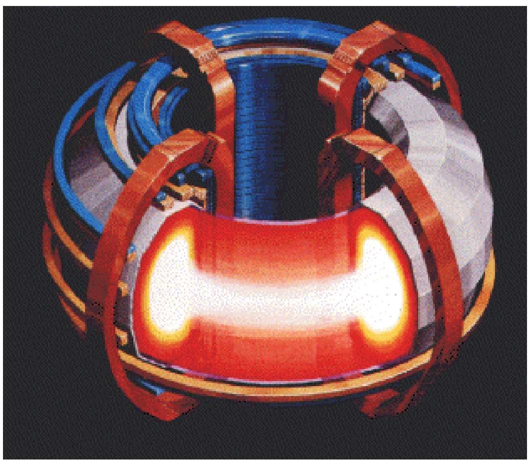
Cambridge Centre of Analysis

Second Mini Project

Linear Spectral Instability of Equilibria of the Relativistic Vlasov-Maxwell System

Author:
Thomas Holding

Supervisor:
Dr. Jonathan Ben-Artzi



A Tokamak reactor

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1 Summary

We consider the linear instability of equilibria of the relativistic Vlasov Maxwell equations. We attempt to extend the work of Lin and Strauss [10, 11] and later Ben-Artzi [2, 1]. In [11] Lin and Strauss proved a sharp linear stability criterion for *monotone* equilibria in a periodic 1.5d regime and in a full 3d regime with cylindrical symmetry where the equilibria has compact support in phase space. Their method for instability is to bundle the entire system into a single family of selfadjoint linear operators depending on the growth rate λ of a hypothetical growing mode. A growing mode would then lie in the kernel of the operator corresponding to its growth rate. The limiting spectral behaviour of this family is analysed in the limits $\lambda \to 0$ and $\lambda \to \infty$, and then a continuity argument on the spectrum shows that an eigenvalue must cross zero as λ increases from 0 to infinity, which corresponds to the existence of a growing mode. In [2] Ben-Artzi extended the instability criterion to *non-monotone* equilibria in the periodic 1.5d regime. This extension relied heavily on the existence of a Poincaré inequality which uses the boundedness of the domain. In this work we try to work around this problem. The cylindrical case considered by Lin and Strauss has an unbounded domain, but the assumption of a *monotone* equilibrium allows the inversion of a linear operator that makes tracking the spectrum of the aforementioned family easier. The novel ideas in the sequel include the consideration of gauge choice, adding a regularising term to make the spectrum discrete, and the observation that the continuity equation is satisfied as a result of the Linearised Vlasov equation whether or not Maxwell's equations hold.

2 Introduction

2.1 Kinetic Theory

Kinetic theory is the study of mathematical models describing the evolution of many particle systems and their derivation. As a simple example consider the evolution of N particles (which without loss of generality can be taken to have unit mass) with positions $(\boldsymbol{x}^i)_{i=1}^N \subset \mathbb{R}^3$ and velocities $(\boldsymbol{v}^i)_{i=1}^N \subset \mathbb{R}^3$ which evolve under the Hamiltonian equations,

$$\begin{aligned} \frac{d\boldsymbol{x}^i}{dt} &= \frac{\partial H}{\partial \boldsymbol{v}^i} = \boldsymbol{v}^i \qquad \frac{d\boldsymbol{v}^i}{dt} = -\frac{\partial H}{\partial \boldsymbol{x}^i} \qquad i = 1, \dots, N \\ H(\boldsymbol{x}, \boldsymbol{v}) &= \frac{1}{2}\sum_{i=1}^{N} |\boldsymbol{v}^i|^2 + \sum_{i<j} \psi(|\boldsymbol{x}^i - \boldsymbol{x}^j|) + \sum_{i=1}^{N} \phi(\boldsymbol{x}^i) \end{aligned} \tag{2.1}$$

where $\frac{1}{2}|\boldsymbol{v}^i|^2$ is the kinetic energy, $\psi(|\boldsymbol{x}^i - \boldsymbol{x}^j|)$ is the interaction potential between two particles i and j, and $\phi(\boldsymbol{x}^i)$ is an external potential. If instead we consider the joint phase space probability density function $f^N(t, \boldsymbol{x}, \boldsymbol{v}) : [0, T) \times \mathbb{R}^{3\times N} \times \mathbb{R}^{3\times N} \to [0, \infty)$, which when integrated over a small volume of phase space gives the probability that the system of N particles will be in that volume, we obtain the Liouville equation,

$$\frac{\partial f^N}{\partial t} + \sum_{i=1}^{N} \left(\frac{\partial H}{\partial \boldsymbol{v}^i} \cdot \frac{\partial f^N}{\partial \boldsymbol{x}^i} - \frac{\partial H}{\partial \boldsymbol{x}^i} \cdot \frac{\partial f^N}{\partial \boldsymbol{v}^i} \right) = 0 \tag{2.2}$$

This is a PDE system on a function f^N of $6N + 1$ variables, so it increases in complexity very fast as we increase N. If we make the additional assumption that for large enough N any two particles are essentially independently distributed, then the joint probability distribution tensorises,

$$f^N = f \otimes f \otimes f \otimes \cdots \otimes f = \bigotimes_{i=1}^{N} f \tag{2.3}$$

so that $f^N(t, \boldsymbol{x}, \boldsymbol{v}) = \prod_{i=1}^{N} f(t, \boldsymbol{x}^i, \boldsymbol{v}^i)$. Under this assumption and an appropriate scaling of the interaction terms, we can take the limit $N \to \infty$ to obtain the PDE on $f(t, \boldsymbol{x}, \boldsymbol{v}) : [0, T) \times \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$,

$$\partial_t f + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{x}} - \frac{\partial \phi}{\partial \boldsymbol{x}} \cdot \frac{\partial f}{\partial \boldsymbol{v}} - \left(\int \frac{\partial \psi(\boldsymbol{x} - \boldsymbol{y})}{\partial \boldsymbol{x}} f(\boldsymbol{y}, \boldsymbol{w}) \, d\boldsymbol{y} d\boldsymbol{w} \right) \cdot \frac{\partial f}{\partial \boldsymbol{v}} = 0 \tag{2.4}$$

which is called the Vlasov equation. Because of the ψ particle-particle interaction term this is a non-linear and non-local equation. Now that we have sketched the derivation of this type of equation we shall move on to our equation of interest.

2.2 The Relativistic Vlasov Maxwell Equations

The relativistic Vlasov-Maxwell equations describe the evolution of the density of a fixed number of species of charged particles moving in space under their induced electromagnetic field according to special relativity and Maxwell's equations. For simplicity we shall present the case when there is no external field and there are two species of opposite charges, and we shall take all physical constants to be 1. In three dimensions the equations are:

$$\begin{gathered} \partial_t f^{\pm} + \hat{\boldsymbol{v}} \cdot \boldsymbol{\nabla}_x f^{\pm} \pm (\boldsymbol{E} + \hat{\boldsymbol{v}} \times \boldsymbol{B}) \cdot \boldsymbol{\nabla}_v f^{\pm} = 0 \\ \boldsymbol{\nabla}_x \cdot \boldsymbol{E} = \rho \qquad \partial_t \boldsymbol{E} = \boldsymbol{\nabla}_x \times \boldsymbol{B} - \boldsymbol{j} \\ \boldsymbol{\nabla}_x \cdot \boldsymbol{B} = 0 \qquad \partial_t \boldsymbol{B} = -\boldsymbol{\nabla}_x \times \boldsymbol{E} \\ \rho = \int f^+ - f^- \, d\boldsymbol{v} \qquad \boldsymbol{j} = \int \hat{\boldsymbol{v}}(f^+ - f^-) \, d\boldsymbol{v} \end{gathered} \tag{RVM 3D}$$

Here $f^{\pm}(t, x, v) : \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}_+$ is the particle density, with f^+ being the density of the positively charged species, and f^- the negatively charged species. $\boldsymbol{E}(t, x), \boldsymbol{B}(t, x) : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}^3$ are respectively the electric and magnetic fields, which evolve according to the charge and current densities $\rho(t, x) : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}_+$ and $\boldsymbol{j}(t, x) : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}$. The $(\boldsymbol{E} + \hat{\boldsymbol{v}} \times \boldsymbol{B})$ term is the Lorentz force acting on the charged particles. Because of special relativity the effective velocity is $\hat{\boldsymbol{v}} \in \mathbb{R}^3$ which is defined as $\hat{\boldsymbol{v}} = \boldsymbol{v} / \langle \boldsymbol{v} \rangle$ where $\langle \boldsymbol{v} \rangle = \sqrt{1 + |\boldsymbol{v}|^2}$. It should be noted that the non-relativistic case can be considered, in which case $\hat{\boldsymbol{v}}$ is replaced by $\boldsymbol{v}$, but this does not simplify the problem. Indeed, many results are first proved for the relativistic case and then extended to the non-relativistic equations. In fact the non-relativistic case does not make physical sense, as it allows the charged particles to move faster than the light waves in the fields. The electric and magnetic potentials can also be defined (up to gauge transformations) to give

$$\boldsymbol{E} = -\boldsymbol{\nabla}_x \phi - \partial_t \boldsymbol{A} \qquad\qquad \boldsymbol{B} = \boldsymbol{\nabla}_x \times \boldsymbol{A} \tag{2.5}$$

where $\phi(t, x) : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}^3$ and $\boldsymbol{A}(t, x) : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}^3$.

The RVM system has a strong link back to the trajectories of individual particles in the form of its characteristics. These are the solutions to the ODE system:

$$\begin{aligned} \dot{\boldsymbol{X}}^{\pm} &= \hat{\boldsymbol{V}}^{\pm} \\ \dot{\boldsymbol{V}}^{\pm} &= \pm(\boldsymbol{E}(t, \boldsymbol{X}^{\pm}) + \hat{\boldsymbol{V}}^{\pm} \times \boldsymbol{B}(t, \boldsymbol{X}^{\pm})) \\ \boldsymbol{X}^{\pm}(0; \boldsymbol{x}, \boldsymbol{v}) &= \boldsymbol{x} \\ \boldsymbol{V}^{\pm}(0; \boldsymbol{x}, \boldsymbol{v}) &= \boldsymbol{v} \end{aligned} \tag{2.6}$$

where for fixed $\boldsymbol{x}, \boldsymbol{v}$, $\boldsymbol{X}^{\pm}, \boldsymbol{V}^{\pm} : \mathbb{R} \to \mathbb{R}^3$ are the trajectories in phase space of the positively and negatively charged particles that start at time 0 at $(\boldsymbol{x}, \boldsymbol{v})$.

2.3 Simplifying symmetry assumptions

Study of the full 3d case of RVM is hindered by the complicated trajectories of the particles. In 3 dimensions a system of ODEs can exhibit chaotic behaviour which is not possible in 2 due to the Poincaré Bendixson theorem. For this reason simplifying symmetries are imposed to reduce the system to fewer dimensions. Because RVM has two 'spatial' coordinates the position and velocity and in the 3d case each are in $\mathbb{R}^3$ we count the dimension as the average dimension of the position and velocity variables. A reduction to one dimension is possible, but due to the structure of Maxwell's equations this removes the magnetic field entirely. To still have a magnetic field we require that the velocity coordinate is at least 2 dimensional. The lowest dimension case is then when the position is one dimensional and the velocity 2 dimensional, making the system 1.5 dimensional.

2.3.1 1.5d case

The 1.5 dimensional case is where the particle densities and fields only depend on a single x coordinate, and two velocity coordinates v_1 and v_2, with v_1 aligned with x. The electric field then only has components in the active velocity directions $\boldsymbol{E}(t, x) = (E_1(t, x), E_2(t, x), 0)$, while the magnetic field is perpendicular to them $\boldsymbol{B}(t,x) = (0,0,B(t,x))$. The equations then become:

$$\begin{aligned} \partial_t f^{\pm} + \hat{v}_1 \partial_x f^{\pm} \pm (E_1 + \hat{v}_2 B)\partial_{v_1} f^{\pm} \pm (E_2 - \hat{v}_2 B)\partial_{v_2} f^{\pm} &= 0 \\ \partial_x E_1 = \rho \qquad \partial_x E_2 = -\partial_t B \qquad \partial_t E_1 = j_1 \qquad \partial_t E_2 &= j_2 + \partial_x B \\ \rho = \int f^+ - f^- \, dv \qquad j_i = \int \hat{v}_i (f^+ - f^-) \, dv & \end{aligned} \tag{RVM 1.5D}$$

2.3.2 Periodic spatial dependence

A further simplification is to assume that the solution is periodic in some of the coordinates. As the v coordinate is a velocity it makes no sense for this to be periodic, the x coordinate can however be assumed periodic. This gives a model for waves extending infinitely in each direction. This is often used to model a Tokamak reactor, in which plasma is magnetically contained in a torus, whose radius is assumed to be large so that the plasma flow is approximately in a straight line.

2.4 Equilibria and Linearisation

To study stability a commonly used technique is to linearise about the equilibrium to be studied. This simplifies the non-linear system into a (not equivalent) linear system. We will use the notation ϕ^0 for the equilibrium electric potential, and similarly for other equilibrium functions. Due to Jean's theorem any equilibria can be expressed in terms of the integrals of motion, functions that are constant along the characteristics. Due to conservation of energy and momentum of the particles the respective quantities

$$e^{\pm} = \langle \boldsymbol{v} \rangle \pm \phi^0(\boldsymbol{x}) \qquad\qquad \boldsymbol{p}^{\pm} = \boldsymbol{v} \pm \boldsymbol{A}^0(\boldsymbol{x}) \tag{2.7}$$

are conserved along the particle trajectories. Due to Jeans' theorem we will look at equilibria that depend only on these so that $\mu^{\pm} = \mu^{\pm}(e^{\pm}, \boldsymbol{p}^{\pm})$, because under the further assumption that the equilibrium fields satisfy the constraint

$$\boldsymbol{\mu}^{\pm}_{\boldsymbol{p}} \cdot \boldsymbol{\nabla}_x (\hat{\boldsymbol{v}} \cdot \boldsymbol{A}^0 - \phi^0) = 0 \tag{2.8}$$

$e^{\pm}$ and $\boldsymbol{p}^{\pm}$ are conserved along the characteristics (see Lemma C.2). The author is unclear on the physical interpretation or origin of this constraint, which reappears later in (4.4). It is satisfied in the 1.5D and cylindrical cases, (which follows from simple calculation). By the notation $\boldsymbol{\mu}^{\pm}_{\boldsymbol{p}}$ we mean the vector of partial derivatives of $\mu^{\pm}(e^{\pm}, \boldsymbol{p}^{\pm})$ with respect to $p_1^{\pm}, p_2^{\pm}$ and $p_3^{\pm}$ evaluated at $(e^{\pm}, \boldsymbol{p}^{\pm})$. We will denote the perturbed quantities with no superscript, i.e. as $f^{\pm}$, etc.. The linearised system is obtained by substituting $\mu^{\pm} + \epsilon f^{\pm}$ into the full system and discarding any terms of order ϵ^2 or above. The only non-linear term in (RVM 3D) is the multiplication between the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ and the $\boldsymbol{v}$ derivatives of the densities. We can thus write the Vlasov equation as

$$(\partial_t + \hat{\boldsymbol{v}} \cdot \boldsymbol{\nabla}_x)(\mu^{\pm} + \epsilon f^{\pm}) + \mathcal{B}(\mu^{\pm} + \epsilon f^{\pm}, \mu^{\pm} + \epsilon f^{\pm}) = 0$$

where $\mathcal{B}$ is a bilinear operator that represents the generation and action of the fields. Expanding, we have,

$$\begin{aligned}(\partial_t + \hat{\boldsymbol{v}} \cdot \boldsymbol{\nabla}_x)\mu^\pm + \mathcal{B}(\mu^\pm, \mu^\pm) + \\ +\epsilon\left[(\partial_t + \hat{\boldsymbol{v}} \cdot \boldsymbol{\nabla}_x)f^\pm + \mathcal{B}(\mu^\pm, f^\pm) + \mathcal{B}(f^\pm, \mu^\pm)\right] + \\ +\epsilon^2\mathcal{B}(f^\pm, f^\pm) = 0\end{aligned}$$

The first line is zero as $\mu^\pm$ is an equilibrium. The third we drop to linearise the system. The $\mathcal{B}(\mu^\pm, f^\pm)$ term represents the action of the fields generated by $\mu^\pm$ on $f^\pm$, while the $\mathcal{B}(f^\pm, \mu^\pm)$ term represents the action of the perturbed fields on $\mu^\pm$. Grouping all the action on $f^\pm$ together, we have

$$(\partial_t + \hat{\boldsymbol{v}} \cdot \boldsymbol{\nabla}_x \pm (\boldsymbol{E}^0 + \hat{\boldsymbol{v}} \times \boldsymbol{B}^0) \cdot \boldsymbol{\nabla}_v)f^\pm = \mp(\boldsymbol{E} + \hat{\boldsymbol{v}} \times \boldsymbol{B}) \cdot \boldsymbol{\nabla}_v\mu^\pm$$

which is the linearised Vlasov equation. When we expand the derivatives of $\mu^\pm$ in terms of the partial derivatives of $\mu^\pm$ with respect to $e^\pm$ and $\boldsymbol{p}^\pm$ we obtain, denoting $\frac{\partial\mu^\pm}{\partial e^\pm} = \mu_e^\pm$ and respectively $\boldsymbol{\mu}_{\boldsymbol{p}}^\pm$,

$$\begin{aligned}\boldsymbol{\nabla}_v\mu^\pm &= \hat{\boldsymbol{v}}\mu_e^\pm + \boldsymbol{\mu}_{\boldsymbol{p}}^\pm \\ \mp(\boldsymbol{E} + \hat{\boldsymbol{v}} \times \boldsymbol{B}) \cdot \boldsymbol{\nabla}_v\mu^\pm &= \mp(\boldsymbol{E} + \hat{\boldsymbol{v}} \times \boldsymbol{B}) \cdot (\hat{\boldsymbol{v}}\mu_e^\pm + \boldsymbol{\mu}_{\boldsymbol{p}}^\pm) \\ &= \mp(\boldsymbol{\nabla}_x\phi + \partial_t\boldsymbol{A}) \cdot \hat{\boldsymbol{v}}\mu_e^\pm \mp (\hat{\boldsymbol{v}} \times (\boldsymbol{\nabla}_x \times \boldsymbol{A})) \cdot \boldsymbol{\mu}_{\boldsymbol{p}}^\pm\end{aligned}$$

where we have introduced the potentials of the perturbation ϕ and $\boldsymbol{A}$.

2.4.1 Monotonicity

We say that the equilibrium $\mu^\pm$ is *monotone* if the partial derivatives $\mu_e^\pm$ are negative whenever $\mu^\pm$ is not zero, (recall that $\mu^\pm$ is non-negative). It is expected due to physical considerations that monotone plasmas are more likely to be stable. However the non-monotone case, by which we mean that for some region where $\mu^\pm > 0$ the derivative $\mu_e^\pm$ is negative, is mathematically more complicated and more sophisticated techniques are required to prove instability.

3 Literature review

3.1 Existence and Uniqueness Theory

The main existence and uniqueness theorem for RVM is the result by Glassey and Strauss[6] stating that given an a priori bound on the maximum velocity, there is a unique classical solution. It should be remembered that these results are for the full non-linear RVM system.

Theorem 3.1 (Glassey, Strauss[6])**.** *Let the Cauchy data $f_0^\pm(x,v) \in C_c^1$ be non-negative, $E_0(x), B_0(x) \in C^2$ be divergence free, with $\rho(0,x)$ and $j(0,x)$ determined by the equations* (RVM 3D)*. Furthermore, let there be a continuous function $\beta(t)$ such that any solution f has compact support in $\{|v| < \beta(t)\}$, i.e. $f^\pm(t,x,v) = 0$ if $|v| > \beta(t)$. Then there is a unique global C^1 solution to* (RVM 3D).

This result is far from satisfactory because in general no such bound is known. However in the 1.5D[4] and 2.5D[5] cases Glassey and Schaeffer exploited symmetries to give the required bound. This 1.5D result is necessary for the consideration of the 1.5D case, as otherwise existence of solutions is not guaranteed.

3.2 (In)Stability theory

The study of the linear stability of Vlasov models of plasmas has developed from the study of simple homogeneous equilibria to strip away such assumptions. In the monotone homogeneous case with vanishing electric and magnetic fields a sharp criterion due to Penrose [12] is applicable. However when non vanishing fields are permitted non-local effects arise and the results are not so simple. A simple model including these effects are BGK waves [3] of the Vlasov-Poisson system, which were studied by Guo and Strauss [7] using perturbation arguments. Later it was proved by Lin [9] that any periodic BGK wave is unstable under perturbations of double the period. The continuation technique used formed the basis of the later papers considered here and also this work. In [10] and [11] Lin and Strauss proved a sharp criterion for monotone equilibria in 1.5d periodic and 3d cylindrical regimes. Later Ben-Artzi [1, 2] extended this result to non-monotone equilibria in the 1.5d periodic case.

We will present a brief summary of the continuation technique and the results obtained. The basic method is to formulate finding a growing mode solution of the linearised system as finding a non-trivial kernel one of a family of selfadjoint operators $(\mathcal{L}^\lambda)_{\lambda\geq 0}$. Here by a growing mode we mean a solution with exponential time dependence, i.e. of the form $(e^{\lambda t}f^\pm, e^{\lambda t}\boldsymbol{E}, e^{\lambda t}\boldsymbol{B})$ where $f^\pm, \boldsymbol{E}, \boldsymbol{B}$ do not depend on t. The family of operators then depend continuously on the growth rate λ. The limiting behaviour as $\lambda \to 0$ and $\lambda \to \infty$ is analysed and used to conclude that as λ increases from 0 to ∞ an eigenvalue of $\mathcal{L}^\lambda$ must cross 0 giving a non-trivial kernel for that λ. The element $u \in \ker(\mathcal{L}^\lambda)$ is then used to construct the growing mode with growth rate λ.

In [9] Lin pioneered this method to show the instability of neutral monotone periodic BGK waves. In this case the limits $\lambda \to 0, \infty$ are relatively simple as the operator $\mathcal{L}^\lambda$ operates only on the electric potential of the perturbation.

In [10] and [11] Lin and Strauss considered the monotone 1.5d periodic case and proved a sharp criterion. The method here was to formulate Gauss's and Ampère's equations as two linear operators $\mathcal{A}_1^\lambda$ and $\mathcal{A}_2^\lambda$ with cross terms $\mathcal{B}^\lambda$. They form a block matrix equation

$$\begin{bmatrix} \mathcal{A}_1^\lambda & \mathcal{B}^\lambda \\ (\mathcal{B}^\lambda)^* & -\mathcal{A}_2^\lambda \end{bmatrix} \begin{bmatrix} \phi \\ \psi \end{bmatrix} = 0$$

but crucially, due to the monotonicity assumption, it is possible to invert $\mathcal{A}_1^\lambda$ on the range of $\mathcal{B}^\lambda$ so that the system can be reduced to

$$\mathcal{L}^\lambda \psi = ((\mathcal{B}^\lambda)^*(\mathcal{A}_1^\lambda)^{-1}\mathcal{B}^\lambda + \mathcal{A}_2^\lambda)\psi = 0$$

Then the criterion is that $\mathcal{L}^0 \geq 0$. As $\lambda \to \infty$, $\mathcal{L}^\lambda$ becomes positive definite, and as $\lambda \to 0$, $\mathcal{L}^\lambda$ converges strongly to $\mathcal{L}^0$. Then the condition of $\mathcal{L}^0$ not being positive semi-definite allows the tracking of a negative eigenvalue as $\lambda \to \infty$ from 0. This eigenvalue must become positive as $\lambda \to \infty$ so it must cross zero.

The actual instability argument in the paper is slightly complicated by having to consider solutions of the form (ψ, b) where $b \in \mathbb{R}$, to deal with the possibility of the electric field of the perturbation having non-zero mean, but as this addition is only one dimensional and has simple limits as $\lambda \to 0, \infty$, it is not a problem. The cylindrical case considered has essentially the same argument, but now there are additional magnetic potentials and many more operator inversions.

