Hölder continuity of solutions to Vlasov-Fokker-Planck type equations with rough coefficients

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A motivation in nonlinear analysis of kinetic equations

- Oldest kinetic equation (Maxwell 1867, Boltzmann 1972): $\partial_t f + v \cdot \nabla_x f = Q(f, f)$
- Nonlinear (quadratic) PDE on $f = f(t, x, v) \geq 0$ with Q bilinear integral operator acting only on v
- For long-distance interactions Q has fractional ellipticity in v
- Global well-posedness major open problem for mathematicians
- Toy (much simpler) model: $\partial_t f + v \cdot \nabla_x f = \rho[f] \nabla_v \cdot (\nabla_v f + v f)$ (where $\rho[f] := \int_v f$) with quadratic nonlinearity and preserving mass
- Surprisingly it seems (to my knowledge) that global well-posedness and propagation of regularity still unknown for this model
- Follow De Giorgi and Nash's strategy when solving Hilbert's 19th problem: consider $\rho[f]$ as a given rough coefficient and try to derive some regularity on the solution

The De Giorgi - Nash - Moser theory (1)

- Hilbert's 19th problem: whether minimizers f of an energy functional

$$\int_U L(\nabla f)\,\mathrm{d}x \qquad \text{are analytic where } L \text{ Lagrangian}$$

- Euler-Lagrange equations for the minimizers take the form

$$\nabla_i \left(a^{ij}(x) \nabla_j f \right) = 0$$

  where $a^{ij} := \partial_{ij} L(\nabla f)$ and conditions on L ensures that the symmetric positive matrice $A = (a^{ij})$ is uniformly bounded above and below
- De Giorgi 1956 and Nash (1958): prove that the above conditions and $f \in L^2$ imply that the solution is L^∞ and Hölder (Nash considered the parabolic case) even when the coefficients are merely measurable
- It solves the problem of Hilbert by bootstrap

The De Giorgi - Nash - Moser theory (2)

- Proof of Nash relies on what is called now "Nash inequality" with an $L \log L$ estimate, and intricate estimates on the fundamental solution (not used here)
- Proof of De Giorgi relies on an iterative gain of integrability and an "isoperimetric lemma" to control oscillations
- Proof of Moser relies on a similar iterative gain of integrability for positive and then "negative" Lebesgue spaces and a clever control of oscillation for relating positive and negative Lebesgue norms by studying the equation on $\ln f$
- We use the iteration in the Moser form and the control of oscillation in the De Giorgi form (but the hypoelliptic nature creates new difficulties)
- Remark that there is a non-divergence form of the theory by Krylov-Safonov (not considered here)

The Hörmander theory of hypoellipticity (1)

- Starting point: 3 pages note of Kolmogorov in Annals of Math. 1934 "*Zufallige Bewegungen (Zur Theorie der Brownschen Bewegung)*"
- Kolmogorov considered $d = 1$ and A constant and with a possible constant drift (thus sometimes called "Kolmogorov equation")

$$\partial_t f + v \cdot \partial_x f + b\partial_v f = a\partial_v^2 f \text{ and compute fundamental solutions}$$

$$G(t,x,v,t',x',v') = \frac{2\sqrt{3}}{\pi a^2 (t'-t)^2} \times \\ \exp\left\{ -\frac{[v'-v-b(t'-t)]^2}{4a(t'-t)} - \frac{3\left[x'-x-\frac{v'+v}{2}(t'-t)\right]^2}{a^3(t'-t)^3} \right\}$$

The Hörmander theory of hypoellipticity (2)

- Hörmander 1967's seminal paper starts from observing the regularisation of this fundamental solution and builds a general theory based on commutator estimates
- Related to Malliavin calculus in probability
- Regularisation Gevrey instead of analytic for parabolic equations
- Simpler case when no first order part and missing directions of diffusion recovered by commutator ("Hörmander type I"): NDGM theory already extended in this case
- Hörmander theory local: global estimates derived under the impulsion of hypocoercivity
- Commutator estimates: $\partial_t f + Bf + A^* A f = 0$, $B = v \cdot \partial_x$, $A = \partial_v$ $[A, B] = C = \partial_x$ missing direction: $\frac{\mathrm{d}}{\mathrm{d}t}\langle Af, Bf\rangle = -\|Cf\|^2 + \dots$

