IMPERIAL COLLEGE LONDON

— *Magistère de Physique Fondamentale* —

<u>Third Year Project</u>

MATHEMATICAL PHYSICS
OF LANDAU DAMPING

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ABSTRACT

In this report, we focus on two aspects of kinetic theory : the Vlasov equation and Landau damping, and we point out the unifying nature of physical and mathematical approaches in this context.

First, we describe and discuss the derivation of the Vlasov equation, and motivate its use in the theory of plasma physics.

Then we define important notions of kinetic theory and functional analysis in order to apply it to the Vlasov equation. We then use all these results to present the Landau damping phenomenon in a linear setting.

We end this report by presenting the recent work of Mouhot and Villani in the context of nonlinear Landau damping.

INTRODUCTION

The present document summarizes the work I realised during my third year project at Imperial College London, with Jonathan Ben-Artzi as my supervisor. It took place within the second year of the *Magistère de Physique Fondamentale*, at the Université Paris-Sud.

My first motivation was to get a general understanding of the world of kinetic theory, from a mathematical physics perspective. It turned out to be a very fruitful project, and I regard it as one of the most enriching knowledge experience I had so far. At the biginning, I became quicky interested in two aspects of Kinetic theory : the *Vlasov equation* and *Landau damping* :

− the Vlasov equation describes an ideal plasma made of many charged particles interacting through long-range forces (i.e. collisions are neglected). This equation is one of the fundamentals of kinetic theory. It gets its name from Anatoly Alexandrovich Vlasov, who first introduced it for the study of plasmas in 1937.

− Landau damping is a relaxation phenomenon for those plasmas which are well-described by the Vlasov equation. A small disturbance in such a plasma at equilibrium generates an electric field which damps away exponentially with time. Landau damping gets its name from Lev Davidovitch Landau, which discovered it in 1946 by a clever mathematical treatment of the Vlasov equation.

The aim of this report is two-fold : presenting the knowledge I have acquired during this project *and* giving the reader a general understanding of the topic, by both presenting the main ideas and giving useful references and sources. This report is divided into three chapters :

− In the first chapter, we introduce and motivate the use of statistical physics to model a system of many interacting particles, in order to describe a plasma. Starting from classical mechanics, we present and discuss the various assumptions leading to the Vlasov equation, which is the central equation of this report.

− In the second chapter, the mathematics of the Vlasov equation are brought forward. We introduce the tools needed for both the study of the Vlasov equation and for Landau damping. This chapter is very theoretical, and is a short summary of what I had to learn to tackle the nonlinear Landau damping problem.

− In the third chapter, we focus on Landau damping in a chronological way. We first describe the seminal works of Vlasov and Landau for the linear Landau damping. Then, based on the work of Backus and Penrose, we point out the limitations of the linear theory and motivate the need for a nonlinear result. Finally, we present the work of Mouhot and Villani, which gave the first result in the direction of nonlinear Landau damping.

Throughout the report, we will use the following notations :

− x and v are position and velocity vectors, respectiely. They will have three real components denoted by $x = (x_1, x_2, x_3)$ and $v = (v_1, v_2, v_3)$ and thus both belong to some subspace of $\mathbb{R}^3$.

− t is the time, and will be positive, i.e. $t \in \mathbb{R}_+$.

− for a function $f = f(t, x, v)$, we will write by ∇_x or ∇_v when refering to position gradients and velocity gradients respectively.

− when a function $W = W(x)$ depends only on one variable $x \in \mathbb{R}^3$, we might write ∇W for $\nabla_x W$ without ambiguity.

FIRST CHAPTER

In this chapter we motivate the use of the Vlasov equation to describe a system of many interacting particles. We discuss the various aspects of its derivation : viewpoint, simplification and generalisation ; and introduce the notations and concepts used in the following two chapters.

1.1 CLASSICAL MECHANICS

Here we introduce the famous N-body problem, and motivate the use of alternative representations of the system than the classical theory of mechanics.

The first problem we will consider consists of describing a system of N bodies interacting through a given force. If the force is an inverse square law (such as the Coulomb or gravitational force) this is known as the *N-body problem*.

The case $N = 2$ has an explicit solution, first given by Kepler : the trajectory of each body is a conic section (parabola, ellipse or hyperbola). But as soon as $N \geqslant 3$, this is a huge problem, and only a few results are available today.

As an example, consider N stars, labelled by $i \in \{1, ... N\}$, of mass m_i and position $x_i \in \mathbb{R}^3$. A star located at x_i is affected by the gravitational pull of a star located at x_j via the force

$$F_{j \to i} = -\mathcal{G} m_i m_j \frac{x_i - x_j}{|x_i - x_j|^3}, \tag{1}$$

where $\mathcal{G}$ stands for Newton's constant and $|\cdot|$ stands for the Euclidian norm in $\mathbb{R}^3$. In order to solve the problem, we must solve Newton's equation of motion for each star :

$$\ddot{x}_i = -\mathcal{G} \sum_{j \neq i} m_j \frac{x_i - x_j}{|x_i - x_j|^3}, \qquad i \in \{1, ... N\} \tag{2}$$

If we consider N particles of charge q_i interacting through the Coulomb force (say in a plasma), the equation of motion would only change by a change of sign. If $\mathcal{K}$ is the Coulomb constant, the equations to be solved are :

$$\ddot{x}_i = \mathcal{K} \sum_{j \neq i} \frac{q_i q_j}{m_i} \frac{x_i - x_j}{|x_i - x_j|^3}, \qquad i \in \{1, ... N\} \tag{3}$$

In both cases, the mathematical description is simply given by $3N$ coupled differential equations such as (2), and can be solved in principle.

The issue comes from the number N itself : for a typical galaxy, N is of the order 10^{11} ; and it is of the order 10^{23} for a gas of charged particles.

Of course, it is not possible to solve that many coupled equations, even with the power of modern computation. In order to make progress, we need to change our point of view.

Different scales

The Newtonian description corresponds to the *microscopic viewpoint* : we focus on each particle and write an equation for each. We work with classical mechanics : the equations are simple but as soon as N becomes large, this is not well adapted. The quantities we work with are positions, forces and momenta.

We can take a big step back and look at the system of interacting particles as a whole. By isolating a virtual cube in space containing enough particles, and applying conservation laws inside this cube, we obtain the *macroscopic viewpoint*. In that case, we will work with hydrodynamics : velocity field, density field, etc.

In between those two scales lies the *mesoscopic viewpoint* : we treat the particles statistically but focus on their collective effects and motion. This allows us to consider the *joint distribution function f_N* of the N particles, which contains all the information about the system at any time t :

$$f_N = f_N(t, x_1, ..., x_N, p_1, ..., p_N). \tag{4}$$

Here, x_i is the position of particle i, and p_i is its momentum. Unlike other viewpoints, the distribution function f_N of N particles is not an observable : no physical experiment can measure it. However, we can measure quantities such as the mean density, the mean temperature, the mean velocity, which are all linked to f_N.

1.2 Statistical Viewpoint

Here we describe the tools used when looking at the system of interacting particle in a mesoscopic context. The use of statistics will be necessary to reduce the complexity of the problem.

The quantity describing the system is now the joint distribution function

f_N, which is a *probability density*, whence

$$\int_{(\Omega \times \mathbb{R}^3)^N} f_N(t, X, P)\mathrm{d}X\mathrm{d}P = 1 \tag{5}$$

with the short hand notation $X = (x_1, ..., x_N)$ and $P = (p_1, ..., p_N)$.

The integration in (5) is carried over the *phase space* $\mathcal{V} = \Omega \times \mathbb{R}^3$, with $\Omega \subseteq \mathbb{R}^3$ the space of positions, and $\mathbb{R}^3$ the space of momenta. Note that Ω will either be the torus $\mathbb{T}_L^3 = \mathbb{R}^3 / L\mathbb{Z}^3$ of length L ; or the whole space $\mathbb{R}^3$, depending on whether the system is confined or not. From now on, we shall write $\mathcal{V} = \Omega \times \mathbb{R}^3$, as the general phase space for one particle, and specify Ω when needed.

The "equation of motion" in the statistical viewpoint is given by Liouville's *equation* :

$$\frac{\partial f_N}{\partial t} + \sum_{i=1}^N \left(\nabla_{p_i} H \cdot \nabla_{x_i} f_N - \nabla_{p_i} f_N \cdot \nabla_{x_i} H \right) = 0 \tag{6}$$

where H is the Hamiltonian of the system of N particles. Even though f_N verifies the deterministic equation (6), the statistics come into play because the quantity $f_N \mathrm{d}X\mathrm{d}P$ represents the "probability of finding particle 1 in the box $[x_1 + \mathrm{d}x_1] \times [p_1 + \mathrm{d}p_1]$, particle 2 in $[x_2 + \mathrm{d}x_2] \times [p_2 + \mathrm{d}p_2]$, and so forth", and also because particles are randomly distributed at $t = 0$.

We will interest ourselves into the case of N particles of mass m_i, position x_i and momentum p_i, and take into account a pairwise interaction between the bodies represented by the potential energy $\phi(x_i - x_j)$. The Hamiltonian will then be :

$$H(X, P) = \sum_i \frac{|p_i|^2}{2m_i} + \sum_{i<j} \phi(x_i - x_j). \tag{7}$$

The system (6)-(7) contains as much information as a system of Newton equations. But instead of having $6N$ simple equations, we have a complicated one, and the unknown f_N depends on $6N + 1$ variables. In that sense, Liouville's equation is unexpoitable. To make progress, we will take advantage of the $N \gg 1$ property of the system.

Firstly, if the collective motion is predominant, there is no reason why some particles should have on average a different motion than others : we will assume that all particles are *identical* and *indistinguishable*. We will also rescale their mass to unity. Since $p_i = m_i v_i$ in a classical context, we will speak in terms of *position* $x_i \in \Omega$ and *velocity* $v_i \in \mathbb{R}^3$ from now on, rather than x_i and p_i.

To reduce the complexity of (6), we will consider the *1-particle distribution function*, defined as the *first marginal* of f_N :

$$f(t, x, v) = \int_{\mathcal{V}^{N-1}} f_N(t, X, V) \mathrm{d}x_2...\mathrm{d}x_N \mathrm{d}v_2...\mathrm{d}v_N \qquad (8)$$

where, since particles are indistinguishable, we have re-labelled x_1, v_1 by x, v. The quantity $f(t, x, v)\mathrm{d}x\mathrm{d}v$ focusses on one particle of the system (they are all equivalent by assumption) : it represents the probability that the particle is located inside the box $[x + \mathrm{d}x] \times [v + \mathrm{d}v] \in \mathcal{V}$, which is reflected in the normalisation :

$$\int_{\mathcal{V}} f(t, x, v)\mathrm{d}x\mathrm{d}v = 1. \qquad (9)$$

This new function f satisfies a Liouville-like equation, which can be found by computing $\partial f / \partial t$ and using equation (6) for f_N. Taking into account the general Hamiltonian (7), one finds for f :

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = (N - 1) \int_{\mathcal{V}} \nabla_x \phi(x - x_2) \cdot \nabla_v f_2 \, \mathrm{d}x_2 \mathrm{d}v_2 \qquad (10)$$

In this equation, f_2 is the *second marginal of f_N*, defined by :

$$f_2(t, x, x_2, v, v_2) = \int_{\mathcal{V}^{N-2}} f_N(t, X, V)\mathrm{d}x_3...\mathrm{d}x_N \mathrm{d}v_3...\mathrm{d}v_N \qquad (11)$$

Equation (10) shows that the behavior of the 1-particle distribution f depends on f_2, the second marginal, which captures the two-particles interactions. If we look for an equation for f_2 using the same procedure (i.e. computing $\partial f_2 / \partial t$ using (6)), we would obtain an equation similar to (10), but with the third marginal f_3 involved.

If we repeat this process, we will obtain the *BBGKY hierarchy*, named after Born, Bogliubov, Green, Kirkwood and Yvon. The behavior of f_k, the kth marginal of f_N, defined by

$$f_k(t, x_1, ..., x_k, v_1, ..., v_k) = \int_{\mathcal{V}^{N-k}} f_N(t, X, V)\mathrm{d}x_{k+1}...\mathrm{d}x_N \mathrm{d}v_{k+1}...\mathrm{d}v_N \qquad (12)$$

is described by a Liouville-like equation, which takes the form :

$$\frac{\partial f_k}{\partial t} = \{H_k, f_k\} + (N - k)\sum_{i=1}^{k} \int_{\mathcal{V}} \nabla_{x_i}\phi(x_i - x_{k+1}) \cdot \nabla_{v_i} f_{k+1}\mathrm{d}x_{k+1}\mathrm{d}v_{k+1} \qquad (13)$$

where $H_k = \sum_{j=1}^{k} v_i^2/2 + \sum_{i<j\leqslant k} \phi(x_i - x_j)$ is the k particles Hamiltonian, and $\{g, h\} = \sum_{i=1}^{N} (\nabla_{x_i}g \cdot \nabla_{v_i}h - \nabla_{v_i}g \cdot \nabla_{x_i}h)$ is the Poisson bracket on $\mathbb{R}^3$. One can check that $k = 1$ with these definitions gives equation (10).

The whole BBGKY hierarchy is equivalent to the system (6)-(7) : we have not reduced the complexity of the description. But as we can see, the equations are ordered in a sense. This will allow us to finally simplify the problem.

1.3 MEAN-FIELD LIMIT

Here, starting from the BBGKY hierarchy, we will simplify the problem by making some assumptions, which we will discuss along the way.

To have an equation on f only, we will make two assumptions, closely linked together :

– *molecular chaos* : in the $N \gg 1$ limit, the particles are uncorrelated in the sense that $f_2 = f_1 \otimes f_1$[1]. We only need the first two BBGKY equations to close the system. With $f_2(t, x, y, v, w) = f(t, x, v) f(t, y, w)$:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = (N-1)\left(\int_{\mathcal{V}} \nabla_x \phi(x-y) f(t, y, w) \mathrm{d}y \mathrm{d}w \right) \cdot \nabla_v f(t, x, v) \quad (14)$$

– *mean field limit* : the potential energy of a particle is the mean potential energy of the cloud divided by N. We let $\phi = W/N$, and call $W = W(x - y)$ the *mean field potential*. This will compensate the $N - 1$ term in the right hand side of (14), which with $N \gg 1$ gives, in first approximation :

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \left(\int_{\mathcal{V}} \nabla_x W(x-y) f(t, y, w) \mathrm{d}y \mathrm{d}w \right) \cdot \nabla_v f = 0. \quad (15)$$

This is the Vlasov equation for the 1-particle distribution $f(t, x, v)$, with mean-field potential W. Now, let us discuss the assumptions behind this formal derivation. As we will see, the molecular chos assumption is closely linked to the $N \gg 1$ limit, and they are far from being completely understood.

Molecular chaos :

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1. In the same way that two independent events A and B have their probabilities unaffected by one another : $P(A \cap B) = P(A) \times P(B)$. If we introduce the correlation function $g(t, x, y, v, w) = f_2(t, x, y, v, w) - f(t, x, v) f(t, y, w)$ (notice $g = 0$ if molecular chaos is assumed), and look for an equation on g, we obtain the Mayer *cluster expansion*.

Clearly the molecular chaos cannot hold true in general : collisions between particles are known to play a great role in physics :

– if we only keep binary colisions into account, for instance in a rarefied gas of neutral particles, BBGKY hierarchy yields the Boltzmann equation :

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q_B(f, f) \tag{16}$$

where Q_B is the Boltzmann collision operator : a nonlinear integral operator whose form depends on how we modelize collisions. Note that (16) is not a mean-field equation : there is no long range interaction here, particles interact only if they are close enough to each other. In that sense the Boltzmann equation is very different from the Vlasov equation.

– in the context of plasma physics, Landau first derived in [17] the equivalent of the Boltzmann equation for a gas of charged particles, by taking into account long range interaction between chaged particles, and hard collisions between neutral particles. This equation is sometimes called the Landau equation :

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + F \cdot \nabla_v f = \frac{\ln \Lambda}{2\pi\Lambda} Q_L(f, f) \tag{17}$$

where F is the mean Coulombian force, Λ is the plasma parameter which takes values between 10^3 and 10^{20} depending on the properties of the plasma and Q_L is the Landau collision operator, taking into account collisions.

