On the Mean-Field and Classical Limit for the N-body Schrödinger Equation

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Motivation

•The Vlasov equation with $C^{1,1}$ interaction potential has been derived from the N-body problem of classical mechanics in the large N, small coupling constant limit (Neunzert-Wick 1973, Braun-Hepp 1977, Dobrushin 1979)

•The Hartree equation with bounded interaction potential has been derived from the N-body linear Schrödinger equation in the same limit (Spohn 80, Bardos-FG-Mauser 2000, Rodnianski-Schlein 09); extension to singular interaction potentials (including Coulomb) by Erdös-Yau 2001, Pickl 2009.

Problem: is the mean-field ($N \to \infty$) limit of the quantum N-body problem uniform in the classical limit ($\hbar \to 0$)? (Graffi-Martinez-Pulvirenti 02, Pezzotti-Pulvirenti 09)

The diagram

$$
\begin{array}{ccc}
\boxed{\textbf{Schrödinger}} & \xrightarrow{N\to\infty} & \boxed{\textbf{Hartree}} \\
\downarrow & & \downarrow \\
\hbar \to 0 & & \hbar \to 0 \\
\downarrow & & \downarrow \\
\boxed{\textbf{Liouville}} & \xrightarrow{N\to\infty} & \boxed{\textbf{Vlasov}}
\end{array}
$$

$\boxed{\text{DOBRUSHIN'S PROOF OF THE MEAN-FIELD LIMIT}}$

The mean-field flow

•Classical N-body problem:

$$\dot{x}_j = \xi_j \,, \quad \dot{\xi}_j = -\frac{1}{N}\sum_{k=1}^{N} \nabla V(x_j - x_k)$$

Here $V \in C^{1,1}(\mathbf{R}^d)$ is even, so that $\nabla V(x_k - x_k) = 0$

•Embed the N-body problem into

$$\dot{X} = \Xi \,, \quad \dot{\Xi} = -\nabla V \star_x \rho(t, X) \,, \quad \rho(t) := \int (X, \Xi)(t) \# f^{in} d\xi$$

where $f^{in} \in \mathcal{P}(\mathbf{R}^d \times \mathbf{R}^d)$, thereby defining a mean-field flow

$$t \mapsto (X, \Xi)(t, x, \xi, f^{in}) \,, \qquad \text{such that } (X, \Xi)(0, x, \xi, f^{in}) = (x, \xi)$$

An important remark

For each probability density f^{in} on $\mathbf{R}^d \times \mathbf{R}^d$, the propagated density $f(t) := (X, \Xi)(t, \cdot, \cdot, f^{in})\# f^{in}$ is the solution of the Vlasov equation

$$(\partial_t + \xi \cdot \nabla_x) f = \nabla V \star_x \rho_f \cdot \nabla_\xi f \,, \quad f\big|_{t=0} = f^{in} \,, \quad \rho_f := \int f d\xi$$

In particular

$$(x_j, \xi_j)(t) = (X, \Xi) \left(t, x_j(0), \xi_j(0), \frac{1}{N} \sum_{k=1}^{N} \delta_{(x_k, \xi_k)(0)} \right)$$

$$\frac{1}{N} \sum_{j=1}^{N} \delta_{(x_j, \xi_j)(t)} \text{ is a weak solution of the Vlasov equation}$$

PBM: Is there an analogue of this property for the quantum problem?

•Estimate $\mathrm{dist_{MK,1}}(f(t),\mu(t))$ with $\mu(t):=(X,\Xi)(t,\cdot,\cdot,\mu^{in})\#\mu^{in}$ where

$$\mu^{in}:=\frac{1}{N}\sum_{k=1}^{N}\delta_{(x_k^{in},\xi_k^{in})}\rightharpoonup f^{in} \text{ weakly in } \mathcal{P}(\mathbf{R}^d\times\mathbf{R}^d)$$

Def. For $\mu,\nu\in\mathcal{P}_p(\mathbf{R}^m)$ (Borel probability measures on $\mathbf{R}^m$ with finite order p moments), Monge-Kantorovich distance of exponent p

$$\mathrm{dist_{MK,p}}(\mu,\nu):=\inf_{\pi\in\Pi(\mu,\nu)}\left(\iint|x-y|^p\pi(dxdy)\right)^{1/p}$$

where $\Pi(\mu,\nu)$=set of Borel probability measures π on $\mathbf{R}^m\times\mathbf{R}^m$ s.t.

