WEAK POINCARÉ AND NASH-TYPE INEQUALITIES VIA DENSITY OF STATES ESTIMATES

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ABSTRACT. For generators of Markov semigroups, it is shown how bounds on the density of states near zero lead to algebraic decay rates. These rates follow from a so-called "weak Poincaré inequality". These results are applied to the heat semigroup, where the optimal decay rate is recovered, as well as to the semigroup generated by the fractional Laplacian.

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CONTENTS

1. INTRODUCTION

In this note we study how the well-known equivalence between spectral gaps, Poincaré inequalities and exponential rates of convergence to equilibrium extends to systems which lack a spectral gap but have a bounded density of states near 0. Our main result relies solely on our ability to "differentiate" the resolution of the identity of a given operator. It is thus quite general, and covers important examples such as Markov semigroups.

Our setup is as follows: we let M be a differential manifold with measure $\mathrm{d}\mu$, $\mathcal{H} = L^2(M;\mathbb{R})$ and $H : D(H) \subset \mathcal{H} \to \mathcal{H}$ be a selfadjoint, non-negative operator, with $-H$ the infinitesimal generator of a Markov semigroup $(P_t)_{t\geq 0}$ whose invariant measure[1] is $\mathrm{d}\mu$. $\{E(\lambda)\}_{\lambda\geq 0}$ is the resolution of the identity of H and the associated Dirichlet form is

$$\mathcal{E}(u) := \int_M (H^{1/2}u)^2 \,\mathrm{d}\mu.$$

We will show that:

[1]See Remark 1.4.

Theorem 1.1. *1. Suppose that there exist a subspace $\mathcal{X} \subset \mathcal{H}$, equipped with a stronger norm, $r > 0$, $C_1 > 0$ and $\alpha > -1$ for which*

$$\left| \frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=\lambda_0} (E(\lambda)u, v)_{\mathcal{H}} \right| \leq C_1 \lambda_0^{\alpha} \|u\|_{\mathcal{X}} \|v\|_{\mathcal{X}}, \qquad \forall u, v \in \mathcal{X},\ \forall \lambda_0 \in (0, r). \tag{1.1}$$

Then H satisfies a Φ_p-weak Poincaré inequality[2] *with $\Phi(u) = \|u\|_{\mathcal{X}}^2$ and $p = \frac{2+\alpha}{1+\alpha}$.*

2. If, in addition, there exist $C_2 \geq 0$ and $\beta \in \mathbb{R}$, such that

$$\|P_t u\|_{\mathcal{X}}^2 \leq \|u\|_{\mathcal{X}}^2 + C_2 t^{\beta}, \qquad \forall t \geq 0 \tag{1.2}$$

then

$$\operatorname{Var}(P_t u) \leq \left(\operatorname{Var}(u)^{\frac{-1}{1+\alpha}} + C_3 \int_0^t \left(\|u\|_{\mathcal{X}}^2 + C_2 s^{\beta}\right)^{\frac{-1}{1+\alpha}} \mathrm{d}s \right)^{-(1+\alpha)} \tag{1.3}$$

where C_3 is given explicitly (and only depends on α, C_1). In particular, $\operatorname{Var}(P_t u)$ satisfies the following decay rates as $t \to \infty$:

$$\operatorname{Var}(P_t u) \leq \begin{cases} O((\log t)^{-(1+\alpha)}) & \beta = 1 + \alpha. \\ O(t^{\beta-(1+\alpha)}) & 0 < \beta < 1 + \alpha. \\ O(t^{-(1+\alpha)}) & C_2 = 0 \text{ or } \beta \leq 0. \end{cases}$$

The definition of a "weak Poincaré inequality" is somewhat ambiguous. This is addressed in further detail in Section 1.4 below. We adopt the following definition, motivated by Liggett [9, Equation (2.3)]:

Definition 1.2 (Weak Poincaré Inequality). *We say that H satisfies a Φ_p-*weak Poincaré inequality (Φ_p-WPI) *if there exist $C > 0$, $1 < p, q < \infty$ with $1/p + 1/q = 1$ and a mapping $\Phi : \mathcal{H} \to \mathbb{R}_+$, such that*

$$\operatorname{Var}(u) \leq C \mathcal{E}(u)^{1/p} \Phi(u)^{1/q}, \qquad \forall u \in D(\mathcal{E}),$$

where C does not depend on u.

We remind that the definition of variance is

$$\operatorname{Var}(u) := \int_M (u - E(\{0\})u)^2 \, \mathrm{d}\mu$$

where $E(\{0\})$ is the projection onto the kernel of H. We discuss the significance of the resolution of the identity of H (and in particular the projection onto its kernel) and its relationship with functional inequalities and decay rates below in Section 1.4. Whenever H is purely a differential operator the kernel consists of the constant functions. Hence if the underlying space is not compact the kernel is typically trivial.

At this point, it is beneficial to remind the reader of the definition of the classical Poincaré inequality (which we will come back to shortly in Section 1.2):

Definition 1.3 (Poincaré Inequality). *We say that H satisfies a* Poincaŕe inequality *if there exists $C > 0$ such that*

$$\operatorname{Var}(u) \leq C \mathcal{E}(u), \qquad \forall u \in D(\mathcal{E}),$$

where C does not depend on u.

