Instability of non-monotone equilibria of the relativistic Vlasov-Maxwell system on unbounded domains

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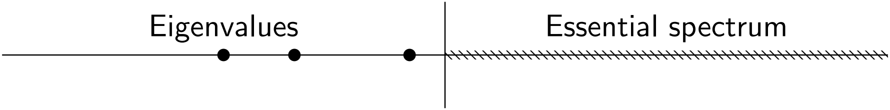
Outline

1. The instability result
   - The linear instability result for Vlasov-Maxwell
   - Conversion to a spectral problem
   - Solving the equivalent problem
     - Tracking eigenvalues
2. Approximations of strongly continuous families of unbounded self-adjoint operators



The Relativistic Vlasov-Maxwell Equations (RVM)

These describe the evolution of a plasma of charged particles evolving under their self-consistent electromagnetic field:

$$\partial_t f^\pm + \hat{\boldsymbol{v}} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} f^\pm \pm (\mathbf{E} + \hat{\boldsymbol{v}} \times \mathbf{B}) \cdot \boldsymbol{\nabla}_{\boldsymbol{v}} f^\pm = 0$$

$$\boldsymbol{\nabla}_{\boldsymbol{x}} \cdot \mathbf{B} = 0 \qquad \partial_t \mathbf{B} = -\boldsymbol{\nabla}_{\boldsymbol{x}} \times \mathbf{E}$$

$$\boldsymbol{\nabla}_{\boldsymbol{x}} \cdot \mathbf{E} = \underbrace{\int_{\mathbb{R}^3} (f^+ - f^-)\, d\boldsymbol{v}}_{\text{charge density } \rho} \qquad \partial_t \mathbf{E} = \boldsymbol{\nabla}_{\boldsymbol{x}} \times \mathbf{B} - \underbrace{\int_{\mathbb{R}^3} \hat{\boldsymbol{v}} (f^+ - f^-)\, d\boldsymbol{v}}_{\text{current density } \mathbf{j}}\,.$$

- f^+ and f^- are the densities of positively and negatively charged particles in phase space: $f^\pm(t, \boldsymbol{x}, \boldsymbol{v}) : [0, T) \times \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$.
- $\mathbf{E}$ and $\mathbf{B}$ are the self-consistent electric and magnetic fields: $\mathbf{E}(t, \boldsymbol{x}), \mathbf{B}(t, \boldsymbol{x}) : [0, T) \times \mathbb{R}^3 \to \mathbb{R}^3$.
- $\hat{\boldsymbol{v}}$ is the relativistic effective velocity given by $\hat{\boldsymbol{v}} = \boldsymbol{v}/\sqrt{1 + |\boldsymbol{v}|^2}$.

Definition (Spectral instability)

We say that a given equilibrium $\mu^{\pm}$ is *spectrally unstable*, if the system linearised around it has a purely growing mode solution of the form

$$\left(e^{\lambda t} f^{\pm}(\boldsymbol{x}, \boldsymbol{v}), e^{\lambda t}\mathbf{E}(\boldsymbol{x}), e^{\lambda t}\mathbf{B}(\boldsymbol{x})\right), \quad \lambda > 0.$$

Consider a compactly supported equilibrium $\mu^{\pm}$ in '1.5d' symmetry or 3d cylindrical symmetry. Then there are self-adjoint Schrödinger operators $\mathcal{L}^0, \mathcal{A}_1^0$ (defined later in the talk) acting on $L^2_{\boldsymbol{x}}$ of the form

$$-\Delta_{\boldsymbol{x}} + \{\text{ bounded perturbation depending on } \mu^{\pm}\}.$$

Theorem (Lin and Strauss '07)

Let $\mu^{\pm}$ be monotone [Defined on next slide]. Then it is spectrally unstable iff $\mathcal{L}^0$ has a negative eigenvalue.

Theorem (Ben-Artzi and H. '15)

Let $\mu^{\pm}$ be non-monotone. Then it is spectrally unstable if $\mathcal{L}^0$ has more negative eigenvalues than $\mathcal{A}_1^0$. [$\mathcal{A}_1^0$ has no negative eigenvalues if $\mu^{\pm}$ is monotone.]

