A BODY MOVING IN A KINETIC SEA

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Joint work with XUWEN CHEN.

OUTLINE:

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Body and Particles

Boundaries have been ignored in much of mathematical kinetic theory. Boundary interactions in kinetic theory are very poorly understood, even when the boundaries are fixed. Free boundaries are even more difficult.

In typical physical scenarios the interaction of particles with a body can be quite complicated. For instance, the boundary may be so rough that a particle may reflect from it in an essentially random way. There could even be some kind of physical or chemical reaction between the particle and the molecules of the body!

The present work, three papers with XUWEN CHEN, therefore considers a fairly general class of boundary conditions. Our work is heavily motivated by a series of four papers by Caprino, Cavallaro, Marchioro, Pulvirenti, Aoki and Tsuji, some of which deal specifically with specular reflection.

Basic assumptions

We consider the one of the simplest free boundary problems: a rigid body colliding with a continuum of particles. We assume

- The body, possibly subject to a constant force E, is initially moving near an equilibrium velocity V_∞.
- The particles are identical, extremely numerous, and either move freely or are subject to a small force G, until they hit the body.
- The particles reflect off the body diffusely, that is, probabilistically with some probability distribution K.
- The motion of the body is one-dimensional (although the motion of the particles may be d-dimensional).

These assumptions will be made more specific later.

Basic result

Does the body approach an equilibrium velocity? In some cases we analyze how it makes that approach. In case the particle force $G = 0$, we find a condition that is sufficient and almost necessary that the collective force of the colliding particles reverses the relative velocity $V(t)$ of the body, that is, changes the sign of $V(t) - V_\infty$, before the body approaches equilibrium. The other possibility is that it does not reverse its approach. Examples of both reversal and irreversal are given. This is in strong contrast with the specular reflection case (for which only reversal can happen if $E \geq 0$ and $V(0) > V_\infty$).

We discover that there are diffusive collision laws that lead to reversal and others that lead to irreversal, no matter what V_∞ and $V(0)$ are, so long as they are close together. These laws are almost exact opposites of each other.

1D Case

In 1D the body is an interval $\Omega(t)$. The body moves with velocity $V(t)$. There may be a constant force $E \geq 0$ acting on the body, as well as the frictional force $F(t)$ due to all the colliding particles at time t. Thus

$$\frac{dX}{dt} = V(t), \quad \frac{dV}{dt} = E - F(t),$$

What is the equilibrium velocity V_∞? If $E = 0$, then the body is at rest in equilibrium ($V_\infty = 0$). If $E \neq 0$, then $V_\infty \neq 0$ is given by

$$F_{00}(V_\infty) = E$$

where $F_{00}(V)$ is the 'fictitious' force, imagining that any particle colliding at time t has not collided at any earlier time and moves with constant velocity between collisions. In order to avoid confusion we shall take $0 \leq V_\infty < V(0)$.

The particle distribution, denoted by $f(t,x,v)$, satisfies $\partial_t f + v\partial_x f = G(t,x)$ in $\Omega^c(t)$. Initially, $f(0,x,v) = f_0(v)$ is an even function. Densities before and after a collision with the body (absorbed/emitted): $f_\pm(t,x,v) = \lim_{\epsilon\to 0^+} f(t\pm\epsilon, x\pm\epsilon x, v)$. The Law of Reflection at the two ends of the interval is

$$f_+(t,x,v) = \int K\left(v - V(t)\,; u - V(t)\right) f_-(t,x;u)du,$$

integrated over $\{(u - V(t))(v - V(t)) \leq 0\}$. The kernel K is even and satisfies a conservation-of-mass condition. There are other assumptions on K and f_0, which include:

$$c\,|u|^p \leqslant \int_{v\geq 0} v^2\, K(v,u)\; dv \;\leqslant C\,|u|^p$$

where $0 < p \leq 2$. In this lecture I will not specify every technical assumption.

