Relativistic Diffusion

Simone Calogero

Chalmers University of Technology, Gothenburg

The Cauchy problem in kinetic theory

Imperial College London, 7-11 September 2015

What is diffusion?

Diffusion is the driving mechanism for the evolution of "particle" systems arising in many different fields of science.

Diffusion is due to random collisions of the particles of the system under study with those of a background medium in thermodynamical equilibrium (thermal bath).

Applications: Heat conduction, Brownian motion, transport of materials within cells, economics, social sciences...

... and now also Cosmology.

The kinetic diffusion equation

The kinetic diffusion equation is

$$\partial_t f + p \cdot \nabla_x f = \sigma \Delta_p f, \quad (*)$$

where $f(t, x, p)$ is the particles density in phase-space and $\sigma > 0$ is the diffusion constant. The solution of (*) with initial datum $f_0(x, p)$ is

$$f(t, x, p) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(t, x - y, p - w) f_0(y, w) \, dy \, dw$$

$$G(t, x, p) = \left(\frac{\sqrt{3}}{2\pi t^2} \right)^3 \exp\left(-\frac{3|x|^2 + 3|x - pt|^2 - t^2|p|^2}{2t^3} \right)$$

Eq. (*) is invariant by Galilean transformations and supports infinite propagation speed.

Some previous works on relativistic diffusion

Relativistic diffusion is an old classical topic. Until recently the emphasis has been in the SDEs that describe diffusion at the microscopic level:

- Dudley: *Lorentz-invariant Markov processes in relativistic phase space.* Ark. Mat. (1966)
- Franchi, Le Jan: *Relativistic Diffusions and Schwarzschild Geometry.* Comm. Pure Appl. Math. (2007)
- Angst, Debbasch, Dunkel, Hänggi, Haba, Hermann,...

The relativistic kinetic diffusion equation

Our first purpose is to discuss a relativistic (Lorentz-invariant) analogue of (*). Since both sides of (*) are Galilelan invariant, we proceed by introducing a Lorentz invariant analogue of each side of the previous equation.

Transport operator: First we replace the transport operator in the left hand side by the relativistic transport operator:

$$\partial_t + p \cdot \nabla_x \to \sqrt{1+|p|^2}\,\partial_t + p \cdot \nabla_x = p^\mu \partial_\mu, \quad \mu = 0, 1, 2, 3$$

with $p^0 = \sqrt{1+|p|^2}$, $p = (p^1, p^2, p^3)$, $\partial_0 = \partial_t$ and $\partial_i = \partial_{x^i}$

Diffusion operator: We look for a Lorentz invariant linear, second order differential operator on the momentum variable. Here we use that the Lorentz transformations in the momentum variable, i.e.,

$$\tilde{p} = p - u\sqrt{1+|p|^2} + \frac{u_0 - 1}{|u|^2} u(u \cdot p), \quad u \in \mathbb{R}^3, \ u_0 = \sqrt{1+|u|^2}$$

are isometries of the hyperbolic metric, which is the Riemannian metric h induced by the Minkowski metric

$$\eta = -dp^0 \otimes dp^0 + dp^1 \otimes dp^1 + dp^2 \otimes dp^2 + dp^3 \otimes dp^3$$

on the hyperboloid $\mathcal{H} = \{(p^0, p^1, p^2, p^3) : p^0 = \sqrt{1+|p|^2}\}$. In components

$$h_{ij} = \delta_{ij} - \hat{p}_i \hat{p}_j, \quad \hat{p} = \frac{p}{\sqrt{1+|p|^2}}$$

The Riemannian manifold $(\mathcal{H}, h)$ has a 6 dimensional group of isometries (Lorentz group + rotations).