[2] See Definition 1.2

Remark 1.4 (Analysis vs Probability). Thus far we have used terminology coming predominantly from *analysis*. For *probabilists*, M is a state space (usually required to be a Polish space, i.e. a separable topological space) where a probability space $(\Omega, \Sigma, \mathbb{P})$ takes its values. The operator $-H$ is the infinitesimal generator of a Markov process X_t on Ω and $\mathbf{P} = (P_t)_{t\geq 0}$ is the associated Markov semigroup. The Dirichlet form is defined as

$$\mathcal{E}(u) = \lim_{t\to 0} \frac{1}{t} \int_M u(I - P_t)u \, \mathrm{d}\mu = \int_M uHu \, \mathrm{d}\mu.$$

Let us also remind that a measure $\mathrm{d}\mu$ is said to be invariant for the semigroup $\mathbf{P}$ if for every u that is bounded and non-negative

$$\int_M P_t u \, \mathrm{d}\mu = \int_M u \, \mathrm{d}\mu, \qquad \forall t \geq 0.$$

Example 1.5. The prototypical example is that of the heat flow, generated by Δ. In this case the Dirchlet form is given by

$$\mathcal{E}(u) = \int_\Omega |\nabla u(x)|^2 \, \mathrm{d}\mu,$$

up to an appropriate constant. We will get back to the heat flow in Section 1.4, and it will be the subject of several examples in Section 3.

1.1. **A generalized theorem: departing from the Hilbertian structure.** Theorem 1.1 demonstrates how estimates on the density of states near 0 may lead to a weak Poincaré inequality and convergence to equilibrium. However it is not essential to restrict oneself to a subspace $\mathcal{X}$ that has compatible structure with the Hilbertian structure of $\mathcal{H}$. In fact, it is often desirable to depart from the Hilbertian structure as it may provide improved estimates and convergence rates. In particular, this makes sense when the operator in question is the generator of a Markov semigroup, and acts on a range of spaces simultaneously. We therefore have the more general theorem:

Theorem 1.6. *Suppose that there exist $r > 0$, subspaces $\mathcal{X} \subset \mathcal{H}_{\mathrm{ac}}(0,r)$ and $\mathcal{Y} \subset \mathcal{H}$ and $\psi_{\mathcal{X},\mathcal{Y}} : (0,r) \to \mathbb{R}_+$ a strictly positive function a.e. on $(0,r)$ that is in $L^1(0,r)$, for which*[3]

$$\left| \frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=\lambda_0} (E(\lambda)u, v)_{\mathcal{H}} \right| \leq \psi_{\mathcal{X},\mathcal{Y}}(\lambda_0) \|u\|_{\mathcal{X}} \|v\|_{\mathcal{Y}}, \qquad \forall u \in \mathcal{X}, \, \forall v \in \mathcal{Y}, \, \forall \lambda_0 \in (0,r). \tag{1.4}$$

Then, defining $\Psi_{\mathcal{X},\mathcal{Y}}(\rho) = \int_0^\rho \psi_{\mathcal{X},\mathcal{Y}}(\lambda) \, \mathrm{d}\lambda$, $\rho \in (0,r)$, there exists $K_0 > 0$ such that H satisfies a weak Poincaré inequality in the following implicit form:

$$(1-K)\Psi^{-1}_{\mathcal{X},\mathcal{Y}} \left(K \frac{\mathrm{Var}(u)}{\|u\|_{\mathcal{X}} \|u\|_{\mathcal{Y}}} \right) \mathrm{Var}(u) \leq \mathcal{E}(u), \qquad \forall K \in (0, K_0), \, \forall u \in \mathcal{X} \cap \mathcal{Y}. \tag{1.5}$$

Remark 1.7. The requirement that $\psi_{\mathcal{X},\mathcal{Y}}$ is strictly positive a.e. on $(0,r)$, for some $r > 0$ (perhaps very small), is quite natural as we are interested in operators that lack a spectral gap. However, one can easily generalise our result even if that is not the case by defining

$$\Psi^{-1}_{\mathcal{X},\mathcal{Y}}(y) = \sup \left\{ x \in (0,r) \mid \Psi_{\mathcal{X},\mathcal{Y}}(x) \leq y \right\}.$$

[3] See Definition 1.9 for a definition of $\mathcal{H}_{\mathrm{ac}}$

Remark 1.8. We refer to the bilinear form $\frac{\mathrm{d}}{\mathrm{d}\lambda}\big|_{\lambda=\lambda_0}(E(\lambda)u,v)_{\mathcal{H}}$ as the density of states (DoS) of H at λ_0. Note that in the Hilbertian case when $u,v\in\mathcal{X}\subset\mathcal{H}$, if the DoS satisfies a bound as in (1.1), it induces an operator $\mathcal{X}\to\mathcal{X}^*$ by the Riesz representation theorem.

