Advances in the modeling of kinetic sheath in plasma

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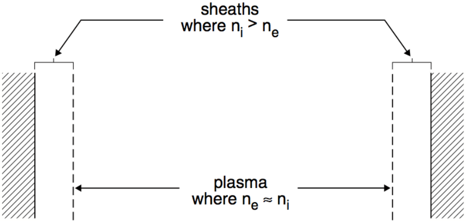
Introduction

Sheaths

• Sheath = boundary layer for a quasi-neutral plasma (i^+, e^-) near metallic wall.
Parameters $\varepsilon \approx \lambda_D << 1$ and $\mu = \frac{m_e}{m_i} << 1$.



From Stangeby : The Plasma Boundary of Magnetic Fusion Devices (2011).

• Applications : Tokamaks (ITER), ionic engine (satellites), RF sheaths, ...
• Condition on the Mach number : $M > 1$.
Kinetic Bohm criterion : $\int_0^\infty \frac{f_i^{bound}}{v^2} dv < 1$.

Some references

• Physics :
Laframboise, Theory of spherical and cylindrical Langmuir probes in a collisionless Maxwellian plasma at rest, 1966
Chen (Introduction to Plasma Physics 74') ;
Review by Riemann (The Bohm criterion and sheath formation 91'),
Manfredi-Devaux, Magnetized plasma-wall transition. Consequences for wall sputtering and erosion, 2008.
Baalrud-Hegna (Kinetic theory of the presheath and the Bohm criterion, 2011.
Stangeby The Plasma Boundary of Magnetic Fusion Devices, 2011.

• Math :
Raviart- Greengard, (A Boundary-Value problem for the stationary Vlasov-Poisson equations : The Plane Diode, 1990)
P-H. Maire, Fluid models with a plasma and b.c., PHD thesis 96' (with Sentis and Golse)
H. Guillard, $M \geq 1$ in fluid models at the SOL (2012).
D. Han-Kwan, Euler-Poisson and related works.

Bi-kinetic model

Equations

Problem : find boundary data f_i^{bound} such that stationary non trivial (i.e. non homogeneous) physically sound sheath solutions exist :

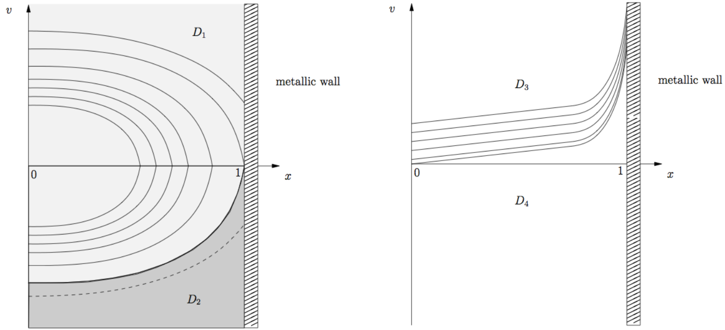
Domain is $(x, v) \in [0, 1] \times \mathbb{R}$, "inside" plasma is $x = 0$, wall is $x = 1$.

$$\begin{cases} v\partial_x f_i - \phi'(x)\partial_v f_i = 0, & \text{Vlasov } i^+ \\ v\partial_x f_e + \frac{1}{\mu}\phi'(x)\partial_v f_e = 0, & \text{Vlasov } e^- \\ -\varepsilon^2\phi''(x) - \int f_i dv + \int f_e dv = 0, & \text{Poisson,} \qquad (\varepsilon \approx \lambda_D) \\ \int v f_i dv - \int v f_e dv = 0, & \text{Ambipolarity,} \quad (J = -\partial_t E = 0) \end{cases}$$

$$\begin{cases} f_i(0, v) = f_i^{\text{bound}}(v), & v > 0, \\ f_i(1, v) = 0, & v < 0, \\ f_e(0, v) = n_0\sqrt{\mu}\ e^{-\frac{\mu v^2}{2}}, & v > 0, \quad \text{Scaled Maxwellian at entrance} \\ f_e(1, v) = \alpha f_e(1, -v), & v < 0, \quad \text{Material dependent reemission of } e^- \\ & \quad \text{with parameter } 0 \le \alpha = \alpha^{bound} < 1, \\ \phi(0) = 0, & \quad \text{Just a convention,} \\ \phi(1) = \phi_w, & \quad \text{Floating potential, unknown at this stage.} \end{cases}$$

Secondary unknowns n_0 and ϕ_w which is the floating potential.