Theorem (1. Approach to V_∞, 1D)

Let $|V(0) - V_\infty| = \gamma$ be sufficiently small. Let $|G(t,x)| = \delta O(t^{-q}|x|^{-m})$ and $f_0(v) = O(|v|^{-\ell})$ with some conditions on the powers (at least $q > \frac{5}{2}, m > \frac{3}{2}, \ell > \frac{7}{2}$) and δ also small. Then there exists at least one solution $(V(t), f(t,x,v))$ of our problem in the following sense. $V \in C^1(\mathbb{R})$ and $f_\pm \in L^\infty$ (where $f_\pm(t,x,v)$ are a.e. defined explicitly in terms of $V(t)$ and $f_0(x,v)$).
For every such solution the body approaches the equilibrium velocity

$$|V(t) - V_\infty| \leq O(t^{-\sigma})$$

where $\frac{1}{\sigma} = \frac{1}{p+1} + \frac{1}{\min(m,q-1)}$.

Uniqueness is an open question.

Theorem (2. Irreversal, 1D)

Let $E \geq 0$, $G \equiv 0$ *and* $0 \leqslant V_\infty < V_0 = V_\infty + \gamma$, *where* γ *is sufficiently small. Assume the Irreversal Criterion*

$$\int_0^\infty K(0,z)\, f_0(z + V_\infty)\, dz > f_0(V_\infty).$$

Every solution satisfies the estimate

$$0 < \gamma e^{-B_0 t} + \frac{c\gamma^{p+1}}{t^{1+p}} \chi_{t \geq t_0+1} < V(t) - V_\infty < \gamma e^{-B_\infty t} + \frac{C\gamma^{p+1}}{(1+t)^{1+p}}$$

for $0 < t < \infty$ *and for some positive constants* c, C, B_0, B_∞, t_0.

(Thus the body's velocity approaches equilibrium from above at a polynomial rate.)

Theorem (3. Reversal, 1D)

Same as Theorem 2, except for the Reversal Criterion:

$$\int_0^\infty K(0,z)\, f_0(z+V_\infty)\, dz < f_0(V_\infty).$$

Then same conclusion is valid except for the estimate

$$\gamma e^{-B_0 t} - \frac{c\gamma^{p+1}}{(1+t)^{1+p}} < V(t) - V_\infty < \gamma e^{-B_\infty t} - \frac{C\gamma^{p+1}}{t^{1+p}} \chi_{t \geqslant t_0+1}$$

for $0 < t < \infty$.

Notice that the negative terms dominate for large t, which means $V(t) < V_\infty < V(0)$ for large t; that is, velocity reversal!

Interpretation

The contrasting criteria have the following interpretation. The body is initially moving to the right. Letting $u = z + V_\infty$, we see that the left side of both inequalities represents the velocity density, after collisions on the left of the body ($u > V_\infty$), of the particles with approximately the same velocity as the body ($v - V_\infty \sim 0$). *These are the particles that are most likely to collide again later.*

In the reversal case there will be fewer such particles that have collided on the left side compared with the particles that have not collided. So the body tends to move more to the left, and $V(t)$ has more of a chance to cross over from being larger than V_∞ to being smaller.

In the irreversal case, there are more such particles, so there are more collisions on the left and the velocity of the body tends to remain larger than V_∞.

Contrast with the opposite case $0 < V_0 < V_\infty$

In case $0 < V_0 < V_\infty$, the body initially moves slower than the equilibrium, so the particles on the right now play the critical role. Adapting to the right side of the cylinder, the Irreversal Criterion becomes

$$\int_{-\infty}^{0} K(0,z)\, f_0(z+V_\infty)\, dz > f_0(V_\infty),$$

while the Reversal Criterion becomes

$$\int_{-\infty}^{0} K(0,z)\, f_0(z+V_\infty)\, dz < f_0(V_\infty).$$

In case $V_\infty = 0$, the criteria on the right and the left coincide due to the evenness of K and f_0.

Examples

Example 1 (Gaussian):

$$f_0(u) = C_1 e^{-\alpha u^2}, \quad K(v,u) = 2\beta e^{-\beta v^2} |u|, \quad p = 1,$$

$|G(t,x)| = \delta O(\langle t \rangle^{-3-\epsilon} \langle x \rangle^{-2-\epsilon})$
A slow body (V_∞ either vanishes or is small enough) satisfies the Reversal Criterion if $\beta < \alpha$. It satisfies the Irreversal Criterion provided $\beta > \alpha$. A faster body with $V_\infty^2 > \frac{\beta}{2\alpha^2}$ satisfies the Reversal Criterion.

Example 2 (also Gaussian):

$$K(v,u) = 2\exp(-v^2/|u|)$$

in which case $p = 3/2$ and $|G(t,x)| = \delta O(\langle t \rangle^{-\frac{8}{3}-\epsilon} \langle x \rangle^{-\frac{5}{3}-\epsilon})$.