Later in [1, 2], Ben-Artzi extended the periodic 1.5d instability criterion to non-monotone equilibria by doing a complete truncation of the operators to finite dimensional approximations. The earlier work had already included truncation, but only as a tool to aid the counting of eigenvalues. Here the truncation is complete and functional analytic tools are used to show that the spectral properties of the truncated operators carry over as the truncation is removed. A key tool used is the spectral gap and discrete spectrum of the Laplacian given by the Poincaré inequality because the domain is bounded.

4 Setup

4.1 Goal and outline

What follows is an attempt to extend the results [2] to unbounded domains. We start by looking at the full 3 dimensional problem and show that inverting the linearised Vlasov equation and obtaining a spectral problem can be done with almost full generality, showing exactly which symmetry assumptions are needed. We then regularise the problem to make the spectrum discrete and use compactness arguments to show that this can be removed. Finally we consider how to make the problem selfadjoint and consider the limits as $\lambda \to 0, \infty$.

4.2 The equations to solve

Recall that after linearisation around an equilibrium $\mu^\pm$ the RVM system becomes,

$$\begin{aligned}
(\partial_t + \mathcal{D}_\pm) f^\pm &= -(\boldsymbol{E} + \hat{\boldsymbol{v}} \times \boldsymbol{B}) \cdot \boldsymbol{\nabla}_v \mu^\pm \\
\mathcal{D}_\pm &= \hat{\boldsymbol{v}} \cdot \boldsymbol{\nabla}_x \pm (\boldsymbol{E}^0 + \hat{\boldsymbol{v}} \times \boldsymbol{B}^0) \cdot \boldsymbol{\nabla}_v \\
\partial_t \boldsymbol{B} = -\boldsymbol{\nabla}_x \times \boldsymbol{E} & \qquad \boldsymbol{\nabla}_x \cdot \boldsymbol{E} = \rho \\
\partial_t \boldsymbol{E} = +\boldsymbol{\nabla}_x \times \boldsymbol{B} - \boldsymbol{j} & \qquad \boldsymbol{\nabla}_x \cdot \boldsymbol{B} = 0 \\
\rho = \int f^+ - f^- \, d\boldsymbol{v} & \qquad \boldsymbol{j} = \int \hat{\boldsymbol{v}} (f^+ - f^-) \, d\boldsymbol{v}
\end{aligned} \tag{4.1}$$

When the ansatz $(e^{\lambda t} f^\pm(\boldsymbol{x}, \boldsymbol{v}), e^{\lambda t} \boldsymbol{E}(\boldsymbol{x}), e^{\lambda t} \boldsymbol{B}(\boldsymbol{x}))$ is substituted into the (4.1) it has the effect of replacing ∂_t on the perturbations with multiplication by the constant λ:

$$\begin{aligned}
(\lambda + \mathcal{D}_\pm) f^\pm &= -(\boldsymbol{E} + \hat{\boldsymbol{v}} \times \boldsymbol{B}) \cdot \boldsymbol{\nabla}_v \mu^\pm \\
\mathcal{D}_\pm &= \hat{\boldsymbol{v}} \cdot \boldsymbol{\nabla}_x \pm (\boldsymbol{E}^0 + \hat{\boldsymbol{v}} \times \boldsymbol{B}^0) \cdot \boldsymbol{\nabla}_v \\
\lambda \boldsymbol{B} = -\boldsymbol{\nabla}_x \times \boldsymbol{E} & \qquad \boldsymbol{\nabla}_x \cdot \boldsymbol{E} = \rho \\
\lambda \boldsymbol{E} = +\boldsymbol{\nabla}_x \times \boldsymbol{B} - \boldsymbol{j} & \qquad \boldsymbol{\nabla}_x \cdot \boldsymbol{B} = 0 \\
\rho = \int f^+ - f^- \, d\boldsymbol{v} & \qquad j = \int \hat{\boldsymbol{v}} (f^+ - f^-) \, d\boldsymbol{v}
\end{aligned} \tag{4.2}$$

If we now introduce the vector and scalar potentials $\phi, \boldsymbol{A}$ we see that ϕ and $\boldsymbol{A}$ must have the same forms $e^{\lambda t}\phi(\boldsymbol{x})$ and $e^{\lambda t}\boldsymbol{A}(\boldsymbol{x})$. Substituting this into the equations we obtain,

$$\begin{aligned}(\lambda + \mathcal{D}_\pm)f^\pm &= \pm(\boldsymbol{\nabla}_x\phi + \lambda\boldsymbol{A} - \hat{\boldsymbol{v}} \times (\boldsymbol{\nabla}_x \times \boldsymbol{A})) \cdot \boldsymbol{\nabla}_v\mu^\pm \\ \rho &= -\Delta\phi - \lambda\boldsymbol{\nabla} \cdot \boldsymbol{A} \\ \boldsymbol{j} &= \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) + \lambda^2\boldsymbol{A} + \lambda\boldsymbol{\nabla}\phi\end{aligned} \tag{4.3}$$

Our goal is to find $(\lambda, f(\boldsymbol{x}, \boldsymbol{v}), \phi(\boldsymbol{x}), \boldsymbol{A}(\boldsymbol{x}))$ which solve this system.

4.3 Gauges

A new idea in respect to this (in)stability argument is the consideration of which gauge to impose in Maxwell's equations. Recall that we have electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ and scalar and vector potentials ϕ and $\boldsymbol{A}$, where

$$\boldsymbol{E} = -\,\mathbf{grad}\,\phi - \partial_t\boldsymbol{A} \qquad\qquad \boldsymbol{B} = \mathbf{curl}\,\boldsymbol{A}$$

Gauges originate because the potential formulation of Maxwell's equations is not complete in the sense that a *gauge transformation* of the form

$$\phi \to \phi - \partial_t\varphi \qquad\qquad \boldsymbol{A} \to \boldsymbol{A} + \mathbf{grad}\,\varphi$$

for some appropriately differentiable φ, leaves the fields $\boldsymbol{E}, \boldsymbol{B}$ unchanged. A gauge is a fixing of this freedom by imposing some condition on the potentials ϕ and $\boldsymbol{A}$. Three commonly used gauges are displayed in the table below, together with the forms that Maxwell's equations take.

Gauge	Condition	Gauss	Ampère
Lorenz	$\operatorname{div}\boldsymbol{A} + \partial_t\phi = 0$	$\Box\phi = \rho$	$\Box\boldsymbol{A} = \boldsymbol{j}$
Coulomb	$\operatorname{div}\boldsymbol{A} = 0$	$-\Delta_x\phi = \rho$	$\Box\boldsymbol{A} + \operatorname{div}\partial_t\phi = \boldsymbol{j}$
Temporal	$\phi = 0$	$-\operatorname{div}\partial_t\boldsymbol{A} = \rho$	$\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{A} + \partial_t^2\boldsymbol{A} = \boldsymbol{j}$

where $\Box = -\boldsymbol{\Delta} + \partial_t^2$. In our case we are looking at growing modes so that ∂_t is replaced with λ. The temporal gauge is interesting to us as it removes ϕ completely, leaving us with just equations for $\boldsymbol{A}$. We will later see that we can recover Gauss's equation from the continuity equation which will be satisfied by construction. This will leave us with just Ampère's equation. An important question is whether imposing this gauge in $L^2(\Omega)$ (with Ω some 3 dimensional domain) restricts our possible fields. The answer is no, as is shown below.

Lemma 4.1. *Suppose that $\phi, \boldsymbol{A} \in L^2(\Omega)$ are potentials of $\boldsymbol{E}, \boldsymbol{B} \in L^2(\Omega)$ in some unspecified gauge, but with the exponential time dependence as in our growing mode with $\lambda > 0$. Then there exists $\phi^{new}, \boldsymbol{A}^{new} \in L^2(\Omega)$ in the temporal gauge with the same form of time dependence.*

Proof. We define $\varphi = \phi/\lambda$. Then applying the gauge transform corresponding to φ we obtain,

$$\phi^{\text{new}} = \phi - \lambda\phi/\lambda = 0 \qquad\qquad \boldsymbol{A}^{\text{new}} = \boldsymbol{A} + \mathbf{grad}\,\varphi = \boldsymbol{A} + \mathbf{grad}\,\phi/\lambda$$

and clearly $\phi^{\text{new}} = 0 \in L^2(\Omega)$ and $\boldsymbol{A}^{\text{new}} = \boldsymbol{A} + \mathbf{grad}\,\phi/\lambda \in L^2(\Omega)$. $\Box$

Something interesting in this proof is that as $\lambda \to 0$ the change we have to make to obtain $\boldsymbol{A}$ in the temporal gauge increases to infinity. This is because the temporal gauge requires time dependence of the vector potential $\boldsymbol{A}$ to achieve a non zero electric field. This comes into play later when we take the strong operator limit topology of Ampère's equation as $\lambda \to 0$. But note also that the term that blows up is a gradient, so that its curl is zero and $\mathbf{curl}\,\boldsymbol{A}$ remains the same (as it must as $\mathbf{curl}\,\boldsymbol{A} = \boldsymbol{B}$).

4.4 The equilibrium

Our equilibrium is $(\mu^{\pm}, \boldsymbol{E}^0, \boldsymbol{B}^0)$ which is assumed to be continuously differentiable and to depend only on the particle energies $e^{\pm}$ and momenta $\boldsymbol{p}^{\pm}$ which have the forms

$$e^{\pm} = \langle v \rangle \pm \phi^0(\boldsymbol{x}) \qquad\qquad \boldsymbol{p}^{\pm} = \boldsymbol{v} \pm \boldsymbol{A}^0(\boldsymbol{x})$$

where ϕ^0 and $\boldsymbol{A}^0$ are the equilibrium scalar and vector potentials, which are time independent and obey

$$\boldsymbol{E}^0 = -\boldsymbol{\nabla}_x \phi^0 \qquad\qquad \boldsymbol{B}^0 = \boldsymbol{\nabla}_x \times \boldsymbol{A}^0$$

We define the weights $w^{\pm}$ which depend only on $e^{\pm}$ by

$$w^{\pm} = c(1 + |e^{\pm}|)^{-\alpha}$$

with some constants $c > 0$ and $\alpha \geq d$ where d is the dimension of the velocity space, so that $\int w^{\pm}\, d\boldsymbol{v} < \infty$. We require that the derivatives of the equilibrium decays faster than $w^{\pm}$ in the sense that

$$|\mu_e^{\pm}| + |\boldsymbol{\mu}_p^{\pm}| \leq w^{\pm}$$

pointwise in $(e^{\pm}, \boldsymbol{p}^{\pm})$, which implies that $\int |\mu_e^{\pm}| + |\boldsymbol{\mu}_p^{\pm}|\, d\boldsymbol{v}$ is finite. For simplicity we will assume that $\mu^{\pm}$ has compact support in all the non-periodic $\boldsymbol{x}$ directions.

4.5 General set-up of spaces

Let $\Omega_x = \Omega_{x_1} \times \Omega_{x_2} \times \Omega_{x_3}$ be our x domain, with each Ω_i being either the real line $\mathbb{R}$ or the torus $\mathbb{T}$. Let $\mathfrak{C}$ be a set of smooth compactly supported functions from $\Omega_x \to \mathbb{R}$ which is closed under differentiation, and for each $u \in \mathfrak{C}$ we require that,

$$\mu_p^{\pm} \cdot \mathbf{grad}_x\, u = 0 \tag{4.4}$$

a condition which we have already imposed on the equilibrium in (2.8).

$$\mathfrak{C} = \left\{ u : \Omega_x \to \mathbb{R} : u \text{ smooth, compactly supported, } \mu_p^{\pm} \cdot \mathbf{grad}_x\, u = 0 \right\} \tag{4.5}$$

We then define the space of scalar fields $\mathfrak{H}$ as the closure of $\mathfrak{C}$ under the standard L^2 norm. We also define $\boldsymbol{\mathfrak{H}} = (\mathfrak{H})^3$ which is a space of vector fields. We denote their norms and inner products by $\|\cdot\|_{\mathfrak{H}}, \|\cdot\|_{\boldsymbol{\mathfrak{H}}}$, $\langle \cdot, \cdot \rangle_{\mathfrak{H}}$ and $\langle \cdot, \cdot \rangle_{\boldsymbol{\mathfrak{H}}}$.

To allow us to consider functions that do not decay at infinity we define the weighted spaces $\mathfrak{L}_{\pm}$ by closing a soon to be described set of smooth compactly supported functions under the weighted L^2 norm given by

$$\|u\|_{\mathfrak{L}_{\pm}}^2 = \int w^{\pm} |u|^2\, d\boldsymbol{v} d\boldsymbol{x}$$

and we denote the inner product by $\langle\cdot,\cdot\rangle_{\mathfrak{L}_\pm}$. The aforementioned space of compactly supported continuous functions is

$$\left\{u(\boldsymbol{x},\boldsymbol{v}):\Omega_x\times\mathbb{R}^3\to\mathbb{R}: u \text{ is smooth with compact support, } \forall\boldsymbol{v},\ u(\cdot,\boldsymbol{v})\in\mathfrak{C}\right\}$$

which means that $\mathfrak{L}_\pm$ inherits any $\boldsymbol{x}$ symmetry properties which were imposed on $\mathfrak{C}$. In particular we can interpret any function $u(\boldsymbol{x})\in\mathfrak{H}$ as being in $\mathfrak{L}_\pm$ by considering u as a function of $(\boldsymbol{x},\boldsymbol{v})$ which does not depend on $\boldsymbol{v}$.

As a notational shortcut we define the bilinear form on $\mathfrak{L}_\pm$,

$$\langle h,k\rangle_{\mu_e^\pm}=\int\mu_e^\pm hk\,d\boldsymbol{v}d\boldsymbol{x}$$

and the bilinear form defined on pairs of functions $h^\pm,k^\pm$ in $\mathfrak{L}_\pm$ respectively by

$$\left\langle h^\pm,k^\pm\right\rangle_{\mu_e}=\sum_\pm\left\langle h^\pm,k^\pm\right\rangle_{\mu_e^\pm}$$

and abuse notation so that $\langle h,k^\pm\rangle_{\mu_e}=\langle h^\pm,k^\pm\rangle_{\mu_e}$ with $h^\pm=h$, and similarly in the other position. We note that although these are not the standard inner products of $\mathfrak{L}_\pm$ (and not inner products at all), operators can still have 'adjoints with respect to $\langle\cdot,\cdot\rangle_{\mu_e^\pm}$'. We will later see that the resolvent operator $\mathcal{R}_\pm^{-\lambda}$ of $\mathcal{D}_\pm$ on $\mathfrak{L}_\pm$ has this property so that for any $h,k\in\mathfrak{L}_\pm$,

$$\left\langle\mathcal{R}_\pm^{-\lambda}h,k\right\rangle_{\mu_e^\pm}=\left\langle h,(\mathcal{R}_\pm^{-\lambda})^*k\right\rangle_{\mu_e^\pm}$$

where $(\mathcal{R}_\pm^{-\lambda})^*$ is the adjoint operator of $\mathcal{R}_\pm^{-\lambda}$ in $\mathfrak{L}_\pm$.

4.5.1 The 1.5d case

We represent the 1.5d case by taking $\Omega_{x_1}=\mathbb{R}$ or $\mathbb{T}$ with $\Omega_{x_2}=\Omega_{x_3}=\mathbb{T}$. The set $\mathfrak{C}$ will consist of smooth compactly supported functions $u(x_1,x_2,x_3)$ which only depend upon x_1. Our equilibrium $\mu^\pm$ will depend only on $p_2^\pm=\hat{v}_2+A_2^0(x_1)$ and $e^\pm=\langle v\rangle\pm\phi^0(x_1)$, so that in particular the requirement $\boldsymbol{\mu}_p^\pm\cdot\mathbf{grad}_x\,u=\mu_{p_2}^\pm\partial_{x_2}u=0$ for $u\in\mathfrak{C}$ holds. We normalise the integration with respect to x_2 and x_3 so that for $u\in\mathfrak{C}$, $\int u\,dx_i=u$ for $i=1,2$ and this integration thus plays no part.

It is interesting to note how the operators div and **curl** behave on $\mathfrak{H}$ in this case. For appropriately differentiable $\boldsymbol{u}=(u_1,u_2,0)\in\mathfrak{H}$ we have the formulae,

$$\begin{aligned}\mathbf{curl}\,\boldsymbol{u}&=(0,0,\partial_{x_1}u_2)\\ \operatorname{div}\boldsymbol{u}&=\partial_{x_1}u_1\\ \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}&=(0,-\partial_{x_1}^2u_2,0)\\ \mathbf{grad}\operatorname{div}\boldsymbol{u}&=(-\partial_{x_1}^2u_1,0,0)\end{aligned}\tag{4.6}$$

so that when we talk about the **curl** we are really talking about the derivatives of u_2 and when we talk about div we are talking about derivatives of u_1.

5 Conversion into a spectral problem

Our goal is to eliminate the linearised Vlasov equation and convert the problem of solving remaining system into finding a λ for which an operator has a non-trivial kernel.

5.1 Inverting the Linearised Vlasov Equation

Some vector calculus (see Lemma C.1) allows us to rewrite the linearised Vlasov equation in (4.3) to obtain,

$$(\lambda+\mathcal{D}_\pm)f^\pm = \pm(\lambda+\mathcal{D}_\pm)(\mu_e^\pm\phi+\boldsymbol{\mu_p^\pm}\cdot\boldsymbol{A})\pm\lambda\mu_e^\pm(-\phi+\hat{\boldsymbol{v}}\cdot\boldsymbol{A})\mp\boldsymbol{\mu_p^\pm}\cdot\boldsymbol{\nabla}_x(-\phi+\hat{\boldsymbol{v}}\cdot\boldsymbol{A}) \tag{5.1}$$

We have assumed that the third term on the right hand side is zero, so we have

$$(\lambda + \mathcal{D}_\pm)f^\pm = \pm(\lambda + \mathcal{D}_\pm)(\mu_e^\pm\phi + \boldsymbol{\mu_p^\pm} \cdot \boldsymbol{A}) \pm \lambda\mu_e^\pm(-\phi + \hat{\boldsymbol{v}} \cdot \boldsymbol{A}) \tag{5.2}$$

We now observe that $\mathcal{D}_\pm$ is skewadjoint on $\mathfrak{L}_\pm$, so its spectrum is purely imaginary and its resolvent set includes all negative real numbers (and indeed all positive real numbers). We denote the resolvent of $\mathcal{D}_\pm$ as

$$\mathcal{R}_\pm^\lambda = (\mathcal{D}_\pm - \lambda)^{-1} \tag{5.3}$$

which is defined at least for all $\operatorname{Im}\lambda \neq 0$ and maps $\mathfrak{L}_\pm \to \mathfrak{L}_\pm$. Applying $\mathcal{R}_\pm^{-\lambda}$ to (5.2) yields,

$$f^\pm = \pm(\mu_e^\pm\phi + \boldsymbol{\mu_p^\pm} \cdot \boldsymbol{A}) \pm \lambda\mathcal{R}_\pm^{-\lambda}[\mu_e^\pm(-\phi + \hat{\boldsymbol{v}} \cdot \boldsymbol{A})] \tag{5.4}$$

As a notational convenience we introduce the 4-potential $\underline{\boldsymbol{A}}$ and the 4-velocity $\underline{\hat{\boldsymbol{v}}}$, (note the underline),

$$\underline{\boldsymbol{A}} = (\phi, A_1, A_2, A_3) \qquad\qquad \underline{\hat{\boldsymbol{v}}} = (-1, \hat{v}_1, \hat{v}_2, \hat{v}_3) \tag{5.5}$$

This is linked to the notion of 4-vectors in relativity. We will not dwell on this, and instead regard it as a neat trick to expose symmetries. We note that $\underline{\hat{\boldsymbol{v}}} \cdot \underline{\boldsymbol{A}} = -\phi + \hat{\boldsymbol{v}} \cdot \boldsymbol{A}$. We can now define the operators $\mathcal{F}_\pm^\lambda : (\mathfrak{H}, \boldsymbol{\mathfrak{H}}) \to \mathfrak{L}_\pm$ which obtain these functions $f^\pm$ from the 4-potential $\underline{\boldsymbol{A}}$ and a given λ.

$$\mathcal{F}_\pm^\lambda\underline{\boldsymbol{A}} = \pm\mu_e^\pm\phi + \pm\boldsymbol{\mu_p^\pm} \cdot \boldsymbol{A} \pm \lambda\mathcal{R}_\pm^{-\lambda}[\mu_e^\pm\underline{\hat{\boldsymbol{v}}} \cdot \underline{\boldsymbol{A}}] \tag{5.6}$$

By the decay of the equilibrium and the boundedness of the resolvent, it is clear that $\mathcal{F}_\pm^\lambda$ is a bounded linear operator.

As a convenience, we define another pair of operators $\mathcal{Q}_\pm^\lambda : \mathfrak{L}_\pm \to \mathfrak{L}_\pm$ as $\lambda\mathcal{R}_\pm^{-\lambda}$ and rewrite $\mathcal{F}_\pm^\lambda$ as

$$\mathcal{F}_\pm^\lambda\underline{\boldsymbol{A}} = \pm\mu_e^\pm\phi \pm \boldsymbol{\mu_p^\pm} \cdot \boldsymbol{A} \pm \mathcal{Q}_\pm^\lambda[\mu_e^\pm\underline{\hat{\boldsymbol{v}}} \cdot \underline{\boldsymbol{A}}] \tag{5.7}$$

5.2 The Current Density and Charge Density Operators

Now we have an expression for $f^\pm$ in terms of $\underline{\boldsymbol{A}}$, we can calculate the charge and current densities ρ and $\boldsymbol{j}$. For $\lambda > 0$, we define the operators $\boldsymbol{\mathcal{J}}^\lambda : (\mathfrak{H}, \boldsymbol{\mathfrak{H}}) \to \boldsymbol{\mathfrak{H}}$ and $\mathcal{P}^\lambda : (\mathfrak{H}, \boldsymbol{\mathfrak{H}}) \to \mathfrak{H}$ which map the candidate vector potential $\underline{\boldsymbol{A}}$ to these densities. They have the rules,

$$
\begin{aligned}
\boldsymbol{\mathcal{J}}^\lambda \underline{\boldsymbol{A}} &= \int \hat{\boldsymbol{v}}(\boldsymbol{\mathcal{F}}_+^\lambda - \boldsymbol{\mathcal{F}}_-^\lambda)\underline{\boldsymbol{A}}\, d\boldsymbol{v} \\
&= \sum_\pm \int \mu_e^\pm \hat{\boldsymbol{v}} \phi \, d\boldsymbol{v} + \sum_\pm \int \hat{\boldsymbol{v}}(\boldsymbol{\mu}_{\boldsymbol{p}}^\pm \cdot \boldsymbol{A})\, d\boldsymbol{v} \\
&\quad + \sum_\pm \int \hat{\boldsymbol{v}} \mathcal{Q}_\pm^\lambda [\mu_e^\pm \underline{\hat{\boldsymbol{v}}} \cdot \underline{\boldsymbol{A}}]\, d\boldsymbol{v}
\end{aligned}
\tag{5.8}
$$

$$
\begin{aligned}
\mathcal{P}^\lambda \underline{\boldsymbol{A}} &= \int (\boldsymbol{\mathcal{F}}_+^\lambda - \boldsymbol{\mathcal{F}}_-^\lambda)\underline{\boldsymbol{A}}\, d\boldsymbol{v} \\
&= \sum_\pm \int \mu_e^\pm \phi \, d\boldsymbol{v} + \sum_\pm \int \boldsymbol{\mu}_{\boldsymbol{p}}^\pm \cdot \boldsymbol{A}\, d\boldsymbol{v} + \sum_\pm \int \mathcal{Q}_\pm^\lambda [\mu_e^\pm \underline{\hat{\boldsymbol{v}}} \cdot \underline{\boldsymbol{A}}]\, d\boldsymbol{v}
\end{aligned}
\tag{5.9}
$$

We now define the 4-current as $\underline{\boldsymbol{j}} = (-\rho, \boldsymbol{j}) = (-\rho, j_1, j_2, j_3)$, and then we can produce the simple expression for $\underline{\boldsymbol{j}}$,

$$
\underline{\boldsymbol{\mathcal{J}}}^\lambda \underline{\boldsymbol{A}} = \left(\sum_\pm \int (\underline{\hat{\boldsymbol{v}}} \otimes (\mu_e^\pm, \boldsymbol{\mu}_{\boldsymbol{p}}^\pm))\, d\boldsymbol{v} \right) \underline{\boldsymbol{A}} + \sum_\pm \int \underline{\hat{\boldsymbol{v}}} \mathcal{Q}_\pm^\lambda [\mu_e^\pm \underline{\hat{\boldsymbol{v}}} \cdot \underline{\boldsymbol{A}}]\, d\boldsymbol{v} \tag{5.10}
$$

If we impose the temporal gauge then $\phi = 0$ and the operators $\mathcal{F}_\pm^\lambda, \boldsymbol{\mathcal{J}}^\lambda$ and $\mathcal{P}^\lambda$ can be viewed as mapping from $\mathfrak{H}$ only. We will take this as the formal definition of the operators, which now have the rules,

$$
\mathcal{F}_\pm^\lambda \boldsymbol{A} = \pm \boldsymbol{\mu}_{\boldsymbol{p}}^\pm \cdot \boldsymbol{A} \pm \lambda \mathcal{R}_\pm^{-\lambda} [\mu_e^\pm \hat{\boldsymbol{v}} \cdot \boldsymbol{A}] \tag{5.11}
$$

$$
\boldsymbol{\mathcal{J}}^\lambda \boldsymbol{A} = \sum_\pm \int \hat{\boldsymbol{v}}(\boldsymbol{\mu}_{\boldsymbol{p}}^\pm \cdot \boldsymbol{A})\, d\boldsymbol{v} + \sum_\pm \int \hat{\boldsymbol{v}} \mathcal{Q}_\pm^\lambda [\mu_e^\pm \hat{\boldsymbol{v}} \cdot \boldsymbol{A}]\, d\boldsymbol{v} \tag{5.12}
$$

$$
\mathcal{P}^\lambda \boldsymbol{A} = \sum_\pm \int \boldsymbol{\mu}_{\boldsymbol{p}}^\pm \cdot \boldsymbol{A}\, d\boldsymbol{v} + \sum_\pm \int \mathcal{Q}_\pm^\lambda [\mu_e^\pm \hat{\boldsymbol{v}} \cdot \boldsymbol{A}]\, d\boldsymbol{v} \tag{5.13}
$$

Although we impose this, the later results Lemma 5.4 and the first part of Lemma 5.5 which shows that the continuity equation is satisfied, will still hold in any gauge.