Trajectories for $E = -\phi' > 0$



e^- $\quad$ i^+

invariant : $\frac{1}{2}v^2 - \frac{1}{\mu}\phi(x) = cst$
character. : $\frac{1}{2}v^2 - \frac{1}{\mu}\phi(x) = -\frac{1}{\mu}\phi_w$

invariant : $\frac{1}{2}v^2 + \phi(x) = cst$
character. : $\frac{1}{2}v^2 + \phi(x) = 0$

Notice that electrons follow a truncated Maxwellian.
The model is collisionless, except for e^- the entrance.

Notice the identity $\sqrt{\mu}\int_{\mathbb{R}} e^{-\mu\frac{(v^2-2\phi/\mu)}{2}}dv = \sqrt{2\pi}e^{\phi}$.

One gets after more computations

$$\begin{cases} n_e(x) & = \int_{\mathbb{R}} f_e dv & = n_0\left(\sqrt{2\pi}e^{\phi(x)} - (1-\alpha)\int_{\sqrt{-2\phi_w}}^{\infty} e^{-\frac{v^2}{2}} \frac{v}{v^2+2\phi(x)}dv\right), \\ \gamma_e & = \int_{\mathbb{R}} vf_e dv & = (1-\alpha)n_0\sqrt{\frac{m_i}{m_e}}e^{\phi_w}, \\ n_i(x) & = \int_{\mathbb{R}} f_i dv & = \int_0^{\infty} f_i^{\mathrm{bound}}(v)\frac{v}{\sqrt{v^2-2\phi(x)}}dv, \\ \gamma_i & = \int_{\mathbb{R}} vf_i dv & = \int_0^{\infty} f_i^{\mathrm{bound}}(v)vdv. \end{cases}$$

One can eliminate the two secondary unknowns with the equations

$$\begin{cases} \gamma_i = \gamma_e & \text{Ambipolarity,} \\ n_i(0) = n_e(0) & \text{Local neutrality at entrance, reasonable but not nec.} \end{cases}$$

A first result : There exists a unique (ϕ_w, n_0) if and only

$$\frac{\int_0^{\infty} f_i^{\mathrm{bound}}(v)vdv}{\int_0^{\infty} f_i^{\mathrm{bound}}(v)dv} \le \sqrt{\frac{2}{\pi\mu}} \times \frac{1-\alpha}{1+\alpha}.$$

Non linear Poisson equation

Now that ϕ_w and n_0 are uniquely determined, one can solves

$$\begin{cases} -\varepsilon^2\phi''(x) = n_i(\phi) - n_e(\phi) = -q'(\phi), \\ \phi(0) = 0 \text{ and } \phi(1) = \phi_w, \end{cases}$$

where the potential q is

$$q(\phi) = \int_0^\infty f_i^{\mathrm{bound}}(v) v \sqrt{v^2 - 2\phi} dv$$

$$+ n_0 \left(\sqrt{2\pi} e^\phi - (1-\alpha) \int_{\sqrt{-2\phi_w}}^\infty e^{-\frac{v^2}{2}} v \sqrt{v^2 + 2\phi} dv \right) \geq 0.$$

Minimization formulation

Define $\phi \in \mathcal{V} = \left\{ \phi \in H^1(0, L),\ 0 = \phi(0) \geq \phi \geq \phi_w = \phi(1) \right\}$ and set

$$J_\varepsilon(\phi) = \int_0^1 \varepsilon^2 \frac{|\phi'(x)|^2}{2} + q(\phi(x)) dx, \quad \phi \in \mathcal{V}$$

Proposition (evident) : There exists a minimum of the problem

$$J_\varepsilon(\phi_\varepsilon) \leq J_\varepsilon(\phi), \quad \phi_\varepsilon \in \mathcal{V},\ \forall \phi \in \mathcal{V}.$$

Existence

Theorem (B-D-CP) : Assume $0 \leq \alpha \leq \alpha_c$, $\rho_0 = n_i(0) - n_0(0) = 0$, $f_i^{\mathrm{bound}} \in \mathcal{I}_{ad}(\rho_0, \alpha)$ and $\varepsilon > 0$. Assume the Bohm criterion

$$\frac{\int_0^\infty f_i^{\mathrm{bound}}(v) \frac{dv}{v^2}}{\int_0^\infty f_i^{\mathrm{bound}}(v) dv} < 1.$$

Then the Vlasov-Poisson-Ampere formulation of the sheath is well-posed.

