Quantitative stability inequalities for Vlasov-Poisson and HMF models

Mohammed Lemou

CNRS, Université of Rennes 1 & ENS Rennes

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General motivation

- Get quantitative stability results of steady states f_0

$$f_0(x) = F\left(\sigma_0(x)\right), \qquad \text{with montonic } F,$$

for evolution equations that essentially preserve the measure (or the rearrangement) and some functional (energy, momentum, etc):

$$\int G(f(t,x))dx = \int G(f(0,x))dx, \qquad \forall G. \qquad f(t)^* = f(0)^*$$

$$\mathcal{H}(f(t)) = \mathcal{H}(f(0)).$$

- Quantitative means that the deviation at time t is controlled by the initial deviation from a steady state q:

$$\|f(t) - f_0\| \leq C\|f(0) - f_0\|, \qquad \forall t. \quad \text{up to symmetries}$$

Examples are: Vlasov-Poisson, HMF and 2D Euler systems.
Most of the known stability results in these contexts are obtained via compactness arguments.

Two key estimates

- Goal here: Quantitative functional inequality of the generic form (up to symmetries of the system ...)

$$\|f - f_0\|_{L^1}^2 \leq C_1 \mathcal{F}(f^*, f_0^*) + C_2 \left(\mathcal{H}(f) - \mathcal{H}(f_0)\right).$$

 Minimal regularity assumptions on the perturbation. For compactly supported steady states f_0:

$$\mathcal{F}(f^*, f_0^*) \leq C\|f^* - f_0^*\|_{L^1}.$$

- First rearrangement inequality $\Longrightarrow$ Stability criteria, under which on has

$$\int \sigma_0(x)[f(x) - f_0(x)]dx \leq \mathcal{H}(f) - \mathcal{H}(f_0) + \mathcal{F}_1(f^*, f_0^*)$$

with $\mathcal{F}_1(f, f) = 0$, up to symmetries of the problem.
- Refined rearangement inequality $\Longrightarrow$ Stability estimate:

$$\|f - f_0\|_{L^1}^2 \leq \int \sigma_0(x)[f(x) - f_0(x)]dx + \mathcal{F}_2(f^*, f_0^*). \quad \text{with} \quad \mathcal{F}_2(f, f) = 0\,\forall f.$$

Plan

- A generalized notion of rearrangement
- An extended Hardy-Littlewood type inequality: leading to a stability criteria on the steady state.
- A refined Hardy-Littelwood type inequality (RHL) for rearrangements: leading to a quantitative control of the whole distribution function.
- Applications to VP and HMF

Equimeasurability and Schwarz rearrangement

- **Equimeasurability:** two nonnegative functions $f, g \in L^1(\Omega)$ are equimeasurable if and only if:

$$\int_\Omega \mathcal{C}(f(x))dx = \int_\Omega \mathcal{C}(g(x))dx, \qquad \forall \mathcal{C}$$

  or equivalently: $\forall \lambda \geq 0$,

$$\mu_f(\lambda) = \text{meas}\{x \in \Omega; f(x) > \lambda\} = \text{meas}\{x \in \Omega; g(x) > \lambda\} = \mu_g(\lambda).$$

  We denote by $\mathrm{Eq}(f)$ the set of functions equimeasurable with f.
- **The standard Schwarz symmetrization.** Let $f \in L^1(\Omega)$, then there exists a unique function $f^* \in L^1(\Omega)$ which is nonincreasing function of $|x|$ such that f^* is equimeasurable with f.
- In other terms: if $f^\sharp$ is the pseudo inverse of of μ_f, then

$$f^*(x) = f^\sharp\left(\text{meas}(B_d(0, |x|) \cap \Omega)\right), \quad \forall\, x \in \Omega.$$

  f^* is essentialy the unique decreasing function of $|x|$ which is equimeasurable with f.

The Hardy-Littlewood inequality

Hardy, Littlewood, Pólya: Inequalities, 1934, Cambridge press.
Lieb and Loss: Analysis.
Burton, Annales de l'IHP (1987), Math. Ann. (1989)

Let Ω be a measurable subset of $\mathbb{R}^d$ and let f and g be two nonnegative measurable functions on Ω. Then

$$\int_\Omega f(x)g(x)dx \leq \int_\Omega f^*(x)g^*(x)dx,$$

In particular

$$\int_\Omega |x|(f(x) - f^*(x))dx \geq 0.$$

A well known tool to prove symmetry properties of minimizers.