5.3 Properties of $\mathcal{Q}_\pm^\lambda$ and $\boldsymbol{\mathcal{J}}^\lambda$

$\mathcal{D}_\pm$ being skew adjoint gives very good control over its resolvent $\mathcal{R}_\pm^\lambda$ and $\mathcal{Q}_\pm^\lambda = \lambda \mathcal{R}_\pm^{-\lambda}$.

Lemma 5.1. *In the respective spaces $\mathfrak{L}_\pm$, $\mathcal{Q}_\pm^\lambda$ obeys:*

(a) $\left\| \mathcal{Q}_\pm^\lambda \right\|_{\mathfrak{L}_\pm} = 1$.

(b) For $\lambda > 0$, $\mathcal{Q}_\pm^\lambda$ is continuous as a function of λ in operator norm.

(c) As $\lambda \to \infty$, $\mathcal{Q}_\pm^\lambda$ converges in the strong operator topology to the identity.

(d) As $\lambda \to 0$, $\mathcal{Q}_\pm^\lambda$ converges strongly to the projection operator onto $\ker(\mathcal{D}_\pm)$.

(e) *For any $h \in \mathfrak{L}_\pm$ and $k \in \mathfrak{D}(\mathcal{D}_\pm)$, we have the relations for $\lambda > 0$,*

$$\mathcal{R}_\pm^{-\lambda}[hk] = k\mathcal{R}_\pm^{-\lambda}h \qquad\qquad \mathcal{Q}_\pm^{\lambda}[hk] = k\mathcal{Q}_\pm^{\lambda}h \tag{5.14}$$

(f) *For any $h, k \in \mathfrak{L}_\pm$, $\mathcal{R}_\pm^{-\lambda}$ has the same adjoint $(\mathcal{R}_\pm^{-\lambda})^*$ (the $\mathfrak{L}_\pm$ adjoint) with respect to the bilinear form $\langle\cdot,\cdot\rangle_{\mu_e^\pm}$. By which we mean that for any $h, k \in \mathfrak{L}_\pm$,*

$$\left\langle \mathcal{R}_\pm^{-\lambda}h, k\right\rangle_{\mu_e^\pm} = \left\langle h, (\mathcal{R}_\pm^{-\lambda})^* k\right\rangle \tag{5.15}$$

This also applies to the form $\langle\cdot,\cdot\rangle_{\mu_e}$.

Proof. As $\mathcal{D}_\pm$ is skew-adjoint, $i\mathcal{D}_\pm$ is selfadjoint, and we may apply the result of [8, V.3.5]. to see that

$$\left\|\mathcal{R}_\pm^{\lambda}\right\|_{\mathfrak{L}_\pm} = \frac{1}{|\lambda|}$$

because the nearest point of the spectrum of $\mathcal{D}_\pm$ is 0. (a) then follows immediately. (b) follows from the analyticity of resolvents as functions of λ. For (c) we compute for $u \in \mathfrak{D}(\mathcal{D}_\pm)$:

$$\begin{aligned}\left\|\mathcal{Q}_\pm^{\lambda}u - u\right\|_{\mathfrak{L}_\pm} &= \left\|\lambda\mathcal{R}_\pm^{-\lambda}u - u\right\|_{\mathfrak{L}_\pm} = \left\|\mathcal{D}_\pm\mathcal{R}_\pm^{-\lambda}u\right\|_{\mathfrak{L}_\pm} \\ &= \left\|\mathcal{R}_\pm^{-\lambda}\mathcal{D}_\pm u\right\|_{\mathfrak{L}_\pm} \leq \frac{1}{\lambda}\left\|\mathcal{D}_\pm u\right\|_{\mathfrak{L}_\pm} \to 0 \quad \text{as } \lambda \to \infty\end{aligned}$$

and extend to arbitrary $u \in \mathfrak{L}_\pm$ by the density of $\mathfrak{D}(\mathcal{D}_\pm)$.

For (d) we introduce the spectral measure of $i\mathcal{D}_\pm$, denoted $M_\pm$. The projection onto $\ker(\mathcal{D}_\pm)$ is then $M_\pm(\{0\}) = \int_{\mathbb{R}} \chi\, dM_\pm$ where $\chi(0) = 1$ and $\chi = 0$ otherwise. Observe that $\mathcal{Q}_\pm^{\lambda} = \lambda\mathcal{R}_\pm^{-\lambda} = \int_{\mathbb{R}} \frac{\lambda}{\lambda+i\alpha}\, dM_\pm(\alpha)$. We compute for $u \in \mathfrak{L}_\pm$,

$$\begin{aligned}\left\|\mathcal{Q}_\pm^{\lambda}u - M_\pm(\{0\})u\right\|_{\mathfrak{L}_\pm}^2 &= \left\|\int_{\mathbb{R}} \frac{\lambda}{\lambda+i\alpha} - \chi(\alpha)\, dM_\pm(\alpha)u\right\|_{\mathfrak{L}_\pm}^2 \\ &= \int_{\mathbb{R}} \left|\frac{\lambda}{\lambda+i\alpha} - \chi(\alpha)\right|^2 d\left\|M_\pm(\alpha)u\right\|_{\mathfrak{L}_\pm}^2\end{aligned}$$

due to orthogonality of spectral projections. This now tends to 0 as $\lambda \to 0$ by dominated convergence.

For (e) we note that $\mathcal{R}_\pm^{-\lambda}[hk]$ is the element $u \in \mathfrak{L}_\pm$ that solves $\mathcal{D}_\pm u + \lambda u = hk$, and in the same way $\mathcal{R}_\pm^{-\lambda}h$ is the element $m \in \mathfrak{L}_\pm$ that solves $\mathcal{D}_\pm m + \lambda m = h$. Multiplying this second equation by k we deduce that $\mathcal{R}_\pm^{-\lambda}[hk] = k\mathcal{R}_\pm^{-\lambda}h$. The equivalent relation for $\mathcal{Q}_\pm^{\lambda}$ follows immediately from the definition of $\mathcal{Q}_\pm^{\lambda}$.

For (f) we recall that $(\mathcal{R}_\pm^{-\lambda})^*$ also has property (e) due to its relation to $\mathcal{R}_\pm^{-\lambda}$. (f) then follows from simple calculation. □

We can now extend most of these properties to the current operator $\boldsymbol{\mathcal{J}}^\lambda$.

Lemma 5.2. *The operator $\boldsymbol{\mathcal{J}}^\lambda$ has the additional properties:*

(i) *The family $(\boldsymbol{\mathcal{J}}^\lambda)_{0<\lambda<\infty}$ is uniformly bounded, i.e. $\sup_{0<\lambda<\infty}\left\|\boldsymbol{\mathcal{J}}^\lambda\right\|_{\mathfrak{H}} < \infty$.*

(ii) For $\lambda > 0$, $\boldsymbol{\mathcal{J}}^\lambda$ is a continuous function of λ in the operator norm topology on $\mathfrak{H}$.

(iii) $\boldsymbol{\mathcal{J}}^\lambda$ as $\lambda \to 0$ has a bounded limit $\boldsymbol{\mathcal{J}}^0$ in the strong operator topology on $\mathfrak{H}$.

Furthermore these properties also hold for $\mathcal{P}^\lambda$ with the corresponding norm and strong topologies.

Proof. We rely on Lemma 5.1. Each of (i), (ii) and (iii) follow from the corresponding results for $\mathcal{Q}^\lambda_\pm$. □

For completeness we now note the rules for the bounded operators $\boldsymbol{\mathcal{J}}^0$ and $\mathcal{P}^0$, where we denote the projection onto $\ker(\mathcal{D}_\pm)$ by $\mathcal{Q}^0_\pm$.

$$\boldsymbol{\mathcal{J}}^0 \boldsymbol{A} = \sum_\pm \int \hat{\boldsymbol{v}}(\boldsymbol{\mu}^\pm_{\boldsymbol{p}} \cdot \boldsymbol{A})\, d\boldsymbol{v} + \sum_\pm \int \hat{\boldsymbol{v}} \mathcal{Q}^0_\pm[\mu^\pm_e \hat{\boldsymbol{v}} \dot{\boldsymbol{A}}]\, d\boldsymbol{v} \tag{5.16}$$

$$\mathcal{P}^0 \boldsymbol{A} = \sum_\pm \int \boldsymbol{\mu}^\pm_{\boldsymbol{p}} \cdot \boldsymbol{A}\, d\boldsymbol{v} + \sum_\pm \int \mathcal{Q}^0_\pm[\mu_e \hat{\boldsymbol{v}} \cdot \boldsymbol{A}]\, d\boldsymbol{v} \tag{5.17}$$

5.3.1 Tightness estimate on $\boldsymbol{\mathcal{J}}^\lambda$

To control the growth of our solutions $\boldsymbol{A}$ as $|\boldsymbol{x}| \to \infty$ we will use a tightness estimate on $\boldsymbol{\mathcal{J}}^\lambda$.

Lemma 5.3. *The operator $\boldsymbol{\mathcal{J}}^\lambda$ maps subsets of $\mathfrak{D}(\boldsymbol{\mathcal{E}})$ whose elements $\boldsymbol{A}$ have $\boldsymbol{A}, \mathbf{curl}\, \boldsymbol{A}, \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{A}, \operatorname{div} \boldsymbol{A}$ all bounded uniformly in $\mathfrak{H}, \mathfrak{H}$ respectively, into tight subsets of $\mathfrak{H}$, and the tightness is uniform for bounded sets of positive λ.*

Proof. μ has compact support in $\boldsymbol{x}$ so the result is immediate as $\boldsymbol{\mathcal{J}}^\lambda \boldsymbol{A}$ will also have compact support in $\boldsymbol{x}$ independent of λ, and $\boldsymbol{\mathcal{J}}^\lambda$ is a bounded operator. □

5.3.2 Relation of $\mathcal{Q}^\lambda_\pm$ to prior work

Operators similar to $\mathcal{Q}^\lambda_\pm$ appeared in the prior paper by Lin [9] and later explicitly named $\mathcal{Q}^\lambda_\pm$ in the papers of Lin and Strauss [10, 11] and then Ben-Artzi [1, 2]. The definition in these works has a different but equivalent definition. When translated into our setup, they define $\mathcal{Q}^\lambda_\pm : \mathfrak{L}_\pm \to \mathfrak{L}_\pm$ by the rule,

$$(\mathcal{Q}^\lambda_\pm h)(\boldsymbol{x}, \boldsymbol{v}) = \int_{-\infty}^0 \lambda e^{\lambda s} h(\boldsymbol{X}^\pm(s; \boldsymbol{x}, \boldsymbol{v}), \boldsymbol{V}^\pm(s; \boldsymbol{x}, \boldsymbol{v}))\, ds \tag{5.18}$$

where $\boldsymbol{X}^\pm, \boldsymbol{V}^\pm$ are the characteristics of the equilibrium system defined by (2.6) with $\boldsymbol{E}, \boldsymbol{B}$ replaced by the equilibrium fields $\boldsymbol{E}^0, \boldsymbol{B}^0$. In functional analytic terms evaluating h at $(\boldsymbol{X}^\pm(-s; \boldsymbol{x}, \boldsymbol{v}), \boldsymbol{V}^\pm(-s; \boldsymbol{x}, \boldsymbol{v}))$ corresponds to the action of the semigroup of bounded linear operators on $\mathfrak{L}_\pm$ generated by the operator $\mathcal{D}_\pm$. That this semigroup exists (and is in fact a one-parameter strongly continuous unitary group) follows from Stone's theorem as $i\mathcal{D}_\pm$ is selfadjoint. Denoting this semigroup by $(e^{s\mathcal{D}_\pm})_{s\in\mathbb{R}}$ the definition (5.18) becomes,

$$\mathcal{Q}^\lambda_\pm h = \int_{-\infty}^0 \lambda e^{\lambda s} e^{-s\mathcal{D}_\pm} h\, ds = \lambda \int_0^\infty e^{-\lambda s} e^{s\mathcal{D}_\pm} h\, ds \tag{5.19}$$

which is λ times the Laplace transform of the semigroup. The existence of the Laplace transform is guaranteed as $e^{s\mathcal{D}_\pm}$ is unitary, so has operator norm 1. The equivalence of this definition to our definition as $\lambda(\mathcal{D}_\pm+\lambda)^{-1}$ now follows from the general fact that for any strongly continuous semigroup the resolvent (at $-\lambda$) of the generator is equal to the Laplace transform (at λ) of the semigroup. Hence for the Q defined by (5.18), we have $\mathcal{Q}^\lambda_\pm = \lambda(\mathcal{D}_\pm-\lambda)^{-1}$ which is exactly our definition.

The reason we never need introduce this more explicit definition of $\mathcal{Q}^\lambda_\pm$ is that for our purposes $\mathcal{Q}^\lambda_\pm$ is viewed as a black box operator whose only properties that matter are those proved in Lemma 5.1 all of which follow from its definition in terms of the resolvent.

5.4 Recovery of the Linearised Vlasov and Continuity Equations

The recovery of the linearised Vlasov equation follows essentially by definition of the operator $\mathcal{F}^\lambda_\pm$. The more interesting thing is that the continuity equation can also be recovered without any reference to Maxwell's equations. This will be of key importance later.

Lemma 5.4. *Let $\boldsymbol{A}\in\boldsymbol{\mathfrak{H}}$, $\lambda>0$ and $f^\pm=\mathcal{F}^\lambda_\pm\boldsymbol{A}$, then the pair $(f^\pm,\boldsymbol{A})$ is a distributional solution of the linearised Vlasov equation:*

$$(\mathcal{D}_\pm+\lambda)f^\pm = \pm(\lambda\boldsymbol{A}-\hat{\boldsymbol{v}}\times\boldsymbol{\nabla}_x\times\boldsymbol{A})\cdot\boldsymbol{\nabla}_v\mu^\pm \tag{5.20}$$

Proof. By Lemma C.1, we have to show that

$$(\mathcal{D}_\pm+\lambda)f^\pm \mp (\mathcal{D}_\pm+\lambda)(\boldsymbol{\mu}^\pm_{\boldsymbol{p}}\cdot\boldsymbol{A}) \mp \lambda\mu_e\hat{\boldsymbol{v}}\cdot\boldsymbol{A}=0 \tag{5.21}$$

in the sense of distributions. But because $f^\pm=\pm\boldsymbol{\mu}^\pm_{\boldsymbol{p}}\cdot\boldsymbol{A}\pm\mu_e\mathcal{Q}^\lambda_\pm[\hat{\boldsymbol{v}}\cdot\boldsymbol{A}]$, the first part of $f^\pm$ cancels exactly the second term in (5.21). Fix a test function $g(\boldsymbol{x},\boldsymbol{v})$, we denote the (unweighted) action of a distribution on a test function by $\langle\cdot,\cdot\rangle$. We compute,

$$\begin{aligned}\pm\left\langle(\mathcal{D}_\pm+\lambda)(\mu_e\mathcal{Q}^\lambda_\pm[\hat{\boldsymbol{v}}\cdot\boldsymbol{A}]),g\right\rangle &= \pm\left\langle(\mathcal{D}_\pm+\lambda)\mathcal{Q}^\lambda_\pm[\hat{\boldsymbol{v}}\cdot\boldsymbol{A}],g\right\rangle_{\mu_e^\pm} &(5.22)\\ &=\pm\lambda\left\langle\hat{\boldsymbol{v}}\cdot\boldsymbol{A},g\right\rangle_{\mu_e^\pm} &(5.23)\end{aligned}$$

which is exactly the action of $\pm\lambda\mu_e\hat{\boldsymbol{v}}\cdot\boldsymbol{A}$ on g, completing the proof. ☐

Lemma 5.5. *Let $\boldsymbol{A}\in\boldsymbol{\mathfrak{H}}$, $\lambda\geq 0$ then $\boldsymbol{A}$ satisfies the continuity equation* $\operatorname{div}\boldsymbol{j}+\lambda\rho=0$ *in the sense of distributions, where $\boldsymbol{j}=\boldsymbol{\mathcal{J}}^\lambda\boldsymbol{A}$ and $\rho=\mathcal{P}^\lambda\boldsymbol{A}$. In particular the operator* $\operatorname{div}\boldsymbol{\mathcal{J}}^\lambda$ *is bounded from $\boldsymbol{\mathfrak{H}}$ to $\mathfrak{H}$ with $\left\|\operatorname{div}\boldsymbol{\mathcal{J}}^\lambda\right\|_{\boldsymbol{\mathfrak{H}}\to\mathfrak{H}}=\lambda\left\|\mathcal{P}^\lambda\right\|_{\boldsymbol{\mathfrak{H}}\to\mathfrak{H}}$ which is uniform over compact sets of λ.*

Proof. It suffices to deal with $\lambda>0$, as we can then deduce the result for $\lambda=0$ by taking limits in the strong operator topology. Indeed, if we have $\operatorname{div}\boldsymbol{\mathcal{J}}^\lambda\boldsymbol{A}=-\lambda\mathcal{P}^\lambda\boldsymbol{A}$ for each $\boldsymbol{A}$, then by taking the strong limit as $\lambda\to 0$ we obtain that $\operatorname{div}\boldsymbol{\mathcal{J}}^0\boldsymbol{A}=0$ for any $\boldsymbol{A}$. In fact, due to the uniform norm bound on $\mathcal{P}^\lambda$ we obtain that $\operatorname{div}\boldsymbol{\mathcal{J}}^\lambda\to 0$ in the norm operator topology.

We apply Lemma 5.4 and integrate (5.20) with respect to $\boldsymbol{v}$ and then sum over $\pm$. The left hand side gives exactly $\operatorname{div}\boldsymbol{\mathcal{J}}^\lambda\boldsymbol{A}+\mathcal{P}^\lambda\boldsymbol{A}$, while the right hand side integrates to zero because $\boldsymbol{A}$ is independent of $\boldsymbol{v}$ and therefore $\hat{\boldsymbol{v}} \times (\boldsymbol{\nabla}_x \times \boldsymbol{A})$ has zero $\boldsymbol{v}$ divergence, so we can express,

$$(\hat{\boldsymbol{v}} \times \boldsymbol{\nabla}_x \times \boldsymbol{A}) \cdot \boldsymbol{\nabla}_v \mu^{\pm} = \boldsymbol{\nabla}_v \cdot ((\hat{\boldsymbol{v}} \times \boldsymbol{\nabla}_x \times \boldsymbol{A})\mu^{\pm}) \tag{5.24}$$

which integrates to zero with respect to $\boldsymbol{v}$.

We now note that $\mathcal{P}^\lambda$ is a bounded operator $\mathfrak{H} \to \mathfrak{H}$ so $\operatorname{div} \boldsymbol{\mathcal{J}}^\lambda = -\lambda \mathcal{P}^\lambda$ is also bounded. The uniform bound for compact sets of λ follows immediately from the uniformity of the bound for $\mathcal{P}^\lambda$ over λ. □

5.5 Ampère's Equation

Ampère's equation is now $\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{A} + \lambda^2 \boldsymbol{A} = \boldsymbol{\mathcal{J}}^\lambda \boldsymbol{A}$, we define formally the operator $\boldsymbol{\mathcal{M}}^\lambda$ for $\lambda > 0$, by

$$\boldsymbol{\mathcal{M}}^\lambda \boldsymbol{A} = \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{A} + \lambda^2 \boldsymbol{A} - \boldsymbol{\mathcal{J}}^\lambda \boldsymbol{A}.$$

Then finding a solution to Ampère's equation is equivalent to finding a non-trivial kernel for $\boldsymbol{\mathcal{M}}^\lambda$. We now give a rigourous definition of $\boldsymbol{\mathcal{M}}^\lambda$.

Lemma 5.6. *The operator* $\mathbf{curl}\,\mathbf{curl} : \mathfrak{H} \to \mathfrak{H}$ *with smooth compactly supported functions in* $\mathfrak{H}$ *as a core is selfadjoint and non-negative.*

Proof. By the properties of derivatives in $\mathfrak{H}$ it suffices to show this for $\mathbf{curl}\,\mathbf{curl} : L^2(\mathbb{R}^3;\mathbb{R}^3) \to L^2(\mathbb{R}^3;\mathbb{R}^3)$. We first define $\mathcal{T}$ as the action of $\mathbf{curl}\,\mathbf{curl}$ on the Schwartz space. Under the Fourier transform $\mathcal{U}$, which is a unitary operator, we can express $\mathcal{T} = \mathcal{U}^{-1}\mathcal{S}\mathcal{U}$ where $\mathcal{S}$ is a real multiplication operator. Real multiplication operators are always essentially selfadjoint, and as $\mathcal{U}$ is a unitary isomorphism, $\mathcal{T}$ is essentially selfadjoint also. Taking the resulting selfadjoint operator as $\mathbf{curl}\,\mathbf{curl}$ we see that clearly the Schwartz space is a core, and hence so are the smooth compactly supported functions. The non-negativity follows from integrating by parts. □

Building $\boldsymbol{\mathcal{M}}^\lambda$ now amounts to adding bounded operators to $\mathbf{curl}\,\mathbf{curl}$.