These solutions are also exact stationary solutions of bi-kinetic+Maxwell equations.

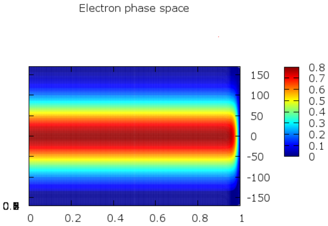
Two difficulties

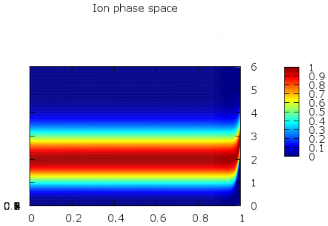
- First one is to prove that the minimizer in of class C^2
- Related difficulty is to understand the compatibility of the two inequalities

$$\begin{cases} \dfrac{\int_0^\infty f_i^{\mathrm{bound}}(v)v dv}{\int_0^\infty f_i^{\mathrm{bound}}(v) dv} \leq \sqrt{\dfrac{2}{\pi\mu}} \times \dfrac{1-\alpha}{1+\alpha}, \\ \\ \dfrac{\int_0^\infty f_i^{\mathrm{bound}}(v)\frac{dv}{v^2}}{\int_0^\infty f_i^{\mathrm{bound}}(v) dv} < 1. \end{cases}$$

Fortunately $\alpha < \alpha_c \approx 0.93$ in a DT fusion plasma yields compatibility.

Electrons and ions in phase space

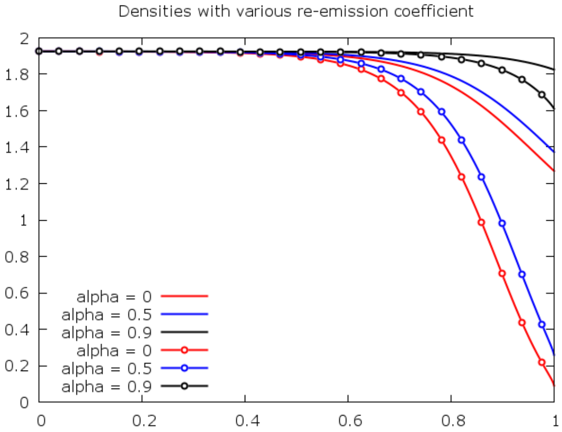




The sheath boundary layer (Debye layer) is visible at the wall.

The electrons are repelled or trapped with high mean velocity (in modulus).
The ions are accelerated and trapped with small mean velocity (in modulus).

Densities in the physical space



Numerical tests show $\alpha \mapsto (n_i, n_e)$ is an increasing function.

Stability (linear stability)

Problem and method

Problem : add a small perturbation to the sheath solution and show linear stability of the time dependent formulation.

Method : Mimic the L^2 technique for homogeneous Vlasov-Poisson equation written in $[0,1]_{\rm per} \times \mathbb{R}$

$$\left\{ \begin{array}{l} \partial_t f + v\partial_x f - E\partial_v f = 0, \\ \partial_x E = \sqrt{2\pi} - \int_{\mathbb{R}} f dv. \end{array} \right. \iff \left\{ \begin{array}{l} \partial_t f + v\partial_x f - E\partial_v f = 0, \\ \partial_t E = \int_{\mathbb{R}} f v dv, \\ + \text{ Gauss law at } t = 0. \end{array} \right.$$

In a nutshell : linearization $f(t,x,v) = e^{-v^2/2} + e^{-v^2/4} h(t,x,v)$ yields

$$\left\{ \begin{array}{ll} \partial_t h + v\partial_x h & = -vEe^{-v^2/4}, \\ \partial_t E & = \int_{\mathbb{R}} vhe^{-v^2/4} dv \end{array} \right.$$

from which the L^2 quadratic estimate proceeds

$$\frac{d}{dt}\left[\int_0^1 \int_{\mathbb{R}} h^2 dxdv + \int_0^1 E^2 dx\right] = 0.$$

Note that Vlasov-Ampere is here convenient.