For references about the Boltzmann and Landau equations, see [42] and [1] respectively and references therein. Taking collisions into account surely depends on what system we want to describe. In our case, the collective motion of particles due to long-range interaction is considered. To justify that such behavior dominates, we must examine the physics of the problem. This was done by Vlasov in [43] who was the first to justify that we could neglect collisions in plasma.

From a mathematical perspective, the "propagation of chaos" assumption is central in kinetic theory : if the initial states of particles are uncorrelated, do they stay so at later times, in the limit $N \to \infty$? The mathematical formulation of this question is still debated today, but some progress has been made since the establishment of Kac's program. For typical references about this, we refer to [25] and references therein.

<u>Mean field limit :</u>

Regarding the mean-field limit, the $1/N$ rescaling provides a limit where the force becomes constant and independent of the large number of particles, which is what we would expect for very large N^2. Also, the $1/N$ rescaling is very natural because it appears in an alternative derivation of the Vlasov equation (equivalent to what we did).

Define the *empirical measure* μ_N of the system of N particles by

$$\mu_t^N = \frac{1}{N}\sum_{i=1}^{N} \delta(x_i(t), v_i(t)). \tag{18}$$

where δ is the usual Dirac distribution. This quantity "counts" particles in phase space at time t. Then, if μ_0^N converges to some measure μ_0 as $N \to \infty$ (i.e. convergence in $N \gg 1$ at initial time) and if W is regular enough, then μ_t^N also converges to some μ_t (i.e. convergence in $N \gg 1$ at any time), and μ_t solves the Vlasov equation (15).

In order to apply this to realistic finite (but large) N particles systems, one could ask how fast this limit (i.e. the Vlasov equation) is reached when $N \to \infty$. In other words, how close are the mean field Vlasov equation and the Newton equation for N particles ? How good is the approximation ? Here is a classical answer to this question :

If ν_t stands for the mean field description ($\nu_t = f(t, x, v)\mathrm{d}x\mathrm{d}v$) and μ_t is the empirical measure (18), then :

$$\mathrm{dist}(\nu_t, \mu_t) \leqslant e^{2ct}\mathrm{dist}(\nu_0, \mu_0) \tag{19}$$

for $c = 1 + \sup|D^2 W|$. We learn two things from this. First, if the potential W is too singular, then $\sup|D^2 W| = \infty$ and the result is not useful at all. In fact, today it is still an open problem for potentials like Coulomb or Newton. Second, if W is not too singular, (i.e. $\sup|D^2 W| < \infty$) then the two descriptions agree as long as e^{2ct} is not too large.

For example : if the particles are chosen randomly at $t = 0$, then $\mathrm{dist}(\nu_0, \mu_0) \simeq 1/\sqrt{N}$ (this is the famous central limit theorem) so by (19) we have $\mathrm{dist}(\nu_t, \mu_t) \leqslant e^{2ct}/\sqrt{N}$: the higher the number of particles, the better the approximation. Note that (19) can also be used to have an idea of the relevant time scale for the approximation. All of the above is particularily well explained in [13], [41] and references therein.

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2. Consider, for instance, a gas of 10^{23} particles. Even if several millions dissapeared at once, the mean force felt by one particle would change ever so slightly, as a few millions is still very small compared to 10^{23}.

1.4 Vlasov-Poisson Equation

Having derived the Vlasov equation, we will now introduce the Vlasov-Poisson system. We will alos discuss some divergence issues and the physics behind it.

<u>Vlasov-Poisson :</u>

Driven by the N-body problem, we have seen how the mean field limit of the classical equations of motion resulted in the Vlasov equation :

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \left(\int_{\mathcal{V}} \nabla_x W(x-y) f(t,y,w) \mathrm{d}y \mathrm{d}w \right) \cdot \nabla_v f = 0 \qquad (20)$$

where f is the 1-particle distribution and W is the potential energy of the mean field interaction. If we take the phase space to be $\mathcal{V} = \Omega \times \mathbb{R}^3$ with $\Omega = \mathbb{R}^3$, and we split the integral in brackets we obtain :

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \nabla_x W * \left(\int_{\mathbb{R}^3} f(t,x,v) \mathrm{d}v \right) \cdot \nabla_v f = 0 \qquad (21)$$

where we have used the $*$ notation for the convolution over $\mathbb{R}^3$:

$$W * \rho : x \mapsto (W * \rho)(x) = \int_{\mathbb{R}^3} W(x-y) \rho(y) \mathrm{d}y \qquad (22)$$

The term in brackets in (21) is called the *microscopic density* $\rho_f(t,x)$ associated to f, and with W, they form the *microscopic force* $F[f]$ defined by

$$F[f] = -\nabla_x W * \rho_f = -\nabla_x W * \left(\int_{\mathbb{R}^3} f(t,x,v) \mathrm{d}v \right). \qquad (23)$$

The microscopic force is a function of t and x and depends on f. It is exactly the force created by the distribution and felt by one particle. Indeed, if we call ψ, the potential energy of one particle due the whole distribution, then for both Coulomb and Newton interaction we have the Poisson equation linking ψ and the source density, in this context ρ_f :

$$\Delta \psi = \gamma 4\pi \rho_f \qquad (24)$$

with $\gamma = +1$ for Newton and $\gamma = -1$ for Coulomb (physical constants have been set to 1). By the Green's method, we can solve (24) explicitely, and obtain :

$$\psi(t,x) = \gamma \int_{\mathbb{R}^3} \frac{\rho(t,y)}{|x-y|} \mathrm{d}y \qquad (25)$$

which, with $F = -\nabla\psi$ and $W(x) = 1/|x|$ is exactly (23).

If the phase space is now $\mathcal{V} = \mathbb{T}_L^3 \times \mathbb{R}^3$, i.e. f is L-periodic in x, then due to (24) we must also have a periodic potential :

$$\psi(r) \propto \sum_{k \in \mathbb{Z}^3} \frac{1}{|r + kL|}, \tag{26}$$

but this series does not converge! Defining the Vlasov-Poisson system in the phase space $\mathcal{V} = \mathbb{T}^3 \times \mathbb{R}^3$, be it Newtonian or Coulombian, seems to lead to divergences. In the plasma case, in order to fix this issue, we have to look deeper into the electrostatic theory of charged particles and take into account *Debye shielding* (see for instance [29]). With this, the Poisson equation for the potential of one particle of charge q is

$$\Delta_r \psi(r) - \frac{1}{\lambda_D^2} \psi(r) = -\delta(r) \tag{27}$$

where λ_D is the caracteristic Debye length, and δ is the Dirac distribution. Then, using Green's method, the solution to (27) is :

$$\psi(r) = \frac{e^{-|r|/\lambda_D}}{4\pi|r|} \tag{28}$$

which now converges if summed in a periodic setting. For the astrophysics case, there is no such thing as shielding. Nevertheless, we can obtain an equation of the type (27) for the gravitational potential, by taking into account the cosmological constant into classical Newton mechanics. There is also the Jeans *swindle*, which might be related to this issue ; we refer to [7] and [26, chap. 2] for additional information about this.

All in all, if we describe the system with the 1-particle distribution f, the problem consists of finding for all $t \in \mathbb{R}$, $x \in \Omega$, $v \in \mathbb{R}^3$, a solution to the following system, called the *Vlasov-Poisson system* :

$$\begin{cases} \dfrac{\partial f}{\partial t} + v \cdot \nabla_x f - \nabla_x \psi \cdot \nabla_v f = 0 \\[2mm] \Delta\psi = \gamma 4\pi \rho_f, \quad \rho_f = \int_{\mathbb{R}^3} f \mathrm{d}v \end{cases} \tag{29}$$

In the context of plasma physics, $\gamma = -1$ and f is the electron distribution. In the context of astrophysics, $\gamma = -1$ and f is the distribution of stars. Later on, we will consider the plasma physics context, and prefer using the following form of the Vlasov-Poisson system, with $F[f]$ and $W = \pm 1/|x|$ instead of ψ. It is of course completely equivalent :

$$\begin{cases} \dfrac{\partial f}{\partial t} + v \cdot \nabla_x f + F[f] \cdot \nabla_v f = 0 \\[2mm] F[f] = -\nabla_x W * \rho_f, \\[2mm] \rho_f = \int_{\mathbb{R}^3} f\, \mathrm{d}v, \quad W = \pm 1/|x| \end{cases} \qquad (30)$$

Generalisation :

If we just look at the first equation in the system (30), we can see that the force between particles can be arbitrary. More generally, we will call "Vlasov equation" any equation of the type :

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + F(t, x) \cdot \nabla_v f = 0 \qquad (31)$$

and call F the force in the context. In this general case a solution to (31) would be given by a couple (f, F). If we add an interaction equation which links F to f (for instance the Poisson equation in (29)) then we will write $F[f]$ to stress this dependance.

The Vlasov-Poisson system (30) with Coulomb potential is the basis for the theory of plasma physics. Nevertheless, (30) is only acurate if we exclude external electric fields, and external and self-generated magnetic fields. If we want to take that into account we have to consider the set of Maxwell equations, as we will see in the third chapter.

As a last word, let us say that we can also generalise the Vlasov-Poisson systems to take into account quantum, relativistic and/or general relativistic effects. There exists a lot of such systems and we refer to [33], [8] and [9] for nice introductions and details on these topics.

SECOND CHAPTER

This chapter is about the mathematical aspects of the Vlasov equation, and its position in the theory of differential equations. We introduce the mathematical tools used in the third chapter, and some of the physics related to it. We refer to [5] and [28] for more details of the analysis.

2.1 DIFFERENTIAL EQUATIONS

This first part is devoted to the concepts underlying differential equation analysis. We will introduce the vocabulary, the notion of Cauchy problem and of Hadamard well-posedness.

Vocabulary

Differential equations are used in every branch of physics, and in all fields of sciences in general : economics, biology, chemistry, etc. Whatever the area it is used in, there is some common vocabulary which we will introduce here.

Consider $x = (x_1, x_2, ..., x_n) \in U$, with U being a subspace of $\mathbb{R}^n$. A *partial differential equation* (PDE) *of order m* is an expression of the type :

$$\mathcal{F}(x, f(x), Df(x), ..., D^m f(x)) = G(x), \qquad x \in U \subseteq \mathbb{R}^n. \tag{32}$$

with $n \geqslant 2$. $\mathcal{F}$ is defined on $U \times \mathbb{R} \times \mathbb{R}^n \times \cdots \times \mathbb{R}^{n^m}$ and is given, $D^i f$ denotes all possible partial derivatives of total order i and G is given. The real valued function f is the *unknown*. If $n = 1$, then this is an *ordinary differential equation* (ODE), and it can be written as

$$\mathcal{F}\big(x, f(x), f'(x), ..., f^{(m)}(x)\big) = G(x), \qquad x \in U \subseteq \mathbb{R}. \tag{33}$$

with this time $\mathcal{F} \colon \mathbb{R}^{m+2} \to \mathbb{R}$.

In both cases, G is the *source term*, and if it is 0 everywhere, then the equation is *homogeneous*. If $G(x) \neq 0$ on U, the equation is *nonhomogeneous*. If there are products of f and/or its derivatives, the PDE is said to be *nonlinear* and if there are no such terms, it is *linear*.

As an example, the full Vlasov-Poisson equation is an homogeneous, nonlinear partial differential equation[3] :

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \nabla_x \bigg(W * \int_{\mathbb{R}^3} f \mathrm{d}v \bigg) \cdot \nabla_v f = 0. \tag{34}$$

The nonlinearity comes from the third term, and $W = \pm 1/|x|$ depending on the nature of the interaction : Coulombian $(+)$ or Newtonian $(-)$. Here we have written the Vlasov-Poisson equation with a convolution product, which comes from solving the Poisson equation, as we saw in the first chapter :

$$(W * \rho_f)(t, x) = \int_{\mathcal{V}} \frac{f(t, y, w)}{|x - y|} \, \mathrm{d}v \mathrm{d}w \tag{35}$$

Cauchy problem.

With a PDE such as (32), we usually prescribe *boundary conditions* : the value of f on some subspace of Ω (usually its boundary). If we can identify one of the variables as the time t, we can prescribe the shape of f at $t = 0$: it is then called an *initial condition*.

For the Vlasov-Poisson equation the initial condition is given by an *initial distribution* f_{in} defined on the phase space $\mathcal{V} = \Omega \times \mathbb{R}^3$:

$$f_{\mathrm{in}}(x, v) = f(0, x, v), \quad (x, v) \in \Omega \times \mathbb{R}^3. \tag{36}$$

The exercise consisting of solving a PDE with given initial conditions is called a "Cauchy *problem*". When we say "solving" a PDE, we don't necessarily mean finding a closed-form expression for the solution : this is rarely possible. In a modern language, "solving" a PDE rather means "finding properties about the solution" : its asymptotics, its size, its regularity, etc.

Well-posedness.

In 1902, Hadamard motivated in [15] the notion of "well-posedness" of a Cauchy problem, by establishing a criterion under which a PDE describes a realistic physical phenomena. This criterion translates the physical notion of causality and under/overdeterminism into mathematics :

A Cauchy problem is said to be well-posed in the sense of Hadamard if a) there exists a unique solution to the problem, and b) this solution depends continuously on the initial data.

Let us examine those two conditions :

---

3. Strictly speaking, the Vlasov-Poisson equation does no belong to the general family of PDE's such as (1) because of the integral term $\int f \mathrm{d}v$. However, we can still apply the systematic vocabulary to it.

a) The existence ensures that the model does not have too much constrains, and that all assumptions are compatible with one another. Also, the solution must be uniqu to prevent underdeterminacy. Indeed, if the same initial conditions were to yield different solutions, various physical scenarii could emerge from the same initial state, and this seems difficult to imagine happening if the system is well-described.

b) The second condition is here to ensure that the solution at later times keeps track of the initial data : if we change the initial conditions very slightly, the system should react continuously and the not be drastically different. Typically, if we quantify by ε the difference between two initial conditions, the two different results should also depend on ε in a continous manner.

Proving the well-posedness[4] of a problem has now become a central topic in PDE analysis, and is has brought a connexion between the mathematics of a physical model, in the sense that the well-posed system are worth studying, since they are realistic to the physicist's eye.

2.2 Tools for PDE's

In this second part we introduce the standard tools for PDE analysis : integral transforms, norms and functionnal spaces. We also review the results available for proving the well-posedness of a Cauchy problem, to motive the next section.

The Fourier and Laplace transform were introduced to convert the differential equations into algebraïc equations, much easier to solve. The conventions introduced here are those used in the next chapter.

Fourier transform.

If $f = f(v)$ is defined on $\mathbb{R}^3$, we define its Fourier transform $\hat{f}$ by

$$\hat{f}(\eta) = \int_{\mathbb{R}^3} f(v) e^{-2i\pi\eta \cdot v} \mathrm{d}v. \tag{37}$$

---

4. The well-posedness of a problem can be shown in a local manner, i.e. only for finite time $t \in [0, T[$, $T < \infty$, and then may be improved to a global result, for which one can actually take $T = \infty$.

Here, $\eta \in \mathbb{R}^3$ and to make sure this intergal exists, we demand that f be integrable : $\int_{\mathbb{R}} |f(v)| \mathrm{d}v < \infty$. With this transform, we have the two properties :

$$\widehat{(\nabla_v f)}(\eta) = 2i\pi\eta \hat{f}(\eta) \quad \text{and} \quad f(x) = \int_{\mathbb{R}^3} \hat{f}(\eta) e^{2i\pi\eta \cdot x} \mathrm{d}\eta. \tag{38}$$

The first one shows that differential operations on f turn into simple products in the Fourier space, and the second gives a way to recover f from its Fourier transform, provided that $\hat{f}$ itself is also integrable[5].