Colliding particles with velocities close to that of the body deflect only a little, while colliding particles with very different velocities reflect with a very wide distribution of velocities. Again, both reversal and irreversal can happen.

Example 3: We can generalize Example 2 to get a continuous range of p. Given $\beta \in [-1, 3)$, we choose

$$K(v, u) = C_2 \, |u|^{\beta} \, e^{-v^2 |u|^{\beta - 1}}.$$

Once again, C_2 is chosen so that mass is conserved during collisions. We then compute

$$C_2 \, |u|^{\beta} \int_0^{\infty} v^2 \, e^{-v^2 |u|^{\beta-1}} dv = C \, |u|^{\frac{3-\beta}{2}}$$

for some constant C. Thus p runs through $(0, 2]$ as β runs through $[-1, 3)$.

Strategy of Proof

Let $F_{00}(t)$ be the force on the body at time t due to collisions at time t without any precollisions and with $G = 0$. Let $F_0(t)$ be the force at time t without any precollisions but with forces G acting on the particles. A family $\mathcal{W}$ of possible body motions $W = W(t)$ is introduced. We split the frictional force due to the possible motion W into three parts as

$$F(t) = F_{00}(W) + [F_0(t, W) - F_{00}(t, W)] - R(t, W).$$

$R(t, W)$ is the force due to all the collisions occurring before time t (“precollisions") Then W generates a new possible motion V_W by the "approximate" equation

$$\frac{dV_W}{dt} = \frac{E - F_{00}(W)}{V_\infty - W}(V_\infty - V_W) + F_{00}(W) - F_0(W) - R(W).$$

Recall that $E = F_{00}(V_\infty)$. Goal: the mapping $W \to V_W$ has a fixed point and the fixed point decays in time. It is the first term that provides the decay.

Total Force

Lemma

In 3D, assuming conservation of mass, the horizontal component of the particles' force on the body is given by the formula

$$F(t) = \int_{\partial\Omega(t)} dS_{\mathbf{x}} \int_{\mathbb{R}^3} d\mathbf{v} \left[(\mathbf{v} - \mathbf{V}(t)) \cdot \mathbf{n}\right] (v_x - V_x)\, f(t, \mathbf{x}, \mathbf{v}).$$

In 1D and with our boundary condition, the total force on the body due to the particles at time t reduces to

$$F(t) = \int_{-\infty}^{\infty} dv\ sgn(V(t) - v)\ L(v - V(t))\ f_{-}(t, x, v),$$

where L is the sum of two terms, one for the absorbed particles and the other for the emitted ones.

$$L(w) = w^2 + \int_0^{\infty} dv\ v^2\ K(v, w).$$

Fictitious Force

Decay comes from the first term. Indeed,

$$F_{00}(W) = \int_{v \leqslant W} sgn(W - v) L(v - W) f_0(v) dv.$$

This is the force on the cylinder if all the collisions occurring before time t, as well as the forces on the particles, were ignored.

The basic properties of this "fictitious" force are as follows. Suppose $f_0(v) \geq 0$ is even, continuous and $\not\equiv 0$. If $L \in C^1$, even, decreasing for $w < 0$ then $F_{00}(W)$ is an increasing C^1 function of W. Indeed, $F_{00}(0) = 0$ and

$$F'_{00}(W) = \int_{\mathbb{R}} sgn(v - W)\, L'(v - W)\, f_0(v)\, dv > 0.$$

So one might expect it to lead to exponential decay. . . but. . .

Proof of Theorem 1

Recall that the scheme is

$$\frac{d(V_W - V_\infty)}{dt} = -\frac{F_{00}(V_\infty) - F_{00}(W)}{V_\infty - W}(V_W - V_\infty) + H(t) - R(W)$$

where $H(t) = F_{00}(t) - F_0(t)$.

The 'coefficient' $\frac{F_{00}(V_\infty)-F_{00}(W)}{V_\infty - W} > \mathit{constant} > 0$,
due to $F'_{00}(V_\infty) > 0$.

We will prove that the other two terms are integrable on $[0, \infty)$.
This leads to Theorem 1.