- Kruzkal-Obermann 56', Antonov 61', and many others.

- D. 2014. Application to Landau damping and link with Morrison integral transform.

The linearized equations

Linearize the non homogeneous sheath solution

$$f_e^{\mathrm{tot}}(t,x,v) = f_e^{\infty}(x,v) + f_e(t,x,v), \qquad E^{\mathrm{tot}}(t,x) = E^{\infty}(x) + E(t,x).$$

with $f_e^{\infty} = f_e^{\mathrm{sheath}}$ and $E^{\infty} = E^{\mathrm{sheath}}$.

The linearized Vlasov-Ampere system is

$$\begin{cases} \partial_t f_e + \left(v\partial_x - \frac{1}{\mu} E^{\infty} \partial_v \right) f_e = \frac{1}{\mu} E \partial_v f_e^{\infty}, & \\ \varepsilon^2 \partial_t E = \int_{\mathbb{R}} f_e v dv, & \\ & \\ f_e(t,0,v>0) = 0, & \text{B. C. } x=0, \\ f_e(t,1,v<0) = \alpha f_e(t,1,-v), & \text{B. C. } x=1, \\ & \\ f_e(0,x,v) = f_e^0(x,v), & \text{init } t=0 \\ E(0,x) = E^0(x) & \text{init } t=0 \end{cases}$$

No ions.

Comments and references

• The ion equation is eliminated.
The fundamental reason is that $f_i^{\mathrm{sheath}}(x, v)$ cannot be written as a monotone function of the microscopic energy $H(x, v) = \frac{v^2}{2} + \phi^{\mathrm{sheath}}(x)$

$$f_i^{\mathrm{sheath}}(x, v) \neq G(H(x, v)), \quad G' \neq 0.$$

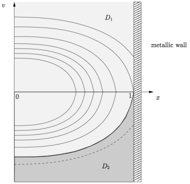
Rein (1994) and Lemou (2012) deal with monotone G.
See recent work Ben-Artzi (2011) for non monotone G.

• Fortunately this is physically relevant since ions are heavy particles and can be considered as "frozen" : confirmed by Garbet (pers. comm. 2015).

• Stability of V.M.+B.C. is addressed in Nguyen-Strauss (2013).

• Nothing in the literature (to our knowledge) about B.C. and non homogeneous stationary profiles.

Linearization of the truncated Maxwellian

- Remind 

, so $\partial_v f_e^\infty = [f_e^\infty]\delta(v - v_e(x)) - \mu v f_e^\infty$.

- Moreover the linearization $f_e^{\rm tot} = f_e^\infty + f_e + o(\dots)$ is similar to linearization of shock waves. So the regularity of f_e is one less than f_e^∞.

- For these two reasons use the Ansatz

$$f_e(t,x,v) = n_e(t,x)\delta(v - v_e(x)) + \sqrt{f_e^\infty} h_e(t,x,v), \quad h_e \text{ a bounded function.}$$

- This is compatible with the structure of

$$\partial_t f_e + \left(v\partial_x - \frac{1}{\mu} E^\infty \partial_v \right) f_e = \frac{1}{\mu} E \partial_v f_e^\infty$$

since $v = v_e(x)$ is characteristic by definition.

A simpler system

$$\begin{cases} \partial_t h_e + \left(v\partial_x - \frac{1}{\mu}E^\infty\partial_v\right)h_e & = -Ev\sqrt{f_e^\infty}, & "v \neq v_e(x)" \\ \partial_t n_e + \partial_x(v_e(x)n_e(x)) & = \frac{[f_e^\infty]}{\mu}E, & "v = v_e(x)" \\ & & \\ \varepsilon^2\partial_t E & = \int_{\mathbb{R}} h_e v\sqrt{f_e^\infty}dv + v_e n_e, & \\ & & \\ g_e(t,0,v>0) = 0, & \text{B. C. } x = 0, & \\ g_e(t,1,v<0) = \alpha g_e(t,1,-v), & \text{B. C. } x = 1, & \\ n_e(t,1) = 0, & \text{B. C. } x = 1, & \\ & & \\ g_e(0,x,v) = g_e^0(x,v), & \text{init } t = 0 & \\ E(0,x) = E^0(x) & \text{init } t = 0 & \end{cases}$$