We remind the reader of the definition of the absolutely continuous subspace:

Definition 1.9 (The subspace $\mathcal{H}_{\mathrm{ac}}(\Omega)\subset\mathcal{H}$). *Let $\Omega\subset\mathbb{R}$ be open. We define $\mathcal{H}_{\mathrm{ac}}(\Omega)$ to be the subspace of $\mathcal{H}$ consisting of all u such that $\lambda\mapsto(E(\lambda)u,u)_{\mathcal{H}}$ is absolutely continuous with respect to the Lebesgue measure on Ω.*[4]

Organization of the paper. Before proceeding to prove our theorems we first discuss both the classical and the weak Poincaré inequalities, and their connection to Markov semigroups. The proofs will follow in Section 2 and we will then demonstrate how to apply these theorems in Section 3. Finally, in Section 4 we will provide some supporting results.

1.2. **The classical Poincaré inequality.** The classical Poincaré inequality reads (in the Hilbertian setting of L^2)

$$\int_\Omega\left|u(x)-\left(\frac{1}{|\Omega|}\int_\Omega u(y)\,\mathrm{d}y\right)\right|^2\mathrm{d}x\le C_\Omega\int_\Omega|\nabla u(x)|^2\,\mathrm{d}x,\tag{1.6}$$

where $|\Omega|$ is the Lebesgue measure of the (bounded) set Ω, and $C_\Omega>0$ is independent of u.

Motivation: the heat semigroup. Let us demonstrate why the quantities appearing in this inequality are natural. Consider the heat semigroup, i.e. solutions of

$$\partial_t u(t,x)=\Delta u(t,x),\qquad x\in\Omega,\ t\in\mathbb{R}_+,$$

with Neumann boundary conditions, and the associated invariant measure

$$\mathrm{d}\mu(x)=\frac{\mathrm{d}x}{|\Omega|}.$$

It is well-known that in this case the spectrum of Δ is discrete and non-positive. In particular, its kernel is separated from the rest of the spectrum. This immediately implies that $P_tu(x)$ converges to the projection onto the kernel, given by

$$P_{\ker}u:=\int_\Omega u(x)\,\mathrm{d}\mu(x).$$

Thus, we are interested in the convergence rate to 0 of

$$\mathcal{V}(P_tu):=\|P_tu-P_{\ker}(P_tu)\|^2_{L^2(\mathrm{d}\mu)}=\|P_tu-P_{\ker}(u)\|^2_{L^2(\mathrm{d}\mu)}.$$

The entropy method. A common method to obtain convergence rates for a distance, or distance-like functional, is the so-called *entropy method*. Given the "relative distance" $\mathcal{V}$ (a Lyapunov functional) we find its *production functional* $\mathcal{E}$ by formally differentiating along the flow of the semigroup:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}(P_tu)=2\langle P_tu,P_tu-P_{\ker}u\rangle=2\int_\Omega P_tu(x)\Delta P_tu(x)\,\mathrm{d}\mu(x)=-2\mathcal{E}(P_tu),\tag{1.7}$$

[4]For further discussion, including proof that this is a closed subspace of $\mathcal{H}$ when $\Omega=\mathbb{R}$, see [8, X-§1.2, Theorem 1.5]. The proof can easily be extended to any open interval $(a,b)\subset\mathbb{R}$, and as such to any open set Ω.

where $\mathcal{E}$ is the associated Dirichlet form. Note that since $P_{\mathrm{ker}} = E(\{0\})$ we can rewrite (1.7) as $\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Var}(P_t u) = -2\mathcal{E}(P_t u)$. Now we seek a pure functional inequality involving $\mathcal{V}$ and $\mathcal{E}$. In particular, one looks for a functional inequality of the form

$$\mathcal{E}(u) \geq \Theta(\mathcal{V}(u))$$

with an explicit Θ. Succeeding in finding such an inequality implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}(P_t u) \leq -2\Theta(\mathcal{V}(P_t u))$$

from which an explicit rate is obtained.

The classical Poincaré inequality. Returning to the heat semigroup, we notice that the classical Poincaré inequality (1.6), is *exactly* the functional inequality associated to this entropy method. Moreover, the linear connection between the variance and the Dirichlet form yields an exponential rate of decay for $\operatorname{Var}(P_t u)$.

1.3. **Relationship to Markov semigroups.** In view of the previous paragraph, it is simple to see how one can extend the notion of a Poincaré inequality to general Markov semigroups. Given a Markov semigroup $\{P_t\}_{t\geq 0}$ on $\mathcal{H} = L^2(M, \mathrm{d}\mu)$ with a generator $-H$, where H is a selfadjoint, non-negative operator, and $\mathrm{d}\mu$ its invariant measure, we define *the variance of* u as $\operatorname{Var}(u) := \int_M (u - E(\{0\})u)^2 \,\mathrm{d}\mu$, and *the Dirichlet form* $\mathcal{E}(u) = \int_M uHu\,\mathrm{d}\mu$, which has its own domain $D(\mathcal{E}) \supset D(H)$. Then the Poincaré inequality, as already defined in Definition 1.3, is

$$\operatorname{Var}(u) \leq C\mathcal{E}(u), \quad \forall u \in D(\mathcal{E}).$$