Generalized rearrangement

Let σ be a nonnegative measurable function of $\Omega \subset \mathbb{R}^d$, $d \geq 1$ such that for all $e \in [0, e_{max})$

$$\text{meas}\{x \in \Omega, \sigma(x) = e\} = 0.$$

Let

$$a_\sigma(e) = \text{meas}\{x \in \Omega, \sigma(x) < e\}, \quad a_\sigma(e_{max}) = |\Omega|.$$

For all $f \in L^1(\Omega)$, we define its rearrangement $f^{*\sigma}$ with respect to σ by

$$f^{*\sigma}(x) = f^\sharp(a_\sigma(\sigma(x))), \quad \forall x \in \Omega,$$

In particular $f^{*\sigma}$ is a decreasing function of $\sigma(x)$ and is equimeasurable with f.

Extended Hardy-Littlewood inequality

Let σ be as above. Then for any nonnegative functions $f, g \in L^1(\Omega)$ we have

$$\int_\Omega f(x)g(x)dx \leq \int_\Omega f^{*\sigma}(x)g^{*\sigma}(x)dx,$$

In particular

$$\int_\Omega \sigma(x)(f(x) - f^{*\sigma}(x))dx \geq 0.$$

Does this nonnegative quantity control some strong norm $\|f - f^{*\sigma}\|$?

- Weak answer: Saturating the inequality $\Longrightarrow$ Compactness

  if $\int_\Omega \sigma(x)(f_n(x) - f_n^{*\sigma}(x))dx \to 0,$ and if $\|f_n^{*\sigma} - f_0\|_{L^1} \to 0$ then

  $$\|f_n - f_0\|_{L^1} \to 0.$$

- In the same spirit as in Burchard-Guo (JFA, 2004) concerning the Riez rearrangement inequality.

Theorem: refined HL inequality (ML)

Let σ be as above and b_σ the pseudo inverse of a_σ. Then for any nonnegative function $f \in L^1(\Omega)$ we have

$$\|f - f^{*\sigma}\|_{L^1}^2 \leq K(f^*, \sigma) \int_\Omega \sigma(x)(f(x) - f^{*\sigma}(x))dx,$$

where $K(f^*, \sigma)$ is a constant depending only on f^* and σ. More generally, for any nonnegative $f, f_0 \in L^1(\Omega)$

$$(\|f - f_0^{*\sigma}\|_{L^1} + \|f_0\|_{L^1} - \|f\|_{L^1})^2 \quad \leq K(f_0^*, \sigma) \left[\int_\Omega \sigma(x)(f(x) - f_0^{*\sigma}(x))dx \right.$$
$$\left. + \int_\Omega b_\sigma[2\mu_{f_0}(s)]\beta_{f^*, f_0^*}(s)ds \right],$$

with $\beta_{f,g}(s) = \text{meas}\{x \in \Omega : f(x) \leq s < g(x)\}$.

A particular case:

Case of Schwarz symmetrization:

Corollary

For all $f \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, $d \geq 1$, and all $0 \leq m \leq d$, we have

$$\int_{\mathbb{R}^d} |x|^m (f(x) - f^*(x)) dx \geq K_d \|f\|_{L^\infty}^{-m/d} \|f\|_{L^1}^{-1+m/d} \|f - f^*\|_{L^1}^2,$$

$$K_d = 2^{-1+m/d} \frac{m^2}{4d^2} |B_d|.$$

This covers the Marchioro-Pulvirenti estimate used for 2D-Euler (1985): $m = 2$, and $d = 2$, and for homogeneous steady states for VP systems.

This estimate was used by Caglioti and Rousset to study long time behavior of some N particles systems (2007): homogeneous steady states to regularized VP, and Euler 2D.

First example: Vlasov-Poisson equations

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \phi_f \cdot \nabla_v f = 0, \qquad f(t=0,x,v) = f_0(x,v)$$

$$\phi_f(t,x) = -\frac{\gamma}{4\pi} \int_{\mathbb{R}^3} \frac{\rho_f(t,y)}{|x-y|} dy, \quad \rho_f(t,x) = \int_{\mathbb{R}^3} f(t,x,v) dv.$$

Poisson equation: $\Delta \phi_f = \gamma \rho_f.$

- Gravitational systems, $\gamma = +1$: galaxies, star clusters, etc.
- Systems of charged particles , $\gamma = -1$: Coulomb interactions. One has to bound the space domain or to confine the system by adding a given potential.