Effect of particle forces on the body (Theorem 1)

For simplicity consider the body to be a point mass located at position $x = X(t)$. We will estimate

$$H(t) \equiv F_{00}(t) - F_0(t) = \int \mathrm{sgn}(V(t) - v)\, L(v - V(t))\, h(t, x, v) dv$$

where $h = f_{NB} - f_0$ and f_{NB} solves the equation without any precollisions before time t. Thus

$$\{\partial_t + v\partial_x + G(t, x)\partial_v\}h = -G(t, x)f_0'(v), \quad h(0, x, v) = 0.$$

The assumptions (G is small and decaying) imply that

$$|h(t, x, v)| \leq \delta \int_0^t ds\, \langle s\rangle^{-q} \langle \check{x}\rangle^{-m} \langle \check{v}\rangle^{-\ell-1},$$

where $\check{x} = \check{x}(s; t, x, v)$, $\check{v} = \check{v}(s; t, x, v)$ is the particle path and δ is a small constant. Also, $|\check{v} - v|$ is bounded by a small constant.

Particle forces: estimate of $H(t)$

We want to prove that $\int_0^\infty |H(t)|dt < \infty$. This is delicate! We will break it into several pieces $H = H_1 + H_{21} + H_{221} + H_{222} + H_{223}$. Let $0 < T = t^\alpha < t$ where $0 < \alpha < 1$ will be chosen later to optimize the estimates. We focus on t being large.

H_1 is the integral over the small $\{|v - V_\infty| \le \frac{bT}{t}\}$ with b to be chosen later. H_1 decays in t because the v interval is small. H_{21} is the portion where $s \in [T, t]$ and so it decays because of the decay in s.

In the other terms we have $0 < s < T < t$. For brevity, for a particle (t, x, v), we denote its trajectory at time T by $(T, \check{x}, \check{v})$ and its trajectory at time s by (s, x^*, v^*).

Particle Forces (cont.)

We have

$$|H(t)| \le \delta C \int_{-\infty}^{\infty} dv \int_0^T ds \frac{|L|}{\langle s\rangle^q \langle x^*\rangle^m \langle v^*\rangle^{\ell+1}}$$

where $|L| \le C(|v - W(t)|^p + |v - W(t)|^2)$.

H_{221} is the portion of the remaining integral with $|v| \le 1$. Using $|v - V_\infty| > \frac{bT}{t}$ and the fact that the trajectory has to travel like $\check{x} \sim x + (t - T)v$, we get $|x^*| > cT$, which provides the decay.

H_{222} is the integral over $\{|x^*| > \frac{t}{2}|v - V_\infty| > \frac{bT}{2}\}$, which once again provides the decay. Recall that $W(t)$ is near V_∞.

Particle Forces (cont.)

The final piece is H_{223}, for which $|x^*| < \frac{t}{2}|v - V_\infty|$ and $\frac{t}{2}|v - V_\infty| > \frac{bT}{2}$. But

$$\check{x} - T\check{v} = t\{V_\infty - \check{v}\} + \{\check{x} - x - (T-t)\check{v}\} + \{x - tV_\infty\}$$

, so that

$$|x^*| \gtrsim |\check{x} - (T-s)\check{v}| \gtrsim t|V_\infty - v| - |\check{x} - x - (T-t)v| - s|\check{v}|$$

so that

$$\frac{t}{2}|v - V_\infty| \gtrsim t|V_\infty - v| - CT - s|\check{v}|$$

or $CT + s|\check{v}| \gtrsim \frac{bT}{2}$. A choice of b leads to $s > CT$, which again provides a decay rate.

Effect of precollisions on the body (Theorem 1)

We want to prove that $\int_0^\infty |R(t)|dt < \infty$. There are particles on the left and on the right. Consider the right side (located at $X(t)$) and a large time t.

$$R(t) = \int_{-\infty}^{W(t)} \mathrm{sgn}(V(t) - v)\, L(v - V(t))[f_-(t,x,v) - f_{NB}(t,x,v)]\, dv.$$

We first prove that $\sup_{x=X(t),|v-V_\infty|<3\gamma} f_+(t,x,v) \leq constant$. This follows from the properties of the collision kernel K and the body velocity W.
Now let a particle collide at (t,x,v) such that t is an isolated collision time (avoiding a null set). Its most recent collision is at (τ,ξ,ν) with $\tau < t$. Then

$$\int_\tau^t \check{v}(s;t,x,v)ds = X(t) - X(\tau) = \int_\tau^t W(s)ds.$$

Effect of precollisions (cont.)