Monotone and non-monotone equilibria

We assume that the equilibrium $f^{0,\pm}$ can be written as a function of the *microscopic energy* $e^{\pm}$ *and momentum* $p^{\pm}$, i.e.

$$f^{0,\pm}(\boldsymbol{x}, \boldsymbol{v}) = \mu^{\pm}(e^{\pm}, p^{\pm})$$

Here $e^{\pm}$ is given by $e^{\pm} = \sqrt{1 + |\boldsymbol{v}|^2} \pm \phi^0(\boldsymbol{x}) \pm \phi^{ext}(\boldsymbol{x})$. $p^{\pm}$ depends on the assumed symmetry, in 1.5d it is $p^{\pm} = v_2 \pm \psi^0(x) \pm \psi^{ext}(x)$.

Definition (Monotone equilibrium)

An equilibrium is *monotone* if

$$\frac{\partial \mu^{\pm}}{\partial e^{\pm}} < 0 \quad \text{whenever } \mu^{\pm} > 0. \tag{1.1}$$

An equilibrium is *non-monotone* if $\frac{\partial \mu^{\pm}}{\partial e^{\pm}} > 0$ at some point where $\mu^{\pm} > 0$, i.e. if it fails to be monotone.

Assuming monotonicity of the equilibrium is very common.
We do not make this assumption.

Symmetries and assumptions on the equilibrium

Symmetries:

- We consider two symmetries.
  1. 3D space with cylindrical symmetry (independence of equilibrium on θ variable in (r, θ, z) cylindrical polar coordinates).
  2. 1.5D *non-periodic* symmetry. Phase space is $(x, v_1, v_2) \in \mathbb{R}^3$.
- For the second, it is more usual to assume *periodicity* in the x variable, i.e. that $(x, v_1, v_2) \in \mathbb{T} \times \mathbb{R}^2$.
- The 1.5D periodic case is easier than either of these due to the discrete spectrum of the Laplacian on $\mathbb{T}$. (See Ben-Artzi '11 for an instability criterion of non-monotone equilibria in this case).

Assumptions on equilibrium:

- $f^{0,\pm}(\boldsymbol{x}, \boldsymbol{v}) = \mu^{\pm}(e^{\pm}, p^{\pm})$ with $\mu^{\pm} \in C^1$.
- $f^{0,\pm}(\boldsymbol{x}, \boldsymbol{v})$ has compact support in $\boldsymbol{x}$, (but not necessarily in $\boldsymbol{v}$).
- Mild algebraic decay of $\mu^{\pm}$ and its first derivatives (which implies $f^{0,\pm} \in L^1_{x,v} \cap C^1$).

Conversion to a spectral problem

After the following steps:

- Linearisation around the equilibrium $\mu^\pm$.
- Inserting the ansatz $(e^{\lambda t} f^\pm(\boldsymbol{x}, \boldsymbol{v}), e^{\lambda t}\mathbf{E}(\boldsymbol{x}), e^{\lambda t}\mathbf{B}(\boldsymbol{x}))$.
- Introducing the scalar and vector potentials $\phi(\boldsymbol{x})$ and $\mathbf{A}(\boldsymbol{x})$ (where $\mathbf{E} = -\nabla_{\boldsymbol{x}}\phi$ and $\mathbf{B} = \nabla_{\boldsymbol{x}} \times \mathbf{A}$) and fixing the Lorenz gauge.

the linearised system becomes

$$(\lambda + \overbrace{\hat{\boldsymbol{v}} \cdot \nabla_{\boldsymbol{x}} \pm (\mathbf{E}^0 + \hat{\boldsymbol{v}} \times \mathbf{B}^0) \cdot \nabla_{\boldsymbol{v}}}^{\mathcal{D}_\pm :=}) f^\pm = \mp(-\nabla_{\boldsymbol{x}}\phi + \hat{\boldsymbol{v}} \times \nabla_{\boldsymbol{x}} \times \mathbf{A}) \cdot \nabla_{\boldsymbol{v}}\mu^\pm$$

$$\lambda^2\phi - \Delta\phi = \int (f^+ - f^-)\, d\boldsymbol{v}$$

$$\lambda^2\mathbf{A} - \boldsymbol{\Delta}\mathbf{A} = \int (f^+ - f^-)\hat{\boldsymbol{v}}\, d\boldsymbol{v}.$$

Inverting the linearised Vlasov equation

- The operators $\mathcal{D}_\pm$ are skew-adjoint, $(\lambda + \mathcal{D}_\pm)^{-1}$ exists for real $\lambda \neq 0$.
- Thus we can invert the linearised Vlasov equation

$$(\lambda + \mathcal{D}_\pm) f^\pm = \{\text{terms involving } \mu^\pm, \phi, \mathbf{A}\}$$

to obtain

$$f^\pm = (\lambda + \mathcal{D}_\pm)^{-1} \{\text{terms involving } \mu^\pm, \phi, \mathbf{A}\}$$