If $f = f(x)$ is now defined on the torus $\mathbb{T}_L^3 = \mathbb{R}^3 / L\mathbb{Z}^3$, (i.e. f is $L-$periodic in the three directions) then the Fourier transform depends on the "length of the box" L, and is defined by

$$\hat{f}^{(L)}(k) = \int_{\mathbb{T}_L^3} f(x) e^{-2i\pi\frac{k}{L} \cdot x} \mathrm{d}x. \tag{39}$$

In this case, $k \in \mathbb{Z}^3$. The integral over $\mathbb{T}_L^3$ means integrating over $[0, L]$ in all three directions. The two properties are similar to (38) :

$$\widehat{\nabla f}(k) = 2i\pi\frac{k}{L}\hat{f}(k) \quad \text{and} \quad f(x) = \frac{1}{L^3} \sum_{k \in \mathbb{Z}^3} \hat{f}^{(L)}(k) e^{2i\pi\frac{k}{L} \cdot x}. \tag{40}$$

For $f = f(x, v)$ with $(x, v) \in \mathbb{T}_L^3 \times \mathbb{R}^3$, we can combine the two and define $\tilde{f} = f(k, \eta)$ with $k \in \mathbb{Z}^3$ and $\eta \in \mathbb{R}^3$:

$$\tilde{f}(k, \eta) = \int_{\mathbb{T}_L^3} \int_{\mathbb{R}^3} f(x, v) e^{-2i\pi\eta \cdot v} e^{-2i\pi(k/L) \cdot x} \mathrm{d}x \mathrm{d}v. \tag{41}$$

We will use these tools in the third chapter, in the context of Landau damping. Every time we work with on $\mathbb{T}_L^3$, changing the size will only change constants in the equation. Thus, we will set $L = 1$ for the sake of simplicity, but the methods and equations are similar for $L \neq 1$.

Last about the Fourier transform of a function : $\hat{f}(k)$ will always decay at infinity (i.e. $\hat{f}(k) \to 0$ as $|k| \to \infty$), this is the "Riemann-Lebesgue's *lemma*". The idea behind this result can already be viewed with equation (38) : if $\widehat{\nabla_v f}$ is bounded (say $\widehat{\nabla_v f} \leqslant C$), then :

$$\left| \hat{f}(\eta) \right| \leqslant \frac{C}{2\pi|\eta|} \xrightarrow[|\eta| \to \infty]{} 0 \tag{42}$$

---

5. For instance, a sufficient condition for $\hat{f}$ to be integrable is that a) f is differentiable and b) f, f' is square-integrable.

Furthermore, the more derivatives f has (i.e. the smoother it is), the faster $\hat{f}$ decays at infinity. This can be viewed by applying n successive Fourier transforms. Once again, with n fixed in $\mathbb{N}$, if $\widehat{\nabla^n f}$ is bounded (say $\widehat{\nabla_v f} \leqslant C$), then :

$$\widehat{\nabla^n f}(\eta) = (2i\pi\eta)^n \hat{f}(\eta) \quad \Rightarrow \quad |\hat{f}(\eta)| = \frac{C}{(2\pi|\eta|)^n} \xrightarrow[|\eta|\to\infty]{} 0$$

and we see that the decay can become faster than any inverse power if f has infinitely differentiable.

Laplace transform.

The Laplace transform is similar to the Fourier tranform, in the sense that this is also an integral transform. The difference comes from the integration over $\mathbb{R}_+$, and from the argument of the transform being a complex number.

If $f = f(t)$ is defined on $\mathbb{R}_+$, we define its Laplace transform $\mathcal{L}(f)$ by :

$$\mathcal{L}(f)(\nu) = \int_0^\infty f(t)e^{-\nu t}\mathrm{d}t, \tag{43}$$

for any $\nu \in \mathbb{C}$ such that the integral is well-defined. For instance, if f is exponentially bounded over $[0, \infty[$, i.e. $|f(t)| \leqslant ae^{bt}$ for some real a, b ; then the Laplace transform exists for all ν such that $\mathrm{Re}(\nu) > b$.

Similarily to the Fourier transform, we have the differential property :

$$\mathcal{L}(f')(\nu) = \nu\mathcal{L}(f)(\nu) - f(0) \tag{44}$$

which will transform time derivatives into simple products in the Laplace space (notice here the term $f(0)$ coming from integration by parts). We also have a way to recover $f(t)$ from $\mathcal{L}(f)(\nu)$ by the inversion formula :

$$f(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{L}(f)(\nu)e^{\nu t}\mathrm{d}\nu. \tag{45}$$

Here, the integration is performed along the line $\mathrm{Re}(\nu) = \sigma$, provided that this line is at the right of all complex poles of the integrand. This formula will be crucial to Landau's first discovery.

Functional spaces, norms

In order to give complete answer to a PDE problem, it is necessary to specify what *functional space* we work in. A functional space is an abstract space made up of all functions having one property. This common property can be of various nature : boundedness, integrability, smoothness, etc, or any combination of those. For instance, the assumption we made for taking the Fourier transform of a function $f\colon\mathbb{R}^3\to\mathbb{R}$ was :

$$\int_{\mathbb{R}^3} |f(v)|\mathrm{d}v = \|f\|_{L^1} < \infty. \tag{46}$$

If we consider the space of all such functions, we obtain the functional space $L^1(\mathbb{R}^3)$ of integrable functions over $\mathbb{R}^3$. The quantity $\|f\|_{L^1}$ defined by (46) is called the L^1 *norm* of f. We can generalise this to the L^p *spaces* (with $p \in [1,\infty]$), using the norm :

$$\|f\|_{L^p} = \left(\int_{\mathbb{R}^3} |f(x)|^p \mathrm{d}x \right)^{1/p} < \infty \tag{47}$$

and if $p=\infty$, then the L^p *norm* is just the common *supremum norm* :

$$\|f\|_{L^\infty} = \sup_{v\in\mathbb{R}^3} |f(v)| \tag{48}$$

The L^p spaces look at the integrability of a function and its powers, but says nothing about the smoothness. For our purposes, the functional spaces used will be about the degree of smoothness, or *regularity*, of a function. For a function $f\colon\mathbb{R}^3\to\mathbb{R}$, typical functional spaces looking at regularity include the following classes of functions :

- C^n functions, where n first derivatives of f are continuous

- C^∞ functions (smooth functions), where for all $n\in\mathbb{N}$, the nth derivative of f is continuous

- C^ω functions (analytic functions), where $f\in C^\infty$ and coïncides with its Taylor series[6]

- H^m Sobolev space , where derivatives of f up to order m are square integrable (i.e. in L^2)

---

6. The Taylor series of a function $f\in C^\infty$ around a point x_0 is defined as :

$$T_{f,x_0}(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

if $f(x) = T_{f,x_0}(x)$ for x in a neigborhood of x_0, then f is analytic at x.

– $W^{m,p}$ Sobolev space (generalisation of $H^m = W^{m,2}$), where derivatives of f up to order m are in L^p

The C^n, C^∞ and C^ω spaces only look at the regularity of f, whereas H^m and $W^{m,p}$ combine regularity and integrability properties.

To each of these spaces we associate a *norm*, which measures the "size" of the function in the context of the functional space. One can think of a norm as a the "microscope" of the mathematician : it is used to look closely at a specific property of a function. Usually, one will need to use several norms to cover all relevant aspects of a function.

When it comes to regularity spaces (e.g. C^n, C^∞ and C^ω), the norm can be given through the use of the Fourier transform. This will come very handy in the analysis of the Vlasov equation. For instance, an analytic function f: $\mathbb{R}^3 \to \mathbb{R}$ can be shown to satisfy $\nabla^n f(v) = O(n!/r^n)$ for some constant $r > 0$. Thus it is natural to set an *analytic norm* as :

$$\|f\|_\lambda = \sum_{n=0}^\infty \frac{\lambda^n}{n!} \|\nabla^n f(v)\|_{L^\infty} \qquad (49)$$

where λ is a parameter interpreted as the width of the analyticity strip of the function f. The L^∞ (i.e. sup) norm measures the size of successive derivatives $\nabla^n f$, but we could for instance measure it with an other L^p norm.

As an example, consider $f(x) = e^{-x}$ and $g(x) = (1+x)^{-1}$, for $x \in \mathbb{R}_+$. From direct computation, we have $\|f\|_\lambda = e^\lambda$ and $\|g\|_\lambda = (1 - \lambda)^{-1}$. We see that λ can be arbitrary large for f, i.e. f is analytic over the strip $[0, \infty[$, whereas for g we recquire $\lambda > 1$ to ensure $\|g\|_\lambda \geqslant 0$: its strip of analyticity is $[1, \infty[$.

Now, if we use formula (38) successively, we obtain an equivalent statement : $f \in C^\omega \iff \hat{f}(k) = O(e^{-2\pi c|k|})$, for some constant $c > 0$. Hence, an other analytic norm for f could be :

$$\|f\|_\mu = \int_{\mathbb{R}^3} e^{2\pi\mu|\eta|} |\hat{f}(\eta)| \mathrm{d}\eta. \qquad (50)$$

Of course, if $f \colon \mathbb{T}^3 \to \mathbb{R}$, then by periodicity the integral in (50) can be replaced by the sum of $e^{2\pi\mu|k|} |\hat{f}(k)|$ over the modes $k \in \mathbb{Z}^3$. These analytic norms will play a major role in the study of nonlinear Landau damping, and we shall refer to the above definitions in the third chapter.

Theorems for odes

In the particular case where one works with an ODE, there are 3 main theorems which give direct results about existence and unicity of solutions. They form the general theory of ODE's, and are the basis of the systematic approach one can adopt when proving well-posedness of systems described by ODE's. To present them, we will consider the following ODE :

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \mathcal{F}(t, f(t)), \quad t \in I \tag{51}$$

where f is the unknown, I is an open interval of $\mathbb{R}$ and $\mathcal{F}: I \times \mathbb{R} \to \mathbb{R}$. These 3 theorems have similarities, and can be summarised into the following table :

Theorem	Regularity of $x \mapsto \mathcal{F}(x, y)$	Regularity of $y \mapsto \mathcal{F}(x, y)$	Unicity of solution f	Regularity of solution f
Cauchy − Kovalevskaya	analytic	analytic	Yes	$f \in C^\omega$
Cauchy − Lipschitz	continuous	Lipschitz −continuous	Yes	$f \in C^1$
Cauchy − Peano	continuous	continuous	No	$f \in C^1$

In the Cauchy-Kovalevskaya theorem, the strong analyticity hypothesis makes it of little use, but it is of historical importance since it was the first to be derived for any general ODE. The Cauchy-Lipschitz theorem is the most interesting one since it gives a existence and uniqueness under weaker regularity assumptions. The Cauchy-Peano theorem of little use in practice because it does not prove uniqueness under such weak assumptions, but it can be used as a prototype for constructing results for PDE's.

It should be noted however that these theorems give local results : the unique solution is defined in any neighborhood of $(t_0, u_0) \in I \times \mathbb{R}$. To get global result in time, one must be careful since the non-linearity of $\mathcal{F}$ comes into play : there may be finite blow-up of the solution, as shows the classical example :

$$\begin{cases} u'(t) = u^2(t) \\ u(0) = u_0 > 0 \end{cases}, \quad t \in \mathbb{R}. \tag{52}$$

The non-trivial solution to this ODE is $u(t) = u_0(1 - u_0 t)^{-1}$, which blows-up at time $t = 1/u_0$. In order to avoid this phenomenon and guarantee global existence of a solution, we can recquire an extra assumption on $\mathcal{F}$, as shows the following proposition :

Proposition 1 : *If F verifies the Cauchy-Lipschitz regularity conditions and if $|\mathcal{F}(t, y)| \leqslant C(1 + |y|)$ for some constant C, then equation (51) has a unique solution defined globally in time.*

This version of the Cauchy-Lipschitz theorem will allow us to prove the well-posedness of the Vlasov equation later on.

2.3 Transport Equations

In this section, we introduce the transport equations, discuss the special case of free transport and the very important notion when we will come to Landau damping : phase mixing.

The above results only applied to ODE's, and are difficult to convert for PDE's. Fortunately, the Vlasov equation inherits the structure of the Liouville equation, in the sense that it is a *transport equation*.

First, if we forget that the force F depends on f, we can write the Vlasov equation as :

$$\begin{cases} \dfrac{\partial f(t, y)}{\partial t} + \omega(t, y) \cdot \nabla_y f(t, y) = 0 \ . \end{cases} \tag{53}$$

if $\omega(t, y) = (v, F(t, y))$ and if $y \in \mathbb{R}^6$ stands for $(x, v) \in \mathcal{V}$. The genral equation (53) is called a *linear transport equation*, and $\omega = (v, F(t, x, v))$ is the *vector field* transporting the initial distribution $f_{\text{in}}(x, v)$ in time, through the phase space $\mathcal{V}$.

So far, the point of view we adopted was Eulerian : we focueds on one spot (x, v) in phase space and watched the particles neighboring around it. We now adopt the Lagrangian point of view : we choose one particle and follow its trajectory along the *flow* of the transport equation.

If at time t the particle is at position (x, v), at later time $s > t$ it will be at $(X(s), V(s))$ in the phase space, as shows the following diagram :
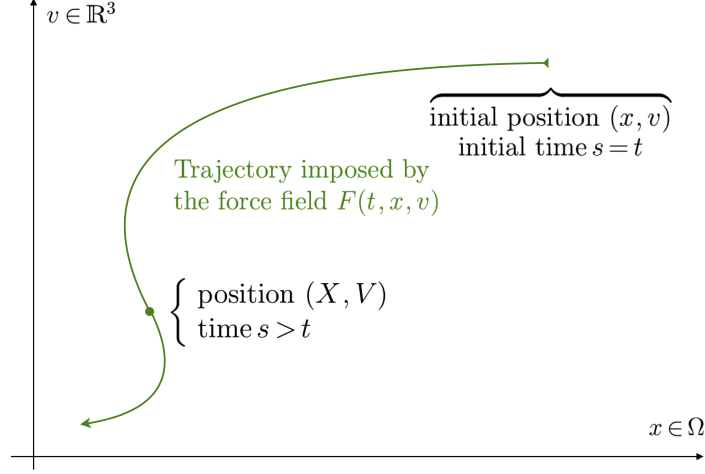

Figure 1. Trajectory of a particle in phase space $\mathcal{V} = \Omega \times \mathbb{R}^3$.

Along this trajectory, the evolution of the variables (X, V) depends only on the single variable s, and (t, x, v) are just "initial conditions", i.e. time and place of when we started looking at the particle. This trajectory is governed by the following system of ODE's :

$$
\begin{cases}
\dfrac{\mathrm{d}X}{\mathrm{d}s}(s, t, x, v) = V(s, t, x, v) \\[2ex]
\dfrac{\mathrm{d}V}{\mathrm{d}s}(s, t, x, v) = F(t, X(s, t, x, v), V(s, t, x, v))
\end{cases}
\tag{54}
$$

with initial conditions $X(0, t, x, v) = x$ and $V(0, t, x, v) = v$. Equation (54) are equivalent to (53), only the point of view changes. The solutions to ODE (54) are called the *characteristics* of the transport equation (53). If we apply proposition 1 to this system we obtain :

Proposition 2 : *If $F(\cdot, x, v) \in C^0(\mathbb{R})$, if $F(t, \cdot, v)$ and $F(t, x, \cdot)$ are Lipschitz continuous on $\mathbb{R}^3$ and if $|F(t, x, v)| \leqslant C(1 + |x| + |v|)$ for some constant C, then equation (54) has a unique solution defined globally in time.*

Since the caracteristics map $(x, v) \mapsto (X(s, t, x, v), V(s, t, x, v))$ takes the particle from (x, v) to (X, V), if the initial time is $t = 0$, then by construction :

$$
f(t, X(t, 0, x, v), V(t, 0, x, v)) = f_{\text{in}}(x, v)
\tag{55}
$$

To get $f(t, x, v)$, we just need to invert this equation. But we can prove that $(x, v) \mapsto (X(s, t, x, v), V(t, s, x, v))$ is a C^1-diffeomorphism : the inverse is given by swapping s and t, i.e. looking at the backwards trajectory of the particle. The Jacobian of this transformation is always positive, so the orientation is preserved as we would expect. Furthermore, if we assume that $\nabla_v F(t, x, v) = 0$, i.e. that $F = F(t, x)$ only, then $\operatorname{Jac}(X, V) = 1$. Thus, for all times t and all $(x, v) \in \mathcal{V}$, we have :

$$f(t, x, v) = f_{\text{in}}(X(0, t, x, v), V(0, t, x, v))$$

With the above assumptions and proposition 2 for the characteristics, we can prove the following proposition, this time for the full equation (53) :

Proposition 3. *Let $F : \mathbb{R} \times \mathbb{R}^3$ be a force field verifying :*

* $t \mapsto F(t, x)$ *is continuous on* $\mathbb{R}$
* $x \mapsto F(t, x)$ *is Lipschitz-continuous on* $\mathbb{R}^3$
* $|F(t, x)| \leqslant C(1 + |x|)$ *for some constant C, and for all $x \in \mathbb{R}^3$*

Then there exist a unique global C^1 solution f to the Cauchy problem :

$$\begin{cases} \dfrac{\partial f(t, x, v)}{\partial t} + v \cdot \nabla_v f(t, x, v) + F(t, x) \cdot \nabla_x f(t, x, v) = 0. \\[2mm] f(0, x, v) = f_{\text{in}}(x, v), \quad \text{with } f_{\text{in}} \in C^1(\mathbb{R}^3 \times \mathbb{R}^3) \end{cases} \tag{56}$$

As a last remark : proposition 3 is for linear transport equations only. It does not apply to the Vlasov-Poisson equation (34), for which F is linked to f, bringing nonlinearity. Nevertheless, we will be able to use this result to prove the wellposedness of the Vlasov-Poisson, by using an iterative scheme in the next section. For now, let us focus on the free tranport equation.