The following well known theorem (see [3]) serves as a motivation for our current investigation:

Theorem 1.10. *The following conditions are equivalent:*

1. *H satisfies a Poincaré inequality with constant C.*
2. *The spectrum of H is contained in $\{0\} \cup \left[\frac{1}{C}, \infty\right)$.*
3. *For every $u \in L^2(M, \mathrm{d}\mu)$ and every $t \geq 0$,*

$$\operatorname{Var}(P_t u) \leq e^{-2t/C}\operatorname{Var}(u).$$

1.4. **The weak Poincaré inequality.** It is natural to ask whether Theorem 1.10 extends to generators which lack a spectral gap. We note that any differential operator acting on functions defined in an unbounded domain (generically) lacks a spectral gap. In this paragraph we provide a review of the literature on this subject.

This topic has a very rich history, in particular in the second half of the 20th century. Nash's celebrated inequality [10] states

$$\|u\|_{L^2}^2 \leq C\left(\|\nabla u\|_{L^2}^2\right)^{\frac{d}{d+2}}\left(\|u\|_{L^1}^2\right)^{\frac{2}{d+2}}, \qquad \forall u \in L^1(\mathbb{R}^d) \cap H^1(\mathbb{R}^d)$$

where $C > 0$ does not depend on u. Estimates of the same spirit are then developed in [7] for example.

The form of the weak Poincaré inequality as appears in our paper first appeared in [9, Equation (2.3)], where it is also shown how such a differential inequality leads to an algebraic decay rate. These ideas were then further developed in [2, 5, 11–15]. We also refer to [1] where the notion of a "weak spectral gap" is introduced.

In fact, in [11] several variants of the WPI were introduced. The most general one is

$$\mathrm{Var}(u) \le \alpha(r)\mathcal{E}(u) + r\Phi(u), \qquad \forall u \in D(\mathcal{E}),\ r > 0,$$

where $\alpha : (0,\infty) \to (0,\infty)$ is decreasing and $\Phi : L^2(\mathrm{d}\mu) \to [0,\infty]$ satisfies $\Phi(cu) = c^2\Phi(u)$ for any $c \in \mathbb{R}$ and $u \in L^2(\mathrm{d}\mu)$. This is equivalent to our Φ_p-WPI whenever $\alpha(r) = Cr^{1-p}$.

For the state-of-the-art of this topic, and other related topics, we refer to [3].

2. Proofs of the theorems

We first prove the more general theorem, Theorem 1.6, and then show how Theorem 1.1 is an immediate consequence. Finally we show how to obtain the rates of convergence in Theorem 1.1. For brevity, we omit the subscripts from the functions $\psi_{\mathcal{X},\mathcal{Y}}$ and $\Psi_{\mathcal{X},\mathcal{Y}}$.

2.1. **Proof of Theorem 1.6.** First we show that an estimate on the density of states near 0 leads to a WPI. Let $r_0 \in (0,r)$ to be chosen later. Let $\{E(\lambda)\}_{\lambda\ge 0}$ be the resolution of the identity of H. Then:

$$\begin{aligned}
\mathcal{E}(u) &= \int_M uHu\,\mathrm{d}\mu = \int_M u\int_{[0,\infty)} \lambda\,\mathrm{d}E(\lambda)u\,\mathrm{d}\mu \\
&\ge \int_M u\int_{[r_0,\infty)} \lambda\,\mathrm{d}E(\lambda)u\,\mathrm{d}\mu \ge r_0\int_M u\int_{[r_0,\infty)} \mathrm{d}E(\lambda)u\,\mathrm{d}\mu \\
&= r_0\int_M u\int_{[0,\infty)} \mathrm{d}E(\lambda)u\,\mathrm{d}\mu - r_0\|E(\{0\})u\|_{\mathcal{H}}^2 - r_0\int_M u\int_{(0,r_0)} \mathrm{d}E(\lambda)u\,\mathrm{d}\mu \\
&= r_0\,\mathrm{Var}(u) - r_0\int_M u\int_{(0,r_0)} \mathrm{d}E(\lambda)u\,\mathrm{d}\mu.
\end{aligned}$$

We now use the estimate on the density of states (1.4) to obtain

$$\begin{aligned}
\int_M u\int_{(0,r_0)} \mathrm{d}E(\lambda)u\,\mathrm{d}\mu &= \int_M\int_{(0,r_0)} \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\mathrm{d}E(\lambda)u,u\right)\mathrm{d}\lambda\,\mathrm{d}\mu \\
&\le \|u\|_{\mathcal{X}}\|u\|_{\mathcal{Y}}\int_{(0,r_0)} \psi(\lambda)\,\mathrm{d}\lambda = \|u\|_{\mathcal{X}}\|u\|_{\mathcal{Y}}\Psi(r_0).
\end{aligned}$$

Hence we have

$$\mathcal{E}(u) \ge r_0\left(\mathrm{Var}(u) - \|u\|_{\mathcal{X}}\|u\|_{\mathcal{Y}}\Psi(r_0)\right).$$

Let $K \in (0,1)$ and define

$$r_0 = \Psi^{-1}\left(K\frac{\mathrm{Var}(u)}{\|u\|_{\mathcal{X}}\|u\|_{\mathcal{Y}}}\right) \qquad \text{so that} \qquad \Psi(r_0) = K\frac{\mathrm{Var}(u)}{\|u\|_{\mathcal{X}}\|u\|_{\mathcal{Y}}}$$

(to satisfy the condition $r_0 < r$ we may need K to be small). Then we get

$$\mathcal{E}(u) \ge r_0(1-K)\,\mathrm{Var}(u)$$

which completes the proof.

