x) \in [0,T] \times \mathbb{R}^3$,

$$\begin{aligned} 4\pi |W_n(t,x) - V_\rho(t,x)| &\leq \int_{\mathbb{R}^3} \frac{|\rho_n(t,y) - \rho(t,y)|}{|x-y|} dy \\ &\leq \int_{|x-y| \leq R} \frac{|\rho_n(t,y) - \rho(t,y)|}{|x-y|} dy + \\ &\quad \int_{|x-y| > R} \frac{|\rho_n(t,y) - \rho(t,y)|}{|x-y|} dy \\ &\leq \|\rho_n - \rho\|_\infty \frac{4\pi}{3} R^3 + \frac{\|\rho_n(t,.) - \rho(t,.)\|_1}{R} \\ &\leq \underbrace{\|\rho_n - \rho\|_\infty \frac{4\pi}{3} R^3 + \frac{4\pi}{3} \frac{\|\rho_n - \rho\|_\infty}{R} (P + TQ_\infty(T))^3}_{\overset{n\to\infty}{\longrightarrow} 0} \end{aligned}$$

[29] Proposition B.4

- *(step 5) Let $0 < T < \delta$, in this step we aime to prove that the function f defined above, has the required regularity to be a solution on $[0,T] \times \mathbb{R}^6$ for the Vlasov-Poisson system.*
  We have, $\forall (t,x) \in [0,T] \times \mathbb{R}^6$:

$$\begin{aligned} \|\nabla_x W_n - \nabla_x W_m\|_\infty &\leq c_1 \|\rho_m - \rho_n\|_\infty^{2/3} \|\rho_m(t,.) - \rho_n(t,.)\|_1^{1/3} \\ &\leq c_1 \frac{4\pi (P + TQ_\infty(T))^3}{3} \|\rho_m - \rho_n\|_\infty \end{aligned}$$

  Let $\varepsilon > 0$, and $C > 0$ a constant that may takes different values, but it doesn't depends on any variable.

$$\begin{aligned} \|\partial_x^2 W_n - \partial_x^2 W_m\|_\infty &\leq C[\|\rho_n - \rho_m\|_\infty + \varepsilon \|\nabla_x \rho_n(t,.) - \nabla_x \rho_m(t,.)\|_\infty + \frac{\|\rho_n(t,.) - \rho_m(t,.)\|_1}{\varepsilon^3} \\ &\leq C[\|\rho_n - \rho_m\|_\infty + \varepsilon + \frac{4\pi \|\rho_n - \rho_m\|_\infty (P + TQ_\infty(T))^3}{3\varepsilon^3}] \end{aligned}$$

  By uniform convergence of (ρ_n), if we take m and n enough big we have:

$$\|\partial_x^2 W_n - \partial_x^2 W_m\|_\infty \leq (\varepsilon^4 + \varepsilon) C$$

  Hence, by completeness of the space of continuous and bounded functions on $[0,T] \times \mathbb{R}^3$, we have that : $(\partial_x^2 W_n)$ and $(\nabla_x W_n)$ converge uniformly on $[0,T] \times \mathbb{R}^3$. Hence,

$$V_\rho, \nabla_x V_\rho, \partial_x^2 V_\rho \in \mathcal{C}([0,T] \times \mathbb{R}^3)$$

  and

$$Z := \lim_n Z_n \in \mathcal{C}^1([0,T] \times \mathbb{R}^3) \tag{2.14}$$

  Thus, taking the limit, we get that $f \in \mathcal{C}^1([0,T] \times \mathbb{R}^3)$ and is in deed a solution for the Vlasov-Poisson system on $[0,T]$. Since this result holds for any $T \in]0,\delta[$, f is a classical solution for the Vlasov-Poisson system on $[0,\delta[$.

- *(step 6) [Uniqueness]*
  Let f and g two classical solutions for the Vlasov-Poisson system with $f(0) = g(0)$ and which both exists on some interval $[0,T]$. According to the previous steps of this proof, f and g are both compactly supported. Since the support depends only on the initial data, f and g are both supported in a compact which can be chosen independent of $t \in [0,T]$. Here, $C > 0$ is a constant which can take different values, but it doesn't depends on the variables. Using **Mean value theorem**, *we have that :*

$$\begin{aligned} \forall t \in [0,T], \forall z \in \mathbb{R}^6,\ |f(t,z) - g(t,z)| &\leq C|Z_f(0,t,z) - Z_g(0,t,z)| \\ &\leq C \int_0^t \|\nabla_x V_{\rho_f}(u,.) - \nabla_x V_{\rho_g}(u,.)\|_\infty du \\ &\leq C \int_0^t \|\rho_f(u,.) - \rho_g(u,.)\|_\infty du \\ &\leq C \int_0^t \|f(u,.) - g(u,.)\|_\infty du \end{aligned}$$

by using **Gronwall's lemma**, *we get that :*

$$\|f(t,.) - g(t,.)\|_\infty = 0$$

and the uniqueness follows.

- *(step 7) [prolongation criterion]*

  – *Let $f \in \mathcal{C}^1([0,T[\times\mathbb{R}^6)$ be the maximally extended classical solution obtained above, and assume that:*

  $$Q_{\infty,0} := \sup_{t\in[0,T[} \sup\{|v|/\ \exists x \in \mathbb{R}^3, (x,v) \in supp(f(t,.,.))\} < \infty$$

  but $T < \infty$.
  The idea of the proof is to take $\hat{t} \leq T$, and considering the Vlasov-Poisson Cauchy problem with initial data $f(\hat{t})$, we will prove that it exists a unique solution on an interval $[\hat{t}, \hat{t}+\delta'[$ such that δ' doesn't depends on $\hat{t}$. Then we can choose $\hat{t}$ enough close to T such that $\hat{t}+\delta' > T$. This is then the desired contradiction.
  Since δ' can be chosen as small as we want , $\hat{t} = T - \frac{\delta'}{2} > 0$ is enough .
  Let's prove that for $\hat{t} \leq T$, it exists a unique solution on an interval $[\hat{t}, \hat{t}+\delta'[$ such that $\delta' > 0$ is independent from $\hat{t}$.
  In fact, let $(Q_{\infty,\hat{t}}, [\hat{t}, \hat{t}+\delta_{max}[)$ the maximal solution for :

  $$y(t) = Q_{\infty,0} + C(f(\hat{t}))\int_{\hat{t}}^{t} y^2(s)ds$$

  with $C(f(\hat{t})) = c_1\|f(\hat{t},.)\|_\infty^{2/3}\|f(\hat{t},.)\|_1^{1/3}$ (c_1 is the constant of the lemma 2.1). By taking the limit where n tends to infinity, it is easy to see that:[30]

  $$\forall t \in [0,T],\ \|f(t,.)\|_\infty = \|f_0\|_\infty \text{ and } \|f(t,.)\|_1 = \|f_0\|_1$$

  and then $C(f_0) = C(f(\hat{t}))$.

  $$Q_{\infty,\hat{t}} : t \to \frac{Q_{\infty,0}}{1 - Q_{\infty,0}C(f(\hat{t}))(t-\hat{t})}$$

  We have $\delta_{max} = \frac{1}{Q_{\infty,0}C(f(\hat{t}))} = \frac{1}{Q_{\infty,0}C(f_0)}$, which gives that δ_{max} doesn't depends on $\hat{t}$. Using the same reasoning above, we have $\forall\delta' < \delta_{max}$, it exists a unique solution on the interval $[\hat{t}, \hat{t}+\delta'[$. δ' is independent from $\hat{t}$.

[30] This property is satisfied for all the terms of the section constructed in this proof; taking the limit when n tends to infinity, we get it for f. A proof is given in the subsection 2.2.

– *Since the second criterion on ρ implies the first one (on the size of the support), it is enough to assume that,*

$$\sup_{t\in[0,T[} \|\rho(t,.)\|_\infty < \infty \tag{2.15}$$

In fact, let's assume (2.15), using lemma (2.1), we have that :

$$\forall z := (x,v) \in supp(f_0), \quad |V(s,0,z)| \leq Q + C(f_0)\int_0^s \|\rho(u,.)\|_1^{1/3}\|\rho(u,.)\|_\infty^{2/3}du \tag{2.16}$$

we have for all $0 \leq T' < T$,[31]

$$\forall t \in [0,T'], \quad \|\rho(t,.)\|_1 = \|\rho(0,.)\|_1$$

hence, by continuity of the function $t \to \|\rho(t,.)\|_1$ *we have that :*

$$\forall t \in [0,T[, \quad \|\rho(t,.)\|_1 = \|\rho(0,.)\|_1$$

with (2.16) we finally get :

$$Q_{\infty,0} \leq Q + C(f_0)T\|\rho(0,.)\|_1^{1/3}\Big(\sup_{t\in[0,T[}\|\rho(t,.)\|_\infty\Big)^{2/3} < \infty$$

⋆ ⋆ ⋆ ⋆

Remark 2.1 *Due to the requirement that initial data is compactly supported, the solutions obtained in the previous theorem enjoy stronger properties than what is required in the definition 2.1. The assumption on initial data can be replaced by a suitable fall off conditions at infinity [9].*

2.2 Conservation laws and a-priori bounds

To improve the local existence result, we need to improve the estimations on the velocity support, by getting more informations about the solution. We will use the notations introduced in the previous section.
Firstly, we notice that the all the properties satisfied by the solution of Vlasov equation still satisfied for the local solution constructed above.

Proposition 2.1 *Let* $f_0 \in \mathcal{C}_c^1(\mathbb{R}^3 \times \mathbb{R}^3)$ *such that* $f_0 \geq 0$*, and* f *the maximal solution for (2.7) given by the theorem 2.1 and* $[0,T[$ *the right maximal existence interval. Then we have :*

1. $f \geq 0$

2. $\forall \Psi \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$*, such that* $\Psi(0) = 0$*.* $\forall t \in [0,T[$ *we have:*

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Psi(f(t,x,v))dxdv = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Psi(f_0(x,v))dxdv$$

[31] See proposition 2.1.

3. $\forall p \in [1, \infty], \forall t \in [0, T[$

$$\|f(t,.,.)\|_{L^p(\mathbb{R}^3\times\mathbb{R}^3)} = \|f_0\|_{L^p(\mathbb{R}^3\times\mathbb{R}^3)}$$

4. *(Volume conservation) Let* $(t, z) \in [0, T[\times\mathbb{R}^6$,*and* $Z(., t, z) := (X(., t, z), V(., t, z))$ *a characteristic for the Vlasov-Poisson system. We recall that the definition of the characteristics in this case is given by (2.14).*
For $s \in [0, T[$, *we define* $J(s, t, z) := det(\partial_z Z(s, t, z))$. *Then we have :*

$$\forall (s, t, z) \in [0, T[\times[0, T[\times\mathbb{R}^6, \quad J(s, t, z) = 1$$

Proof 2.1 1. *By the proposition 1.5, we have:*

$$\forall n \in \mathbb{N}, f_n \geq 0$$

The uniform convergence implies the punctual convergence. Then, taking the limit when n *tends to infinity we get:*

$$f \geq 0$$

2. *Let* $\Psi \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$, *such that* $\Psi(0) = 0$, $0 \leq T' < T$, *and* $t \in [0, T']$.
Using proposition 1.5 , we have that :

$$\forall n \in \mathbb{N}, \ \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Psi(f_n(t, x, v))dxdv = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Psi(f_0(x, v))dxdv$$

Furthermore, we have :

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Psi(f_n(t, x, v))dxdv \to \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Psi(f(t, x, v))dxdv$$

when n *tends to infinity. In fact:*

- *We can notice first that:*

$$\forall n \in \mathbb{N}, \quad \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Psi(f_n(t, x, v))dxdv = \int_{|v|\leq Q_\infty(T')}\int_{|x|\leq P+\int_0^{T'} Q_\infty(s)ds} \Psi(f_n(t, x, v))dxdv$$

- *By continuity of* Ψ *we have :*

$$\forall (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3, \quad \lim_n \Psi(f_n(t, x, v)) = \Psi(f(t, x, v))$$

- *Let* $S := \{(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 / |v| \leq Q_\infty(T')$ *and* $|x| \leq P + \int_0^{T'} Q_\infty(s)ds\}$.
We have:

$$\forall n \in \mathbb{N}, \quad \|f_n(t,.,.)\|_\infty = \|f_0\|_\infty =: \alpha$$

Thus,

$$\forall n \in \mathbb{N}, \forall (x, v) \in S, \quad |\Psi(f_n(t, x, v))| \leq \underbrace{\sup_{s\in[0,\alpha]} |\Psi(s)|}_{\textit{integrable on } S}$$

Using **the dominated convergence theorem***, and taking the limit when n tends to infinity, we get finally:*

$$\forall t \in [0,T'], \quad \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Psi(f(t,x,v))dxdv = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Psi(f_0(x,v))dxdv$$

this result being satisfied for all $0 \leq T' < T$*, and the function* $t \to \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Psi(f(t,x,v))dxdv$ *continuous* [32]*, hence :*

$$\forall t \in [0,T[, \quad \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Psi(f(t,x,v))dxdv = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Psi(f_0(x,v))dxdv$$

3. - *For* $p \in [1,\infty[$ *we use the previous result with* $\Psi : y \to y^p$.
   - *Moreover, by uniform convergence we have:*

$$\forall t \in [0,T[, \ \|f(t,.,.)\|_\infty = \|f_0\|_\infty$$

4. *(Volume conservation) Let* $(s,t,z) \in [0,T[\times[0,T[\times\mathbb{R}^6$*, using the construction of the solutions given in the previous theorem we have:*

$$\forall n \in \mathbb{N}, \quad J_n(s,t,z) := det(\partial_z Z_n(s,t,z)) = 1$$

Using the fact that $\partial_z Z_n$ *converges uniformly in n to* $\partial_z Z$*, the continuity of the determinant and taking the limit when n tends to infinity, we get the claimed result.*

⋆ ⋆ ⋆ ⋆

Remark 2.2 *In the case* $p = 1$*, the previous result gives* **the global conservation of mass /charge** [33] *:*

$$\forall t \in [0,T[, \ \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(t,x,v)dxdv = \int_{\mathbb{R}^d} \rho(t,x)dx = \int_{\mathbb{R}^d} \rho(0,x)dx$$

We can get the local version of this law just by integrating the Vlasov-Poisson equation with respect to v.
$\forall t \in [0,T[, \forall x \in \mathbb{R}^3$

$$\begin{aligned}
\int_{\mathbb{R}^3} \partial_t f(t,x,v)dv &= -\int_{\mathbb{R}^3} v.\nabla_x f(t,x,v)dv + \int_{\mathbb{R}^3} \nabla_x V_\rho(t,x)\nabla_v f(t,x,v)dv \\
&= -\int_{\mathbb{R}^3} \nabla_x[f(t,x,v)v]dv + \underbrace{\int_{\mathbb{R}^3} \nabla_v(\nabla_x V_\rho(t,x)f(t,x,v))dv}_{\text{equal to 0 by Green's theorem}} \\
&= -\int_{\mathbb{R}^3} \nabla_x[f(t,x,v)v]dv
\end{aligned}$$

[32] This is a consequence of Lebesgue's theorem. Since the support is not bounded on $[0,T[$, we prove the domination on any compact set of $[0,T[$.

[33] Depending on which case : Coulomb interaction or gravitational interactions

Since f is compactly supported on any compact set of $[0,T[$, *we can move the derivatives inside and outside the integrals using* **integral derivation theorem**. *Then we get* **the local conservation of mass / charge** *:*

$$\partial_t \rho + \nabla_x j = 0 \tag{2.17}$$

With j is the mass flux, defined as:

$$\forall (t,x) \in [0,T[\times \mathbb{R}^3,\ j(t,x) = \int_{\mathbb{R}^3} f(t,x,v) v dv$$

Since, there is no dissipative phenomenon, the total energy of the system must be conserved.

Proposition 2.2 (Total energy conservation) *If $f_0 \in \mathcal{C}_c^1(\mathbb{R}^3 \times \mathbb{R}^3)$ such that $f_0 \geq 0$, and f is the maximal solution for (2.7) defined on $[0,T[$, we have that:*

$$\begin{aligned}\forall t \in [0,T[,\ \mathcal{E}(t) &:= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} (\frac{|v|^2}{2} + \frac{V_\rho(t,x)}{2}) f(t,x,v) dx dv \\ &= \mathcal{E}(0) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} (\frac{|v|^2}{2} + \frac{V_\rho(t,x)}{2}) f_0(x,v) dx dv\end{aligned}$$

Proof 2.2 *Let $t \in [0,T[$, multiplying the equation satisfied by f by $|v|^2/2$ and integrating with respect to x and v we get:*

$$\begin{aligned}&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \partial_t f(t,x,v) \frac{|v|^2}{2} dx dv + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|v|^2}{2} v.\nabla_x f(t,x,v) dx dv \\ &- \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|v|^2}{2} \nabla_x V_\rho(t,x).\nabla_v f(t,x,v) dx dv = 0\end{aligned}$$

thus,

$$\begin{aligned}&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \partial_t f(t,x,v) \frac{|v|^2}{2} dx dv + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \nabla_x (\frac{|v|^2}{2} f(t,x,v).v) dx dv \\ &- \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|v|^2}{2} \nabla_x V_\rho(t,x).\nabla_v f(t,x,v) dx dv = 0\end{aligned}$$

Using **Green's theorem** *we have that :*

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \nabla_x (\frac{|v|^2}{2} f(t,x,v).v) dx dv = 0$$

Moreover, let:

$$I := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \partial_t f(t,x,v) \frac{|v|^2}{2} dx dv - \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|v|^2}{2} \nabla_x V_\rho(t,x).\nabla_v f(t,x,v) dx dv$$

Let $0 \leq T' < T$, and $t \in [0,T']$, performing an integration by part with respect to v, and using the fact that f is compactly supported, we get that

$$\begin{aligned}I &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \partial_t f(t,x,v) \frac{|v|^2}{2} dx dv + \int_{\mathbb{R}^3} \nabla_x V_\rho(t,x) \int_{\mathbb{R}^3} f(t,x,v) v dv dx \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \partial_t f(t,x,v) \frac{|v|^2}{2} dx dv + \int_{\mathbb{R}^3} \nabla_x V_\rho(t,x) j(t,x) dx\end{aligned}$$

Notice that $j(t,.)$ is also compactly supported in $\mathbb{R}^3$. Then, by another integration by part with respect to x, and using (2.17), we get :

$$\begin{aligned} I &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \partial_t f(t,x,v)\frac{|v|^2}{2}dxdv - \int_{\mathbb{R}^3} V_\rho(t,x)\nabla_x j(t,x)dx \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \partial_t f(t,x,v)\frac{|v|^2}{2}dxdv + \int_{\mathbb{R}^3} V_\rho(t,x)\partial_t \rho(t,x)dx \end{aligned}$$

Using **Fubini's theorem** *we can notice that :*

$$\int_{\mathbb{R}^3} V_\rho(t,x)\partial_t\rho(t,x)dx = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} V_\rho(t,x)\partial_t f(t,x,v)dxdv$$

Since, f is compactly supported we can move the derivatives inside and outside the integrals. It's a consequence of **the integral derivation theorem**. *Furthermore, using* **Fubini's theorem** *we have that :*

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} V_\rho(t,x)f(t,x,v)dxdv &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} V_\rho(t,x)\partial_t f(t,x,v)dxdv + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \partial_t V_\rho(t,x)f(t,x,v)dxdv \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} V_\rho(t,x)\partial_t f(t,x,v)dxdv \\ &\quad + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\Big(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\partial_t f(t,x,w)}{|x-y|}dydw\Big)f(t,x,v)dxdv \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} V_\rho(t,x)\partial_t f(t,x,v)dxdv \\ &\quad + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\partial_t f(t,x,w)}{|x-y|}f(t,x,v)dydwdxdv \\ &= 2\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} V_\rho(t,x)\partial_t f(t,x,v)dxdv \end{aligned}$$

Combining all these equation we finally the conservation of energy on $[0,T']$:

$$\frac{d}{dt}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}[\frac{|v|^2}{2}+\frac{1}{2}V_\rho(t,x)]f(t,x,v)dxdv = 0$$

Hence,

$$\forall T' \in [0,T[, \forall t\in[0,T'], \quad \mathcal{E}(t)=\mathcal{E}(0)$$

by continuity of energy [34] *we get :*

$$\forall t\in[0,T[, \quad \mathcal{E}(t)=\mathcal{E}(0)$$

⋆ ⋆ ⋆ ⋆

Remark 2.3 *The total energy conservation can also be written in the following form:*

$$\forall t\in[0,T[, \ \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|v|^2}{2}f(t,x,v)dxdv - \frac{\gamma}{8\pi}\int_{\mathbb{R}^3}|\nabla_x V_\rho(t,x)|^2dx = \mathcal{E}(0) \qquad (2.18)$$

[34] Using Lebesgue's theorem on any compact set of $[0,T[$.

In deed, using the previous proposition we have:

$$\forall t \in [0,T[,\ \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|v|^2}{2}f(t,x,v)dxdv + \frac{1}{2}\int_{\mathbb{R}^3}\rho(t,x)V_\rho(t,x)dx = \mathcal{E}(0)$$

Thus, using Poisson's equation, and an integration part with respect to x, we get:

$$\begin{aligned}\forall T' \in [0,T[, \forall t \in [0,T'],\quad \mathcal{E}(0) &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|v|^2}{2}f(t,x,v)dxdv + \frac{1}{2}\int_{\mathbb{R}^3}\frac{-\gamma}{4\pi}V_\rho(t,x)\Delta V_\rho(t,x)dx\\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|v|^2}{2}f(t,x,v)dxdv - \frac{\gamma}{8\pi}\int_{\mathbb{R}^3}V_\rho(t,x)\nabla_x\nabla_x V_\rho(t,x)dx\\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|v|^2}{2}f(t,x,v)dxdv - \frac{\gamma}{8\pi}\int_{\mathbb{R}^3}|V_\rho(t,x)|^2dx\end{aligned}$$

By continuity of the energy we get the claimed result on $[0,T[$.
The kinetic and the potential energy of state are defined respectively by the following expressions :

$$\forall t \in [0,T[,\ \mathcal{E}_{kin}(t) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|v|^2}{2}f(t,x,v)dxdv \tag{2.19}$$

$$\forall t \in [0,T[,\ \mathcal{E}_{pot}(t) := -\frac{\gamma}{8\pi}\int_{\mathbb{R}^3}|V_\rho(t,x)|^2dx = \frac{1}{2}\int_{\mathbb{R}^3}\rho(t,x)V_\rho(t,x)dx \tag{2.20}$$

The equality :

$$-\frac{\gamma}{8\pi}\int_{\mathbb{R}^3}|V_\rho(t,x)|^2dx = \frac{1}{2}\int_{\mathbb{R}^3}\rho(t,x)V_\rho(t,x)dx$$

remains in the general case (ρ not compactly supported) using a density argument.

Remark 2.4 *In (2.18) we can see that in the plasma case where $\gamma = -1$, the total energy is positive. Then, $\mathcal{E}_{pot}$ and $\mathcal{E}_{kin}$ remain bounded.*
In the gravitational case ($\gamma = 1$), it is concevable that the kinetic and potential energy can become unbounded in finite time, but the sum remains constant. We will prove later, that even for the gravitational case the kinetic and potential energy remains bounded.

We can notice that most quantities that we are trying to estimate here, are on the following forms :

$$\forall t \in [0,T[, \forall x \in \mathbb{R}^3,\ m_k(f)(t,x) = \int_{\mathbb{R}^3}|v|^k f(t,x,v)dv$$

or

$$\forall t \in [0,T[,\ M_k(f)(t) = \int m_k(f)(t,x)dx = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|v|^k f(t,x,v)dxdv$$

These quantities above are called respectively , the k^{th} order moment density and the k^{th} order moment in velocity. The following lemma gives a general result about bounds of moments, it is very technical, but it will be used later to prove that $\mathcal{E}_{kin}$ and $\mathcal{E}_{pot}$ remains bounded even in the gravitational case.