Basic properties of VP

- Conservation of the energy: $\mathcal{H}(f) = E_{kin}(f) - \gamma E_{pot}(f)$

$$E_{kin}(f) = \frac{1}{2}\int_{\mathbb{R}^6} |v|^2 f dx dv, \qquad E_{pot}(f) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla_x \phi_f|^2 dx$$

- Conservation of the Casimir functionals $\int_{\mathbb{R}^6} G(f) dx dv$.
- Isotropic galactic models:

$$f_0(x,v) = F\left(\frac{|v|^2}{2} + \phi_{f_0}(x)\right).$$

- Relative Hamiltonian:

$$\mathcal{H}(f) - \mathcal{H}(f_0) = \int_{\mathbb{R}^6}\left(\frac{|v|^2}{2} + \phi_0(x)\right)(f - f_0) dx dv - \frac{\gamma}{2}\int_{\Omega} |\nabla\phi_f(x) - \nabla\phi_0(x)|^2 dx$$

Known stability results: Vlasov-Poisson system

- Linear stability Physics literature: Gardner, Antonov, Lynden-Bell (1960'), , Kandrup-Signet (1980'), Aly-Perez (1990'), Chavanis (2000'), ..., see also Binney-Tremaine. Most rigorous result: Doremus-Baumann-Feix (1970'). Key assumption: F decreasing.
- Non linear stability: Essentially a Mathematics literature in the two last decades.
  - Stability of a subclass of steady states under general perturbations: Wolansky, Guo, Rein, Dolbeault, Lin, Hadzic, Sanchez, Soler, Lemou-Méhats-Raphaël, Rigault ...
  - Stability of the whole class of steady states $Q_0(x,v) = F\left(\frac{|v|^2}{2} + \phi_0(x)\right)$, with F decreasing: Lemou-Méhats-Raphaël 2012.
  - A different important context: If periodic domain in space: Homogeneous steady states:

    $$f(x,v) = g_0(|v|).$$

    Asymptotic stability under Penrose conditions and regularity assumptions: Landau damping, Mouhot-Villani 2011.

Statement of stability inequalities for VP

The energy space

$$\mathcal{E} = \{f \in L^\infty : f \geq 0, \ \|(1+|v|^2)f\|_{L^1} < \infty\}.$$

Theorem: Quantitative stability (ML).

We have the following

i) There exist a constant $K_0 > 0$ depending only on f_0 such that or all $f \in \mathcal{E}$

$$\|f - f_0\|_{L^1} \leq \quad \|f^* - f_0^*\|_{L^1} + K_0 \left[\mathcal{H}(f) - \mathcal{H}(f_0) + 2|\phi_{f_0}(0)|\|f^* - f_0^*\|_{L^1} + \|\nabla\phi_f - \nabla\phi_{f_0}\|_{L^2}^2\right]^{1/2}.$$

ii) There exist constants $K_0, R_0 > 0$ depending only on f_0 such that, for all $f \in \mathcal{E}$ satisfying

$$\inf_{z\in\mathbb{R}^3} \left(\|\phi_f - \phi_{f_0}(.-z)\|_{L^\infty} + \|\nabla\phi_f - \nabla\phi_{f_0}(.-z)\|_{L^2}\right) < R_0,$$

there holds:

$$\|f - f_0(.-z_{\phi_f})\|_{L^1} + \|\nabla\phi_f - \nabla\phi_{f_0}(.-z_{\phi_f})\|_{L^2} \leq \quad \|f^* - f_0^*\|_{L^1} + K_0\left[\mathcal{H}(f) - \mathcal{H}(f_0) + K_0\|f^* - f_0^*\|_{L^1}\right]^{1/2}.$$

Second example: HMF model

The Hamiltonian Mean Field (HMF) model is a kinetic model describing particles on a unit circle interacting via an inifinite range attractive cosine potential: $f(t,\theta,v), \theta \in [0,2\pi], v \in \mathbb{R}$.