The only maximally symmetric, linear, second order operator acting on scalar functions is the Laplace-Beltrami operator:

$$\Delta_p^{(h)} f = \frac{1}{\sqrt{\det h}} \partial_{p^i} \left(\sqrt{\det h}\, (h^{-1})^{ij} \partial_{p^j} f \right).$$

Hence we are led to the relativistic kinetic diffusion equation in the form

$$p^\mu \partial_\mu f = \sigma \Delta_p^{(h)} f, \quad \sigma > 0$$

i.e.,

$$\partial_t f + \hat{p} \cdot \nabla_x f = \sigma \partial_{p^i} \left(D^{ij} \partial_{p^j} f \right), \quad D^{ij} = \frac{\delta^{ij} + p^i p^j}{\sqrt{1 + |p|^2}} \quad (**)$$

D is the relativistic diffusion matrix, while the constant σ is the kinetic diffusion coefficient.

Some properties of the relativistic kinetic diffusion equation

I) Let us rewrite the relativistic kinetic diffusion equation in the form ($\sigma = 1$)

$$\partial_t f = Lf, \quad Lf = -\hat{p}\cdot\nabla_x f + \partial_{p^i}\left(D^{ij}\partial_{p^j} f\right) = \left(\sum_{i=1}^{3} A_{(i)}^2 + A_0\right) f,$$

where A_μ denote the vector fields

$$A_0 f = (\partial_{p^j} a^{ij}) a_i{}^k \partial_{p^k} f - \hat{p}^i \partial_{x^i} f, \quad A_{(i)} f = a^k{}_{(i)} \partial_{p^k} f$$

and $a = \sqrt{D}$. It can be shown that the vector fields A_μ satisfy the first rank Hörmander condition, that is the vector fields

$$A_{(i)},\ [A_0, A_{(i)}], \quad i = 1, 2, 3$$

are linearly independent at each point $(x, p) \in \mathbb{R}^6$.

This implies two facts:

(i) L is a hypoelliptic operator

(ii) There exists a smooth function $K : (0, \infty) \times \mathbb{R}^6 \times \mathbb{R}^6 \to (0, \infty)$ such that the solution of the initial value problem

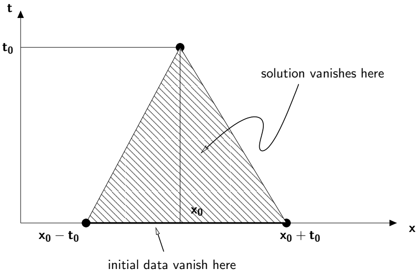
$$\partial_t f = Lf, \quad f(0, x, p) = f_0(x, p)$$

is given by

$$f(t, x, p) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} K(t, x, p, y, w) f_0(y, w) \, dy \, dw$$

However, as opposed to the non-relativistic case, an explicit formula for the kernel K is not known.

II) The relativistic kinetic diffusion equation (**) satisfies the compact domain of dependence property: Let $f \geq 0$ solves (**) with initial datum $f_0(x,p)$ supported on the ball $|x - x_0| \leq t_0$; then $f(t,x,p)$ vanishes on the cone



$$\Lambda(t_0, x_0) = \{(t,x) \in [0,t_0] \times \mathbb{R}^n : |x - x_0| \leq (t_0 - t)\}$$

for all $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^3$.

III) The current density vector and the energy-momentum tensor, given by

$$J^{\mu}(t,x)=\int_{\mathbb{R}^3} f(t,x,p)\,p^{\mu}\,\frac{dp}{\sqrt{1+|p|^2}},$$
$$T^{\mu\nu}(t,x)=\int_{\mathbb{R}^3} f(t,x,p)\,p^{\mu}p^{\nu}\,\frac{dp}{\sqrt{1+|p|^2}},$$

satisfy the identity

$$\partial_{\mu}J^{\mu}=0,\quad \partial_{\mu}T^{\mu\nu}=3\sigma J^{\nu}.$$

Define the energy and the mass as

$$\mathcal{E} = \int_{\mathbb{R}^6} \sqrt{1+|p|^2}\, f\, dp\, dx, \quad M = \int_{\mathbb{R}^6} f\, dp\, dx$$

Then $M = const.$, and

$$\mathcal{E}(t) = \mathcal{E}(0) + 3\sigma M t$$

The energy grows linearly in time. This is consistent with the physical interpretation of diffusion: particles undergo random collisions with the molecules of a background medium, which is assumed to be thermodynamical equilibrium. Hence no energy is transferred to the background.

