WEAK POINCARÉ INEQUALITIES IN ABSENCE OF SPECTRAL GAPS

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Abstract. For generators of Markov semigroups which lack a spectral gap, it is shown how bounds on the density of states near zero lead to algebraic temporal decay rates. These rates follow from a so-called "weak Poincaré inequality" (WPI), originally introduced by Liggett [Ann. Probab., 1991]. Applications to the heat semigroup and to the semigroup generated by the fractional Laplacian are studied, where the optimal decay rates are recovered. The classical Nash inequality appears as a special case of the WPI for the heat semigroup.

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Contents

1. Introduction and statement of results

In this note we study how the well-known equivalence between spectral gaps, Poincaré inequalities and exponential rates of decay to equilibrium extends to systems which lack a spectral gap but have a bounded density of states near 0. Our main result relies solely on our ability to "differentiate" the resolution of the identity of a given operator. It is thus quite general, and covers important examples such as Markov semigroups.

Our setup is as follows: we let M be a manifold with Borel measure $\mathrm{d}\mu$, $\mathcal{H} = L^2(M, \mathrm{d}\mu; \mathbb{R})$ equipped with scalar product $(\cdot,\cdot)_{\mathcal{H}}$. We assume that $H : D(H) \subset \mathcal{H} \to \mathcal{H}$ is a self-adjoint, non-negative operator, so that $-H$ is the infinitesimal generator of a Markov semigroup $(P_t)_{t\geq 0}$, whose invariant measure is $\mathrm{d}\mu$, i.e. for every u that is bounded and non-negative $\int_M P_t u \,\mathrm{d}\mu = \int_M u \,\mathrm{d}\mu$ for any $t \geq 0$. Let $\{E(\lambda)\}_{\lambda\geq 0}$ be the resolution of the identity of H

and let the associated Dirichlet form be

$$\mathcal{E}(u) := \int_M (H^{1/2}u)^2 \, \mathrm{d}\mu.$$

As stated above, instead of assuming a spectral gap, we assume the opposite: H has continuous spectrum in a neighborhood of 0 (and 0 itself is possibly an eigenvalue). We show that an appropriate estimate of the density of the spectrum near 0 leads to a weaker version of the Poincaré inequality (also known as a weak Poincaré inequality, defined below in Definition 1.3). This, in turn, leads to an algebraic decay rate for the associated semigroup.

In this paper we employ the following definition for the variance of a given function $u \in \mathcal{H}$:

$$\mathrm{Var}(u) := \int_M (u - E(\{0\})u)^2 \, \mathrm{d}\mu$$

where $E(\{0\})$ is the projection onto the kernel of H. In the case where the kernel only consists of constant functions and μ is a probability measure, this definition coincides with the standard definition, see [3, §4.2.1]. We discuss the significance of the resolution of the identity of H (and in particular the projection onto its kernel) and its relationship with functional inequalities and decay rates below in Section 2.3.

We can now recall the classical Poincaré inequality (again, see [3, §4.2.1]):

Definition 1.1 (Poincaré Inequality). *We say that H satisfies a* Poincaré inequality *if there exists $C > 0$ such that*

$$\mathrm{Var}(u) \leq C\mathcal{E}(u), \qquad \forall u \in D(\mathcal{E}),$$

where C does not depend on u.

Remark 1.2. The topology of $D(\mathcal{E})$ is the graph norm topology generated by $\|\cdot\|_{\mathcal{H}}^2 + \mathcal{E}(\cdot)$, see [3, §3.1.4].

The definition of a "weak Poincaré inequality" is somewhat ambiguous. This is addressed in further detail in Section 2.3 below. We adopt the following definition, motivated by Liggett [9, Equation (2.3)]:

Definition 1.3 (Weak Poincaré Inequality). *Let $\Phi : \mathcal{H} \to [0, \infty]$ satisfy $\Phi(u) < \infty$ on a dense subset of $D(\mathcal{E})$. Let $p \in (1, \infty)$. We say that H satisfies a* (Φ, p)-weak Poincaré inequality ((Φ, p)-WPI) *if there exists $C > 0$ such that*

$$\mathrm{Var}(u) \leq C\mathcal{E}(u)^{1/p}\Phi(u)^{1/q}, \qquad \forall u \in D(\mathcal{E}), \tag{1.1}$$

where C does not depend on u and where $1/p + 1/q = 1$.

Remark 1.4. Note that (1.1) is meaningful only on a dense subset of $D(\mathcal{E})$ where $\Phi < +\infty$.

