The quasineutral limit of the Vlasov–Poisson system

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The quasineutral limit of Vlasov-Poisson

$$\begin{cases} \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + E_\varepsilon \cdot \nabla_v f_\varepsilon = 0, \quad t \geq 0, (x, v) \in \mathbb{T}^d \times \mathbb{R}^d \\ E_\varepsilon = -\nabla_x U_\varepsilon, \\ U_\varepsilon - \varepsilon^2 \Delta_x U_\varepsilon = \displaystyle\int_{\mathbb{R}^d} f_\varepsilon \, dv - 1, \\ f_\varepsilon|_{t=0} = f_{0,\varepsilon} \geq 0, \ \displaystyle\int_{\mathbb{T}^d \times \mathbb{R}^d} f_{0,\varepsilon} \, dx \, dv = 1. \end{cases}$$

• f_ε describes the dynamics of ions, in a background of massless electrons following a linearized Maxwell-Boltzmann law :

$$n_e = e^{U_\varepsilon} \sim 1 + U_\varepsilon.$$

• The parameter $\varepsilon \in (0, 1]$ is the ratio between the **Debye length** and the observation length. In practice, $\varepsilon \ll 1$.

• **Quasineutral limit** : $\varepsilon \to 0$.

Vlasov-Dirac-Benney

Taking $\varepsilon = 0$ yields

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x \rho \cdot \nabla_v f = 0, \\ \rho = \displaystyle\int_{\mathbb{R}^d} f \, dv, \\ f|_{t=0} = f_0 \geq 0, \quad \displaystyle\int_{\mathbb{R}^d} f_0 \, dv = 1. \end{cases}$$

- ... a system called **Vlasov-Dirac-Benney** by Bardos.

- **Loss of derivative ?** The force $-\nabla_x \rho$ is one derivative less regular than f.

- Is Vlasov-Dirac-Benney a good approximation of Vlasov-Poisson when $\varepsilon \to 0$?

More on Vlasov-Dirac-Benney

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x \rho \cdot \nabla_v f = 0, \\ f\big|_{t=0} = f_0 \geq 0, \quad \displaystyle\int_{\mathbb{R}^d} f_0 \, dv = 1. \end{cases}$$

Existence of solutions is known for

- **analytic** initial data (Cauchy-Kowlevski type result) ;
- in $d = 1$, Sobolev initial data that, for all x, have the shape of one bump [Bardos, Besse 2013], through a water-bag rep. ;
- Penrose stable Sobolev initial data [DHK, Rousset 2015].

There are equilibria around which the linearized equations have **unbounded unstable spectrum** [Bardos, Nouri 2012]. This implies **illposedness properties** : the flow map around these equilibria is very irregular [DHK, T. Nguyen 2015].

Quasineutral limit and large time behavior

• For all $\varepsilon \in (0,1]$, the Cauchy theory is very well understood (Arsenev, Ukai-Okabe, Pfaffelmoser, Schaeffer, Lions-Perthame, Batt-Rein,...), but does not provide useful **uniform** estimates.

Using these only yield a weak form of the limit with **defect measures** [Brenier, Grenier '94], [Grenier '95].

• The change of variables $(t, x, v) \mapsto (\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, v)$ gives the **unscaled** Vlasov-Poisson system.

Quasineutral limit $\rightarrow$ **Large Time Behavior** problem

• As we shall see, **the stability or instability** of homogeneous equilibria play a decisive role in the derivation of Vlasov-Dirac-Benney in the quasineutral limit.

Plan of the talk

Invalidity of Vlasov-Dirac-Benney in the quasineutral limit

- [Grenier '99], [DHK, Hauray 2015]

Validity of Vlasov-Dirac-Benney in the quasineutral limit

- Uniform analytic regularity [Grenier '96]
- Zero-temperature limit [Brenier 2000], [DHK 2011]
- **General Penrose stable data** [DHK, Rousset 2015]

Nonlinear instability

Penrose instability conditions ensure the spectral instability of homogeneous equilibria of Vlasov-Poisson (**two-stream instabilities**). In [Guo, Strauss '95], it is proved that spectral instability implies **nonlinear instability** as well.