- Putting this into Maxwell's equations produces

$$\lambda^2 \phi - \Delta \phi = \{\text{terms involving } \mu^\pm, \phi, \mathbf{A}\}$$
$$\lambda^2 \mathbf{A} - \mathbf{\Delta} \mathbf{A} = \{\text{terms involving } \mu^\pm, \phi, \mathbf{A}\}.$$

- A typical term looks like:

$$\int \frac{\partial \mu^\pm}{\partial e^\pm} \lambda (\lambda + \mathcal{D}_\pm)^{-1} \phi \, d\boldsymbol{v} = \int \frac{\partial \mu^\pm}{\partial e^\pm} \int_0^\infty \lambda e^{-\lambda t} \phi(\boldsymbol{X}^\pm(t; \boldsymbol{x}, \boldsymbol{v})) \, dt d\boldsymbol{v}$$

where $e^\pm$ is the infinitesimal particle energy, and $(\boldsymbol{X}^\pm(t; \boldsymbol{x}, \boldsymbol{v}), \boldsymbol{V}^\pm(t; \boldsymbol{x}, \boldsymbol{v}))$ are the trajectories of $\mathcal{D}_\pm$ starting from $(\boldsymbol{x}, \boldsymbol{v})$.

- After writing the system in an appropriate system of coordinates, and *flipping the sign of one of the equations*, we can write the system in block matrix form as

$$\boldsymbol{\mathcal{M}}^{\lambda}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} := \begin{bmatrix} \mathcal{A}_2^{\lambda} & \left(\mathcal{B}^{\lambda}\right)^* \\ \mathcal{B}^{\lambda} & -\mathcal{A}_1^{\lambda} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0.$$

- Due to the assumed symmetries, this is a self-adjoint operator on L^2, (the self-adjointness is somewhat miraculous).
- The operator $\boldsymbol{\mathcal{M}}^{\lambda}$ has the form

$$\boldsymbol{\mathcal{M}}^{\lambda} = \begin{bmatrix} -\Delta + \lambda^2 & 0 \\ 0 & \boldsymbol{\Delta} - \lambda^2 \end{bmatrix} + \boldsymbol{\mathcal{K}}^{\lambda}$$

where $\boldsymbol{\mathcal{K}}^{\lambda}$ is a bounded symmetric operator, which takes an ergodic average over the trajectories of $\mathcal{D}_{\pm}$.

An equivalent spectral problem

Find $\lambda > 0$ such that $\ker(\boldsymbol{\mathcal{M}}^{\lambda})$ is non-trivial.

Solving the equivalent problem

An equivalent spectral problem

Find $\lambda > 0$ such that $\ker(\boldsymbol{\mathcal{M}}^\lambda)$ is non-trivial, where

$$\boldsymbol{\mathcal{M}}^\lambda = \begin{bmatrix} \mathcal{A}_2^\lambda & (\mathcal{B}^\lambda)^* \\ \mathcal{B}^\lambda & -\mathcal{A}_1^\lambda \end{bmatrix} = \begin{bmatrix} -\Delta + \lambda^2 & 0 \\ 0 & \boldsymbol{\Delta} - \lambda^2 \end{bmatrix} + \boldsymbol{\mathcal{K}}^\lambda.$$

Lemma (Spectral continuity)

$\lambda \mapsto \boldsymbol{\mathcal{M}}^\lambda$ is norm-resolvent continuous, i.e. the resolvent $(\boldsymbol{\mathcal{M}}^\lambda + i)^{-1}$ is a continuous function of λ in the operator norm topology. This implies that the spectrum of $\boldsymbol{\mathcal{M}}^\lambda$ depends continuously (as a set) upon λ.

Method of solution

Compare the spectrum of $\boldsymbol{\mathcal{M}}^\lambda$ at $\lambda = 0$ and as $\lambda \to \infty$. Then use *continuity* of the spectrum to find a $\lambda > 0$ where an eigenvalue crosses 0.