2.2. **Proof of Theorem 1.1.** Now we proceed to prove Theorem 1.1 which is split into two parts. First, in Lemma 2.1 we show how the implicit form of the WPI in (1.5) leads to a Φ_p-WPI. Then we complete the proof by showing the different convergence rates we get, depending on α and on the behavior of $\Phi(P_t u)$.

Lemma 2.1. *When $\mathcal{X} = \mathcal{Y}$ and $\psi(\lambda) = C\lambda^\alpha$, $\alpha > -1$, the implicit form of the WPI* (1.5) *as in Theorem 1.6 reduces to the standard Φ_p-WPI as in Theorem 1.1 .*

Proof. Let $\psi(\lambda) = C\lambda^\alpha$, $\alpha > -1$. Then

$$\Psi(\rho) = C\int_0^\rho \lambda^\alpha \,\mathrm{d}\lambda = \frac{C}{\alpha+1}\rho^{\alpha+1}$$

so that

$$\Psi^{-1}(e) = \left(\frac{\alpha+1}{C}\right)^{\frac{1}{\alpha+1}} e^{\frac{1}{\alpha+1}}.$$

Hence

$$\Psi^{-1}\left(K\frac{\mathrm{Var}(u)}{\|u\|_{\mathcal{X}}^2}\right) = \left(\frac{\alpha+1}{C}\right)^{\frac{1}{\alpha+1}} \left(K\frac{\mathrm{Var}(u)}{\|u\|_{\mathcal{X}}^2}\right)^{\frac{1}{\alpha+1}}.$$

Plugging this into (1.5) we have

$$\begin{aligned}
\mathcal{E}(u) &\geq (1-K)\Psi^{-1}\left(K\frac{\mathrm{Var}(u)}{\|u\|_{\mathcal{X}}^2}\right)\mathrm{Var}(u)\\
&= (1-K)\left(\frac{\alpha+1}{C}\right)^{\frac{1}{\alpha+1}} \left(K\frac{\mathrm{Var}(u)}{\|u\|_{\mathcal{X}}^2}\right)^{\frac{1}{\alpha+1}} \mathrm{Var}(u)\\
&= C' \,\mathrm{Var}(u)^{\frac{\alpha+2}{\alpha+1}} \left(\|u\|_{\mathcal{X}}^2\right)^{-\frac{1}{\alpha+1}}.
\end{aligned}$$

This leads to

$$\mathrm{Var}(u) \leq C''\mathcal{E}(u)^{\frac{\alpha+1}{\alpha+2}} \left(\|u\|_{\mathcal{X}}^2\right)^{\frac{1}{\alpha+2}}$$

which is a Φ_p-WPI with $\Phi(u) = \|u\|_{\mathcal{X}}^2$ and $p = \frac{\alpha+2}{\alpha+1}$. □

Convergence rates. Finally, we show that the growth rate assumption (1.2) leads to a decay of the variance as in (1.3). We have:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Var}(P_t u) = -2\mathcal{E}(P_t u) &\leq -2C'\,\mathrm{Var}(P_t u)^{\frac{\alpha+2}{\alpha+1}} \left(\|P_t u\|_{\mathcal{X}}^2\right)^{-\frac{1}{\alpha+1}}\\
&\leq -2C'\,\mathrm{Var}(P_t u)^{\frac{\alpha+2}{\alpha+1}} \left(\|u\|_{\mathcal{X}}^2 + C_2 t^\beta\right)^{-\frac{1}{\alpha+1}}
\end{aligned}$$

which is an ordinary differential equation for $\mathrm{Var}(P_t u)$ of the form

$$\dot{y} \leq -Ay^{1+a}(B + Ct^b)^{-c},$$

for $a, c, A, B > 0$, $b \in \mathbb{R}$, and $C \geq 0$. It yields the bound

$$y(t) \leq \left(y(0)^{-a} + aA\int_0^t \left(B + Cs^b\right)^{-c} \mathrm{d}s\right)^{-1/a}$$

which gives us the desired bound (1.3). Asymptotically, we immediately see that

$$y(t) \leq O(t^{-1/a}) \text{ as } t \to \infty, \qquad \text{if } C = 0 \text{ or } b \leq 0.$$

Otherwise, it is easy to see that $bc = 1$ leads to logarithmic decay, while $bc < 1$ leads to polynomial decay. The precise rates are

$$y(t) \leq \begin{cases} O((\log t)^{-1/a}) \text{ as } t \to \infty, & bc = 1. \\ O(t^{-(1-bc)/a}) \text{ as } t \to \infty, & bc < 1. \end{cases}$$

This completes the proof of Theorem 1.1.