From the particle equation for $\check{v}$ and the properties of W, we prove $|v - V_\infty| \leq \frac{\delta}{t}$ for some small δ. So we obtain

$$|R(t)| \leq C \int_{v_\infty - \frac{\delta}{t}}^{W(t)} |L(v - W(t))| \, dv = C \left(\frac{\gamma}{t} \right)^{p+1}$$

for small γ. This is integrable since $p > 0$.

Existence

Existence Proof: Let $\mathcal{K} = \{W \in \mathcal{W} \mid \sup(|W(t)| + |W'(t)| \leq L\}$ for some L large enough. $\mathcal{K}$ is a compact convex set in $C([0,\infty))$. Then $\mathcal{A} : W \to V_W$ maps $\mathcal{K}$ into itself. We will apply the Schauder fixed point theorem (This will prove existence but not uniqueness.)

It suffices to prove that $\mathcal{A}$ is continuous in the topology of $C([0,\infty))$. To do that, we let $W_j \to W$ in $C([0,\infty))$, where $W_j \in \mathcal{K}$. First fix $T > 0$ so large that the interval (T,∞) provides a negligible contribution due to the uniform decay in time. It suffices to prove that $R_{W_j}(t) \to R_W(t)$ uniformly in $[0, T]$.

Existence (concluded)

Let N be a positive integer. Define A_j^N be the set of all pairs (x, v_x) such that no trajectory passing through (T, x, v_x) has collided more than N times in the time interval $[0, T]$. Let B_j^N be its complement. We write $R_{W_j}(t) = R_{W_j}(t; A_j^N) + R_{W_j}(t; B_j^N)$. By the main estimate we have $|R_{W_j}(t; B_j^N)| \leq (C\gamma^{p+1})^N$ for $t \leq T$. Also $R_{W_j}(t; A_j^N) \to R_W(t; A^N)$ in $C([0, T])$ by iterating the collision boundary condition N times. It follows that $R_{W_j}(t) \to R_W(t)$ uniformly in $[0, T]$. $\square$

If (V, f) is an arbitrary solution in the sense of Theorem, then it is a fixed point of $\mathcal{A}$, so that the inequalities are valid for it. We need only check that the strict inequalities

$$\gamma e^{-B_0 t} + \frac{A_+ \gamma^{p+1}}{t^{p+d}} \chi\{t \geqslant t_0 + 1\} < V(t) - V_\infty < \gamma e^{-B_\infty t} + \frac{A_- \gamma^{p+1}}{(1+t)^{p+d}}, \tag{1}$$

are valid for small $t > 0$. Indeed, note that at $t = 0$ we have $\gamma = V(0) - V_\infty < \gamma + C\gamma^{p+1}$ and $V'(0) = E - F(0) = F_0(V_\infty) - F_0(V_\infty + \gamma) < -\gamma \min(F_0') \leq -\gamma B_\infty < 0$. Therefore (1) is valid for very small $t > 0$ and hence for all $t > 0$. $\square$

Reversal Case (Theorem 3; no particle forces)

The key step is to prove both upper and lower bounds of $R(t, W)$). In the reversal case, because $E \geq 0$, the collisions on the right begin to dominate but, after the velocity reversal, eventually those on the right and the left balance each other and the body tends to its equilibrium speed from below.

Definition (Class of possible motions)

- $W : [0, \infty) \to \mathbb{R}$ is Lipschitz and $W(0) = V_\infty + \gamma$.
- W is decreasing over the interval $[0, t_0]$ for $t_0 = K_0 \,|\ln \gamma|$ with $\frac{1}{B_0} \leqslant K_0 \leqslant \frac{2}{B_0}$, where $B_0 = \max_{V \text{ near } V_\infty} F_{00}'(V), \quad B_\infty = \min_{V \text{ near } V_\infty} F_{00}'(V)$.
- The basic desired inequality holds for W:

$$\gamma h(t, \gamma) \leqslant V_\infty - W(t) \leqslant \gamma g(t, \gamma),$$

$$-g(t, \gamma) = e^{-B_0 t} - \frac{\gamma^p A_+}{\langle t \rangle^{p+d}}, \quad -h(t, \gamma) = e^{-B_\infty t} - \frac{\gamma^p A_-}{t^{p+d}} \chi_{t \geqslant t_0 + 1}$$