The monotone case (Lin and Strauss '07)

An equivalent spectral problem

Find $\lambda > 0$ such that $\ker(\boldsymbol{\mathcal{M}}^\lambda)$ is non-trivial, where

$$\boldsymbol{\mathcal{M}}^\lambda = \begin{bmatrix} \mathcal{A}_2^\lambda & \left(\mathcal{B}^\lambda\right)^* \\ \mathcal{B}^\lambda & -\mathcal{A}_1^\lambda \end{bmatrix} = \begin{bmatrix} -\Delta + \lambda^2 & 0 \\ 0 & \boldsymbol{\Delta} - \lambda^2 \end{bmatrix} + \boldsymbol{\mathcal{K}}^\lambda.$$

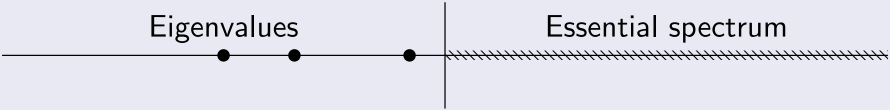
When the equilibrium is monotone, the operator $\mathcal{A}_1^\lambda$ is invertible for all $\lambda \geq 0$, and some simple algebra reduces the problem to finding a non-trivial kernel of

$$\mathcal{L}^\lambda := \mathcal{A}_2^\lambda + \left(\mathcal{B}^\lambda\right)^* \left(\mathcal{A}_1^\lambda\right)^{-1} \mathcal{B}^\lambda = -\Delta + \lambda^2 + \mathcal{J}^\lambda$$

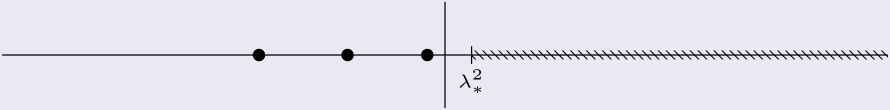
which is *semi-bounded* (its spectrum is bounded below).

Find $\lambda > 0$ such that $\ker(\mathcal{L}^\lambda)$ is non-trivial, where $\mathcal{L}^\lambda = -\Delta + \lambda^2 + \mathcal{J}^\lambda$.

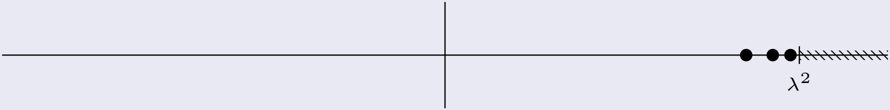
The spectrum at $\lambda = 0$. ($\mathcal{L}^\lambda$ is self-adjoint, so it's spectrum is real.)



The spectrum at $\lambda = \lambda_* > 0$ (small).



The spectrum for large λ.



The non-monotone case

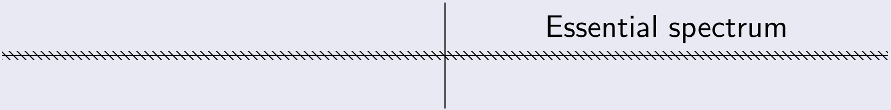
The operator $\mathcal{A}_1^\lambda$ is not necessarily invertible for all $\lambda \geq 0$, so we cannot look at $\mathcal{L}^\lambda$.

An equivalent spectral problem

Find $\lambda > 0$ such that $\ker(\boldsymbol{\mathcal{M}}^\lambda)$ is non-trivial, where

$$\boldsymbol{\mathcal{M}}^\lambda = \begin{bmatrix} \mathcal{A}_2^\lambda & \left(\mathcal{B}^\lambda\right)^* \\ \mathcal{B}^\lambda & -\mathcal{A}_1^\lambda \end{bmatrix} = \begin{bmatrix} -\Delta + \lambda^2 & 0 \\ 0 & \boldsymbol{\Delta} - \lambda^2 \end{bmatrix} + \boldsymbol{\mathcal{K}}^\lambda.$$

The spectrum at $\lambda = 0$. ($\boldsymbol{\mathcal{M}}^\lambda$ is self-adjoint, so it's spectrum is real.)



An equivalent spectral problem

Find $\lambda > 0$ such that $\ker(\boldsymbol{\mathcal{M}}^\lambda)$ is non-trivial, where

$$\boldsymbol{\mathcal{M}}^\lambda = \begin{bmatrix} \mathcal{A}_2^\lambda & \left(\mathcal{B}^\lambda\right)^* \\ \mathcal{B}^\lambda & -\mathcal{A}_1^\lambda \end{bmatrix} = \begin{bmatrix} -\Delta + \lambda^2 & 0 \\ 0 & \boldsymbol{\Delta} - \lambda^2 \end{bmatrix} + \boldsymbol{\mathcal{K}}^\lambda.$$

Assumption of the theorem

$\mathcal{L}^0 = \mathcal{A}_2^0 + \left(\mathcal{B}^0\right)^* \left(\mathcal{A}_1^0\right)^{-1} \mathcal{B}^0$ has more negative eigenvalues than $\mathcal{A}_1^0$.