An hypoelliptic extension of the NDGM theory

Theorem (Imbert-Golse-CM-Vasseur)

We consider the following Vlasov-Fokker-Planck type equation

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (A(t,x,v)\nabla_v f)$$

where the $d \times d$ symmetric matrix A satisfies the ellipticity condition $0 < \lambda Id \leq A \leq \Lambda Id$ but is, besides that, merely measurable.
We define for $z = (t,x,v)$ the cube $Q_r(z) = B_{r^3}(x) \times B_r(v) \times (t - r^2, t]$.
Then for $0 < r_1 < r_0$, if f is a solution in $Q_{r_0(z_0)}$ then

$$\|f\|_{L^\infty(Q_{r_1}(z_0))} + \|f\|_{C^\alpha(Q_{r_1}(z_0))} \leq C\|f\|_{L^2(Q_{r_0}(z_0))}$$

where C depends on $z_0, r_0, r_1, \lambda, \Lambda, d$ and α only depends on λ, Λ, d.

Gain of L^∞ done before in [Pascucci-Polidoro 2004]
C^α claimed in [Wang-Zhang 2011] (I don't understand their proof)

Gain of integrability - The elliptic case (following Moser)

- We consider, with $f = f(t,v)$ and g source term nicely behaved:

$$\nabla_v \left(A(t,v) \nabla_v f \right) = g$$

- Central energy estimate:

$$\|f\|_{H^1(Q_{r_1})} \lesssim \frac{1}{(r_0 - r_1)^2} \|f\|_{L^2(Q_{r_0})}$$

- Sobolev embedding translates the gain H^1 into L^p, $p > 2$
- Iteration by applying the argument to any subsolution $f^{p/2}$, $p \geq 2$, for a sequence of radii $r_n \to r_\infty > 0$, to get finally L^∞ in Q_{r_∞}
- It relies crucially on the ellipticity of the operator in all variables

The parabolic case (following Moser)

- We now consider the parabolic case of the classical theory as it closer to the hypoelliptic we want to treat, with $f = f(t, v)$:

$$\partial_t f = \nabla_x \left(A(t, v) \nabla_v f \right)$$

- Central energy estimate:

$$\left(\int_{v \in B_{r_1}} f^2 \, \mathrm{d}v \right)_{t=T} + \int_{T - r_1^2}^{T} |\nabla_v f|^2 \, \mathrm{d}v \, \mathrm{d}t \lesssim \frac{1}{(r_0 - r_1)^2} \int_{T - r_0^2}^{T} \int_{v \in B_{r_0}} f^2 \, \mathrm{d}v \, \mathrm{d}t$$

- Similar iteration argument in both variables t, v
- Again it relies crucially on the ellipticity of the operator in all variables

Difficulties here

- Coming to our equation $\partial_t f + v \cdot \nabla_x f = \nabla_v (A \nabla_v f)$ we can derive the central energy estimate as above:

$$\left(\int_{x \in B_{r_1^3},\ v \in B_{r_1}} f^2 \,\mathrm{d}x\,\mathrm{d}v \right)_{t=T} + \int_{T-r_1^2}^{T} |\nabla_v f|^2 \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t$$

$$\lesssim \frac{1}{(r_0 - r_1)^2} \int_{T-r_0^2}^{T} \int_{x \in B_{r_0^3},\ v \in B_{r_0}} f^2 \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t$$

- Problem: control only on v-gradients, not x-gradients
- Key tool in kinetic theory to remedy this: averaging lemma [Golse-Perthame-Sentis 1985]
- Second problem: for the iteration we need to work on subsolutions (cf. $f^{p/2}$) for which averaging lemma do not hold

Strategy

- Averaging lemma: solutions to

$$\partial_t f + v \cdot \nabla_x f = \nabla_v^k g, \quad f, g \in L^2, \quad k \geq 0$$

  have more regularity on the partial v-averages: $\int_v f \, \mathrm{d}v \in H^s_{t,x}$, s small
- Averages "transversal" to cancellations of symbol of the hyperbolic transport operator (gain of regularity limited by order 1 of operator)
- It degenerates if RHS g not controlled $\Rightarrow$ problem for subsolutions

$$\partial_t f + v \cdot \nabla_x f \leq \nabla_v \left(A \nabla_v f \right) + \cdots = \nabla_v \cdot H_0 + H_1 \text{ with } H_0, H_1 \in L^2$$