$$\partial_t f + v\,\partial_\theta f - \partial_\theta \phi_f\,\partial_v f = 0, \qquad f(t=0,\theta,v) = f_0(\theta,v)$$

$$\phi_f(t,\theta) = -\int_0^{2\pi} \rho_f(t,\theta')\cos(\theta-\theta')d\theta', \quad \rho_f(t,\theta) = \int_{\mathbb{R}} f(t,\theta,v)dv.$$

- Magnetization vector:

$$M_f = \int_0^{2\pi} \rho_f(\theta)u(\theta)d\theta, \quad u(\theta) = (\cos\theta, \sin\theta)^T.$$

$$\phi_f(\theta) = -M_f \cdot u(\theta).$$

- Hamiltonian:

$$\mathcal{H}(f) = \frac{1}{2}\iint v^2 f(\theta,v)d\theta dv - \frac{1}{2}|M_f|^2.$$

steady states of the HMF model

- Consider the following class of steady states

$$f_0(\theta, v) = F(e_0(\theta, v)), \quad \text{with} \quad e_0(\theta, v) = \frac{v^2}{2} + \phi_0(\theta),$$

$$\phi_0(\theta) = -m_0 \cos\theta, \qquad m_0 > 0.$$

Note that f_0 may have an unbounded support.

- The relative Hamiltonian writes

$$\mathcal{H}(f) - \mathcal{H}(f_0) = \iint \left(\frac{v^2}{2} + \phi_0(x)\right)(f - f_0)d\theta dv - \frac{1}{2}|M_f - M_{f_0}|^2$$

- No stability result in this case. Formal linear stability criteria has been given by Barré et al 2011 and more explicitly by Ogawa, 2013.

Non linear stability Criteria for HMF

- Steady states of the form

$$f_0(\theta, v) = F\left(e_0(\theta, v)\right), \quad e_0(\theta, v) = \frac{v^2}{2} - m_0 \cos\theta, \quad m_0 > 0.$$

- Assume F is decreasing and let

$$\kappa_0 = \int_0^{2\pi}\int_{-\infty}^{+\infty} \left|F'(e_0(\theta, v))\right| \left(\frac{\int_{\mathcal{D}} (\cos\theta - \cos\theta')(e_0(\theta, v) - \phi_0(\theta'))^{-1/2} d\theta'}{\int_{\mathcal{D}} (e_0(\theta, v) - \phi_0(\theta'))^{-1/2} d\theta'} \right)^2 d\theta dv.$$

  We prove that if $\kappa_0 < 1$, then f_0 is orbitally stable.
  The same criteria has been obtained formally for the linear stability by Barré et al (2011) and more explicitly by Ogawa (2013).

- f_0 may have an unbounded support, but in this case we assume

$$\|f_0\|_{L^p} < +\infty, \qquad \forall\, 0 < p < 1/3.$$

Non linear stability for HMF (L, Luz, Méhats)

There exists $\delta > 0$ such that, for all $f \in L^1\left((1+|v|^2)d\theta dv\right)$ satisfying $|M_f - M_{f_0(\cdot - \theta_f)}| < \delta$, we have

$$\|f - f_0(\cdot - \theta_f)\|_{L^1}^2 \leq C \left(\mathcal{H}(f) - \mathcal{H}(f_0) + C(1 + \|f\|_{L^1})\|f^* - f_0^*\|_{L^1} \right.$$
$$\left. + C \int_0^{+\infty} s^2 \left(f_0^\sharp(s) - f^\sharp(s)\right)_+ ds + C \int_0^{+\infty} \mu_{f_0}(s)^2 \beta_{f^*, f_0^*}(s) ds \right),$$

where
$\beta_{f^*, f_0^*}(s) = \text{meas}\left\{(\theta, v) \in [0, 2\pi] \times \mathbb{R} : f^*(\theta, v) \leq s < f_0^*(\theta, v)\right\}$, for all $s \geq 0$, and where C is a positive constant depending only on f_0. The parameter θ_f is defined by $M_f = |M_f|(\cos\theta_f, \sin\theta_f)^T$. In particular, if f_0 is a compactly supported steady state, then

$$\|f - f_0(\cdot - \theta_f)\|_{L^1}^2 \leq C \left(\mathcal{H}(f) - \mathcal{H}(f_0) + C(1 + \|f\|_{L^1})\|f^* - f_0^*\|_{L^1} \right).$$

Main lines of the proof: i) reduce to a functional of M_f.