Lemma 5.7. *The operator* $\boldsymbol{\mathcal{M}}^\lambda = \mathbf{curl}\,\mathbf{curl} + \lambda^2 - \boldsymbol{\mathcal{J}}^\lambda$ *defined for* $\lambda \geq 0$ *is closed with the same domain as* $\mathbf{curl}\,\mathbf{curl}$*. Furthermore the family* $(\boldsymbol{\mathcal{M}}^\lambda)_{\lambda \geq 0}$ *is continuous in the strong operator topology when restricted to* $\mathfrak{D}(\boldsymbol{\mathcal{M}}^\lambda)$*.*

Proof. The closure and domain follow immediately as $\boldsymbol{\mathcal{J}}^\lambda$ is a bounded operator. The continuity in the strong operator topology follows from the continuity in strong operator topology of $(\boldsymbol{\mathcal{J}}^\lambda)_{\lambda \geq 0}$ proved in Lemma 5.2. □

5.6 Recovery of Gauss's Equation

Our first use of the continuity equation obtained by Lemma 5.5 is to show that Gauss' equation is satisfied automatically whenever we have Ampère's equation.

Recall that in the temporal gauge Gauss's equation is:

$$\lambda \operatorname{div} \boldsymbol{A} = -\rho \tag{5.25}$$

Lemma 5.8. *Let* $\lambda > 0$ *and assume that* $\boldsymbol{A} \in \mathfrak{H}$ *is in the kernel of* $\boldsymbol{\mathcal{M}}^\lambda$*. Then* (5.25) *holds in the sense of weak derivatives, where* $\rho = \mathcal{P}^\lambda \boldsymbol{A}$*.*

Proof. By taking the divergence of $\boldsymbol{\mathcal{M}}^\lambda \boldsymbol{A} = 0$ we obtain using $\operatorname{div} \mathbf{curl} = 0$,

$$\lambda^2 \operatorname{div} \boldsymbol{A} - \operatorname{div} \boldsymbol{\mathcal{J}}^\lambda \boldsymbol{A} = 0 \tag{5.26}$$

in the sense of distributions. But as $\operatorname{div} \boldsymbol{\mathcal{J}}^\lambda$ is a bounded operator, and $\lambda > 0$, $\operatorname{div} \boldsymbol{A}$ exists in the sense of weak derivatives. Applying Lemma 5.5, we obtain the result after dividing by λ. □

5.7 Regularisation

We have now shown that if we can find $\lambda > 0$ and a solution $\boldsymbol{A} \in \boldsymbol{\mathfrak{H}}$ of $\boldsymbol{\mathcal{M}}^\lambda \boldsymbol{A} = 0$, then we can recover all the equations of linearised RVM, with this in mind we now turn all our attention to $\boldsymbol{\mathcal{M}}^\lambda$. We would like $\boldsymbol{\mathcal{M}}^\lambda$ to have discrete spectrum, but this is in general not true. Instead we consider the operator $\boldsymbol{\mathcal{M}}_\epsilon^\lambda$ which we define for $\epsilon, \lambda > 0$ by the rule

$$\boldsymbol{\mathcal{M}}_\epsilon^\lambda = \mathbf{curl}\,\mathbf{curl} + \lambda^2 + \epsilon \boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{J}}^\lambda$$

where $\boldsymbol{\mathcal{E}}$ is a perturbation to make $\boldsymbol{\mathcal{M}}_\epsilon^\lambda$ have discrete spectrum. We define $\boldsymbol{\mathcal{E}}$ in the following manner. Let $W(\boldsymbol{x}) \in L^1_{loc}(\Omega_x)$ be a smooth scalar function that has $W \geq 1$ and $|\,\mathbf{grad}\, W| \leq 1$ and if Ω_x is not $\mathbb{T}^3$ then $W \to \infty$ as $|\boldsymbol{x}| \to \infty$. If $\Omega_x = \mathbb{T}^3$ we simply take $W = 0$, as we do not need it as $H^2(\mathbb{T}^3)$ is compactly contained in $L^2(\mathbb{T}^3)$. Further, let W have the same symmetry properties as $\mathfrak{H}$, by which we mean that if $\varphi \in \mathfrak{H}$ with compact support, then $W\varphi \in \mathfrak{H}$. We now define

$$\boldsymbol{\mathcal{E}} = -\boldsymbol{\Delta} + W$$

where $\boldsymbol{\Delta}$ is the vector Laplacian. Recall that the vector Laplacian in 3 dimensions is defined by

$$-\boldsymbol{\Delta} = \mathbf{curl}\,\mathbf{curl} - \mathbf{grad}\operatorname{div} \tag{5.27}$$

By a simple extension of [14, Theorem XIII.67.] and [8, VI.4.3.] to the case of the vector Laplacian, we obtain that on $L^2(\Omega_x; \mathbb{R}^3) \to L^2(\Omega_x; \mathbb{R}^3)$, $\boldsymbol{\mathcal{E}}$ is selfadjoint with compact resolvent and discrete spectrum, and that smooth compactly supported functions are a core of $\boldsymbol{\mathcal{E}}$. By restriction to $\boldsymbol{\mathfrak{H}}$ using the assumption that $\boldsymbol{\Delta}$ maps (appropriately differentiable) functions in $\mathfrak{H}$ to $\mathfrak{H}$, we obtain that $\boldsymbol{\mathcal{E}} : \mathfrak{H} \to \mathfrak{H}$ is selfadjoint with discrete spectrum (as Ω_x has no boundary), and again smooth compactly supported functions in $\boldsymbol{\mathfrak{H}}$ form a core. Its domain is thus the closure of the smooth compactly supported functions in $\boldsymbol{\mathfrak{H}}$ under its graph norm

$$\|\boldsymbol{u}\|_{\boldsymbol{\mathcal{E}}}^2 = \|\boldsymbol{u}\|_{\boldsymbol{\mathfrak{H}}}^2 + \|\boldsymbol{\mathcal{E}} \boldsymbol{u}\|_{\boldsymbol{\mathfrak{H}}}^2 .$$

We must now show that indeed $\boldsymbol{\mathcal{M}}_\epsilon^\lambda$ is closed with discrete spectrum. The method of proof is slightly perverse. We consider $\boldsymbol{\mathcal{M}}_\epsilon^\lambda$ as a perturbation of $\boldsymbol{\mathcal{E}}$, with the size of the perturbation increasing as $\epsilon \to 0$, and blowing up as it does so. This is not a problem as we only want that $\boldsymbol{\mathcal{M}}_\epsilon^\lambda$ has discrete spectrum for small positive ϵ. We don't care at all about what the graph norm of $\boldsymbol{\mathcal{M}}_\epsilon^\lambda$ looks like, or any other such properties. Indeed, after we have proved that $\boldsymbol{\mathcal{M}}_\epsilon^\lambda$ has pure point spectrum, ϵ becomes a hinderence, producing terms that must be bounded independently from ϵ, despite their ϵ dependence.

Lemma 5.9. *For $0 < \epsilon < 1$ and $\lambda \geq 0$, $\boldsymbol{\mathcal{M}}_\epsilon^\lambda$ is closed with discrete spectrum, compact resolvent, and has the same domain as $\boldsymbol{\mathcal{E}}$.*

Proof. We now wish to show that $\boldsymbol{\mathcal{M}}_\epsilon^\lambda$ has the same essential spectrum (i.e. none) as $\boldsymbol{\mathcal{E}}$. To start with we show that $\mathbf{curl}\,\mathbf{curl} + \epsilon\boldsymbol{\mathcal{E}}$ is relatively bounded with respect to $\boldsymbol{\mathcal{E}}$, with bound less than 1. Expanding the vector Laplacian,

$$\mathbf{curl}\,\mathbf{curl} + \epsilon\boldsymbol{\mathcal{E}} = (1-\epsilon)\,\mathbf{curl}\,\mathbf{curl} - \epsilon\,\mathbf{grad}\,\mathrm{div} + \epsilon W \tag{5.28}$$

we immediately see that we may take the bound to be $1-\epsilon$. Thus by Theorem B.2, $\mathbf{curl}\,\mathbf{curl} + \epsilon\boldsymbol{\mathcal{E}}$ has discrete spectrum, is selfadjoint and has the same domain as $\boldsymbol{\mathcal{E}}$. Now we move on to $\boldsymbol{\mathcal{J}}^\lambda$. We will show that $\boldsymbol{\mathcal{J}}^\lambda$ is relatively compact with respect to $\mathbf{curl}\,\mathbf{curl} + \epsilon\boldsymbol{\mathcal{E}}$, which is the same thing as $\boldsymbol{\mathcal{J}}^\lambda\mathcal{R}(\zeta;\boldsymbol{\mathcal{E}})$ being compact for some ζ where $\mathcal{R}(\zeta;\boldsymbol{\mathcal{E}})$ is the resolvent of $\boldsymbol{\mathcal{E}}$. But this is clearly the case as $\boldsymbol{\mathcal{E}}$ has compact resolvent and $\boldsymbol{\mathcal{J}}^\lambda$ is bounded. Then we apply Theorem B.3 to see that indeed $\mathbf{curl}\,\mathbf{curl} + \epsilon\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{J}}^\lambda$ has discrete spectrum and has the same domain as $\boldsymbol{\mathcal{E}}$. Finally we add the λ^2 which trivially does not change the domain or the lack of essential spectrum, to show that $\boldsymbol{\mathcal{M}}_\epsilon^\lambda$ has discrete spectrum and the same domain as $\boldsymbol{\mathcal{E}}$, and as it has the same domain as $\boldsymbol{\mathcal{E}}$ its resolvent must be compact as it maps into a relatively compact subset of $\boldsymbol{\mathfrak{H}}$. □

6 Estimates on Ampère's Equation

We now have to show that we can remove the regularisation. We want to find solutions to $\boldsymbol{\mathcal{M}}_\epsilon^\lambda \boldsymbol{A} = 0$ for a sequence of positive $\epsilon_n \to 0$ and then use compactness to find a solution for the unregularised problem. To do this we will need many estimates on Ampère's equation.

6.1 The general method of estimation

We start by standardising some notation and assumptions. We will be dealing with bounded intervals of positive λ, and positive ϵ bounded above.

$$\text{There are finite positive } \lambda_*, \lambda^*, \epsilon, \text{ such that } \lambda \in [\lambda_*, \lambda^*] \text{ and } \epsilon \in (0, \epsilon^*] \tag{6.1}$$

We will use the notation $a \lesssim b$ to mean that there is a constant c which may depend on $\lambda_*, \lambda^*, \epsilon^*$, but is independent of λ, ϵ and anything in a or b, such that $a \leq bc$.

The main method of estimation is to take inner products of $\boldsymbol{\mathcal{M}}_\epsilon^\lambda \boldsymbol{A} = 0$ with some element of $\boldsymbol{\mathfrak{H}}$, for example $\boldsymbol{A}$. Recall the standard identities and adjointness relations, which hold in the sense of weak (and strong) derivatives,

$$\begin{aligned} \mathrm{div}^* &= \mathbf{grad} & \mathrm{div}\,\mathbf{curl} &= 0 \\ \mathbf{curl}^* &= \mathbf{curl} & \mathbf{curl}\,\mathbf{grad} &= 0 \\ (-\boldsymbol{\Delta})^* &= -\boldsymbol{\Delta} & -\boldsymbol{\Delta} &= \mathbf{curl}\,\mathbf{curl} - \mathbf{grad}\,\mathrm{div} \end{aligned} \tag{6.2}$$

which mean that **curl**s are orthogonal to **grad**s. We will also make use of the product rules for div and **curl**, which are

$$\mathrm{div}(\varphi \boldsymbol{A}) = \mathbf{grad}\,\varphi \cdot \boldsymbol{A} + \varphi\,\mathrm{div}\,\boldsymbol{A} \tag{6.3}$$

$$\mathbf{curl}(\varphi \boldsymbol{A}) = (\mathbf{grad}\,\varphi) \times \boldsymbol{A} + \varphi\,\mathbf{curl}\,\boldsymbol{A} \tag{6.4}$$

and the identity for $\boldsymbol{u} \in L^2$,

$$\|\mathbf{grad}\,\boldsymbol{u}\|_{L^2}^2 = \|\mathrm{div}\,\boldsymbol{u}\|_{L^2}^2 + \|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{\mathfrak{H}}}^2 \tag{6.5}$$

which also holds in $\boldsymbol{\mathfrak{H}}$.

6.2 Technical lemmas for product rules

The introduction of $\boldsymbol{\mathcal{E}}$ and in particular the function W mean we need a few technical lemmas.

Lemma 6.1. *Let γ be smooth and non-negative, such that if $\varphi \in \mathfrak{H}$ with compact support, then $\varphi\gamma \in \mathfrak{H}$ and $\gamma^* = \sup_{\boldsymbol{x}} |\operatorname{\mathbf{grad}} \gamma(\boldsymbol{x})| < \infty$. Then for any $\boldsymbol{u} \in \boldsymbol{\mathfrak{H}}$ with $\boldsymbol{\Delta u} \in \boldsymbol{\mathfrak{H}}$, we have the estimates*

$$\|\sqrt{\gamma}\operatorname{\mathbf{curl}}\boldsymbol{u}\|_{\boldsymbol{\mathfrak{H}}}^2 \leq \langle\operatorname{\mathbf{curl}}\operatorname{\mathbf{curl}}\boldsymbol{u}, \gamma\boldsymbol{u}\rangle_{\boldsymbol{\mathfrak{H}}} + \gamma^* \|\boldsymbol{u}\|_{\boldsymbol{\mathfrak{H}}} \|\operatorname{\mathbf{curl}}\boldsymbol{u}\|_{\boldsymbol{\mathfrak{H}}} \tag{6.6}$$

$$\|\sqrt{\gamma}\operatorname{div}\boldsymbol{u}\|_{\mathfrak{H}}^2 \leq -\langle\operatorname{\mathbf{grad}}\operatorname{div}\boldsymbol{u}, \gamma\boldsymbol{u}\rangle_{\boldsymbol{\mathfrak{H}}} + \gamma^* \|\boldsymbol{u}\|_{\boldsymbol{\mathfrak{H}}} \|\operatorname{div}\boldsymbol{u}\|_{\mathfrak{H}} \tag{6.7}$$

Which are interpreted in the sense that if the right hand side makes sense and is finite then the left hand side makes sense and is finite.

Proof. First we prove it for γ which additionally has compact support. We expand using the product rule for **curl**,

$$\begin{aligned}\langle\operatorname{\mathbf{curl}}\operatorname{\mathbf{curl}}\boldsymbol{u}, \gamma\boldsymbol{u}\rangle_{\boldsymbol{\mathfrak{H}}} &= \langle\operatorname{\mathbf{curl}}\boldsymbol{u}, \operatorname{\mathbf{curl}}(\gamma\boldsymbol{u})\rangle_{\boldsymbol{\mathfrak{H}}} &(6.8)\\ &= \langle\operatorname{\mathbf{curl}}\boldsymbol{u}, \gamma\operatorname{\mathbf{curl}}\boldsymbol{u}\rangle_{\boldsymbol{\mathfrak{H}}} + \langle\operatorname{\mathbf{curl}}\boldsymbol{u}, \boldsymbol{u}\times\operatorname{\mathbf{grad}}\gamma\rangle_{\boldsymbol{\mathfrak{H}}} &(6.9)\end{aligned}$$

Hence,

$$\|\sqrt{\gamma}\operatorname{\mathbf{curl}}\boldsymbol{u}\|_{\boldsymbol{\mathfrak{H}}}^2 \leq \langle\operatorname{\mathbf{curl}}\operatorname{\mathbf{curl}}\boldsymbol{u}, \gamma\boldsymbol{u}\rangle_{\boldsymbol{\mathfrak{H}}} + \gamma^* \|\operatorname{\mathbf{curl}}\boldsymbol{u}\|_{\boldsymbol{\mathfrak{H}}} \|\boldsymbol{u}\|_{\boldsymbol{\mathfrak{H}}} \tag{6.10}$$

The bound for div is much the same.

$$\begin{aligned}-\langle\operatorname{\mathbf{grad}}\operatorname{div}\boldsymbol{u}, \gamma\boldsymbol{u}\rangle_{\boldsymbol{\mathfrak{H}}} &= \langle\operatorname{div}\boldsymbol{u}, \operatorname{div}(\gamma\boldsymbol{u})\rangle_{\mathfrak{H}} &(6.11)\\ &= \|\sqrt{\gamma}\operatorname{div}\boldsymbol{u}\|_{\mathfrak{H}}^2 + \langle\operatorname{div}\boldsymbol{u}, \boldsymbol{u}\cdot\operatorname{\mathbf{grad}}\gamma\rangle_{\mathfrak{H}} &(6.12)\end{aligned}$$

Hence,

$$\|\sqrt{\gamma}\operatorname{div}\boldsymbol{u}\|_{\mathfrak{H}}^2 \leq -\langle\operatorname{\mathbf{grad}}\operatorname{div}\boldsymbol{u}, \gamma\boldsymbol{u}\rangle_{\boldsymbol{\mathfrak{H}}} + \gamma^* \|\operatorname{div}\boldsymbol{u}\|_{\mathfrak{H}} \|\boldsymbol{u}\|_{\boldsymbol{\mathfrak{H}}} \tag{6.13}$$

For the general result we approximate γ with a sequence γ_k which are compactly supported and have the same properties and gradient bound as γ with $\gamma_k \uparrow \gamma$ and then use Fatou's lemma. ☐

6.3 Bounds for curl, div and curl curl

Given a sequence of $\boldsymbol{A}_n$ which solve $\boldsymbol{\mathcal{M}}_{\epsilon_n}^{\lambda_n}\boldsymbol{A}_n = 0$ for $\epsilon_n \to 0$, we wish to establish the weak relative compactness of $\boldsymbol{A}_n$, $\operatorname{\mathbf{curl}}\boldsymbol{A}_n$, $\operatorname{div}\boldsymbol{A}_n$ and $\operatorname{\mathbf{curl}}\operatorname{\mathbf{curl}} A_n$ so that we can take a weak limit (of a subsequence) to obtain an $\boldsymbol{A}$ and a λ for which $\boldsymbol{\mathcal{M}}^\lambda\boldsymbol{A} = 0$. We can do this immediately by scaling each $\boldsymbol{A}_n$ so that

$$\|\boldsymbol{A}_n\|_{\boldsymbol{\mathfrak{H}}} + \|\operatorname{\mathbf{curl}}\boldsymbol{A}_n\|_{\boldsymbol{\mathfrak{H}}} + \|\operatorname{\mathbf{curl}}\operatorname{\mathbf{curl}}\boldsymbol{A}_n\|_{\boldsymbol{\mathfrak{H}}} = 1$$

and noting that bounded sets in a such a Hilbert space are relatively weakly compact. However, this would allow the possibility that

$$\|\boldsymbol{A}_n\|_{\boldsymbol{\mathfrak{H}}} \to 0 \quad \text{as } n \to \infty$$

which would mean that $\boldsymbol{A}_n \to 0$. For this reason we seek to bound $\|\operatorname{\mathbf{curl}}\boldsymbol{A}_n\|_{\boldsymbol{\mathfrak{H}}}$, $\|\operatorname{div}\boldsymbol{A}_n\|_{\mathfrak{H}}$ and $\|\operatorname{\mathbf{curl}}\operatorname{\mathbf{curl}}\boldsymbol{A}_n\|_{\boldsymbol{\mathfrak{H}}}$ with $\|\boldsymbol{A}_n\|_{\boldsymbol{\mathfrak{H}}}$. Then we could scale $\boldsymbol{A}_n$ so that $\|\boldsymbol{A}_n\|_{\boldsymbol{\mathfrak{H}}} = 1$ and still have the relative weak compactness. This is still not quite enough, but more on that later.

Lemma 6.2. *Suppose* (6.1) *and* $\boldsymbol{\mathcal{M}}_\epsilon^\lambda \boldsymbol{A} = 0$*. Then*

$$\begin{aligned} \|\mathbf{curl}\,\boldsymbol{A}\|_{\mathfrak{H}} &\lesssim \|\boldsymbol{A}\|_{\mathfrak{H}} \\ \|\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{A}\|_{\mathfrak{H}} &\lesssim \|\boldsymbol{A}\|_{\mathfrak{H}} \\ \|\mathrm{div}\,\boldsymbol{A}\|_{\mathfrak{H}} &\lesssim \|\boldsymbol{A}\|_{\mathfrak{H}} \end{aligned} \tag{6.14}$$

Proof. We will exploit that $\boldsymbol{A}$ solves $\boldsymbol{\mathcal{M}}_\epsilon^\lambda \boldsymbol{A} = 0$, i.e.

$$\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{A} + \lambda^2 \boldsymbol{A} - \epsilon \boldsymbol{\mathcal{E}} \boldsymbol{A} = \boldsymbol{\mathcal{J}}^\lambda \boldsymbol{A}$$

For the **curl** estimate, we take the inner product with $\boldsymbol{A}$. We obtain,

$$\langle \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{A}, \boldsymbol{A}\rangle_{\mathfrak{H}} + \lambda^2 \|\boldsymbol{A}\|_{\mathfrak{H}}^2 + \epsilon \langle \boldsymbol{\mathcal{E}} \boldsymbol{A}, \boldsymbol{A}\rangle_{\mathfrak{H}} = \left\langle \boldsymbol{\mathcal{J}}^\lambda \boldsymbol{A}, \boldsymbol{A} \right\rangle_{\mathfrak{H}} \tag{6.15}$$

Noting that **curl** is symmetric, we see that $\langle \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{A}, \boldsymbol{A}\rangle_{\mathfrak{H}} = \|\mathbf{curl}\,\boldsymbol{A}\|_{\mathfrak{H}}^2$ All the terms on the left hand side of (6.15) are non-negative, (recall $\boldsymbol{\mathcal{E}}$ is positive definite), so we may drop all but the $\mathbf{curl}\,\boldsymbol{A}$ term to obtain the inequality

$$\|\mathbf{curl}\,\boldsymbol{A}\|_{\mathfrak{H}}^2 \leq \left\langle \boldsymbol{\mathcal{J}}^\lambda \boldsymbol{A}, \boldsymbol{A} \right\rangle_{\mathfrak{H}} \leq \left\| \boldsymbol{\mathcal{J}}^\lambda \right\|_{\mathfrak{H}} \|\boldsymbol{A}\|_{\mathfrak{H}}^2 \tag{6.16}$$

But recall from Lemma 5.2 that $\left\| \boldsymbol{\mathcal{J}}^\lambda \right\|_{\mathfrak{H}}$ is bounded uniformly in λ, which proves the first inequality.

For the **curl curl** estimate, we perform a similar procedure, this time with $\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{A}$. We obtain, after expanding $\boldsymbol{\mathcal{E}}\boldsymbol{A}$,

$$\begin{aligned} (1+\epsilon) \|\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{A}\|_{\mathfrak{H}}^2 + \lambda^2 \langle \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{A}, \boldsymbol{A}\rangle_{\mathfrak{H}} + \\ + \epsilon \langle W \boldsymbol{A}, \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{A}\rangle_{\mathfrak{H}} = \left\langle \boldsymbol{\mathcal{J}}^\lambda \boldsymbol{A}, \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{A} \right\rangle_{\mathfrak{H}}. \end{aligned} \tag{6.17}$$

Using Lemma 6.1 we can bound

$$\langle W \boldsymbol{A}, \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{A}\rangle_{\mathfrak{H}} \gtrsim - \|\boldsymbol{A}\|_{\mathfrak{H}} \|\mathbf{curl}\,\boldsymbol{A}\|_{\mathfrak{H}}.$$

Hence, after dropping the non-negative terms on the left hand side of (6.17), the inequality

$$\begin{aligned} \|\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{A}\|_{\mathfrak{H}}^2 &\leq \left\langle \boldsymbol{\mathcal{J}}^\lambda \boldsymbol{A}, \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{A} \right\rangle_{\mathfrak{H}} + C \|\boldsymbol{A}\|_{\mathfrak{H}} \|\mathbf{curl}\,\boldsymbol{A}\|_{\mathfrak{H}} \\ &\leq \left\| \boldsymbol{\mathcal{J}}^\lambda \right\|_{\mathfrak{H}} \|\boldsymbol{A}\|_{\mathfrak{H}} \|\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{A}\|_{\mathfrak{H}} + C' \|\boldsymbol{A}\|_{\mathfrak{H}}^2 \end{aligned} \tag{6.18}$$

We now use Young's inequality to obtain,

$$\|\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{A}\|_{\mathfrak{H}}^2 \leq \tfrac{1}{2} \left\| \boldsymbol{\mathcal{J}}^\lambda \right\|_{\mathfrak{H}}^2 \|\boldsymbol{A}\|_{\mathfrak{H}}^2 + \tfrac{1}{2} \|\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{A}\|_{\mathfrak{H}}^2 + C' \|\boldsymbol{A}\|^2 \tag{6.19}$$

and hence,

$$\|\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{A}\|_{\mathfrak{H}}^2 \lesssim \left\| \boldsymbol{\mathcal{J}}^\lambda \right\|_{\mathfrak{H}}^2 \|\boldsymbol{A}\|_{\mathfrak{H}}^2 + \|\boldsymbol{A}\|_{\mathfrak{H}}^2 \tag{6.20}$$

and recalling that $\left\| \boldsymbol{\mathcal{J}}^\lambda \right\|_{\mathfrak{H}}$ has uniform bound we obtain the second required inequality.