- Comparison principle: $0 \leq f \leq F$ with a true solution F of previous equation on which energy estimate $L^2_{t,x} H^1_v$ plus averaging lemma $H^s_{t,x} L^1_v$ imply some $H^{s'}_{t,x,v}$ $(0 < s' < s)$ and thus some gain of integrability on F by Sobolev embedding that is inherited by f

The iteration

- The previous argument ends up with the following: there is $\kappa > 1$ such that for all $q > 1$:

$$\|(f^q)^\kappa\|^2_{L^2(Q_{r_1})} \lesssim \left(\frac{1}{(r_0 - r_1)^2} + \frac{1}{r_0(r_1 - r_0)}\right) \|f^q\|^{2\kappa}_{L^2(Q_{r_0})}$$

- Choose $q = q_n = 2\kappa^n$ and $R_{n+1} = R_n - \frac{1}{a(n+1)^2}$ (a small)
- The constant at step n is $C_n \sim c(a^2 n^4 + bn^2)^\kappa$ and

$$\prod_{n=0}^{+\infty} C_n^{\frac{1}{2\kappa^n}} < +\infty, \qquad R_n \to R_\infty > 0$$

which implies the convergence of the iteration

Regularity of non-negative subsolutions revisited (1)

- With the $L^2 - L^\infty$ gain at hand we can now revisit the question of the regularity of subsolutions to $\partial_t f + v \cdot \nabla_x f \leq \nabla_v \cdot H_0 + H_1$, which we write

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot H_0 + H_1 - \mu$$

  with a measure $\mu \geq 0$ and $H_0, H_1 \in L^2$ and $0 \leq f \in L^\infty_{t,x,v} \cap L^2_{t,x} H^1_v$
- The estimate on the mass $\int_{t,x,v} f\phi$ (for a cutoff function ϕ) yields

$$\|\mu\|_{M^1(Q_{r_1})} \lesssim \|f\|_{L^2(Q_{r_0})} + \|H_0\|_{L^2(Q_{r_0})} + \|H_1\|_{L^2(Q_{r_0})}$$

  which is a surprisingly strong control on the unknown error μ

Regularity of non-negative subsolutions revisited (2)

- We then write $-\mu = (1-\Delta_x)^{1/4} g$ with $g \in L^p$, $p \in (1,2)$ by ellipticity of the fractional Laplacian, and apply refined averaging lemma on

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot H_0 + H_1 + (1-\Delta_x)^{1/4} g$$

  in L^p to get some $W^{s,p}_{t,x} L^1_v$ regularity for a small s
- We then interpolate with the L^∞ regularity to deduce some $H^{s'}_{t,x} L^1_v$ regularity
- Finally we combine it with the energy estimate $L^2_{t,x} H^1_v$ to get $f \in H^{s''}_{t,x,v}$ for some (possibly very small) $s'' > 0$
- Note that because of the interpolation with L^∞ to "bring back" the regularity obtained in L^2 this argument does not supersede the previous comparison principle in the $L^2 - L^\infty$ iteration, which is still needed

Gain of integrability of $\nabla_v f$ (1)

- Another property on subsolutions $\partial_t f + v \cdot \nabla_x f \leq \nabla_v (A \nabla_v f)$ which is (surprisingly) as strong as in the parabolic case

Theorem

There is $\varepsilon > 0$ universal so that

$$\int_{Q_{r_1}} |\nabla_v f|^{2+\varepsilon} \, \mathrm{d}t \, \mathrm{d}x \, \mathrm{d}v \lesssim_{r_0, r_1, \lambda, \Lambda, d} \left(\int_{Q_{r_0}} |\nabla_v f|^2 \, \mathrm{d}t \, \mathrm{d}x \, \mathrm{d}v \right)^{\frac{2+\varepsilon}{2}}$$