3. Examples

In this section we will see how one can apply our main theorems to obtain explicit decay rates of the variance under the flow of the Laplace and the fractional-Laplace operators.

3.1. The heat semigroup. Consider the Laplace operator $H = -\Delta$ acting in $\mathcal{H} = L^2(\mathbb{R}^d, \mathrm{d}x)$. $-H$ is the generator of the *heat semigroup*, whose evolution equation is given by

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x), & t \in \mathbb{R}_+,\ x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d. \end{cases}$$

One can show that for this operator $E(\{0\}) = 0$, and as such

$$\mathrm{Var}(u(t,\cdot)) = \|u(t,\cdot)\|_{\mathcal{H}}^2.$$

It is well established that $\|u(t,\cdot)\|_{\mathcal{H}}$ has a decay rate of $t^{-d/4}$. Our goal here is to compare this with our methods.

Example 3.1 (Applying Theorem 1.1). Since H is unitarily equivalent to multiplication by $|\xi|^2$ via a Fourier transform, we can express its resolution of the identity $E(\lambda)$ via the following bilinear form:

$$(E(\lambda)u, v)_{\mathcal{H}} = \int_{|\xi|^2 \leq \lambda} \widehat{u}(\xi)\overline{\widehat{v}(\xi)}\,\mathrm{d}\xi. \tag{3.1}$$

Differentiating in λ we get

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=\lambda_0} (E(\lambda)u, v)_{\mathcal{H}} = \frac{1}{2\sqrt{\lambda_0}} \int_{|\xi|=\sqrt{\lambda_0}} \widehat{u}(\xi)\overline{\widehat{v}(\xi)}\,\mathrm{d}\sigma_{\sqrt{\lambda_0}} \tag{3.2}$$

where $\mathrm{d}\sigma_{\sqrt{\lambda_0}}$ is the Lebesgue (uniform) surface measure on the $d-1$-dimensional sphere of radius $\sqrt{\lambda_0}$. An evaluation of the L^2 functions $\widehat{u}$ and $\widehat{v}$ on a hypersurface only makes sense if they in fact belong to any Sobolev space $H^s(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$ with $s > 1/2$ by the trace lemma. Since the functions $\widehat{u}$ and $\widehat{v}$ belong to $H^s(\mathbb{R}^d)$ if and only if u and v belong to $L^{2,s}(\mathbb{R}^d)$, defined as

$$L^{2,s}(\Omega) := \left\{ u \in L^2(\Omega) \ : \ \|u\|_{L^{2,s}(\Omega)}^2 := \int_{\Omega} |u(x)|^2 (1+|x|^2)^s \,\mathrm{d}x < \infty \right\},$$

we conclude that we can bound (3.2) using the $L^{2,s}$-norms of u and v, which are stronger than their $\mathcal{H}$-norms. The immediate bound,

$$\left| \frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=\lambda_0} (E(\lambda)u, v)_{\mathcal{H}} \right| \leq \frac{C}{\sqrt{\lambda_0}} \|u\|_{L^{2,s}} \|v\|_{L^{2,s}}, \tag{3.3}$$

coupled with Theorem 1.1 yield the inequality

$$\mathrm{Var}(u) \leq C\mathcal{E}(u)^{1/3} \left(\|u\|_{L^{2,s}}^2\right)^{2/3} = C\|\nabla u\|_{\mathcal{H}}^{2/3} \|u\|_{L^{2,s}}^{4/3} \tag{3.4}$$

where the constant C has changed, but does not depend on u. In case where $s \in (0,1]$, Lemma 4.1 below shows the following decay of the $L^{2,s}$ norm:

$$\|u(t,\cdot)\|_{L^{2,s}}^2 \le \|u_0\|_{L^{2,s}}^2 + 2s(2s+d-2)\|u_0\|_{\mathcal{H}}^2 t. \tag{3.5}$$

This growth bound, together with (3.4) doesn't fall under the cases of the rates of the decay in our theorem. However, in the appendix of [4], the improved estimate

$$\int_{|\xi|=\rho} \widehat{u}(\xi)\overline{\widehat{v}(\xi)}\,\mathrm{d}\sigma_\rho \le C\min(\rho^{s-\frac{1}{2}},1)^2 \|\widehat{u}\|_{H^s}\|\widehat{v}\|_{H^s},$$

where C only depends on s and d, was shown to be true when $s \in (1/2, 1]$. We therefore obtain (with $\rho = \sqrt{\lambda_0}$)

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=\lambda_0}(E(\lambda)u,v)_{\mathcal{H}}\right| &\le \min(\lambda_0^{\frac{s}{2}-\frac{1}{4}},1)^2\frac{C}{2\sqrt{\lambda_0}}\|\widehat{u}\|_{H^s}\|\widehat{v}\|_{H^s}\\ &= \min(\lambda_0^{s-\frac{1}{2}},1)\frac{C}{2\sqrt{\lambda_0}}\|u\|_{L^{2,s}}\|v\|_{L^{2,s}}.\end{aligned}$$