The kinetic diffusion constant measures the energy gained by the particle per unit of time.

The kinetic diffusion equation on a curved manifold

Suppose now that the background Minkowski spacetime is replaced by a smooth Lorentzian, time-oriented manifold (M, g).
For the moment the metric g is assumed to be given.

Let x denote an arbitrary point of M and x^μ a (local) system of coordinates on an open set $U \subset M$, $x \in U$, with $x^0 \equiv t$ being timelike. The vectors ∂_{x^μ} form a basis of the tangent space T_xM and the components of $p \in T_xM$ in this basis will be denoted by p^μ. (x^μ, p^ν) provides a system of coordinates on $TU \subset TM$, where TM denotes the tangent bundle of M.

The (future) mass-shell for unit mass particles is the 7-dimensional submanifold of the tangent bundle defined as

$$\Pi M = \{(x,p) \in TM : g(x)(p,p) = -1,\ p \text{ future directed}\}.$$

On the subset $\Pi U = \{(x,p) \in \Pi M : x \in U\}$ of the mass-shell, the condition $g(x)(p,p) = -1$ is equivalent to $g_{\mu\nu} p^\mu p^\nu = -1$ (where $g_{\mu\nu} = g_{\mu\nu}(x^\alpha)$), which can be used to express p^0 in terms of p^1, p^2, p^3, precisely:

$$p^0 = -\frac{1}{g_{00}} \left[g_{0j} p^j + \sqrt{(g_{0j} p^j)^2 - g_{00}(1 + g_{ij} p^i p^j)} \right],$$

where the choice of the positive root reflects the condition that p is future directed.

The particles density in phase-space is a function $f : \Pi M \to [0, \infty)$ on the mass-shell.

The Transport part: Let $\widetilde{L}$ denote the geodesic flow vector field on the tangent bundle. In the vector fields basis $(\partial_{x^\mu}, \partial_{p^\nu})$ it is given by

$$\widetilde{L} = p^\mu(\partial_{x^\mu} - \Gamma^\nu{}_{\mu\alpha} p^\alpha \partial_{p^\nu}).$$

The Liouville, or Vlasov, operator L is defined as the projection of $\widetilde{L}$ on the mass shell. It has the following coordinates representation

$$L = p^\mu(\partial_{x^\mu} - \Gamma^i{}_{\mu\alpha} p^\alpha \partial_{p^i}).$$

The first fundamental change compared to the flat case is the definition of the "free-transport operator". It is now assumed that in the absence of diffusion the particles move along the geodesics of (M, g), i.e., f to solve the Vlasov equation:

$$Lf = 0, \quad \text{i.e.,} \quad p^\mu \partial_{x^\mu} f - \Gamma^i{}_{\mu\nu} p^\mu p^\nu \partial_{p^i} f = 0.$$

To transform the Vlasov equation into a diffusion equation we need to add a diffusion operator on the right hand side.

Diffusion part: Let $\Pi_x M$ denote the fiber over $x \in M$ of the mass-shell and $\pi_x : \Pi M \to \Pi_x M$ the canonical projection onto it. The quadratic form $g(x)$ induces a Riemannian metric $h(x)$ on $\Pi_x M$.

In analogy with the diffusion operator on the mass-shell of Minkowski space defined earlier, we now define the action of the diffusion operator on f by the formula

$$\mathcal{D}_p f = \Delta_p^{(h)}(f \circ \pi_x),$$

where $\Delta_p^{(h)}$ is the Laplace-Beltrami operator of the Riemannian metric h on $\Pi_x M$.