Free Transport :

Let's assume that our particles are free-moving, in the sense that no force field $F[f]$ acts on them. The evolution of the distribution f is given by :

$$\begin{cases} \dfrac{\partial f}{\partial t} + v \cdot \nabla_x f = 0 \\[2mm] f(0, x, v) = f_{\text{in}}(x, v) \end{cases} \tag{57}$$

where we have prescribed the initial data f_{in}. This is the transport equation (53) we met previously with $F = 0$, so we could solve it by the method of characteristics. But let's use a physical reasoning first.

If one particle is at position x with velocity v at time $t = 0$, then at a time $t \geqslant 0$, it will have moved to position $x + vt$. Furthermore, velocity can only vary if a force is applied to the particle, so here the velocity must remain the same along the trajectory. Hence, we expect for all $(t, x, v) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^3$,

$$f(t, x, v) = f_{\text{in}}(x - vt, v). \tag{58}$$

Indeed, taking the Fourier transform of (57) in x, and treating the obtained equation as an ODE for $\hat{f}$ in time, we have :

$$\frac{\partial \hat{f}}{\partial t} + 2i\pi k \cdot v \hat{f} = 0 \quad \Rightarrow \quad \hat{f}(t, k, v) = \hat{f}(0, k, v) e^{-2i\pi k \cdot vt} \tag{59}$$

for any fixed k and v. If we absorb the exponential term in (59) into the integral defining $\hat{f}(0, k, v)$, and make the change of variables $x \to x - vt$, we get in terms of f_{in} :

$$\hat{f}(t, k, v) = \int_{\mathbb{R}^3} f_{\text{in}}(x - vt, v) e^{-2i\pi k \cdot x} \mathrm{d}x \tag{60}$$

Comparing equations (60) with the very definition of $\hat{f}(t, k, v)$, we conclude by uniqueness of the Fourier transform that indeed, our reasoning (58) was right. Here we learn that the free transport equation is equivalent to translational invariance of the system in the phase space.

Phase mixing

Notice also in equation (59) the *phase mixing* property : as a function of v, $\hat{f}(t, k, v)$ oscilates with a frequency $\propto |k| t$. These oscillations become wilder with time, and modes of higher order k oscillate faster as well. Thus, an initially rather smooth velocity distribution function will become not so smooth after some time, in the sense that the successive derivatives $\nabla_v^n \hat{f}$ will become larger and larger.

Phase mixing is also connected to the effect of *filamentation*. To illustrate this property, consider a continuous distribution of electrons in the phase space $\mathcal{V} = \mathbb{T}_L^3 \times \mathbb{R}^3$ such as :
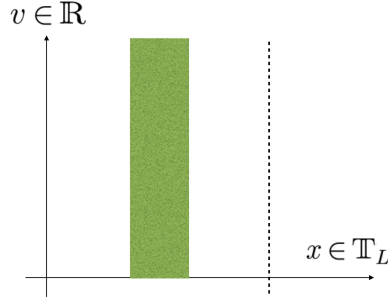
Figure 2. Initial distribution of electrons. The dashed line is $x = L$.

As time passes, electrons located at (x, v), will move to $(x + vt, v)$ at time t, so the faster the velocity the further away it moves. As a consquence, the initial ditribution will be stretched in the phase space, as follows :
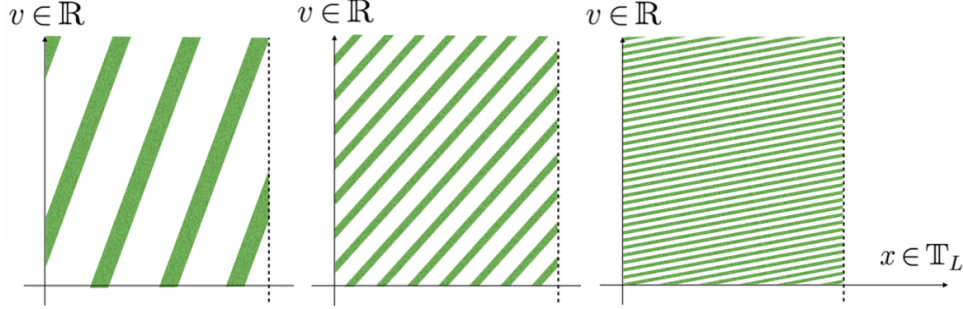

Figure 3. The distortion of the initial distribution creates filamentation as time passes from left to right.

The volume in phase space is conserved (i.e. the green areas are equal for each picture), but the distribution is distorted : this is the *filamentation*. As an ultimate consequence, the distribution will *homogenise* in phase space.

Now, not only do we have the solution to the free-transport equation, but we can also predicts its asymptotic behavior in time. This behavior is closely linked to the phase mixing property of the transport equation. Let's introduce the space-velocity Fourier transform defined by (41) (here with $L = 1$) and apply it to equation (57) to obtain :

$$\tilde{f}(t, k, \eta) = \tilde{f}_{\text{in}}(k, \eta + kt). \tag{61}$$

Notice the presence of t in $\eta + kt$, the velocity argument of f_{in}. If we take the limit $t \to \infty$, then $\eta + kt \to \infty$ for any η and $k \neq 0$. Since $\tilde{f}_{\text{in}}$ is Fourier transform, the Rieman-Lebesgue's lemma applies and gives $\tilde{f}_{\text{in}} \to_{t \to \infty} 0$. Hence by equation (61), $\tilde{f} \to 0$ with time.

Moreover, the rate at which $\tilde{f}(t,k,\eta)$ goes to zero is given by the regularity of f_{in} in the velocity variable, and the bigger the spatial mode $|k|$, the faster the decay. In particular, if f_{in} is analytic, then the convergence is exponential (cf discussion above (50)) :

$$\tilde{f}(t,k,\eta) = O\big(e^{-c|\eta+kt|}\big), \text{ for some } c > 0 \tag{62}$$

A nice way to summarize all the above is by saying that the structure of the free transport equation converts the regularity of f_{in} into decay. This phenomena is fundamental when it comes to Landau Damping.

Here, if $k \in \mathbb{Z}^3$ (i.e. $\Omega = \mathbb{T}$) the result is true for all $k \neq 0$. But if we had chosen $x \in \mathbb{R}^3$ (and thus $k \in \mathbb{R}^3$), $|k|$ could be as small as we want, and some modes would not have enough time to decay. The assumption $\Omega = \mathbb{T}$ is necessary to have *uniform* bounds.

However, we could choose $x \in \mathbb{R}^3$ and add decide to add a cutoff $k_{\min}$ such that $|k|$ should never be smaller than $k_{\min}$. This turns into an upper bound for the wavelength of the field, justified for instance by the length of the plasma container. In any case, some confinement mechanism (cutoff or $\mathbb{T}$) is necessary to constrain k and have uniform bounds. We will discuss this assumption in the third chapter.

2.4 THE VLASOV-POISSON EQUATION

In this last section, we will apply results of the previous parts to the well-posedned of the Vlasov-Poisson equation. We will then discuss equilibria and their stability for the Vlasov equation.

Well-posedness :

For the Vlasov-Poisson system, the well-posedness is established under various assumptions : size of the initial data, phase-space dimension, geometry of the problem, etc. For generic initial data f_{in}, there are two main results :

− the result of Pfaffelmoser [32], who proved global well-posedness of the problem in the case of initial data $f_{\text{in}} \in C^1$ and compactly supported. This result is of theoretical importance, since it shows well-posedness but it is unsatisfactory for we can't see why the solution should be compactly supported from a physical point of view.

− the result of Perthame and Lions [21], who proved that unique global solution exists even if f is not compactly supported, but satisfies some other conditions. The proof relies on the analysis of velocity moments of order greater than 3.

In both results, the main goal is to *control* (i.e. get bounds on) the influence of large velocities. We refer to the original papers and to [11] for a general and more detailed introduction to Vlasov systems well-posedness. However, there is an important part of the proofs that we will discuss : the use of an iterative scheme.

Iterative scheme :

So far, we have given a result for the well-posedness of the Vlasov equation (proposition 3) with a general force field $F(t, x)$. By doing that, we avoided the nonlinearity of the Vlasov equation :

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \left(\nabla_x W * \int f \mathrm{d}v\right) \cdot \nabla_v f = 0 \tag{63}$$

We now set $W = \pm 1 / |x|$. The iterative scheme is the most natural approach to overcome the nonlinearity. It is an equivalent way of writing (63), but as an infinite system of linear equations. We define the iterative scheme as follow :

Step 0 : define $F^0 = -\nabla_x W * \int f^0 \mathrm{d}v$ and $f^0(t, x, v) = f_{\mathrm{in}}(x, v)$,

Step 1 : define $F^1 = -\nabla_x W * \rho[f^1]$ and f^1 as the solution to

$$\frac{\partial f^1}{\partial t} + v \cdot \nabla_x f^1 + F^0 \cdot \nabla_v f^1 = 0, \tag{64}$$

Step n : define $F^n = -\nabla_x W * \rho[f^n]$ and f^n as the solution to

$$\frac{\partial f^n}{\partial t} + v \cdot \nabla_x f^n + F^{n-1} \cdot \nabla_v f^n = 0 \tag{65}$$

Note that all these equations are linear in f^n. Now, for each step, one tries to find *a priori estimates*. An a priori estimate is a bound found only by exploiting the structure of the PDE. Since this is done in the early stage of the analysis, without knowing beforehand if we can actually construct solutions, we call it "a priori". This is a key aspect of PDE analysis.

Once this is done, one obtains two sequences $(f^n)_{n \in \mathbb{N}}$ and $(F^n)_{n \in \mathbb{N}}$ for which we know the size (by the a priori estimates) at each step of the scheme. The next goal is to show that the sequences converge to f and F as $n \to \infty$, and that they verify the assumptions of proposition 3. This is how we show the well-posedness of the Vlasov-Poisson system, and overcome the nonlinearity, and this is done for many other PDE systems.

Even if for the Vlasov-Poisson system, the well-posedness is well covered, it is not so regarding other Vlasov systems. We refer again to [11] for more informations about the Vlasov-Maxwell system for instance.

Steady states :

Let's get back to our original Vlasov equation (63), and look for an equilibrium solution (also called a steady state), i.e. a solution f such that $\partial f / \partial t = 0$. We obtain :

$$v \cdot \nabla_x f(x,v) = \nabla_x \left(W * \int_{\mathbb{R}^3} f(x,v) \mathrm{d}v \right) \cdot \nabla_v f(x,v) \qquad (66)$$

where the convolution property $(\nabla_x f) * g = \nabla_x (f * g)$ has been used. From there it is easy to see that there can be many equilibria for the Vlasov equation.

$-Homogeneous\ equilibria$: consider any distribution depending only on $v : f = f^o(v)$. Then, $\nabla_x f^o = 0$ and using $\nabla_x (f * g) = f * \nabla_x g$ in the right hand side of (66) we obtain :

$$\nabla_x \left(W * \int_{\mathbb{R}^3} f(v) \mathrm{d}v \right) = W * (\nabla_x (\mathrm{cst})) = 0 \qquad (67)$$

so we conclude that any homogeneous $f^o(v)$ is an equilibrium for the Vlasov equation (for any potential W). A typical example of such equilibrium is the Gaussian distribution, ecountered frequently in physics :

$$f^o(v) = C \exp(-|v - v_0|^2 / \bar{v}^2) \qquad (68)$$

where C, v_0 and $\bar{v}$ are constants characterising the shape of the Gaussian. The Gaussian equilibrium is very common because it is also the *only equilibrium* when one takes collisions into account (i.e. the only steady state of Boltzmann's equation).

A natural question is to ask wheither a perturbation of this equilibrium is stable or not. The answer is generally yes, and this will lead us to Landau Damping in the next chapter.

$-BGK$ *equilibria* : along with homogeneous equilibria, there is an other family of equilibrium solutions which are not homogeneous. Those are known as BGK solutions, named after Bernstein, Greene and Kruskal who studied these inhomogenous equilibria in [6].

If we define the *macroscopic energy* of a particle $E = E(x, v)$ by :

$$E(x, v) = \frac{|v|^2}{2} + W * \rho_f, \qquad \rho_f = \int_{\mathbb{R}^3} f \mathrm{d}v \qquad (69)$$

then, any function $f(x, v) = g(E(x, v))$ is a steady state solution. Indeed, applying the chain rule gives :

$$\nabla_v f(x, v) = g'(E) v \cdot v \quad \text{and} \quad \nabla_x f(x, v) = g'(E) \nabla_x (W * \rho_f) \qquad (70)$$

and plugging (70) into (66) shows that $f = g(E)$ is an equilibrium, and this for any given interaction potential W. The nice think about these equilibira is that they are not homogeneous, which my be appealing from the physical point of view (an equilibrium need not be homogeneous).

Note that these BGK solutions are often called BGK *waves* because they can be viewed as a superposition of waves travelling in opposite directions, i.e. they are stationnary waves. Thus, it is customary to speak of period and speed of BGK solutions, even though they are steady states.

Regarding the stability of BGK equilibria it is widely open, and only a few results are known to this day (see [18] and [19] for example).

Stability :

Equilibrium solution are important in physics because we expect a system's behavior to be time-independent after a while : if we wait long enough, all forces and all perturbation should have balanced out, and the system should be in an equilibrium state.

The *stability* of an equilibrium depends on the evolution of a system after a small perturbation :

– if the system reacts to the perturbation by returning to equilibrium, we have a *stable equilibrium*. The perturbation is *damped* : there should be[7] a mechanism that dissipates the energy brought by the perturbation.

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7. Or there may not! In fact, we will see that Landau damping is on of the rare case of physical relaxation to equilibrium in the abscense of a dissipation mechanism. The correct interpretation of this is still a debated problem among physicists and mathematicians.

– if the system, once perturbed, does not return to equilibrium, then we have an *unstable equilibrium*. In this case the perturbation does not dissapear, or worse, it grows with time, leading to divergences in mathematics, and recquiring a change of model in physics.

Let's formalise this, using the Vlasov-Poisson equation in plasma physics as a basis. We will take a plasma in homogeneous equilibrium $f^o(v)$, and add a perturbation to the distribution of electrons $h(t, x, v)$ (e.g. by applying a very brief electric field). This perturbation should be small in some sense, i.e. with a choice of norm.

Right after the perturbation, the distribution f inside the plasma is given by $f(t, x, v) = f^o(v) + h(t, x, v)$. The Vlasov equation is then :

$$\frac{\partial(f^o + h)}{\partial t} + v \cdot \nabla_x(f^o + h) + F[f^o + h] \cdot \nabla_v(f^o + h) = 0 \qquad (71)$$

Now, of course $\partial_t f^o = \nabla_x f^o = 0$. Also, notice that by linearity of F we have $F[f^o + h] = F[f^o] + F[h]$. But $F[f^o] = 0$ as the convolution property showed us already in (66). Thus (71) becomes

$$\frac{\partial h}{\partial t} + v \cdot \nabla_x h + F[h] \cdot \nabla_v f^o = -F[h] \cdot \nabla_v h \qquad (72)$$

The right hand side is quadratic in h, and since we assumed h to be small with respect to some norm, we expect it to be smaller than all linear terms. We will thus neglect it. This assumption is very strong, and we will study it in more details in the next chapter.