Lemma 2.2 *Let $f : \mathbb{R}^6 \longrightarrow [0,\infty[$ a measurable function. The moments of f are defined on the same way as above.*
Let $(p,q) \in [1,\infty]$, such that $\frac{1}{p}+\frac{1}{q}=1$, $0 \le k' \le k < \infty$, and :

$$r = \frac{k+3/q}{k'+3/q+(k-k')/p}$$

If $f \in L^p(\mathbb{R}^6)$ with, $M_k(f) < \infty$, then $m_{k'}(f) \in L^r(\mathbb{R}^3)$ and :

$$\|m_{k'}(f)\|_r \le C\|f\|_p^{(k-k')/(k+3/q)} M_k(f)^{(k'+3/q)/(k+3/q)}$$

with C a constant depending only on k, k' and p.

Proof 2.2 *In this proof $C > 0$ is a constant that could changes value from line to line, but it depends only on k,k' and p.*

Let $R > 0$, and $x \in \mathbb{R}^3$,

$$\begin{aligned} m_{k'}(f)(x) &= \int_{|v|\le R} |v|^{k'} f(x,v)dv + \int_{|v|>R} |v|^{k'} f(x,v)dv \\ &\le \underbrace{\|f(x,.)\|_p(\int_{|v|\le R} |v|^{qk'}dv)^{1/q}}_{Holder's\ inequality} + R^{k'-k} m_k(f)(x) \\ &\le \|f(x,.)\|_p(\frac{4\pi}{qk'+3})^{1/q} R^{k'+3/q} + R^{k'-k} m_k(f)(x) \end{aligned}$$

taking

$$R := (m_k(f)(x)/\|f(x,.)\|_p)^{1/(k+3q)}$$

we get:

$$m_{k'}(f)(x) \le C\|f(x,.)\|_p^{(k-k')/(k+3/q)}(m_k(f)(x))^{(k'+3/q)/(k+3/q)}$$

Thus,

$$\|m_{k'}(f)\|_r^r \le C\int_{\mathbb{R}^3}[(\|f(x,.)\|_p^p)^{\frac{r(k-k')/p}{k+3/q}}(m_k(f)(x))^{\frac{r(k'+3/q)}{k+3/q}}]dx$$

Using the definition of r if $\theta := \frac{r(k-k')/p}{k+3/q}$ then $\frac{r(k'+3/q)}{k+3/q} = 1-\theta$.
Hence,

$$\begin{aligned} \|m_{k'}(f)\|_r^r &\le C\int_{\mathbb{R}^3}[(\|f(x,.)\|_p^p)^\theta(m_k(f)(x))^{1-\theta}]dx \\ &\le \underbrace{(\int_{\mathbb{R}^3}\|f(x,.)\|_p^p dx)^\theta(\int_{\mathbb{R}^3} m_k(f)(x)dx)^{1-\theta}}_{Holder's\ inequality\ with\ \frac{1}{\theta}\ and\ \frac{1}{1-\theta}} \\ &\le \|f\|_p^{p\theta} M_k(f)^{1-\theta} \end{aligned}$$

by replacing θ and $1-\theta$ by their values we get the claimed result.

⋆ ⋆ ⋆ ⋆

Here's another lemma which allows to bound the potential energy with the macroscopic density of particles.

Lemma 2.3 *If* $\rho \in L^{6/5}(\mathbb{R}^3)$ *, then*

$$E_\rho : x \to \frac{\gamma}{4\pi} \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \rho(y) dy \quad (\gamma = \pm 1)$$

blongs to $L^2(\mathbb{R}^3)$*, and :*

$$\frac{1}{8\pi} \int_{\mathbb{R}^3} |E_\rho(x)|^2 dx \leq C \|\rho\|_{6/5}^2$$

with $C > 0$ *a constant.*

Proof 2.3 *We apply Young's weak inequality* [35] *for* $r = 2, p = 6/5, q = 3/2$ *and* $f = \rho$, $g : x \to \frac{x}{|x|^3}$ *and* $d = 3$*. We have:*

$$\forall \tau > 0, \ \{x \in \mathbb{R}^3 / |g(x)| > \tau\} = B(0, \frac{1}{\sqrt{\tau}})$$

Thus,

$$\|g\|_{3/2,w} \leq 1$$

⋆ ⋆ ⋆ ⋆

Remark 2.5 *The previous lemma prove that for the local solution, the potential energy remains bounded by* ρ*. In deed, if* $f_0 \in \mathcal{C}_c^1(\mathbb{R}^3 \times \mathbb{R}^3)$*, and* f *the maximal solution for (2.7) on* $[0, T[$*, then for* $t \in [0, T[$, $\rho_f(t, .) \in L^{6/5}(\mathbb{R}^3)$ *and :*

$$\forall t \in [0, T[, \ |\mathcal{E}_{pot}(t)| = \frac{1}{8\pi} \int_{\mathbb{R}^3} |\nabla_x V_\rho(t, x)|^2 dx \leq C \|\rho(t, .)\|_{6/5}^2$$

with $C > 0$ *a constant.*

Using the two previous lemmas we can now prove that in the two cases $(\gamma = \pm 1)$, the potential and kinetic energy remain bounded for all the times.

Proposition 2.3 *Let* $f_0 \in \mathcal{C}_c^1(\mathbb{R}^3 \times \mathbb{R}^3)$ *and* f *the maximal solution for (2.7) on* $[0, T[$*. Then,* $\mathcal{E}_{pot}$*,* $\mathcal{E}_{kin}$ *are bounded and :*

$$\forall t \in [0, T[, \ \|\rho_f(t, .)\|_{5/3} \leq C$$

with $C > 0$ *is a constant.*

Proof 2.3 *The constant* C *in the proof can get many values but it doesn't depend on the variables.*

[35] Theorem B.3.

- *If $\gamma = -1$, the potential and kinetic energy are both bounded. Using lemma 2.2 with $k = 2, k' = 0, p = \infty, q = 1, r = 5/3$, we have that:*

$$\begin{aligned} \forall t \in [0,T[,\ \|\rho(t,.)\|_{5/3} &\leq C\|f(t,.)\|_\infty^{2/5}\mathcal{E}_{kin}(t)^{3/5} \\ &\leq C\|f_0\|_\infty^{2/5}\|\mathcal{E}_{kin}\|_\infty^{3/5} \end{aligned} \tag{2.21}$$

- *If $\gamma = 1$, combining the lemma 2.2 with $k = 2, k' = 0, p = 9/7, r = 6/5$ and the lemma 2.3 we have that :*

$$\begin{aligned} \forall t \in [0,T[,\ |\mathcal{E}_{pot}(t)| &\leq C\|\rho(t,.)\|_{6/5}^2 \\ &\leq C\|f(t,.)\|_{9/7}^{3/2}\mathcal{E}_{kin}(t)^{1/2} \\ &\leq C\|f_0\|_{9/7}^{3/2}\mathcal{E}_{kin}(t)^{1/2} \\ &\leq C\mathcal{E}_{kin}(t)^{1/2} \end{aligned} \tag{2.22}$$

  hence,

$$\begin{aligned} \forall t \in [0,T[,\ \mathcal{E}_{kin}(t) - C\mathcal{E}_{kin}(t)^{1/2} &\leq \mathcal{E}_{kin}(t) + \mathcal{E}_{pot}(t) \\ &\leq \mathcal{E}_{kin}(0) + \mathcal{E}_{pot}(0) \end{aligned}$$

  We deduce that $\mathcal{E}_{kin}$ is bounded. Using (2.21) and (2.22) we get the claimed result.

⋆ ⋆ ⋆ ⋆

Remark 2.6 *Using the previous proposition we can get now a better estimation of the force. In deed using the estimate of lemma 2.1 with $p = 5/3$ we get (using the notations of the previous section):*

$$\begin{aligned} \forall t \in [0,T[,\quad \|\nabla_x V_\rho(t)\|_\infty &\leq C\|\rho(t,.)\|_\infty^{4/9} \qquad (2.23) \\ &\leq C_0 Q_\infty(t)^{4/3} \end{aligned}$$

with $C_0 > 0$ a constant.

2.3 Global existence for the Vlasov-Poisson system

The aim of the section is to prove the following theorem.

Theorem **2.2 (Global existence and uniqueness)** *If $f_0 \in \mathcal{C}_c^1(\mathbb{R}^3 \times \mathbb{R}^3)$ and $f_0 \geq 0$. The Vlasov-Poisson system (2.7) has a unique global classical solution.*

Proof for the Theorem **2.2** *Let $[0,T[$ the maximal existence interval of the local solution f provided by the theorem 2.1. We assume that $T < \infty$. The idea of the proof is to use the prolongation criterion given in the theorem 2.1. In fact, to* *prove global existence, it is enough to prove that the velocities support is bounded on any interval* $[0, \tau[$, *with* $\tau \in \mathbb{R}_+^*$. *We define here:*

$$\forall t \in [0, T[, \quad Q(t) := \sup\{|v| \;/\; \exists x \in \mathbb{R}^3, (x, v) \in supp(f(s, ., .)), 0 \leq s \leq t\}$$

and make it increasing. According to the theorem 2.1, if Q *is bounded on every bounded time interval, the local solution is global.*
To prove this result, there is two different approches. The first one is given by Pffafelmoser with a simplified version due to Schaeffer, and the second one is given by P.-L. Lions and B. Perthame. Pfaffelmoser's idea is to split the phase space into suitably chosen sets, and estimate the growth of velocities on each of these sets. Since this approach is more elementary and gives better estimates on the possible growth of the solution, we discuss only the Pffafelmoser / Schaeffer approche in this paper.

The Pfaffelmoser/Schaeffer proof

The idea of this approach is to follow each particle along the trajectory, and see how the velocity is growing by estimating the variation of velocity along the characteristic in a small time interval. **In all this proof $C > 0$ is a constant that could change value from line to line, but doesn't depends on the variables.**
We fix a characteristic $Z = (X, V)$ *of the system such that* $Z(0) \in supp(f_0)$. *And for fixed* $t \in [0, T[$ *we estimate the increase of velocity. Using the definition of characteristic we have that :*

$$\begin{aligned} |V(t-\Delta) - V(t)| &\leq \int_{t-\Delta}^{t} \int_{\mathbb{R}^3} \frac{\rho_f(u, y)}{|X(u) - y|^2} dydu \\ &\leq \int_{t-\Delta}^{t} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{f(u, y, w)}{|X(u) - y|^2} dwdydu \end{aligned} \tag{2.24}$$

with $\Delta \leq t$. *We recall that the definition of the characteristics in this case is given by (2.14).*

We perform the change of variable:

$$y = X(u, t, x, v) \;\; and \;\; w = V(u, t, x, v)$$

and by the property of volume conservation given in proposition 2.1 , we get:

$$|V(t-\Delta) - V(t)| \leq \int_{t-\Delta}^{t} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{f(t, x, v)}{|X(u) - X(u, t, x, v)|^2} dvdxdu$$

To control the increasing of velocity, we split the integration domain on tree sets, with fixed parameters, $0 < p \leq \inf_{t \in [0,T[} Q(t)$ *and* $r > 0$, *that will be specified later :*

- *The good set :*

$$S_g := \{(s, x, v) \in [t, t+\Delta] \times \mathbb{R}^3 \times \mathbb{R}^3 / |v| \leq p \;\; or \;\; |v - V(t)| \leq p\}$$

- *The bad set:*

$$S_b := \quad \{(s,x,v) \in [t,t+\Delta] \times \mathbb{R}^3 \times \mathbb{R}^3/[|v| > p \text{ and } |v - V(t)| > p] \\ \text{and } [|X(s,t,x,v) - X(s)| \leq r|v|^3 \text{ or } |X(s,t,x,v) - X(s)| \leq r|v - V(t)|^3]\}$$

- *The ugly set:*

$$S_u := \quad \{(s,x,v) \in [t,t+\Delta] \times \mathbb{R}^3 \times \mathbb{R}^3/[|v| > p \text{ and } |v - V(t)| > p] \\ \text{and } [|X(s,t,x,v) - X(s)| > r|v|^3 \text{ and } |X(s,t,x,v) - X(s)| > r|v - V(t)|^3]\}$$

It is obvious that

$$S_u \cup S_g \cup S_b = [t, t+\Delta] \times \mathbb{R}^3 \times \mathbb{R}^3$$

Moreover, we choose a suitable Δ so that, the characteristics don't go so far from their value at the instant t. Using (2.23), we have that :

$$\begin{aligned}
\forall s \in [t-\Delta, t],\ |V(s,t,x,v) - v| &\leq |\int_s^t \partial_s V(u,t,x,v)du| \\
&\leq |\int_{t-\Delta}^t \partial_s V(u,t,x,v)du| \\
&\leq \Delta C_0 Q(t)^{4/3}
\end{aligned}$$

we choose [36]*:*

$$\Delta := \min(t, \frac{p}{4C_0 Q(t)^{4/3}})$$

Hence,

$$\forall s \in [t-\Delta, t],\ |V(s,t,x,v) - v| \leq \frac{p}{4}$$

We estimate now the growth of the velocities on the different sets.

- *On the good set :*
  $\forall (u,x,v) \in S_g$ *we have:*

  $$|w| < 2p \quad or \quad |V(u) - w| < 2p$$

  We start by writing:

  $$\int_{S_g} \frac{f(u,y,v)}{|X(u)-y|^2} dvdydu \leq \int_{t-\Delta}^t \int \frac{\rho_{S_g}(u,y)}{|X(u)-y|^2} dydu$$

  Where:

  $$\begin{aligned}
  \rho_{S_g}(u,y) &:= \int_{|w|<2p \ or \ |w-V(u)|<2p} f(u,y,w)dw \\
  &\leq |f_0|_\infty [\int_{|w|<2p} dw + \int_{|w-V(u)|<2p} dw] \\
  &\leq Cp^3
  \end{aligned}$$

[36] We recall, that t is fixed.

Using (2.23), with an adaptation for S_g, we get :

$$\begin{aligned} \int_{S_g} \frac{f(u,y,v)}{|X(u)-y|^2} dvdydu &\leq \int_{t-\Delta}^{t} \|\nabla_x V_\rho(u,.)\|_{\infty, S_{g,x,v,u}} du \\ &\leq C \int_{t-\Delta}^{t} \|\rho_{S_g}(u,y)\|_{\infty}^{4/9} du \\ &\leq \Delta C p^{4/3} \end{aligned} \tag{2.25}$$

with $S_{g,x,v,u} := \{(x,v) \in \mathbb{R}^3 / (u,x,v) \in S_g\}$.

- *On the bad set :*
  $\forall (u,x,v) \in S_g$ *we have that:*

$$\frac{p}{2} < |w| < 2|v| \quad \textit{and} \quad \frac{p}{2} < |V(u) - w| < 2|v - V(t)|$$

and

$$|y - X(u)| < 8r|w|^{-3} \quad \textit{or} \quad |y - X(u)| < 8r|w - V(u)|^{-3}$$

We can also notice that, if $w \in supp(f(s,y,.))$ *for some* $s \in [0, T[$ *we have that :*

$$|w| \leq Q(t) \quad \textit{and} \quad |V(s) - w| \leq 2Q(t)$$

If $I := \int_{S_b} \frac{f(u,y,v)}{|X(u)-y|^2} dvdydu$, *using* $\|f(u,.)\|_\infty = \|f_0\|_\infty$ *we have that :*

$$\begin{aligned} I &\leq \int_{t-\Delta}^{t} \int_{\frac{p}{2}<|w|\leq Q(t)} \int_{|y-X(u)|<8r|w|^{-3}} \frac{f(u,y,v)}{|X(u)-y|^2} dydwdu \\ &\quad + \int_{t-\Delta}^{t} \int_{\frac{p}{2}<|V(u)-w|\leq 2Q(t)} \int_{|y-X(u)|<8r|w-V(u)|^{-3}} \frac{f(u,y,v)}{|X(u)-y|^2} dydwdu \\ &\leq \|f_0\|_\infty \int_{t-\Delta}^{t} \int_{\frac{p}{2}<|w|\leq Q(t)} \int_{|y-X(u)|<8r|w|^{-3}} \underbrace{\frac{1}{|X(u)-y|^2}}_{=\eta} dydwdu \\ &\quad + \|f_0\|_\infty \int_{t-\Delta}^{t} \int_{\frac{p}{2}<|V(u)-w|\leq 2Q(t)} \int_{|y-X(u)|<8r|w-V(u)|^{-3}} \underbrace{\frac{1}{|X(u)-y|^2}}_{=\eta} dydwdu \\ &\leq \|f_0\|_\infty \Delta 32\pi r \int_{\frac{p}{2}<|w|\leq Q(t)} \underbrace{\frac{1}{|w|^3}}_{=\eta} dw \\ &\quad + \|f_0\|_\infty \int_{t-\Delta}^{t} 32\pi r \int_{\frac{p}{2}<|V(u)-w|\leq 2Q(t)} \underbrace{\frac{1}{|V(u)-w|^3}}_{=\eta} dwdy \\ &\leq \|f_0\|_\infty \Delta 32\pi r \ln(\frac{2Q(t)}{p}) \\ &\quad + \|f_0\|_\infty \Delta 32\pi r \ln(\frac{4Q(t)}{p}) \end{aligned}$$

Thus,

$$\int_{S_b} \frac{f(u,y,v)}{|X(u)-y|^2} dvdydu \leq \Delta Cr \ln(\frac{4Q(t)}{p}) \tag{2.26}$$

- *On the ugly set :*
  On the ugly set, $|X(u)-y|$ *is not bounded and we can't perform an integration as for the previous case. The idea of the proof here is to analyse the behavior of the function* $d : u \to X(u) - X(u,t,x,v)$ *on* $[t-\Delta, t]$.
  Let,

  $$|d(u_0)| := \min_{u \in [t-\Delta,t]} |d(u)|$$

  and

  $$\forall u \in [t-\Delta, t],\ D(u) := d(u_0) + (u-u_0)\dot{d}(u_0)$$

  $\forall u \in [t-\Delta, t]$, *we have that :*

  $$\begin{aligned} |\ddot{d}(u)| &= |\dot{V}(u,t,x,v) - \dot{V}(u)| \\ &\leq 2\|\nabla_x V_\rho(u,.)\|_\infty \\ &\leq 2CQ^{4/3}(t) \end{aligned}$$

  Using **Taylor-Lagrange formula**, *we get :*

  $$\begin{aligned} \forall u \in [t-\Delta, t],\ |d(u) - D(u)| &\leq CQ(t)^{4/3}(u-u_0)^2 \\ &\leq CQ(t)^{4/3}|u-u_0|\Delta \\ &\leq \frac{p}{4}|u-u_0| \\ &< \frac{1}{4}|V(t)-v||u-u_0| \end{aligned} \tag{2.27}$$

  Furthermore,

  $$|\dot{d}(u_0)| = |V(u_0,t,x,v) - V(u_0)| \geq |v - V(u_0)| - \frac{p}{2} > \frac{1}{2}|v - V(u_0)| \tag{2.28}$$

  Moreover, for all $u \in [t-\Delta, t]$ *:*

  $$(u-u_0)d(u).\dot{d}(u_0) \geq 0 \tag{2.29}$$

  In deed, we have:

  $$|d(u_0)|^2 = \min_{u \in [t-\Delta,t]} |d(u)|^2 := \min_{u \in [t-\Delta,t]} q(u)$$

  - *If* $u_0 \in]t-\Delta, t[,\ \dot{q}(u_0) = 2d(u_0).\dot{d}(u_0) = 0$
  - *if* $u_0 = t-\Delta,\ \dot{q}(u_0) \geq 0$
  - *if* $u_0 = t,\ \dot{q}(u_0) \leq 0$

Thus, using (2.28) and (2.29):

$$\begin{aligned}\forall u \in [t-\Delta, t],\ |D(u)|^2 &= |d(u_0)|^2 + (u-u_0)^2|\dot{d}(u_0)|^2 + 2(u-u_0)\dot{d}(u_0).d(u_0) \\ &\geq \frac{1}{4}(u-u_0)^2|v-V(u_0)|^2\end{aligned}$$

Then combining with (2.27),

$$\begin{aligned}\forall u \in [t-\Delta, t],\ |d(u)| &\leq |D(u)| - \frac{1}{4}|V(t)-v||u-u_0| \\ &\geq \frac{1}{4}|V(t)-v||u-u_0| \qquad (2.30)\end{aligned}$$

Note that to establish this result we used only: $|v - V(t)| > p$. *Now we fixe* v *and we define:*

$$\sigma_1(\alpha) = \begin{cases} \alpha^{-2} & \text{if } \alpha > r|v|^{-3} \\ r|v|^{-3} & \text{si } \alpha \leq r|v|^{-3} \end{cases}$$

$$\sigma_2(\alpha) = \begin{cases} \alpha^{-2} & \text{if } \alpha > r|v-V(t)|^{-3} \\ r|v-V(t)|^{-3} & \text{si } \alpha \leq r|v-V(t)|^{-3} \end{cases}$$

these functions will allow us to avoid the singularity near to 0.

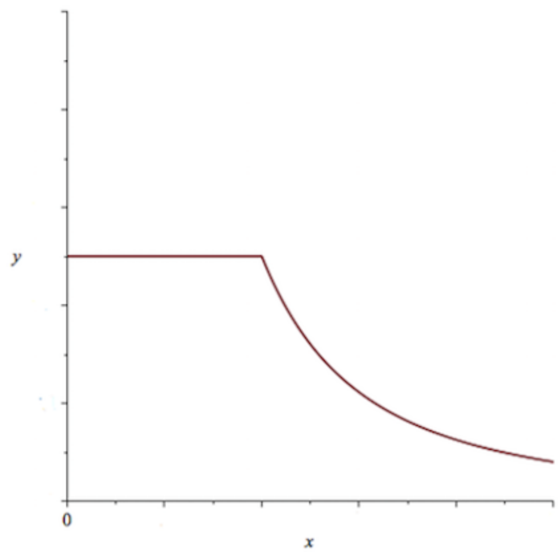

Figure 10: The general form of the function σ_i.

We can now estimate the time integral. The estimate (2.30) implies for $i \in \{1, 2\}$ *and* $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$, *that:*

$$\begin{aligned}\int_{t-\Delta}^{t} |d(u)|^{-2}\chi_{S_u}(u, x, v)du &\leq \int_{t-\Delta}^{t} \sigma_i(|d(u)|)du \\ &\leq \int_{t-\Delta}^{t} \sigma_i(\frac{1}{4}|V(t)-v||u-u_0|)du\end{aligned}$$

splitting the integral to distinguish the cases ($u \geq u_0$ and $u \leq u_0$) and performing a change of variable we finally get :

$$\begin{aligned}
\int_{t-\Delta}^{t} |d(u)|^{-2}\chi_{S_u}(u,x,v)du &\leq \frac{8}{|V(t)-v|}\int_0^\infty \sigma_i(u)du \\
&\leq \begin{cases} \frac{16}{r|V(t)-v|}|v|^3 & \textit{for } i=1 \\ \\ \frac{16}{r|V(t)-v|}|V(t)-v|^3 & \textit{for } i=2 \end{cases} \\
&\leq \frac{16}{r|V(t)-v|}\min(|v|,|V(t)-v|)^3 \\
&\leq \frac{16}{r}\min(|v|,|V(t)-v|)^2 \\
&\leq \frac{16}{r}|v|^2
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_{S_u}\frac{f(u,x,v)}{|X(u)-X(u,t,x,v)|^2}dudxdv &\leq \int\int f(u,x,v)\int_{t-\Delta}^{t}|d(u)|^{-2}\chi_{S_u}(u,x,v)dudxdv \\
&\leq Cr^{-1}\underbrace{\int\int |v|^2 f(u,x,v)dxdv}_{=\mathcal{E}_{kin}}
\end{aligned}$$

and since the kinetic energy is bounded, we get :

$$\int_{S_u}\frac{f(u,x,v)}{|X(u)-X(u,t,x,v)|^2}dudxdv \leq Cr^{-1} \tag{2.31}$$

Combining the estimates (2.25), (2.26) and (2.31) we get finally:

$$|V(t-\Delta)-V(t)| \leq C[p^{4/3} + r\ln(\frac{4Q(t)}{p}) + \frac{1}{r\Delta}]\Delta$$

We choose the parameters p and r in order to make the termes in the right in the estimate above, with the same order in $Q(t)$. We choose :

$$p = Q(t)^{4/11} \quad \textit{and} \quad r = Q(t)^{16/33}$$

Replacing $Q(t)$ by $Q(t)+1$, we can always assume that $Q(t) \geq 1$. Doing so, $p = Q(t)^{4/11} \leq Q(t)$. Hence, p is well defined.
We recall that by the definition of Δ we have: $\Delta(t)^{-1} = \max(1/t, 4C_0Q(t)^{32/33})$
Since Q is no-decreasing and by theorem 2.1, if $T < \infty$ then

$$\lim_{t\to T} Q(t) = +\infty$$

Hence, it exists $T' \in [0,T[$ such that,

$$\forall t \geq T',\ 1/t \leq 4C_0Q(t)^{32/33}$$

Then, $\forall t \geq T'$, we have that,

$$|V(t-\Delta) - V(t)| \leq CQ(t)^{16/33} \ln(Q(t))\Delta$$

Hence, $\forall \varepsilon > 0$ it exists a constant $C > 0$ such that :

$$\forall t \geq T', \ |V(t-\Delta) - V(t)| \leq CQ(t)^{16/33+\varepsilon}\Delta \tag{2.32}$$

We fix $t > T'$ and we define a sequence : $t_0 := t$ and $t_{i+1} := t_i - \Delta(t_i)$ we do so as long as the iteration $t_{i+1} > T'$. Since Q is non-increasing, we have:

$$\Delta(t_{i+1}) \geq \Delta(t_0)$$

Then there exists $k \in \mathbb{N}^\star$, such that:

$$t_k < T' \leq t_{k-1} < ... < t_0$$

Using (2.32), we get :

$$\begin{aligned}
|V(t_k) - V(t)| &\leq \sum_{i=0}^{k-1} |V(t_i) - V(t_{i+1})| \\
&\leq C \sum_{i=0}^{k-1} Q(t_i)^{16/33+\varepsilon}(t_i - t_{i+1}) \\
&\leq CQ(t)^{16/33+\varepsilon} \sum_{i=0}^{k-1}(t_i - t_{i+1}) \\
&\leq CQ(t)^{16/33+\varepsilon}(t - t_k) \\
&\leq CQ(t)^{16/33+\varepsilon} t
\end{aligned}$$

Hence, using the definition of Q we have that :

$$Q(t) \leq Q(t_k) + CQ(t)^{16/33+\varepsilon} t$$

Then for any $\eta > 0$ there exists a constant $C > 0$ such that :

$$\forall t \in [0, T[, \ Q(t) \leq C(1+t)^{33/17+\eta}$$

The theorem 2.1 complete the proof.