$$\iint(\phi(x)+\psi(y)\pi(dxdy)=\int\phi(x)\mu(dx)+\int\psi(y)\nu(dy)$$

Key observation

•By integrating along characteristics, compute

$$(X,\Xi)(t,x,\xi,f^{in})-(X,\Xi)(t,y,\eta,\mu^{in})=(X,\Xi)(t)-(Y,H)(t)$$

and integrate against an arbitrary coupling π^{in} of f^{in} and μ^{in}.
•The difference of force fields satisfies

$$\begin{aligned}&\nabla V\star\rho_f(t,X)-\nabla V\star\rho_\mu(t,Y)\\&=\int(\nabla V(x-\bar{x})-\nabla V(y-\bar{y}))\pi(t,d\bar{x}d\bar{\xi}d\bar{y}d\bar{\eta})\end{aligned}$$

where $\pi(t)=((X,\Xi),(Y,H))(t)\#\pi^{in}\in\Pi(f(t),\mu(t))$
•One arrives at an integral inequality mastered by Gronwall's lemma

AN EULERIAN ANALOGUE OF DOBRUSHIN'S ARGUMENT

An alternative strategy

•Seek to estimate

$$\mathrm{dist}_{\mathrm{MK},2}(f(t)^{\otimes n}, F_N^{\mathbf{n}}(t))$$

where F_N is the solution of the N-body Liouville equation and

$$F_N^{\mathbf{n}}(t) := \int F_N(t) dy_{n+1} d\eta_{n+1} \dots dy_N d\eta_N$$

instead of

$$\mathrm{dist}_{\mathrm{MK},1}\left(f(t), \frac{1}{N}\sum_{k=1}^{N} \delta_{(x_k,\xi_k)(t)}\right)$$

•Look for an Eulerian analogue of the Dobrushin argument, avoiding the use of classical trajectories

•All the steps in the estimate should have clear quantum analogues

•Initial data for Vlasov's equation: $f^{in} \in \mathcal{P}_2(\mathbf{R}^d \times \mathbf{R}^d)$

•Initial data for N-body Liouville $F_N^{in} \in \mathcal{P}_2^s((\mathbf{R}^d \times \mathbf{R}^d)^N)$ symmetric in the phase-space variables

Notation:

$$X_N := (x_1, \ldots, x_N)\,, \quad \Xi_N := (\xi_1, \ldots, \xi_N)$$
$$Y_N := (y_1, \ldots, y_N)\,, \quad H_N := (\eta_1, \ldots, \eta_N)$$

For each $\sigma \in \mathfrak{S}_N$, set

$$\sigma \cdot X_N := (x_{\sigma(1)}, \ldots, x_{\sigma(N)})$$

Initial coupling: $P^{in} \in \Pi^s((f^{in})^{\otimes N}, F_N^{in})$ — s means invariant by

$$(X_N, \Xi_N, Y_N, H_N) \mapsto (\sigma \cdot X_N, \sigma \cdot \Xi_N, \sigma \cdot Y_N, \sigma \cdot H_N)\,, \quad \sigma \in \mathfrak{S}_N$$

Vlasov vs Liouville dynamics

Vlasov equation:

$$(\partial_t + \xi \cdot \nabla_x) f - \nabla V \star_x \rho_f \cdot \nabla_\xi f = 0\,, \quad f\big|_{t=0} = f^{in}$$

Hence

$$(\partial_t + \Xi_N \cdot \nabla_{X_N}) f^{\otimes N} = \sum_{j=1}^{N} \nabla V \star_x \rho_f(t, x_j) \cdot \nabla_{\xi_j} f^{\otimes N}$$

Liouville equation

$$(\partial_t + H_N \cdot \nabla_{Y_N}) F_N = \frac{1}{N} \sum_{j,k=1}^{N} \nabla V(y_j - y_k) \cdot \nabla_{\eta_j} F_N\,, \quad F_N\big|_{t=0} = F_N^{in}$$