The expression of $\mathcal{D}_p$ in local coordinates is given as before by

$$\mathcal{D}_p f = \frac{1}{\sqrt{\det h}} \partial_{p^i} \left(\sqrt{\det h}\, (h^{-1})^{ij} \partial_{p^j} f \right),$$

where now $h_{ij} = h_{ij}(x^\alpha, p^k)$ are given by

$$h_{ij} = g_{ij} - \frac{p_i}{p_0} g_{0j} - \frac{p_j}{p_0} g_{0i} + g_{00} \frac{p_i p_j}{(p_0)^2}.$$

The kinetic diffusion equation on the curved spacetime (M, g) is then given by

$$p^\mu \partial_{x^\mu} f - \Gamma^i{}_{\mu\nu} p^\mu p^\nu \partial_{p^i} f = \sigma \mathcal{D}_p f.$$

Example: Diffusion on a conformally flat spacetime

The metric is given by

$$g = e^{2\phi}(-dt \otimes dt + dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3)$$

where $\phi = \phi(t, x)$, $t \in \mathbb{R}$, $x = (x^1, x^2, x^3) \in \mathbb{R}^3$. We obtain

$$\partial_t f + \frac{p}{\sqrt{e^{2\phi} + |p|^2}} \cdot \nabla_x f - \frac{e^{2\phi}\nabla_x \phi}{\sqrt{e^{2\phi} + |p|^2}} \cdot \nabla_p f = \sigma \partial_{p^i} \left(D[\phi]^{ij} \partial_{p^j} f \right).$$

where

$$D[\phi]^{ij} = \frac{e^{4\phi}\delta^{ij} + e^{2\phi} p^i p^j}{\sqrt{e^{2\phi} + |p|^2}}$$

Note that the diffusion matrix depends on time, which gives rise to the following interesting behavior.

Because of the time dependence of the diffusion matrix the particle density f may have a non-trivial asymptotic profile *even in the absence of friction*.

Consider for example the spatially homogeneous equation on $f = f(t,p)$

$$\partial_t f = e^{2\phi}\partial_{p^i}\left(\frac{e^{2\phi}\delta^{ij} + p^i p^j}{\sqrt{e^{2\phi}+|p|^2}}\partial_{p^j} f\right), \quad t>0,\ p\in\mathbb{R}^3 \qquad (*)$$

Suppose $\phi(t)$ is a given function such that

$$\lim_{t\to+\infty}\phi(t) = -\infty, \quad e^{\phi}\in L^2(0,\infty)$$

Theorem (J. A. Alcántara, S. C., S. Pankavich (2014))

Let $0 \leq f_{\rm in} \in H^1(\mathbb{R}^3), |p| f_{\rm in} \in L^1(\mathbb{R}^3)$. *There exists a unique solution of (*)* $0 \leq f \in L^\infty((0,\infty); H^1(\mathbb{R}^3))$ *such that* $f(0) = f_{\rm in}$. *Moreover there exist* $C, \varepsilon, \alpha > 0$ *such that*

$$\|f(t)\|_{H^1} \leq C, \quad \||\cdot|f(t)\|_1 \leq C,$$

and

$$\mu(f(t,p) > \varepsilon) > \alpha,$$

where μ *denotes the Lebesgue measure.*

A non-linear example

Let now (f,ϕ) satisfy the Vlasov-Nördstrom- Fokker-Planck (VNFP) system:

$$\partial_t f + \frac{p}{\sqrt{e^{2\phi}+|p|^2}}\cdot\nabla_x f - \frac{e^{2\phi}\nabla_x\phi}{\sqrt{e^{2\phi}+|p|^2}}\cdot\nabla_p f = \partial_{p^i}\left(D[\phi]^{ij}\partial_{p^j} f\right)$$

$$\partial_t^2\phi - \Delta_x\phi = -e^{2\phi}\int_{\mathbb{R}^3}\frac{f}{\sqrt{e^{2\phi}+|p|^2}}\,dp,\quad t>0,\ x\in\mathbb{R}^3,\ p\in\mathbb{R}^3,$$

In the spatially homogeneous case, $\phi=\phi(t)$, $f=f(t,p)$ satisfy

$$\partial_t f = e^{2\phi}\partial_{p^i}\left(\frac{e^{2\phi}\delta^{ij}+p^ip^j}{\sqrt{e^{2\phi}+|p|^2}}\partial_{p^j} f\right),\quad \ddot{\phi} = -e^{2\phi}\int_{\mathbb{R}^3}\frac{f}{\sqrt{e^{2\phi}+|p|^2}}\,dp$$