Main a priori estimate

Set

$$\mathcal{E}(t)=\frac{\mu}{[f_e^\infty]}\int_0^1 |v_e(x)|n_e^2dx+\int_0^1\int_{\mathbb{R}} h_e^2dxdv+\varepsilon^2\int_0^1 E^2dx.$$

Proposition : One has formerly

$$\frac{d}{dt}\mathcal{E}=-\frac{\mu}{[f_e^\infty]}(v_e(x)n_e)(t,0)^2$$

$$-(1-\alpha)\int_{\mathbb{R}^+} h_e(t,1,v)^2vdv+\int_{\mathbb{R}^-} h_e(t,0,v)^2vdv\leq 0.$$

Proof

$$\frac{d}{dt}\left(\frac{\mu}{[f_e^\infty]}\int_0^1 |v_e| n_e^2 dx\right) = 2\frac{\mu}{[f_e^\infty]}\int_0^1 (-v_e) n_e \left(\frac{[f_e^\infty]}{\mu}E - \partial_x(v_e n_e)\right) dx$$
$$= -2\int_0^1 v_e n_e E dx + \frac{\mu}{[f_e^\infty]}\left[(v_e n_e)^2(x=1) - (v_e n_e)^2(x=0)\right]$$
$$= -2\int_0^1 v_e n_e E dx - \frac{\mu}{[f_e^\infty]}(v_e n_e)^2(x=0).$$

$$\frac{d}{dt}\left(\int_0^1 \int_{\mathbb{R}} h_e^2 dxdv\right) = 2\int_0^1 \int_{\mathbb{R}} h_e \left(-Ev\sqrt{f_e^\infty} - \left(v\partial_x - \frac{1}{\mu}E^\infty \partial_v\right) h_e\right) dxdv$$
$$= -2\int_0^1 \int_{\mathbb{R}} Ev\sqrt{f_e^\infty} h_e dxdv$$
$$-(1-\alpha)\int_{\mathbb{R}^+} h_e(t,1,v)^2 vdv + \int_{\mathbb{R}^-} h_e(t,0,v)^2 vdv$$

$$\frac{d}{dt}\left(\varepsilon^2 \int_0^1 E^2 dx\right) = 2\int_0^1 E\left(\int_{\mathbb{R}} h_e v\sqrt{f_e^\infty} dv + v_e n_e\right) dx$$
$$= 2\int_0^1 \int_{\mathbb{R}} Ev\sqrt{f_e^\infty} h_e dxdv + 2\int_0^1 v_e n_e E dx.$$

Only the dissipative B.C. contributions remain.

Spaces

- Some spaces are $L^2_{v_e}(0,1) = \left\{ u,\ \int_0^1 \frac{1}{|v_E|} u^2 dx < \infty \right\}$,

$$H^1_{v_e}(0,1) = \left\{ u \in L^2(0,1),\ \sqrt{|v_e|}u' \in L^2(0,1), \right\} \hookrightarrow C^0[0,1] \quad \left(\text{since } \int_0^1 \frac{dx}{|v_e|} < \infty \right)$$

$$H^1_{v_e,0}(0,1) = \left\{ u \in H^1_{v_e}(0,1),\ u(1) = 0 \right\}$$

and $W^2_\alpha = \left\{ h \in L^2_{xv},\ Dh \in L^2_{xv} \text{ and } B.C. \right\}$.

Why these spaces ?
Because the velocity vanishes and is non Lipschitz at the wall.

- One can prove that $w_e = v_e n_e \in H^1_{v_e,0}(0,1)$ in general.