⋆ ⋆ ⋆ ⋆

3 The Vlasov-Maxwell system

For some situations the Vlasov-Poisson description of a plasma is not sufficient. Charged particles in the plasma are also creating a magnetic field, and when the speed of charged particles considered in the system is sufficiently large, the magnetic force should be taken into account. Thus, the electric and magnetic field both act on particles via Lorentz's force :

$$F(t,x,v) := E(t,x) + v \times B(t,x)$$

with E, v and B are respectively the electric field, the speed of the particle and the magnetic field. Observe that,

$$\nabla_v F = 0$$

We recall that for simplicity, we give up the constraint of global neutrality and consider only the case of a single species of charged particles. In this model, we are taking into account the electromagnetic field created by identical charged particles. Thus, we should consider also Maxwell's equations :

$$\begin{cases} \nabla_x E = \rho & \text{(Maxwell-Gauss)} \\ \nabla_x B = 0 & \text{(Flux conservation)} \\ \nabla_x \times E = -\partial_t B & \text{(Maxwell-Faraday)} \\ \nabla_x \times B = j + \partial_t E & \text{(Maxwell-Ampere)} \end{cases}$$

as for the previous section all the physical constants are taken equal to 1. ρ and j are respectively the macroscopic density and the charges flux. Having

$$\partial_t \nabla_x E = \nabla_x \partial_t E$$

we can also combine Maxwell-Ampere and Maxwell-Gauss equations to get the **local conservation of charge** :

$$\nabla_x \cdot (\nabla_x \times B) = \nabla_x j + \partial_t \rho = 0$$

It is known that in systems of charged particles in a electromagnetic field, the particles can reach a very hight velocities. Thus, we should take into account the relativistic effects. Hence, for each particle, the characteristic system is :

$$\begin{cases} \dot{x} = \hat{v} \\ \dot{v} = E + \hat{v} \times B \end{cases}$$

with $\hat{v}$, v and x are respectively, the velocity, the momentum and position of the particle. We recall that:

$$\hat{v} := \frac{v}{\sqrt{1+|v|^2}}$$

The combination of Vlasov's and Maxwell's equations gives the following system called the relativistic Vlasov-Maxwell system :

$$\begin{cases} \partial_t f + \hat{v}.\nabla_x f + (E(t,x) + \hat{v} \times B(t,x))\nabla_v f = 0 \\ \rho_f(t,x) = \int_{\mathbb{R}^3} f(t,x,v)dv \\ j_f(t,x) = \int_{\mathbb{R}^3} f(t,x,v)\hat{v}(v)dv \\ \nabla_x E = \rho_f \\ \nabla_x B = 0 \\ \nabla_x \times E = -\partial_t B \\ \nabla_x \times B = j_f + \partial_t E \end{cases} \tag{3.1}$$

the unknown is the triple (E, B, f). The system (3.1) is supplemented with the initial data (E_0, B_0, f_0), such that :

$$\begin{cases} f(0,.,.) = f_0 \\ E(0,.) = E_0 \\ B(0,.) = B_0 \\ \rho_0(x) = \int_{\mathbb{R}^3} f_0(x,v)dv \\ \nabla_x E = \rho_0 \\ \nabla_x B = 0 \end{cases} \tag{3.2}$$

There are global existence theorems for solutions of the Vlasov-Maxwell equations with small initial data, for solutions with certain symmetry properties and for solutions in two space dimensions. However, unlike the Vlasov-Poisson system there is no general result of existence of solutions for Vlasov-Maxwell system.
In this section we use the following notations for coordinates of the electromagnetic field. If $X \in \mathbb{R}^3$ then :

$$X = (X^1, X^2, X^3)$$

3.1 LOCAL EXISTENCE AND UNIQUENESS THEOREM

Having a local solution, we can get a-priori bounds on the fields and the solution, which are very useful for any other investigation. The following result is proved by Wollman in [17]. This theorem is produced by generalizing a theorem of Kato [10, Theorem II, p: 195] on quasi-linear symmetric and hyperbolic systems. Let's consider the initial value problem for Vlasov-Maxwell system, with initial data (E_0, B_0, f_0). Such that :

$$\begin{cases} \forall s \geq 5, \quad f_0 \in H^s(\mathbb{R}^3 \times \mathbb{R}^3) \\ \\ \forall s \geq 5, \quad E_0, B_0 \in H^s(\mathbb{R}^3) \\ \\ f_0 \text{ is compactly supported} \\ \\ \exists M > 0, \quad \|f_0\|_{H^s(\mathbb{R}^3 \times \mathbb{R}^3)} \leq M \\ \\ \exists N > 0, \quad \|E_0\|_{H^s(\mathbb{R}^3)} + \|B_0\|_{H^s(\mathbb{R}^3)} \leq N \end{cases} \tag{3.3}$$

then we have the following result.

<u>Theorem</u> **3.1 (S.Wollman, 1983)** *Let $s \geq 5$, and the initial conditions (E_0, B_0, f_0) satisfy (3.3). Then, there is a unique $T > 0$ and unique function :*

$$f \in \mathcal{C}([0,T[, H^s(\mathbb{R}^3 \times \mathbb{R}^3)) \cap \mathcal{C}^1([0,T[, H^{s-1}(\mathbb{R}^3 \times \mathbb{R}^3))$$

such that f is a solution for the Vlasov Maxwell-system with initial data (E_0, B_0, f_0).

Proof for the Theorem **3.1** *The proof of this theorem is based on a generalization of a local existence and uniqueness theorem of Kato [10, Theorem II, p: 195] for quasi-linear, symmetric and hyperbolic system of equations. The explicit proof of this theorem is given by Wollman in [17].*

⋆ ⋆ ⋆ ⋆

3.2 Glassey-Strauss' global existence theorem

With an additional assumption on the size of the velocity support, we can prove a global existence and uniqueness. However, this result remains not general, since the assumption made here is not proved. In fact, there is a-priori no reason for the velocity support to be bounded.
The aim of this subsection is to prove the following theorem :

<u>Theorem</u> **3.2 (Glassey-Strauss, 1986)** *Let $f_0 \in \mathcal{C}^1_c(\mathbb{R}^3 \times \mathbb{R}^3)$ such that $f_0 \geq 0$, and $E_0, B_0 \in \mathcal{C}^2$ satisfying the conditions (3.2). If there exists a continuous function Q such that:*

$$\forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \ \textit{and} \ \ |v| > Q(t), \ f(t,x,v) = 0$$

then the system (3.1) has a unique global solution.

Proof for the Theorem **3.2 (Sketch of the proof)** *The proof of this theorem is the aim of all this subsection. The approach used here, is the one given in [4] and [5]. In this proof we proceed as follows :*

- *Representation of the fields.*
- *A priori bounds on the fields and gradient of the fields.*
- *Bounds on the derivatives of f.*
- *Proof of uniqueness using the representation of the fields.*
- *Proof of existence using a iterative scheme and the a priori bounds.*

We define the following norms on the functional space $\mathcal{C}^1([0,T[\times\mathbb{R}^3\times\mathbb{R}^3)\cap W^{1,\infty}([0,T[\times\mathbb{R}^3\times\mathbb{R}^3)$,

$$|g(t,.,.)|_1 := \sup_{x\in\mathbb{R}^3}\sup_{v\in\mathbb{R}^3}\Big(|\partial_t g(t,x,v)| + \sum_i \Big(|\partial_{x_i} g(t,x,v)| + |\partial_{v_i} g(t,x,v)|\Big)\Big)$$

and

$$\|g\|_{1,\infty} = \sup_{t\in[0,T[} |g(t,.,.)|_1$$

By the same way we define on $\mathcal{C}^1([0,T[\times\mathbb{R}^3)\cap W^{1,\infty}([0,T[\times\mathbb{R}^3)$ the norms :

$$|g(t,.)|_1 := \sup_{x\in\mathbb{R}^3}\Big(|\partial_t g(t,x)| + \sum_i |\partial_{x_i} g(t,x)|\Big)$$

and

$$\|g\|_{1,\infty} = \sup_{t\in[0,T[} |g(t,.)|_1$$

3.2.1 The fields representation

Let's collect some facts from wave equation theory, which will be very useful to find the representation of the fields.

Lemma 3.1 (Inversion of wave operator) *We define the wave operator (or d'Alembert operator) $\mathcal{W}$ as:*

$$\mathcal{W} := \partial_t^2 - \Delta$$

And we consider the initial value problem:

$$\begin{cases} \mathcal{W}u(t,x) = F(t,x) \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^3 \\ u(0,.) = f \quad \partial_t u(0,.) = g \end{cases} \tag{3.4}$$

with $f\in\mathcal{C}^3$ and F and g in $\mathcal{C}^2$. Then, the following expression define a $\mathcal{C}^2$ solution for the problem (3.4)

$$u(t,x) = u_0(t,x) + \frac{1}{4\pi}\int_{|y|\le t} F(t-|y|, x-y)\frac{dy}{|y|} \tag{3.5}$$

with u_0 is unique solution for the homogenous problem :

$$\begin{cases} \mathcal{W}u(t,x) = 0 \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^3 \\ u(0,.) = f \quad \partial_t u(0,.) = g \end{cases} \tag{3.6}$$

Furthermore we have:

$$\forall(t,x)\in\mathbb{R}^+\mathbb{R}^3,\quad u_0(t,x) = \frac{1}{4\pi t^2}\int_{|x-y|=t}[tg(y) + f(y) - \nabla_x f(y).(x-y)]dS(y) \tag{3.7}$$

Proof 3.1 *We present here only some elements of the proof. The complete proof of this facts could be found in [14, pages: 1-9] and [3, pages: 65-85].*

- *We start by proving first existence and uniqueness of solutions for (3.6). We prove that*

$$u_0(t,x) = \frac{1}{4\pi t^2}\int_{|x-y|=t}\Big[tg(y) + f(y) - \nabla_x f(y).(x-y)\Big]dS(y)$$

  is the unique solution for (3.6). The proof of this fact is given in [14, pages: 1-3] and [3, pages: 71-73].

- *We notice that if v is a solution for :*

$$\begin{cases} \mathcal{W}v(s,t,x) = 0 & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3 \\ v(s,0,x) = 0 & \partial_t v(s,0,x) = F(s,x) \end{cases}$$

  then

$$u(t,x) := \int_0^t v(s,t,x)ds$$

  is a solution for :

$$\begin{cases} \mathcal{W}u(t,x) = F(t,x) & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3 \\ u(0,.) = 0 & \partial_t u(0,.) = 0 \end{cases}$$

Combining this two results and by linearity of the wave operator we get the claimed result using superposition principale.

⋆ ⋆ ⋆ ⋆

Remark 3.1 (Weak solutions) *Theorem 3.1 extends to the case where F is only continuous. In that case we require that $f \in \mathcal{C}^1$ and $g \in \mathcal{C}$. The unique weak solution, in this case, is given by the expressions (3.5) and (3.7).*[37]

<u>Theorem</u> **3.3** *We define an operator S as :*

$$S := \partial_t + \sum_{k=1}^{3} \hat{v}_k \partial_{x_k}$$

We also call:

$$\partial_t E(0,.) := E_1 \text{ and } \partial_t B(0,.) := B_1$$

Let f a local solution for (3.1) *defined on $[0,T[$. Under the assumptions of the theorem (3.2), and if we assume that f, E_0, B_0, E_1 and B_1 are enough regular*[38]*, the fields admits the following representations :*

$\forall (t,x) \in [0,T[\times\mathbb{R}^3$, $\forall i \in \{1,2,3\}$,

$$\begin{aligned} 4\pi E^i(t,x) &= (E^i)_0(t,x) + E^i_T(t,x) + E^i_S(t,x) \\ 4\pi B^i(t,x) &= (B^i)_0(t,x) + B^i_T(t,x) + B^i_S(t,x) \end{aligned} \tag{3.8}$$

[37] See [14, page: 11].

[38] Enough regular to be able to inverse the wave operator (in the classical or weak sens) in the proof.

with

$$\begin{aligned}
E_T^i(t,x) &= -\int_{|y-x|\le t}\int_{\mathbb{R}^3} \frac{(\omega_i+\hat{v}_i)(1-|\hat{v}|^2)}{(1+\hat{v}.\omega)^2} f(t-|y-x|,y,v)dv\frac{dy}{|y-x|^2} \\
E_S^i(t,x) &= -\int_{|y-x|\le t}\int_{\mathbb{R}^3} \frac{(\omega_i+\hat{v}_i)}{(1+\hat{v}.\omega)}(Sf)(t-|y-x|,y,v)dv\frac{dy}{|y-x|} \qquad (3.9)
\end{aligned}$$

and

$$\begin{aligned}
B_T^i(t,x) &= -\int_{|y-x|\le t}\int_{\mathbb{R}^3} \frac{(\omega\times\hat{v})_i(1-|\hat{v}|^2)}{(1+\hat{v}.\omega)^2} f(t-|y-x|,y,v)dv\frac{dy}{|y-x|^2} \\
B_S^i(t,x) &= -\int_{|y-x|\le t}\int_{\mathbb{R}^3} \frac{(\omega\times\hat{v})_i}{(1+\hat{v}.\omega)}(Sf)(t-|y-x|,y,v)dv\frac{dy}{|y-x|} \qquad (3.10)
\end{aligned}$$

with

$$\omega = \frac{y-x}{|y-x|}$$

PROOF FOR THE THEOREM **3.3** *If we assume that E and B are $\mathcal{C}^2$. Then, combining Maxwell's equations and using* **Schwartz's theorem** *to switch time and space derivatives, the electromagnetic field must satisfy the following wave equations:*

$$(\partial_t^2-\Delta)E = -(\nabla_x\rho_f+\partial_t j_f)$$

$$(\partial_t^2-\Delta)B = \nabla_x\times j_f$$

Let's consider now this two equations and let's solve them using the results given in lemma 3.1 and the remark 3.1.

In all this section, we note :

$$P(t) := \max(0, Q(t))$$

$$Q_{T'} := \sup_{t\in[0,T']} P(t)$$

Notice that, $\forall x\in\mathbb{R}^3, \forall t\in[0,T[$,

$$\int f(t,x,v)dv + \int \hat{v}(v)f(t,x,v)dv = \int_{|v|\le Q_{T'}} f(t,x,v)dv + \int_{|v|\le Q_{T'}} \hat{v}(v)f(t,x,v)dv$$

- *Let's start with the electric field equation. Using the* **integral derivation theorem** *we can switch the derivatives inside and outside the integrals*[39] *:*

$$\partial_{x_i}\rho_f + \partial_t j_f = \int \partial_{x_i} f(.,.,v)dv + \int \hat{v}(v)\partial_t f(.,.,v)dv$$

[39] In fact, for all $0<T'<T$, $x\in B(0,\eta)\subset\mathbb{R}^3$ and $t\in[0,T]$:
$\forall i\in\{1,2,3\}$, $\forall v\in B(0,C_{T'})$,

$$\begin{aligned}
\partial_{x_i} f(t,x,v) &\le \|\partial_{x_i} f\|_{\infty,[0,T']\times B(0,\eta)\times B(0,C_{T'})} \\
\|\hat{v}\|\partial_t f(t,x,v) &\le \|\partial_t f\|_{\infty,[0,T']\times B(0,\eta)\times B(0,C_{T'})}
\end{aligned}$$

Hence,

$$\forall i \in \{1,2,3\}, \quad (\partial_t^2 - \Delta)E^i = -\int \Big(\partial_{x_i} f(.,.,v) + \hat{v}\partial_t f(.,.,v)\Big)dv \qquad (3.11)$$

We have to solve now the unhomogeneous wave equation (3.11).
Let $i \in \{1,2,3\}$, if we assume that f, E_0 and E_1 are enough smooth so that the assumption of lemma 3.1 or remark 3.1 are all satisfied, we get then, $\forall (t,x) \in [0,T[\times\mathbb{R}^3$:

$$4\pi E^i(t,x) = (E^i)'_0(t,x) - \underbrace{\int_{|y-x|\le t}\int \Big(\partial_{x_i} f(t-|y-x|,y,v) + \hat{v}\partial_t f(t-|y-x|,y,v)\Big)dv\frac{dy}{|y-x|}}_{:=I_E}$$

with $(E^i)'_0$ is a function depending on the initial data (E_0, E_1).
According to the formula above, we need two derivatives on f to estimate the gradient of the field. To avoid this difficulty, the idea of Glassey-Strauss is to introduce the operator S defined above, and an operator T, defined with the following expression :

$$T = \nabla_x f - \omega\partial_t f$$

Notice that the operator T satisfy the following property:

$$\partial_{y_i}\Big[f(t-|y-x|,y,v)\Big] = (T_i f)(t-|y-x|,y,v) \qquad (3.12)$$

Furthermore, the relations between the old and the new operators are given by:

$$\partial_t = \frac{S - \hat{v}.T}{1+\hat{v}.\omega}$$

and

$$\begin{aligned}\partial_{x_i} &= T_i + \frac{\omega_i}{1+\hat{v}.\omega}(S - \hat{v}.T)\\ &= \frac{\omega_i S}{1+\hat{v}.\omega} + \sum_j(\delta_{i,j} - \frac{\omega_i\hat{v}_i}{1+\hat{v}.\omega})T_j\end{aligned}$$

Hence,

$$\partial_{x_i} + \hat{v}\partial_t = \frac{(\omega_i+\hat{v}_i)S}{1+\hat{v}.\omega} + \sum_j(\delta_{i,j} - \frac{(\omega_i+\hat{v}_i)\hat{v}_j}{1+\hat{v}.\omega})T_j$$

We will now use this relations to prove the claimed result. We have that,

$$\begin{aligned}I_E &= \int_{|y-x|\le t}\int \Big[\frac{(\omega_i+\hat{v}_i)S}{1+\hat{v}.\omega} + \sum_j(\delta_{i,j} - \frac{(\omega_i+\hat{v}_i)\hat{v}_j}{1+\hat{v}.\omega})T_j\Big](f)(t-|y-x|,y,v)dv\frac{dy}{|y-x|}\\ &= \int_{|y-x|\le t}\int \frac{(\omega_i+\hat{v}_i)}{1+\hat{v}.\omega}(Sf)(t-|y-x|,y,v)dv\frac{dy}{|y-x|}\\ &\quad + \int_{|y-x|\le t}\int \sum_j(\delta_{i,j} - \frac{(\omega_i+\hat{v}_i)\hat{v}_j}{1+\hat{v}.\omega})(T_j f)(t-|y-x|,y,v)dv\frac{dy}{|y-x|}\\ &= E_S^i(t,x) + \underbrace{\int_{|y-x|\le t}\int \sum_j(\delta_{i,j} - \frac{(\omega_i+\hat{v}_i)\hat{v}_j}{1+\hat{v}.\omega})(T_j f)(t-|y-x|,y,v)dv\frac{dy}{|y-x|}}_{=I'_E}\end{aligned}$$

Hence, using (3.12),

$$\begin{aligned} I'_E &= \int_{|y-x|\le t}\int\sum_j \underbrace{(\delta_{i,j} - \frac{(\omega_i+\hat{v}_i)\hat{v}_j}{1+\hat{v}.\omega})}_{=A_j}\partial_{y_j}[f(t-|y-x|,y,v)]dv\frac{dy}{|y-x|} \\ &= \int_{|y-x|\le t}\int\sum_j \frac{A_j}{|y-x|}\partial_{y_j}[f(t-|y-x|,y,v)]dvdy \end{aligned}$$

performing an **integration by part**[40] *with respect to y, we get :*

$$\begin{aligned} I'_E &= \sum_j\int_{|y-x|=t}\int\frac{A_j}{|y-x|}f(0,y,v)dvN_j(y)dS(y) \\ &\quad -\sum_j\int_{|y-x|\le t}\int\partial_{y_j}[\frac{A_j}{|y-x|}]f(t-|y-x|,y,v)dvdy \\ &= \sum_j\int_{|y-x|=t}\int\frac{\omega_jA_j}{|y-x|}f(0,y,v)dvdS(y) \\ &\quad -\sum_j\int_{|y-x|\le t}\int\partial_{y_j}[\frac{A_j}{|y-x|}]f(t-|y-x|,y,v)dvdy \end{aligned}$$

By an elementary but lengthy calculations[41]*, we have that :*

$$\sum_j\partial_{y_j}[\frac{A_j}{|y-x|}] = \frac{(\omega_i+\hat{v}_i)(|\hat{v}|^2-1)}{|x-y|^2(1+\hat{v}.\omega)^2}$$

Hence, if

$$(E^i)_0 = (E^i)'_0 + \sum_j\int_{|y-x|=t}\int\frac{\omega_jA_j}{|y-x|}f(0,y,v)dvdS(y)$$

we get the claimed result.

- *To get the formula of the magnetic field, we follow the same steps. Since we can move the derivatives inside and outside the integrals* [42] *we get :*

$$\begin{aligned} (\partial_t^2-\Delta)B^1 &= \int\partial_{x_2}\hat{v}_3f(v) - \partial_{x_3}\hat{v}_2f(v)dv \\ (\partial_t^2-\Delta)B^2 &= \int\partial_{x_3}\hat{v}_1f(v) - \partial_{x_1}\hat{v}_3f(v)dv \\ (\partial_t^2-\Delta)B^3 &= \int\partial_{x_1}\hat{v}_2f(v) - \partial_{x_2}\hat{v}_1f(v)dv \end{aligned}$$

We only solve the first equation since the second and the third equation are obtained only by a circular permutation of $\hat{v}$ and B indices.

[40] Theorem B.11.
[41] See details in [5, page: 64].
[42] Consequence of the integral derivation theorem. The proof is the same as for the electric case above.