Hence (3.3) is improved to:

$$\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=\lambda_0}(E(\lambda)u,v)_{\mathcal{H}}\right| \le C\lambda_0^{s-1}\|u\|_{L^{2,s}}\|v\|_{L^{2,s}}, \qquad \forall\lambda_0 \in (0,1],\ s \in (1/2,1].$$

Together with the decay estimate (3.5) of $\|u(t,\cdot)\|_{L^{2,s}}$, and taking $s = 1$, we conclude from the second part of Theorem 1.1 (with $\beta = 1$, $\alpha = 0$) that

$$\mathrm{Var}(u(t,\cdot)) \le \left(\mathrm{Var}(u_0)^{-1} + C\log\left(1 + \frac{2d\|u_0\|_{\mathcal{H}}^2}{\|u_0\|_{L^{2,1}}^2}\,t\right)\right)^{-1}.$$

This is a decay rate of $O((\log t)^{-1})$ which is very far from optimal (and in particular the dependence on the dimension d is very weak). We obtain the optimal rate in the next example, using L^1 norms.

Example 3.2 (Applying Theorem 1.6). We consider again $H = -\Delta$ acting in $\mathcal{H} = L^2(\mathbb{R}^d)$. Now we let

$$\mathcal{Y} = \mathcal{X} = \mathcal{H} \cap L^1(\mathbb{R}^d) \quad \text{equipped with the } L^1 \text{ norm.}$$

Recalling (3.1), we find that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=\lambda_0}(E(\lambda)u,v)_{\mathcal{H}} = \frac{1}{2\sqrt{\lambda_0}}\sqrt{\lambda_0}^{\,d-1}\int_{\mathbb{S}^{d-1}}\widehat{u}(\sqrt{\lambda_0}\sigma)\overline{\widehat{v}(\sqrt{\lambda_0}\sigma)}\,\mathrm{d}\sigma$$

where $\mathrm{d}\sigma$ is the uniform measure on the unit sphere. Using the fact that the Fourier transform of an L^1 function is bounded and continuous, we get

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=\lambda_0}(E(\lambda)u,v)_{\mathcal{H}}\right| &\le \frac{1}{2}\left|\mathbb{S}^{d-1}\right|\lambda_0^{\frac{d}{2}-1}\|\widehat{u}\|_{L^\infty}\|\widehat{v}\|_{L^\infty}\\ &\le \frac{1}{2}\left|\mathbb{S}^{d-1}\right|\lambda_0^{\frac{d}{2}-1}\|u\|_{L^1}\|v\|_{L^1}\end{aligned}$$

We deduce that $u, v \in \mathcal{H}_{\mathrm{ac}}$ so that Theorem 1.6 is applicable, with

$$\psi(\lambda) = \frac{1}{2}\left|\mathbb{S}^{d-1}\right|\lambda^{\frac{d}{2}-1}. \tag{3.6}$$

Since $\psi(\lambda)$ is of the form $C\lambda^\alpha$, we can use the second part of the simpler Theorem 1.1 (see Lemma 2.1 below) with $\alpha = \frac{d}{2} - 1$ and $\Phi(u) = \|u\|_{L^1(\mathbb{R}^d)}$. Using the fact that the L^1 norm of solutions to the heat equation decays monotonically (see Lemma 4.2 below), we have $C_2 = 0$, where C_2 is the constant appearing in (1.2). Then (1.3) becomes

$$\mathrm{Var}(u(t,\cdot)) \leq \left(\mathrm{Var}(u_0)^{-\frac{2}{d}} + \frac{2s}{d}\left(\frac{d}{2|\mathbb{S}^{d-1}|}\right)^{\frac{2}{d}} \|u_0\|_{L^1}^{-\frac{4}{d}} t\right)^{-\frac{d}{2}}$$

and we conclude that

$$\mathrm{Var}(u(t,\cdot)) = \|u(t,\cdot)\|_{\mathcal{H}}^2 \leq O(t^{-\frac{d}{2}}), \qquad \text{as } t \to \infty,$$

which is the optimal rate.

Remark 3.3 (The Nash Inequality). We point out that the functional inequality we obtain here, using Theorem 1.6 with $\mathcal{X} = L^1(\mathbb{R}^d)$, is exactly Nash's inequality. This demonstrates how our methodology gives a general framework for many known important inequalities. We shall see this again in the next example as well, see Remark 3.5.

3.2. **The fractional Laplacian.** For our final example, we turn to the fractional Laplacian – a nonlocal operator which has received significant interest in recent years.