1.1. The Hilbertian case.

In this subsection we consider subspaces which respect the Hilbert structure of $\mathcal{H}$, such as Sobolev spaces or weighted spaces. Our basic assumption is:

Assumption A1. There exists a dense subspace $\mathcal{X} \subset \mathcal{H}$ such that

(1) $\mathcal{X} \cap D(\mathcal{E})$ is dense in $D(\mathcal{E})$ (in the topology of $D(\mathcal{E})$),
(2) for some constants $r > 0$, $C_1 > 0$ and $\alpha > -1$,

the mapping $\lambda \mapsto \frac{\mathrm{d}}{\mathrm{d}\lambda}(E(\lambda)u, v)_{\mathcal{H}}$ is continuous on $(0, r)$ for every $u, v \in \mathcal{X}$ and satisfies

$$\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}(E(\lambda)u, v)_{\mathcal{H}}\right| \leq C_1 \lambda^{\alpha} \|u\|_{\mathcal{X}} \|v\|_{\mathcal{X}}, \qquad \forall u, v \in \mathcal{X},\ \forall \lambda \in (0, r). \tag{1.2}$$

Remark 1.5. We refer to the bilinear form $\frac{\mathrm{d}}{\mathrm{d}\lambda}(E(\lambda)\cdot, \cdot)_{\mathcal{H}}$ as the density of states (DoS) of H at λ. Note that if the DoS satisfies a bound as in (1.2) and $\mathcal{X}$ has a norm compatible with (and stronger than) the norm on $\mathcal{H}$ then it induces an operator $\mathcal{X} \to \mathcal{X}^*$ by the Riesz representation theorem.

We can finally state our main results on how (1.2) leads to a (Φ, p)-WPI (Theorem 1.6) and, in turn, an explicit rate of decay (Theorem 1.7). Theorem 1.6 will be further generalized below in Theorem 1.9. The decay rates presented in Theorem 1.7 apply to the Markov semigroup generated by H.

Theorem 1.6. *If Assumption A1 holds then H satisfies a (Φ, p)-weak Poincaré inequality with $\Phi(u) = \|u\|_{\mathcal{X}}^2$ (and $\Phi(u) = +\infty$ if $u \in \mathcal{H} \setminus \mathcal{X}$) and $p = \frac{2+\alpha}{1+\alpha}$.*

Theorem 1.7. *Let Assumption A1 hold. Let $u \in \mathcal{X}$ and suppose that there exist $C_2 = C_2(u) \geq 0$ and $\beta \in \mathbb{R}$, such that the Markov semigroup satisfies*

$$\|P_t u\|_{\mathcal{X}}^2 \leq \|u\|_{\mathcal{X}}^2 + C_2 t^{\beta}, \qquad \forall t \geq 0. \tag{1.3}$$

Then

$$\operatorname{Var}(P_t u) \leq \left(\operatorname{Var}(u)^{\frac{-1}{1+\alpha}} + C_3 \int_0^t (\|u\|_{\mathcal{X}}^2 + C_2 s^{\beta})^{\frac{-1}{1+\alpha}}\, \mathrm{d}s\right)^{-(1+\alpha)} \tag{1.4}$$

where C_3 is given explicitly (and only depends on α, C_1). In particular, $\operatorname{Var}(P_t u)$ *satisfies the following decay rates as $t \to \infty$:*

$$\operatorname{Var}(P_t u) \leq \begin{cases} O((\log t)^{-(1+\alpha)}) & \beta = 1+\alpha. \\ O(t^{\beta-(1+\alpha)}) & 0 < \beta < 1+\alpha. \\ O(t^{-(1+\alpha)}) & C_2 = 0 \text{ or } \beta \leq 0. \end{cases}$$

Remark 1.8. 1. The choice of space $\mathcal{X}$ is motivated by (1.3): it is beneficial to choose $\mathcal{X}$ that is invariant under the Markov semigroup.

2. Clearly $C_2(u)$ is subject to quadratic scaling, for example it can be $C\|u\|_{\mathcal{H}}^2$ or $C\|u\|_{\mathcal{X}}^2$, but the explicit form is not important.

1.2. **A generalized theorem: departing from the Hilbert structure.** Theorems 1.6 and 1.7 demonstrate how estimates on the density of states near 0 imply a weak Poincaré inequality and a rate of decay to equilibrium. However it is not essential to restrict oneself to a subspace $\mathcal{X}$. In fact, it is often desirable to deal with functional spaces that are not contained in $\mathcal{H}$, as it may provide improved estimates and decay rates. In particular, this makes sense when the operator in question is the generator of a Markov semigroup, and acts on a range of spaces simultaneously. Hence we replace Assumption A1 by a more general one:

Assumption A2. There exist Banach spaces $\mathcal{X}, \mathcal{Y}$ of functions on M, a constant $r > 0$ and a function $\psi_{\mathcal{X},\mathcal{Y}} \in L^1(0,r)$ that is strictly positive a.e. on $(0,r)$, such that

(1) $\mathcal{X} \cap \mathcal{Y} \cap D(\mathcal{E})$ is dense in $D(\mathcal{E})$ (in the topology of $D(\mathcal{E})$).