With all this, we obtain the *linearised* Vlasov *equation*, where the unknown is the perturbation $h = h(t, x, v)$:

$$\begin{cases} \dfrac{\partial h}{\partial t} + v \cdot \nabla_x h + F[h] \cdot \nabla_v f^o = 0 \\[2mm] F[h] = -\nabla W * \rho_h, \quad \rho_h = \int_{\mathbb{R}^3} h \mathrm{d}v \end{cases} \qquad (73)$$

If we prescribe the initial profile for the perturbation $h_{\mathrm{in}} = h(0, x, v)$, then (73) is well-posed Cauchy problem, and will be the starting point for the stability analysis, which will lead to Landau damping in the next chapter.

In the context of astrophysics, the problem of stability was nicely studied by Lynden-Bell in [22] who introduced the concept of "violent relaxation". There is a difference compared to plasma physics : a limiting length scale (the *Jeans* length) which plays a fundamental role. More information about all this can be found in the very detailed textbook [7].

Finally, let us mention that there has been new mathematical results about the long-standing problem of *orbital stability*, for the models of galactic-dynamics. This problems tries to give a meaningful answer to questions such as "Why are observed galaxies the way they are ?". These results are closely related to the Vlasov-Poisson system, and are very-well described in the survey [27].

THIRD CHAPTER

In this chapter, we describe the Landau damping phenomenon, from the first equations obtained by Vlasov, to the recent work of Mouhot and Villani. We point out the strong correlation between mathematics and physics of the Landau damping, and make use of the tools and concepts introduced in the first two chapters.

3.1 DISCOVERY OF LANDAU DAMPING

In this first part we will discuss the seminal analysis of Vlasov and expose Landau's discovery. We will introduce notions related to Landau Damping and point out the limitations of these early treatments.

Vlasov's treatment :

In 1935, A.A. Vlasov was the first to justify that at large scale, we can neglect collisions in the study of gases of charged particles (see [43]). He also stresses the importance of long ranged forces between such particles. In his own words :

" [...] *in the theory of such systems, an essential role must be played by the interaction forces, particularly at large distances and, hence, a system of charged particles is, in essence, not a gas but a distinctive system coupled by long-range forces*".

He considered the Boltzmann equation, which governs the behavior of a gas and removed[8] all contribution from collisions : this is exactly the Vlasov equation. Vlasov considered the non-relativistic[9] Vlasov-Maxwell system with several particles species. Fro the sake of simplicity, we will just define the Vlasov-Maxwell for a system with only one species (say electrons), but this can be generalised easily (see [11] for instance). We set all physical constants to 1.

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8. In astrophysics, the Vlasov equation is still called the collisionless Boltzmann equation.

9. The non-relativistic version of the Maxwell system (which in itself seems contradisctory since Maxwell's equation are relativistic) is by assuming that velocities are slow with respect to the speed of light c, or equivalently that $\gamma = (1 + v^2/c^2)^{-1/2} \simeq 1$. The relativistic Vlasov-Maxwell system is defined by replacing v by $\hat{v} = \gamma v$ in both the Lorentz force and the $v \cdot \nabla_x f$ term in the Vlasov equation. Details on this non-relativistic limit can be found in [11] and references therein.

The Vlasov-Maxwell equation is :

$$\begin{cases} \dfrac{\partial f}{\partial t} + v \cdot \nabla_x f + F[f] \cdot \nabla_v f = 0, \\[2mm] F[f] = E_f + v \times B_f \end{cases} \tag{74}$$

where $F[f] = E_f + (\hat{v}/c) \times B_f$ is the Lorentz force and the electromagnetic fields E_f and B_f are given by Maxwell equations :

$$\nabla_x \cdot E_f = \rho_f, \qquad \nabla_x \cdot B_f = 0,$$
$$\nabla_x \times E_f = -\frac{\partial B_f}{\partial t}, \qquad \nabla_x \times B_f = j_f + \frac{\partial E_f}{\partial t}. \tag{75}$$

The definition of current and charge density then closes the Vlasov-Maxwell system (73),(74) and (75) :

$$\rho_f(t, x) = \int_{\mathbb{R}^3} f(t, x, v) \mathrm{d}v, \quad j_f(t, x) = \int_{\mathbb{R}^3} v f(t, x, v) \mathrm{d}v. \tag{76}$$

We will show the main ideas behind Vlasov's treatment. First, we linearise the equation by setting $f = f^o(v) + h(t, x, v)$, where $f^o(v)$ is a homogeneous equilibrium and $h(t, x, v)$ is the perturbation, taken smaller in some sense[10] than the equilibrium. At first order in h, equation (74) writes

$$\frac{\partial h}{\partial t} + v \cdot \nabla_x h + (E_h + v \times B_h) \cdot \nabla_v f^o = 0. \tag{77}$$

The shape of Maxwell's equation does not change under the linearisation, since the fields take into account contribution from both f^o and h. However, since the homogeneous equilibrium recquires global neutrality, we just focus on the current and charge densities from the perturbation :

$$\rho_h = \int_{\mathbb{R}^3} h(t, x, v) \mathrm{d}v, \quad j_h = \int_{\mathbb{R}^3} v h(t, x, v) \mathrm{d}v \tag{78}$$

To solve the system (75), (77) and (78), Vlasov used a normal mode approach, by looking for periodic solutions for the perturabtion, which would then induce the same behavior for the E and B field. We thus set

$$h = \bar{h}(v) e^{i(k \cdot x - \omega t)}, \quad E = E_0 e^{i(k \cdot x - \omega t)}, \text{ and } B = B_0 e^{i(k \cdot x - \omega t)}, \tag{79}$$

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10. This sense will be given by the choice of the norm used, and, as pointed out by Backus years later, fundamental for the physical interpretation of the result. In Vlasov's and Landau's treatment, the smallness of h is not quantified, and the linearity is assumed to work.

for some constant vectors E_0 and B_0. Pluging ansatz (79) into the Maxwell equations, we find that k is parallel to E_0, resulting in $B_h = 0$ since $B \propto k \times E$ for periodic fields. Introducing the electric potential ϕ such that $E = -ik\phi$, equation (77) becomes :

$$\bar{h}(t, x, v) = \phi(t, x) \frac{k \cdot \nabla_v f^o(v)}{v \cdot k - \omega}, \tag{80}$$

and the first Maxwell equation gives the potential in terms of the distributions :

$$\phi(t, x) = \frac{1}{k^2} \int_{\mathbb{R}^3} \bar{h}(t, x, v) \mathrm{d}v. \tag{81}$$

Together, equations (80) and (81) imply a relation between k and ω :

$$k^2 = \int_{\mathbb{R}^3} \frac{k \cdot \nabla_v f^o(v)}{v \cdot k - \omega} \, \mathrm{d}v \tag{82}$$

Vlasov found equation (82) and saw it as a dispersion relation : he concluded that periodic electric waves (fully characterised by k, ω) existed in the gas of electrons if equation (82) is fullfilled. This result, although very interesting, is not satisfying for several reasons :

– there is a pole at $\omega = v \cdot k$ in (82), which makes the integral divergent if no assumption is made,

– the dispersion relation (82) is a non-linear integral expression with k and ω, and it can allow several frequencies for a given wave number : there is non-uniqueness of the wave produced,

– the linearisation of the Vlasov equation in the first place is not justified, because we should check that the neglected terms are indeed small. Furthermore, the filamentation effect could make the neglected term grow in time through large velocity gravients.

The analysis of Vlasov was re-done by Landau 10 years later. Landau managed to fix some of the issues encountered by Vlasov by approaching the problem with a different perspective.

Landau's treatment :

Landau's treatment (see [16]) of the linearised Vlasov equation is essentially the same as that of Vlasov. The main difference is that Landau's approach is based on the following argument : given a initially perturbed equilibrium, we want to know its evolution. This is a Cauchy problem.

Landau's treatment is not denuated of handwavy asumptions, which will be later made rigorous by Backus and Penrose. Nonetheless, the Cauchy problem approach will result in the discovery of short time vanishing of the electric field : the Landau damping.

Once again we linearise around an homogenous equilibrium $f^o(v)$, the perturbation h is $f - f^o$ as before. The perturbed electron density is $\rho_h = \int h(t, x, v)\mathrm{d}v$. We start from the linearised Vlasov-Poisson equation with initial distribution $h_{\mathrm{in}}(x, v) = h(0, x, v)$, and electric potential ϕ :

$$\frac{\partial h}{\partial t} + v \cdot \nabla_x h + \nabla_x \phi \cdot \nabla_v f^o = 0, \tag{83}$$

where the electric potential ϕ results from the perturbed density of electrons. In order to solve the problem, we want to derive an expression for h. This is done (at least formally) as follows :

We take the Fourier transform in position. The Vlasov equation gives

$$\frac{\partial \hat{h}}{\partial t} + 2i\pi k \cdot v\hat{h} + 2i\pi k\hat{\phi} \cdot \nabla_v f^o = 0. \tag{84}$$

Now we use the Poisson equation to find $\hat{\phi}$ in terms of $\hat{\rho_h}$:

$$\Delta\phi = \rho_h \quad \Rightarrow \quad -4\pi^2|k|^2\hat{\phi}(t, k) = \hat{\rho_h} \tag{85}$$

Once again, we will chose the x_1-axis aligned along k. Then combining (84) and (85) gives

$$\frac{\partial \hat{h}}{\partial t} + 2i\pi k v_1 \hat{h} = -\frac{\hat{\rho_h}}{2i\pi k}\frac{\partial f^o}{\partial v_1}. \tag{86}$$

A Laplace transform in time will allow us to factorise the left hand side and express $\mathcal{L}(\hat{h})$ as :

$$\mathcal{L}(\hat{h}) = \frac{\hat{h}_{\mathrm{in}}}{\nu + 2i\pi k v_1} - \frac{1}{\nu + 2i\pi k v_1}\frac{\mathcal{L}(\hat{\rho_h})}{2i\pi k}\frac{\partial f^o}{\partial v_1}. \tag{87}$$

Finally, integrating in veocity gives a closed-form expression for the Laplace transform of ρ_h :

$$\mathcal{L}(\hat{\rho_h}) = \int_{\mathbb{R}^3}\frac{\hat{h}_{\mathrm{in}}\,\mathrm{d}v}{\nu + 2i\pi k v_1} \times \left(1 + \frac{1}{2i\pi k}\int_{\mathbb{R}^3}\frac{\partial f^o}{\partial v_1}\frac{\mathrm{d}v}{\nu + 2i\pi k v_1}\right)^{-1} \tag{88}$$

Now it is just a matter of inverting the time-Laplace transform and the space-Fourier transform, and the problem is technically solved. Assuming that we can perform the inverse Laplace transform as in chapter 2, we get $\hat{\rho}_h$ as

$$\hat{\rho}_h(t,k) = \frac{1}{2i\pi} \int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{L}(\hat{\rho}_h)(\nu,k)e^{\nu t}\mathrm{d}\nu, \tag{89}$$

given that σ is large enough. In order to evaluate the integral in (89), we can just integrate along this original line $\mathrm{Re}(\nu) = \sigma$. But we will use contour integration in a clever way, as Landau did : we shift the original path to the left (contour Γ below), still at the right of all n poles $(\nu_i) \in \mathbb{C}^n$, here drawn with $n = 3$:
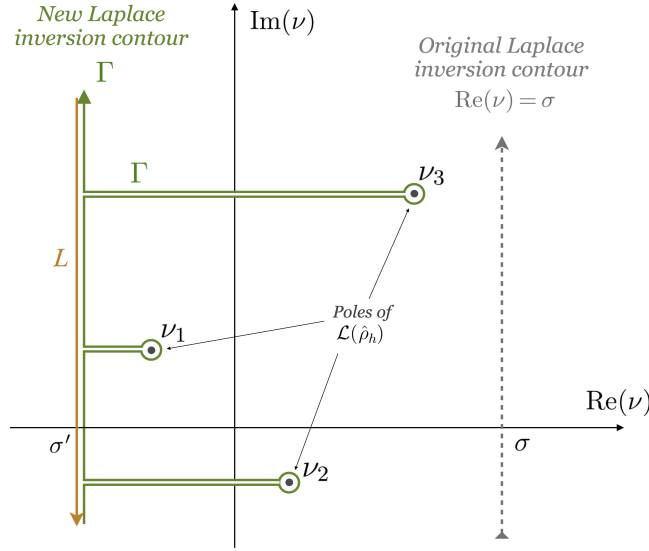
Figure 4. Original and new contour of intergation, still at the righ of all the poles. The path L and the vertical part of Γ both lie on $\mathrm{Re}(\nu) = \sigma'$.

The shifting can be done by virtue of Cauchy's theorem, assuming that $\mathcal{L}(\hat{\rho}_h)$ is analytic inside this whole region. We now have :

$$\hat{\rho}_h(t,k) = \frac{1}{2i\pi} \int_\Gamma \mathcal{L}(\hat{\rho}_h)(\nu,k)e^{\nu t}\mathrm{d}\nu \tag{90}$$

with Γ as in Figure 1. To evaluate this integral, we apply the residue theorem to the closed (infinite) contour $\Gamma \cup L$, where L is the straight line $\mathfrak{R}(\nu) = \sigma'$ (see Figure 1). With (90), this gives :

$$\hat{\rho}_h(t,k) + \frac{1}{2i\pi} \int_L \mathcal{L}(\hat{\rho}_h)(\nu,k)e^{\nu t}\mathrm{d}\nu = \sum_{i=1}^{n} \mathrm{res}(\mathcal{L}(\hat{\rho}_h), \nu_i)e^{\nu_i t} \tag{91}$$

where we have used ; $\mathrm{res}(\mathcal{L}(\hat{\rho}_h)e^{\nu t}, \nu_i) = \mathrm{res}(\mathcal{L}(\hat{\rho}_h), \nu_i)e^{\nu_i t}$. The integral on the path L has $\nu = \sigma' + iy$, with fixed $\sigma' < 0$ and y varying from $-\infty$ to $+\infty$. If σ' is negative, the exponential factor $e^{\sigma' t}$ will make the integral small enough so that we can neglect it in large time.

The sum on the right is a linear combination of complex exponentials $e^{\nu_i t}$ (remember that ν_i's are complex poles here). Among those ν_i's, let's call ν_p the one with the largest real part. Then in the asymptotic regime there are three possible scenarii :

– if $\mathrm{Re}(\nu_p) < 0$, all $e^{\nu_i t}$ terms are exponentially decreasing : we have an exponential decay of the density $\hat{\rho}_h$ and hence of the electric field itself in vertue of the Poisson equation. This decay of the field is what we call "Landau *damping*". Notice however that it does not tell us anything about stability a priori.

– if $\mathrm{Re}(\nu_p) > 0$, then there is at least one exponentially growing term, which means that $\hat{\rho}_h$ has growing modes and there is no damping possible, and thus no stability.

– if $\mathrm{Re}(\nu_p) = 0$, then we only know that $\hat{\rho}_h(t, k)$ is bounded, but that does not tell us anything about stability either.

Analysis of $\mathcal{L}(\hat{\rho}_h)$:

We will study the integrand $\mathcal{L}(\hat{\rho}_h)$ in equation (91) for two reasons : 1) see if it can be analytically continued in such a way that we can change the contours and perform Laplace inversion, and 2) to see where the poles lie in the ν-plane.

We want to extend $\mathcal{L}(\hat{\rho}_h)$ as an analytic function over the complex plane. To do that, we just need to extend the numerator and the denominator in (88) in a similar fashion. We only need to focus on the integrals, which define a function of ν. These are respectively :

$$\int_{\mathbb{R}} \frac{\hat{h}_{\text{in}} \, \mathrm{d}u}{\nu + 2i\pi k u} \quad \text{and} \quad \int_{\mathbb{R}} \frac{\mathrm{d}g^o}{\mathrm{d}u} \frac{\mathrm{d}u}{\nu + 2i\pi k u} \tag{92}$$

where $g^o(u) = \int f^o(u, v_2, v_3)\mathrm{d}v_2\mathrm{d}v_3$, i.e. we have performed the integration with respect to v_2, v_3, re-labelled v_1 by u.

First, notice that as soon as $\hat{h}_{\text{in}}$ and $(g^o)'$ are integrable, these integrals define analytic functions in the whole ν-plane, except on $\text{Re}(\nu) = 0$, where there is a simple pole at $u = i\nu/2\pi k$. The goal is to define these integrals for $\text{Re}(\nu) = 0$. We will do this only for the first integral, it is exactly the same for the second.