The first equation gives:

$$
\begin{aligned}
(\partial_t^2 - \Delta)B^1 &= \int T_2\hat{v}_3 f dv - \int T_3\hat{v}_2 f dv + \int \frac{(\omega_2\hat{v}_3 - \omega_3\hat{v}_2)}{1+\hat{v}.\omega}(Sf - \hat{v}.Tf)dv \\
&= \int T_2\hat{v}_3 f dv - \int T_3\hat{v}_2 f dv + \int \frac{(\omega\times\hat{v})_1}{1+\hat{v}.\omega}(Sf - \hat{v}.Tf)dv
\end{aligned}
$$

Let $(t,x) \in [0,T[\times\mathbb{R}^3$, solving this equation using lemma 3.1, we get :

$$
\begin{aligned}
B^1(t,x) &= (B^1)'_0(t,x) + B^1_S(t,x) + \int_{|x-y|\le t}\int T_2.\hat{v}_3 f(t-|x-y|,y,v)dv\frac{dy}{|x-y|} \\
&- \int_{|x-y|\le t}\int T_3.\hat{v}_2 f(t-|x-y|,y,v)dv\frac{dy}{|x-y|} \\
&- \int_{|x-y|\le t}\int \frac{(\omega\times\hat{v})_1}{1+\hat{v}.\omega}\hat{v}.T[f(t-|x-y|,y,v)]dv\frac{dy}{|x-y|}
\end{aligned}
$$

We call:

$$
I'_B := B^1(t,x) - (B^1)'_0(t,x) - B^1_S(t,x)
$$

Using (3.12) , we get :

$$
\begin{aligned}
I'_B &= \int_{|x-y|\le t}\int \hat{v}_3\partial_{y_2}[f(t-|x-y|,y,v)]dv\frac{dy}{|x-y|} \\
&- \int_{|x-y|\le t}\int \hat{v}_2\partial_{y_3}[f(t-|x-y|,y,v)]dv\frac{dy}{|x-y|} \\
&- \int_{|x-y|\le t}\int \frac{(\omega\times\hat{v})_1}{1+\hat{v}.\omega}\hat{v}.\nabla_y[f(t-|x-y|,y,v)]dv\frac{dy}{|x-y|}
\end{aligned}
$$

Performing an integration by part and including the integral on the surface in $(B^1)_0$, as we did for E^i, we get the claimed result.

⋆ ⋆ ⋆ ⋆

3.2.2 Representation of the gradient of the field

<u>Theorem</u> **3.4** *Under the assumptions and notations of the theorem 3.3, we have the following results:*

- *For all $(i,j) \in \{1,2,3\}$ and $(t,x) \in [0,T[\times\mathbb{R}^3$:*

$$
\begin{aligned}
\partial_{x_k}E^i(t,x) &= (\partial_{x_k}E^i)_0(t,x) + \int_{|x-y|\le t}\int a(\omega,\hat{v})f(t-|x-y|,y,v)dv\frac{dy}{|x-y|^3} \\
&+ \int_{|x-y|\le t}\int b(\omega,\hat{v})(Sf)(t-|x-y|,y,v)dv\frac{dy}{|x-y|^2} \\
&+ \int_{|x-y|\le t}\int c(\omega,\hat{v})(S^2 f)(t-|x-y|,y,v)dv\frac{dy}{|x-y|} \\
&+ \int_{|\omega|=1}\int d(\omega,\hat{v})f(t,x,v)dvdS(\omega)
\end{aligned}
$$

with $(\partial_{x_k} E^i)_0$ *is an integral of derivatives of the initial data. The functions (kernels), a, b ,c and d are* $\mathcal{C}^\infty$ *except at* $\omega.\hat{v} = -1$ *and have algebraic singularities in such points. Furthermore, for all* $(t,x,v) \in [0,T[\times\mathbb{R}^3 \times \mathbb{R}^3$*:*

$$\int_{|\omega|=1}\int d(\omega,\hat{v})f(t,x,v)dvdS(\omega) = \mathcal{O}(1)$$

and

$$\int_{|\omega|=1} a(\omega,\hat{v})dS(\omega) = 0$$

and for a suitably smooth f*, the integral:*

$$\int_{|x-y|\leq t}\int a(\omega,\hat{v})f(t-|x-y|,y,v)dv\frac{dy}{|x-y|^3}$$

is convergent.

- *There is a similar representation of the gradient of the magnetic field* B *with kernels* a_B , b_B, c_B *and* d_B *which satisfy the same properties.*

Proof for the Theorem **3.4** *We note here :*

$$\hat{f}(t,x,y,v) := f(t-|x-y|,y,v)$$

- *Using the representation of the electric field given in the theorem 3.3 we get for a fixed* $(i,k) \in \{1,2,3\}$ *and* $(t,x) \in [0,T[\times\mathbb{R}^3$*:*

$$\begin{aligned}
4\pi\partial_{x_k}E^i(t,x) &= (\partial_{x_k}E^i)'_0(t,x) \\
&\quad - \underbrace{\int_{|x-y|\leq t}\int \frac{(\omega_i+\hat{v}_i)(1-|\hat{v}|^2)}{(1+\hat{v}.\omega)^2}\partial_{x_k}\hat{f}(t,x,y,v)dv\frac{dy}{|y-x|^2}}_{=I_1} \\
&\quad - \underbrace{\int_{|y-x|\leq t}\int \frac{(\omega_i+\hat{v}_i)}{1+\hat{v}.\omega}\partial_{x_k}\Big[(Sf)(t-|x-y|,y,v)\Big]dv\frac{dy}{|y-x|}}_{=I_2}
\end{aligned}$$

Thus,

$$I_1 = \int_{|x-y|\leq t}\int \frac{(\omega_i+\hat{v}_i)(1-|\hat{v}|^2)}{(1+\hat{v}.\omega)^2}\Big[\sum_j(\delta_{j,k} - \frac{\omega_k\hat{v}_j}{1+\hat{v}.\omega})T_j + \frac{\omega_k}{1+\hat{v}.\omega}S\Big]\hat{f}(t,x,y,v)dv\frac{dy}{|y-x|^2}$$

and :

$$\begin{aligned}
I_2 &= \int_{|x-y|\leq t}\int \frac{(\omega_i+\hat{v}_i)}{1+\hat{v}.\omega}\Big[\sum_j(\delta_{j,k} - \frac{\omega_k\hat{v}_j}{1+\hat{v}.\omega})T_j + \frac{\omega_k}{1+\hat{v}.\omega}S\Big](Sf)(t-|x-y|,y,v)dv\frac{dy}{|y-x|} \\
&= \sum_j\int_{|x-y|\leq t}\int \frac{(\omega_i+\hat{v}_i)}{1+\hat{v}.\omega}(\delta_{j,k} - \frac{\omega_k\hat{v}_j}{1+\hat{v}.\omega})T_j[(Sf)(t-|x-y|,y,v)]dv\frac{dy}{|y-x|} \\
&\quad + \int_{|x-y|\leq t}\int \frac{(\omega_i+\hat{v}_i)\omega_k}{(1+\hat{v}.\omega)^2}(S^2f)(t-|x-y|,y,v)dv\frac{dy}{|y-x|}
\end{aligned}$$

using (3.12) and performing an integration by parts, we get:

$$\begin{aligned}
I_2 &= \sum_j \int_{|x-y|\le t}\int \frac{(\omega_i+\hat{v}_i)}{1+\hat{v}.\omega}(\delta_{j,k}-\frac{\omega_k\hat{v}_j}{1+\hat{v}.\omega})\partial_{y_j}[(Sf)(t-|x-y|,y,v)]dv\frac{dy}{|y-x|}\\
&\quad+\int_{|x-y|\le t}\int \frac{(\omega_i+\hat{v}_i)\omega_k}{(1+\hat{v}.\omega)^2}(S^2f)(t-|x-y|,y,v)dv\frac{dy}{|y-x|}\\
&= -\sum_j \int_{|x-y|\le t}\int \partial_{y_j}\Big[\frac{(\omega_i+\hat{v}_i)}{|y-x|(1+\hat{v}.\omega)}(\delta_{j,k}-\frac{\omega_k\hat{v}_j}{1+\hat{v}.\omega})\Big](Sf)(t-|x-y|,y,v)dvdy\\
&\quad+\int_{|x-y|\le t}\int \frac{(\omega_i+\hat{v}_i)\omega_k}{(1+\hat{v}.\omega)^2}(S^2f)(t-|x-y|,y,v)dv\frac{dy}{|y-x|}\\
&\quad+\frac{1}{t}\int_{|x-y|=t}\int \sum_j \frac{\omega_j(\omega_i+\hat{v}_i)}{(1+\hat{v}.\omega)}(\delta_{j,k}-\frac{w_k\hat{v}_j}{1+\hat{v}.\omega})(Sf)(0,y,v)dvdS(y)
\end{aligned}$$

moreover :

$$\begin{aligned}
\sum_j \frac{\omega_j(\omega_i+\hat{v}_i)}{(1+\hat{v}.\omega)}(\delta_{j,k}-\frac{w_k\hat{v}_j}{1+\hat{v}.\omega}) &= \frac{(\omega_i+\hat{v}_i)}{(1+\hat{v}.\omega)^2}[\sum_j \omega_j\delta_{j,k}(1+\hat{v}.\omega)-\omega_k\hat{v}.\omega]\\
&= \frac{(\omega_i+\hat{v}_i)\omega_k}{(1+\hat{v}.\omega)^2}
\end{aligned}$$

Hence,

$$\begin{aligned}
I_2 &= -\sum_j \int_{|x-y|\le t}\int \partial_{y_j}\Big[\frac{(\omega_i+\hat{v}_i)}{|y-x|(1+\hat{v}.\omega)}(\delta_{j,k}-\frac{\omega_k\hat{v}_j}{1+\hat{v}.\omega})\Big](Sf)(t-|x-y|,y,v)dvdy\\
&\quad+\int_{|x-y|\le t}\int \underbrace{\frac{(\omega_i+\hat{v}_i)\omega_k}{(1+\hat{v}.\omega)^2}}_{:=c(\omega,\hat{v})}(S^2f)(t-|x-y|,y,v)dv\frac{dy}{|y-x|}\\
&\quad+\frac{1}{t}\int_{|x-y|=t}\int \frac{(\omega_i+\hat{v}_i)\omega_k}{(1+\hat{v}.\omega)^2}(Sf)(0,y,v)dvdS(y)
\end{aligned}$$

The last term depends only on the initial data and it will be included in $(\partial_{x_k}E^i)_0$*. To find the other kernels, we have to put together the terms with* f *and* Sf*. After a lengthy but elementary computations we have that* [43]*:*

$$\partial_{y_j}\Big[\frac{(\omega_i+\hat{v}_i)}{|y-x|(1+\hat{v}.\omega)}(\delta_{j,k}-\frac{\omega_k\hat{v}_j}{1+\hat{v}.\omega})\Big]=\frac{\beta(\omega,\hat{v})}{|x-y|^2}$$

with

$$\begin{aligned}
\beta(\omega,\hat{v}) &:= (1+\hat{v}.\omega)^{-3}\Big[(\omega_i+\hat{v}_i)\{2\omega_k(|\hat{v}|^2-1)+\omega_k(1+\hat{v}.\omega)-2\hat{v}(1+\hat{v}.\omega)\}\\
&\quad (1+\hat{v}.\omega)\{(1+\hat{v}.\omega)(\delta_{i,k}-\omega_i\omega_k)-\omega_k(\hat{v}_i-\hat{v}.\omega\omega_i)\}\Big]
\end{aligned}$$

[43] See [5, Appendix]

Combining the (Sf) part of I_1 and the (Sf) part of I_2 we get :

$$b(\omega,\hat{v}) := \beta(\omega,\hat{v}) + \Big[\frac{\omega_k(\omega_i+\hat{v}_i)(1-|\hat{v}|^2)}{(1+\hat{v}.\omega)^3}\Big]$$

Let's consider now the first part left of I_1,

$$I_1' := \int_{|x-y|\le t}\int \frac{(\omega_i+\hat{v}_i)(1-|\hat{v}|^2)}{(1+\hat{v}.\omega)^2}\sum_j(\delta_{j,k}-\frac{\omega_k\hat{v}_j}{1+\hat{v}.\omega})T_j\hat{f}(t,x,y,v)dv\frac{dy}{|y-x|^2}$$

we can also write:

$$I_1' := \lim_{\varepsilon\to 0}\int_{\varepsilon\le|x-y|\le t}\int \frac{(\omega_i+\hat{v}_i)(1-|\hat{v}|^2)}{(1+\hat{v}.\omega)^2}\sum_j(\delta_{j,k}-\frac{\omega_k\hat{v}_j}{1+\hat{v}.\omega})T_j\hat{f}(t,x,y,v)dv\frac{dy}{|y-x|^2}$$

by an integration by parts and using (3.12) we get for all $\epsilon\le t$,

$$\begin{aligned}
&\int_{\varepsilon\le|x-y|\le t}\int \frac{(\omega_i+\hat{v}_i)(1-|\hat{v}|^2)}{(1+\hat{v}.\omega)^2}\sum_j(\delta_{j,k}-\frac{\omega_k\hat{v}_j}{1+\hat{v}.\omega})T_j\hat{f}(t,x,y,v)dv\frac{dy}{|y-x|^2}\\
&=\sum_j\int_{|x-y|=t}\int \frac{\omega_j(\omega_i+\hat{v}_i)(1-|\hat{v}|^2)}{t^2(1+\hat{v}.\omega)^2}(\delta_{j,k}-\frac{\omega_k\hat{v}_j}{1+\hat{v}.\omega})f(0,y,v)dvdS(y)\\
&\underbrace{-\sum_j\int_{|x-y|=\varepsilon}\int \frac{\omega_j(\omega_i+\hat{v}_i)(1-|\hat{v}|^2)}{\varepsilon^2(1+\hat{v}.\omega)^2}(\delta_{j,k}-\frac{\omega_k\hat{v}_j}{1+\hat{v}.\omega})f(t-\varepsilon,y,v)dvdS(y)}_{=I_{1,2}'}\\
&\underbrace{-\sum_j\int_{\varepsilon\le|x-y|\le t}\int \partial_{y_j}\Big[\frac{(\omega_i+\hat{v}_i)(1-|\hat{v}|^2)}{|y-x|^2(1+\hat{v}.\omega)^2}(\delta_{j,k}-\frac{\omega_k\hat{v}_j}{1+\hat{v}.\omega})\Big]f(t-|x-y|,y,v)dvdy}_{=I_{1,3}'}
\end{aligned}$$

- *The first term depends only on the initial data and it will be included in $(\partial_{x_k}E^i)_0$.*
- *Moreover,*

$$\lim_{\varepsilon\to 0} I_{1,2}' = \int_{|\omega|=1}\int \underbrace{\sum_j\frac{\omega_j(\omega_i+\hat{v}_i)(1-|\hat{v}|^2)}{(1+\hat{v}.\omega)^2}(\delta_{j,k}-\frac{\omega_k\hat{v}_j}{1+\hat{v}.\omega})}_{:=d(\omega,\hat{v})}f(t,x,v)dvdS(\omega)$$

In fact, performing a spherical change of variable :

$$\begin{aligned}
&\sum_j\int_{|x-y|=\varepsilon}\int \frac{\omega_j(\omega_i+\hat{v}_i)(1-|\hat{v}|^2)}{\varepsilon^2(1+\hat{v}.\omega)^2}(\delta_{j,k}-\frac{\omega_k\hat{v}_j}{1+\hat{v}.\omega})f(t-\varepsilon,y,v)dvdS(y)\\
&=\sum_j\int_{[0,\pi]\times[0,2\pi]}\int \frac{\omega_j(\omega_i+\hat{v}_i)(1-|\hat{v}|^2)}{\varepsilon^2(1+\hat{v}.\omega)^2}(\delta_{j,k}-\frac{\omega_k\hat{v}_j}{1+\hat{v}.\omega})f(t-\varepsilon,y,v)dv\varepsilon^2\sin(\theta)d\theta d\phi\\
&=\sum_j\int_{\partial B(0,1)}\int \frac{\omega_j(\omega_i+\hat{v}_i)(1-|\hat{v}|^2)}{(1+\hat{v}.\omega)^2}(\delta_{j,k}-\frac{\omega_k\hat{v}_j}{1+\hat{v}.\omega})f(t-\varepsilon,x+\varepsilon\omega,v)dvdS(\omega)
\end{aligned}$$

it exists $\eta > 0$ such that if $\varepsilon \leq \eta$ we get :

$$|x+\varepsilon\omega| \leq 1+|x|$$

Thus, for ε enough small and $T' \in [0,T[$ we have :

$$|f(t-\varepsilon, x+\varepsilon\omega, v)| \leq \sup_{t\in[0,T']} \sup_{u\in B(0,1+|x|)} \sup_{|v|\leq Q_{T'}} |f(t,u,v)|$$

hence by **the dominated convergence theorem** *we get the claimed result.*

– *For the last term, the ε allows to avoid the singularity. After lengthy but elementary computation we get* [44] *:*

$$\begin{aligned}
&\sum_j \partial_{y_j}\Big[\frac{(\omega_i+\hat{v}_i)(1-|\hat{v}|^2)}{|x-y|^2(1+\hat{v}.\omega)^2}(\delta_{j,k}-\frac{\omega_k\hat{v}_j}{1+\hat{v}.\omega})\Big] \\
&= \frac{-3(\omega_i+\hat{v}_i)[\omega_k(1-|\hat{v}|^2)+\hat{v}_k(1+\hat{v}.\omega)]+(1+\hat{v}.\omega)^2\delta_{i,k}}{|x-y|^3(1+|v|^2)(1+\hat{v}.\omega)^4} \\
&= \frac{a(\omega,\hat{v})}{|x-y|^3}
\end{aligned}$$

Hence we have:

$$\begin{aligned}
a(\omega,\hat{v}) &= \frac{-3(\omega_i+\hat{v}_i)[\omega_k(1-|\hat{v}|^2)+\hat{v}_k(1+\hat{v}.\omega)]+(1+\hat{v}.\omega)^2\delta_{i,k}}{(1+|v|^2)(1+\hat{v}.\omega)^4} \\
b(\omega,\hat{v}) &= \Big[\frac{\omega_k(\omega_i+\hat{v}_i)(1-|\hat{v}|^2)}{(1+\hat{v}.\omega)^3}\Big]+\beta(\omega,\hat{v}) \\
c(\omega,\hat{v}) &= \frac{(\omega_i+\hat{v}_i)\omega_k}{(1+\hat{v}.\omega)^2} \\
d(\omega,\hat{v}) &= \sum_j \frac{\omega_j(\omega_i+\hat{v}_i)(1-|\hat{v}|^2)}{(1+\hat{v}.\omega)^2}(\delta_{j,k}-\frac{\omega_k\hat{v}_j}{1+\hat{v}.\omega})
\end{aligned}$$

according to the expressions above all the kernels have in deed an algebraic singularity in the points satisfying the equation $1+\omega.\hat{v}=0$.
With the assumption on the velocity support of f, notice that the kernel $d(\omega,\hat{v})$ is bounded. Since f is also bounded, we have that :

$$\int_{|\omega|=1}\int d(\omega,\hat{v})f(t,x,v)dvdS(\omega) = \mathcal{O}(1)$$

Furthemore, We can also write the final result on the following form:

$$\partial_{x_k}E^i = (\partial_{x_k}E^i)_0 + \partial_{x_k}E^i{}_{TT} - \partial_{x_k}E^i{}_{TS} + \partial_{x_k}E^i{}_{ST} - \partial_{x_k}E^i{}_{SS} + \mathcal{O}(1)$$

[44] For details, see [5, Appendix]

for $(i,k) \in \{1,2,3\}$*, and with :*

$$\begin{aligned}
\partial_{x_k} E^i_{TT} &= \int_{|x-y|\le t}\int \sum_j \partial_{y_j}\Big[\frac{(\omega_i+\hat{v}_i)}{(1+|v|^2)(1+\hat{v}.\omega)^2}(\delta_{j,k}-\frac{\omega_k\hat{v}_j}{1+\hat{v}.\omega}).\frac{1}{|x-y|^2}\Big]\hat{f}(t,x,y,v)dvdy \\
\partial_{x_k} E^i_{TS} &= \int_{|x-y|\le t}\int \frac{(\omega_i+\hat{v}_i)\omega_k}{(1+|v|^2)(1+\hat{v}.\omega)^3|x-y|^3}(Sf)(t-|x-y|,y,v)dvdy \\
\partial_{x_k} E^i_{ST} &= \int_{|x-y|\le t}\int \sum_j \partial_{y_j}\Big[\frac{(\omega_i+\hat{v}_i)}{(1+\hat{v}.\omega)|x-y|}(\delta_{j,k}-\frac{\omega_k\hat{v}_j}{1+\hat{v}.\omega})\Big](Sf)(t-|x-y|,y,v)dvdy \\
\partial_{x_k} E^i_{SS} &= \int_{|x-y|\le t}\int \frac{(\omega_i+\hat{v}_i)\omega_k}{(1+\hat{v}.\omega)^2|x-y|}(S^2f)(t-|x-y|,y,v)dydv \qquad (3.13)
\end{aligned}$$

We can write B *on a similar form to. For the fact that :*

$$\int_{|\omega|=1} a(\omega,\hat{v})d\omega = 0$$

a proof is given in [4, pages: 146-147].

- *The method for calculating* $\partial_{x_k}B^i$ *is similar as above. See [5, pages: 69-71] for more details.*

⋆ ⋆ ⋆ ⋆

We have also a similar result for the time derivatives.

Theorem **3.5** *Under the assumptions and notations of the theorem 3.3, we have the following results:*

- *For all* $(i,j) \in \{1,2,3\}$ *and* $(t,x) \in [0,T[\times\mathbb{R}^3$*:*

$$\begin{aligned}
\partial_t E^i(t,x) &= (\partial_t E^i)_0(t,x) + \int_{|x-y|\le t}\int \hat{a}(\omega,\hat{v})f(t-|x-y|,y,v)dv\frac{dy}{|x-y|^3} \\
&+ \int_{|x-y|\le t}\int \hat{b}(\omega,\hat{v})(Sf)(t-|x-y|,y,v)dv\frac{dy}{|x-y|^2} \\
&+ \int_{|x-y|\le t}\int \hat{c}(\omega,\hat{v})(S^2f)(t-|x-y|,y,v)dv\frac{dy}{|x-y|} \\
&+ \int_{|\omega|=1}\int \hat{d}(\omega,\hat{v})f(t,x,v)dvd\omega
\end{aligned}$$

with $(\partial_t E^i)_0$ *is an integral of derivatives of the initial data. The kernels,* $\hat{a}$*,* $\hat{b}$*,* $\hat{c}$ *and* $\hat{d}$ *are* $\mathcal{C}^\infty$ *except at* $\omega.\hat{v} = -1$ *and have algebraic singularities in such points. Furthermore, for all* $v \in \mathbb{R}^3$*:*

$$\int_{|\omega|=1} \hat{a}(\omega,\hat{v})d\omega = 0$$

and

$$\int_{|\omega|=1} \int d(\omega, \hat{v}) f(t,x,v) dv dS(\omega) = \mathcal{O}(1)$$

and for a suitably smooth f, the integral:

$$\int_{|x-y|\leq t} \int \hat{a}(\omega, \hat{v}) f(t-|x-y|, y, v) dv \frac{dy}{|x-y|^3}$$

is convergent.

- *There is a similar representation for the time derivative of the magnetic field B with kernels $\hat{a}_B$, $\hat{b}_B$, $\hat{c}_B$ and d_B which satisfy the same properties above.*

Proof for the Theorem **3.5** *The proof is very similar to the proof of the theorem 3.4. The details of the proof are given in [5, page: 71].*
As for the previous theorem we can write the fields on the following form :

$$\partial_t E^i = (\partial_t E^i)_0 + \partial_t E^i{}_{TT} - \partial_t E^i{}_{TS} + \partial_t E^i{}_{ST} - \partial_t E^i{}_{SS} + \mathcal{O}(1)$$

for $i \in \{1,2,3\}$.