Lemma (Key Estimate on the Collision term $R_W(t)$)

Assume in addition that the kernel K and the initial data f_0 verify the Reversal Criterion. Then for small enough γ there exist ϵ, c_1, C_2, and $C > 0$ such that for all $W \in \mathcal{W}$, the force due to all the precollisions is bounded below and above:

$$R_W(t) \geqslant \left(\frac{c_1 \gamma^{p+1}}{t^{p+d}} - \frac{C_2 \gamma^{(1+\varepsilon)(p+1)} A_+^{p+1}}{t^{p+d}} \right) \chi_{t \geqslant t_0} \geqslant 0$$

and

$$R_W(t) \leqslant \frac{C \left(\gamma + \gamma^{p+1} A_+\right)^{p+1}}{(1+t)^{p+d}}.$$

Lemma (Iteration)

Assume the Key Estimate above. Then for small enough γ, we can choose A_+ and A_- in the definition of $\mathcal{W}$ such that, for any $W \in \mathcal{W}$, the solution V_W to the iteration equation

$$\frac{dV_W}{dt} = Q(t)\,(V_\infty - V_W) - R_W\,(t)\,, \quad Q(t) = \frac{F_0(V_\infty) - F_0(W(t))}{V_\infty - W(t)},$$

satisfies

$$-\gamma e^{-B_\infty t} + \frac{A_-\gamma^{p+1}\chi_{t\geqslant t_0+1}}{t^{p+d}} \leqslant V_\infty - V_W(t) \leqslant -\gamma e^{-B_0 t} + \frac{A_+\gamma^{p+1}}{(1+t)^{p+d}}.$$

In other words, for every $W \in \mathcal{W}$, we have $V_W \in \mathcal{W}$.

Proof of Iteration Lemma:

$$\frac{d\left(V_\infty - V_W\right)}{dt} = -Q(t)\left(V_\infty - V_W\right) + R_W\left(t\right)$$

where $V_\infty - V_W(0) = -\gamma < 0$. The point is that the quotient $Q(t)$ is positive and bounded away from zero. So we should have exponential decay and remain negative. However, it is spoiled by the force $R_W(t)$.

$$V_\infty - V_W\left(t\right) = -\gamma e^{-\int_0^t Q(r)dr} + \int_0^t \left(e^{-\int_s^t Q(r)dr}\right) R_W(s)ds$$

Upper bound:

$$\begin{aligned} V_\infty - V_W\left(t\right) &\leqslant -\gamma e^{-B_0 t} + \int_0^t e^{-B_\infty(t-s)} \frac{C\left(\gamma + \gamma^{p+1} A_+\right)^{p+1}}{(1+s)^{p+d}} ds \\ &= -\gamma e^{-B_0 t} + C\left(\gamma + \gamma^{p+1} A_+\right)^{p+1} I, \end{aligned}$$

$$\begin{aligned} I &= \int_0^{\frac{t}{2}} e^{-B_\infty(t-s)} \frac{1}{(1+s)^{p+d}} ds + \int_{\frac{t}{2}}^t e^{-B_\infty(t-s)} \frac{1}{(1+s)^{p+d}} ds \\ &\leqslant \int_0^{\frac{t}{2}} e^{-B_\infty(t-s)} ds + \frac{C}{(1+t)^{p+d}} \int_{\frac{t}{2}}^t e^{-B_\infty(t-s)} ds \\ &\leqslant \frac{1}{B_\infty}\left(e^{-\frac{B_\infty t}{2}} - e^{-B_\infty t}\right) + \frac{C}{(1+t)^{p+d}} \leqslant \frac{C'}{(1+t)^{p+d}}. \end{aligned}$$

That is,

$$V_\infty - V_W(t) \leqslant -\gamma e^{-B_0 t} + \frac{C' \left(\gamma + \gamma^{p+1} A_+\right)^{p+1}}{(1+t)^{p+d}}.$$

Letting $A_+ > C'$, we have $C' \left(1 + \gamma^p A_+\right)^{p+1} \leqslant A_+$ for small γ, so that

$$V_\infty - V_W(t) \leqslant -\gamma e^{-B_0 t} + \frac{A_+ \gamma^{p+1}}{(1+t)^{p+d}}.$$