(2) The mapping $\lambda \mapsto \frac{\mathrm{d}}{\mathrm{d}\lambda}(E(\lambda)u,v)_{\mathcal{H}}$ is continuous on $(0,r)$ for every $u \in \mathcal{X} \cap \mathcal{H}$ and $v \in \mathcal{Y} \cap \mathcal{H}$ and satisfies

$$\left| \frac{\mathrm{d}}{\mathrm{d}\lambda}(E(\lambda)u,v)_{\mathcal{H}} \right| \le \psi_{\mathcal{X},\mathcal{Y}}(\lambda) \|u\|_{\mathcal{X}} \|v\|_{\mathcal{Y}}, \qquad \forall \lambda \in (0,r). \tag{1.5}$$

We can now state the following more general theorem:

Theorem 1.9. *Let the conditions of Assumption A2 hold, and define* $\Psi_{\mathcal{X},\mathcal{Y}}(\rho) = \int_0^\rho \psi_{\mathcal{X},\mathcal{Y}}(\lambda)\, \mathrm{d}\lambda$, $\rho \in (0,r)$. *Then:*

a. There exists $K_0 \in (0,1)$ *such that the following functional inequality holds:*

$$(1-K)\Psi^{-1}_{\mathcal{X},\mathcal{Y}}\left(K \frac{\mathrm{Var}(u)}{\|u\|_{\mathcal{X}}\|u\|_{\mathcal{Y}}} \right) \mathrm{Var}(u) \le \mathcal{E}(u), \qquad \forall K \in (0,K_0),\, \forall u \in D(\mathcal{E}) \tag{1.6}$$

where $\|u\|_{\mathcal{X}} = +\infty$ *if* $u \notin \mathcal{X}$ *and similarly for* $\mathcal{Y}$.

b. If $\mathcal{X} = \mathcal{Y}$ *and* $\psi(\lambda) = C_1 \lambda^\alpha$, $\alpha > -1$, *the estimate* (1.6) *reduces to the* (Φ, p)*-WPI as in Definition 1.3 with* $\Phi(u) = \|u\|^2_{\mathcal{X}}$ *and* $p = \frac{\alpha+2}{\alpha+1}$.

c. If, in addition, $\mathcal{X} = \mathcal{Y} \subset \mathcal{H}$ *then we obtain Theorem 1.6.*

Remark 1.10. The inequality (1.6) can be viewed as an implicit form of the weak Poincaré inequality. Note that setting $K = 0$ (which is excluded in the theorem) leads to the Poincaré inequality.

The power of this result is demonstrated in the following corollary, where the celebrated Nash inequality is obtained as a simple consequence. This simple derivation is discussed in Remark 4.3 below.

Corollary 1.11. *When* $H = -\Delta : H^2(\mathbb{R}^d) \subset L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ *and* $\mathcal{Y} = \mathcal{X} = L^1(\mathbb{R}^d)$ *the inequality* (1.6) *is precisely Nash's inequality [10]:*

$$\|u\|^2_{L^2} \le C \left(\|\nabla u\|^2_{L^2} \right)^{\frac{d}{d+2}} \left(\|u\|^2_{L^1} \right)^{\frac{2}{d+2}}, \qquad \forall u \in L^1(\mathbb{R}^d) \cap H^1(\mathbb{R}^d), \tag{1.7}$$

where $C > 0$ *does not depend on* u.

Proof. The proof of this simple corollary is provided in Remark 4.3 below. □

Remark 1.12. The requirement that $\psi_{\mathcal{X},\mathcal{Y}}$ is strictly positive a.e. on $(0,r)$, for some $r > 0$ (perhaps very small), is quite natural as we are interested in operators that lack a spectral gap. However, one can easily generalize our result even if that is not the case by defining

$$\Psi^{-1}_{\mathcal{X},\mathcal{Y}}(y) = \sup \left\{ x \in (0,r) \mid \Psi_{\mathcal{X},\mathcal{Y}}(x) \le y \right\}.$$

Organization of the paper. Before proceeding to prove our theorems we first discuss both the classical and the weak Poincaré inequalities, and their connection to Markov semigroups in Section 2. The proofs will follow in Section 3 and we then present various applications of these theorems in Section 4, where we shall also prove Corollary 1.11. Finally, in Section 5 we provide some supporting results.

2. Poincaré inequalities

In this section we recall the famous Poincaré inequality, its connection to Markov semigroups, and we discuss its "weak" variant, the so-called "weak Poincaré inequality."

2.1. **The classical Poincaré inequality.**

When M is a compact Riemannian manifold or a bounded domain of $\mathbb{R}^d$, the classical L^2 Poincaré inequality reads [3, §4.2.1]

$$\int_M \left|\varphi(x) - \left(\frac{1}{|M|}\int_M \varphi(y)\,\mathrm{d}y\right)\right|^2 \mathrm{d}x \le C_M \int_M |\nabla\varphi(x)|^2\,\mathrm{d}x, \tag{2.1}$$

where $|M|$ is the volume of M, and $C_M > 0$ is independent of u.

Motivation: the heat semigroup. Let us illustrate why the quantities appearing in this inequality are natural. Let $M \subset \mathbb{R}^d$ be a bounded, connected and smooth domain. Consider the heat semigroup, i.e. solutions of

$$\partial_t u(t,x) = \Delta_x u(t,x), \qquad x \in M,\, t \in \mathbb{R}_+,$$

subject to Neumann boundary conditions with initial data $u(0,x) = u_0(x)$. The associated invariant measure is $\mathrm{d}\mu(x) = \frac{\mathrm{d}x}{|M|}$. It is well-known that in this case the spectrum of Δ_x is discrete and non-positive. In particular, its kernel is separated from the rest of the spectrum. This immediately implies that $u(t,x) = P_t u_0(x)$ converges to the projection onto the kernel, given by

$$P_{\ker} u_0 := \int_M u_0(x)\,\mathrm{d}\mu(x).$$

Thus, we are interested in the decay rate as $t \to +\infty$ of

$$\mathcal{V}(P_t u_0) := \|P_t u_0 - P_{\ker}\left(P_t u_0\right)\|^2_{L^2(\mathrm{d}\mu)} = \|P_t u_0 - P_{\ker} u_0\|^2_{L^2(\mathrm{d}\mu)}.$$