Theorem ([1])

Given $\phi_{\rm in}, \psi_{\rm in} \in \mathbb{R}$ *and* $0 \leq f_{\rm in} \in L^1 \cap H^2(\mathbb{R}^3)$ *such that*

$$\int_{\mathbb{R}^3} |p|(f_{\rm in} + |\nabla f_{\rm in}|^2)\, dp < \infty,$$

there exists a unique solution

$$0 \leq f \in L^\infty((0,\infty); L^1 \cap H^1) \cap L^\infty_{\rm loc}([0,\infty); H^2),$$

$$\phi \in C^1((0,\infty)) \cap W^{2,\infty}_{\rm loc}([0,\infty))$$

such that $f(0,p) = f_{\rm in}(p)$ *and* $(\phi(0), \dot{\phi}(0)) = (\phi_{\rm in}, \psi_{\rm in})$. *Moreover there exist constants* $\alpha, \beta, C > 0$ *such that*

$$-C - \alpha\, t \leq \phi(t) \leq C - \beta\, t, \quad |\dot{\phi}(t)| < C, \quad -Ce^{-\alpha\, t} \leq \ddot{\phi}(t) \leq 0$$

[1] J. A. Alcántara, S. C., S. Pankavich. JDE 2014

Coupling with Einstein equations

Let f solve

$$p^\mu \partial_{x^\mu} f - \Gamma^i{}_{\mu\nu} p^\mu p^\nu \partial_{p^i} f = \sigma \mathcal{D}_p f.$$

The current density J^μ and the energy-momentum tensor $T^{\mu\nu}$ are defined by

$$J^\mu = \sqrt{|g|} \int_{\Pi_x U} f \, \frac{p^\mu}{-p_0} \, dp^1 dp^2 dp^3,$$

$$T^{\mu\nu} = \sqrt{|g|} \int_{\Pi_x U} f \, \frac{p^\mu p^\nu}{-p_0} \, dp^1 dp^2 dp^3$$

and satisfy

$$\nabla_\mu J^\mu = 0, \quad \nabla_\mu T^{\mu\nu} = 3\sigma J^\nu.$$

The Einstein equations are

$$R_{\mu\nu} - g_{\mu\nu} R/2 = T_{\mu\nu}$$

Since $T_{\mu\nu}$ is not divergence-free, then, by the Bianchi identities, the Einstein equation has no solutions.

This incompatibility can be solved by assuming that there exist other matter fields in spacetime. *These additional matter fields make up the background medium in which the particles undergoing diffusion.*

Hence, *as opposed to the non-relativistic and the special relativistic case, a consistent theory of diffusion in general relativity cannot neglect completely the interaction with the background medium*

Let $\mathcal{T}_{\mu\nu}$ denote the energy-momentum tensor of the background medium. The Einstein equations now read

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = T_{\mu\nu} + \mathcal{T}_{\mu\nu}$$

By the Bianchi identities,

$$\nabla_\mu \mathcal{T}^{\mu\nu} = -\nabla_\mu T^{\mu\nu} = -3\sigma J^\nu. \quad (*)$$

The simplest choice for $\mathcal{T}_{\mu\nu}$ is the cosmological scalar field term

$$\mathcal{T}_{\mu\nu} = -\phi g_{\mu\nu},$$

where ϕ is a scalar field on spacetime (vacuum energy).

This choice leads to the following Einstein equations:

$$R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu} + \phi g_{\mu\nu} = T_{\mu\nu}.$$

The evolution equation for ϕ is

$$\nabla_\mu \phi = 3\sigma J_\mu \Rightarrow \begin{cases} \Box\phi = 0, \\ J^\mu \partial_\mu \phi = 3(J^\mu J_\mu) < 0 \end{cases}$$

Hence the cosmological scalar field propagates through space-time in form of waves without dissipation and it is decreasing along the matter flow (which can be interpreted as energy being transferred to the particles). In the absence of diffusion ($\sigma = 0$), $\phi = \Lambda$ is constant. In this case the model reduces to the Einstein-Vlasov system with cosmological constant.