- It follows (iteration) from "Gehring lemma": given $q > 1$ if there is θ small enough s.t. for all $z \in \Omega$

$$\fint_{Q_r(z)} g^q \leq C_\theta \left(\fint_{Q_{8r}(z)} g \, \mathrm{d}z \right)^q + \theta \fint_{Q_{8r}(z)} g^q \, \mathrm{d}z$$

then $(\fint_{Q_r} g^{q+\varepsilon} \, \mathrm{d}z)^{1/(q+\varepsilon)} \lesssim (\fint_{Q_{4r}} g^q \, \mathrm{d}z)^{1/q})$

Gain of integrability of $\nabla_v f$ (2)

- Proof of Gehring lemma is based on the following inequalities

$$(1) \qquad \int_{\mathcal{Q}_r} |\nabla_v f|^2 \, \mathrm{d}z \le \frac{C}{r^2} \int |f - \tilde{f}_{2r}|^2 \, \mathrm{d}z$$

$$(2) \qquad \sup_{t \in (T - r^2, T]} \int_{\mathcal{Q}_r^t} |f - \tilde{f}_r|^2 \, \mathrm{d}z \le C r^2 \int_{Q_{3r}} |\nabla_v f|^2 \, \mathrm{d}z$$

$$(3) \qquad \left(\int |f - \tilde{f}|^{q+\varepsilon} \right)^{1/(2+\varepsilon)} \mathrm{d}z \lesssim \left(\int |\nabla_v f|^2 \, \mathrm{d}z \right)^{1/2}$$

  proved by the energy estimate (written removing the x, v-average $\tilde{f}_{2r}$), fractional Poincaré in x, v, Sobolev embedding and the $H^s_{x,v}$ regularity for subsolutions (averages $\tilde{f}_{\dots}$ and cubes $\mathcal{Q}_{\dots}$ along free flow)
- Then boostrap the estimate on $\nabla_v f$ using $\int |f - \tilde{f}|^2$ as pivot

Control of oscillation: the classical theory (1)

- Clear observation: bad idea to differentiate the PDE
- Best to relate local suprema and infima (oscillation), i.e. control the modulus of continuity
- In view of the previous gain of integrability it is natural to continue in "L^∞" setting, i.e. Hölder regularity
- In the parabolic case, it takes time for the diffusive effect to manifest, and time delays must be taken into when comparing suprema and infima
- Moser's strategy: gains $L^\varepsilon - L^\infty$ and $L^{-\infty} - L^{-\varepsilon}$ and then compare L^ε and $L^{-\varepsilon}$ by studying the equation for $\ln f$ and using Poincaré inequality $\Rightarrow$ not clear how to follow this approach by lack of Poincaré inequality
- Moser arrives at comparing suprema and infima which is an independent property called Harnack inequality (see later)
- Harnack inequality implies Hölder regularity but not the opposite

Control of oscillation: the classical theory (2)

- De Giorgi's strategy: always consider oscillation as a whole without separating controls suprema and infima, and control decrease of oscillation when reducing the size of the cube considered
- Again time delays must be taken into account for the decrease to take place when considering our time-evolution case
- Main Lemma of decrease of oscillations: for f solution in Q_2 with $|f| \leq 1$ then $\mathrm{osc}_{Q_{1/2}} f \leq 2 - \delta$ for some $\delta > 0$
- It implies Hölder regularity at the point at which cubes shrink
- It is implied by the following Lemma of decrease of supremum bound: for f solution in Q_2 with $|f| \leq 1$ and $|\{f \leq 0\} \cap Q_1| \geq (1/2)|Q_1|$ then $\sup_{Q_{1/2}} f \leq 1 - \delta$
- This decrease of the supremum bound follows from the *isoperimetric De Giorgi argument*

De Giorgi isoperimetric argument (1)

- Original statement is proved by constructive isoperimetric calculation:

Theorem

Consider $f \in H^1$ on Q_2 with $f \leq 1$ and

$$\left| \left\{ f \geq \frac{1}{2} \right\} \cap Q_1 \right| \geq \delta_1 \quad \textit{and} \quad |\{f \leq 0\} \cap Q_1| \geq \delta_2$$

then there is $\alpha > 0$ depending on δ_1, δ_2 and the H^1 norm so that

$$\left| \left\{ 0 < f < \frac{1}{2} \right\} \cap Q_1 \right| \geq \alpha$$

- Here we don't have H^1 in all variables which seems critical in the calculation
- We argue by contradiction and compactness

De Giorgi isoperimetric argument (2)

- We want to prove (a source term could also be included...)