The entropy method. A common method to obtain decay rates of this type is the so-called *entropy method.* Given the "relative distance" $\mathcal{V}$ (a Lyapunov functional) we find its *production functional* $\mathcal{E}$ by formally differentiating along the flow of the semigroup:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}(P_t u_0) = 2\left(\partial_t P_t u_0, P_t u_0 - P_{\ker} u_0\right)_{L^2(\mathrm{d}\mu)} = 2\int_M P_t u_0(x)\Delta_x P_t u_0(x)\,\mathrm{d}\mu(x) = -2\mathcal{E}(P_t u_0), \tag{2.2}$$

where $\mathcal{E}$ turns out to be the associated Dirichlet form. Note that since $P_{\ker} = E\left(\{0\}\right)$ we can rewrite (2.2) as $\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Var}(P_t u_0) = -2\mathcal{E}(P_t u_0)$. Now we seek a pure functional inequality involving $\mathcal{V}$ and $\mathcal{E}$. In particular (see, for example, [12, Chapter 3, §3.2]), one looks for a functional inequality of the form

$$\mathcal{E}(u) \ge \Theta(\mathcal{V}(u)), \qquad \forall u \in D(\mathcal{E}), \tag{2.3}$$

with an appropriate nonnegative function Θ. Succeeding in finding such an inequality entails, in view of (2.2),

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}(P_t u_0) \le -2\Theta(\mathcal{V}(P_t u_0))$$

from which an explicit rate is derived.

Returning to the heat semigroup, we notice that the classical Poincaré inequality (2.1) is *exactly* a functional inequality of the form of (2.3). Moreover, the linear connection between the variance and the Dirichlet form yields an exponential rate of decay for $\mathrm{Var}(P_t u_0)$.

2.2. **Relationship to Markov semigroups.** In view of Subsection 2.1, there is a natural extension of the notion of a Poincaré inequality to general Markov semigroups. Let $\{P_t\}_{t\geq 0}$ be a Markov semigroup on $\mathcal{H} = L^2(M, \mathrm{d}\mu)$ with a generator $-H$, where H is a self-adjoint, non-negative operator, and $\mathrm{d}\mu$ its invariant measure. Then the Poincaré inequality, as already defined (Definition 1.1), is

$$\mathrm{Var}(u) \leq C\mathcal{E}(u), \qquad \forall u \in D(\mathcal{E}).$$

The following well known theorem (see [3, Theorem 4.2.5]) serves as a motivation for our current investigation:

Theorem 2.1. *The following conditions are equivalent:*

(1) *H satisfies a Poincaré inequality with constant C.*
(2) *The spectrum of H is contained in $\{0\} \cup \left[\frac{1}{C}, \infty\right)$.*
(3) *For every $u \in L^2(M, \mathrm{d}\mu)$ and every $t \geq 0$,*

$$\mathrm{Var}(P_t u) \leq e^{-2t/C}\,\mathrm{Var}(u).$$

2.3. **The weak Poincaré inequality (WPI).** It is natural to ask whether one can obtain a generalization of Theorem 2.1 to generators which lack a spectral gap. We note that a differential operator acting on functions defined in an unbounded domain (generically) lacks a spectral gap. Our Theorems 1.6 and 1.9 provide an answer to this question, where the Poincaré inequality is replaced by some form of a weak Poincaré inequality. In the following we provide a brief review of the existing literature on variants of the weak Poincaré inequality.

This topic has a very rich history, in particular in the second half of the 20th century. As was hinted in Corollary 1.11, a closely related example is Nash's celebrated inequality [10]:

$$\|u\|_{L^2}^2 \leq C\left(\|\nabla u\|_{L^2}^2\right)^{\frac{d}{d+2}}\left(\|u\|_{L^1}^2\right)^{\frac{2}{d+2}}, \qquad \forall u \in L^1(\mathbb{R}^d) \cap H^1(\mathbb{R}^d) \tag{2.4}$$

where $C > 0$ does not depend on u. Estimates of the same spirit are then developed in [7] for example.

The form of the weak Poincaré inequality which we consider (Definition 1.3) first appeared in [9, Equation (2.3)], where it is also shown how such a differential inequality leads to an algebraic decay rate. These ideas were then further developed in [2, 5, 11, 13–16]. We also refer to [1] where the notion of a "weak spectral gap" is introduced.

In fact, in [11] several variants of the WPI were introduced. The most general one is

$$\mathrm{Var}(u) \leq \alpha(r)\mathcal{E}(u) + r\Phi(u), \qquad \forall u \in D(\mathcal{E}),\ r > 0,$$

where $\alpha : (0, \infty) \to (0, \infty)$ is decreasing and $\Phi : L^2(\mathrm{d}\mu) \to [0, \infty]$ satisfies $\Phi(cu) = c^2\Phi(u)$ for any $c \in \mathbb{R}$ and $u \in L^2(\mathrm{d}\mu)$. This is equivalent to our (Φ, p)-WPI whenever $\alpha(r) = Cr^{1-p}$.