We follow Landau's prescription : we set $\text{Re}(\nu) = 0$, and assume that $\text{Im}(\nu) < 0$. Then we apply the residus theorem in the u-complex plane to the function $u \mapsto \hat{h}_{\text{in}}(u)(\nu + 2i\pi k u)^{-1}$ with the following contour :
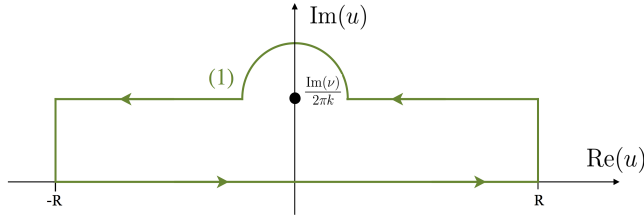
Figure 5. Laudau's prescription for analytic continuation, $R > 0$.

where the contour *(1)* is the upper horizontal line with semicircle around the pole. Since the vertical parts cancel each other out, the limit $R \to \infty$ gives :

$$\int_{\mathbb{R}} \frac{\hat{h}_{\text{in}}\, \mathrm{d}u}{\nu + 2i\pi k u} = 2i\pi \hat{h}_{\text{in}}(k, i\nu/2\pi k) - \int_{(1)_{\infty}} \frac{\hat{h}_{\text{in}}\, \mathrm{d}u}{\nu + 2i\pi k u} \qquad (93)$$

where *(1)$_\infty$* now denotes the infinite extension of *(1)*. In equation (93), the right hand side is analytic even when $\text{Re}(\nu) = 0$. Hence for all ν such that $\text{Re}(\nu) \neq 0$, expression (92) defines an anlytic function, and if $\text{Re}(\nu) = 0$ we use (93) to define the same function, also analytic on that path. The same argument applies to the case $\text{Im}(\nu) > 0$.

In the end, $\mathcal{L}(\hat{\rho}_h)$ is the ratio of two functions which can be analytically continued over the whole ν-plane. The Laplace inversion integration is thus justified[11]. Now let's turn to the poles of $\mathcal{L}(\hat{\rho}_h)$. Since we have showed that the numerator and denominator are analytic, the poles can only come from the zeros of the denominator, i.e. :

$$\int_{\mathbb{R}} \frac{\mathrm{d}g^o}{\mathrm{d}u} \frac{\mathrm{d}u}{u - i\nu/2\pi k} = (2\pi k)^2 \qquad (94)$$

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11. However this does not justifies the use of Laplace tranform in the first place : having an analytic continuation is not sufficient to be exponentially bounded. This will be done first rigorously by Backus years later, who shows the exponential bound for the linearised Vlasov equation.

where $g^o(u) = \int_{\mathbb{R}^2} f^o(u,v,w)\,\mathrm{d}v\mathrm{d}w$. We recover the "dispersion" relation (10) obtained by Vlasov, up to 2π factors due to the definition of Fourier transform, and the usual notation of the Laplace transform variable as a frequency $\omega = i\nu$.

In Vlasov's treatment, it was not clear how to deal with the divergent integral, but here we can see why the "dispersion" relation is still a relevant feature of the linear theory : the divergent integral of Vlasov is the average of the two integrals along path $(1)_\infty$ and along the real line.

The solutions to equation (94) will lead to poles, that will then induce growing or decaying modes, depending on the sign of their real part. Landau ends up with a prediction of the rate at which this damping occurs for a Gaussian equilibrium of electrons. This can be found in his paper [16].

Let us state the results obtained so far : if the Fourier/Laplace transform of f^o, h and ρ_h are well defined and if both $u \mapsto g^o(u)$ and $\hat{h}_{\mathrm{in}}$ are integrable or square integrable, then the following criterion holds :

– if equation (94) has at least one solution ν with $\mathrm{Re}(\nu) > 0$, then the density ρ_h grows exponentially in time until the linear theory breaks down : there is instability.

– if equation (94) has solutions with $\mathrm{Re}(\nu) \leqslant 0$ only, then the density decays exponentially in time, and we have Landau damping. We cannot conclude as to the stability though.

Landau's treatment was purely mathematical, and very little physical discussion about the physical phenomenon can be found in his paper. Nonetheless, analytic continuation of the integrals and the use of complex integration has now become standard in plasma physics. However, even if the Cauchy problem approach was a good start, there is no quantification of the smallness of the perturbation h, using norms, which will be crucial as we can expect.

3.2. Rigor and Interpretation

In this section, we will show how mathematicians have established rigorous results on the Landau damping, and point out the strong assumptions made by Vlasov and Landau in their treatment. We will first take a look at the experiments made after Landau's predictions.

Experiments :

The work of Vlasov, Landau and others formed a basis for the linear theory of plasma physics, and the damping of electrostatic waves allowed physicists to verify Landau's prediction about the decay rate of the field. Among many references, [10], [23], [30] and [38] demonstrated the validity of Landau's prediction for Gaussian equilibria :
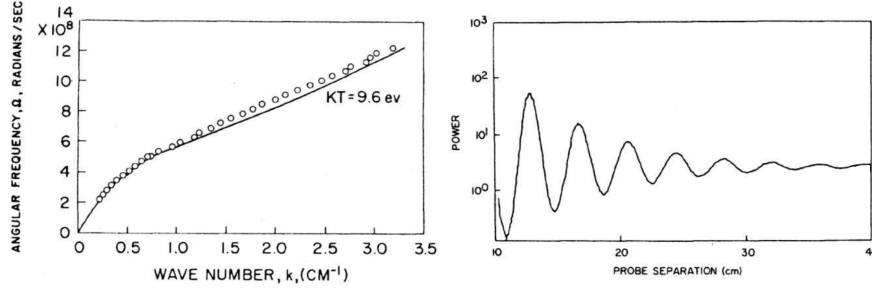

Figure 6. From Malmberg and Wharton, [23].

Thse two figures can be found in. The left one shows the measured (circles) dispersion relation for the wave compared with Landau's prediction in [16] (hard line). The second figure shows both the oscillations and the Landau damping of the electric field. This is done by exciting the plasma at one spot with a electric pulse and measuring its value after propagation, at different places.

Nowadays, we can also use computer simulations to check the Landau damping. The following picture is from [26] and shows the exponential damping (hence linear in log scale), with the oscillations of the electric field :
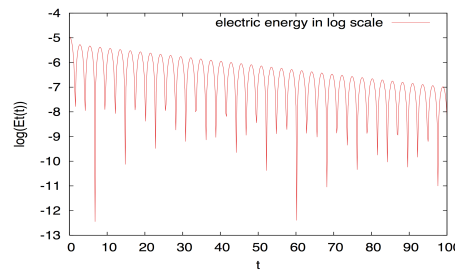
Figure 7. Time-evolution of the norm of the field, for electrostatic interactions.

But even with all this experimental evidence, there was still a debate as to the physical process in play with Landau damping. The physical origin remained quite a mystery for physicists, and even today, it is not settled. Indeed, the damping of the electric field is a relaxation to equilibrium, yet there is no dissipation mechanism (we excluded collisions) involved in the theory !

<u>Interpretation :</u>

Back in the sixties, the interpretation was based on the conservation of energy exchanged between the electrons and the electrostatic wave : the electrostatic wave moves through the plasma with velocity v_w, and electrons have a continuous distribution of velocity given by $v \mapsto f(t, x, v)$. Among them, three types can be distinguished :

a) Electrons with velocity $|v| \ll v_w$ or $|v| \gg v_w$: as the wave passes through these slow electrons, they will gain energy by ascending the front of the wave, and lose that same amount by moving down the tail of the wave. There is no net gain nor net loss because the interaction with the wave takes little time, as the relative speed $|v| - v_w$ at which it occurs if small.

Electrons with velocity $|v| \sim v_w$ (electrons that are nearly resonnant with the wave) will suffer net gain of loss of energy by interacting with the wave, since the exchange process takes a longer time :

b) those that have $|v| \lesssim v_w$ will gain energy from the wave by ascending the front of the wave : the wave loses energy

c) those that have $|v| \gtrsim v_w$ will lose energy to the wave as they ascend the tail of the wave : the wave gains energy

We conclude that if the number of slower electrons is greater than the number of faster electrons, then the wave loses energy and damps away.

This slow/fast electron ratio depends strongly on the slope of the equilibrium, and explains the $\mathrm{d}g^o/\mathrm{d}u$ term in the "dispersion" relation (94). This interpretation of Landau damping is often refered to as the "surfer" interpretation, for obvious reasons.

<u>Plasma wave echo</u>

Among other applications, there is one that will retain our attention here, for it will be used at the end of the chapter and play a crucial role in the nonlinear Landau damping : the plasma wave echo phenomenon. I was both predicted (see [14]) and discovered experimentally (see [24]) in 1967.

In the previous study, we saw that if there was Landau damping, then the the density ρ_h associated to the perturbation damped away. But what about the distribution $h(t, x, v)$ itself ? We know that if there is Landau damping, the perturbed density ρ_h goes to zero. Hence by equation (86),

$$\frac{\partial \hat{h}}{\partial t} + 2i\pi kv \cdot \hat{h} \xrightarrow[t\to\infty]{} 0 \tag{95}$$

In other words, we will recover the free transport equation for large time, as expected. Thus, if we wait long enough, the kth mode of the perturbation will be :

$$\hat{h}(t, k, v) = \hat{h}(0, k, v)e^{-2i\pi k \cdot vt} \tag{96}$$

where we wrote $\hat{h}(0, k, v)$ for $\hat{h}_{\text{in}}(k, v)$. This was for any electric perturbation, which we expanded in Fourier space as a superposition of modes. Now, with this in mind, suppose we do the following experiment :

– we impose at $t = 0$ an electric perturbation of wavenumber K (something like $\phi \propto e^{iKx}$ for the potential). The Fourier transform will turn this periodic wave into a Dirac distribution $\delta(K - k)$, which will force the electron distribution to oscillate at the same frequency.

– we wait for the damping of the wave. Following the above reasoning, for large t, the first order perturbed distribution will be like :

$$\hat{h}_1(t, v) = \hat{h}_1(0, v)e^{-2i\pi K \cdot vt}. \tag{97}$$

It is the velocity integral of (97) that makes the asosciated density to go to zero, this is exactly phase mixing.

– we impose at $t = \tau$ (new "initial" time) an second electric perturbation of wavenumber L. It induces a new perturbation h_2, and wait again for the wave to damp away. We can think of it as a second layer of perturbation added to the first one. From the viewpoint of this new perturbation h_2, after damping it has the form :

$$\hat{h}_2(t, v) = \hat{h}_2(\tau, v)e^{-2i\pi L \cdot v(t-\tau)}. \tag{98}$$

the velocity integral of (98) phase mixes to zero once again. But from the viewpoint of the first distribution, for $t \gg \tau$ it has the form :

$$h_1(t, v) = \left(\hat{h}_1 e^{-2i\pi K \cdot vt}\right)\left(\hat{h}_2 e^{-2i\pi L \cdot v(t-\tau)}\right) \propto e^{-2i\pi(Kt + L(t-\tau)) \cdot v} \tag{99}$$

of course if h is small, then (99) is negligeable : it is a second-order perturbation. Nevertheless it has its (macroscopic) importance since its velocity integral does not phase mix to zero when the argument of the exponential is zero, i.e. when :

$$t = t_e, \quad \text{with} \quad t_e = \frac{L}{K + L}\tau \tag{100}$$

The time t_e is called the echo time. At this very time, there is no phase mixing, and an electric field should re-appear slightly in the plasma before phase mixing away. Its properties depend on the first two pulses, and it appears at later times, whence the name "plasma echo".

The echoes here appear at successive times (they are *time-echoes*), but one can also create *spatial-echoes* if the plasma is excited at one place only, which is easier to verify experimentally. We refer to [14] for more details about this.

Plasma wave echoes have been verified experimentally (see figure 1 in [24]), and is now used as a verifaction that indeed collisions are negligeable in the plasma (remember that if collisions play a significant role, the damping rate should be different since collisions dissipate energy)

The plasma wave echo experiment shows two important things :

– there is an interaction between the layers of the perturbations : plasma wave echo is a nonlinear effect of the theory, in the sense that it involves products of small perturbations, but separated by long enough times. If we were to add more perturbations, there would be more echoes.

– even if Landau damping seemed to be contradicting the reversibility of the Vlasov equation, there is no such thing. The information of the distribution is hidden in the fast velocities (by phase mixing) but can be recovered, as shows the plasma wave echo experiment.

Penrose criterion :

The work of Penrose (see [31]) in the theory of linear Landau damping is driven by two goals. First, obtaining a criterion for for f^o on whether or not (94) has solutions with $\mathrm{Re}(\nu) > 0$. Second, verify that the plasma is indeed stable when there are no such solutions (there could be linearly growing perturbation in time).

First let's express equation (94) as

$$Z\left(\frac{i\nu}{2\pi k}\right) = (2\pi k)^2, \quad \text{with} \quad Z(s) = \int_{\mathbb{R}} \frac{\mathrm{d}g^o}{\mathrm{d}u} \frac{\mathrm{d}u}{u-s} \tag{101}$$

where k is real, ν is complex, and Z is a function of the complex variable s.

It is clear that a solution ν to (101) with $\mathrm{Re}(\nu) > 0$ (i.e. growing mode) exists if and only if $Z(s)$ takes a positive value for some given s such that $\mathrm{Im}(s) > 0$. Thus, all we need is to see whether *the function Z takes positive values in the upper half plane*. If it does there exist growing modes and no chance of Landau damping. If it does not, then we need to check that the plasma is stable, but we can be sure there are no exponentially growing modes.

Since Z is a complex valued function, we can write its real and imaginary part. Setting $s = \sigma + i\tau$, we have :

$$Z(\sigma + i\tau) = \int_{\mathbb{R}} \frac{\mathrm{d}g^o}{\mathrm{d}u} \frac{(u-\sigma)\,\mathrm{d}u}{(u-\sigma)^2 + \tau^2} + i\tau \int_{\mathbb{R}} \frac{\mathrm{d}g^o}{\mathrm{d}u} \frac{\mathrm{d}u}{(u-\sigma)^2 + \tau^2}. \tag{102}$$

As we can see, the only hope for a real value of Z is in the limit $\tau \to 0$, which will make the imaginary part of Z go to zero. But this needs to be done carefully. This limit is given by the Plemelj formula

$$\lim_{\tau \to 0} Z(\sigma + i\tau) = \mathrm{pv}\left(\int_{\mathbb{R}} \frac{\mathrm{d}g^o}{\mathrm{d}u} \frac{\mathrm{d}u}{u - \sigma} \right) - i\pi (g^o)'(\sigma) \tag{103}$$

where "pv" stands for the principal value[12] of the integral. As we can see, taking $\tau \to 0$ in (102) does not just remove the imaginary part. This way of dealing with singular integrals has been common in physics and is the right way to interpret a divergent expression like this.

Now, if we look for positive real values of the expression in (103), we will have two recquirements : the real part has to be positive, and the imaginary part must vanish, i.e. g^o must reach an extremum. This new criterion has a very nice geometric interpreation in the complex Z-plane, since the integral in (103) is linked to the winding number of a curve. It can be viewed in detail in Penrose's original paper [31].

In the end, we have a new criterion, now stated on g^o : there exists some growing modes if and only if there exists some σ_o such that

$$\frac{\mathrm{d}g^o}{\mathrm{d}x}(\sigma_o) = 0 \quad \Rightarrow \quad \mathrm{pv}\left(\int_{\mathbb{R}} \frac{\mathrm{d}g^o}{\mathrm{d}x} \frac{\mathrm{d}x}{x - \sigma_o} \right) > 0 \tag{104}$$

This is the Penrose criterion. Let's make a few comments on it.

– The regularity of the initial perturbation for this criterion to hold can be very weak as showed by Lin and Zeng in [20] (at this point the only assumption made by Penrose on h_{in} is that it is in L^1 or L^2, nothing on regularity).

– Regarding the equilibrium's regularity, assuming $(g^o)' \in L^1$ and $(g^o)'$ be Lipschitz-continuous[13] is enough.