For the third inequality we will use that $\operatorname{div} \boldsymbol{\mathcal{J}}^\lambda$ is a bounded linear operator, with uniform bound for compact sets of positive λ, see Lemma 5.5. We take the inner product with $-\operatorname{\mathbf{grad}} \operatorname{div} \boldsymbol{A}$ to obtain,

$$\begin{aligned}
-\left\langle \operatorname{\mathbf{curl}} \operatorname{\mathbf{curl}} \boldsymbol{A}, \operatorname{\mathbf{grad}} \operatorname{div} \boldsymbol{A} \right\rangle_{\mathfrak{H}} - \lambda^2 \left\langle \boldsymbol{A}, \operatorname{\mathbf{grad}} \operatorname{div} \boldsymbol{A} \right\rangle_{\mathfrak{H}} + \\
-\epsilon \left\langle \boldsymbol{\mathcal{E}} \boldsymbol{A}, \operatorname{\mathbf{grad}} \operatorname{div} \boldsymbol{A} \right\rangle_{\mathfrak{H}} &= \left\langle \boldsymbol{\mathcal{J}}^\lambda \boldsymbol{A}, \operatorname{\mathbf{grad}} \operatorname{div} \boldsymbol{A} \right\rangle_{\mathfrak{H}} \\
&= -\left\langle \operatorname{div} \boldsymbol{\mathcal{J}}^\lambda \boldsymbol{A}, \operatorname{div} \boldsymbol{A} \right\rangle_{\mathfrak{H}}
\end{aligned} \tag{6.21}$$

The first term is zero as $\operatorname{div} \operatorname{\mathbf{curl}} = 0$. The second becomes $\lambda^2 \left\| \operatorname{div} \boldsymbol{A} \right\|_{\mathfrak{H}}^2$ which is what we want to bound. For the third we use Lemma 6.1 to bound,

$$\begin{aligned}
-\left\langle \boldsymbol{\mathcal{E}} \boldsymbol{A}, \operatorname{\mathbf{grad}} \operatorname{div} \boldsymbol{A} \right\rangle_{\mathfrak{H}} &= \left\| \operatorname{\mathbf{grad}} \operatorname{div} \boldsymbol{A} \right\|_{\mathfrak{H}}^2 - \left\langle W \boldsymbol{A}, \operatorname{\mathbf{grad}} \operatorname{div} \boldsymbol{A} \right\rangle_{\mathfrak{H}} \\
&\gtrsim -\left\| \boldsymbol{A} \right\|_{\mathfrak{H}} \left\| \operatorname{div} \boldsymbol{A} \right\|_{\mathfrak{H}}
\end{aligned} \tag{6.22}$$

Hence,

$$\begin{aligned}
\lambda^2 \left\| \operatorname{div} \boldsymbol{A} \right\|_{\mathfrak{H}}^2 &\lesssim -\left\langle \operatorname{div} \boldsymbol{\mathcal{J}}^\lambda \boldsymbol{A}, \operatorname{div} \boldsymbol{A} \right\rangle_{\mathfrak{H}} + \epsilon \left\| \boldsymbol{A} \right\|_{\mathfrak{H}} \left\| \operatorname{div} \boldsymbol{A} \right\|_{\mathfrak{H}} \\
&\lesssim \left\| \boldsymbol{A} \right\|_{\mathfrak{H}} \left\| \operatorname{div} \boldsymbol{A} \right\|_{\mathfrak{H}} \sup_{\lambda \in [\lambda_*, \lambda^*]} \left\| \operatorname{div} \boldsymbol{\mathcal{J}}^\lambda \right\|_{\mathfrak{H} \to \mathfrak{H}} + \epsilon^* \left\| \boldsymbol{A} \right\|_{\mathfrak{H}} \left\| \operatorname{div} \boldsymbol{A} \right\|_{\mathfrak{H}}
\end{aligned} \tag{6.23}$$

where we have used that $\operatorname{div} \boldsymbol{\mathcal{J}}^\lambda : \mathfrak{H} \to \mathfrak{H}$ is uniformly bounded over compact sets of λ due to Lemma 5.5. We then divide through by the lower bound on λ and then use Young's inequality to obtain the final inequality. □

6.4 Estimates using epsilon

Later on we will need to use the regularisation term to give strong compactness.

Lemma 6.3. *Assume* (6.1)*, and that* $\left\langle \boldsymbol{\mathcal{M}}_\epsilon^\lambda \boldsymbol{A}, \boldsymbol{A} \right\rangle_{\mathfrak{H}} \lesssim \left\| \boldsymbol{A} \right\|_{\mathfrak{H}}^2$*. Then,*

$$\left\| \operatorname{div} \boldsymbol{A} \right\|_{\mathfrak{H}}^2 + \left\| \operatorname{\mathbf{curl}} \boldsymbol{A} \right\|_{\mathfrak{H}}^2 + \left\| \sqrt{W} \boldsymbol{A} \right\|_{\mathfrak{H}}^2 \lesssim (1/\epsilon) \left\| \boldsymbol{A} \right\|_{\mathfrak{H}}^2 \tag{6.24}$$

Proof. Expanding $\left\langle \boldsymbol{\mathcal{M}}_\epsilon^\lambda \boldsymbol{A}, \boldsymbol{A} \right\rangle$, we obtain, after dropping all the other positive terms from the left hand side,

$$\epsilon \left\| \operatorname{\mathbf{curl}} \boldsymbol{A} \right\|_{\mathfrak{H}}^2 + \epsilon \left\| \operatorname{div} \boldsymbol{A} \right\|_{\mathfrak{H}} + \epsilon \left\| \sqrt{W} \boldsymbol{A} \right\|_{\mathfrak{H}}^2 \lesssim \left\| \boldsymbol{\mathcal{J}}^\lambda \right\|_{\mathfrak{H}} \left\| \boldsymbol{A} \right\|_{\mathfrak{H}}^2 + \left\| \boldsymbol{A} \right\|_{\mathfrak{H}}^2 \tag{6.25}$$

and the result follows from the uniform bound on $\boldsymbol{\mathcal{J}}^\lambda$. □

Lemma 6.4. *Assume* (6.1)*, and that* $\left\langle \boldsymbol{\mathcal{M}}_\epsilon^\lambda \boldsymbol{A}, \boldsymbol{A} \right\rangle_{\mathfrak{H}} \lesssim \left\| \boldsymbol{A} \right\|_{\mathfrak{H}}^2$ *for all* $\boldsymbol{A} \in \mathfrak{A} \subset \mathfrak{H}$ *with* ϵ *fixed and* $\mathfrak{A}$ *bounded in* $\mathfrak{H}$*. Then,* $\mathfrak{A}$ *is strongly relatively compact.*

Proof. By Lemma 6.3, both the **curl**s and the divs of elements in $\mathfrak{A}$ are bounded uniformly in norm. Hence $\mathfrak{A} \subset H^1$. Since also $\left\| \sqrt{W} \boldsymbol{A} \right\|$ is bounded uniformly for $\boldsymbol{A} \in \mathfrak{A}$, and $\sqrt{W} \to \infty$ as $|\boldsymbol{x}| \to \infty$, we can use Lemma A.1 and Lemma A.3 to deduce the strong relative compactness. □

6.5 Tightness

To show that $\boldsymbol{A}_n \not\to 0$ we will show that $\boldsymbol{A}_n$ are strongly relatively compact. If our domain were bounded, then the bounds on $\operatorname{div} \boldsymbol{A}_n$ and $\mathbf{curl}\, \boldsymbol{A}_n$ would be sufficient. As our domain is unbounded we must use the same tightness idea as in Lemma 6.4, but we cannot use the regularisation as that depends on ϵ which we send to 0. Instead we will exploit the current operator $\boldsymbol{\mathcal{J}}^\lambda$.

Lemma 6.5. *Assume* (6.1)*, and that* $\mathfrak{A}$ *is a bounded subset of* $\boldsymbol{\mathfrak{H}}$ *with* $\boldsymbol{\mathcal{M}}_\epsilon^\lambda \boldsymbol{A} = 0$ *for all* $\boldsymbol{A} \in \mathfrak{A}$ *and some* $\epsilon \in (0, \epsilon^*)$ *depending on that particular* $\boldsymbol{A}$*. Then* $\mathfrak{A}$ *is strongly relatively compact.*

Proof. By Lemma A.3, Lemma A.1 and Lemma 6.2 it is enough to show that for some positive scalar function $\gamma(\boldsymbol{x}) \in \mathfrak{H}$ with $\gamma(\boldsymbol{x}) \to \infty$ as $|\boldsymbol{x}| \to \infty$, for which $\left\|\sqrt{\gamma}\boldsymbol{A}\right\|_{\boldsymbol{\mathfrak{H}}}$ is finite and bounded uniformly. By Lemma A.3 and Lemma 5.3 there is such a function γ for which $\left\|\gamma \boldsymbol{\mathcal{J}}^\lambda \boldsymbol{A}\right\|_{\boldsymbol{\mathfrak{H}}}$ is finite and bounded uniformly. By making γ smaller, we may assume that $\sup_{\boldsymbol{x}} |\mathbf{grad}\, \gamma(\boldsymbol{x})| < 1$, and that γ is smooth. In a similar way to the proof of Lemma 6.1 we will consider a sequence γ_k of smooth compactly supported functions in $\mathfrak{H}$ which converge pointwise from below to γ, and obey the same bound $\sup_{\boldsymbol{x}} |\mathbf{grad}\, \gamma_k(\boldsymbol{x})| < 1$. For each k, we take the inner product of $\boldsymbol{\mathcal{M}}_\epsilon^\lambda \boldsymbol{A} = 0$ with $\gamma_k \boldsymbol{A}$ to obtain,

$$\langle \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{A}, \gamma_k \boldsymbol{A}\rangle_{\boldsymbol{\mathfrak{H}}} + \lambda^2 \langle \boldsymbol{A}, \gamma_k \boldsymbol{A}\rangle_{\boldsymbol{\mathfrak{H}}} + \epsilon \langle \boldsymbol{\mathcal{E}} \boldsymbol{A}, \gamma_k \boldsymbol{A}\rangle_{\boldsymbol{\mathfrak{H}}} = \left\langle \boldsymbol{\mathcal{J}}^\lambda \boldsymbol{A}, \gamma_k \boldsymbol{A}\right\rangle_{\boldsymbol{\mathfrak{H}}} \tag{6.26}$$

Expanding $\boldsymbol{\mathcal{E}}$ and discarding the positive $\langle W\boldsymbol{A}, \boldsymbol{A}\gamma_k\rangle_{\boldsymbol{\mathfrak{H}}}$ term yields,

$$\begin{aligned} &(1+\epsilon) \langle \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{A}, \gamma_k \boldsymbol{A}\rangle_{\boldsymbol{\mathfrak{H}}} - \epsilon \langle \mathbf{grad}\, \operatorname{div} \boldsymbol{A}, \gamma_k \boldsymbol{A}\rangle_{\boldsymbol{\mathfrak{H}}} + \\ &\quad + \lambda^2 \|\sqrt{\gamma_k}\boldsymbol{A}\|^2_{\boldsymbol{\mathfrak{H}}} \leq \left\langle \boldsymbol{\mathcal{J}}^\lambda \boldsymbol{A}, \gamma_k \boldsymbol{A}\right\rangle_{\boldsymbol{\mathfrak{H}}} \leq \left\|\gamma_k \boldsymbol{\mathcal{J}}^\lambda \boldsymbol{A}\right\|_{\boldsymbol{\mathfrak{H}}} \|\boldsymbol{A}\|_{\boldsymbol{\mathfrak{H}}} \lesssim \|\boldsymbol{A}\|^2_{\boldsymbol{\mathfrak{H}}} \end{aligned} \tag{6.27}$$

where the constant in '$\lesssim$' is independent of k. Now we use Lemma 6.1 and Lemma 6.2 to bound the first two terms below:

$$\begin{aligned} \langle \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{A}, \gamma_k \boldsymbol{A}\rangle_{\boldsymbol{\mathfrak{H}}} &\gtrsim - \|\boldsymbol{A}\|_{\boldsymbol{\mathfrak{H}}} \|\mathbf{curl}\, \boldsymbol{A}\|_{\boldsymbol{\mathfrak{H}}} \gtrsim - \|\boldsymbol{A}\|^2_{\boldsymbol{\mathfrak{H}}} \\ - \langle \mathbf{grad}\, \operatorname{div} \boldsymbol{A}, \gamma_k \boldsymbol{A}\rangle_{\boldsymbol{\mathfrak{H}}} &\gtrsim - \|\boldsymbol{A}\|_{\boldsymbol{\mathfrak{H}}} \|\operatorname{div} \boldsymbol{A}\|_{\mathfrak{H}} \gtrsim - \|\boldsymbol{A}\|^2_{\boldsymbol{\mathfrak{H}}} \end{aligned} \tag{6.28}$$

Hence, we have

$$\lambda^2 \|\sqrt{\gamma_k}\boldsymbol{A}\|^2 \lesssim (1+\epsilon) \|\boldsymbol{A}\|^2_{\boldsymbol{\mathfrak{H}}} \lesssim \|\boldsymbol{A}\|^2_{\boldsymbol{\mathfrak{H}}} \tag{6.29}$$

Using the lower bound on λ and applying Fatou's lemma as $k \to \infty$ gives

$$\|\sqrt{\gamma}\boldsymbol{A}\|_{\boldsymbol{\mathfrak{H}}} \lesssim \|\boldsymbol{A}\|^2_{\boldsymbol{\mathfrak{H}}}$$

and we can conclude. ☐

6.6 Removing the regularisation

We now have enough to show we can remove the regularisation.

Proposition 6.1. *Let $\lambda_*, \lambda^*, \epsilon^*$ be fixed positive real numbers, with $\lambda_* < \lambda^*$, and suppose that for any $\epsilon \in (0, \epsilon^*)$ there is a $\lambda_\epsilon \in (\lambda_*, \lambda^*)$ and a non-trivial $\boldsymbol{A}_\epsilon \in \mathfrak{D}(\boldsymbol{\mathcal{E}})$ such that $\boldsymbol{\mathcal{M}}_\epsilon^{\lambda_\epsilon} \boldsymbol{A}_\epsilon = 0$. Then there is a $\lambda \in (\lambda_*, \lambda^*)$ and a non-trivial $\boldsymbol{A} \in \mathfrak{D}(\boldsymbol{\mathcal{M}}^\lambda)$ for which $\boldsymbol{\mathcal{M}}^\lambda \boldsymbol{A} = 0$.*

Proof. We take a sequence $\epsilon_n \to 0$ as $n \to \infty$, and denote the corresponding λ and $\boldsymbol{A}$ with n subscripts. By linearity we may scale the sequence $\boldsymbol{A}_n$ so that $\|\boldsymbol{A}_n\|_{\mathfrak{H}} = 1$. Then by Lemma 6.2 we have the bounds, independent of n,

$$\|\mathbf{curl}\, \boldsymbol{A}_n\| + \|\mathbf{curl}\,\mathbf{curl}\, \boldsymbol{A}_n\|_{\mathfrak{H}} \lesssim \|\boldsymbol{A}\|_{\mathfrak{H}} \tag{6.30}$$

This means that each of $\boldsymbol{A}_n, \mathbf{curl}\, \boldsymbol{A}_n, \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{A}_n$ are bounded in $\mathfrak{H}$-norm uniformly in n and are all relatively weakly compact in $\mathfrak{H}$. Further, by Lemma 6.5 the sequence $\boldsymbol{A}_n$ is relatively strongly compact in $\mathfrak{H}$. Also as $\lambda_n \in [\lambda_*, \lambda^*]$, the sequence λ_n is relatively compact in $[\lambda_*, \lambda^*]$. Hence we may pass to successive subsequences to obtain a new sequence, still denoted $\boldsymbol{A}_n$ for simplicity, a $\lambda \in [\lambda_*, \lambda^*]$ and an $\boldsymbol{A} \in \mathfrak{H}$ for which,

$$\begin{gathered} \boldsymbol{A}_n \to \boldsymbol{A} \quad \text{strongly in } \mathfrak{H} \\ \mathbf{curl}\, \boldsymbol{A}_n \to \mathbf{curl}\, \boldsymbol{A} \quad \text{weakly in } \mathfrak{H} \\ \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{A}_n \to \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{A} \quad \text{weakly in } \mathfrak{H} \\ \lambda_n \to \lambda \end{gathered} \tag{6.31}$$

By the strong convergence $\boldsymbol{A}_n \to \boldsymbol{A}$, and the normalisation $\|\boldsymbol{A}_n\|_{\mathfrak{H}} = 1$, we conclude that $\|\boldsymbol{A}\|_{\mathfrak{H}} = 1$, so that crucially $\boldsymbol{A} \neq 0$. The weak convergences (and that of λ) now allow us to pass to the limit $\epsilon \to 0$ in the equation,

$$\mathbf{curl}\,\mathbf{curl}\, \boldsymbol{A}_n + \epsilon \boldsymbol{\mathcal{E}} \boldsymbol{A}_n + \lambda_n^2 \boldsymbol{A}_n = \boldsymbol{\mathcal{J}}^{\lambda_n} \boldsymbol{A}_n \tag{6.32}$$

Indeed, fix a smooth compactly supported test function $\boldsymbol{\Phi} \in \mathfrak{H}$, then taking the inner product with (6.32), we can move $\boldsymbol{\mathcal{E}}$ across onto $\boldsymbol{\Phi}$ as $\boldsymbol{\mathcal{E}}$ is selfadjoint, to obtain

$$\langle \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{A}_n, \boldsymbol{\Phi} \rangle_{\mathfrak{H}} + \epsilon \langle \boldsymbol{A}_n, \boldsymbol{\mathcal{E}} \boldsymbol{\Phi} \rangle_{\mathfrak{H}} + \lambda_n^2 \langle \boldsymbol{A}_n, \boldsymbol{\Phi} \rangle_{\mathfrak{H}} = \left\langle \boldsymbol{\mathcal{J}}^{\lambda_n} \boldsymbol{A}_n, \boldsymbol{\Phi} \right\rangle_{\mathfrak{H}} \tag{6.33}$$

When we take $\epsilon \to 0$, the second term converges to 0 as $\boldsymbol{\mathcal{E}}\boldsymbol{\Phi}$ is constant and $\boldsymbol{A}_n$ is bounded in norm, while the other terms converge due to the various weak and strong convergences, to give,

$$\langle \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{A}, \boldsymbol{\Phi} \rangle_{\mathfrak{H}} + \lambda^2 \langle \boldsymbol{A}, \boldsymbol{\Phi} \rangle_{\mathfrak{H}} = \left\langle \boldsymbol{\mathcal{J}}^{\lambda} \boldsymbol{A}, \boldsymbol{\Phi} \right\rangle_{\mathfrak{H}} \tag{6.34}$$

As $\boldsymbol{\Phi}$ was an arbitrary test function, we can conclude that indeed,

$$\mathbf{curl}\,\mathbf{curl}\, \boldsymbol{A} + \lambda^2 \boldsymbol{A} = \boldsymbol{\mathcal{J}}^\lambda \boldsymbol{A} \qquad \text{in } \mathfrak{H}$$

which is exactly the equation $\boldsymbol{\mathcal{M}}^\lambda \boldsymbol{A} = 0$. This is exactly what we wanted and completes the proof of the proposition. □

7 Back to the main problem

7.1 Tracking the spectrum

Recall our aim was to find a λ for which $\boldsymbol{\mathcal{M}}_\epsilon^\lambda$ has a non-trivial kernel. This is equivalent to $\boldsymbol{\mathcal{M}}^\lambda$ having 0 as an eigenvalue. The idea is now to see what happens as $\lambda \to 0$ and $\lambda \to \infty$ and deduce by spectral continuity results that an eigenvalue must travel across and hit zero for some intermediate λ. The natural result to use is the continuity of the spectrum of selfadjoint operators as they vary in operator norm.

7.2 Selfadjointness

The problem is that $\boldsymbol{\mathcal{M}}^\lambda$ is not selfadjoint. This is because $\boldsymbol{\mathcal{J}}^\lambda$ is not symmetric. To understand why, it is helpful to look at the 1.5d case. In this case we only have x_1, v_1, v_2 dependence and A_3 is constrained to be 0. We will for simplicity forget completely about x_2, x_3 and consider $\mathfrak{H} = L^2(\Omega_1; \mathbb{R})$. The matrix $\boldsymbol{\mathcal{M}}^\lambda$ can be viewed as a 2×2 block matrix acting on $(A_1, A_2)^T$ and is of the form

$$\boldsymbol{\mathcal{M}}^\lambda = \begin{bmatrix} \lambda^2 & 0 \\ 0 & \lambda^2 - \partial_{x_1}^2 \end{bmatrix} - \begin{bmatrix} \mathcal{J}_{11}^\lambda & \mathcal{J}_{12}^\lambda \\ \mathcal{J}_{21}^\lambda & \mathcal{J}_{22}^\lambda \end{bmatrix} \tag{7.1}$$

where

$$\mathcal{J}_{ij}^\lambda A_j = A_j \sum_\pm \int \hat{v}_i \mu_{p_j}^\pm \, d\boldsymbol{v} + \sum_\pm \int \hat{v}_i \mathcal{Q}_\pm^\lambda [\mu_e^\pm \hat{v}_j A_j] \, d\boldsymbol{v} \tag{7.2}$$

The first part of $\mathcal{J}_{ij}^\lambda$ is diagonal as the equilibrium $\mu^\pm$ (and its derivatives) depend only on $p_2^\pm = v_2 + A_2^0(x_1)$ and $e^\pm = \langle \boldsymbol{v} \rangle$ so is an even function of v_1, so in fact only the $2,2$ component is non-zero. The second term is where the problem comes from. In [1, Lemma 2.5.(5)] it is proved that

$$\left\langle \mathcal{Q}_\pm^\lambda [h(x_1, v_1, v_2)], k(x_1, v_1, v_2) \right\rangle_{\mathfrak{L}_\pm} = \left\langle h(x_1, -v_1, v_2), \mathcal{Q}_\pm^\lambda [k(x_1, -v_1, v_2)] \right\rangle_{\mathfrak{L}_\pm}$$

So $\mathcal{Q}_\pm^\lambda$ is not selfadjoint, and in fact we find (noting that $\mu_e^\pm$ is unchanged by $\mathcal{Q}_\pm^\lambda$ and can be moved in and out), $\mathcal{J}_{ii}^\lambda$, $i = 1, 2$, are symmetric, but $(\mathcal{J}_{12}^\lambda)^* = -\mathcal{J}_{21}^\lambda$ so that $\boldsymbol{\mathcal{J}}^\lambda$ is not symmetric and hence $\boldsymbol{\mathcal{M}}^\lambda$ is not selfadjoint.