For a recent account of the notions discussed here, and in particular the relationship between functional inequalities and Markov semigroups, we refer to the book [3].

3. Proofs of the theorems

We first prove the more general Theorem 1.9, and then show how Theorem 1.6 is a straightforward corollary. Finally we show how to obtain the decay rates in Theorem 1.7. For brevity, we omit the subscripts from the functions $\psi_{\mathcal{X},\mathcal{Y}}$ and $\Psi_{\mathcal{X},\mathcal{Y}}$.

3.1. **Proof of Theorem 1.9a.** First we show that an estimate on the density of states near 0 leads to the WPI (1.6). Let $r_0 \in (0, r)$ to be chosen later. Let $\{E(\lambda)\}_{\lambda\geq 0}$ be the resolution of the identity of H. Let $u \in D(\mathcal{E}) \cap \mathcal{X} \cap \mathcal{Y}$. Then:

$$\begin{aligned}
\mathcal{E}(u) &= \int_M uHu \,\mathrm{d}\mu = \int_M u \int_{[0,\infty)} \lambda \,\mathrm{d}E(\lambda) u \,\mathrm{d}\mu \\
&\geq \int_M u \int_{[r_0,\infty)} \lambda \,\mathrm{d}E(\lambda) u \,\mathrm{d}\mu \geq r_0 \int_M u \int_{[r_0,\infty)} \mathrm{d}E(\lambda) u \,\mathrm{d}\mu \\
&= r_0 \int_M u \int_{[0,\infty)} \mathrm{d}E(\lambda) u \,\mathrm{d}\mu - r_0 \|E(\{0\})u\|^2_{\mathcal{H}} - r_0 \int_M u \int_{(0,r_0)} \mathrm{d}E(\lambda) u \,\mathrm{d}\mu \\
&= r_0 \operatorname{Var}(u) - r_0 \int_M u \int_{(0,r_0)} \mathrm{d}E(\lambda) u \,\mathrm{d}\mu.
\end{aligned}$$

We now use the estimate on the density of states (1.5) to obtain

$$\begin{aligned}
\int_M u \int_{(0,r_0)} \mathrm{d}E(\lambda) u \,\mathrm{d}\mu &= \int_{(0,r_0)} \frac{\mathrm{d}}{\mathrm{d}\lambda} (E(\lambda)u, u)_{\mathcal{H}} \,\mathrm{d}\lambda \\
&\leq \|u\|_{\mathcal{X}} \|u\|_{\mathcal{Y}} \int_{(0,r_0)} \psi(\lambda) \,\mathrm{d}\lambda = \|u\|_{\mathcal{X}} \|u\|_{\mathcal{Y}} \Psi(r_0).
\end{aligned}$$

Hence we have

$$\mathcal{E}(u) \geq r_0 \left(\operatorname{Var}(u) - \|u\|_{\mathcal{X}} \|u\|_{\mathcal{Y}} \Psi(r_0) \right).$$

Let $K \in (0,1)$ and define

$$r_0 = \Psi^{-1}\left(K \frac{\operatorname{Var}(u)}{\|u\|_{\mathcal{X}} \|u\|_{\mathcal{Y}}} \right) \qquad \text{so that} \qquad \Psi(r_0) = K \frac{\operatorname{Var}(u)}{\|u\|_{\mathcal{X}} \|u\|_{\mathcal{Y}}}$$

(to satisfy the condition $r_0 < r$ we may need K to be small). Then we get

$$\mathcal{E}(u) \geq r_0 (1-K) \operatorname{Var}(u)$$

which completes the proof.

3.2. **Proof of Theorem 1.9 b & c (and Theorem 1.6).** The proofs follow from the following lemma where we show how (1.6) leads to a (Φ, p)-WPI.

Lemma 3.1. *When $\mathcal{X} = \mathcal{Y}$ and $\psi(\lambda) = C_1 \lambda^\alpha$, $\alpha > -1$, the inequality* (1.6) *reduces to the (Φ, p)-WPI with $\Phi(u) = \|u\|^2_{\mathcal{X}}$ and $p = \frac{\alpha+2}{\alpha+1}$. Furthermore, if $\mathcal{X} = \mathcal{Y} \subset \mathcal{H}$ we recover Theorem 1.6.*

Proof. Let $\psi(\lambda) = C_1 \lambda^\alpha$, $\alpha > -1$. Then

$$\Psi(\rho) = C_1 \int_0^\rho \lambda^\alpha \,\mathrm{d}\lambda = \frac{C_1}{\alpha+1} \rho^{\alpha+1}$$

so that

$$\Psi^{-1}(\tau) = \left(\frac{\alpha+1}{C_1} \right)^{\frac{1}{\alpha+1}} \tau^{\frac{1}{\alpha+1}}.$$

Hence

$$\Psi^{-1}\left(K\frac{\mathrm{Var}(u)}{\|u\|_{\mathcal{X}}^2}\right)=\left(\frac{\alpha+1}{C_1}\right)^{\frac{1}{\alpha+1}}\left(K\frac{\mathrm{Var}(u)}{\|u\|_{\mathcal{X}}^2}\right)^{\frac{1}{\alpha+1}}.$$