Theorem

For all $\delta_1, \delta_2 > 0$ *and* $f \le 1$ *solution of our equation on* Q_2 *and*

$$\left| \left\{ f \ge \frac{1}{2} \right\} \cap Q_1 \right| \ge \delta_1 \quad \text{and} \quad |\{ f \le 0 \} \cap Q_1| \ge \delta_2$$

there is $\alpha > 0$ *depending on* δ_1, δ_2

$$\left| \left\{ 0 < f < \frac{1}{2} \right\} \cap Q_1 \right| \ge \alpha$$

- We consider a contradiction sequence f_k, A_k (the diffusion matrix must be let depending on n, i.e. oscillating in order to prove something universal and independent of scaling-zooming)

De Giorgi isoperimetric argument (3)

- The sequence satisfies $\lambda \mathrm{Id} \leq A_k \leq \Lambda \mathrm{Id}$ and $f_k \leq 1$

$$\left| \left\{ f_k \geq \frac{1}{2} \right\} \cap Q_1 \right| \geq \delta_1 \quad \text{and} \quad |\{ f_k \leq 0 \} \cap Q_1| \geq \delta_2$$

$$\left| \left\{ 0 < f_k < \frac{1}{2} \right\} \cap Q_1 \right| \xrightarrow[k\to\infty]{} 0$$

- The positive part f_k^+ is a subsolution, it has a weak limit in $L^2_{t,x} H^1_v$ by the energy estimate and a strong limit in $L^2_{t,x,v}$ by the H^s regularity
- Consider $T = 1/2$ in $[1/2, 1]$ smooth, then the limit $g = T(f^+)$ satisfies $0 \leq g \leq 1$ and $\left| \left\{ 0 < g < \frac{1}{2} \right\} \cap Q_1 \right| = 0$
- And since

$$\int T''(f_n^+) A_n \nabla_v f_n \cdot \nabla_v f_n \phi \xrightarrow[k\to\infty]{} 0$$

by the gain of integrability on $\nabla_v f_n$, g is still a subsolution

De Giorgi isoperimetric argument (4)

- Hence we have built a subsolution g that only the values 0 and 1/2 but also inherits

$$\left| \left\{ g \geq \frac{1}{2} \right\} \cap Q_1 \right| \geq \delta_1 \quad \text{and} \quad |\{g \leq 0\} \cap Q_1| \geq \delta_2$$

- Observe that for almost every x, by the classical isoperimetric lemma in v (using the H^1_v control) it is constant in v, and $\nabla_v g = 0$
- It remains to identify the limit (product of weak limits)

$$h_k = A_k \nabla_v g_k \rightharpoonup_{L^2(Q_1)} h$$

- Using the equation integrated against $g_k \phi$ we have

$$\int h_k \cdot \nabla_v (g_k \phi) \to \int h \nabla_v (g \phi)$$

which implies since $\nabla_v g = 0$ and $g_k \to g$ strongly in L^2 that

$$\int h_k \cdot (\nabla_v g_k) \phi \to 0$$

De Giorgi isoperimetric argument (5)

- We deduce that

$$\int |h|^2 \phi \le \liminf \int |A_k \nabla_v g_k|^2 \phi$$

$$\le \Lambda \lim \int h_k \cdot \nabla_v g_k = 0$$

and therefore $h = 0$

- Finally we end up with $\partial_t g + v \cdot \nabla_x g \le 0$ with some mass at 0 and 1/2 which implies a contradiction by studying the trajectories of free transport
- Note that this proof is not quantitative and finding a quantitative proof would be very interesting

From the isoperimetric estimate to the supremum

- Use the $L^2 - L^\infty$ gain locally: if not enough L^2 mass then supremum is low
- Iterate $f_{k+1} = 2f_k - 1$ which preserves $f_k \leq 1$ and prove that for some k_0 large enough (finite) then $|\{f_{k_0} \geq 0\} \cap Q_1|$ is small enough to get $\|f_{k_0}\|_{L^\infty(Q_{1/2})} \leq 1/2$
- This means back on f: $f \leq 1 - 2^{-1-k_0}$
- k_0 has to exists since mass α necessary each time too much mass above...