Theorem A

Assume that the potential V is even with $\nabla V \in W^{1,\infty}(\mathbf{R}^d)$. Let $f(t)$ be the solution of the Vlasov equation with initial data f^{in} and F_N be the solution of the Liouville equation with initial data F_N^{in}. Then

$$\frac{1}{n}\operatorname{dist}_{\mathrm{MK},2}(f(t)^{\otimes n}, F_N^{\mathbf{n}}(t))^2 \leq \frac{1}{N}\operatorname{dist}_{\mathrm{MK},2}((f^{in})^{\otimes N}, F_N^{in})^2 e^{\Lambda t} + \frac{(2\|\nabla V\|_{L^\infty})^2}{N}\frac{e^{\Lambda t}-1}{\Lambda}$$

for all $t \geq 0$ and $n = 1, \ldots, N$, with

$$\Lambda = 2 + 1 \vee 2\operatorname{Lip}(\nabla V)^2$$

Lemma 1 Let $t \mapsto P(t) \in \mathcal{P}((\mathbf{R}^d \times \mathbf{R}^d)^2)$ satisfy $P\big|_{t=0} = P^{in}$ and

$$(\partial_t + \Xi_N \cdot \nabla_{X_N} + H_N \cdot \nabla_{Y_N})P$$

$$= \sum_{j=1}^{N} \left(\nabla V \star_x \rho_f(t, x_j) \cdot \nabla_{\xi_j} + \frac{1}{N} \sum_{k=1}^{N} \nabla V(y_j - y_k) \cdot \nabla_{\eta_j} \right) P$$

Then $P(t) \in \Pi^s(f(t)^{\otimes N}, F_N(t))$ for each $t \geq 0$, i.e.

$$\int P(t) dY_N dH_N = f(t)^{\otimes N}, \qquad \int P(t) dX_N d\Xi_N = F_N(t)$$

Proof: Integrate both sides of the equation for P in (Y_N, H_N) and in (X_N, Ξ_N), and use the uniqueness property for the Vlasov and the Liouville equations

The quantity $D_N(t)$

Definition For each $P^{in} \in \Pi^s((f^{in})^{\otimes N}, F_N^{in})$, set

$$D_N(t) := \int \frac{1}{N} \sum_{j=1}^{N} (|x_j - y_j|^2 + |\xi_j - \eta_j|^2) P(t)$$

Lemma 2

$$D_N(t) \geq \frac{1}{n} \mathrm{dist}_{\mathrm{MK},2}(f(t)^{\otimes n}, F_N^{\mathbf{n}}(t))^2$$

Proof: By symmetry of $P(t)$, one has

$$D_N(t) := \int (|x_j - y_j|^2 + |\xi_j - \eta_j|^2) P(t) \quad \text{for all } j = 1, \ldots, N$$

$$\geq \int \frac{1}{n} \sum_{j=1}^{n} (|x_j - y_j|^2 + |\xi_j - \eta_j|^2) P(t)$$

Bound on $\mathrm{dist}_{\mathrm{MK},2}(f^{\otimes n}, F_N^{\mathbf{n}}) =$ moment bound for a 1st order PDE

The dynamics of $D_N(t)$

Notation for $Y_N = (y_1, \ldots, y_N)$, we set

$$\mu_{Y_N} := \frac{1}{N}\sum_{j=1}^{N} \delta_{y_j}$$

•Multiplying each side of the equation for P by

$$\frac{1}{N}(|X_N - Y_N|^2 + |\Xi_N - H_N|^2)$$

and integrating in all variables

$$\dot{D}_N(t) = \int \frac{1}{N}\sum_{j=1}^{N}(\xi_j \cdot \nabla_{x_j} + \eta_j \cdot \nabla_{y_j})|x_j - y_j|^2 P(t)$$

$$+\int \frac{1}{N}\sum_{j=1}^{N}\left(\nabla V \star_x \rho_f(x_j)\cdot\nabla_{\xi_j} + \nabla V \star \mu_{Y_N}(y_j)\nabla_{\eta_j}\right)|\xi_j - \eta_j|^2 P(t)$$

Thus

$$\dot{D}_N(t) = \int \frac{2}{N} \sum_{j=1}^{N} (\xi_j - \eta_j) \cdot (x_j - y_j) P(t)$$

$$+ \int \frac{2}{N} \sum_{j=1}^{N} (\nabla V \star_x \rho_f(x_j) - \nabla V \star \mu_{Y_N}(y_j)) \cdot (\xi_j - \eta_j) P(t)$$

so that

$$\dot{D}_N(t) \leq D_N(t)$$

$$+ \int \frac{2}{N} \sum_{j=1}^{N} (\nabla V \star_x \rho_f(x_j) - \nabla V \star \mu_{X_N}(x_j)) \cdot (\xi_j - \eta_j) P(t)$$