To solve the spectral problem we proceed as follows:

1. Move the assumption from $\lambda = 0$ to some (small) positive λ_*.
2. Pretend that $\boldsymbol{\mathcal{M}}^\lambda$ is a *finite dimensional matrix* and solve the problem.
3. Hope that there is an approximation argument that implies this is sufficient. (Part II of talk.)
4. Note that the lower bound λ_* carries over to the approximations - growth rate can't be zero in the limit.

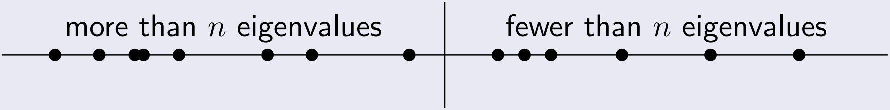
Solving the approximate problems

Lemma

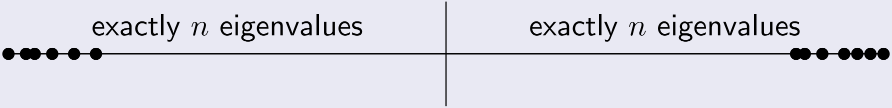
The finite dimensional approximate operators M^λ *(of dimension* $2n$*) can be taken to satisfy, [*$\mathrm{neg}(\mathcal{A})$ $=$ *number of negative eigenvalues of* $\mathcal{A}\}$*],*

$$\mathrm{neg}(M^{\lambda_*}) \geq \mathrm{neg}(\mathcal{L}^0) + n - \mathrm{neg}(\mathcal{A}_1^0) \quad (> n \textit{ by assumption}).$$

The spectrum at $\lambda = \lambda_*$.



The spectrum for large λ.



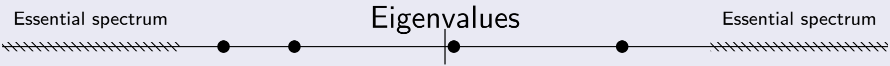
Part II

The Problem

Consider the family of self-adjoint operators $\{\mathcal{M}^\lambda\}_{\lambda\in[0,1]}$ given by

$$\mathcal{M}^\lambda = \mathcal{A} + \mathcal{K}^\lambda = \begin{bmatrix} -\Delta + 1 & 0 \\ 0 & \Delta - 1 \end{bmatrix} + \begin{bmatrix} \mathcal{K}^\lambda_{++} & \mathcal{K}^\lambda_{+-} \\ \mathcal{K}^\lambda_{-+} & \mathcal{K}^\lambda_{--} \end{bmatrix}, \quad \lambda \in [0,1] \tag{2.1}$$

with $\mathcal{K}$ symmetric and strongly continuous, and for each λ is relatively compact w.r.t. $\mathcal{A}$. The spectrum of $\mathcal{M}^\lambda$ schematically looks like:



Is it possible to construct explicit finite-dimensional symmetric approximations of $\mathcal{M}^\lambda$ whose spectrum in $(-1,1)$ converges to that of $\mathcal{M}^\lambda$ for all λ simultaneously?

Remarks

- Solving this problem completes the instability proof in Part I.
- Much is known about constructing approximations of *fixed* operators (as opposed to families of operators), often from a numerical perspective. See Hansen '08 for a review of various numerical techniques, M. Strauss '14 for a recent result, among many to numerous to mention here.
- In the case of fixed semi-bounded operators this is related to computing the discrete spectra of Schrödinger operators - a central problem in mathematical physics, which has been studied extensively (Reed, Simon, Kato, Davies, ...).
- In the case of a family of operators the literature is smaller. A somewhat similar problem was considered by Kumar, Namboodiri, and Serra-Capizzano '14 for holomorphic families of bounded operators.
- The central problem is *spectral pollution*: the appearance of spurious eigenvalues in the approximations that do not correspond to spectrum of the original operator.