3.2.3 Estimates on the gradient of f

Let's start with a compactly supported initial data f_0 as in the theorem 3.2. We assume that, $supp(f_0) \subset \{(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3 / |x| \leq P$ and $|v| \leq P\}$, with $P > 0$. To get estimations on the gradient of f, we need to control the size of the support in the space of possible positions. To do this, we use the characteristics of the system, given by relativistic Newton's equations :

$$\begin{cases} \dot{X} = \hat{V} \\ \dot{V} = E(t,X) + \hat{V} \times B(t,X) \end{cases}$$

we recall that:

$$\hat{V} = \frac{V}{\sqrt{1+|V|^2}}$$

Let $T' \in [0, T[$, in all this section, $M_{T'} > 0$ is a constant that could change value from line to line, and depends only on T', and $C > 0$ is a constant that could change value from line to line, but doesn't depends on the variables.
Let's assume that f is a maximal local solution for the problem defined on $[0, T[$. f has to be constant along the characteristics. Thus :

$$f(t,x,v) = f(0, X(0,t,x,v), V(0,t,x,v)) = f_0(X(0,t,x,v), V(0,t,x,v))$$

This gives that, for all $t \in [0, T[$:

$$\|f(t,.,.)\|_\infty \leq \|f_0\|_\infty$$

Using the expression of f above we can notice that, for all $(t, x) \in [0, T[\times\mathbb{R}^3$:

$$f(t, x, v) = 0 \text{ if } |v| \geq P + t$$

In fact, for $(t, x) \in [0, T[\times\mathbb{R}^3$:

$$|X(0, t, x, v) - x| = |\int_0^t \hat{V}(s, t, x, v)ds| \leq t$$

thus, if $|v| \geq P + t$, we get:

$$\begin{array}{rcl} |X(0, t, x, v)| & \geq & |x| - t \\ & \geq & P \end{array}$$

which gives $f(t, x, v) = f_0(X(0, t, x, v), V(0, t, x, v)) = 0$.

Using this remark we get :

$$\rho(t, x) = j(t, x) = 0 \text{ if } |x| \geq P + t$$

for any $t \in [0, T[$.

- Let $i \in \{1, 2, 3\}$. Let's assume that f is enough regular so that the following expressions have a sens, and all the derivatives commutates. Thus, by differentiating the equation satisfied by f, we get :

  $$(\partial_t + \hat{v}.\nabla_x + F(t, x, v).\nabla_v)(\partial_{x_i} f) = -\partial_{x_i} F(t, x, v).\nabla_v f$$

  for $(t, x, v) \in [0, T[\times\mathbb{R}^3 \times \mathbb{R}^3$.

  Hence, for all $(s, t, x, v) \in [0, T[\times[0, T[\times\mathbb{R}^3 \times \mathbb{R}^3$ we get the following,

  $$\frac{d}{ds}\partial_{x_i} f(s, X(s, t, x, v), V(s, t, x, v)) = -(\partial_{x_i} F.\nabla_v f)(s, X(s, t, x, v), V(s, t, x, v))$$

  Thus, for all $(t, x, v) \in [0, T[\times\mathbb{R}^3 \times \mathbb{R}^3$,

  $$\begin{array}{rcl} |\partial_{x_i} f(t, x, v)| & \leq & |\partial_{x_i} f(0, X(0, t, x, v), V(0, t, x, v))| \\ & & + \displaystyle\int_0^t |(\partial_{x_i} F.\nabla_v f)(s, X(s, t, x, v), V(s, t, x, v))|ds \end{array}$$

  Hence,

  $$\|\partial_{x_i} f(t,.,.)\|_\infty \leq \|\partial_{x_i} f_0\|_\infty + 3\int_0^t (|E(s,.)|_1 + |B(s,.)|_1)|f(s,.,.)|_1 ds$$

- The same estimate holds for ∂_{v_i}. In fact, we have that :

$$(\partial_t f + \hat{v}.\nabla_x f + F.\nabla_v)(\partial_{v_i} f) = -(\partial_{v_i}\hat{v}).\nabla_x f - [(\partial_{v_i}\hat{v}) \times B].\nabla_v f$$

Using the same reasoning as above, and since $\partial_{v_i}\hat{v}$ is bounded [45], we get for all $t \in [0, T']$:

$$\begin{aligned}
\|\partial_{v_i} f(t,.,.)\|_\infty &\leq \|\partial_{v_i} f_0\|_\infty \\
&\quad + \int_0^t \Big|(\partial_{v_i}\hat{v}).\nabla_x f)(s, X(s,t,x,v), V(s,t,x,v))\Big| ds \\
&\quad + \int_0^t \Big|([(\partial_{v_i}\hat{v}) \times B].\nabla_v f)(s, X(s,t,x,v), V(s,t,x,v))\Big| ds \\
&\leq \|\partial_{v_i} f_0\|_\infty + M_{T'}. \int_0^t |f(s,.,.)|_1 ds \\
&\quad + \int_0^t \Big|([(\partial_{v_i}\hat{v}) \times B].\nabla_v f)(s, X(s,t,x,v), V(s,t,x,v))\Big| ds \\
&\leq \|\partial_{v_i} f_0\|_\infty + M_{T'}. \int_0^t (1 + \|B(s,.)\|_\infty)|f(s,.,.)|_1 ds
\end{aligned}$$

Combining all these results we get :

$$|f(t,.,.)|_1 \leq C + M_{T'} \int_0^t \Big(1 + \|B(s,.)\|_\infty + |B(s,.)|_1 + |E(s,.)|_1\Big)|f(s,.,.)|_1 ds \tag{3.14}$$

for all $t \in [0, T']$.

3.2.4 Bounds on the field

Let's start with the expressions of the fields given in the theorem 3.3. Under the assumptions of theorem 3.2 we have:

$$\begin{aligned}
4\pi E &= (E)_0 + E_T + E_S \\
4\pi B &= (B)_0 + B_T + B_S
\end{aligned}$$

Let $0 \leq T' < T$, the aim of this subsection is to prove that, for all $t \in [0, T']$:

$$\|E(t,.)\|_\infty + \|B(t,.)\|_\infty \leq M_{T'} \tag{3.15}$$

with $M_{T'} > 0$ is constant depending only on T', and E_T,E_S, B_T and B_S are given in (3.9) and (3.10).
We recall that, for all $t \in [0, T']$,

$$P(t) \leq Q_{T'} = \|P\|_{\infty,[0,T']}$$

and

$$P(t) := \max(0, Q(t))$$

[45] This is a consequence of the assumption on the velocity support of f. In general, this is not satisfied.

In all this section, $M_{T'} > 0$ is a constant that could change value from line to line, and depends only on T', and $C > 0$ is a constant that could change value from line to line, but doesn't depends on the variables.

Using **Cauchy-Schwartz inequality** and since $C_{T'} > 0$ we get :

$$|\hat{v}.\omega| \leq |\hat{v}| \leq \frac{Q_{T'}}{\sqrt{1+Q_{T'}^2}} < 1$$

thus, for each $i \in \{1,2,3\}$

$$\Big|\frac{\omega_i + \hat{v}_i}{(1+\|\hat{v}\|^2)(1+\hat{v}.\omega)^2}\Big| \leq Q_{T'}$$

Notice that, using the same technic here, we can prove that all the kernels and there derivatives are bounded. Hence, for all $(t,x) \in [0,T'] \times \mathbb{R}^3$:

- For the first term

$$\begin{aligned} |E_T(t,x)| &\leq M_{T'} \int_{|x-y|\leq t} \int_{|v|\leq Q_{T'}} f(t-|x-y|,y,v)dv \frac{dy}{|x-y|^2} \\ &\leq M_{T'} \int_0^t \int_{|y|\leq 1} \int_{|v|\leq Q_{T'}} f(s,x+(t-s)y,v)dvdyds \\ &\leq M_{T'} \int_0^t \|f(s,.,.)\|_\infty ds \end{aligned}$$

- For the second

$$\begin{aligned} |E_S(t,x)| &= \Big|\int_{|x-y|\leq t} \int_{|v|\leq Q_{T'}} \frac{\omega_i+\hat{v}_i}{(1+\hat{v}.\omega)}(Sf)(t-|x-y|,y,v)dv\frac{dy}{|x-y|}\Big| \\ &= \Big|\int_{|x-y|\leq t} \int_{|v|\leq Q_{T'}} \frac{\omega_i+\hat{v}_i}{(1+\hat{v}.\omega)}\nabla_v(F(t-|x-y|,y)f(t-|x-y|,y,v))dv\frac{dy}{|x-y|}\Big| \end{aligned}$$

performing an integration by parts and using the assumption on the velocity support of f, we get :

$$|E_S(t,x)| \leq \int_{|x-y|\leq t} \int_{|v|\leq Q_{T'}} \Big|\nabla_v[\frac{\omega_i+\hat{v}_i}{(1+\hat{v}.\omega)}].F(t-|x-y|,y)f(t-|x-y|,y,v)\Big|dv\frac{dy}{|x-y|}$$

hence, since the gradient factor is bounded, we get :

$$|E_S(t,x)| \leq \|f_0\|_\infty M_{T'} \int_{|x-y|\leq t} \int_{|v|\leq Q_{T'}} |F(t-|x-y|,y,v)|dv\frac{dy}{|x-y|}$$

using the properties of the Lebesgue measure on the unit sphere [46] we get:

$$\begin{aligned} |E_S(t,x)| &\leq \|f_0\|_\infty C_{T'} \int_0^t \int_{|y|=1} \int (t-s)|F(s,x-(s-t)y,v)|dvdS(y)ds \\ &\leq \|f_0\|_\infty C_{T'} \int_0^t (\|E(s,.)\|_\infty + \|B(s,.)\|_\infty)ds \end{aligned}$$

[46] Theorem B.9.

Since $(E)_0$ depends only on the initial data, and since all the integration domains are bounded, $(E)_0$ is bounded.
Hence, we can write :

$$\|E(t,.)\|_\infty \leq \|(E)_0\|_\infty + M_{T'} \int_0^t (\|f(s,.,.)\|_\infty + \|E(s,.)\|_\infty + \|B(s,.)\|_\infty) ds$$

which can also been written on the following form :

$$\|E(t,.)\|_\infty \leq M_{T'} + M_{T'} \int_0^t (\|E(s,.)\|_\infty + \|B(s,.)\|_\infty) ds$$

Using the representation of the magnetic field, and following exactly the same steps, we get finally for a suitable constant $M_{T'} > 0$:

$$\|E(t,.)\|_\infty + \|B(t,.)\|_\infty \leq M_{T'} + M_{T'} \int_0^t (\|E(s,.)\|_\infty + \|B(s,.)\|_\infty) ds \tag{3.16}$$

using Gronwall's lemma, we get finally the claimed result.

3.2.5 Bounds on the gradient of the field

<u>Theorem</u> **3.6** *Let $0 < T' < T$ and let's consider the function, Ψ given by :*

$$\Psi(x) := x.\chi_{]-\infty,1]}(x) + (1 + \ln(x))\chi_{]1,+\infty[}(x)$$

then,

$$|E(t,.)|_1 + |B(t,.)|_1 \leq C_{T'}[1 + \Psi(\sup_{\tau \leq t} |f(t,.,.)|_1)]$$

for all $t \in [0, T']$ and with $C_{T'} > 0$ a constant that depends only on T'.

Proof for the Theorem **3.6 In all this proof, $M_{T'} > 0$ is a constant that could change value from line to line, and depends only on T', and $C > 0$ is a constant that could change value from line to line, but doesn't depends on the variables.**
Since the steps of the proof for the electric and magnetic fields are similar, we only prove the result for the electric field. Let's start by the expressions (3.13) given in the theorem (3.4). For all $(k, i) \in \{1, 2, 3\}^2$

$$\partial_{x_k} E^i = (\partial_{x_k} E^i)_0 + \partial_{x_k} {E^i}_{TT} - \partial_{x_k} {E^i}_{TS} + \partial_{x_k} {E^i}_{ST} - \partial_{x_k} {E^i}_{SS} + \mathcal{O}(1)$$

- *The data terms $(\partial_{x_k} E^i)_0$, with $k \in \{1, 2, 3\}$ are bounded, since they depend only on initial data and since the integration domains are bounded.*

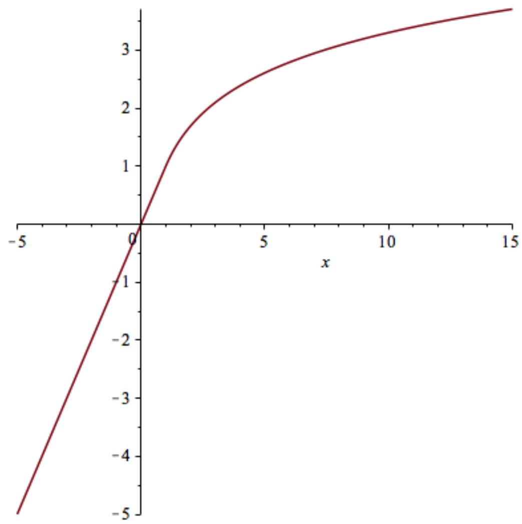

Figure 11: The plot of the function Ψ

- *Let* $(k, i) \in \{1, 2, 3\}^2$*, we start with the most singular term* $\partial_{x_k} E^i{}_{TT}$*. Using integration theorems, we can write* $\partial_{x_k} E^i{}_{TT}$ *as a time integral*

$$\begin{aligned}
\partial_{x_k} E^i{}_{TT} &= \int_{|x-y|\leq t} \int a(\omega, \hat{v}) f(t - |x-y|, y, v) dv \frac{dy}{|x-y|^3} \\
&= \int_{|y|\leq t} \int a(\omega, \hat{v}) f(t - |y|, x + y, v) dv \frac{dy}{|y|^3} \\
&= \int_0^t \int_{|y|=1} \int a(\omega, \hat{v}) f(t - s|y|, x + sy, v) dv s^2 \frac{dy}{s^3 |y|} ds \\
&= \int_0^t \underbrace{\frac{1}{t-s} \int_{|\omega|=1} \int a(\omega, \hat{v}) f(s, x + (t-s)\omega, v) dv d\omega}_{=\mathcal{S}} ds \quad (3.17)
\end{aligned}$$

Now, the kernel $a(\omega, \hat{v})$ *is bounded since* $|v| \leq C_{T'}$*. The integral given in (3.17) has a singularity in* t*. Therefore, we have to split the integral and estimate each of the terms independently, for all* $0 < d < t$*, we have that :*

$$|\int_0^{t-d} \frac{1}{t-s} \int_{|\omega|=1} \int a(\omega, \hat{v}) f(s, x + (t-s)\omega, v) dv d\omega ds| \leq M_{T'} \ln(\frac{t}{d})$$

with $M_{T'} > 0$ *is a constant which depends only on* T'*. Moreover, since* $\int_{|\omega|=1} a(\omega, \hat{v}) d\omega = 0$*, we have that :*

$$\begin{aligned}
|\int_{t-d}^t \mathcal{S} ds| &= \int_{t-d}^t \int_{|\omega|=1} \int |a(\omega, \hat{v})[\frac{f(s, x + (t-s)\omega, v) - f(s, x, v)}{t-s}]| dv d\omega ds \\
&\leq \sup_{0\leq\tau\leq t} |f(t, ., .)|_1 M_{T'} \int_{t-d}^t \int_{|\omega|=1} \int dv d\omega ds \\
&\leq M_{T'} d \sup_{0\leq\tau\leq t} |f(t, ., .)|_1
\end{aligned}$$

Hence

$$|\partial_{x_k} E^i{}_{TT}| \leq M_{T'}[\ln(\frac{t}{d}) + d \sup_{0\leq\tau\leq t} |f(t,.,.)|_1]$$

Notice that if $f \neq 0$ *then* $\sup_{0\leq\tau\leq t} |f(t,.,.)|_1 \neq 0$ *for all* $t \geq 0$.

Taking

$$d^{-1} = N \sup_{0\leq\tau\leq t} |f(t,.,.)|_1$$

for suitable $N \in \mathbb{N}$, *such that :*

$$\frac{1}{N \sup_{0\leq\tau\leq t} |f(t,.,.)|_1} < t$$

we get :

$$|\partial_{x_k} E^i{}_{TT}| \leq M_{T'}[1 + \Psi(\sup_{0\leq\tau\leq t} |f(t,.,.)|_1)] \tag{3.18}$$

Another useful estimations could be deduced from the fact that:

$$\int_{|\omega|=1} a(\omega, \hat{v}) d\omega = 0$$

In fact, starting from the first expression, we can directly write:

$$|\partial_{x_k} E^i_{TT}| = \int_0^t \int_{|\omega|=1} \int |a(\omega,\hat{v})[\frac{f(s, x+(t-s)\omega, v) - f(s,x,v)}{t-s}]| dv d\omega ds$$

thus,

$$|\partial_{x_k} E^i_{TT}| = M_{T'} \int_0^t |f(s,.,.)|_1 ds \tag{3.19}$$

- *For the* (Sf) *term, we perform an integration by part. Using the fact that the* v*-support of* f *is bounded, we have that :*

$$\begin{aligned}
\int_{|x-y|\leq t} \int b(\omega,\hat{v})(Sf) dv \frac{dy}{|x-y|^2} &= -\int_{|x-y|\leq t} \int b(\omega,\hat{v}) \nabla_v (F.f) dv \frac{dy}{|x-y|^2} \\
&= \underbrace{\int_{|x-y|\leq t} \int \nabla_v b(\omega,\hat{v}) F.f dv \frac{dy}{|x-y|^2}}_{=\mathcal{I}}
\end{aligned}$$

the functions Sf *and* $F.f$ *above are evaluated in* $(t - |x-y|, y, v)$.

Moreover,

$$\begin{aligned}
|\mathcal{I}| &\leq M_{T'} |\int_0^t \int_{|y|=1} \int F.f(s, x+(s-t)y, v) dv dS(y) ds| \\
&\leq M_{T'} \int_0^t (\|E(s,.)\|_\infty + \|B(s,.)\|_\infty) ds
\end{aligned}$$

Thus,

$$|\int_{|x-y|\leq t} \int b(\omega,\hat{v})(Sf)(t-|x-y|, y, v) dv \frac{dy}{|x-y|^2}| \leq M_{T'} \tag{3.20}$$

- *Before estimating the $S^2 f$ term, we commence by developing the expression of $S^2 f$. We have that :*

$$\begin{aligned}
S^2 f &= -S[\nabla_v(F.f)] = -\sum_j (\partial_t + \hat{v}.\nabla_x).\partial_{v_j}(f.F^j) \\
&= \sum_j [-\partial_{v_j}[(\partial_t + \hat{v}.\nabla_x)(f.F_j)] + \partial_{v_j}\hat{v}.\nabla_x(F_j.f)] \\
&= -\nabla_v[S(F.f)] + \sum_j \sum_i \frac{\delta_{i,j} - \hat{v}_i.\hat{v}_j}{\sqrt{1+|v|^2}}(f\partial_{x_i}F^j + F^j\partial_{x_i}f)
\end{aligned}$$

Using these expressions and performing an integration by parts with respect to v we get [47] :

$$\begin{aligned}
\partial_{x_k}{E^i}_{SS} &= \int_{|x-y|\leq t}\int c(\omega,\hat{v})S^2 f dv \frac{dy}{|x-y|} \\
&= -\int_{|x-y|\leq t}\int c(\omega,\hat{v})\nabla_v[S(F.f)]dv\frac{dy}{|x-y|} \\
&\quad + \sum_j\sum_i \int_{|x-y|\leq t}\int c(\omega,\hat{v})\frac{\delta_{i,j} - \hat{v}_i.\hat{v}_j}{\sqrt{1+|v|^2}}\Big(f\partial_{x_i}F^j + F^j\partial_{x_i}f\Big)dv\frac{dy}{|x-y|} \\
&= \int_{|x-y|\leq t}\int \nabla_v c(\omega,\hat{v})[S(F.f)]dv\frac{dy}{|x-y|} \\
&\quad + \sum_j\sum_i \int_{|x-y|\leq t}\int \underbrace{c(\omega,\hat{v})\frac{\delta_{i,j} - \hat{v}_i.\hat{v}_j}{\sqrt{1+|v|^2}}}_{=\hat{c}_{i,j}}\Big(f\partial_{x_i}F^j + F^j\partial_{x_i}f\Big)dv\frac{dy}{|x-y|}
\end{aligned}$$

the functions inside the integrals are evaluated in $(t-|x-y|, y, v)$.
Notice that :

$$S(F.f) = F.(Sf) + f.(SF) = -F.\nabla_v(F.f) + f.(SF)$$

Hence, we can write $\partial_{x_k}{E^i}_{SS}$ as a sum of tree terms $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$ and $\mathcal{I}_4$ such that:

$$\begin{aligned}
\mathcal{I}_1 &:= -\int_{|x-y|\leq t}\int \nabla_v c(\omega,\hat{v})F.\nabla_v(F.f)dv\frac{dy}{|x-y|} \\
&= \int_{|x-y|\leq t}\int \nabla_v(\nabla_v c(\omega,\hat{v})F)F.f dv\frac{dy}{|x-y|}
\end{aligned}$$

$$\mathcal{I}_2 := \int_{|x-y|\leq t}\int \nabla_v c(\omega,\hat{v})f.(SF)dv\frac{dy}{|x-y|}$$

$$\mathcal{I}_3 := \sum_j\sum_i \int_{|x-y|\leq t}\int \hat{c}_{i,j}(\omega,\hat{v})f\partial_{x_i}F^j dv\frac{dy}{|x-y|}$$

[47] In the integration by part, we use the fact that the v-support of f is bounded.

$$\mathcal{I}_4 := \sum_j \sum_i \int_{|x-y|\leq t} \int \hat{c}_{i,j}(\omega,\hat{v}) F^j \partial_{x_i} f dv \frac{dy}{|x-y|}$$

1. *Since the v-derivatives of the kernel c are bounded and $\nabla_v F = 0$ we get using (3.15) :*

$$\begin{aligned} |\mathcal{I}_1| &\leq M_{T'} \int_0^t \int_{|y|=1} |\nabla_v(\nabla_v c(\omega,\hat{v}) F(s, x + t(s-t)y, v)) F(s, x+(s-t)y, v)| dv dS(y) d \\ &\leq M_{T'} \int_0^t (\|E(s,.)\|_\infty + \|B(s,.)\|_\infty)^2 ds \\ &\leq M_{T'} \end{aligned}$$

2. *For the second term, we get :*

$$\begin{aligned} |\mathcal{I}_2| &\leq M_{T'} \int_0^t \int_{|y|=1} \int |(SF)(s, x+(t-s)y, v)| dv dS(y) ds \\ &\leq M_{T'} \int_0^t (|E(s,.)|_1 + |B(s,.)|_1) ds \end{aligned}$$

3. *Since $\hat{c}_{i,j}$ is bounded, we get the same estimate as above. In fact :*

$$\begin{aligned} |\mathcal{I}_3| &\leq M_{T'} \int_0^t \int_{|y|=1} \int \sum_i \sum_j |\partial_{x_i} F^j(s, x+(t-s)y, v)| dv dS(y) ds \\ &\leq M_{T'} \int_0^t (|E(s,.)|_1 + |B(s,.)|_1) ds \end{aligned}$$

4. *For the last terme the problem, comes from the term with the f derivative. To avoid this problem, we split $\partial_{x_i} f$.*

$$\mathcal{I}_4 = \mathcal{I}_{4,1} + \mathcal{I}_{4,2}$$

with

$$\begin{aligned} \mathcal{I}_{4,1} &:= -\sum_j \sum_i \int_{|x-y|\leq t} \int \hat{c}_{i,j}(\omega,\hat{v}) F^j (\frac{\omega_i \nabla_v (Ff)}{1+\hat{v}.\omega}) dv \frac{dy}{|x-y|} \\ \mathcal{I}_{4,2} &:= \sum_j \sum_i \sum_p \int_{|x-y|\leq t} \int \hat{c}_{i,j}(\omega,\hat{v}) F^j (\delta_{i,p} - \frac{\omega_i \hat{v}_p}{1+\hat{v}.\omega}) T_p f dv \frac{dy}{|x-y|} \end{aligned}$$

the functions inside the integrals are evaluated in $(t-|x-y|, y, v)$. Using (3.12) we can write $\mathcal{I}_{4,2}$ on the following form:

$$\mathcal{I}_{4,2} = \sum_j \sum_i \sum_p \int_{|x-y|\leq t} \int \hat{c}_{i,j}(\omega,\hat{v}) F^j (\delta_{i,p} - \frac{\omega_i \hat{v}_p}{1+\hat{v}.\omega}) \partial_{y_p} f dv \frac{dy}{|x-y|}$$