We write the relative Hamiltonian in the following form ($\phi_f(\theta) = M_f \cdot u(\theta)$)

$$\mathcal{D} = \mathcal{H}(f) - \mathcal{H}(f_0) = \iint \left(\frac{v^2}{2} + \phi_f(x)\right)(f - f_0)d\theta dv \ + \frac{1}{2}|M_f - M_0|^2.$$

We take : $\sigma(\theta, v) = \frac{v^2}{2} + \phi + \max\phi$ and denote $f^{*\sigma} = f^{*\phi}$. Note that: $f_0 = f_0^{*\phi_0}$.

$$\mathcal{D} = \iint \left(\frac{|v|^2}{2} + \phi + \max\phi\right)(f - f^{*\phi})d\theta dv + \iint \left(\frac{v^2}{2} + \phi\right)(f^{*\phi} - f_0^{*\phi})d\theta dv$$
$$+ \iint \left(\frac{v^2}{2} + \phi\right)(f_0^{*\phi} - f_0)d\theta dv + \frac{1}{2}|M_f - M_0|^2.$$

We get

$$\mathcal{J}(|M_f|) - \mathcal{J}(|M_{f_0}|) \leq \mathcal{H}(f) - \mathcal{H}(f_0) + \int_0^{+\infty} s^2\left(f_0^\sharp(s) - f^\sharp(s)\right)_+ ds + C(1 + \|f\|_{L^1})\|f^* - f_0^*\|_{L^1} \ .$$

$$\mathcal{J}(|M_f|) = \iint \left(\frac{v^2}{2} + \phi\right) f_0^{*\phi} d\theta dv + \frac{1}{2}|M_f|^2.$$

ii) Stability criteria for HMF

- The stability criteria: the quadratic form

$$\mathcal{J}''(m_0) = 1 - \kappa_0 > 0, \qquad \phi_0(\theta) = -m_0 cos\theta.$$

- In case of VP, this leads to a criteria which was proved to hold true in L-Méhats-Raphaël, 2012, thanks to a new Poincaré type inequality.
- We have (locally)

$$\mathcal{J}(m) - \mathcal{J}(m_0) \geq C(m - m_0)^2, \qquad \forall\ m > 0.$$

iii) Control of $\|f - f_0\|_{L^1}$

Now we first write the relative Hamiltonian in the following form

$$\mathcal{H}(f) - \mathcal{H}(f_0) = \iint \left(\frac{v^2}{2} + \phi_0(\theta)\right)(f - f_0)d\theta dv \ - \frac{1}{2}|M_f - M_{f_0}|^2.$$

$$\sigma(x,v) = \frac{v^2}{2} + \phi_0(\theta) + m_0, \qquad f^{*\sigma} \equiv f^{*\phi_0}, \qquad f_0^{*\phi_0} = f_0.$$

$$a_\sigma(e) = \int_0^{2\pi} (e + m_0 \cos\theta)_+^{1/2}\, d\theta.$$

$$\|f - f_0\|_{L^1}^2 \leq \quad C\left(\mathcal{H}(f) - \mathcal{H}(f_0) + C(1 + \|f\|_{L^1})\|f^* - f_0^*\|_{L^1} + \frac{1}{2}|M_f - M_0|^2\right)$$
$$+C\int_0^{+\infty} a_\sigma^{-1}(2\mu_0(s))\mathrm{meas}\{f^* \leq s < f_0^*\}\, ds.$$

In summary

There exists $\delta > 0$ such that:

If $f(t)$ is the solution the HMF model with initial data f_{init} then we have

$$\|f(t) - f_0(\cdot - \theta_{f(t)})\|_{L^1}^2 \leq C \Big[\mathcal{H}(f_{init}) - \mathcal{H}(f_0) + C(1 + \|f_{init}\|_{L^1}) \|f_{init}^* - f_0^*\|_{L^1}$$

$$+ \int_0^{+\infty} s^2 \left(f_0^\sharp(s) - f_{init}^\sharp(s) \right)_+ ds + \int_0^{+\infty} \mu_0(s)^2 \mathrm{meas}\{ f_{init}^* \leq s < f_0^* \}\, ds. \Big]$$

provided that

$$|M_{f_{init}} - M_{f_0(.+\theta_0)}| < \delta.$$

THANK YOU !