Theorem 1 (*DHK, Hauray, 2015*)

Let $\mu(v)$ be a **smooth** *Penrose unstable equilibrium. For all $n \geq 0$, there is $\theta > 0$ such that, for all $\delta > 0$, there is a solution $g(t)$ of Vlasov-Poisson with*

$$\|g(0) - \mu\|_{W^{n,1}_{x,v}} \leq \delta$$

but

$$\sup_{t \in [0, t_\delta]} \|g(t) - \mu\|_{W^{-n,1}_{x,v}} \geq \theta > 0$$

with $t_\delta = O(|\log \delta|)$ as $\delta \to 0$.

A non-derivation result

Combining the previous **nonlinear instability theorem** and the change of variables $(t, x, v) \mapsto \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, v\right)$ (**high frequency** regime), we deduce that Vlasov-Dirac-Benney is not a good approximation near unstable equilibria.

Theorem 2 (*DHK, Hauray 2015*)

Let $\mu(v)$ be a smooth Penrose unstable equilibrium. For all $n, k \geq 0$, there exists a sequence of solutions $(f_\varepsilon(t))$ such that

$$\|f_\varepsilon(0) - \mu\|_{W^{n,1}_{x,v}} \leq \varepsilon^k,$$

but

$$\liminf_{\varepsilon \to 0} \sup_{t \in [0, \varepsilon^{1/2}]} \|f_\varepsilon(t) - \mu\|_{W^{-n,1}_{x,v}} > 0.$$

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Derivation in Analytic regularity

• In [Grenier, '96], it is shown that two-stream instabilities have no effect for solutions with uniform **analytic regularity**.

• Loosely speaking, the principle of his proof is to write the distribution function f_ε as the **superposition of layers of fluids**.

For some fixed probability space $(M, \mu(d\theta))$, write the decomposition

$$f_\varepsilon(t,x,v) = \int_M \rho_\varepsilon^\theta(t,x)\delta_{v=u_\varepsilon^\theta(t,x)}(v)\mu(d\theta),$$

Derivation in Analytic regularity

This leads to the study of a **system of coupled Burgers eq.** :

$$\begin{cases} \partial_t \rho_\varepsilon^\theta + \nabla_x \cdot (\rho_\varepsilon^\theta u_\varepsilon^\theta) = 0, \\ \partial_t u_\varepsilon^\theta + u_\varepsilon^\theta \cdot \nabla_x u_\varepsilon^\theta = -\nabla_x U_\varepsilon, \\ U_\varepsilon - \varepsilon^2 \Delta_x U_\varepsilon = \int_M \rho_\varepsilon^\theta \, \mu(d\theta) - 1. \end{cases}$$

Theorem 3 (*Grenier, '96*)

Assume that for f_0 with analytic regularity ($\| \cdot \|$ is a norm that is analytic in x)

$$\sup_v \| f_{\varepsilon,0} - f_0 \| \to 0.$$

Then there is a finite time interval on which f_ε weakly converges to a weak solution to Vlasov-Dirac-Benney with initial condition f_0.

In [DHK, Iacobelli, 2015] : still true for **exponentially small but rough** perturbations of such data ($d \leq 3$).

Derivation in stable cases ?

• Is it possible to say something under an assumption of **Penrose stability** on the initial condition ?

• The first result in this direction is due to [Brenier, 2000] where the **Modulated Energy method** was introduced (see also [Yau, '94], [Golse, 2000]) .

For **monokinetic data**

$$f(t,x,v) = \rho(t,x)\delta_{v=u(t,x)},$$

note that f satisfies Vlasov-Dirac-Benney iff (ρ, u) satisfies the **isentropic Euler system** (with $\gamma = 2$) :

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0, \\ \partial_t u + u \cdot \nabla_x u + \nabla_x \rho = 0, \\ (\rho, u)|_{t=0} = (\rho_0, u_0). \end{cases}$$

Derivation in the zero–temperature limit

- Consider

$$f_{0,\varepsilon} \rightharpoonup \rho_0(x)\delta_{v=u_0(x)}$$

(zero-temperature limit, extremal case of a Maxwellian (**stable**)).

- Following [Brenier, 2000], introduce

$$\mathcal{H}_\varepsilon(t) := \frac{1}{2}\int f_\varepsilon |v - u(t,x)|^2 \, dvdx \\ + \frac{1}{2}\int (U_\varepsilon - \rho(t,x))^2 \, dx + \frac{\varepsilon^2}{2}\int |E_\varepsilon(t,x)|^2 \, dx.$$

where (ρ, u) solves the **isentropic Euler system** on $[0, T]$.