Performing an integration by part with respect to v and with the same technics used for the other integrals above, we get:

$$|\mathcal{I}_{4,1}| \leq M_{T'} \int_0^t (\|E(s,.)\|_\infty + \|B(s,.)\|_\infty)^2 \|f(s,.,.)\|_\infty ds \leq M_{T'}$$

For the last terme, we also perform an integration by parts with respect to y using the fact that the v-support of f is bounded :

$$
\begin{aligned}
|\mathcal{I}_{4,2}| &= \Big|\sum_j\sum_i\sum_p\int_{|x-y|\le t}\int \hat{c}_{i,j}F^j(\delta_{i,p}-\frac{\omega_i\hat{v}_p}{1+\hat{v}.\omega})\partial_{y_p}f dv\frac{dy}{|x-y|}\Big| \\
&\le \Big|\sum_j\sum_i\sum_p\int_{|x-y|\le t}\int \partial_{y_p}[\frac{\hat{c}_{i,j}F^j}{\|x-y\|}(\delta_{i,p}-\frac{\omega_i\hat{v}_p}{1+\hat{v}.\omega})]f dvdy\Big| \\
&\le \Big|\sum_j\sum_i\sum_p\int_{|x-y|\le t}\int F^j\partial_{y_p}[\frac{\hat{c}_{i,j}}{\|x-y\|}(\delta_{i,p}-\frac{\omega_i\hat{v}_p}{1+\hat{v}.\omega})]f dvdy\Big| \\
&\quad +\Big|\sum_j\sum_i\sum_p\int_{|x-y|\le t}\int (\partial_{y_p}F^j).[\frac{\hat{c}_{i,j}}{\|x-y\|}(\delta_{i,p}-\frac{\omega_i\hat{v}_p}{1+\hat{v}.\omega})]f dvdy\Big|
\end{aligned}
$$

Using now the same technics as for the other integrals, we get :

$$
\begin{aligned}
|\mathcal{I}_{4,2}| &\le M_{T'}\sum_j\sum_i\sum_p\int_0^t\int_{|y|=1}\int |F^j(s,x+(t-s)y,v)\|f(s,.,.)\|_\infty dvdS(y)ds| \\
&\quad +M_{T'}\sum_j\sum_i\sum_p\int_0^t\int_{|y|=1}\int |\partial_{y_p}F^j(s,x+(t-s)y,v)\|f(s,.,.)\|_\infty dvdS(y)ds| \\
&\le M_{T'}\int_0^t(|E(s,.)|_1+|B(s,.)|_1+\|E(s,.)\|_\infty+\|B(s,.)\|_\infty)\|f(s,.,.)\|_\infty ds
\end{aligned}
$$

Combining now all these estimations, we can get an estimation of $\partial_{x_k}{E^i}_{SS}$:

$$
\begin{aligned}
|\partial_{x_k}{E^i}_{SS}| &\le M_{T'}\int_0^t\Big[(\|E(s,.)\|_\infty+\|B(s,.)\|_\infty)^2+(\|E(s,.)\|_\infty+\|B(s,.)\|_\infty)\Big]\|f(s)\|_\infty ds \\
&\quad +M_{T'}\|f_0\|_\infty\int_0^t(|E(s,.)\|_1+|B(s,.)\|_1)ds
\end{aligned}
\tag{3.21}
$$

Using, (3.18), (3.20) and (3.21) we finally get for all $t\in[0,T']$:

$$|E(t,.)|_1\le M_{T'}[1+\Psi(\sup_{0\le\tau\le t}|f(t,.,.)|_1)+\int_0^t(|E(s,.)|_1+|B(s,.)|_1)ds]$$

Following exactly the same approche we can prove a similar result for the magnetic field B. Thus, combining the two results we get:

$$|E(t,.)|_1+|B(t,.)|_1\le M_{T'}[1+\Psi(\sup_{0\le\tau\le t}|f(t,.,.)|_1)+\int_0^t(|E(s,.)|_1+|B(s,.)|_1)ds] \tag{3.22}$$

for all $t\in[0,T']$. By an application of the **Gronwall lemma** *we get the claimed result.*

⋆ ⋆ ⋆ ⋆

Remark 3.2 *Combining (3.14) and (3.15), we get for all $t\in[0,T']$:*

$$|f(t,.,.)|_1\le C+M_{T'}\int_0^t(1+|E(s,.)|_1+|B(s,.)|_1)|f(s,.,.)|_1 ds$$

using the theorem 3.6 we get :

$$|f(t,.,.)|_1 \leq C + M_{T'} \int_0^t (1+\Psi(\sup_{0\leq\tau\leq s} |f(\tau,.,.)|_1))|f(s,.,.)|_1 ds$$

hence, if $\alpha(t) := \sup_{0\leq\tau\leq t} |f(\tau,.,.)|_1$ *we get:*

$$\alpha(t) \leq \underbrace{C + M_{T'} \int_0^t [1+\Psi(\alpha(s))]\alpha(s)ds}_{:=\beta(t)}$$

with $C > 0$ *and* $M_{T'} > 0$ *constants. Hence,*

$$\int_C^{\beta(t)} \frac{d\beta}{\beta(1+\Psi(\beta))} \leq M$$

with $M > 0$ *a constant. Thus,* β *is bounded, and so is* α *and hence* $t :\to |f(t,.,.)|_1$. *Using the theorem 3.6, we get:*

$$|E(t,.)|_1 + |B(t,.)|_1 + |f(t,.,.)|_1 \leq M_{T'}$$

for all $t \in [0,T']$ *and* $M_{T'} > 0$ *a constant depending only on* T'.

3.2.6 Proof of existence and uniqueness

The a-priori estimates for the field and the gradient of the field is the key for proving the existence in theorem 3.2. Uniqueness is proved using the representation of the fields.

Proof of uniqueness :

Let $(E^{(1)}, B^{(1)}, f^{(1)})$ and $(E^{(2)}, B^{(2)}, f^{(2)})$ two classical solutions of the Vlasov-Poisson system with the same initial data (E_0, B_0, f_0).
And we define,

$$F^{(i)} = E^{(i)} + \hat{v} \times B^{(i)}$$

$$F = F^{(1)} - F^{(2)} = E + \hat{v} \times B$$

and

$$f = f^{(1)} - f^{(2)}$$

By linearity of the equations, (E, B, f) satisfies:

$$S(f) = (\partial_t + \hat{v}.\nabla_x)(f) = -\nabla_v(F^{(1)}.f^{(1)} - F^{(2)}f^{(2)}) = -\nabla_v(F.f^{(1)} + F^{(2)}f)$$

$$(\partial_t^2 - \Delta)E = -(\nabla_x \rho_f + \partial_t j_f)$$

$$(\partial_t^2 - \Delta)B = \nabla_x \times j_f$$

with $\rho = \int f dv$ and $j_f = \int \hat{v}.f dv$. By uniqueness of weak solutions (see remark 3.1), and since initial data is zero, we get:

$$\begin{aligned} 4\pi E^i &= E_T + E_S \\ 4\pi B^i &= B_T + B_S \end{aligned}$$

and for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^3$

$$\begin{aligned} E_T(t,x) &= -\int_{|y-x|\leq t}\int_{\mathbb{R}^3} \frac{(\omega+\hat{v})(1-|\hat{v}|^2)}{(1+\hat{v}.\omega)^2} f(t-|y-x|,y,v)dv\frac{dy}{|y-x|^2} \\ E_S(t,x) &= \int_{|y-x|\leq t}\int_{\mathbb{R}^3} \frac{(\omega+\hat{v})}{(1+\hat{v}.\omega)}\nabla_v(F.f^{(1)}+F^{(2)}f)(t-|y-x|,y,v)dv\frac{dy}{|y-x|} \end{aligned}$$

and

$$\begin{aligned} B_T(t,x) &= -\int_{|y-x|\leq t}\int_{\mathbb{R}^3} \frac{(\omega\times\hat{v})(1-|\hat{v}|^2)}{(1+\hat{v}.\omega)^2} f(t-|y-x|,y,v)dv\frac{dy}{|y-x|^2} \\ B_S(t,x) &= \int_{|y-x|\leq t}\int_{\mathbb{R}^3} \frac{(\omega\times\hat{v})}{(1+\hat{v}.\omega)}\nabla_v(F.f^{(1)}+F^{(2)}f)(t-|y-x|,y,v)dv\frac{dy}{|y-x|} \end{aligned}$$

using an integration by part with respect to v, we get :

$$\begin{aligned} E_S(t,x) &= -\int_{|y-x|\leq t}\int_{\mathbb{R}^3} \nabla_v[\frac{(\omega+\hat{v})}{(1+\hat{v}.\omega)}](F.f^{(1)}+F^{(2)}f)(t-|y-x|,y,v)dv\frac{dy}{|y-x|} \\ B_S(t,x) &= -\int_{|y-x|\leq t}\int_{\mathbb{R}^3} \nabla_v[\frac{(\omega+\hat{v})}{(1+\hat{v}.\omega)}](F.f^{(1)}+F^{(2)}f)(t-|y-x|,y,v)dv\frac{dy}{|y-x|} \end{aligned}$$

Let $0 < T' < T$, we will use here the same technics used before to estimate these expressions on $[0,T']$. **In all this proof $M_{T'} > 0$ is a constant that could change value from line to line and depends only on T'.** Since f is compactly supported in v, $1+\hat{v}.\omega$ is bounded away from 0.
Hence, for $(t,x) \in [0,T'] \times \mathbb{R}^3$:

$$\begin{aligned} |E_T(t,x)| &\leq M_{T'}\int_{|y-x|\leq t}\int f(t-|x-y|,y,v)dv\frac{dy}{|x-y|} \\ &\leq M_{T'}\int_0^t\int_{|y|\leq 1}\int f(s,x+(t-s)y,v)dvdyds \\ &\leq M_{T'}\int_0^t \|f(s,.,.)\|_\infty ds \end{aligned}$$

$$\begin{aligned} |E_S(t,x)| &\leq M_{T'}\int_{|y-x|\leq t}\int (F.f^{(1)}+F^{(2)}f)(t-|y-x|,y,v)dv\frac{dy}{|x-y|} \\ &\leq M_{T'}\int_0^t\int_{|y|\leq 1}\int (F.f^{(1)}+F^{(2)}f)(s,x+(t-s)y,v)dvdyds \\ &\leq M_{T'}\int_0^t \|f^{(1)}(s,.,.)\|_\infty(\|E(s,.)\|_\infty+\|B(s,.)\|_\infty)ds \\ &\quad +M_{T'}\int_0^t \|f(s,.,.)\|_\infty(\|E^{(2)}(s,.)\|_\infty+\|B^{(2)}(s,.)\|_\infty)ds \end{aligned}$$

It is easy to see that we have a similar estimation for the magnetic field B. Hence, for all $t \in [0, T']$

$$\begin{aligned}\|E(t,.)\|_\infty + \|B(t,.)\|_\infty &\leq M_{T'} \int_0^t \|f(s,.,.)\|_\infty (1 + \|E^{(2)}(s,.)\|_\infty + \|B^{(2)}(s,.)\|_\infty) ds \\ &\quad + M_{T'} \int_0^t \|f^{(1)}(s,.,.)\|_\infty (\|E(s,.)\|_\infty + \|B(s,.)\|_\infty) ds\end{aligned}$$

Since $f^{(1)}$, $E^{(2)}$ and $B^{(2)}$ are bounded, we can write the estimate above on the following form:

$$\|E(t,.)\|_\infty + \|B(t,.)\|_\infty \leq M_{T'} \int_0^t (\|E(s,.)\|_\infty + \|B(s,.)\|_\infty + \|f(s,.,.)\|_\infty) ds \quad (3.23)$$

Moreover, using the characteristics of the equation:

$$\partial_t f + \hat{v}.\nabla_x f + F^{(2)}.\nabla_v f = -F.\nabla_v f^{(1)}$$

defined by the equations :

$$\dot{X} = V, \qquad \dot{V} = F^{(2)}, \qquad \dot{f} = -F.\nabla_v f^{(1)}$$

we get, for all $t \in [0, T']$:

$$f(t, X(t), V(t)) - f(0, X(0), V(0)) = f(t, X(t), V(t)) = -\int_0^t F.\nabla_v f^{(1)}(s, X(s), V(s)) ds$$

Thus, for all $t \in [0, T']$:

$$\|f(t,.,.)\|_\infty \leq \int_0^t \|F.\nabla_v f^{(1)}(s,.,.)\|_\infty ds$$

Since, $\nabla_v f$ is bounded, we get for all $t \in [0, T']$:

$$\|f(t,.,.)\|_\infty \leq \int_0^t (\|E(s,.)\|_\infty + \|B(s,.)\|_\infty) ds \quad (3.24)$$

adding (3.23) and (3.24), we get for all $t \in [0, T']$:

$$\|f(t,.,.)\|_\infty + \|E(t,.)\|_\infty + \|B(t,.)\|_\infty \leq M_{T'} \int_0^t \|f(s,.,.)\|_\infty + \|E(s,.)\|_\infty + \|B(s,.)\|_\infty ds$$

using **Gronwal's lemma** [48] we get :

$$(E, B, f) = 0$$

which proves uniqueness assertion in theorem 3.6.

[48] See the Appendix.

Proof of existence :

For simplicity we take smooth initial data $f_0 \in \mathcal{C}_c^2$, E_0 and B_0 in $\mathcal{C}^3$ and E_1 and B_1 in $\mathcal{C}^2$. We recall that E_1 and B_1 are the initial values for the time derivative of the field :

$$\partial_t E(0,.) = E_1 \quad , \quad \partial_t B(0,.) = B_1$$

We define an iteration scheme $(E^{(n)}, B^{(n)}, f^{(n)})$ defined with the following :

$$f^{(0)} = f_0, \quad E^{(0)} = E_0, \quad B^{(0)} = B_0$$

Given the $(n-1)$ iteration, $f^{(n)}$ is the solution for :

$$\begin{cases} \partial_t f^{(n)} + \hat{v}.\nabla_x f^{(n)} + (E^{(n-1)} + \hat{v} \times B^{(n-1)}).\nabla_v f^{(n)} = 0 \\ f^{(n)}(0,.,.) = f_0 \end{cases} \tag{3.25}$$

The iteration schemes allows us to avoid the problem of non-linearity [49]. In fact, given the $(n-1)$ iteration, $f^{(n)}$ is a solution for a linear Vlasov problem. Hence, theorem (1.4) gives that $f^{(n)}$ is $\mathcal{C}^2$, has a compact support in v and

$$\|f^{(n)}\|_\infty \leq \|f_0\|_\infty$$

Thus, the following quantities are well defined:

$$\rho^{(n)} := \int f^{(n)} ds \quad , \quad j^{(n)} := \int f^{(n)} \hat{v} dv$$

and the fields of the next iteration are given by :

$$\begin{cases} (\partial_t^2 - \Delta)E^{(n)} = -\nabla_x \rho^{(n)} - \partial_t j^{(n)} \\ (\partial_t^2 - \Delta)B^{(n)} = \nabla_x \times j^{(n)} \\ E^{(n)}(0,.) = E_0, \quad \partial_t E^{(n)}(0,.) = E_1 \\ B^{(n)}(0,.) = B_0, \quad \partial_t B^{(n)}(0,.) = B_1 \end{cases} \tag{3.26}$$

- Regularity: Once we define the iterations, the first question to ask is : what is the regularity of these iterations ? The regularity of $f^{(n)}$ is already known and the following lemma gives an answer for the regularity of the fields.

Lemma 3.2 *Let $n \in \mathbb{N}$. If $f^{(n)}$ is a $\mathcal{C}^2$ solution for (3.25) and $E^{(n)}$, $B^{(n)}$ solutions for (3.26), then $E^{(n)}$ and $B^{(n)}$ are also $\mathcal{C}^2$ functions.*

Proof 3.2 *We proceed by induction on (n). The result is obviously satisfied for $n = 0$. We assume that the result is satisfied for an iteration $(n-1)$. Since the proof is similar for the electric and magnetic fields, the proof given is only for E.*
In the system of equations (3.26), the right hand side terms are $\mathcal{C}^1$. Hence, one of the conditions for the inversion of wave operator given in theorem (3.1) is not satisfied. But, despite this let's start with the expressions given in theorem (3.3). According

[49] The force is depending on the particle density f.

to Remark 3.1, these expressions are in deed solutions of the wave equation in the weak sens.

$$4\pi E^{(n)} = (E)_0 + E_T^{(n)} + E_S^{(n)}$$

with, for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^3$

$$\begin{aligned} E_T^{(n)}(t,x) &= -\int_{|y-x|\le t}\int_{\mathbb{R}^3} \frac{(\omega+\hat{v})(1-|\hat{v}|^2)}{(1+\hat{v}.\omega)^2} f^{(n)}(t-|y-x|,y,v)dv\frac{dy}{|y-x|^2} \\ E_S^{(n)}(t,x) &= \int_{|y-x|\le t}\int_{\mathbb{R}^3} \frac{(\omega+\hat{v})}{(1+\hat{v}.\omega)} \nabla_v(F^{(n-1)}f^{(n)})(t-|y-x|,y,v)dv\frac{dy}{|y-x|} \end{aligned}$$

the key step, is performing an integration by parts with respect to v*. Hence,*

$$\begin{aligned} E_T^{(n)}(t,x) &= -\int_{|y-x|\le t}\int_{\mathbb{R}^3} \frac{(\omega+\hat{v})(1-|\hat{v}|^2)}{(1+\hat{v}.\omega)^2} f^{(n)}(t-|y-x|,y,v)dv\frac{dy}{|y-x|^2} \\ E_S^{(n)}(t,x) &= -\int_{|y-x|\le t}\int_{\mathbb{R}^3} \nabla_v[\frac{(\omega+\hat{v})}{(1+\hat{v}.\omega)}](F^{(n-1)}f^{(n)})(t-|y-x|,y,v)dv\frac{dy}{|y-x|} \end{aligned}$$

$f^{(n)}$ and $F^{(n-1)}$ are $\mathcal{C}^2$, therefore $E^{(n)}$ is $\mathcal{C}^2$ and $E^{(n)}$ defined with the expressions above do satisfy the system (3.26). The same holds for the magnetic field.

⋆ ⋆ ⋆ ⋆

- Bounds: Since the fields representation given in theorem 3.3 remains for the iterates defined here in this proof, all the estimates found above will remain satisfied too, for an iterate $n \in \mathbb{N}^\star$, and on a time interval $[0,T']$ with $0 < T' < T$. But we should take into account the fact that the force in the equation satisfied by the (n) iterate is depending on the fields $E^{(n-1)}$ and $B^{(n-1)}$. Hence, the estimate (3.14) gives:

$$|f^{(n)}(t,.,.)|_1 \le C+M_{T'}\int_0^t(1+\|B^{(n-1)}(s,.)\|_\infty+|B^{(n-1)}(s,.)|_1+|E^{(n-1)}(s,.)|_1)|f(s,.,.)^{(n)}|_1 ds$$

using the fields representations given in the lemma above and since $f^{(n)}$ is bounded, we can notice that $B^{(n)}$ is also bounded. Thus,

$$|f^{(n)}(t,.,.)|_1 \le C+M_{T'}\int_0^t(1+|B^{(n-1)}(s,.)|_1+|E^{(n-1)}(s,.)|_1)|f(s,.,.)^{(n)}|_1 ds \quad (3.27)$$

Using **Gronwall's lemma** we get:

$$|f^n(t,.,.)|_1 \le M_{T'}\exp(\int_0^t M_{T'}(|B^{(n-1)}(s,.)|_1 + |E^{(n-1)}(s,.)|_1)ds) \quad (3.28)$$

with C, $M_{T'} > 0$ are constants, $M_{T'}$ could change value and depends only on T'.

Moreover, the analogue estimate for the bounds on the field (3.16) will be :

$$\|E(t,.)^{(n)}\|_\infty + \|B(t,.)^{(n)}\|_\infty \le M_{T'} + M_{T'}\int_0^t(\|E^{(n-1)}(s,.)\|_\infty + \|B^{(n-1)}(s,.)\|_\infty)ds \quad (3.29)$$

with $M_{T'} > 0$ is a constant depending only on T'. Hence,

$$\|E(t,.)^{(n)}\|_\infty + \|B(t,.)^{(n)}\|_\infty \leq M_{T'}(1 + M_{T'}t + ... + \frac{M_{T'}^n t^n}{n!}) \leq M_{T'}\exp(M_{T'}t)$$

Thus, the fields are pointwise bounded in time, and uniformly bounded in n.

The analogue estimate of (3.22) is:

$$|E^{(n)}(t,.)|_1 + |B^{(n)}(t,.)|_1 \leq M_{T'}[1 + \Psi(\sup_{0\leq\tau\leq t}|f^{(n)}(t,.,.)|_1) + \int_0^t (|E^{(n-1)}(s,.)|_1 + |B^{(n-1)}(s,.)|_1)ds] \tag{3.30}$$

If $\sup_{0\leq\tau\leq t}|f^{(n)}(t,.,.)|_1 \leq 1$ then :

$$\Psi(\sup_{0\leq\tau\leq t}|f^{(n)}(t,.,.)|_1) \leq 1 + \int_0^t (|E^{(n-1)}(s,.)|_1 + |B^{(n-1)}(s,.)|_1)ds$$

If not,

$$\Psi(\sup_{0\leq\tau\leq t}|f^{(n)}(t,.,.)|_1) \leq 1 + \ln(M_{T'}) + M_{T'}\int_0^t (|E^{(n-1)}(s,.)|_1 + |B^{(n-1)}(s,.)|_1)ds$$

Thus, combining (3.30) with (3.28), we get:

$$|E^{(n)}(t,.)|_1 + |B^{(n)}(t,.)|_1 \leq M_{T'} + M_{T'}\int_0^t (|E^{(n-1)}(s,.)|_1 + |B^{(n-1)}(s,.)|_1)ds]$$

with $M_{T'} > 0$ is a constant that could change value from line to line and depends only on T'. This estimate is iterated as above to get a uniform bound in n.

$$\|E^{(n)}\|_1 + \|B^{(n)}\|_1 \leq M_{T'}\exp(M_{T'}t)$$

Thus, using (3.28) we get a uniform bound on $\|f\|_1$ on $[0,T']$.

- Convergence : Using the bounds proved above, we aim to proof the convergence of the sequence $((E^{(n)}, B^{(n)}, f^{(n)}))_{n\in\mathbb{N}}$ in enough strong sens so that the limit define a classical solution for Vlasov-Maxwell system. To get optimal results, we show that the sequences are Cauchy in the norm $\|.\|_1$. Let's fixe $(m,n) \in \mathbb{N}^2$, and define on $[0,T']$:

$$\mathcal{F}_1^{m,n}(t) := |E^{(n)}(t,.) - E^{(m)}(t,.)|_1 + |B^{(n)}(t,.) - B^{(m)}(t,.)|_1$$
$$\mathcal{F}_\infty^{m,n}(t) := \|E^{(n)}(t,.) - E^{(m)}(t,.)\|_\infty + \|B^{(n)}(t,.) - B^{(m)}(t,.)\|_\infty$$
$$f_1^{m,n}(t) := |f^{(n)}(t,.,.) - f^{(m)}(t,.,.)|_1$$
$$f_\infty^{m,n}(t) := \|f^{(n)}(t,.,.) - f^{(m)}(t,.,.)\|_\infty$$

With exactly the same steps as for (3.23) and (3.24) we get for $(m,n) \in \mathbb{N}^\star \times \mathbb{N}^\star$ and $t \in [0,T']$:

$$\mathcal{F}_\infty^{m,n}(t) \leq M_{T'}\int_0^t [\mathcal{F}_\infty^{m-1,n-1}(s) + f_\infty^{m,n}(s)]ds$$

$$f_\infty^{m,n}(t) \leq M_{T'} \int_0^t \mathcal{F}_\infty^{m-1,n-1}(s)ds$$

combining these to last estimates and interchanging the order of integration we get:

$$\mathcal{F}_\infty^{m,n}(t) \leq M_{T'} \int_0^t \mathcal{F}_\infty^{m-1,n-1}(s)ds$$

hence, by iteration, we get for $k \leq \min(m,n)$:

$$\mathcal{F}_\infty^{m,n}(t) \leq M_{T'}^k \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} \mathcal{F}_\infty^{m-k,n-k}(s)ds$$

thus, since the fields are uniformly bounded in m we get :

$$\begin{aligned}
\mathcal{F}_\infty^{n,m+n}(t) &\leq M_{T'}^n \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} \mathcal{F}_\infty^{0,m}(s)ds \\
&\leq A.M_{T'}^n \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} ds \\
&\leq A \frac{(M_{T'}T')^n}{n!}
\end{aligned}$$

Hence, $(E^{(n)})_{n\in\mathbb{N}}$ and $(B^{(n)})_{n\in\mathbb{N}}$ are uniformly Cauchy sequences on $[0,T']$ with the norm $\|.\|_\infty$, so that they converge uniformly.