7.2.1 A solution

To fix this we have to reformulate the equations into a selfadjoint operator which has the same kernel. In the 1.5d case this is easy, we simply switch the sign of the first row of $\boldsymbol{\mathcal{M}}^\lambda$ to produce,

$$\widehat{\boldsymbol{\mathcal{M}}^\lambda} = \begin{bmatrix} -\lambda^2 & 0 \\ 0 & \lambda^2 - \partial_{x_1}^2 \end{bmatrix} - \widehat{\boldsymbol{\mathcal{J}}^\lambda} = \begin{bmatrix} -\lambda^2 & 0 \\ 0 & \lambda^2 - \partial_{x_1}^2 \end{bmatrix} - \begin{bmatrix} -\mathcal{J}_{11}^\lambda & -\mathcal{J}_{12}^\lambda \\ \mathcal{J}_{21}^\lambda & \mathcal{J}_{22}^\lambda \end{bmatrix} \tag{7.3}$$

Clearly this operator has the same kernel. Then by the remarks about adjoints of the components of $\boldsymbol{\mathcal{J}}^\lambda$, it is also clear that $\widehat{\boldsymbol{\mathcal{J}}^\lambda}$ and also $\widehat{\boldsymbol{\mathcal{M}}^\lambda}$ are symmetric. When we add the regularisation we obtain,

$$\widehat{\boldsymbol{\mathcal{M}}_\epsilon^\lambda} = \begin{bmatrix} -\lambda^2 & 0 \\ 0 & \lambda^2 - \partial_{x_1}^2 \end{bmatrix} + \epsilon \begin{bmatrix} \partial_{x_1}^2 - W & 0 \\ 0 & -\partial_{x_1}^2 + W \end{bmatrix} - \begin{bmatrix} -\mathcal{J}_{11}^\lambda & -\mathcal{J}_{12}^\lambda \\ \mathcal{J}_{21}^\lambda & \mathcal{J}_{22}^\lambda \end{bmatrix} \tag{7.4}$$

Lemma 7.1. *The operator $\widehat{\boldsymbol{\mathcal{M}}_\epsilon^\lambda} : \mathfrak{H} \to \mathfrak{H}$ defined by the formula (7.4) with the smooth compactly supported functions in $\mathfrak{H}$ as a core is selfadjoint and has compact resolvent and discrete spectrum.*

Proof. This is essentially the same as Lemma 5.9. $\square$

7.3 Truncation

As $\lambda \to \infty$ the spectrum of $\widehat{\boldsymbol{\mathcal{M}}_\epsilon^\lambda}$ has both positive and negative parts of infinite cardinality, so we cannot say there are 'more' or 'less' positive eigenvalues than negative eigenvalues, and we cannot compare this to the spectrum as $\lambda \to 0$. Instead we will truncate the operator $\widehat{\boldsymbol{\mathcal{M}}_\epsilon^\lambda}$ to a finite dimensional approximation, for which we can count eigenvalues and compare the spectrum as $\lambda \to \infty$ to that as $\lambda \to 0$. Then we will have to remove the truncation and somehow recover a solution to the untruncated problem.

For a fixed $\epsilon > 0$ we define $\widehat{\boldsymbol{\mathcal{M}}_n^\lambda} = \boldsymbol{\mathcal{G}}_n \widehat{\boldsymbol{\mathcal{M}}_\epsilon^\lambda} \boldsymbol{\mathcal{G}}_n$ where $\boldsymbol{\mathcal{G}}_n$ is some unspecified projection of finite dimension for each n which maps into the domain of $\widehat{\boldsymbol{\mathcal{M}}_\epsilon^\lambda}$ and with $\boldsymbol{\mathcal{G}}_n \to 1$ strongly as $n \to \infty$. We now ask in what sense does $\widehat{\boldsymbol{\mathcal{M}}_n}$ converge to $\widehat{\boldsymbol{\mathcal{M}}_\epsilon^\lambda}$, or more generally, in what sense does $\boldsymbol{\mathcal{G}}_n \mathcal{T} \boldsymbol{\mathcal{G}}_n$ converge to $\mathcal{T}$ for operators $\mathcal{T}$ for which this makes sense?

Lemma 7.2. *Let $\mathcal{B}$ be a bounded operator on a Hilbert space, and $(\boldsymbol{\mathcal{G}}_n)_{n\geq 0}$ be a family of projections on this Hilbert space with $\boldsymbol{\mathcal{G}}_n \to 1$ strongly as $n \to \infty$. Then $\boldsymbol{\mathcal{G}}_n \mathcal{B} \boldsymbol{\mathcal{G}}_n \to \mathcal{B}$ strongly as $n \to \infty$.*

Proof. Fix an element u of the space. Then we have,

$$\begin{aligned} \|(\boldsymbol{\mathcal{G}}_n \mathcal{B} \boldsymbol{\mathcal{G}}_n - \mathcal{B})u\| &= \|\boldsymbol{\mathcal{G}}_n \mathcal{B}(\boldsymbol{\mathcal{G}}_n - 1)u + (\boldsymbol{\mathcal{G}}_n - 1)\mathcal{B}u\| \\ &\leq \|\boldsymbol{\mathcal{G}}_n\| \, \|\mathcal{B}\| \, \|(\boldsymbol{\mathcal{G}}_n - 1)u\| + \|(\boldsymbol{\mathcal{G}}_n - 1)(\mathcal{B}u)\| \to 0 \end{aligned} \tag{7.5}$$

by the strong convergence of $\boldsymbol{\mathcal{G}}_n$. □

In fact this proof is simple to extend to when $\mathcal{B}_n \to \mathcal{B}$ strongly and are uniformly bounded in operator norm.

For unbounded (selfadjoint) operators we cannot hope to have strong convergence as they are not defined on the whole space. Instead we might hope for generalised strong convergence (also known as convergence in the strong resolvent sense). A sufficient condition for generalised strong convergence is given by Theorem B.4, we merely require the strong convergence when restricted to a common core.

Lemma 7.3. *Suppose that $(\mathcal{K}^\lambda)_{\lambda\geq 0}$ is a strongly continuous family of bounded linear operators on a Hilbert space $\mathfrak{X}$ which is uniformly bounded for compact sets of λ. Let $\mathcal{T}$ be a selfadjoint operator on $\mathfrak{X}$ such that $\mathcal{T}^\lambda = \mathcal{T} + \mathcal{K}^\lambda$ is selfadjoint with common core $\mathfrak{T}$. Define $\mathcal{T}_n^\lambda = \boldsymbol{\mathcal{G}}_n \mathcal{T}^\lambda \boldsymbol{\mathcal{G}}_n$ where $\boldsymbol{\mathcal{G}}_n$ is a family of projections such that $\boldsymbol{\mathcal{G}}_n \to 1$ strongly as $n \to \infty$ and $\boldsymbol{\mathcal{G}}_n$ maps into $\mathfrak{T}$. Suppose that for some $\lambda_0 \geq 0$ we have $\mathcal{T}_n^{\lambda_0} u \to \mathcal{T}^{\lambda_0} u$ strongly as $n \to \infty$ for all $u \in \mathfrak{T}$. Then if $\lambda_n \to \lambda$ as $n \to \infty$ we have that for all $u \in \mathfrak{T}$,*

$$\mathcal{T}_n^{\lambda_n} u \to \mathcal{T}^\lambda u \qquad n \to \infty \tag{7.6}$$

and hence $\mathcal{T}_n^{\lambda_n} \to \mathcal{T}^\lambda$ as $n \to \infty$ in the strong generalised sense.

Proof. Fix $u \in \mathfrak{T}$. Then,

$$\left\|\mathcal{T}_n^{\lambda_n} u - \mathcal{T}^\lambda u\right\| \leq \left\|\mathcal{T}_n^{\lambda_n} u - \mathcal{T}_n^\lambda u\right\| + \left\|\mathcal{T}_n^\lambda u - \mathcal{T}^\lambda u\right\| \tag{7.7}$$

The first term on the right is equal to $\left\|\mathcal{K}_n^{\lambda_n} u - \mathcal{K}_n^{\lambda} u\right\|$ (where $\mathcal{K}_n^{\lambda}$ is defined in the same way as $\mathcal{T}_n^{\lambda}$), which goes to zero by the extension to Lemma 7.2. For the second we continue,

$$\left\|\mathcal{T}_n^{\lambda} u - \mathcal{T}^{\lambda} u\right\| \leq \left\|(\mathcal{T}_n^{\lambda_0} u - \mathcal{T}_n^{\lambda} u) - (\mathcal{T}^{\lambda_0} u - \mathcal{T}^{\lambda} u)\right\| + \left\|\mathcal{T}_n^{\lambda_0} u - \mathcal{T}^{\lambda_0} u\right\| \tag{7.8}$$

The second term on the right goes to zero by assumption. The first term on the right is equal to

$$\left\|\boldsymbol{\mathcal{G}}_n(\mathcal{K}^{\lambda_0} - \mathcal{K}^{\lambda})\boldsymbol{\mathcal{G}}_n u - (\mathcal{K}^{\lambda_0} - \mathcal{K}^{\lambda}) u\right\| \tag{7.9}$$

which again goes to zero by Lemma 7.2 as $\mathcal{K}^{\lambda_0} - \mathcal{K}^{\lambda}$ is a bounded operator. □

Corollary 7.1. *Suppose that $\boldsymbol{\mathcal{G}}_n$ is a family of projections on $\mathfrak{H}$ with $\boldsymbol{\mathcal{G}}_n \to 1$ strongly as $n \to \infty$ and $\boldsymbol{\mathcal{G}}_n$ maps into a common core $\mathfrak{M}$ of $(\widehat{\boldsymbol{\mathcal{M}}_\epsilon^{\lambda}})_{\lambda \geq 0}$. Further suppose that for some $\lambda_0 \geq 0$, we have the convergence $\widehat{\boldsymbol{\mathcal{M}}_n^{\lambda_0}} \boldsymbol{u} \to \widehat{\boldsymbol{\mathcal{M}}_\epsilon^{\lambda_0}} \boldsymbol{u}$ as $n \to \infty$ for all $\boldsymbol{u} \in \mathfrak{M}$. Then if $\lambda_n \to \lambda$ as $n \to \infty$, $\widehat{\boldsymbol{\mathcal{M}}_n^{\lambda_n}} \to \widehat{\boldsymbol{\mathcal{M}}_\epsilon^{\lambda}}$ as $n \to \infty$ in the strong generalised sense.*

A suitable $\boldsymbol{\mathcal{G}}_n$ for which this condition holds would be a sequence of eigenprojections onto the eigenspace of $\widehat{\boldsymbol{\mathcal{M}}_\epsilon^{\lambda_0}}$ for some λ_0. As we then have (for $u \in \mathfrak{M}$) $\boldsymbol{u} = \sum_{j=1}^{\infty} a_j \boldsymbol{e}_j$ and $\widehat{\boldsymbol{\mathcal{M}}_\epsilon^{\lambda_0}} \boldsymbol{u} = \sum_{j=1}^{\infty} b_j \boldsymbol{e}_j$, where $\boldsymbol{e}_j$ are the eigenfunctions of $\widehat{\boldsymbol{\mathcal{M}}_\epsilon^{\lambda_0}}$ in some order. We deduce that $b_j = \alpha_j a_j$ where α_j are the eigenvalues of $\widehat{\boldsymbol{\mathcal{M}}_\epsilon^{\lambda_0}}$. Then we note that $\widehat{\boldsymbol{\mathcal{M}}_n^{\lambda_0}} \boldsymbol{u} = \sum_{j=1}^{n} b_j \boldsymbol{e}_j$ and hence,

$$\left\|\widehat{\boldsymbol{\mathcal{M}}_n^{\lambda_0}} \boldsymbol{u} - \widehat{\boldsymbol{\mathcal{M}}_\epsilon^{\lambda_0}} \boldsymbol{u}\right\|_{\mathfrak{H}} = \sum_{j=n+1}^{\infty} |b_j|^2 \to 0 \quad \text{as } n \to \infty \tag{7.10}$$

It should be noted that the key property that made this work was that $\boldsymbol{\mathcal{G}}_n$ commuted with $\widehat{\boldsymbol{\mathcal{M}}^{\lambda_0}}$, so that $\widehat{\boldsymbol{\mathcal{M}}_n^{\lambda_0}} = \boldsymbol{\mathcal{G}}_n \widehat{\boldsymbol{\mathcal{M}}_\epsilon^{\lambda_0}} \boldsymbol{\mathcal{G}}_n = \boldsymbol{\mathcal{G}}_n \widehat{\boldsymbol{\mathcal{M}}_\epsilon^{\lambda_0}}$.

We will now show how to recover a non-trivial kernel as we remove the truncation.

Lemma 7.4. *Let $\mathcal{A}_n \to \mathcal{A}$ as $n \to \infty$ in the strong resolvent sense, and $\mathcal{A}_n u_n = 0$ for u_n with unit norm. Additionally assume that $(u_n)_{n=1}^{\infty}$ is strongly relatively compact. Then there is a u of unit norm in* $\ker(\mathcal{A})$.

Proof. Due to the relative compactness we can extract a subsequence, (still denoted u_n), which converges strongly to u which will also have unit norm. Let $a < 0 < b$ be arbitrarily chosen, but not in the pure point spectrum (if there is one) of $\mathcal{A}$. Let $\chi = 1_{(a,b)}$. Then by Theorem B.5(b) we have the strong operator convergence $\chi(\mathcal{A}_n) \to \chi(\mathcal{A})$. We now claim that $\chi(\mathcal{A}_n)u \to u$ as $n \to \infty$. Indeed,

$$\begin{aligned} \|\chi(\mathcal{A}_n)u - u\| &= \|\chi(\mathcal{A}_n)(u - u_n) - (u - u_n)\| \\ &\leq (\|\chi(\mathcal{A}_n)\| + 1) \|u - u_n\| = 2\|u - u_n\| \to 0 \end{aligned} \tag{7.11}$$

where in the first line we have used that as $\mathcal{A}_n u_n = 0$ we have $\chi(\mathcal{A}_n) u_n = u_n$. Therefore $\chi(\mathcal{A}_n)u \to u$ also, and u lies in the eigenspace associated with the (a, b) section of the spectrum of $\mathcal{A}$. Taking a sequence of intervals (a, b) approaching 0, which is possible as the pure point spectrum cannot be continuous, we deduce by dominated convergence that u is in the eigenspace associated with $\{0\}$, and hence in $\ker(\mathcal{A})$. □

So it is now sufficient to show that a sequence of eigenfunctions of $\widehat{\boldsymbol{\mathcal{M}}_n^\lambda}$ is strongly relatively compact.

Lemma 7.5. *Let there be a sequence $\boldsymbol{u}_n$ in the domains of $\widehat{\boldsymbol{\mathcal{M}}_n^\lambda}$ viewed as finite dimensional operators, of unit norm, and a sequence of $\lambda_n \to \lambda \geq 0$ as $n \to \infty$, for which $\widehat{\mathcal{M}_n^{\lambda_n}} u_n = 0$. Then the sequence is relatively strongly compact.*

Proof. As $\boldsymbol{u}_n$ is in the domains of $\widehat{\mathcal{M}_n^{\lambda_n}}$ when viewed as finite dimensional operators, we have $\boldsymbol{\mathcal{G}}_n \boldsymbol{u}_n = \boldsymbol{u}_n$. We will use the ϵ regularisation part, which we are allowed to do as ϵ is fixed. As $\widehat{\mathcal{M}_n^{\lambda_n}} \boldsymbol{u}_n = 0$, we also have $\mathcal{M}_n^{\lambda_n} \boldsymbol{u}_n = 0$, with $\boldsymbol{\mathcal{M}}_n^\lambda$ defined in the obvious manner, so we may drop the hats. We take the inner product of the equation $\boldsymbol{\mathcal{M}}_n^\lambda \boldsymbol{u}_n = 0$ with $\boldsymbol{u}_n$ to obtain

$$\begin{aligned} \langle \boldsymbol{\mathcal{G}}_n \, \mathbf{curl}\, \mathbf{curl}\, \boldsymbol{\mathcal{G}}_n \boldsymbol{u}_n, \boldsymbol{u}_n \rangle_{\mathfrak{H}} + \lambda_n \langle \boldsymbol{\mathcal{G}}_n \boldsymbol{u}_n, \boldsymbol{u}_n \rangle_{\mathfrak{H}} + \epsilon \langle \boldsymbol{\mathcal{G}}_n \boldsymbol{\mathcal{E}} \boldsymbol{\mathcal{G}}_n \boldsymbol{u}_n, \boldsymbol{u}_n \rangle_{\mathfrak{H}} + \\ - \left\langle \boldsymbol{\mathcal{G}}_n \boldsymbol{\mathcal{J}}^{\lambda_n} \boldsymbol{\mathcal{G}}_n \boldsymbol{u}_n, \boldsymbol{u}_n \right\rangle_{\mathfrak{H}} = 0 \end{aligned} \tag{7.12}$$

Using that $\boldsymbol{\mathcal{G}}_n$ is selfadjoint we deduce that $\left\langle \boldsymbol{\mathcal{M}}_{\epsilon_n}^{\lambda_n} \boldsymbol{\mathcal{G}}_n \boldsymbol{u}_n, \boldsymbol{\mathcal{G}}_n \boldsymbol{u}_n \right\rangle = 0$ and hence $\left\langle \boldsymbol{\mathcal{M}}_{\epsilon_n}^{\lambda_n} \boldsymbol{u}, \boldsymbol{u} \right\rangle = 0$, i.e. for the untruncated $\boldsymbol{\mathcal{M}}_\epsilon^\lambda$. Then we may apply Lemma 6.3 to prove this lemma. ☐

Proposition 7.1. *Suppose that the conditions on $\boldsymbol{\mathcal{G}}_n$ given in Corollary 7.1 hold. Fix $0 < \lambda_* < \lambda^* < \infty$ and suppose that for each sufficiently large n, there is a $\lambda_n \in (\lambda_*, \lambda^*)$ and a $\boldsymbol{u}_n$ of unit norm in the kernel of $\widehat{\boldsymbol{\mathcal{M}}_n^{\lambda_n}}$ viewed as an operator from $\boldsymbol{\mathcal{G}}_n \boldsymbol{\mathfrak{H}}$ to $\boldsymbol{\mathcal{G}}_n \boldsymbol{\mathfrak{H}}$. Then there is a $\lambda \in [\lambda_*, \lambda^*]$ for which $\widehat{\boldsymbol{\mathcal{M}}_\epsilon^\lambda}$ (and hence $\boldsymbol{\mathcal{M}}_\epsilon^\lambda$) has non-trivial kernel.*

Proof. As the λ_n are bounded, we may pass to a subsequence were they converge to some $\lambda \in [\lambda_*, \lambda^*]$. We can now apply Lemma 7.5, Corollary 7.1 and Lemma 7.4 to prove the proposition. ☐

7.4 Limit as $\lambda \to \infty$

Lemma 7.6. *There is a positive real λ^* independent of ϵ, such that for all $\lambda > \lambda^*$, the untruncated operator $\widehat{\boldsymbol{\mathcal{M}}_\epsilon^\lambda}$ (in the 1.5d case) satisfies for all non-zero $A_1, A_2 \in \mathfrak{H}$,*

$$\begin{aligned} \left\langle \widehat{\boldsymbol{\mathcal{M}}_\epsilon^\lambda} \begin{bmatrix} A_1 \\ 0 \end{bmatrix}, \begin{bmatrix} A_1 \\ 0 \end{bmatrix} \right\rangle_{\mathfrak{H}} < 0 \\ \left\langle \widehat{\boldsymbol{\mathcal{M}}_\epsilon^\lambda} \begin{bmatrix} 0 \\ A_2 \end{bmatrix}, \begin{bmatrix} 0 \\ A_2 \end{bmatrix} \right\rangle_{\mathfrak{H}} > 0 \end{aligned} \tag{7.13}$$

Proof. This is quite straightforward as the diagonal of $\widehat{\boldsymbol{\mathcal{M}}_\epsilon^\lambda}$ is comprised of positive and negative definite operators plus bounded parts. Indeed,

$$\left\langle \widehat{\boldsymbol{\mathcal{M}}_\epsilon^\lambda} \begin{bmatrix} A_1 \\ 0 \end{bmatrix}, \begin{bmatrix} A_1 \\ 0 \end{bmatrix} \right\rangle_{\mathfrak{H}} = \left\langle (-\lambda^2 + \epsilon \partial_{x_1}^2 - \epsilon W) A_1, A_1 \right\rangle_{\mathfrak{H}} + \left\langle \mathcal{J}_{11}^\lambda A_1, A_1 \right\rangle_{\mathfrak{H}} \tag{7.14}$$

The operator $(-\lambda^2+\epsilon\partial_{x_1}^2-\epsilon W)$ is bounded above by $-\lambda^2$, so to make the whole expression negative we just need to make λ^2 larger than the uniform bound on $\boldsymbol{\mathcal{J}}^\lambda$, which is independent of ϵ.

The computation for A_2 is identical, except that the operator is now $(\lambda^2 - (1+\epsilon)\partial_{x_1}^2 + \epsilon W)$ which is bounded below by λ^2. $\square$

7.5 Limit as $\lambda \to 0$

Alas, due to multiple dead ends and other hazards, the author has not been able to complete this work in the time allotted. Instead of stopping here, the author sketches his current thoughts on the matter, which should not be taken as proofs.

The problem is to obtain the limiting spectral behaviour as $\lambda \to 0$ in a way independent of the regularisation parameter ϵ. More precisely the required lower bound on the growth rate λ_* obtained in the obvious fashion depends upon ϵ and so could go to zero as we send $\epsilon \to 0$.

The limiting behaviour as $\lambda \to 0$ is more complicated than that as $\lambda \to \infty$. In [2, Lemma 3.4.] it is proved that the strong limit of $\mathcal{Q}_\pm^\lambda$ as $\lambda \to 0$, which we shall denote $\mathcal{Q}_\pm^0$, and which we proved in Lemma 5.1 to be the orthogonal projection onto $\ker(\mathcal{D}_\pm)$, preserves parity with respect to the variable v_1 in the 1.5d case. In other words, if h is an odd function of v_1, then so is $\mathcal{Q}_\pm^0 h$, and if k is an even function of v_1 then so is $\mathcal{Q}_\pm^0 k$. In the 1.5d case the equilibrium and its derivatives are even functions of v_1, so we deduce that $\mathcal{J}_{12}^0 = \mathcal{J}_{21}^0 = 0$ and $\widehat{\boldsymbol{\mathcal{M}}^0}$ is diagonal. Furthermore, due to Lemma 5.5 and (4.6) we deduce that $\mathcal{J}_{11}^0$ maps into the constant functions, and as the only constant function which is square integrable over $\mathbb{R}$ is 0, $\mathcal{J}_{11}^0 = 0$ also. In summary,

$$\widehat{\boldsymbol{\mathcal{M}}^0} = \begin{bmatrix} 0 & 0 \\ 0 & -\partial_{x_1}^2 - \mathcal{J}_{22}^0 \end{bmatrix} \tag{7.15}$$

This means that we can extract very little information about the spectrum in the limit, as we care about the sign of the eigenvalues, which is lost as they become zero.