- One proves that

$$\frac{d}{dt}\mathcal{H}_\varepsilon(t) \lesssim \mathcal{H}_\varepsilon(t) + o(1)$$

so that roughly

$$f_{0,\varepsilon} \rightharpoonup \rho_0(x)\delta_{v=u_0(x)} \implies \forall t \in [0, T], f_\varepsilon(t) \rightharpoonup \rho(t,x)\delta_{v=u(t,x)}.$$

May one generalize the modulated energy method ?

• A natural idea would be to adapt this method to handle other stable initial conditions.
• [DHK, Hauray, 2015] : works for stationary $\mu(v)$ satisfying

$$\nearrow \text{ on } (-\infty, 0]\ ,\ \searrow \text{ on } [0, +\infty) \text{ and } \textbf{even}$$

• Fails to handle other stable initial data one would like to consider, for **symmetry** and **rigidity** reasons.
The modulated energy method requires that the solution of the limit system is the **minimizer of some entropy** and thus satisfies

$$f \equiv g(t, x, -|v - v(t,x)|^2).$$

We prove that such solutions of Vlasov-Dirac-Benney are necessarily stationary...

Derivation result for stable data

- We say that $\mathbf{f}(v)$ satisfies the c_0 **Penrose stability condition** if

$$\inf_{(\gamma,\tau,\eta)\in\mathbb{R}_*^+\times\mathbb{R}\times\mathbb{R}_*^d}\left|1-\int_0^{+\infty}e^{-(\gamma+i\tau)s}\,\frac{i\eta}{1+|\eta|^2}\cdot(\mathcal{F}_v\nabla_v\mathbf{f})(\eta s)\,ds\right|\geq c_0.$$

(Recall that this also appears for Landau Damping [Mouhot, Villani, 2011].)

- Introduce also for $k\in\mathbb{N}$, $r\in\mathbb{R}$, the weighted Sobolev norms

$$\|f\|_{\mathcal{H}_r^k}:=\left(\sum_{|\alpha|+|\beta|\leq k}\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}(1+|v|^2)^r|\partial_x^\alpha\partial_v^\beta f|^2\,dvdx\right)^{1/2}$$

and the regularity indices

$$k_0=4+d,\quad r_0=\max(d,2+\frac{d}{2}).$$

Derivation result for stable data

Theorem 4 (*DHK, Rousset 2015*)

Let $2m > k_0$, $2r > r_0$. *Let* $M_0 > 0, c_0 > 0$. *Assume that for all* $\varepsilon \in (0,1]$, $\|f_{0,\varepsilon}\|_{\mathcal{H}^{2m}_{2r}} \leq M_0$ *and for all* $x \in \mathbb{T}^d$, $f_{0,\varepsilon}(x,\cdot)$ *satisfies the* c_0 *Penrose stability condition. Assume that* $f_{0,\varepsilon} \to f_0$ *in* L^2. *Then there is* $T > 0$ *such that*

$$\sup_{[0,T]} \|f_\varepsilon(t) - f(t)\|_{L^2} \to 0,$$

where $f(t)$ *satisfies Vlasov-Dirac-Benney with initial data* f_0.

As a by-product we get well-posedness (i.e. existence + uniqueness) in the class of such data for Vlasov-Dirac-Benney.

Sketch of the proof

Recall $\|f_{0,\varepsilon}\|_{\mathcal{H}^{2m}_{2r}} \leq M_0$. Introduce

$$\mathcal{N}_{2m,\,2r}(t, f_\varepsilon) := \|f_\varepsilon\|_{L^\infty((0,t),\mathcal{H}^{2m-1}_{2r})} + \|\rho_\varepsilon\|_{L^2((0,t),H^{2m})},$$

with $\rho_\varepsilon = \int_{\mathbb{R}^d} f_\varepsilon \, dv$. The main task is to find $T > 0$, $R > 0$ such that

$$\forall \varepsilon \in (0,1], \quad \sup_{[0,T]} \mathcal{N}_{2m,\,2r}(t, f_\varepsilon) \leq R.$$

The proof is based on a bootstrap argument. By a standard energy estimate, we see that the key quantity to be controlled is actually $\|\rho_\varepsilon\|_{L^2((0,t),H^{2m})}$.