We aim to prove now that the same is valid for the $\mathcal{C}^1$ norm: $|.|_1$. To simplify the notations, let's denote by ∂ any first derivative of E or B, and $r = |x-y|$ and for $i \in \{1,2,3\}$ and $n \in \mathbb{N}$ we note :

$$(E^{(n)})^i = E^{(n)} \ , \quad (B^{(n)})^i = B^{(n)}$$

Since, the proofs for the electric and magnetic fields are exactly the same, we only consider now the electric field. Using the representation of the gradient of the fields (theorems 3.4 and 3.5), and since the initial data is the same for all the iterates, we can write:

$$\partial E^{(n)} = (\partial E)_0 + \partial E^{(n)}{}_{TT} - \partial E^{(n)}{}_{TS} + \partial E^{(n)}{}_{ST} - \partial E^{(n)}{}_{SS} + \mathcal{O}(1)$$

Now, using the the gradient of the field estimations, we can see what happens for the Cauchy sequence of each of these terms. Let's fixe $(n,m) \in \mathbb{N}^\star \times \mathbb{N}^\star$ such that $m \neq n$ and $t \in [0,T']$.

- As for (3.19) we have that:

$$\|\partial E_{TT}^{(m)}(t,.) - \partial E_{TT}^{(n)}(t,.)\|_\infty \leq M_{T'} \int_0^t f_1^{m,n}(s)ds$$

- For the (Sf) terms, performing an integration by parts with respect to v we get:

$$\begin{aligned}
\int_{r\leq t}\int k(\omega,\hat{v})S(f^{(n)}-f^{(m)})dv\frac{dy}{r^2} &= \int_{r\leq t}\int k(\omega,\hat{v}).\nabla_v[F^{(m-1)}.f^{(m)}-F^{(n-1)}.f^{(n)}]dv\frac{dy}{r^2}\\
&= \int_{r\leq t}\int k(\omega,\hat{v}).\nabla_v[(F^{(m-1)}-F^{(n-1)}).f^{(n)}]dv\frac{dy}{r^2}\\
&\quad +\int_{r\leq t}\int k(\omega,\hat{v}).\nabla_v[F^{(m-1)}.(f^{(m)}-f^{(n)})]dv\frac{dy}{r^2}\\
&= -\int_{r\leq t}\int \nabla_v k(\omega,\hat{v}).(F^{(m-1)}-F^{(n-1)}).f^{(n)}dv\frac{dy}{r^2}\\
&\quad -\int_{r\leq t}\int \nabla_v k(\omega,\hat{v}).F^{(m-1)}.(f^{(m)}-f^{(n)})dv\frac{dy}{r^2}
\end{aligned}$$

with k is the kernel b or $\hat{b}$ in the theorems 3.4 and 3.5). Since the derivative of the kernel k, $(f^{(n)})_{n\in\mathbb{N}}$ and $(F^{(n)})_{n\in\mathbb{N}}$ are bounded, we deduce that the TS and ST terms are estimated by:

$$M_{T'}\int_0^t(\mathcal{F}_\infty^{m-1,n-1}(s)+f_\infty^{m,n}(s))ds$$

- Following the same procedures as for (3.21), we get:

$$\|\partial E_{SS}(t,.)-\partial E_{SS}(t,.)\|_\infty \leq M_{T'}\int_0^t(\mathcal{F}_1^{m-1,n-1}(s)+f_\infty^{m,n}(s))ds$$

Combining now all these estimates we get:

$$|E^{(m)}(t,.)-E^{(n)}(t,.)|_1 \leq M_{T'}\int_0^t(\mathcal{F}_1^{m-1,n-1}(s)+f_1^{m,n}(s)+f_\infty^{m,n}(s)+\mathcal{F}_\infty^{m-1,n-1}(s))ds$$

Furthermore, since all the iterates have the same initial data,

$$E^{(m)}(t,.)-E^{(n)}(t,.)=\int_0^t\partial_t(E^{(m)}(s,.)-E^{(n)}(s,.))ds$$

and

$$f^{(m)}(t,.)-f^{(n)}(t,.)=\int_0^t\partial_t(f^{(m)}(s,.)-f^{(n)}(s,.))ds$$

Hence, for all $t\in[0,T']$,

$$f_\infty^{m,n}(t)\leq f_1^{m,n}(t)$$

and

$$\mathcal{F}_\infty^{m-1,n-1}(t)\leq\mathcal{F}_1^{m-1,n-1}(t)$$

Thus,

$$|E^{(m)}(t,.)-E^{(n)}(t,.)|_1 \leq M_{T'}\int_0^t(\mathcal{F}_1^{m-1,n-1}(s)+f_1^{m,n}(s))ds$$

Following exactly the same steps for the magnetic field B we get :

$$\mathcal{F}_1^{m,n}(t) \leq M_{T'} \int_0^t (\mathcal{F}_1^{m-1,n-1}(s) + f_1^{m,n}(s))ds \tag{3.31}$$

To estimate the $f^{(n)}$ derivatives we use the characteristics system. We recall that the characteristics system (X_n, V_n) taking the value (x,v) when $s = u$ of $f^{(n)}$ is given by Newton's equations [50]:

$$\dot{X}_n(s,u,x,v) = \hat{V}_n(s,u,x,v) \quad , \quad \dot{V}_n(s) = F^{(n-1)}(s, X_n(s,u,x,v))$$

with $F^{(n-1)} = E^{(n-1)} + \hat{v}_n \times B^{(n-1)}$.
Let $(u,x,v) \in [0,T'] \times \mathbb{R}^3 \times \mathbb{R}^3$. The first characteristics equation gives :

$$|(X_m - X_n)(t)| = |\int_u^t (\hat{V}_m - \hat{V}_n)(s)ds| \leq \int_0^t |(V_m - V_n)(s)|ds$$

The second characteristics equation gives for $s \in [0,T']$:

$$\begin{aligned} |(V_m - V_n)(t)| &\leq \int_0^t |F^{(m-1)}(s, X_m(s)) - F^{(n-1)}(s, X_n(s))|ds \\ &\leq \int_0^t |F^{(m-1)}(s, X_m) - F^{(m-1)}(s, X_n)|ds \\ &\quad + \int_0^t |F^{(n-1)}(s, X_n(s)) - F^{(m-1)}(s, X_n(s))|ds \end{aligned}$$

Since the $\mathcal{C}^1$ norm of $F^{(n)}$ is uniformly bounded, we get:

$$\begin{aligned} |(V_m - V_n)(t)| &\leq M_{T'} \int_0^t |X_m(s) - X_n(s)|ds + \int_0^t |F^{(n-1)}(s, X_n(s)) - F^{(m-1)}(s, X_n(s))|ds \\ &\leq M_{T'} \int_0^t |X_m(s) - X_n(s)|ds + \int_0^t \mathcal{F}_\infty^{m-1,n-1}(s)ds \end{aligned}$$

Hence,

$$\begin{aligned} |X_n(t) - X_m(t)| + |V_n(t) - V_m(t)| &\leq M_{T'} \int_0^t (|X_m(s) - X_n(s)| + |(V_m - V_n)(s)|)ds \\ &\quad + \int_0^t \mathcal{F}_\infty^{m-1,n-1}(s)ds \end{aligned}$$

Using the estimates of $\mathcal{F}_\infty^{m,n}$ given above, we get:

$$\begin{aligned} |X_n(t) - X_{m+n}(t)| + |V_n(t) - V_{m+n}(t)| &\leq M_{T'} \int_0^t (|X_{m+n}(s) - X_n(s)| + |(V_{m+n} - V_n)(s)|)ds \\ &\quad + M_{T'} \frac{(M_{T'}T')^n}{n!} \end{aligned} \tag{3.32}$$

using **Gronwall's lemma**,we get :

$$|X_n(t) - X_{m+n}(t)| + |V_n(t) - V_{m+n}(t)| \leq M_{T'} \frac{(M_{T'}T')^n}{n!} e^{M_{T'}T'} \tag{3.33}$$

[50]The characteristics exists according to theorem 1.1, since $F^{(n)}$ is bounded.

Hence $(X_n)_{n\in\mathbb{N}}$ and $(V_n)_{n\in\mathbb{N}}$ converge uniformly on $[0,T']$. Notice that the convergence is also uniform in the parameters (u,x,v). Now we use the characteristics to estimate the derivatives of $f^{(n)}$. Since $f^{(n)}$ is $\mathcal{C}^2$ we can notice that:

$$(\partial_t + \hat{v}.\nabla_x + F^{(n-1)}.\nabla_v)(\partial f^{(n)}) = -\nabla_v f^{(n)} \partial F^{(n-1)}$$

Hence,

$$\partial f^{(n)}(t,x,v) = \partial f^{(n)}(0,X_n(0),V_n(0)) - \int_0^t [\nabla_v f^{(n)}(s,X_n(s),V_n(s)).\partial F^{(n-1)}(s,X_n(s),V_n(s))]ds$$

Thus,

$$|\partial f^{(n)}(t,x,v) - \partial f^{(m)}(t,x,v)| \leq |\partial f_0(X_n(0),V_n(0)) - \partial f_0(X_m(0),V_m(0))| +$$
$$\underbrace{\int_0^t |\nabla_v f^{(n)}(s,X_n,V_n).\partial F^{(n-1)}(s,X_n,V_n) - \nabla_v f^{(m)}(s,X_m,V_m).\partial F^{(m-1)}(s,X_m,V_m)|ds}_{=u_{m,n}}$$

By the hypothesis on f_0 and using the previous estimates on the characteristics, it is easy to see that the first term satisfies :

$$|\partial f_0(X_n(0),V_n(0)) - \partial f_0(X_{m+n}(0),V_{m+n}(0))| \leq M_{T'}\frac{(M_{T'}T')^n}{n!}$$

For the seconde term, we split the integral :

$$\begin{aligned} u_{m+n,n} \leq\ & \underbrace{\int_0^t |\partial F^{(n-1)}(s,X_n,V_n)(\nabla_v f^{(n)}(s,X_n,V_n) - \nabla_v f^{(n)}(s,X_{m+n},V_{m+n}))|ds}_{:=I_1} \\ & + \underbrace{\int_0^t |(\partial F^{(n-1)}(s,X_n,V_n) - \partial F^{(m+n-1)}(s,X_{m+n},V_{m+n})).\nabla_v f^{(n)}(s,X_{m+n},V_{m+n})|ds}_{:=I_2} \\ & + \underbrace{\int_0^t |\partial F^{(m+n-1)}(s,X_{m+n},V_{m+n})(\nabla_v f^{(n)}(s,X_{m+n},V_{m+n}) - \nabla_v f^{(m+n)}(s,X_{m+n},V_{m+n}))|ds}_{:=I_3} \end{aligned}$$

Using the $\mathcal{C}^1$ uniform bounds and (3.33), the first term tends to 0 when n tends to infinity. The second term and the last term are respectively dominated by $\int_0^t \mathcal{F}_1^{m+n-1,n-1}(s)ds$ and $\int_0^t f_1^{m+n,n}(s)ds$. Combining all these estimates, we get :

$$f_1^{m+n,n} \leq \eta_{m+n,n} + M_{T'}\int_0^t (\mathcal{F}_1^{m+n-1,n-1}(s) + f_1^{m+n,n}(s))ds \tag{3.34}$$

with $\lim_n \eta_{m+n,n} = 0$ uniformly on $[0,T']$.Let $G_{m,n}$ the function defined by:

$$G_{m,n}(t) = \int_0^t f_1^{m,n}(s)ds$$

this function satisfies:

$$\dot{G}_{m+n,n}(t) - M_{T'}G_{m,n}(t) \leq \eta_{m+n,n} + M_{T'}\int_0^t \mathcal{F}_1^{m+n-1,n-1}(s)ds$$

thus,

$$\frac{d}{dt}[G_{m+n,n}(t).e^{-M_{T'}t}+\frac{\eta_{m+n,n}}{M_{T'}}e^{-M_{T'}t}]\leq M_{T'}\int_0^t \mathcal{F}_1^{m+n-1,n-1}(s)dse^{-M_{T'}t}$$

hence,

$$G_{m+n,n}(t)\leq\frac{\eta_{m+n,n}}{M_{T'}}(e^{M_{T'}T'}-1)+M_{T'}\int_0^t\int_0^u \mathcal{F}_1^{m+n-1,n-1}(s)dse^{-M_{T'}u}du.e^{M_{T'}t}$$

Using (3.34), we get:

$$f_1^{m+n,n}\leq\eta'_{m+n,n}+M_{T'}\int_0^t \mathcal{F}_1^{m+n-1,n-1}(s)ds \tag{3.35}$$

with $\lim_n \eta'_{m+n,n}=0$ uniformly on $[0,T']$. Combining, (3.35) and (3.31) we get:

$$\mathcal{F}_1^{m+n,n}(t)\leq M_{T'}\int_0^t \mathcal{F}_1^{m+n-1,n-1}(s)ds+M_{T'}\eta'_{m+n,n}$$

by iterations we finally get:

$$\mathcal{F}_1^{m+n,n}(t)\leq A\frac{(M_{T'}T')^n}{n!}+M_{T'}\eta'_{m+n,n}e^{M_{T'}T'}$$

(f^n) , (E^n) and (B^n) are Cauchy sequences with the $\mathcal{C}^1$ and the uniform norms. Hence, (f^n) , (E^n) and (B^n) converge [51] with this norms and we note f, E and B the respective limits with the $\mathcal{C}^1$ and the uniform norms [52]. Taking the limit in (3.25) and (3.26), (E,B,f) satisfies the Vlasov equation coupled with the wave equations with the initial conditions (E_0,E_1,B_0,B_1,f_0).

This proves the existence for smooth initial data (E_0,E_1,B_0,B_1,f_0). We recall, that we made some assumptions on the regularity of initial data in this proof. In the theorem we assume weaker regularity on the initial data. We approximate these functions by smoother functions, and use the estimations to proof the uniform convergence as we did here. Passing to the limit, that proves existence for theorem 3.2.

⋆ ⋆ ⋆ ⋆

[51] The space of $\mathcal{C}^1$ function on $[0,T']\times\mathbb{R}^6$, such that all derivatives are bounded, is a complete space with the norm $\|.\|_1$. It's a consequence of proposition B.4 (Appendix).

[52] Since the convergence is uniform on $[0,T']$ the limits with the two norms are identical.

A Appendix : Notation

A.1 Geometric notation

- For $d \in \mathbb{N}^\star$, $\mathbb{R}^d$ is d-dimensional real Euclidean space and $|.|$ is the Euclidean norm associated to the the usual scalar product:

$$x.y = \sum_{i=1}^{d} x_i y_i$$

- In $\mathbb{R}^3$, $x \times y$ denotes the vectorial product :

$$x \times y := \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}$$

- We define also on $\mathbb{R}^d$ the norme:

$$\|.\|_\infty := \max_{1 \leq i \leq d} |x_i|$$

- For $x \in \mathbb{R}^d$ and $r > 0$,

$$\begin{aligned} B(x,r) &:= \{y \in \mathbb{R}^d : |x-y| < r\} \\ \overline{B(x,r)} &:= \{y \in \mathbb{R}^d : |x-y| \leq r\} \\ \partial B(x,r) &:= \{y \in \mathbb{R}^d : |x-y| = r\} \end{aligned}$$

- $e_i = (0,..0,1,0,..,0)$ is the i^{th} standard coordinate vector.
- For $X \subset \mathbb{R}^d$, $\overline{X}$ and $\overset{\circ}{X}$ are respectively the topological closer, and interior of X in $\mathbb{R}^d$.

A.2 Notation for functions

- χ_A is the indicator function of the set A.

$$\chi_A : x \longrightarrow \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

- If w is a function from A in B, and $C \subset A$, then $w_{/C}$ is the restriction of w on C.
- If w is a measurable function :

$$\int_A w := \int_A w(x)dx$$

- If w is a function from $\mathbb{R}^m$ to $\mathbb{R}^n$,we not $supp(w)$ the support of w,

$$supp(w) := \overline{\{x \in \mathbb{R}^m \ / \ w(x) \neq 0\}}$$

  we say that w is compactly supported if $supp(w)$ is a compact set.

- Let Ψ a function defined on a set $A_1 \times A_2 \times .. \times A_p$. For $(x_1, .., x_{k-1}, x_{k+1}, .., x_p) \in A_1 \times .. \times A_{k-1} \times A_{k+1} \times .. \times A_p$, $\Psi(x_1, .., x_{k-1}, ., x_{k+1}, .., x_p)$ is the function defined on A_k, given by :

$$s \in A_k \overset{\Psi_k}{\longmapsto} \Psi(x_1, .., x_{k-1}, s, x_{k+1}, .., x_p)$$

  Furthermore, if B_k is a subset of A_k, $\Psi(x_1, .., x_{k-1}, B_k, x_{k+1}, .., x_p)$ is the image of B_k by the function Ψ_k.

- **Function spaces :** Ω denotes an open set of $\mathbb{R}^d$ and $p \in \mathbb{N}^\star$.
  - $\mathcal{F}(A, B)$ is the space of functions from A to B.
  - $\mathcal{C}(\Omega) = \{\varphi : \Omega \to \mathbb{R}^p \ / \ \varphi \text{ continuous } \}$
  - $\mathcal{C}^k(\Omega) = \{\varphi : \Omega \to \mathbb{R}^p \ / \ \varphi \text{ is k-times continuously differentiable } \}$
  - $\mathcal{C}^\infty(\Omega) = \{\varphi : \Omega \to \mathbb{R}^p \ / \ \varphi \text{ is infinitely differentiable } \}$
  - $\mathcal{C}_c(\Omega)$, $\mathcal{C}_c^k(\Omega)$ and $\mathcal{C}_c^\infty(\Omega)$, denote respectively the set of functions in $\mathcal{C}(\Omega)$, $\mathcal{C}^k(\Omega)$ and $\mathcal{C}^\infty(\Omega)$ with compact support.
  - $L^p(\Omega) = \{\varphi : \Omega \to \mathbb{R}^p \ / \ \varphi \text{ is Lebesgue measurable, and } \|\varphi\|_{L^p(\Omega)} < \infty\}$, where
$$\|\varphi\|_{L^p(\Omega)} := \Big(\int_\Omega |\varphi(x)|^p dx\Big)^{\frac{1}{p}}$$
  - $L^p(\Omega) = \{\varphi : \Omega \to \mathbb{R}^p \ / \ \varphi \text{ is Lebesgue measurable, and } \|\varphi\|_{L^p(\Omega)} < \infty\}$, where
$$\|\varphi\|_{L^p(\Omega)} := \Big(\int_\Omega |\varphi(x)|^p dx\Big)^{\frac{1}{p}}$$
  - $L^\infty(\Omega) = \{\varphi : \Omega \to \mathbb{R}^p \ / \ \varphi \text{ is Lebesgue measurable, and } \|\varphi\|_{L^\infty(\Omega)} < \infty\}$, where
$$\|\varphi\|_{L^\infty(\Omega)} := \inf\{C \in [0, \infty] : |\varphi| \leq C, \text{ almost everywhere } \}$$
  - $L^p_{loc}(\Omega) = \{\varphi : \Omega \to \mathbb{R}^p \ / \ \varphi \in L^p(K), \text{ for every compact set } K \subset \Omega\}$
  - $L^\infty_{loc}(\Omega) = \{\varphi : \Omega \to \mathbb{R}^p \ / \ \varphi \in L^\infty(K), \text{ for every compact set } K \subset \Omega\}$
  - (Sobolev spaces) $W^{k,p}(\Omega)$ is the space of functions of $L^p(\Omega)$ such that all the k-derivatives in the weak sens belong to $L^p(\Omega)$.
  - $H^k(\Omega) := W^{k,2}(\Omega)$

- If A is a measurable set we note $\mu(A)$ the volume of A with Lebesgue measure:

$$\mu(A) := \int \chi_A(x) dx$$

- For φ and ψ two measurable functions, $\varphi * \psi$ denote the convolution product (if it exists) given by:

$$\varphi * \psi(x) := \int \varphi(x-t)\psi(t)dt$$

- If φ is a bounded function on $X \subset \mathbb{R}^d$ we define:

$$\|\varphi\|_\infty := \sup_{x \in X} |\varphi(x)|$$

To specify the domain, we can also note :

$$\|\varphi\|_{\infty,X} := \sup_{x \in X} |\varphi(x)|$$

A.3 Notation for differential operators

- If g is a function defined on a interval of $\mathbb{R}$, we note the derivative

$$\dot{g} \quad \text{or} \quad \frac{dg}{dt}$$

- If Ψ is a function of time $t \in \mathbb{R}$ position $x \in \mathbb{R}^d$ and velocity $v \in \mathbb{R}^d$. We note:

$$\begin{aligned}
\partial_t \Psi(s,.,.) &:= \lim_{h \to 0} \frac{\Psi(s+h,.,.) - \Psi(s,.,.)}{h} \\
\partial_{x_i} \Psi(.,u,.) &:= \lim_{h \to 0} \frac{\Psi(.,u+he_i,.) - \Psi(.,u,.)}{h} \\
\partial_{v_i} \Psi(.,.,w) &:= \lim_{h \to 0} \frac{\Psi(.,.,w+he_i) - \Psi(.,.,w)}{h}
\end{aligned}$$

- A specific notation for the characteristics $X(s,t,z)$, because of the two time variables.

$$\begin{aligned}
\partial_s X(a,.,.) &:= \lim_{h \to 0} \frac{X(a+h,.,.) - X(a,.,.)}{h} \\
\partial_t X(.,a,.) &:= \lim_{h \to 0} \frac{X(.,a+h,.) - X(.,a,.)}{h} \\
\partial_{z_i} X(.,.,u) &:= \lim_{h \to 0} \frac{X(.,.,u+he_i) - X(.,.,u)}{h}
\end{aligned}$$

- If φ is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$ such that:

$$\varphi(x) = (\varphi_1(x), \varphi_2(x), .., \varphi_m(x))$$

then,

$$\partial_x \varphi := \begin{pmatrix} \partial_{x_1}\varphi_1 & \partial_{x_2}\varphi_1 & \cdots & \partial_{x_n}\varphi_1 \\ \partial_{x_1}\varphi_2 & \partial_{x_2}\varphi_2 & \cdots & \partial_{x_n}\varphi_2 \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{x_1}\varphi_m & \partial_{x_2}\varphi_m & \cdots & \partial_{x_n}\varphi_m \end{pmatrix}$$

– If m=1,

$$\nabla_x \varphi := \begin{pmatrix} \partial_{x_1}\varphi \\ \vdots \\ \partial_{x_n}\varphi \end{pmatrix}$$

– If $m = n$,

$$\nabla_x \varphi := \sum_{i=1}^{n} \partial_{x_i}\varphi_i$$

– If $m = n = 3$,

$$\nabla_x \times \varphi := \begin{pmatrix} \partial_{x_2}\varphi_3 - \partial_{x_3}\varphi_2 \\ \partial_{x_3}\varphi_1 - \partial_{x_1}\varphi_3 \\ \partial_{x_1}\varphi_2 - \partial_{x_2}\varphi_1 \end{pmatrix}$$

– If φ is enough regular, for $i \in \{1, .., n\}$,

$$\partial^2_{x_i}\varphi := \partial_{x_i}\partial_{x_i}\varphi$$

–

$$\Delta\varphi := \begin{pmatrix} \sum_{i=1}^{n} \partial^2_{x_i}\varphi_1 \\ \vdots \\ \sum_{i=1}^{n} \partial^2_{x_i}\varphi_m \end{pmatrix}$$

A.4 Notation for Matrices

- The Kronecker indice is defined by :

$$\delta_{i,j} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

- $\mathcal{M}_d(\mathbb{R})$ is the set of real $d \times d$ matrices.
- For a matrice A in $\mathcal{M}_d(\mathbb{R})$, $det(A)$ is the determinant of A.
- If $A \in \mathcal{M}_n(\mathbb{R})$ such that:

$$A := \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix}$$

then,

$$Tr(A) := \sum_{i=1}^{n} a_{i,i}$$

and ${}^tA$ the transpose of the matrice A.