- Note that $w_e = (1-x)^\alpha \in H^1_{v_e,0}(0,1)$ for all $\alpha > \frac{1}{4}$. In this case

$$n_e = \frac{w_e}{v_e} \approx C(1-x)^\beta, \quad \beta > -\frac{1}{4}.$$

Well-posedness

Set

$$H = L^2_{v_e}(0,1) \times L^2\left((0,1)\times\mathbb{R}\right) \times L^2(0,1)$$

and

$$G = H^1_{v_e,0}(0,1) \times W^2_\alpha \times L^2(0,1).$$

A first theoretical result (Badsi) : For all $(w_e, h_e, E) \in G$, there exists a unique strong solution

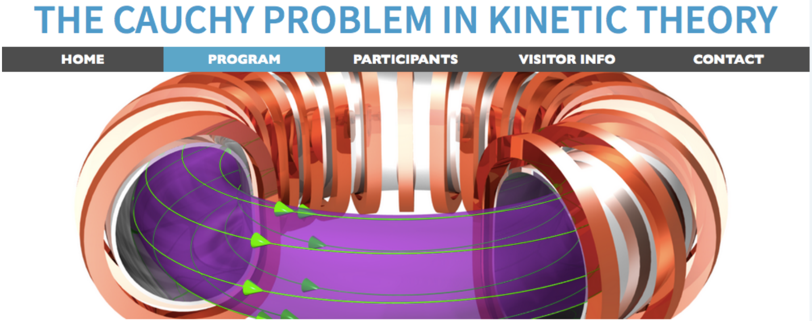
$$(w_e, h_e, E) \in C^0([0,\infty); G) \cap C^1([0,\infty); H).$$

proof : Hille-Yosida theorem and ask Mehdi.

Notice that the E does not require additional regularity between H and G in this version which says nothing about the Gauss law.

With magnetic field $\mathbf{B}_0$

Chodura sheath,
Stangeby (The Plasma Boundary of Magnetic Fusion Devices, 2011).



Basics

Consider a potential electric field and a constant tilted magnetic field

$$\mathbf{E} = \left(-\varphi', 0, 0\right), \quad \mathbf{B}_0 = \omega_c(\cos\theta, \sin\theta, 0).$$

The 1d-3v model with $m = 1$ and $e = 1$ is

$$\left\{ \begin{array}{l} \frac{d}{dt}\mathbf{x} = \mathbf{v}, \\ \frac{d}{dt}\mathbf{v} = \mathbf{E} + \mathbf{v} \times \mathbf{B}_0 \end{array} \right. \Longrightarrow \left\{ \begin{array}{l} \frac{d}{dt}x = v_1, \\ \frac{d}{dt}v_1 = -\varphi'(x) - \omega_c \sin\theta v_3, \\ \frac{d}{dt}v_2 = \omega_c \cos\theta v_3, \\ \frac{d}{dt}v_3 = -\omega_c \cos\theta v_2 + \omega_c \sin\theta v_1. \end{array} \right.$$

Our potential method is based on invariants : the energy is invariant (as usual)

$$I_1 = \frac{1}{2}(v_1^2 + v_2^2 + v_3^2) + \varphi(x).$$

Other invariants

Are they other invariants ?
Yes for $\theta = \frac{\pi}{2}$ which means tangential $\mathbf{B}_0$

$$\begin{cases} \frac{d}{dt}x = v_1, \\ \frac{d}{dt}v_1 = -\varphi'(x) - \omega_c v_3, \\ \frac{d}{dt}v_2 = 0, \\ \frac{d}{dt}v_3 = \omega_c v_1. \end{cases}$$

One finds

$$I_2 = v_2$$

and

$$I_3 = v_3 - \omega_c x.$$

With that information, representation of the solutions of the various kinetic equations is possible, and so the construction of the stationary sheath (with tangential $\mathbf{B}_0$) is possible.

Surprisingly, at least to me, the theory of tangential magnetic fields is considered as less advanced in Stangeby text book (last chapter).

Conclusion

- A sheath solution has been constructed, starting from stationary solutions of two species Vlasov + Poisson (to appear in KRM 2015).
- Restriction is that $x = 0$ is the pre-sheath, not the center of the plasma.
- The method is constructive with natural numerical algorithms.

- Progress on the stability analysis around this non homogeneous profile with physical non trivial B.C. : see Mehdi Badsi for preprint (soon available).
- Restriction of our stability analysis is : ions not addressed.

- It is, in principle, possible to generalize the method to other equations :
  - the 1d-3v model,
  - and the theory of Langmuir probes with finite curvature (with fascinating applications to Rosetta-Tchoury mission).