$$+ \int \frac{2}{N} \sum_{j=1}^{N} (\nabla V \star \mu_{X_N}(x_j) - \nabla V \star \mu_{Y_N}(y_j)) \cdot (\xi_j - \eta_j) P(t)$$

$$=: D_N(t) + I_N(t) + J_N(t)$$

Controling I_N and J_N

Since ∇V is Lipschitz continuous

$$J_N(t) \le \int \frac{1}{N}\sum_{j=1}^{N}(|\nabla V \star (\mu_{X_N}(x_j) - \mu_{Y_N}(y_j))|^2 + |\xi_j - \eta_j|^2)P(t)$$
$$\le \frac{1}{N}\int \sum_{j=1}^{N}(2\,\mathrm{Lip}(\nabla V)^2|x_j - y_j|^2 + |\xi_j - \eta_j|^2)P(t)$$
$$\le (1 \vee 2\,\mathrm{Lip}(\nabla V)^2)D_N(t)$$

Likewise

$$I_N(t) \le \int \frac{1}{N}\sum_{j=1}^{N}(|\nabla V \star (\rho_f(x_j) - \mu_{X_N}(x_j))|^2 + |\xi_j - \eta_j|^2)P(t)$$
$$\le \int \frac{1}{N}\sum_{j=1}^{N}|\nabla V \star (\rho_f - \mu_{X_N})(x_j))|^2\rho_f(t)^{\otimes N} + D_N(t)$$

Lemma 3

$$\int |\nabla V \star (\rho_f - \mu_{X_N})(x_1)|^2 \rho_f(t)^{\otimes N} \le \frac{(2\|\nabla V\|_{L^\infty})^2}{N}$$

Proof Setting $\mathcal{V}(z) := \nabla V \star \rho_f(x_1) - \nabla V(x_1 - z)$, one has

$$|\nabla V \star (\rho - \mu_{X_N})(x_1))|^2 = \frac{1}{N^2} \sum_{k,l=1}^{N} \mathcal{V}(x_j) \cdot \mathcal{V}(x_k)$$

and

$$j \neq k \Rightarrow \int \mathcal{V}(x_j) \cdot \mathcal{V}(x_k) \rho^{\otimes N} = 0$$

Apply Gronwall's lemma to the differential inequality

$$\dot{D}_N(t) \le \Lambda D_N(t) + \frac{(2\|\nabla V\|_{L^\infty})^2}{N}$$

ADAPTATION TO THE QUANTUM PROBLEM

Schrödinger vs Hartree

•The N-body wave function $\Psi_N \equiv \Psi_N(t, x_1, \ldots, x_N) \in \mathbf{C}$ satisfies the linear N-body Schrödinger's equation

$$i\hbar\partial_t\Psi_N = \mathcal{H}_N\Psi_N\,, \quad \mathcal{H}_N := \sum_{j=1}^{N} -\tfrac{1}{2}\hbar^2\Delta_{x_j} + \frac{1}{N}\sum_{j,k=1}^{N} V(x_j - x_k)$$

•Symmetry property of Ψ_N: for all $t \geq 0$ and all $\sigma \in \mathfrak{S}_N$, one has

$$\Psi_N(t,\cdot) = U_\sigma\Psi_N(t,\cdot)\,, \qquad \text{where} \qquad U_\sigma\Psi_N(t, X_N) := \Psi_N(t, \sigma\cdot X_N)$$

•The 1-body wave function $\psi \equiv \psi(t,x)$ satisfies the nonlinear Hartree equation

$$i\hbar\partial_t\psi = \mathbf{H}_{|\psi\rangle\langle\psi|(t)}\psi\,, \quad \mathbf{H}_{\rho(t)} := -\tfrac{1}{2}\hbar^2\Delta_x + \int V(x-y)\rho(t,y,y)dy$$

Schrödinger vs Hartree for density matrices

•The N-body density operator $\rho_N(t)$ satisfies the linear N-body Heisenberg equation

$$i\hbar\partial_t\rho_N = [\mathcal{H}_N, \rho_N] , \quad \rho_N\Big|_{t=0} = \rho_N^{in}$$

•Symmetry property of ρ_N: for all $t \geq 0$ and all $\sigma \in \mathfrak{S}_N$, one has

$$\rho_N(t) = U_\sigma^* \rho_N(t) U_\sigma$$

•The 1-body density operator $\rho \equiv \rho(t)$ satisfies the nonlinear Hartree equation

$$i\hbar\partial_t\rho(t) = [\mathbf{H}_{\rho(t)}, \rho(t)] , \quad \rho\Big|_{t=0} = \rho^{in}$$