Plugging this into (1.6) we have

$$\begin{aligned}\mathcal{E}(u)&\geq(1-K)\Psi^{-1}\left(K\frac{\mathrm{Var}(u)}{\|u\|_{\mathcal{X}}^2}\right)\mathrm{Var}(u)\\&=(1-K)\left(\frac{\alpha+1}{C_1}\right)^{\frac{1}{\alpha+1}}\left(K\frac{\mathrm{Var}(u)}{\|u\|_{\mathcal{X}}^2}\right)^{\frac{1}{\alpha+1}}\mathrm{Var}(u)\\&=C'\,\mathrm{Var}(u)^{\frac{\alpha+2}{\alpha+1}}\left(\|u\|_{\mathcal{X}}^2\right)^{-\frac{1}{\alpha+1}}.\end{aligned}$$

This leads to

$$\mathrm{Var}(u)\leq C''\mathcal{E}(u)^{\frac{\alpha+1}{\alpha+2}}\left(\|u\|_{\mathcal{X}}^2\right)^{\frac{1}{\alpha+2}}$$

which is a (Φ,p)-WPI with $\Phi(u)=\|u\|_{\mathcal{X}}^2$ and $p=\frac{\alpha+2}{\alpha+1}$. □

3.3. **Proof of Theorem 1.7.** We show that the growth rate assumption (1.3) leads to a decay of the variance as in (1.4). Using (2.2), the (Φ,p)-WPI and (1.3), we have:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Var}(P_tu)=-2\mathcal{E}(P_tu)&\leq-2C'\,\mathrm{Var}(P_tu)^{\frac{\alpha+2}{\alpha+1}}\left(\|P_tu\|_{\mathcal{X}}^2\right)^{-\frac{1}{\alpha+1}}\\&\leq-2C'\,\mathrm{Var}(P_tu)^{\frac{\alpha+2}{\alpha+1}}\left(\|u\|_{\mathcal{X}}^2+C_2t^{\beta}\right)^{-\frac{1}{\alpha+1}}\end{aligned}$$

where C' is as in the proof of Lemma 3.1. This is an ordinary differential inequality for $\mathrm{Var}(P_tu)$ of the form

$$\dot{y}\leq-Ay^{1+a}(B+Ct^b)^{-c},$$

for $a,c,A,B>0$, $b\in\mathbb{R}$, and $C\geq0$. We readily obtain

$$y(t)\leq\left(y(0)^{-a}+aA\int_0^t\left(B+Cs^b\right)^{-c}\mathrm{d}s\right)^{-1/a}$$

which yields the bound (1.4). Asymptotically, we have

$$y(t)=O(t^{-1/a})\text{ as }t\to\infty,\qquad\text{if }C=0\text{ or }b\leq0.$$

Otherwise, it is easy to see that $bc=1$ leads to logarithmic decay, while $bc<1$ leads to polynomial decay. The precise rates are

$$y(t)=\begin{cases}O((\log t)^{-1/a})\text{ as }t\to\infty,& bc=1.\\O(t^{-(1-bc)/a})\text{ as }t\to\infty,& bc<1.\end{cases}$$

This completes the proof of Theorem 1.7.

Remark 3.2 (The constant C_3). It is beneficial to provide a detailed computation of the constant C_3 appearing in (1.4). The following computations are performed up to a constant C which does not depend on $\alpha,M,\mathcal{H},\mathcal{X}$ or any other fundamental quantity.

Considering the proof of Theorem 1.7, we see that C_3 is denoted aA where $a=\frac{1}{\alpha+1}$ and $A=2C'$ with $C'=\underbrace{(1-K)K^{\frac{1}{1+\alpha}}}_{C}\left(\frac{\alpha+1}{C_1}\right)^{\frac{1}{\alpha+1}}$ where C_1 and α appear in the bound (1.2).

We readily obtain

$$C_3 = 2C\frac{1}{\alpha+1}\left(\frac{\alpha+1}{C_1}\right)^{\frac{1}{\alpha+1}} = 2C\,(\alpha+1)^{\frac{-\alpha}{\alpha+1}}\,C_1^{\frac{-1}{\alpha+1}}. \tag{3.1}$$

4. Examples

In this section we show how the theorems presented above lead to explicit decay rates of the variance under the flow of the Laplace and the fractional-Laplace operators.

4.1. The heat semigroup.

Consider the Laplace operator $H = -\Delta$ acting in $\mathcal{H} = L^2(\mathbb{R}^d, \mathrm{d}x)$. $-H$ is the generator of the *heat semigroup*, whose evolution equation is given by

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x), & t \in \mathbb{R}_+,\, x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d. \end{cases}$$

In this case $E(\{0\}) = 0$, and as such $\mathrm{Var}(u(t,\cdot)) = \|u(t,\cdot)\|_{\mathcal{H}}^2$. It is well established that $\|u(t,\cdot)\|_{\mathcal{H}}$ has a decay rate of $t^{-d/4}$. Our goal here is to use this fundamental model in order to illustrate how the various choices of spaces in Theorems 1.6 & 1.9 affect the resulting decay rates in Theorem 1.7. In fact, we show that an application of Theorem 1.6 gives rise to a sub-optimal decay rate, while a better choice of space in Theorem 1.9 leads to the optimal decay rate. We keep in mind that the main ingredients are:

- a carefully chosen functional space $\mathcal{X}$,
- an estimate of the density of states, as in (1.2) or (1.5),
- a bound of the semigroup, as in (1.3).