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12. The principal value of a divergent integral is taken as the symetry limit :

$$\mathrm{pv}\left(\int_{\mathbb{R}} \frac{\mathrm{d}g^o}{\mathrm{d}x} \frac{\mathrm{d}x}{x - \sigma} \right) = \lim_{\epsilon \to 0^+} \left(\int_0^{\sigma - \epsilon} \frac{\mathrm{d}g^o}{\mathrm{d}x} \frac{\mathrm{d}x}{x - \sigma} + \int_{\sigma - \epsilon}^{\infty} \frac{\mathrm{d}g^o}{\mathrm{d}x} \frac{\mathrm{d}x}{x - \sigma} \right)$$

– Even if Penrose criterion for the existence of growing modes seems very sharp (and it is!), it does not give any information about the stability of the considered equilibria. This has to be done in a different manner, which we will study in the next part. Before that, le's apply this criterion to the case of a Gaussian equilibrium centered around $v_0 = (v_{0x}, v_{0y}, v_{0z})$:

$$f^o(v) = e^{-(v-v_0)^2} \quad \Rightarrow \quad g^o(u) = Ce^{-(u-v_{0x})^2} \tag{105}$$

for some constant C. The shape of $g^o(u)$ is also Gaussian centered around $v_{0x} \in \mathbb{R}$, say positive. The only maximum is at v_{0x} so we just need to check the sign of the following integral in (105) :

$$\int_{\mathbb{R}} \frac{\mathrm{d}g^o}{\mathrm{d}u} \frac{\mathrm{d}u}{u - v_{0x}} = -2C \int_{\mathbb{R}} e^{-(u-v_{0x})^2} \mathrm{d}u \tag{106}$$

which is always negative. By the Penrose criterion, no exponential growing modes can exist when the equilibrium is Gaussian.

Backus' analysis :

G. Backus, in an attempt to justify Landau's work and bring rigor to Landau damping, published a paper (see [2]) in the same year as Penrose. He proved that inverting the Laplace transform was justified, by showing an exponential bound for the perturbation h (sufficient condition to take the Laplace transform). Here is a sketch of how he did it.

We will keep a general wave vector and not set it along v_1 for the sake of generality. The inhomogeneous transport equation (86) can be written as :

$$\frac{\partial \hat{h}}{\partial t} + 2i\pi k \cdot v\hat{h} = S(t, k, v), \quad S(t, k, v) = -\frac{\hat{\rho}_h}{2i\pi k} \nabla_v f^o \tag{107}$$

with $\rho_h = \int h \mathrm{d}v$. Using Duhamel's principle[14], and integrating in velocity we can rewrite this differential expression as an integral equation :

$$\hat{\rho}_h(t, k) = \int_{\mathbb{R}^3} \hat{h}_{\mathrm{in}}(x - vt, v)\mathrm{d}v + \int_0^t \hat{\rho}_h(\tau, k) K^o(t - \tau, k)\mathrm{d}\tau, \tag{108}$$

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13. The Lipschitz continuity is a sufficient condition for the Plemelj formula to hold.

14. Duhamel's principle is linked to the method of characteristics (see ref) : if f is a solution to the inhomogeous transport equation $\partial_t f + v \cdot \nabla_x f = S$ with some source term $S = S(t, x, v)$, then f is given by the formula :

$$f(t, x, v) = f(0, x - vt, v) - \int_0^t S(\tau, x - v(t - \tau), v)\mathrm{d}\tau.$$

where the kernel K^o is defined by :

$$K^o(t,k) = -\frac{1}{2i\pi k}\int_{\mathbb{R}^3} \nabla_v f^o \, e^{-2i\pi k \cdot vt} \mathrm{d}v. \tag{109}$$

Then, Backus showed that if $\hat{\rho}_h$ satisfies equations (108)-(109), then there is a bound for $|\hat{\rho}_h|$ of the type :

$$|\hat{\rho}_h(t,k)| \leqslant \int_{\mathbb{R}^3} |\hat{h}_{\text{in}}(k,v)| \mathrm{d}v \times \cosh(\omega_p t). \tag{110}$$

We see that as soon as $\int_{\mathbb{R}^3} |\hat{h}_{\text{in}}(k,v)| \mathrm{d}v < \infty$, no exponential disturbance can grow at a faster rate than ω_p. Hence the exponential bound and the legitimacy of using the Laplace transform. We refer to Backus' very clear paper [6] for the proof of this result.

The second goal of Backus (which was also Penrose motivation) was to go further than a criterion for the existence of growing modes, and get a stability criterion. The idea is this :

– the Penrose criterion is only a necessary condition for the stability : if exponentially growing modes exists, there can be no hope for stability. Yet if Penrose criterion is satisfied, that does not prevent linear instability from happening. In other words, Penrose criterion is only sensible to exponentially growing instabilities.
– but the expression (91) shows that if $\hat{\rho}_h(t,k)$ is uniquely defined by its inverse Laplace transform, then we only need to worry about these exponentially growing modes.

Thus all boils down to show the uniqueness of the solution of the Vlasov equation via Laplace transform. To do this, it was crucial to quantify the size of the perturbation using norms, which until then was neglected by Vlasov and Landau. Penrose did this by studying equation (108) and writing it as

$$\hat{\rho}_h(t) = a(t) + \int_0^t K^o(t-\tau)\hat{\rho}_h(\tau)\mathrm{d}\tau \tag{111}$$

with K^o defined as in (109) and $a(t) = \int_{\mathbb{R}^3} \hat{h}_{\text{in}}(x - vt, v)\mathrm{d}v$. This type of equation is called a Volterra equation, and under some assumptions on a and K it has a unique solution. These assumptions involve the size of the perturbation and the equilibrium, i.e. a choice of norm.

Backus used a similar argument, but went further in the following sense : he showed that with alternative norms (finer measure of the perturbation size) their could be growing perturbations, even for the Gaussian equilibrium. This showed that Landau damping, and stability, crucially depends on the choice of functional space we work with, and the regularity and size assumptions of the initial data h_{in} and the equilibrium f^o.

Backus and Penrose work brought this rigor (choice of norm, counter examples and limits of linear theory) to the world of Landau damping, and at this point only this became a rigorous mathematical topic. However, even if, quantitatively, things became rigorous, there was still the problem of going beyond linear Landau damping.

Conclusion :

At this point of history, even if mathematicians and physicists had worked on their own, they established a few results for the Vlasov equation and Landau damping :

 – a rigorous theory for the linearisation around homogeneous equilibria,
 – a sharp criterion for linear stability, and thus for Landau damping,
 – a physical interpretation which gives some answer to the mechanism behind the asymptotic behavior in time.

Unfortunately, there are also many issues which started to worry physisicts and mathematiciand regarding :

 – the linearisation : it is based on the assumption that we can neglect the quadratic term :

$$F[h]\nabla_v h \ll F[h]\nabla_v f^0 \qquad (112)$$

since h is assumed to be small. But already in the sixties people showed that this was not true for very natural norms, and Backus was the first to made it explicit : for a typical plasma the linearisation breakdowns after a few microseconds. Moreover, we expect filamentation, i.e. very large velocity gradients of h, which seems hardly compatible with (112).

 – there seem to be an irreversible behavior of the system : the electric field goes to zero, and yet the Vlasov equation itself is time-reversible. In a similar picture, the phase mixing property of the transport equation did not seem compatible with the return to homogeneous equilibria. Furthermore, the "surfer" interpretation of the damping mechanism started to show its limit when analysed carefully.

In other words : people could not see[15] how to properly handle nonlinearity, yet it was known that the linear theory was limited, and the time-paradoxes associated to Landau damping did not find any satisfactory answer in the linearised theory, nor in the early works regarding the full nonlinear Vlasov equation.

3.3 NONLINEAR LANDAU DAMPING

This last section will be devoted to the nonlinear Landau damping theory. We will focus on the result of C.Mouhot and C.Villani and show the strategy used to overcome nonlinearity. At the very end, we will point out some more recents developpements, and state the

The answers to all above questions were brought by the paper [26] of C.Mouhot and C.Villani in 2011, in their treatment for nonlinear Landau damping. Among all their results, the real breakthrough was proving that Landau damping still holds for the full nonlinear Vlasov equation, using new mathematical tools.

To understand their motivation, let us summarise the key concepts around the Vlasov equation. We will consider the general form :

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + F[f] \cdot \nabla_v f = 0 \tag{113}$$

$$F[f](t, x) = -\int_{\mathcal{V}} \nabla W(x - y) f(t, y, w) \mathrm{d}y \mathrm{d}w \tag{114}$$

with W the potential energy associated to the interaction[16] between particles and $\mathcal{V} = \mathbb{T}^3 \times \mathbb{R}^3$ the phase space[17] of one particle.

The two key concepts we have seen so far for equation (113) are :
– as a tranport equation, it has an underlying phase mixing property, which creates time decay from velocity regularity,

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15. Actually, there were some attempts : the so-called quasilinear theory, or using the Van Kampen modes [39] to construct a nonlinear stability theory, but none of this seemed convincing.

16. Until now, we had set $W(x) = \pm 1 / |x|$ but we could have taken any potential for the linear theory. In the nonlinear picture, we will have more restrictions, but damping with the Coulomb/Nexton interaction will still hold. We will stick to the case $W(x) = \pm 1 / |x|$ for the discussion.

17. As we have seen, there needs to be a confining mechanism. Furthermore, we can construct exact undamped solutions to the Vlasov equation in $\mathbb{R}^3$, as shown by Glassey and Shaeffer in [12].

– after linearisation, there is Landau damping : exponential decay of the perturbation density ρ_h, and hence of the force $F[h]$.

Now, if we believe in Landau damping for the nonlinear equation, then the force term should go to zero and in the asymptotic regime, we should recover a free transport equation : in simple minded terms :

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + F \cdot \nabla_v f = 0 \quad \xrightarrow[F \to 0]{\text{L. damping}} \quad \frac{\partial f}{\partial t} + v \cdot \nabla_x f = 0, \qquad (115)$$

Furthermore, as we have seen before, the phase mixing property converts regularity of f^o and f_{in} into time decay of the force in the first place. We are led to the conclusion that phase mixing should play an important role in Landau damping. Also, the exponential nature of the decay (both in free transport and Landau's treatment) came an analyticity assumption, which should also play a role, at least in the first approach.

<u>Gliding analytic norms :</u>

To overcome non-linearity, the goal is to compare the solution of (113) to the solution of the free transport equation, and see that the "distance" between the two solutions becomes smaller and smaller as time goes on. This "distance to the free transport" will be measured by a specific norm with as few as 4^{18} indices : the gliding, hybrid, analytic norm $\mathcal{Z}_\tau^{\lambda, \mu; p}$, defined as follows :

For any $\lambda, \mu \geqslant 0, \tau \in \mathbb{R}$ and $p \in [1, \infty]$:

$$\|f\|_{\mathcal{Z}_\tau^{\lambda, \mu; p}} = \sum_{k \in \mathbb{Z}^3} \sum_{n \in \mathbb{N}^3} e^{2\pi\mu|k|} \frac{\lambda^n}{n!} \left\|(\nabla_v + 2i\pi\tau k)^n \hat{f}(k, v)\right\|_{L^p(\mathrm{d}v)} \qquad (116)$$

This norm is only complicated in appearance : it is indeed very natural when trying to solve the nonlinear Landau damping problem, as the following explanation will demonstrate.

– The μ and λ indices are here to quantify analyticity. In the position variable, $x \in \mathbb{T}^3$, we use the Fourier version, and in the velocity variable, $v \in \mathbb{R}^3$, we use the standard version of analytic norms (equations (49) and (50) respectiviely). One then mixes (or "hybridizes") the two by applying it on $\hat{f}(k, v)$, to obtain :

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18. Strictly speaking, the general norm used in [26] contains 5 indices : $\mathcal{Z}_\tau^{\lambda, (\mu, \gamma); p}$, where the added γ index counts the derivative of f in the position variable. It is refered to as "Sobolev correction" in [26]. However, it is not important for the general discussion here, and we shall leave it aside.

$$\|f\|_{\mu,\lambda} = \sum_{n\in\mathbb{N}^3}\sum_{k\in\mathbb{Z}^3} e^{2\pi\mu|k|} \frac{\lambda^n}{n!} \sup_{v\in\mathbb{R}^3} \left|\nabla_v^n \hat{f}(k,v)\right| \tag{117}$$

– The p index is mainly of technical importance : instead of measuring the size of $\nabla_v^n \hat{f}(k,v)$ using the "sup" norm (which remember is the L^p norm with $p = \infty$), one generalises to any L^p norm to take into account integrability :

$$\|f\|_{\mu,\lambda,p} = \sum_{n\in\mathbb{N}^3}\sum_{k\in\mathbb{Z}^3} e^{2\pi\mu|k|} \frac{\lambda^n}{n!} \left\|\nabla_v^n \hat{f}(k,v)\right\|_{L^p(\mathrm{d}v)} \tag{118}$$

where $L^p(\mathrm{d}v)$ stresses that the L^p norm is taken for $v \mapsto \nabla_v^n \hat{f}(k,v)$.

– The last index is the time-shift parameter τ, and is of great importance. Since the phase mixing introduces fast oscillations in the velocity variable (and hence $v \mapsto f(x,v)$ becomes less regular, we called it filamentation in the second chapter), we take it directly into the norm and work with it. It is defined by :

$$\|f(x,v)\|_{\mathcal{Z}_\tau} = \|f(x-v\tau,v)\|_{\mathcal{Z}} \tag{119}$$

where the reference to free transport is obvious. The difference between the $\mathcal{Z}$ and the $\mathcal{Z}_\tau$ norm is that the second follows the size of f in time and keeps track of phase mixing : this is the "gliding" property of $\mathcal{Z}_\tau^{\lambda,\mu;p}$. Pluging expression (119) in (118), we obtain the final shape of the norm (116).

Before describing what comes next, let us mention that even though this (complicated) norm is used in the mathematical proof, in the final form of the theorem, one does not need to assume that $f \in \mathcal{Z}_\tau^{\lambda,\mu;p}$ for the result to hold. Indeed, the $\mathcal{Z}$ norms can be related to more standard norms entirely based on Fourier space (see chapter 4 of [26]).

Strategy :

Now that one has the main tool for the study, what is the strategy for proving nonlinear Landau damping ? First, let us recall what the problem is :

Suppose a plasma is prepared in an homogeneous equilibrium state $f^o(v)$. Starting from $t = 0$, we perturb it such that the distrubution $f(t,x,v)$ is now governed by the full Vlasov equation :

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + F[f] \cdot \nabla_v f = 0 \tag{120}$$

with the initial perturbation $f(0, x, v) = f_{\text{in}}(x, v)$ small compared to the equuilibrium. The question to be answered is then : in those conditions, does the plasma relaxes to a (possibly) new equilibrium ? In other words, does the perturbation $h = f^o - f_{\text{in}}$ go to zero as $t \to \infty$?

First, one uses an iterative scheme : the Newton method. At each step n of the scheme, the approximate solution, f^n, will solve a linear equation much easier to deal with.

<u>Classic Newton scheme :</u>

Suppose we want to solve the equation $G(x) = 0$ for som function G: $\mathbb{R} \to \mathbb{R}$ on a given interval $I \subseteq \mathbb{R}$. We denote by x_s the unique solution to this equation in I. If we have a first approximate solution $x_s \simeq x_0$, then we can linearise around x_0 (i.e approximate G by it tangent at x_0) and get a new approximation x_1 :

$$G(x) \simeq G'(x_0)(x - x_0) + G(x_0) = 0 \quad \Rightarrow \quad x_1 = x_0 - \frac{G(x_0)}{G'(x_0)} \tag{121}$$

If the new solution is still in I, we can do this again, now linearising the equation around x_1, and get a new approximation $x_s \simeq x_2$. Do this n times to obtain a sequence of approximations $(x_n)_{n \in \mathbb{N}}$ defined by :

$$x_{n+1} = x_n - \frac{G(x_n)}{G'(x_n)}. \tag{122}$$

We thus have a sequence $(x_n)_{n \in \mathbb{N}}$ that converges very quickly to x_s, as shows the following proposition

If x_0 is close enough to the "true" solution (i.e. like $|x_s - x_0| < \varepsilon$), if G is twice continuously differentiable on I, ($F \in C^2(I)$) and if $G(x) \neq 0$ for all $x \in I$, then for each $n \in \mathbb{N}$ we have :

$$\sup_{x \in I} |x_n - x_s| \leqslant C^n \delta^{2^n} \tag{123}$$

for some constant $C \geqslant 0$ and $0 \leqslant \delta < 1$.