Quantum couplings and pseudo-distance

•Density operators on a Hilbert space $\mathfrak{H}$:

$$\rho \in \mathcal{D}(\mathfrak{H}) \Leftrightarrow \rho = \rho^* \geq 0\,, \quad \mathrm{tr}(\rho) = 1$$

•Couplings between two density operators $\rho_1, \rho_2 \in \mathcal{D}(\mathfrak{H})$:

$$\rho \in \mathcal{D}(\mathfrak{H} \otimes \mathfrak{H}) \text{ s.t. } \left\{ \begin{array}{l} \mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}((A \otimes I)\rho) = \mathrm{tr}_{\mathfrak{H}}(A\rho_1) \\ \mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}((I \otimes A)\rho) = \mathrm{tr}_{\mathfrak{H}}(A\rho_2) \end{array} \right.$$

for all $A \in \mathcal{L}(\mathfrak{H})$; the set of all such ρ will be denoted $\mathcal{Q}(\rho_1, \rho_2)$
•For $\rho_1, \rho_2 \in \mathcal{D}(L^2(\mathbf{R}^d))$, define

$$MK_2^\hbar(\rho_1, \rho_2) = \inf_{\rho \in \mathcal{Q}(\rho_1, \rho_2)} \mathrm{tr}\left(\sum_{j=1}^{d} ((x_j - y_j)^2 - \hbar^2(\partial_{x_j} - \partial_{y_j})^2)\rho \right)^{1/2}$$

Let $R_N^{in} \in \mathcal{Q}((\rho^{in})^{\otimes N}, \rho_N^{in})$ and let $t \mapsto R_N(t)$ be the solution of

$$i\hbar\partial_t R_N = [\mathbf{H}_{\rho(t)} \otimes I + I \otimes \mathcal{H}_N, R_N] , \quad R_N\Big|_{t=0} = R_N^{in}$$

Then $R_N(t) \in \mathcal{Q}((\rho(t))^{\otimes N}, \rho_N(t))$ for each $t \geq 0$. Define

$$D_N(t) = \mathrm{tr}\left(\frac{1}{N}\sum_{j=1}^N (Q_j^* Q_j + P_j^* P_j) R_N(t)\right)$$

with

$$Q_j = x_j - y_j \, , \quad P_j := \tfrac{\hbar}{i}(\nabla_{x_j} - \nabla_{y_j}) \, , \quad P_j^* := \tfrac{\hbar}{i}(\mathrm{div}_{x_j} - \mathrm{div}_{y_j})$$

Classical/Quantum Dictionary	
Monge-Kantorovich $\mathrm{dist}_{\mathrm{MK},2}$	Pseudo-distance $MK_2^\hbar$
$\int a\,\mathrm{div}(fu) = -\int (u\cdot\nabla a)f$	$\mathrm{tr}(A[H,\rho]) = -\mathrm{tr}([H,A]\rho)$
Cauchy-Schwarz inequality and Young's inequality	$\mathrm{tr}((A^*B+B^*A)\rho)$ $\le \mathrm{tr}((\lvert A\rvert^2+\lvert B\rvert^2)\rho)$

Theorem B

Assume that the potential V is even and satisfies $\nabla V \in W^{1,\infty}(\mathbf{R}^d)$. Let $\rho_\hbar(t)$ be the solution of Hartree's equation with initial data $\rho_\hbar^{in}$, and let $\rho_{N,\hbar}(t)$ be the solution of Heisenberg's equation with initial data $\rho_{N,\hbar}^{in}$ satisfying the symmetry $\rho_{N,\hbar}^{in} = U_\sigma^* \rho_{N,\hbar}^{in} U_\sigma$ for all $\sigma \in \mathfrak{S}_N$. Then, for each $n = 1, \ldots, N$, and each $t \geq 0$

$$\begin{aligned}\frac{1}{n} MK_2^\hbar(\rho_\hbar(t)^{\otimes n}, \rho_{N,\hbar}^{\mathbf{n}}(t))^2 \leq & \frac{1}{N} MK_2^\hbar((\rho_\hbar^{in})^{\otimes N}, \rho_{N,\hbar}^{in})^2 e^{Lt} \\ & + \frac{8}{N} \|\nabla V\|_{L^\infty}^2 \frac{e^{Lt}-1}{L}\end{aligned}$$