Lower bound:

$$R_W(t) \geqslant \frac{c\gamma^{p+1}}{t^{p+d}} \chi\{t \geqslant t_0\} \geqslant 0$$

for small γ. Thus

$$\begin{aligned} V_\infty - V_W(t) &\geqslant -\gamma e^{-B_\infty t} + \int_0^t e^{-B_0(t-s)} \chi\{s \geqslant t_0\} \frac{c\gamma^{p+1}}{s^{p+d}} ds \\ &= -\gamma e^{-B_\infty t} + c\gamma^{p+1} II, \end{aligned}$$

where

$$II \geqslant \int_{t-1}^{t} e^{-B_0} \frac{1}{s^{p+d}} ds \geqslant e^{-B_0} \frac{1}{t^{p+d}},$$

as long as $1 + t_0 < t$. Hence

$$V_\infty - V_W(t) \geqslant -\gamma e^{-B_\infty t} + \frac{c'^{p+1} \chi\{t \geqslant t_0 + 1\}}{t^{p+d}}.$$

Therefore, selecting $A_- \leqslant c'$ yields

$$V_\infty - V_W(t) \geqslant -\gamma e^{-B_\infty t} + \frac{A_- \gamma^{p+1} \chi\{t \geqslant t_0 + 1\}}{t^{p+d}}.$$

Proof of Key Estimate on $R_W(t)$ in Reversal Case

To prove

$$R_W(t) \geqslant \left(\frac{c_1 \gamma^{p+1}}{t^{p+d}} - \frac{C_2 \gamma^{(1+\varepsilon)(p+1)} A_+^{p+1}}{t^{p+d}} \right) \chi_{t \geqslant t_0} \geqslant 0$$

and

$$R_W(t) \leqslant \frac{C\left(\gamma + \gamma^{p+1} A_+\right)^{p+1}}{(1+t)^{p+d}}.$$

We write $R_W = R_W^L + R_W^R$ is the force on the body due to collisions on the left side and the right side of the body $\Omega(t) = [a(t), b(t)]$. Here

$$\begin{aligned} R_W^L(t) &= \int_{W(t)}^{\infty} \ell(u - W(t)) \{f_0(u) - f_-(t, x, u)\} \, du \\ &= \int_{W(t)}^{\langle W \rangle_t} \ell(u - W(t)) \{f_0(u) - f_+(\tau, a(\tau), u)\} du \end{aligned}$$

where $\langle W \rangle_t = \frac{1}{t} \int_0^t W(s) ds$.

Now a particle can collide at two times $\tau < t$ only if its velocity is $v = \frac{1}{t-\tau}\int_\tau^t W(s)ds < \langle W\rangle_t$. The assumptions on W imply

$$\frac{C\gamma}{t}\chi_{\{t\geq t_0\}} \leq \langle W\rangle_t - W(t) \leq C\frac{\gamma + \gamma^{p+1}A_+}{1+t}.$$

Now

$$f_+(t,x,v) = \int_{W(t)}^\infty K(v - W(t), u - W(t))$$
$$\{f_+(\tau, a(\tau), u)\chi_1(t,u) + f_0(u)\chi_0(t,u)\}du$$

where τ is the time of the preceding collision at the left endpoint $a(\tau)$. The last term represents the particles that did not previously collide. Now the Reversal Criterion says

$$\int_{-\infty}^0 K(0,z)\, f_0(z + V_\infty)\, dz < f_0(V_\infty).$$

Thus $\sup f_+ < C\gamma(\sup f_+) + f_0(V_\infty)$ so that $\sup f_+ < f_0(V_\infty) - \epsilon$ for small enough γ.

We put this into $R_W^L(t)$ as follows:

$$R_W^L(t) \geq \int_{W(t)}^{\langle W \rangle_t} \ell(u - W(t))\{f_0(u) - f_0(V_\infty) + \epsilon\} du$$

$$\geq C\frac{\epsilon}{2}(\langle W \rangle_t - W(t))^{p-1} \geq C(\frac{\gamma}{t})^{p-1}$$

for $t \geq t_0$. In a similar fashion, we get

$$R_W^L(t) \leq C\left(\frac{\gamma + \gamma^{p+1} A_+}{1+t}\right)^{p+1}$$

and

$$|R_W^R(t)| \leq C\left(\frac{\gamma^{1+\epsilon} A_+}{t}\right)^{p+1} \chi_{\{t \geq t_0\}}.$$

$\square$