Sketch of the proof

• Natural idea : up to commutators, $\partial_x^{2m} f_\varepsilon$ evolves according to the linearized equation about f_ε, that is

$$\partial_t \partial_x^{2m} f_\varepsilon + v \cdot \nabla_x \partial_x^{2m} f_\varepsilon + \partial_x^{2m} E_\varepsilon \cdot \nabla_v f_\varepsilon + E_\varepsilon \cdot \nabla_v \partial_x^{2m} f_\varepsilon = S,$$

where S should involve remainder terms only.

• When $f_\varepsilon \equiv \mu(v)$ does not depend on t and x, then the linearized equation reduces to

$$\partial_t g + v \cdot \nabla_x g + E_g \cdot \nabla_v \mu(v) = S,$$

yielding an integral equation for $\rho_g = \int_{\mathbb{R}^d} g \, dv$ by solving the free transport equation and integrating in v [Mouhot, Villani, 2011].
By Fourier analysis, under a Penrose stability condition for $\mu(v)$, one may estimate ρ_g in $L^2_{t,x}$.

Sketch of the proof

• However, when applying this strategy, there are subprincipal terms which involve $2m$ derivatives of f :

$$\partial_x E_\varepsilon \cdot \nabla_v \partial_x^{2m-1} f_\varepsilon.$$

• Applying more general vector fields would also generate bad subprincipal terms. Instead : consider powers of relevant **second order operators**, yielding $(f_{i,j})_{1 \leq i,j \leq d}$ that satisfy two key properties.

- They control ρ_ε in the sense that

$$\int_{\mathbb{R}^d} f_{i,j} \, dv = \partial_x^{2m} \rho_\varepsilon + R,$$

  where R is a good remainder.
- $f_{i,j}$ satisfies

$$\partial_t f_{i,j} + v \cdot \nabla_x f_{i,j} + E_\varepsilon \cdot \nabla_v f_{i,j} + E_{f_{i,j}} \cdot \nabla_v f_\varepsilon = S_{i,j}.$$

  where $S_{i,j}$ is a good remainder.

Sketch of the proof

We thus study

$$\partial_t g + v \cdot \nabla_x g + E_g \cdot \nabla_v f_\varepsilon + E_\varepsilon \cdot \nabla_v g = S.$$

As f_ε depends on x, E_ε is not trivial.

However, we can use a **near identity change of variables to straighten the vector field** and come down to the equation

$$\partial_t g + \Phi(t, x, v) \cdot \nabla_x g = S$$

where $\Phi(t, x, v)$ is close to v for small times.

Integrating along characteristics and integrating in v, we end up with the study, for **small times**, of...

Sketch of the proof

...the integral equation

$$\rho = K_{\nabla_v f_{0,\varepsilon}} (I - \varepsilon^2 \Delta)^{-1} \rho + R,$$

with

$$K_{\nabla_v f_{0,\varepsilon}}(G) = \int_0^t \int_{\mathbb{R}^d} (\nabla_x G)(s, x - (t-s)v) \cdot \nabla_v f_{0,\varepsilon}(x, v) \, dv \, ds.$$

Note that $K_{\nabla_v f_{0,\varepsilon}}$ may seem to feature a loss of derivative. However, we have

Proposition 1

$K_{\nabla_v \mu}$ is a bounded operator on L^2 if μ is sufficiently smooth.

This is an effect in the spirit of **averaging lemmas** ([Golse, Lions, Perthame, Sentis '88]).

Sketch of the proof

$$\rho = K_{\nabla_v f_{0,\varepsilon}}(I - \varepsilon^2 \Delta)^{-1}\rho + R$$

We finally relate $K_{\nabla_v f_{0,\varepsilon}}(I - \varepsilon^2 \Delta)^{-1}$ to a **semi-classical pseudodifferential operator**, of symbol

$$\int_0^{+\infty} e^{-(\gamma + i\tau)s} \frac{i\eta}{1 + |\eta|^2} \cdot (\mathcal{F}_v \nabla_v f_{0,\varepsilon})(x, \eta s)\, ds.$$

Thus, the c_0 Penrose condition implies the **ellipticity of the symbol** associated to $I - K_{\nabla_v f_{0,\varepsilon}}$.

We can finally use a **semi-classical pseudodifferential calculus with parameter** in order to invert $I - K_{\nabla_v f_{0,\varepsilon}}$ up to a **small remainder**, which yields an estimate for $\partial_x^{2m}\rho$ in $L^2_{t,x}$.
This allows to close the bootstrap argument.

Any question?