Example 3.4. Consider now the operator $H = (-\Delta)^p$, $p \in (0,1)$, acting in $\mathcal{H} = L^2(\mathbb{R}^d)$. In analogy to (3.1), the resolution of the identity of H satisfies

$$(E(\lambda)u, v)_{\mathcal{H}} = \int_{|\xi|^{2p} \leq \lambda} \widehat{u}(\xi)\overline{\widehat{v}(\xi)}\,\mathrm{d}\xi.$$

Following the arguments that lead to (3.6), and letting $\mathcal{Y} = \mathcal{X} = \mathcal{H} \cap L^1(\mathbb{R}^d)$ as before, we now obtain

$$\psi(\lambda) = \frac{1}{2p}\left|\mathbb{S}^{d-1}\right| \lambda^{\frac{d}{2p}-1}.$$

Now, from [6] we know that $\|u(t,\cdot)\|_{L^1} \leq \|u_0\|_{L^1}$ and as such, much like the previous example, we conclude that

$$\mathrm{Var}(u(t,\cdot)) \leq \left(\mathrm{Var}(u_0)^{-\frac{2p}{d}} + \frac{2p}{d}\left(\frac{d}{2p|\mathbb{S}^{d-1}|}\right)^{\frac{2p}{d}} \|u_0\|_{L^1}^{-\frac{4p}{d}} t\right)^{-\frac{d}{2p}}$$

and hence the asymptotic decay rate is

$$\mathrm{Var}(u(t,\cdot)) \leq O(t^{-\frac{d}{2p}}), \qquad \text{as } t \to \infty.$$

Remark 3.5. The functional inequality we obtain in this example, using our main theorem, is the Nash-type inequality

$$\|u\|_{L^2}^2 \leq C\left(\|\,(-\Delta)^{\frac{p}{2}}\, u\|_{L^2}^2\right)^{\frac{d}{d+2p}} \left(\|u\|_{L^1}^2\right)^{\frac{2p}{d+2p}}.$$

4. Additional results and lemmas

In this last section we prove two auxiliary lemmas that were used above.

Lemma 4.1 ($L^{2,s}$ norm of solutions to the heat equation). *Let $u(t,x)$ be a solution to the heat equation in $\mathbb{R}^d$ with initial condition $u(0,x) = u_0(x)$. Let $s \in (0,1]$. Then*

$$\|u(t,\cdot)\|_{L^{2,s}}^2 \leq \|u_0\|_{L^{2,s}}^2 + 2s(2s-2+d)\|u_0\|_{L^2}^2 t.$$

Proof. Differentiating the expression $\|u(t,\cdot)\|^2_{L^{2,s}} = \int_{\mathbb{R}^d} u(t,x)^2(1+|x|^2)^s \,\mathrm{d}x$ in time (using the fact that solutions to the heat equation are smooth) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t,\cdot)\|^2_{L^{2,s}} = 2\int_{\mathbb{R}^d} u(t,x)\Delta u(t,x)(1+|x|^2)^s \,\mathrm{d}x.$$

Integration by parts yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t,\cdot)\|^2_{L^{2,s}} &= -2\int_{\mathbb{R}^d} \nabla u(t,x)\cdot\big(\nabla u(t,x)(1+|x|^2)^s + 2s(1+|x|^2)^{s-1}xu(t,x)\big)\,\mathrm{d}x \\ &\le -4s\underbrace{\int_{\mathbb{R}^d} \nabla u(t,x)\cdot(1+|x|^2)^{s-1}xu(t,x)\,\mathrm{d}x}_{I(t)}.\end{aligned}$$

Integrating $I(t)$ by parts again gives us

$$\begin{aligned}I(t) &= \int_{\mathbb{R}^d}\sum_{j=1}^d x_j\nabla_j u(t,x)(1+|x|^2)^{s-1}u(t,x)\,\mathrm{d}x \\ &= -d\int_{\mathbb{R}^d} u(t,x)^2(1+|x|^2)^{s-1}\mathrm{d}x - I(t) - 2(s-1)\int_{\mathbb{R}^d} u(t,x)^2|x|^2(1+|x|^2)^{s-2}\,\mathrm{d}x\end{aligned}$$

so that

$$I(t) = -\left(s-1+\frac{d}{2}\right)\|u(t,\cdot)\|^2_{L^{2,s-1}(\mathbb{R}^d)} + (s-1)\,\|u(t,\cdot)\|^2_{L^{2,s-2}(\mathbb{R}^d)}.$$

Returning to $\frac{\mathrm{d}}{\mathrm{d}t}\|u(t,\cdot)\|^2_{L^{2,s}}$, using the fact that $s>0$ but $s-1\le 0$, we conclude that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t,\cdot)\|^2_{L^{2,s}} &\le 4s\left(s-1+\frac{d}{2}\right)\|u(t,\cdot)\|^2_{L^{2,s-1}} - 4s\,(s-1)\,\|u(t,\cdot)\|^2_{L^{2,s-2}} \\ &\le 2s\,(2s-2+d)\,\|u(t,\cdot)\|^2_{L^2}\end{aligned}$$

which completes the proof. □

Lemma 4.2 (L^1 norm of solutions to the heat equation). *Let $u(t,x)$ be a solution to the heat equation in $\mathbb{R}^d$ with initial condition $u(0,x)=u_0(x)$. Then*

$$\|u(t,\cdot)\|_{L^1} \le \|u_0\|_{L^1}.$$

Proof. This is seen by decomposing u_0 into its positive and negative parts $u_0 = u_{0,+} - u_{0,-}$, $u_{0,\pm}\ge 0$, and using the fact that the mass of a non-negative function is preserved under the heat flow. □

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