Example 4.1 (Applying Theorems 1.6 & 1.7). Since H is unitarily equivalent to multiplication by $|\xi|^2$ via a Fourier transform, we can express its resolution of the identity $\{E(\lambda)\}_{\lambda\in[0,\infty)}$ via the following bilinear form:

$$(E(\lambda)u,v)_{\mathcal{H}} = \int_{|\xi|^2\le\lambda} \widehat{u}(\xi)\overline{\widehat{v}(\xi)}\,\mathrm{d}\xi. \tag{4.1}$$

Differentiating in λ we get

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=\lambda_0} (E(\lambda)u,v)_{\mathcal{H}} = \frac{1}{2\sqrt{\lambda_0}} \int_{|\xi|=\sqrt{\lambda_0}} \widehat{u}(\xi)\overline{\widehat{v}(\xi)}\,\mathrm{d}\sigma_{\sqrt{\lambda_0}} \tag{4.2}$$

where $\mathrm{d}\sigma_{\sqrt{\lambda_0}}$ is the Lebesgue (uniform) surface measure on the $d-1$-dimensional sphere of radius $\sqrt{\lambda_0}$. An evaluation of the L^2 functions $\widehat{u}$ and $\widehat{v}$ on a hypersurface only makes sense if they belong to any Sobolev space $H^s(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$ with $s > 1/2$ by the trace lemma. The functions $\widehat{u}$ and $\widehat{v}$ belong to $H^s(\mathbb{R}^d)$ if and only if u and v belong to $L^{2,s}(\mathbb{R}^d)$, defined as

$$L^{2,s}(\mathbb{R}^d) := \left\{ u \in L^2(\mathbb{R}^d) \;:\; \|u\|_{L^{2,s}(\mathbb{R}^d)}^2 := \int_{\mathbb{R}^d} |u(x)|^2(1+|x|^2)^s\,\mathrm{d}x < \infty \right\}.$$

We therefore conclude that we can bound (4.2) using the $L^{2,s}$-norms of u and v, which are stronger than their $\mathcal{H}$-norms. The immediate bound,

$$\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=\lambda_0} (E(\lambda)u,v)_{\mathcal{H}}\right| \le \frac{C}{\sqrt{\lambda_0}}\|u\|_{L^{2,s}}\|v\|_{L^{2,s}}, \tag{4.3}$$

(where C does not depend on λ_0) coupled with Theorem 1.6 (where $\mathcal{X} = L^{2,s}$) yield the inequality

$$\mathrm{Var}(u) \le C\mathcal{E}(u)^{1/3}\left(\|u\|^2_{L^{2,s}}\right)^{2/3} = C\|\nabla u\|^{2/3}_{\mathcal{H}}\|u\|^{4/3}_{L^{2,s}}$$

where the constant C has changed, but does not depend on u. This establishes our (Φ, p)-WPI. Next, to obtain decay of the variance we require an estimate of the type (1.3). For $s \in (0, 1]$, Lemma 5.1 below shows the following estimate of the $L^{2,s}$ norm:

$$\|u(t,\cdot)\|^2_{L^{2,s}} \le \|u_0\|^2_{L^{2,s}} + 2s(2s + d - 2)\|u_0\|^2_{\mathcal{H}}t. \tag{4.4}$$

In the notation of Theorem 1.7, this means $\beta = 1$ and $\alpha = -1/2$, so that $1 = \beta > 1+\alpha = 1/2$. This is outside the scope of the cases appearing in Theorem 1.7.

We rectify this by improving our α, as follows. In the appendix of [4], the improved estimate

$$\int_{|\xi|=\rho} \widehat{u}(\xi)\overline{\widehat{v}(\xi)}\,\mathrm{d}\sigma_\rho \le C\min(\rho^{s-\frac{1}{2}}, 1)^2\|\widehat{u}\|_{H^s}\|\widehat{v}\|_{H^s},$$

where C only depends on s and d, was shown to be true when $s \in (1/2, 1]$. We therefore obtain (with $\rho = \sqrt{\lambda_0}$)

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=\lambda_0}(E(\lambda)u, v)_{\mathcal{H}}\right| &\le \min(\lambda_0^{\frac{s}{2}-\frac{1}{4}}, 1)^2\frac{C}{2\sqrt{\lambda_0}}\|\widehat{u}\|_{H^s}\|\widehat{v}\|_{H^s}\\ &= \min(\lambda_0^{s-\frac{1}{2}}, 1)\frac{C}{2\sqrt{\lambda_0}}\|u\|_{L^{2,s}}\|v\|_{L^{2,s}}.\end{aligned}$$