This turns out to be an effect of imposing the temporal gauge, which has strange behaviour in the limit $\lambda \to 0$, as it relies on time dependence of the vector potentials to allow the scalar potential to be zero. We can however, rescale the operators to obtain a non-trivial limit. We will consider the rescaled operator given by

$$\widetilde{\boldsymbol{\mathcal{M}}^\lambda} = \begin{bmatrix} 1/\lambda & 0 \\ 0 & 1 \end{bmatrix} \widehat{\boldsymbol{\mathcal{M}}^\lambda} \begin{bmatrix} 1/\lambda & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1+\lambda^{-2}\mathcal{J}_{11}^\lambda & \lambda^{-1}\mathcal{J}_{12}^\lambda \\ -\lambda^{-1}\mathcal{J}_{21}^\lambda & \lambda^2 - \partial_{x_1}^2 - \mathcal{J}_{22}^\lambda \end{bmatrix} \tag{7.16}$$

For any positive λ this will have the same positive and negative eigenspaces and is still selfadjoint with the same domain. We will now compute what the rescaled $\mathcal{J}$ operators are. Recall that

$$\mathcal{J}_{11}^\lambda h = \sum_\pm \int \mu_e^\pm \hat{v}_1 \mathcal{Q}_\pm^\lambda [h\hat{v}_1]\, d\boldsymbol{v} \tag{7.17}$$

and note that if $h = \frac{dH}{dx_1}$ for some function H then $h\hat{v}_1 = \hat{v}_1\partial_{x_1}H = \mathcal{D}_\pm H$, so that $\mathcal{Q}_\pm^\lambda[h\hat{v}_1] = \lambda\mathcal{R}_\pm^{-\lambda}(\mathcal{D}_\pm + \lambda)H - \lambda\mathcal{Q}_\pm^\lambda[H] = \lambda H - \lambda\mathcal{Q}_\pm^\lambda[H]$. Fixing $h, k \in \mathfrak{H}$

such that $h = \frac{dH}{dx_1}$ and $k = \frac{dK}{dx_1}$ we deduce that,

$$\begin{aligned}\left\langle \mathcal{J}_{11}^\lambda h, k\right\rangle_{\mathfrak{H}} &= \lambda \left\langle H - \mathcal{Q}_\pm^\lambda H, \hat{v}_1 k\right\rangle_{\mu_e} \\ &= \lambda \left\langle H, \mathcal{D}_\pm K\right\rangle_{\mu_e} + \lambda \left\langle H, \mathcal{Q}_\pm^\lambda[\mathcal{D}_\pm K]\right\rangle_{\mu_e} \\ &= \lambda \left\langle H, \mathcal{D}_\pm K\right\rangle_{\mu_e} + \lambda^2 \left\langle H, K - \mathcal{Q}_\pm^\lambda K\right\rangle_{\mu_e} \\ &= \lambda \left\langle H, \mathcal{D}_\pm K\right\rangle_{\mu_e} + \lambda^2 \left\langle H, K\right\rangle_{\mu_e} - \lambda^2 \left\langle H, \mathcal{Q}_\pm^\lambda K\right\rangle_{\mu_e}\end{aligned} \tag{7.18}$$

The $\lambda \left\langle H, \mathcal{D}_\pm K\right\rangle_{\mu_e}$ term looks worryingly non-symmetric, but when we expand it, we see that it vanishes as $\mu_e^\pm$ are even functions of v_1. Indeed,

$$\lambda \left\langle H, \mathcal{D}_\pm K\right\rangle_{\mu_e} = \sum_\pm \int Hk \int \hat{v}_1 \mu_e^\pm \, d\boldsymbol{v} d\boldsymbol{x} = \sum_\pm \int Hk \cdot 0 \, d\boldsymbol{x} = 0 \tag{7.19}$$

The next thing to note is that the expression in (7.18) does not depend on the exact functions H, K we choose. If we send $K \to K + c$ for some constant c then the expression is unchanged, as $\mathcal{Q}_\pm^\lambda c = c$, and by symmetry this extends to H. If we now assume $\mu^\pm$ has compact support in x contained in $[-R, R]$ we can define (purely formally) the operator $\mathcal{I} : \mathfrak{H} \to \mathfrak{H}$ by defining $\mathcal{I}h$ as the k solving

$$\begin{gathered}\partial_{x_1} k = h \text{ in } [-R, R] \\ \text{with some fixed boundary condition}\end{gathered} \tag{7.20}$$

Where due to the independence of our expressions with respect to how we choose k, we don't care about the boundary condition as long as it is consistent. A quick computation that we omit shows that $\mathcal{I}$ is bounded and produces functions that are absolutely continuous on $[-R, R]$ and supported on the same. Hence it is compact. An integration by parts shows that $\mathcal{I}$ has an adjoint on $\mathfrak{H}$, which is of the same form but negative with a different boundary condition. We hence deduce that

$$\mathcal{J}_{11}^\lambda h = -\lambda^2 \left(\sum_\pm \int \mu_e^\pm \, d\boldsymbol{v}\right) \mathcal{I}^* \mathcal{I} h + \lambda^2 \mathcal{I}^* \left[\sum_\pm \int \mu_e^\pm \mathcal{Q}_\pm^\lambda [\mathcal{I}h] \, d\boldsymbol{v}\right] \tag{7.21}$$

so that $\lambda^{-2} \mathcal{J}_{11}^\lambda$ has a non-trivial strong limit as $\lambda \to 0$.

A similar series of computations for $\mathcal{J}_{21}^\lambda$ yield,

$$\mathcal{J}_{21}^\lambda h = \lambda \left(\sum_\pm \int \mu_e^\pm \hat{v}_2 \, d\boldsymbol{v}\right) \mathcal{I} h - \lambda \sum_\pm \int \mu_e^\pm \hat{v}_2 \mathcal{Q}_\pm^\lambda [\mathcal{I}h] \, d\boldsymbol{v} \tag{7.22}$$

and $\mathcal{J}_{12}^\lambda$ can be recovered by the relation $\mathcal{J}_{12}^\lambda = -\left(\mathcal{J}_{21}^\lambda\right)^*$. So both have non-trivial strong limits as $\lambda \to 0$.

Further computation allows the extraction of essentially the same operators as in [2] acting on H instead of h.

This means that the zero limit is not an insurmountable problem. The real issue is trying to obtain the lower bound on the growth rate λ_* independent of ϵ. To illustrate the problem, we will consider a simplified situation. Suppose we have two strongly continuous (in the generalised sense) families of Schroedinger operators $(\mathcal{A}_i^\lambda)_{\lambda \geq 0}$ with essential spectrum $[0, \infty)$ and a uniformly bounded finite number of negative eigenvalues. Suppose that at $\lambda = 0$ the number of negative eigenvalues of $\mathcal{A}_1^0$ is greater than the number of negative eigenvalues of $\mathcal{A}_2^0$. We wish to find a $\lambda_* > 0$ such that this is still true. In general this would be impossible as generalised strong convergence does not give good control over spectra. However in our case we know (by computations that we omit) that all the negative eigenfunctions of $\mathcal{A}_i^\lambda$ for *any* λ lie in the same compact set (independent of λ). This means that the truncated operator $1_{(-\infty,0)}(\mathcal{A}_i^\lambda)$ is norm continuous in λ and we can hence find a $\lambda_* > 0$ such that the number of negative eigenvalues of $\mathcal{A}_i^{\lambda_*}$ is not less than the number at $\lambda = 0$. That is, none have escaped. What we cannot do is say whether any additional negative eigenvalues have appeared from the continuous spectrum. This would make our comparison between $\mathcal{A}_1^\lambda$ and $\mathcal{A}_2^\lambda$ impossible. In the case of a bounded domain, the spectra are discrete so there is no essential spectrum and the problem does not arise. If the operators $\mathcal{K}_i^\lambda$ produce functions which are compactly supported (uniformly in λ) then intuitively the essential spectrum should not be able to do this as soon as $\lambda > 0$, as the essential spectrum corresponds to 'eigenfunctions' supported outside the support of $\mathcal{K}_i^\lambda$ which should not be able to become 'eigenfunctions' of negative eigenvalues.

7.5.1 A possible solution

In this section the author will sketch his current thoughts on fixing the issue described above.

We shall as before suppose that $\mu^\pm$ each have compact x-support inside $[-R, R]$ for some finite $R > 0$. Recalling (RVM 1.5D), Maxwell's equations in the 1.5d case are,

$$\begin{aligned} \partial_x E_1 &= \rho & \partial_x E_2 &= -\lambda B_3 \\ \lambda E_1 &= j_1 & \lambda E_2 &= j_2 + \partial_x B_3 \end{aligned} \tag{7.23}$$

First by the definition of the operator $\mathcal{F}_\pm^\lambda$ we deduce that $f^\pm$ have the same support $[-R, R]$. By conservation laws of the linearised Vlasov equation we can deduce that for any growing mode the total perturbed charge $\int \rho\, dx$ is constant, and because as $t \to -\infty$ the perturbation tends to 0, we deduce that the total perturbed charge is zero. We hence deduce that for any solution E_1 is zero outside $[-R, R]$. That this is possible is due to the fact that in the 1.5D case Maxwell's equations only allow for light waves in the x_1 direction, meaning that any such waves will have zero E_1 component. Strictly speaking a solution could exist with a non-zero E_1 outside $[-R, R]$, but this field would not be generated by the perturbation of the particles and would have to have an external source. We can then define a 'scalar potential' ϕ such that

$$E_1 = \begin{cases} -\partial_x \phi & \text{if } x \in [-R, R] \\ 0 & \text{if } x \notin [-R, R] \end{cases} \tag{7.24}$$

and aim to find such a potential in $H^2[-R, R]$ whose derivative vanishes at $\pm R$ so that E_1 is continuous and differentiable almost everywhere and hence in $H^1(\mathbb{R})$. We then follow the method of [2], defining operators $\mathcal{A}_1^\lambda, \mathcal{A}_2^\lambda, \mathcal{B}^\lambda$, forming the matrix operator

$$\boldsymbol{\mathcal{M}}^\lambda = \begin{bmatrix} -\mathcal{A}_1^\lambda & \mathcal{B}^\lambda \\ (\mathcal{B}^\lambda)^* & \mathcal{A}_2^\lambda \end{bmatrix}. \tag{7.25}$$

Here the first row is Gauss's equation and the first coloumn acts on ϕ. We now define the space $L^2_0[-R,R]$ as those functions u in $L^2[-R,R]$ which satisfy $\int_{-R}^{R} u\,dx = 0$, equipped with the standard L^2 norm and inner product. We let $\mathcal{A}_1^\lambda$ act on $L^2_0[-R,R]$ with Neumann boundary conditions for the Laplacian. By the formulae for $\mathcal{B}^\lambda$ we deduce that it maps $L^2(\mathbb{R})$ to $L^2_0[-R,R]$. So we can deduce that $\boldsymbol{\mathcal{M}}^\lambda : L^2_0[-R,R] \times L^2(\mathbb{R}) \to L^2_0[-R,R] \times L^2(\mathbb{R})$ is selfadjoint with domain $H^2_{0,N}[-R,R] \times H^2(\mathbb{R})$, where $H^2_{0,N}$ is the domain of the Laplacian with Neumann boundary conditions on $L^2_0[-R,R]$. Because its domain is bounded, the operator $\mathcal{A}_1^\lambda$ has discrete spectrum, and we assume (as in [2]) that its has trivial kernel, so we can find a $\lambda_* > 0$ such that its number of negative eigenvalues has not increased by the same kind of arguments used in [2].

This solves the problem as described before, as we can already track the escape of negative eigenvalues of $\mathcal{A}_2^\lambda$. To handle the limit as $\lambda \to \infty$, (which is not so simple any more as there is no λ^2 term in $\mathcal{A}_1$), we perform the reverse of the calculation in the previous subsection, expressing the action of $\mathcal{A}_1^\lambda$ in terms of the derivative of its argument. This brings back the λ^2 term and makes the limit simple again. To handle the essential spectrum of $\mathcal{A}_2^\lambda$, a regularisation of the form applied in subsection 5.7 can be used.

8 Conclusion

We have investigated the problem of spectral linear instability of the relativistic Vlasov-Maxwell system. Although no definite result has been achieved, many insights have been gained. Further work would be to flesh out the idea in subsubsection 7.5.1.

A Strong Compactness in unbounded domains

Tightness is a concept commonly seen in probability theory, where it is used to show the weak relative compactness of probability measures. Here we shall use the concept to give strong relative compactness in $L^2(\mathbb{R}^3;\mathbb{R}^3)$. The usual method of bounding derivatives and using the Rellich theorem fails because the domain is not bounded. However, if in addition to the derivatives being bounded in L^2, the sequence is *tight*, then strong relative compactness is obtained. Because the main tools for proving tightness come from probability theory, we shall use these methods here.

Definition A.1 (Tightness)**.** *Let $1 \le p \le \infty$, n be a positive integer, and $\mathbb{X}$ be a metric space. Then a subset A of $L^p(\mathbb{X};\mathbb{R}^n)$ is said to be tight iff for any $\epsilon > 0$ there is a compact $K_\epsilon \subset \mathbb{X}$ with*

$$\|\mathbf{1}(\mathbb{X}\setminus K_\epsilon) f\|_{L^p(\mathbb{X};\mathbb{R}^n)} < \epsilon \qquad \textit{for all } f \in A$$

where $\mathbf{1}(\mathbb{X}\setminus K_\epsilon)$ is the indicator function of the set $\mathbb{X}\setminus K_\epsilon$.

This definition is somewhat hard to work with. Instead a different criterion may be used.

Definition A.2. *Let $1 \le p \le \infty$, n be a positive integer, and $\mathbb{X}$ be a metric space. Suppose that w is a positive scalar function such that $w(x) \to \infty$ as*

$|x| \to \infty$. *Then a subset A of $L^p(\mathbb{X};\mathbb{R}^n)$ is said to be w-tight iff for all $f \in A$, $\{wf : f \in A\}$ is bounded in $L^p(\mathbb{X};\mathbb{R}^n)$. We also say that a function f is w-tight if $\{f\}$ is w-tight. Saying f is tight is facetious as all functions in L^p are w-tight for some w.*

Lemma A.1. *Let $1 \leq p < \infty$ and n, m be positive integers and let $A \subset L^p(\mathbb{R}^n;\mathbb{R}^m)$ be bounded. Then A is tight iff A is w-tight for some w.*

Proof. By considering $|f|^p$ it is sufficient to prove it for $L^1(\mathbb{R}^n;\mathbb{R})$, and we may renormalise so that $\|f\|_{L^1} = 1$ for all $f \in A$. By considering $\inf_{|y|\geq x} w(y)$ we may take w to be radially symmetric and increasing. Under these conditions $f \in A$ are probability densities on $\mathbb{R}^n$. To prove the if part, we apply Markov's inequality to the random variable X defined by $f \in A$, for $x > 0$,

$$\mathbb{P}(|X| > x) \leq \mathbb{P}(w(X) > w(x)) \leq \frac{1}{w(x)}\mathbb{E}[w(X)] \to 0 \text{ as } x \to \infty \text{ uniformly in } f$$

For the other direction we apply the definition of tightness, for any $1/k^3 > 0$ there is a compact set $K \subset \mathbb{R}^n$ for which $\mathbb{P}(X \notin K) < \epsilon$, we take a ball that contains K of some radius R_k. We may assume that R_k is increasing, and we define $\tilde{w}(x) = k$ for $R_k \leq |x| < R_{k+1}$ where we take $R_0 = 0$. If $(R_k)_{k\geq 0}$ is bounded above we define $\tilde{w}(x) = \infty$ for $|x|$ larger than this bound. (This corresponds to the case of all $f \in A$ having mutual compact support). We now estimate,

$$\begin{aligned}\mathbb{E}[\tilde{w}(X)] &\leq \sum_{k=0}^{\infty} \mathbb{E}[\tilde{w}(X)1(R_k \leq |X| < R_{k+1})] \\ &\leq \sum_{k=0}^{\infty} k\mathbb{P}(|X| \geq R_k) \leq \sum_{k=0}^{\infty} 1/k^2 < \infty\end{aligned} \tag{A.1}$$

and note the bound is uniform in f. We now take w as some smooth positive radially unbounded function that is less than $\tilde{w}$. $\square$

Lemma A.2. *Let $1 \leq p < \infty$ and n, m be positive integers, then relatively compact subsets of $L^p(\mathbb{R}^n;\mathbb{R}^m)$ are tight.*

Proof. In the same way as in the proof of Lemma A.1 we may consider probability densities. Recall Prokhorov's theorem, which states that tightness of a set of probability measures is equivalent their weak (in the probabilist sense) compactness. As our probability densities are strongly relatively compact they are weakly relatively compact as probability measures. Therefore by Prokhorov's theorem they are tight as probability measures, which corresponds exactly to their densities being tight as functions. $\square$

Lemma A.3. *Let n, m be positive integers, and A be a bounded subset of $H^\alpha(\mathbb{R}^n;\mathbb{R}^m)$ for some fixed $\alpha > 0$. Then A is strongly relatively compact in $L^2(\mathbb{R}^n;\mathbb{R}^m)$ iff it is tight in $L^2(\mathbb{R}^n;\mathbb{R}^m)$.*

Proof. One direction was proved by Lemma A.2, leaving only that tightness implies relative compactness. By Lemma A.1 we obtain a smooth function w such that $\|fw\|_{L^2} \leq 1$ for all $f \in A$ and $w \to \infty$ as $|x| \to \infty$. We may take w to be radially symmetric. We define the weighted Sobolev space H^α_w as the closure of smooth compactly supported functions using the norm $\|f\|^2_{H^\alpha_w} = \|f\|^2_{H^\alpha} +$

$\|fw\|_{L^2}^2$. We then define the operators $\mathcal{T}_k$ from H_w^α to L^2 by $(\mathcal{T}_k f)(x) = x$ if $|x| \leq k$ and 0 otherwise. Then each $\mathcal{T}_k$ is a compact operator by the Rellich theorem. We then claim that the sequence $\mathcal{T}_n$ converges to the inclusion map $\mathcal{I} : H_w^\alpha \to L^2$ in operator norm. Indeed, take $\|f\|_{H_w^\alpha} = 1$,

$$\|(\mathcal{I} - \mathcal{T}_k)f\|_{L^2} = \int_{|x|>k} |f(x)|^2 \, dx \leq \frac{1}{w(k)} \|fw\|_{L^2}^2 \to 0 \tag{A.2}$$

with the convergence uniform over such f. Then we deduce that $\mathcal{I}$ is compact as the compact operators form a closed set in the operator norm topology. □

We will now extend some of these results to bounded linear operators.

Definition A.3. *Let n, m be positive integers, $\mathfrak{X}$ be a Banach space, and $\mathcal{L} \in \mathfrak{B}(\mathfrak{X}; L^2(\mathbb{R}^n; \mathbb{R}^m))$. Then we say that $\mathcal{L}$ is w-tight iff it maps bounded subsets of $\mathfrak{X}$ to w-tight subsets of $L^2(\mathbb{R}^n; \mathbb{R}^m)$. If we do not wish to specify w we just say that $\mathcal{L}$ is tight. (Due to Lemma A.1 tightness is equivalent to w-tightness for some w).*

Lemma A.4. *In a similar way to compactness, tightness is preserved by (bounded) operator composition. But, this only applies to right composition, so that if $\mathcal{T}$ is tight and $\mathcal{A}$ is bounded then $\mathcal{T}\mathcal{A}$ is tight, but $\mathcal{A}\mathcal{T}$ need not be.*

B Crash course on spectral theory

In this section we give an overview of spectral theory, defining notation and presenting results used in the prequel. Readers who wish to go into more detail are encouraged to consult for example the book of Kato[8], or the series of books of Reed and Simon[13].

We start with bounded linear operators.

Definition B.1 (Bounded Linear Operator)**.** *The space of bounded linear operators between two Hilbert spaces $\mathfrak{H}$ and $\mathfrak{H}'$ is denoted $\mathfrak{B}(\mathfrak{H}, \mathfrak{H}')$ and is comprised of all linear maps $\mathcal{A} : \mathfrak{H} \to \mathfrak{H}'$ that are bounded in the sense that there is a fixed constant $C \geq 0$ depending only on $\mathcal{A}$ such that for all $u \in \mathfrak{H}$, $\|\mathcal{A}u\|_{\mathfrak{H}'} \leq C \|u\|_{\mathfrak{H}}$. The norm of $\mathcal{A}$ is defined as the smallest possible constant C for which this holds.*

Definition B.2 (Convergence of Operators)**.** *There are several types of convergence for elements of $\mathfrak{B}(\mathfrak{H}, \mathfrak{H}')$.*

(i) $\mathcal{A}_n \to \mathcal{A}$ *in norm, iff* $\|\mathcal{A}_n - \mathcal{A}\|_{\mathfrak{B}(\mathfrak{H},\mathfrak{H}')} \to 0$.

(ii) $\mathcal{A}_n \to \mathcal{A}$ *strongly (written* $\mathcal{A}_n \xrightarrow{s} \mathcal{A}$*) iff for all* $u \in \mathfrak{H}$, $\mathcal{A}_n u \to \mathcal{A}u$ *in* $\mathfrak{H}'$.

(iii) $\mathcal{A}_n \to \mathcal{A}$ *weakly (written* $\mathcal{A}_n \xrightarrow{w} \mathcal{A}$*) iff for all* $u \in \mathfrak{H}$, $\mathcal{A}_n u \to \mathcal{A}u$ *weakly in* $\mathfrak{H}'$*, i.e. for all* $v \in \mathfrak{H}'$, $\langle \mathcal{A}_n u, v \rangle \to \langle \mathcal{A}u, v \rangle$.

Clearly (i) $\implies$ (ii) $\implies$ (iii). Also, norm convergence is uniform strong convergence, in the sense that if $\mathcal{A}_n u \xrightarrow{s} \mathcal{A}u$ uniformly over $u \in \mathfrak{H}$, then $\mathcal{A}_n \to \mathcal{A}$ in norm.

Definition B.3 (Compact Operator). *An operator $A : \mathfrak{H} \to \mathfrak{H}'$ is compact iff it maps into a relatively compact set of $\mathfrak{H}'$. In other words, for any sequence $(u_n)_{n\in\mathbb{N}} \subset \mathfrak{H}$, the sequence $(\mathcal{A}u_n)_{n\in\mathbb{N}} \subset \mathfrak{H}'$ has a convergent subsequence. A compact operator is sometimes called completely continuous.*

Lemma B.1. *The composition of a compact operator and any other bounded operator is compact, i.e. if $\mathcal{T} : \mathfrak{H} \to \mathfrak{H}'$, $\mathcal{A} : \mathfrak{H}' \to \mathfrak{H}''$, $\mathcal{S} : \mathfrak{H}'' \to \mathfrak{H}'''$ are bounded linear operators with $\mathcal{A}$ compact, then $\mathcal{AT}$ and $\mathcal{SA}$ are compact.*

Definition B.4 (Adjoint). *Let $\mathcal{A} \in \mathfrak{B}(\mathfrak{H})$, then the adjoint of $\mathcal{A}$ is the unique bounded operator $\mathcal{A}^* \in \mathfrak{B}(\mathfrak{H})$ such that for all $u, v \in \mathfrak{H}$,*

$$\langle \mathcal{A}u, v\rangle = \langle u, \mathcal{A}^* v\rangle$$

Some caution with this definition is necessary as it is not the same as the adjoint of $\mathcal{A}$ viewed as a bounded operator from $\mathfrak{H} \to \mathfrak{H}$ with $\mathfrak{H}$ viewed as a Banach space as opposed to a Hilbert space. In particular, we have $(a\mathcal{A})^* = \bar{a}\mathcal{A}^*$ where $a \in \mathbb{C}$ and $\bar{a}$ is its complex conjugate. In the Banach space case $(a\mathcal{A})^* = a\mathcal{A}^*$. We shall only deal with the Hilbert space adjoint.

Sometimes it is helpful to consider operators which are either not bounded, or not defined on the whole space.

Definition B.5 (Unbounded Linear Operator). *Let $\mathcal{A}$ be an linear operator from $\mathfrak{H}$ to $\mathfrak{H}'$, which is defined on some subspace of $\mathfrak{H}$ which we denote $\mathfrak{D}(\mathcal{A})$. Then $\mathcal{A}$ maps $\mathfrak{D}(\mathcal{A})$ to some subspace $\mathfrak{R}(\mathcal{A})$ of $\mathfrak{H}'$. If $\mathfrak{D}(\mathcal{A})$ is dense in $\mathfrak{H}$, we say that $\mathcal{A}$ is densely defined.*

Remark B.1. *All bounded linear operators $\mathcal{A} : \mathfrak{H} \to \mathfrak{H}'$ can also be viewed as unbounded operators with $\mathfrak{D}(\mathcal{A}) = \mathfrak{H}$ and range $\mathfrak{R}(\mathcal{A}) = \mathfrak{H}'$. If $\mathfrak{H} = \mathfrak{H}'$ then the graph norm (defined below) of $\mathcal{A}$ is equivalent to the usual $\mathfrak{H}$ norm.*

The domain of an unbounded operator is not merely a technicality. It is intimately connected with the spectral properties of the operator. Two operators can have the same 'rule' but different domains and spectral properties. One instance this comes up in is the imposition of boundary conditions on differential operators. Because of these issues we often need to have more properties on the domain.