- We note $I_n \in \mathcal{M}(\mathbb{R})$ the matrice defined:

$$I_n := \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

B Appendix : Analysis therorems toolbox

B.1 Ordinary differential equations theorems

<u>Theorem</u> **B.1 (Cauchy-Lipschitz)** *Let I be an interval of $\mathbb{R}$ and $\mathcal{O}$ an open subset of a normed and finite dimension space E. If $f \in \mathcal{C}^1(I \times \mathcal{O})$ then, for all $(x_0, y_0) \in \times\mathcal{O}$ the Cauchy problem:*

$$\begin{cases} \frac{dy}{ds}(s) = f(s, y(s)) \\ y(x_0) = y_0 \end{cases}$$

has a unique maximal solution (y, J) such that $J \subset I$ is the maximal interval of definition of y.

Element of proof **B.1**

- *If a solution y exists, we have in a neighborhood of x_0: $y(s) = y_0 + \int_{x_0}^{s} f(u, y(u))du$. We prove that the application:*

$$\phi \longmapsto (s \longmapsto y_0 + \int_{x_0}^{s} f(u, \phi(u))du)$$

is Lipschitz in a neighborhood of (x_0, y_0). Then we apply the fixed point theorem.

- *The application of the fixed point theorem gives a unique solution which is only local. To get the maximal solution we piece the local solutions together.*

⋆ ⋆ ⋆ ⋆

Proposition B.1 (Gronwall's lemma) *Let γ be a continuous function from an interval I of $\mathbb{R}$ in $\mathbb{R}^+$. Let $x_0 \in I$ such that,*

$$\exists (A, B) \in \mathbb{R}^+ \times \mathbb{R}^+, \quad \forall s \in I, \quad \gamma(s) \leq A + B|\int_{x_0}^{s} \gamma(t)dt|$$

Then $\forall s \in I$ we have:

$$\gamma(s) \leq Ae^{B|s-x_0|}$$

Proof B.1 *If $B = 0$ we have directly the result. Let's assume that $B > 0$. We have for all $s \in I \cap [x_0, +\infty[$;*

$$\gamma(s)e^{-B(s-x_0)} - Ae^{-B(s-x_0)} - B\int_{x_0}^{s} \gamma(t)dte^{-B(s-x_0)} \leq 0$$

then:

$$\frac{d}{ds}(\int_{x_0}^{s}\gamma(t)dte^{-B(s-x_0)}+\frac{A}{B}e^{-B(s-x_0)})\leq 0$$

Which means that the function $s\to\int_{x_0}^{s}\gamma(t)dte^{-B(s-x_0)}+\frac{A}{B}e^{-B(s-x_0)}$ *is decreasing on* $I\cap[x_0,+\infty[$*. Thus:*

$$\int_{x_0}^{s}\gamma(t)dte^{-B(s-x_0)}+\frac{A}{B}e^{-B(s-x_0)}\leq\frac{A}{B}$$

Then,

$$\gamma(s)e^{-B(s-x_0)}\leq B\int_{x_0}^{s}\gamma(t)dte^{-B(s-x_0)}+Ae^{-B(s-x_0)}\leq A$$

For $s\in I\cap]-\infty,x_0]$ *the result still true by the same reasoning.*

⋆ ⋆ ⋆ ⋆

Proposition B.2 (Gronwall's Lemma / Generalization) *Let* γ *,*ζ*,* φ *continuous functions from an interval* I *of* $\mathbb{R}$ *in* $\mathbb{R}^+$*. Let* $x_0\in I$ *such that,*

$$\forall s\in I,\quad \gamma(s)\leq\varphi(s)+|\int_{x_0}^{s}\gamma(t)\zeta(t)dt|$$

Then $\forall s\in I$ *we have:*

$$\gamma(s)\leq\varphi(s)+|\int_{x_0}^{s}\varphi(u)\zeta(u)\exp(|\int_{u}^{s}\zeta(v))dv|)du|$$

Proof B.2

- *(*$I\cap[x_0,+\infty[$*).*

$$F: s\in I\cap[x_0,+\infty[\to\int_{x_0}^{s}\varphi(s)\gamma(s)ds$$

If we make a multiplication by ζ *in the inequality, we have:*

$$\forall s\in I\cap[x_0,+\infty[,\ F'(s)\leq\varphi(s)\zeta(s)+\zeta(s)F(s)$$

Thus , if :

$$G: s\in I\cap[x_0,+\infty[\to F(s)\exp(-\int_{x_0}^{s}\zeta(u)du)$$

We have:

$$\forall s\in I\cap[x_0,+\infty[,\ G'(s)\leq\varphi(s)\zeta(s)\exp(-\int_{x_0}^{s}\zeta(u)du)$$

Integrating this inequality we get:

$$\forall t\in I\cap[x_0,+\infty[,\ G(t)\leq\int_{x_0}^{t}\varphi(s))\zeta(s)\exp(-\int_{x_0}^{s}\zeta(u)du)ds$$

Using the assumption, which can be written:

$$forallt\in I\cap[x_0,+\infty[,\ \gamma(t)\leq\varphi(t)+G(t)\exp(-\int_{x_0}^{t}\zeta(u)du)ds$$

We finally get the desired inequality.

- *For $t \in I\cap]-\infty, x_0]$ the result still true by the same reasoning.*

⋆ ⋆ ⋆ ⋆

THEOREM **B.2 (Derivation relatively to the initial condition)** *Let ϕ be a function from $[0,T]\times\mathbb{R}^d$ in $\mathbb{R}^d$. We assume that : $\phi \in \mathcal{C}([0,T]\times\mathbb{R}^d)$ and $\partial_x\phi$ exists and belong to $\mathcal{C}([0,T]\times\mathbb{R}^d)$. We assume that the unique solution $X(.,t,x)$ for:*

$$\begin{cases} \frac{dy}{ds}(s) = \phi(s,y) \\ y(t) = x \end{cases}$$

is defined on $[0,T]\times[0,T]\times\mathbb{R}^d$.
The function X:

$$(s,t,x) \to X(s,t,x)$$

is called the **the characteristic flow**.
Under these assumptions, $\partial_x X$ exists and belongs to $\mathcal{C}([0,T]\times[0,T]\times\mathbb{R}^d)$, furthermore $\partial_x X$ is the unique solution in $\mathcal{C}^1([0,T],\mathcal{M}_d(\mathbb{R}))$ for the linear differential equation:

$$\begin{cases} \frac{dM}{ds}(s) = \partial_x\phi(s,X(s,t,x)).M \\ M(t) = I_d \end{cases} \tag{B.1}$$

ELEMENT OF PROOF **B.2**

- *Let $Y(.,t,x)$ be the unique solution for the linear differential equation (B.1). Since the equation is linear this solution is the defined for all the times $s \in [0,T]$. The idea of the proof is to show that $Y(.,t,x)$ is in deed $\partial_x X(.,t,x)$.*

- *Let $(s,t,x) \in [0,T]^2\times\mathbb{R}^3$ and $\varepsilon > 0$, we need to prove that for $|h|$ enough small we have:*

$$|X(s,t,x+h) - X(s,t,x) - Y(s,t,x).h| \leq \varepsilon|h|$$

Let

$$Z_h :\ s \to X(s,t,x+h) - X(s,t,x) - Y(s,t,x).h$$

Let $r > 0$, by continuity of X: $\exists\eta > 0$, such that if $|h| \leq \eta$ we have

$$|X(s,t,x+h) - X(s,t,x)| \leq r$$

Furthermore, we have:

$$\begin{aligned}
\dot{Z}_h(s) &= \phi(s,t,x+h) - \phi(s,t,x) - \partial_x\phi(s,X(s,t,x))Y(s,t,x).h \\
&= [\phi(s,X(s,t,x+h)) - \phi(s,X(s,t,x)+Y(s,t,x).h)] \\
&\quad +[-\phi(s,X(s,t,x) + \phi(s,t,X(s,t,x)+Y(s,t,x).h) - \partial_x\phi(s,X(s,t,x))Y(s,t,x).h] \\
&= \underbrace{[\phi(s,X(s,t,x+h)) - \phi(s,X(s,t,x)+Y(s,t,x).h)]}_{A} \\
&\quad + \underbrace{\int_0^1 [\partial_x\phi(s,X(s,t,x)+uY(s,t,x).h) - \partial_x\phi(s,X(s,t,x))]Y(s,t,x).hdu}_{B}
\end{aligned}$$

Let

$$K = \bigcup_{(s,t)\in[0,T]^2} \overline{B(X(s,t,x),r)}$$

and

$$\gamma = \sup\{\frac{|\phi(s,\alpha)-\phi(s,\beta)|}{|\alpha-\beta|}/s\in[0,T],(\alpha,\beta)\in K,\alpha\neq\beta\}$$

K is a compact set. In fact, K is the image of $[0,T]^2\times\overline{B(0,r)}$ with the continuous application : $(s,t,v)\to X(s,t,x)+v$. If $|h|\leq\frac{r}{M+1}$, with $M=\max_{(s,t)\in[0,T]^2}|Y(s,t,x)|$, we have $X(s,t,x)+Y(s,t,x).h\in K$.
Then, for $|h|\leq\min(\eta,r/M+1)$ we have :

$$|A|\leq\gamma|Z_h(s)|$$

Furthermore, if

$$\omega(\rho):=\sup\{|\partial_x\phi(s,\alpha)-\partial_x\phi(s,\beta)|/(\alpha,\beta)\in K,|\alpha-\beta|\leq\rho\}$$

We have :

$$|B|\leq\omega(M|h|)M|h|$$

And finally:

$$|\dot{Z}_h(s)|\leq\gamma|Z_h(s)|+\omega(M|h|)M|h|$$

Using Growall's Lemma :

$$|Z_h(s)|\leq\omega(M|h|)M|h|e^{\gamma T}$$

By uniforme continuity of $\partial_x\phi(s,.)$ on K. $\exists\eta'>0$ such that if $|h|\leq\eta'$ we have: $\omega(M|h|)\leq\varepsilon$. And then:

$$|Z_h(s)|\leq\varepsilon M|h|e^{\gamma T}$$

$\star \quad \star \quad \star \quad \star$

B.2 Inequalities toolbox

Proposition B.3 (Young's inequality) *Let p,q, r in $[1,\infty]$ satisfying :*

$$\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}$$

If $f\in L^p(\mathbb{R}^d)$ and $g\in L^q(\mathbb{R}^d)$, $f*g<\infty$ almost every where, and:

$$\|f*g\|_r\leq\|f\|_p\|g\|_q$$

Proof B.3 *Let α, β in $[0,1]$ and p',q' such that :*

$$\frac{1}{p'} + \frac{1}{q'} + \frac{1}{r} = 1$$

. Let $x \in \mathbb{R}^d$

$$\begin{aligned}
|f * g(x)| &\leq \int |f(y)g(x-y)|dy \\
&\leq \int |f(y)|^\alpha |g(x-y)|^\beta |f(y)|^{1-\alpha} |g(x-y)^{1-\beta}|dy \\
&\leq (\int |f(y)|^{p'\alpha} dy)^{1/p'} (\int |g(x-y)^{q'\beta}|dy)^{1/q'} (\int |f(y)|^{r(1-\alpha)} |g(x-y)^{r(1-\beta)}|dy)^{1/r} \\
&\leq \|f\|^\alpha_{p'\alpha} \|g\|^\beta_{q'\beta} (\int |f(y)|^{r(1-\alpha)} |g(x-y)^{r(1-\beta)}|dy)^{1/r}
\end{aligned}$$

Thus,

$$\begin{aligned}
\|f * g\|_r &\leq \|f\|^\alpha_{p'\alpha} \|g\|^\beta_{q'\beta} (\int\int |f(y)|^{r(1-\alpha)} |g(x-y)^{r(1-\beta)}|dydx)^{1/r} \\
&\leq \|f\|^\alpha_{p'\alpha} \|g\|^\beta_{q'\beta} (\int |f(y)|^{r(1-\alpha)} \int |g(\underbrace{x-y}_{=u})^{r(1-\beta)}|dxdy)^{1/r} \\
&\leq \|f\|^\alpha_{p'\alpha} \|g\|^\beta_{q'\beta} \|f\|^{1-\alpha}_{r(1-\alpha)} \|g\|^{1-\beta}_{r(1-\beta)}
\end{aligned}$$

We choose :

$$\alpha = 1 - \frac{p}{r} \quad p' = \frac{p}{1-\frac{p}{r}} \quad \beta = 1 - \frac{q}{r} \quad q' = \frac{q}{1-\frac{q}{r}}$$

α, β, p' and q' are all positive numbers, and $(\alpha, \beta) \in [0,1]^2$ since $p \leq r$ and $q \leq r$ [53], and we can verify easily that with this definition:

$$\frac{1}{p'} + \frac{1}{q'} + \frac{1}{r} = 1 \text{ and } p'\alpha = r(1-\alpha) = p \text{ and } q'\beta = r(1-\beta) = q$$

replacing p', q', α and β by there new values we finally get the claimed result.

⋆ ⋆ ⋆ ⋆

<u>Theorem</u> **B.3 (Weak Young's inequality)** *Let $p, q, r \in]1, \infty[$ with $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. If $f \in L^p(\mathbb{R}^d)$ and $g \in L^q_w(\mathbb{R}^d)$, then $f * g \in L^r(\mathbb{R}^d)$ and :*

$$\|f * g\|_r \leq C \|f\|_p \|g\|_{q,w}$$

with C a constant which depends only on p, q and d.
By definition: $g \in L^q_w(\mathbb{R}^d)$ if and only if

$$\|g\|_{q,w} = \sup_{\tau > 0} \tau.[vol(\{x \in \mathbb{R}^d / |g(x)| \geq \tau\}]^{1/q} < \infty$$

This expression does not define a norm.

[53] $\max(p,q) \leq p + q = \frac{r}{r+1} \leq r$

B.3 Some completeness results

Let E a vectorial normed space.

Proposition B.4 (Completness with $\|.\|_\infty$) *Let $\Omega \subset E$, and*

$$\mathcal{B}(A,\mathbb{R}) := \{\varphi : A \to \mathbb{R}/\varphi \textit{ is continuous and bounded on } A\ \}$$

The following application define a norm on $\mathcal{B}(A,\mathbb{R})$:

$$\forall g \in \mathcal{B}(A,\mathbb{R}), \quad \|g\|_\infty := \sup_{x\in A} |g(x)|$$

and $(\mathcal{B}(A,\mathbb{R}), \|.\|_\infty)$ is a Banach space (complete normed space).

Element of proof **B.3** *Let (g_n) a Cauchy sequence in $(\mathcal{B}(A,\mathbb{R}), \|.\|_\infty)$ and $p \in \mathbb{N}^\star$. Then, we have that :*

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall m \geq N, \forall x \in A, \quad |f_m(x) - f_{m+p}(x)| \leq \varepsilon$$

For $x \in A$, $(f_n(x))$ is a Cauchy sequence in $\mathbb{R}$. Hence, $(f_n(x))$ converges to a limite noted $f(x)$. Let,

$$f : x \in A \longrightarrow f(x)$$

- *We have that :*

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall m \geq N, \forall x \in A, \quad |f_m(x) - f_{m+p}(x)| \leq \varepsilon$$

  taking the limit when p tens to infinity we get :

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall m \geq N, \forall x \in A, \quad |f_m(x) - f(x)| \leq \varepsilon$$

  hence,

$$\lim_n \|f_n - f\|_\infty = 0$$

- *Since (f_n) converges uniformly on A, f is continous on A*

- *For $\varepsilon = 1$, it exists $N \in \mathbb{N}$ such that :*

$$\forall x \in A, \quad |f(x)| \leq 1 + \|f_N\|_\infty$$

  Hence, $f \in \mathcal{B}(A,\mathbb{R})$.

B.4 Differential calculus toolbox

Let E and E' two vectoriel normed spaces.

Theorem **B.4 (Implicit function theorem)** *Let f a $\mathcal{C}^1$ function defined on an open subset Ω of $E \times E'$ that takes values in E'. Let $(x_0, x_0') \in E \times E'$ such that:*

$$f(x_0, x_0') = 0 \quad \textit{and} \quad df_{(x_0,x_0')/\{0\}\times E'} \textit{ is an isomorphism}$$

then, it existes V a neighborhood of x_0 and a unique function $\varphi \in \mathcal{C}^1(V, E')$ such that:

$$\varphi(x_0) = x_0' \quad \textit{and} \quad \forall x \in V, \quad f(x, \varphi(x)) = 0$$

THEOREM **B.5 (Mean value theorem)** *Let f a $\mathcal{C}^1$ function defined on an open subset Ω of $E \times E'$ that takes values in E', then :*

$$\forall (x,y) \in \Omega^2, \quad [x,y] \subset \Omega \Longrightarrow \|f(x)-f(y)\|_{E'} \leq \|x-y\|_E.\Big(\sup_{z\in[x,y]}\sup_{\|x\|_E\leq 1}\|\partial f(z).x\|_{E'}\Big)$$

B.5 Integration toolbox

Let E and F two topological spaces and μ,ν are respectively, the Lebesgue's measure defined on E and F. We note $\mu \otimes \nu$ the product measure on $E \times F$.

THEOREM **B.6 (Dominated convergence theorem)** *Let (f_n) be a function sequence such that $\forall n \in \mathbb{N}$, $\int |f_n| d\mu < \infty$. We assume that :*

1. *It exists a measurable function f such that :*

$$f_n(x) \longrightarrow f(x) \quad \mu \text{ almost everywhere}$$

2. *(Domination) It exists a function $g : E \longrightarrow \mathbb{R}^+$, such that $\int g d\mu < \infty$ and for all $n \in \mathbb{N}$,*

$$|f_n| \leq g \quad \mu \text{ almost everywhere}$$

Then, $\int |f| d\mu < \infty$ and,

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu$$

and

$$\lim_{n\to\infty} \int |f_n - f| d\mu = 0$$

THEOREM **B.7 (Theorem of Lebesgue)** *Let $u_0 \in E$, $f : U \times E \to \mathbb{R}$ with (U,d) a metric space. Let's assume that :*

1. *for all $u \in U$, $x \to f(u,x)$ is measurable;*
2. *$u \to f(u,x)$ is continuous in u_0, μ almost everywhere.*
3. *(Domination) It exists an integrable function on X such that, for all $u \in U$,*

$$|f(u,x)| \leq g(x) \quad \mu(dx) \text{ almost everywhere}$$

then, $u \to \int f(u,x)\mu(dx)$ is defined for any $u \in U$ and continuous in u_0.

THEOREM **B.8 (Integral derivation theorem)** *Let I an interval of $\mathbb{R}$ and $u_0 \in I$. We assume that:*

1. *$\forall u \in I$, the function $x \to f(u,x)$ is integrable.*
2. *The function $x \to f(u,x)$ is derivable in u_0, μ almost everywhere in x.*

3. *(Domination)It exists an integrable and positive function g such that, for all $u \in I$,*
$$|f(u,x) - f(u_0,x)| \leq g(x)|u - u_0| \quad \mu\text{almost everywhere}$$

Then the function $F(u) := \int f(u,x)d\mu(x)$ is derivable in u_0, and

$$F'(u_0) = \int \frac{\partial f}{\partial u}(u_0,x)d\mu(x)$$

<u>Theorem</u> **B.9** *Let $\eta > 0$. If f is a measurable function from $\mathbb{R}^d$ in $\mathbb{R}^+$,*

$$\int_{B(0,\eta)} f(x)dx = \int_0^\eta \int_{\partial B(0,1)} f(rz)r^{d-1}drdS(z)$$

<u>Theorem</u> **B.10 (Fubini-Lebesgue)** *Let φ an integrable function on $E\times F$. Then,*

1. *The function $y \to f(x,y)$ is integrable on F, μ almost everywhere.*
2. *The function $x \to f(x,y)$ is integrable on E, ν almost everywhere.*
3. *The functions, $x \to \int f(x,y)d\nu(y)$ and $x \to\in f(x,y)d\mu(x)$, are well defined except on an measure-zero set, and belong respectively to $L^1(E)$ and $L^1(F)$.*
4. *Moreover,*
$$\int_{E\times F} fd\mu\otimes\nu = \int_E\Big(\int_F f(x,y)d\nu(y)\Big)d\mu(x) = \int_F\Big(\int_E f(x,y)d\mu(x)\Big)d\nu(y)$$

<u>Theorem</u> **B.11 (Green's formula)** *Let Ω a $\mathcal{C}^1$ open set of $\mathbb{R}^3$, and v, w two functions in $\mathcal{C}^1(\overline{\Omega})$, then*

$$\int_\Omega w(x).\nabla v(x)dx = \int_{\partial\Omega} v(x).w(x)N(x)dS(x) - \int_\Omega v(x).\nabla w(x)dx$$

with $N(x)$ the normal vector to $\partial\Omega$ in x. In particular we get:

$$\int_\Omega \nabla v(x)dx = \int_{\partial\Omega} v(x)N(x)dS(x)$$

<u>Theorem</u> **B.12** *Let Ω a measurable set of $\mathbb{R}^d$ and $w \in \mathcal{C}(\Omega)$. We have that:*

$$\Big(\forall\varphi\in\mathcal{C}^1_c(\Omega), \quad \int_\Omega w\varphi = 0\Big) \Rightarrow w = 0$$

References

[1] Biney, J. & Tremaine, S.: *Galactic dynamics* Princeton University Press; Second edition (27 Jan. 2008)

[2] Dolbeaut, J.: *An introduction to kinetic equations the Vlasov-Poisson system and the Boltzmann equation,* Lecture notes, Université Paris IX-Dauphine, (7 Mai. 1999)

[3] Evans, L. C.: *Partial Differential equations.* American Mathematical Society, Volume 19, 1998

[4] Glassey, R. T.: *The Cauchy problem in kinetic theory.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1996

[5] Glassey, R.T. & Strauss, W.A. : *Singularity formation in a collisionless plasma could only occur at high velocities.* Arch. Rat. Mech. Anal, 92 (1986)

[6] Golse, F. : *Mean Field Kinetic Equations,* Lecture notes, 2013

[7] Golse, F. : *On the Dynamics of Large Particle Systems in the Mean Field Limit,* Ecole Polythechnique, Journées Équations aux dérivées partielles Forges-les-Eaux, 2-6 juin 2003

[8] Hérau, F.: *Introduction aux équations cinétiques. Solutions faibles pour l'équation de Vlasov-Poisson.* French postgraduate course, Université de Reims, 2009

[9] Horst, E.: *On the classical solutions of the initial value problem for the unmodified non-linear Vlasov equation I.* Math. Methods Appl. Sci. 3, 229-248 (1981)

[10] Kato, T.: *The Cauchy problem for quasi-linear symmetric hyperbolic systems.* Archive for Rational Mechanics and Analysis, Volume 58, Issue 3 , pp 181-205

[11] Mouhot, C.: *Kinetic theory lecture notes*, University of Cambridge.

[12] Mouhot, C.: *PDE lecture notes*, University of Cambridge.

[13] Rein, G. : *Collisionless Kinetic Equations from Astrophysics -The Vlasov-Poisson System*, Lecture notes, University of Bayreuth, Germany.

[14] Sogge, C.D.: *Lectures on non-linear wave equations.* International Press, University of California, 2008

[15] Villani, C.: *A review of mathematical topics in collisional kinetic theory.* In Handbook of Mathematical Fluid Dynamics, S. Friedlander and D. Serre, Eds, Elsevier Science, 2002

[16] Vlasov, A.A.: *On Vibration Properties of Electron Gas* . J. Exp. Theor. Phys. (in Russian). 8 (3): 291.

[17] Wollman, S. : *An existence and uniqueness theorem for the Vlasov-Maxwell system.* Communications on Pure and Applied Mathematics, Volume 37, Issue 4, pages 457-462, July 1984