This idea can be adapted to the Vlasov equation : the unknown is now the family of functions $(f(t, x, v))_{t \geqslant 0}$, for each $(x, v) \in \mathbb{T}^3 \times \mathbb{R}^3$, and the equation to solve is the Vlasov equation $\partial_t f + v \cdot \nabla_x f + F[f] \cdot \nabla_v f = 0$, with $F = -\nabla W * \int f \mathrm{d}v$.
Without going into further details, let us state the equivalent of equation (123) in this context is : if we write $f^n = f^o + h^1 + \cdots + h^n$ as the approximation of f at the nth step, with $f^o(v)$ the equilibrium, then :

$$\frac{\partial h^1}{\partial t} + v \cdot \nabla_x h^1 + F[h^1] \cdot \nabla_v f^o = 0 \tag{124}$$

where $h^1 = f_{\text{in}} - f^0$, and for each $n \geqslant 1$, f^n solves

$$(\partial_t + v \cdot \nabla_x)h^{n+1} + F[f^n] \cdot \nabla_v h^{n+1} + F[h^{n+1}] \cdot \nabla_v f^n + F[h^n] \cdot \nabla_v h^n = 0 \tag{125}$$

with the initial condition $h^{n+1}(0, x, v) = 0$. Note that (124) is exactly the linearised Vlasov equation we have dealt with before. In the end, we have reduced the Vlasov equation to an infinite amount of linear equation in f^n. The goal now is to use the structure of (125) to show that the perturbations h^n remain small, in the same way that the error $|x - x_s|$ was in (123). Leaving the technical proofs aside, which is well explained in [24] and [40], we only state the result for the smallness of h^n :

If the interaction verifies $\nabla W \in L^1(\mathbb{T}^3)$, if $\|f^o\|_{\mathcal{Z}^{\lambda_0;1}} \leqslant C_0$ for some $\lambda_0 > 0$ and if the perturbation is small in the sense $\|f_{\text{in}} - f^o\|_{\mathcal{Z}^{\lambda,\mu}} \leqslant \varepsilon$, then for all $n \in \mathbb{N}$, there is a time T such that :

$$\|h^n(t)\|_{\mathcal{Z}^{\lambda,\mu;1}} \leqslant C\varepsilon^{a^n}, \quad t \in [0, T]. \tag{126}$$

for some $a \in]1, 2[$. This estimate holds only for finite, i.e. with $T < \infty$.

If we further assume that $\hat{W}(k) = O(|k|^{-2})$ and that the equilibrium satisfies the Penrose stability criterion[19], then we can have a long-time estimate :

$$\text{for all } t \geqslant 0, \quad \|h^n(t)\|_{\mathcal{Z}_t^{\lambda,\mu;1}} \leqslant C\varepsilon^{a^n}. \tag{127}$$

Once all these bounds are established (and this is the most tricky part), then we can sum them up and state a result on the quantity $\|f^o - f_{\text{in}}\|_{\mathcal{Z}}$ itself (i.e. the size of the perturbation), and finally on the force term : they all remain small and, the force damps exponentially if all data is analytic.

Even if the mathematics behind all this are complex, and already very-well described in [24] and [40], there is one capital step of the proof which is worth mentioning, and relates to the plasma wave echo experiment.

Plasma wave echo :

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19. The Penrose criterion in [26] is slightly more general than what we have shown, because it takes into account any potential W and generalises to higher dimensions.

The focus of the proof is now on equation (125), which can be viewed as the nonlinear equivalent of the linear equation (124) we had in linear Landau damping theory. With the free transport term $(\partial_t + v \cdot \nabla_x)h^{n+1}$, equation (125) combines 3 types of terms :

– a forcefield term : $F[h^n] \cdot \nabla_v h^{n+1}$ which we have already encountered in an iterative scheme ; the force field at step n curves the trajectories of particles at step $n+1$.

– a quadratic term : $F[h^n] \cdot \nabla_v h^n$, which is new in the nonlinear context,

– a reaction term : $F[h^{n+1}] \cdot \nabla_v h^n$ which is expected to grows linearly in time (in linear theory it was $F[h] \cdot \nabla_v f^o$). For this term, the plasma wave echo will play a great role.

The first two terms will be shown to remain small at each step of the scheme, however the reaction term will demand to be treated carefully since it is expected to dominate. Formally speaking, equation (125) behaves like

$$\partial_t h^{n+1} + v \cdot \nabla_x h^{n+1} + F[h^{n+1}] \cdot \nabla_v f^n = 0 \tag{128}$$

since the other two terms are small. Equation (128) is called by the authors of [24] a reaction equation. This equation is similar to the linearised Vlasov equation, except that f^n is not homogeneous anymore : $f^n = f^n(t, x, v)$.

Earlier in this chapter, we encountered the plasma wave echo phenomenon. We saw that it was the result of the interaction between the layers of a perturbation : it appears here in the interaction between layers n and $n+1$. In the linearised theory, the Vlasov equation decoupled into an infinite amount of independant equations. Based on (111), we have :

$$\left| \hat{\rho}_h(t, k) - \int_0^t K^o(t - \tau, k)\hat{\rho}_h(\tau, k)\mathrm{d}\tau \right| \leqslant |a(k, t)| \tag{129}$$

where $a(t)$ and K^o are defined as in (109). In this equation there is no "interaction" between the modes k and $k + 1$ or $k - 1$, hence no appearence of plasma wave echoes. However, in the nonlinear setting, (129) becomes a bit more complicated. If we write ρ for the density at step n induced by the force at step $n+1$ (remember the reaction term $F[h^{n+1}] \cdot \nabla_v f^n$) there are interactions between the different modes of ρ_h and the kernel quantifies this. Instead of (129) one has something like :

$$\left\| \hat{\rho}(t) - \int_0^t K^o(t - \tau)\hat{\rho}_h(\tau)\mathrm{d}\tau \right\|_{\mathcal{A}} \leqslant A(t) + \int_0^t K(t, \tau)\|\hat{\rho}(\tau)\|_{\mathcal{B}}\,\mathrm{d}\tau \tag{130}$$

where $A(t)$ results is the contribution of the initial datum, as $a(k,t)$ in (129), and $\mathcal{A}, \mathcal{B}$ are some gliding analytic norms, whose exact form do not matter for the discussion. The kernel $K(t,\tau)$ on the righ hand side of (130) is new : it takes into account the interactions between all the modes. Some modes will interact with each other at specific times by echoes (the time echoes t_e as in (100)). For instance, in a simplified picture :

$$K(t,\tau) = \sup_{k,l}\left(\frac{|k|(t-\tau)e^{-2\pi(\bar{\lambda}-\lambda)|k(t-\tau)+l\tau|}e^{-2\pi(\bar{\mu}-\mu)|k|}}{1+|k-l|^{\gamma}} \right) \tag{131}$$

which in view of (100) should be clear that some echoes are occuring, at least from the formalism. The shape of the kernel will become more and more peaked at specific times : as time goes on, the kernel will become less bulky and concentrate like Dirac distributions around echo-times, as shows the following simulation :
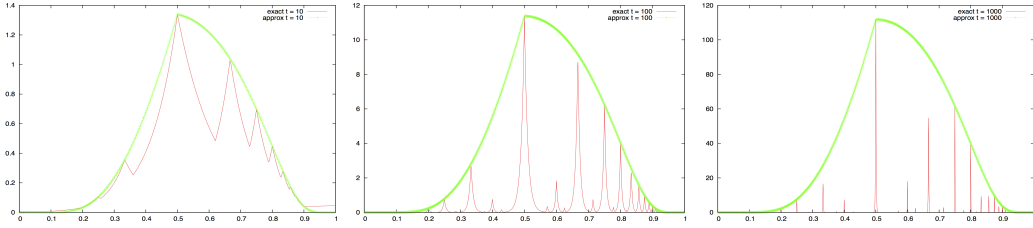
Figure 8. Concentration of the original kernel (green) into Dirac distributions (red) due to mode interaction via echoes in the plasma, at 3 successive times.

Even though there is an infinite amount of such echoes, a precise analysis of the kernel shows that the overall response of the plasma for the reaction terms is still controlable. If the interaction potential W is analytic, then the analysis of (131) alone allows to conclude. But if $W = \pm 1/|x|$, which is the most interesting case, then it has to be refined.

Under some assumptions, (130) can be made more precise : the analysis of echoes gives for each mode $\rho(t,k)$ a coupled inequilaty :

$$\bar{\rho}(t,k) \leqslant a(t,k) + \frac{\bar{\rho}(t',k')}{(k+1)^2}Ct, \quad \text{with} \begin{cases} t' = \dfrac{kt}{k+1} \\ k' = k+1 \end{cases} \tag{132}$$

where $a(t,\,k)$ is the contribution from the initial datum, C is a constant and $\bar{\rho}(t,\,k) = e^{2\pi(\lambda t+\mu)|k|}|\hat{\rho}_h(t,\,k)|$, because we are in gliding regularity. This equation on the modes themselves allows to have the same conclusions for analytic W, and for $W(x) = \pm 1/|x|$.

It is quite remarkable that the physical phenomenon of plasma wave echo plays such a great role in the mathematical proof of nonlinear Landau damping. This retroaction of the plasma on itself, quantified by the kernel K above can be viewed as a resonnance phenomena, which does not destruct the phase mixing property because echoes are well-separated in time. A precise study of this phenomenon is crucial for the mathematical proof.

Discussion :

The analysis was made in the phase space $\mathcal{V} = \mathbb{T}^3 \times \mathbb{R}^3$. There are various reasons for this : we have seen that Landau damping is closely related to phase mixing, which only occur if the system is bounded in position (i.e. we take $\Omega = \mathbb{T}^3$). Alternatively, one could make an infrared cutoff hypothesis, by limiting the values of k to some bounded interval. Anyway there should be some confinement mechanism for the damping to occur.

If we take $\Omega = \mathbb{R}^3$, then the large time behavior of the system is dominated by dissipation at infinity. In fact, in a very natural norm setting, we can construct undamped perturbations even for Gaussian equilibrium and Coulomb interaction ! Also, since for all t we have :

$$f(t, x, v)\mathrm{d}x\mathrm{d}v = f(t, x+L, v)\mathrm{d}x\mathrm{d}v \tag{133}$$

this means that the probability of finding one particle between x and $x + \mathrm{d}x$ is identical to that of finding one particle between $x + L$ and $x + L + \mathrm{d}x$. As soon as one particle leaves the box on one side, then an other particle enters it from the opposite side. This should be true on average for dense plasmas, for which many particles are moving in all directions. Nonetheless it would be interesting to investigate this assumptions from a physics point of view.

Finally, even in the seminal paper of Landau [16], the damping rate (i.e. the γ coefficient in $e^{-\gamma t}$) of a Gaussian, in Coulomb interaction is something like $\gamma \propto e^{-1/k^2}$: it is clear that arbitrary small values of k can become a problem, as the damping rate becomes arbitrary large.

The work of Mouhot and Villani also provides a new interpretation for the Landau damping : this is a relaxation by *mixing*, *confinement* and *smoothness*. The transport nature of the Vlasov equation equation converts regularity in the v variable into smoothness of the force in the x variable. The gradient structure of this force then implies time decay.

To conclude, we refer the reader to pages 17 tot 21 of [26] for the Theorem of Mouhot and Villani, which was the main inspiration and motivation to the author for this report. Hopefully, with all the discussion above, the reader will be able to appreciate and understand the result.

<u>Further results :</u>

To end this chapter, let us discuss a few things that could not be dealt with in this report for timing reasons, but are nonetheless worth mentionning.

In their paper on nonlinear Landau damping [26], Mouhot and Villani showed that the analytic regularity is not a necessary condition for Landau damping : their proof also work (up to some details) for a less regular class of equilibria : the Gevrey class $\mathcal{G}^\nu$, with $\nu \geqslant 1$ defined by

$$f \in \mathcal{G}^\nu(\mathbb{R}) \iff \hat{f}(k) = O\big(\exp\big(-c|k|^{1/\nu}\big)\big) \tag{134}$$

for some constant $c > 0$. The $\mathcal{G}^1$ class corresponds to analytic functions (simple exponential decay). The $\mathcal{G}^\nu$ class lies between the C^ω and C^∞ classes.

A few years before Mouhot and Villani's work, Lin and Zeng showed that to have Landau damping, one has to assume some regularity on the equilibrium distribution. Indeed, in [20] they show that there exist BGK waves of arbitrary period and arbitrary traveling speed (i.e. solutions that are not damped) around any equilibrium that is in $W^{m,p}$ with any $p > 1$ and $m < 1 + 1/p$ (reminder : $f \in W^{m,p}$: derivatives of f up to order m are in L^p).

They also showed that the Penrose criterion was very sharp, and that only slight improvements could be made regarding stability criteria, showing that the linear theory is well-understood.

The proof of nonlinear Landau damping of Mouhot and Villani was used by Masmoudi and Bedrossian in [3] to show similar result for the 2-dimensional Euler equation. Given an initial perturbation of the Couette flow (small in some regularity space), the velocity converges strongly to a shear flow which is also close to the Couette flow. This convergence of the velocity field is called *inviscid damping*. The linear result of inviscid damping for 2D-Euler equation goes back to Kelvin in 1887. This result allowed Mouhot, Masmoudi and Bedrossian to work together and simplify the proof of Landau damping for Gevrey regularity, details can be found in [4].

At last, we mention the strong links between Landau damping and KAM theory. This subject is described in the original paper [26]. Due to lack of time, we have not been able to indtroduce it, but it is definitely an important aspect of the result : it establishes a new bridge between two different mathematical physics theories,. Perhaps will it be the basis of new and improved results regarding stability of physical systems.

Conclusions

The conlusion of this report is split into two parts. First a brief summary of what this report has been about, serving as a conclusion for each chapter. Then, one word emphasising on the mathematical physics nature of the project, which was also a motivation for doing this project.

The main objective of this project was to present some aspects of the Vlasov equation and Landau damping from a mathematical physics point of view. The general lines can be summarised as follows :

– We first studied the derivation of the Vlasov equation. We saw that it was the mean field limit of the classical N-body problem, studied in a statistical picture. We also saw that the Vlasov equation was just one aspect of kinetic theory, but already led to major open problems in mathematical physics : understanding the mean field limits and the molecular chaos remain very chalenging problems in kinetic theory.

– Then, we presented the mathematics of the Vlasov equations : PDE analysis, norms and functional spaces, and transport equations. We have seen that proving the well-posedness of the Vlasov system relied on its transport structure and on the control of large velocities. We also introduced the very important notion of phase mixing and filamentation, and its link to the regularity of the particle distribution via Fourier transofrm.

– Ultimately, we presented Landau damping in a chronological way. Vlasov's first analysis showed the existence of periodic electrostatic waves in a plasma after perturbation. The lack of rigor in its treatment made him miss the Landau damping, which was discovered several years later by Landau. Years later, Backus and Penrose brought rigor into the linear theory, with a sharp criterion for plasma stability related to Landau damping. More recently Mouhot and Villani generalises Landau damping to the full nonlinear Vlasov equation by exploiting the phase mixing structure of the Vlasov equation in a clever fashion.

As a conclusion the theory of Landau damping is the result of many years of trial and error, both from physics and mathematics communities.

– In its early developpements, it was hard to believe that relaxation without dissipation could be a thing, putting Landau's prediction aside from plasma physics theory. Experiments then came to value this prediction, and Landau damping was becoming a physical phenomenon in its entirety. Today it is a fundamental in collisionless kinetic theory: complex analysis and Plemelj formulas have become a standard in plasma physics.

– On the opposite side, the physical interpretation of Landau damping using the surfer picture did not make any sense from the mathematical point of view. Only a fine analysis of the Vlasov equation reveals the phase mixing property, which is both the modern physical interpretation for Landau damping, and the natural approach to overcome nonlinearity in the Vlasov equation.

All in all, Landau damping is a proof that mathematics and physics are deeply related, and that trips back and forth into the abstract world of mathematics and the sensible physical world allows to make great discoveries, or to bridge a priori unrelated areas of science.

Acknowledgement

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