Assumptions

$$\mathcal{M}^\lambda = \mathcal{A}^\lambda + \mathcal{K}^\lambda = \begin{bmatrix} \mathcal{A}^\lambda_+ & 0 \\ 0 & -\mathcal{A}^\lambda_- \end{bmatrix} + \begin{bmatrix} \mathcal{K}^\lambda_{++} & \mathcal{K}^\lambda_{+-} \\ \mathcal{K}^\lambda_{-+} & \mathcal{K}^\lambda_{--} \end{bmatrix}, \quad \lambda \in [0,1]$$

- $\mathcal{A}^\lambda_\pm$ is an (unbounded) self-adjoint holomorphic family of operators.
- **Gap:** $\mathcal{A}^\lambda_\pm > 1$ for all $\lambda \in [0,1]$.
- **Compactness:** There exists a bounded operator

$$\mathcal{P} = \begin{bmatrix} \mathcal{P}_+ & 0 \\ 0 & \mathcal{P}_- \end{bmatrix}$$

  for which $\mathcal{K}^\lambda \mathcal{P} = \mathcal{K}^\lambda$ for all $\lambda \in [0,1]$, and $\mathcal{P}_\pm$ are *relatively compact* w.r.t. $\mathcal{A}^\lambda_\pm$ for all $\lambda \in [0,1]$.
- **Compactification of the resolvent:** There exist self-adjoint holomorphic family of operators $\mathcal{W}^\lambda_\pm$, such that

$$\mathcal{A}^\lambda_\varepsilon := \mathcal{A}^\lambda + \varepsilon \mathcal{W}^\lambda$$

  has discrete spectrum, and $\mathcal{W}^\lambda_\pm \geq 0$ for all $\lambda \in [0,1]$.

$$\mathcal{M}_\varepsilon^\lambda = \mathcal{A}^\lambda + \varepsilon\mathcal{W}^\lambda + \mathcal{K}^\lambda = \begin{bmatrix} \mathcal{A}_+^\lambda + \varepsilon\mathcal{W}_+^\lambda & 0 \\ 0 & -\mathcal{A}_-^\lambda - \varepsilon\mathcal{W}_-^\lambda \end{bmatrix} + \begin{bmatrix} \mathcal{K}_{++}^\lambda & \mathcal{K}_{+-}^\lambda \\ \mathcal{K}_{-+}^\lambda & \mathcal{K}_{--}^\lambda \end{bmatrix}$$

Order the eigenfunctions of $\mathcal{A}_\varepsilon^\lambda$, then define $\mathcal{M}_{\varepsilon,n}^\lambda$ as the restriction of $\mathcal{M}_\varepsilon^\lambda$ onto the eigenspace associated with the first $2n$ eigenfunctions. It is a $2n \times 2n$ symmetric matrix.

Theorem (Ben-Artzi and H. '15)

The functions Σ and Σ_ε are continuous, where

$$\Sigma : [0,1] \times [0,\varepsilon^*] \to (\textit{subsets of } (-1,1), \textit{Hausdorff distance})$$
$$\Sigma(\lambda,\varepsilon) = (-1,1) \cap \mathrm{sp}(\boldsymbol{\mathcal{M}}_\varepsilon^\lambda)$$

and for fixed $\varepsilon > 0$,

$$\Sigma_\varepsilon : [0,1] \times (\mathbb{N} \cup \{\infty\}) \to (\textit{subsets of } (-1,1), \textit{Hausdorff distance})$$
$$\Sigma_\varepsilon(\lambda,n) = (-1,1) \cap \mathrm{sp}(\boldsymbol{\mathcal{M}}_{\varepsilon,n}^\lambda)$$

where we use the convention that $\mathcal{M}_{\varepsilon,\infty}^\lambda := \mathcal{M}_\varepsilon^\lambda$.

Thank you for your attention!

Theorem (Ben-Artzi and H. '15)

Let $\mu^{\pm}$ be a non-monotone compactly supported equilibrium of (RVM) in 1.5d non-periodic symmetry or cylindrical symmetry. Then it is spectrally unstable if $\mathcal{L}^0$ has more negative eigenvalues than $\mathcal{A}_1^0$.

Jonathan Ben-Artzi and Thomas Holding.
Approximations of strongly continuous families of unbounded operators.
Submitted. Preprint: arXiv:1403.3963, pages 1–20, 2015.

Jonathan Ben-Artzi and Thomas Holding.
Instabilities of the relativistic Vlasov-Maxwell system on unbounded domains.
Submitted. Preprint: arXiv:1505.05672, pages 1–49, 2015.