Applications to Cosmology

There is overwhelming observational evidence that, on very large scales, the Universe is

- Spacetime is spatially homogeneous
- Spacetime is spatially isotropic
- Spacetime is spatially flat

Under these assumptions the spacetime metric takes the form

$$g = -dt \otimes dt + a(t)^2 (dx \otimes dx + dy \otimes dy + dz \otimes dz)$$

Moreover $f = F(t, q)$, where $q = (p_1^2 + p_2^2 + p_3^2)^{1/2}$.

The Einstein equations becomes

$$\dot{H} = \frac{1}{3}\phi - \frac{1}{6}(\rho + 3\mathcal{P}) - H^2,$$
$$H^2 := \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}(\rho + \phi),$$

where $H = \dot{a}/a$ is the Hubble function and

$$\rho(t) = \frac{4\pi}{a(t)^4}\int_0^\infty q^2 F(t,q)\sqrt{a(t)^2 + q^2}\,dq,$$

$$\mathcal{P}(t) = \frac{4\pi}{3a(t)^4}\int_0^\infty \frac{q^4 F(t,q)}{\sqrt{a(t)^2 + q^2}}\,dq.$$

The kinetic diffusion equation on $f = F(t, q)$ becomes

$$\partial_t f = \partial_{p_i}\left[\left(\frac{a(t)^2\delta_{ij} + p_i p_j}{p^0}\right)\partial_{p_j} f\right], \quad p^0 = \sqrt{1 + \frac{q^2}{a(t)^2}}$$

while the cosmological scalar field satisfies

$$\dot{\phi}(t) = -\frac{\sigma}{a(t)^3}.$$

How can we interpret this model?

Main ingredients of the Universe

The Universe is believed to be made up of:

- 5% of ordinary matter (barionic matter)
- 25% Dark matter
- 70% Dark energy

Cosmological constant

The most popular model for the dark energy is the cosmological constant $\Lambda > 0$. This amounts to change the Einstein equations into

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = T_{\mu\nu}$$

The cosmological constant assumptions poses some problems, among which:

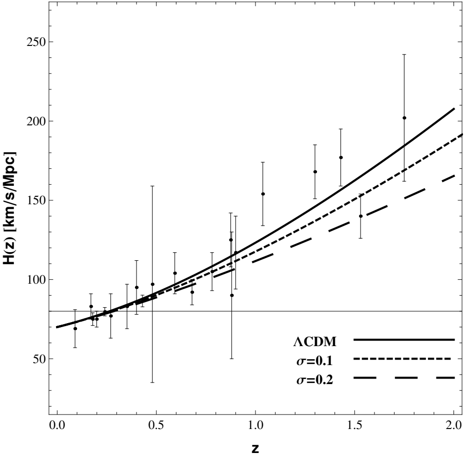
- The theoretical value and the measured value differs of 120 order of magnitude!
- Coincidence problem

Both these problems suggest that dark energy should be described by a dynamical field decreasing along the matter flow.

This is exactly what the diffusion model predicts.

Validation of the model

The model is consistent with the current cosmological observations



S.C., H.Vetten: *Cosmological diffusion*. J. Cosm. Astrop. Phys. (2013)

Future asymptotic behavior

There exist $\phi_{**} > \phi_* > 0$ such that

- For $\phi_0 > \phi_{**}$ the spacetime is singularity free and forever expanding in the future
- For $0 \leq \phi_0 < \phi_*$ the metric develops a singularity in finite time in the future (Big Crunch)

The singularity scenario is absent in the diffusion-free case.

Past asymptotic behavior

Two scenario are possible:

- Either a singularity forms in the past (Big-Bang)
- Or the Entropy becomes zero while spacetime is still regular. At this time the solution can be matched to the vacumm De-Sitter solution in the past

Only the first scenario is present in the diffusion-free case.