Definition B.6 (Graph Norm and Closed Operators). *The domain of an unbounded linear operator $\mathcal{A} : \mathfrak{H} \to \mathfrak{H}'$ can be equipped with the graph norm $\|\cdot\|_{\mathcal{A}}$ of $\mathcal{A}$. This is defined for $u \in \mathfrak{D}(\mathfrak{A})$ by*

$$\|u\|_{\mathcal{A}}^2 = \|u\|_{\mathfrak{H}}^2 + \|\mathcal{A}u\|_{\mathfrak{H}'}^2$$

If $\mathfrak{D}(\mathcal{A})$ is complete when equipped with this norm and hence forms a Hilbert space we say that $\mathcal{A}$ is a closed operator.

A closed operator $\mathcal{A} : \mathfrak{H} \to \mathfrak{H}'$ can be viewed as a bounded operator in $\mathfrak{B}(\mathfrak{D}(\mathcal{A}), \mathfrak{H}')$ with operator norm 1 (by definition!).

Definition B.7 (Extensions and Closure of an unbounded operator). *If $\mathcal{A}, \mathcal{A}'$ are operators on $\mathfrak{H}$ such that $\mathfrak{D}(\mathcal{A}) \subset \mathfrak{D}(\mathcal{A}')$ and $\mathcal{A}u = \mathcal{A}'u$ for all $u \in \mathfrak{D}(\mathcal{A})$ then we say that $\mathcal{A}'$ is an extension of $\mathcal{A}$. We define the closure of $\mathcal{A}$ as the minimal closed extension of $\mathcal{A}$. Not all operators have any closed extensions, operators which have such an extension are called closable.*

We can extend the notion of bounded operators to operators bounded relative to some other operator.

Definition B.8 (Relatively bounded)**.** *Let $\mathcal{T}, \mathcal{A}$ be operators on a Hilbert space $\mathfrak{H}$ with $\mathfrak{D}(\mathcal{T}) \subset \mathfrak{D}(\mathcal{A})$. Then we say that $\mathcal{A}$ is $\mathcal{T}$-bounded iff there are $a, b \geq 0$ such that for all $u \in \mathfrak{D}(\mathcal{T})$ we have the bound*

$$\|\mathcal{A}u\|_{\mathfrak{H}} \leq a\,\|u\|_{\mathfrak{H}} + b\,\|\mathcal{T}u\|_{\mathfrak{H}} \tag{B.1}$$

Then the greatest lower bound on such possible b is called the $\mathcal{T}$-bound of $\mathcal{A}$.

Remark B.2. *Clearly all bounded operators are $\mathcal{T}$ bounded with $\mathcal{T}$-bound 0 for any operator $\mathcal{T}$.*

Sometimes dealing with the whole domain of an operator is tricky, or sometimes two operators may have different domains which makes comparing them hard. For these purposes it is useful to have a concept of the important part of the domain.

Definition B.9 (Core)**.** *Let $\mathcal{A}$ be a closed operator. Then a subset $\mathfrak{A}$ of $\mathfrak{D}(\mathcal{A})$ is called a core of $\mathcal{A}$ iff the closure of $\mathcal{A}'$ defined by $\mathcal{A}'u = \mathcal{A}u$ for $u \in \mathfrak{D}(\mathcal{A}') = \mathfrak{A}$ is equal to $\mathcal{A}$.*

Definition B.10. *An operator $\mathcal{A}$ is symmetric iff for all $u, v \in \mathfrak{D}(\mathcal{A})$, $\langle \mathcal{A}u, v\rangle = \langle u, \mathcal{A}v\rangle$.*

Despite not all operators being closable, all symmetric operators are. For infinite dimensional Hilbert spaces, symmetric is not the same property as selfadjoint which is much stronger.

Definition B.11. *For a closed densely defined operator $\mathcal{A}$ we define an adjoint of $\mathcal{A}$ as an operator $\mathcal{B}$ such that for all $u \in \mathfrak{D}(\mathcal{A})$,*

$$\langle \mathcal{A}u, v\rangle = \langle u, \mathcal{B}v\rangle \tag{B.2}$$

with the domain $\mathfrak{D}(\mathcal{B})$ being those v for which an element $w = \mathcal{B}v$ exists with this property. We then denote $\mathcal{B} = \mathcal{A}^*$.

It is essential that $\mathcal{A}$ is closed and densely defined for this to give a unique adjoint $\mathcal{A}^*$.

Definition B.12 (Selfadjoint)**.** *An operator $\mathcal{A}$ is said to be selfadjoint iff $\mathcal{A} = \mathcal{A}^*$. If the closure of $\mathcal{A}$ is selfadjoint then we say that $\mathcal{A}$ is essentially selfadjoint.*

Selfadjointness is equivalent to symmetry and $\mathfrak{D}(\mathcal{A}) = \mathfrak{D}(\mathcal{A}^*)$. A key property of selfadjoint operators is that their spectrum is real.

Definition B.13 (Relatively Compact Operator)**.** *An operator $\mathcal{A} : \mathfrak{H} \to \mathfrak{H}'$ is relatively compact with respect to $\mathcal{T} : \mathfrak{H} \to \mathfrak{H}'$, iff*

- *$\mathfrak{D}(\mathcal{T}) \subset \mathfrak{D}(\mathcal{A})$.*
- *For every bounded sequence $(u_n)_{n=1}^{\infty}$ for which $(\mathcal{T}u_n)_{n=1}^{\infty}$ is bounded, $(\mathcal{A}u_n)_{n=1}^{\infty}$ has a convergent subsequence.*

An equivalent definition if $\mathcal{T}$ is closed is that $\mathcal{A}$ is relatively compact with respect to $\mathcal{T}$ if the restriction of $\mathcal{A}$ to $\mathfrak{D}(\mathcal{T})$ is a compact operator from $\mathfrak{D}(\mathcal{T})$ to $\mathfrak{H}'$ where $\mathfrak{D}(\mathcal{T})$ is as usual equipped with the graph norm of $\mathcal{T}$.

Definition B.14 (Resolvent and resolvent set)**.** *The resolvent of an operator $\mathcal{A} : \mathfrak{H} \to \mathfrak{H}'$ is the complex analytic, bounded operator valued function*[1] $\mathcal{R}(\lambda; \mathcal{A}) = (\mathcal{A} - \lambda)^{-1}$*, which is defined where this inverse is in $\mathfrak{B}(\mathfrak{H}', \mathfrak{H})$. The domain in the complex plane where $\mathcal{R}(\lambda; \mathcal{A})$ is defined is called the resolvent set and denoted $\rho(\mathcal{A})$. The complement of this set in the complex plane is the spectrum of $\mathcal{A}$ and is denoted $\Sigma(\mathcal{A})$.*

That the resolvent is complex analytic in λ is a non trivial property that requires proof. For brevity we shall not do so here.

It is easy to prove that the resolvent $\mathcal{R}(\lambda; \mathcal{A})$ commutes with $\mathcal{A}$ in the sense that for all $u \in \mathfrak{D}(\mathcal{A})$,

$$\mathcal{R}(\lambda; \mathcal{A})\mathcal{A}u = \mathcal{A}\mathcal{R}(\lambda; \mathcal{A})u$$

Furthermore resolvents of the same operator commute, i.e. for any $\lambda, \zeta \in \rho(\mathcal{A})$,

$$\mathcal{R}(\lambda; \mathcal{A})\mathcal{R}(\zeta; \mathcal{A}) = \mathcal{R}(\zeta; \mathcal{A})\mathcal{R}(\lambda; \mathcal{A})$$

A further useful identity is the *resolvent equation*. For any $\lambda, \zeta \in \rho(\mathcal{A})$,

$$\mathcal{R}(\lambda; \mathcal{A}) - \mathcal{R}(\zeta; \mathcal{A}) = (\zeta - \lambda)\mathcal{R}(\zeta; \mathcal{A})\mathcal{R}(\lambda; \mathcal{A})$$

6 An immediate consequence of this is that if $\mathcal{R}(\lambda; \mathcal{A})$ is compact for some λ, then $\mathcal{R}(\lambda; \mathcal{A})$ is compact for all $\lambda \in \rho(\mathcal{A})$.

We shall now describe some more notions of spectra. For simplicity we will only consider the spectra of selfadjoint operators.

Theorem B.1 (Spectral Theorem)**.** *Let $\mathcal{A}$ be a selfadjoint operator on $\mathfrak{H}$. Then there is a unique projection valued measure (defined below) μ called the spectral measure of $\mathcal{A}$, such that,*

$$\begin{aligned} \mathcal{A} &= \int_{\mathbb{R}} \lambda \, d\mu(\lambda) \\ \mathcal{P}_\Omega &= \int_{\mathbb{R}} 1_\Omega \, d\mu \\ f(\mathcal{A}) &= \int_{\mathbb{R}} f \, d\mu \end{aligned} \tag{B.3}$$

where $\Omega \subset \mathbb{R}$ is Borel and $\mathcal{P}_\Omega$ is the projection onto the eigenspace of Ω associated with the operator $\mathcal{A}$, and f is a real-valued Borel function on $\mathbb{R}$. In addition $f(\mathcal{A})$ is selfadjoint with domain

$$\mathfrak{D}(f(\mathcal{A})) = \left\{ u \in \mathfrak{H} : \int_{\mathbb{R}} |f(\lambda)|^2 \, d\langle u, \mu(\lambda)u \rangle < \infty \right\}$$

Furthermore, any projection valued measure is associated with a selfadjoint operator.

In particular,

$$\mathcal{R}(\zeta; \mathcal{A}) = \int_{\mathbb{R}} \frac{d\mu}{\zeta - \lambda}$$

[1]Some authors instead define the resolvent as $(\lambda - \mathcal{A})^{-1}$.

Definition B.15 (Projection valued measure)**.** *A projection valued measure is a map $\mu(\Omega) = \mathcal{P}_\Omega$ from the Borel sets in $\mathbb{R}$ to orthogonal projections on $\mathfrak{H}$ such that,*

$$\begin{aligned} \mu(\emptyset) &= 0 \\ \mu(\mathbb{R}) &= 1 \\ \mu\left(\bigcup_{n=1}^{\infty} \Omega_n\right) &= \sum_{n=1}^{\infty} \mu(\Omega_n) \\ \mu(\Omega_1)\mu(\Omega_2) &= \mu(\Omega_1 \cap \Omega_2) \end{aligned} \tag{B.4}$$

where the union is disjoint and the sum converges in the strong operator topology.

We can now give a definition of the essential spectrum of a selfadjoint operator.

Definition B.16 (Essential Spectrum)**.** *Let $\mathcal{A}$ be a selfadjoint operator on $\mathfrak{H}$. Then $\lambda \in \mathbb{R}$ is in the essential spectrum of $\mathcal{A}$ written $\lambda \in \Sigma_{ess}(\mathcal{A})$ iff for any $\epsilon > 0$, the range of the spectral projection $\mathcal{P}_{(-\epsilon+\lambda,\lambda+\epsilon)} = 1_{(-\epsilon+\lambda,\lambda+\epsilon)}(\mathcal{A})$ is infinite dimensional.*

now we have the basics defined we can state some results.

Theorem B.2. *Let $\mathcal{T}$ be a selfadjoint operator in a Hilbert space. Let $\mathcal{A}$ be a symmetric operator on the same space which is $\mathcal{T}$-bounded with $\mathcal{T}$-bound less than 1. Then $\mathcal{T} + \mathcal{A}$ is selfadjoint, and if $\mathcal{T}$ has compact resolvent then so does $\mathcal{T} + \mathcal{A}$.*

Proof. This is a combination of [8, IV. Thm 3.17 and V. Thm 4.3.]. □

The reader should note that having compact resolvent is a stronger property than having discrete spectrum.

Theorem B.3. *The essential spectrum of closed operators is conserved under perturbation by relatively compact operators.*

Proof. This is exactly [8, IV. Thm 5.35]. □

Definition B.17 (Convergence in the Strong Resolvent Sense)**.** *Let $\mathcal{A}_n$, $n = 1, 2, 3, \dots$ and $\mathcal{A}$ be selfadjoint operators. Then we say that $\mathcal{A}_n \to \mathcal{A}$ in the strong resolvent sense, if for all $\operatorname{Im} \lambda \neq 0$, the resolvents $\mathcal{R}_n^\lambda$ of $\mathcal{A}_n$ converge strongly to the resolvents $\mathcal{R}^\lambda$ of $\mathcal{A}$ as $n \to \infty$.*

Theorem B.4. *Let $\mathcal{A}_n$, $n = 1, 2, 3, \dots$ and $\mathcal{A}$ be selfadjoint operators with common core $\mathfrak{A}$. Then if for all $\varphi \in \mathfrak{A}$ we have the convergence $\mathcal{A}_n\varphi \to \mathcal{A}\varphi$, then $\mathcal{A}_n \to \mathcal{A}$ in the strong resolvent sense.*

Proof. See for example [13, Theorem VIII.25(a)]. □

Theorem B.5. *Let $\mathcal{A}_n$, $n = 1, 2, 3, \dots$ and $\mathcal{A}$ be selfadjoint operators and $\mathcal{A}_n \to \mathcal{A}$ in the strong resolvent sense. Let E_n, E be the spectral measures of $\mathcal{A}_n, \mathcal{A}$ respectively. Then*

(a) If $a, b \in \mathbb{R}$ with $a < b$, and $(\mathcal{A}, b) \cap \Sigma(\mathcal{A}_n) = \emptyset$ for all n, then $(\mathcal{A}, b) \cap \Sigma(\mathcal{A}) = \emptyset$. That is, if $\lambda \in \Sigma(\mathcal{A})$ then there is a sequence $\lambda_n \in \Sigma(\mathcal{A}_n)$ that converges to λ.

(b) *If $a, b \in \mathbb{R}$ with $a < b$, and $a, b \notin \Sigma_{pp}(\mathcal{A})$ (That is a, b are not in the pure point spectrum of $\mathcal{A}$), then the spectral projections $1_{(a,b)}(\mathcal{A}_n) = \int 1_{(a,b)}\, dE_n$ converge strongly as bounded operators as $n \to \infty$ to the spectral projection $1_{(a,b)}(\mathcal{A}) = \int 1_{(a,b)}\, dE$.*

Proof. See for example [13, Theorem VIII.24]. $\square$

C Auxiliary calculations

Lemma C.1. *Let $\boldsymbol{A}(\boldsymbol{x}), \phi(\boldsymbol{x})$ be distributions. Then the equation* (C.1) *below holds in the sense of distributions.*

$$\begin{aligned} &\pm(\boldsymbol{\nabla}_x\phi + \lambda\boldsymbol{A} - \hat{\boldsymbol{v}} \times (\boldsymbol{\nabla}_x \times \boldsymbol{A})) \cdot \boldsymbol{\nabla}_v \mu^{\pm} = \\ &\quad = \pm(\mathcal{D}_{\pm} + \lambda)(\mu_e^{\pm}\phi + \boldsymbol{\mu}_{\boldsymbol{p}}^{\pm} \cdot \boldsymbol{A}) + \\ &\quad \pm \lambda\mu_e^{\pm}(-\phi + \hat{\boldsymbol{v}} \cdot \boldsymbol{A}) + \\ &\quad \pm \boldsymbol{\mu}_{\boldsymbol{p}}^{\pm} \cdot \boldsymbol{\nabla}_x(\phi - \hat{\boldsymbol{v}} \cdot \boldsymbol{A}) \end{aligned} \tag{C.1}$$

Proof. This is just a long stream of vector calculus. The only thing to note is that $\mathcal{D}_{\pm}\phi = \hat{\boldsymbol{v}} \cdot \boldsymbol{\nabla}_x\phi$ as ϕ depends only on $\boldsymbol{x}$, and that $\mathcal{D}_{\pm}$ commutes with multiplication by $\mu^{\pm}$ or its derivatives.

$$\begin{aligned} \boldsymbol{\nabla}_v\mu^{\mp} &= \hat{\boldsymbol{v}}\mu_e^{\pm} + \boldsymbol{\mu}_{\boldsymbol{p}}^{\pm} \\ \pm(\boldsymbol{\nabla}_x\phi + \lambda\boldsymbol{A}) \cdot \boldsymbol{\nabla}_v\mu^{\mp} &= \pm(\boldsymbol{\nabla}_x\phi + \lambda\boldsymbol{A}) \cdot (\mu_e^{\pm}\hat{\boldsymbol{v}} + \boldsymbol{\mu}_{\boldsymbol{p}}^{\pm}) \\ &= \pm\mu_e^{\pm}((\hat{\boldsymbol{v}} \cdot \boldsymbol{\nabla}_x)\phi + \lambda\phi) \pm \lambda\mu_e^{\pm}(-\phi + \hat{\boldsymbol{v}} \cdot \boldsymbol{A}) \\ &\quad \pm \boldsymbol{\mu}_{\boldsymbol{p}}^{\pm} \cdot ((\hat{\boldsymbol{v}} \cdot \boldsymbol{\nabla}_x)\boldsymbol{A} + \lambda\boldsymbol{A}) + \\ &\quad \pm \boldsymbol{\mu}_{\boldsymbol{p}}^{\pm} \cdot (\boldsymbol{\nabla}_x\phi - (\hat{\boldsymbol{v}} \cdot \boldsymbol{\nabla}_x)\boldsymbol{A}) \\ &= \pm(\mathcal{D}_{\pm} + \lambda)(\mu_e^{\pm}\phi) \pm (\mathcal{D}_{\pm} + \lambda)(\boldsymbol{\mu}_{\boldsymbol{p}}^{\pm} \cdot \boldsymbol{A}) + \\ &\quad \pm \lambda\mu_e^{\pm}(-\phi + \hat{\boldsymbol{v}} \cdot \boldsymbol{A}) + \\ &\quad \pm \boldsymbol{\mu}_{\boldsymbol{p}}^{\pm} \cdot (\boldsymbol{\nabla}_x\phi - (\hat{\boldsymbol{v}} \cdot \boldsymbol{\nabla}_x)\boldsymbol{A}) \\ \mp(\hat{\boldsymbol{v}} \times (\boldsymbol{\nabla}_x \times \boldsymbol{A})) \cdot \boldsymbol{\nabla}_v\mu^{\mp} &= \mp(\hat{\boldsymbol{v}} \times (\boldsymbol{\nabla}_x \times \boldsymbol{A})) \cdot (\hat{\boldsymbol{v}}\mu_e^{\pm} + \boldsymbol{\mu}_{\boldsymbol{p}}^{\pm}) \\ &= \mp(\hat{\boldsymbol{v}} \times \hat{\boldsymbol{v}}) \cdot (\boldsymbol{\nabla}_x \times \boldsymbol{A}) + \\ &\quad \mp (\hat{\boldsymbol{v}} \times (\boldsymbol{\nabla}_x \times \boldsymbol{A})) \cdot \boldsymbol{\mu}_{\boldsymbol{p}}^{\pm} \\ &= \mp(\hat{\boldsymbol{v}} \times (\boldsymbol{\nabla}_x \times \boldsymbol{A})) \cdot \boldsymbol{\mu}_{\boldsymbol{p}}^{\pm} \end{aligned} \tag{C.2}$$

We now use the standard identity,

$$\boldsymbol{\nabla}(\boldsymbol{X} \cdot \boldsymbol{Y}) = (\boldsymbol{X} \cdot \boldsymbol{\nabla})\boldsymbol{Y} + (\boldsymbol{Y} \cdot \boldsymbol{\nabla})\boldsymbol{X} + \boldsymbol{X} \times (\boldsymbol{\nabla} \times \boldsymbol{Y}) + \boldsymbol{Y} \times (\boldsymbol{\nabla} \times \boldsymbol{X})$$

to obtain that,

$$\begin{aligned} \boldsymbol{\nabla}_x(\hat{\boldsymbol{v}} \cdot \boldsymbol{A}) &= \hat{\boldsymbol{v}} \cdot \boldsymbol{\nabla}_x\boldsymbol{A} + \boldsymbol{A} \cdot \boldsymbol{\nabla}_x\hat{\boldsymbol{v}} + \hat{\boldsymbol{v}} \times (\boldsymbol{\nabla}_x \times \boldsymbol{A}) + \boldsymbol{A} \times (\boldsymbol{\nabla}_x \times \hat{\boldsymbol{v}}) \\ &= \hat{\boldsymbol{v}} \cdot \boldsymbol{\nabla}_x\boldsymbol{A} + \hat{\boldsymbol{v}} \times (\boldsymbol{\nabla}_x \times \boldsymbol{A}). \end{aligned} \tag{C.3}$$

So we can combine two of the terms to obtain (C.1). $\square$

Lemma C.2. *Let* $e^{\pm}, \boldsymbol{p}^{\pm}$ *be defined by* (2.7) *and assume that* (2.8) *holds. Then* $e^{\pm}, \boldsymbol{p}^{\pm}$ *are conserved quantities of* (RVM 3D)*, by which we mean that*

$$\mathcal{D}_{\pm}e^{\pm} = 0 \qquad\qquad \mathcal{D}_{\pm}p_i^{\pm} = 0, \quad i = 1,2,3 \tag{C.4}$$

Proof. For $e^{\pm}$ we note that $\boldsymbol{\nabla}_v e^{\pm} = \hat{\boldsymbol{v}}$ and $\boldsymbol{\nabla}_x e^{\pm} = \pm \boldsymbol{\nabla}_x \phi^0$. Hence,

$$\begin{aligned} \mathcal{D}_{\pm}e^{\pm} &= \hat{\boldsymbol{v}} \cdot \boldsymbol{\nabla}_x e^{\pm} \pm (-\boldsymbol{\nabla}_x \phi^0 + \hat{\boldsymbol{v}} \times (\boldsymbol{\nabla}_x \times \boldsymbol{A}^0)) \cdot \boldsymbol{\nabla}_v e^{\pm} \\ &= (\hat{\boldsymbol{v}} \times (\boldsymbol{\nabla}_x \times \boldsymbol{A}^0)) \cdot \hat{\boldsymbol{v}} = 0 \end{aligned} \tag{C.5}$$

Similarly for $\boldsymbol{p}^{\pm}$ we consider $\mathcal{D}_{\pm}\boldsymbol{p}^{\pm} = (\mathcal{D}_{\pm}p_1^{\pm}, \mathcal{D}_{\pm}p_2^{\pm}, \mathcal{D}_{\pm}p_3^{\pm})$. We note that $(\hat{\boldsymbol{v}} \cdot \boldsymbol{\nabla}_x)\boldsymbol{p}^{\pm} = \pm(\hat{\boldsymbol{v}} \cdot \boldsymbol{\nabla}_x)\boldsymbol{A}^0$, and $\boldsymbol{\nabla}_v \boldsymbol{p}^{\pm}$ is the identity matrix. Hence,

$$\mathcal{D}_{\pm}\boldsymbol{p}^{\pm} = \pm \hat{\boldsymbol{v}} \cdot \boldsymbol{\nabla}_x \boldsymbol{A}^0 \pm (-\boldsymbol{\nabla}_x \phi^0 + \hat{\boldsymbol{v}} \times (\boldsymbol{\nabla}_x \times \boldsymbol{A}^0)) \tag{C.6}$$

We then use (C.3) to equate this to,

$$\mathcal{D}_{\pm}\boldsymbol{p}^{\pm} = \pm \boldsymbol{\nabla}_x(\hat{\boldsymbol{v}} \cdot \boldsymbol{A}^0 - \phi^0) \tag{C.7}$$

Then by (2.8) we deduce that in the directions were $\mu^{\pm}$ depends on $\boldsymbol{p}^{\pm}$ this is zero, and in the other directions $\mu^{\pm}$ is independent anyway. $\square$

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