with

$$L := 3 + 4\,\mathrm{Lip}(\nabla V)^2$$

Theorem C: properties of $MK_2^\hbar$

(a) First $MK_2^\hbar$ is **not a distance**: for all $\rho_1, \rho_2 \in \mathcal{D}(L^2(\mathbf{R}^d))$, one has

$$MK_2^\hbar(\rho_1, \rho_2)^2 \geq 2d\hbar$$

(b) For all $\rho_1, \rho_2 \in \mathcal{D}(L^2(\mathbf{R}^d))$, one has

$$MK_2^\hbar(\rho_1, \rho_2)^2 \geq \mathrm{dist}_{\mathrm{MK},2}(\tilde{W}_\hbar[\rho_1], \tilde{W}_\hbar[\rho_2])^2 - 2d\hbar$$

where $\tilde{W}_\hbar[\rho]$ is the Husimi transform of ρ at scale $\hbar$

(c) Let ρ_j be the Töplitz operators at scale $\hbar$ with symbol $(2\pi\hbar)^d \mu_j$, with $\mu_j \in \mathcal{P}_2(\mathbf{C}^d)$ for $j = 1, 2$; then

$$MK_2^\hbar(\rho_1, \rho_2)^2 \leq \mathrm{dist}_{\mathrm{MK},2}(\mu_1, \mu_2)^2 + 2d\hbar$$

Theorem B'

Under the same assumptions as in Theorem B, assume that $\rho_\hbar^{in}$ and $\rho_{N,\hbar}^{in}$ are Töplitz operators, with symbols $(2\pi\hbar)^d\mu_\hbar^{in}$ and $(2\pi\hbar)^{dN}\mu_{N,\hbar}^{in}$

Then, for each $n=1,\ldots,N$, and each $t\ge 0$

$$\frac{1}{n}MK_2^\hbar(\rho_\hbar(t)^{\otimes n},\rho_{N,\hbar}^{\mathbf{n}}(t))^2\le\frac{1}{N}\operatorname{dist}_{\mathrm{MK},2}((\mu_\hbar^{in})^{\otimes N},\mu_{N,\hbar}^{in})^2e^{Lt}$$
$$+2d\hbar e^{Lt}+\frac{8}{N}\|\nabla V\|_{L^\infty}^2\frac{e^{Lt}-1}{L}$$

Wigner and Husimi transforms

•**Wigner transform** at scale $\hbar$ of an operator $\rho \in \mathcal{D}(L^2(\mathbf{R}^d))$:

$$W_\hbar[\rho](x,\xi) = \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} e^{-i\xi\cdot y} \rho(x + \tfrac{1}{2}\hbar y, x - \tfrac{1}{2}\hbar y) dy$$

•**Husimi transform** at scale $\hbar$

$$\tilde{W}_\hbar[\rho](x,\xi) = e^{\hbar\Delta_{x,\xi}/4} W_\hbar[\rho] \geq 0$$

Töplitz quantization

•**Coherent state** with $q, p \in \mathbf{R}^d$:

$$|q+ip,\hbar\rangle = (\pi\hbar)^{-d/4} e^{-|x-q|^2/2\hbar} e^{ip\cdot x/\hbar}$$

•With the identification $z = q + ip \in \mathbf{C}^d$

$$\mathrm{OP}^T(\mu) := \tfrac{1}{(2\pi\hbar)^d} \int_{\mathbf{C}^d} |z,\hbar\rangle\langle z,\hbar|\mu(dz)\,, \quad \mathrm{OP}^T(1) = I$$

•Fundamental properties:

$$\mu \geq 0 \Rightarrow \mathrm{OP}^T(\mu) \geq 0\,, \quad \mathrm{tr}(\mathrm{OP}^T(\mu)) = \tfrac{1}{(2\pi\hbar)^d} \int_{\mathbf{C}^d} \mu(dz)$$

•Important formulas:

$$W_\hbar[\mathrm{OP}^T(\mu)] = \tfrac{1}{(2\pi\hbar)^d} e^{\hbar\Delta_{q,p}/4}\mu\,, \quad \tilde{W}_\hbar[\mathrm{OP}^T(\mu)] = \tfrac{1}{(2\pi\hbar)^d} e^{\hbar\Delta_{q,p}/2}\mu$$

The most important message of this talk...

$\boxed{\text{BEST WISHES TO WALTER, BOB AND JACK!}}$