Hence (4.3) is improved to:

$$\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=\lambda_0}(E(\lambda)u, v)_{\mathcal{H}}\right| \le C\lambda_0^{s-1}\|u\|_{L^{2,s}}\|v\|_{L^{2,s}}, \qquad \forall\lambda_0 \in (0, 1],\ s \in (1/2, 1].$$

Together with the bound (4.4) of $\|u(t,\cdot)\|_{L^{2,s}}$, and taking $s = 1$, we conclude from Theorem 1.7 (with $\beta = 1$, $\alpha = 0$) that

$$\mathrm{Var}(u(t,\cdot)) \le \left(\mathrm{Var}(u_0)^{-1} + C\log\left(1 + \frac{2d\|u_0\|^2_{\mathcal{H}}}{\|u_0\|^2_{L^{2,1}}}\,t\right)\right)^{-1}.$$

This is a decay rate of $O((\log t)^{-1})$ which is very far from optimal (and in particular the dependence on the dimension d is very weak). We obtain the optimal rate in the next example, using L^1 norms.

Example 4.2 (Applying Theorems 1.9 & 1.7). We consider again $H = -\Delta$ acting in $\mathcal{H} = L^2(\mathbb{R}^d)$. Now we let

$$\mathcal{Y} = \mathcal{X} = L^1(\mathbb{R}^d).$$

Recalling (4.2), we find that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=\lambda_0}(E(\lambda)u, v)_{\mathcal{H}} = \frac{1}{2\sqrt{\lambda_0}}\sqrt{\lambda_0}^{\,d-1}\int_{\mathbb{S}^{d-1}}\widehat{u}(\sqrt{\lambda_0}\sigma)\overline{\widehat{v}(\sqrt{\lambda_0}\sigma)}\,\mathrm{d}\sigma$$

where $\mathrm{d}\sigma$ is the uniform measure on the unit sphere. Using the fact that the Fourier transform of an L^1 function is bounded and continuous, we get

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=\lambda_0}(E(\lambda)u, v)_{\mathcal{H}}\right| &\le \frac{1}{2}\left|\mathbb{S}^{d-1}\right|\lambda_0^{\frac{d}{2}-1}\|\widehat{u}\|_{L^\infty}\|\widehat{v}\|_{L^\infty}\\ &\le \frac{1}{2}\left|\mathbb{S}^{d-1}\right|\lambda_0^{\frac{d}{2}-1}\|u\|_{L^1}\|v\|_{L^1}.\end{aligned}$$

Hence Theorem 1.9b is applicable, with

$$\psi(\lambda) = \frac{1}{2}\left|\mathbb{S}^{d-1}\right|\lambda^{\frac{d}{2}-1}. \tag{4.5}$$

Now we restrict to $\mathcal{X}\cap\mathcal{H}$. Since $\psi(\lambda)$ is of the form $C\lambda^\alpha$, we can apply Theorems 1.9c and 1.7 with $\alpha = \frac{d}{2}-1$ and $\Phi(u) = \|u\|^2_{L^1(\mathbb{R}^d)}$. Using the fact that the L^1 norm of solutions to the heat equation decays monotonically (see Lemma 5.2 below), we have $C_2 = 0$, where C_2 is the constant appearing in (1.3). Using the expression (3.1) for C_3, with $C_1 = \frac{1}{2}\left|\mathbb{S}^{d-1}\right|$ the constant coming from the estimate (1.2), the bound (1.4) becomes

$$\begin{aligned}
\mathrm{Var}(u(t,\cdot)) &\le \left(\mathrm{Var}(u_0)^{\frac{-1}{1+\alpha}} + 2C\,(\alpha+1)^{\frac{-\alpha}{\alpha+1}}\,C_1^{\frac{-1}{\alpha+1}}\int_0^t \|u_0\|_{L^1}^{\frac{-2}{1+\alpha}}\,\mathrm{d}s\right)^{-(1+\alpha)}\\
&= \left(\mathrm{Var}(u_0)^{-\frac{2}{d}} + 2C\left(\frac{d}{2}\right)^{\frac{2}{d}-1}\left|\mathbb{S}^{d-1}\right|^{-\frac{2}{d}}\|u_0\|_{L^1}^{-\frac{4}{d}}t\right)^{-\frac{d}{2}}\\
&\le \tilde{C}\left(\frac{d}{2}\right)^{\frac{d}{2}-1}|\mathbb{S}^{d-1}|\|u_0\|^2_{L^1}t^{-\frac{d}{2}}
\end{aligned}$$

and we conclude that for every $u_0 \in \mathcal{X}\cap\mathcal{H} = L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$

$$\|u(t,\cdot)\|^2_{\mathcal{H}} = \mathrm{Var}(u(t,\cdot)) = O(t^{-\frac{d}{2}}), \qquad \text{as } t\to+\infty,$$

which is the optimal rate. This can be extended to any $u_0 \in \mathcal{X} = L^1(\mathbb{R}^d)$ by density.

Remark 4.3 (The Nash Inequality). We point out that the functional inequality we obtain here, using Theorem 1.9b with $\mathcal{X} = L^1(\mathbb{R}^d)$, is exactly Nash's inequality (2.4). Indeed, with $\alpha = \frac{d}{2}-1$ we have that $p = \frac{\alpha+2}{\alpha+1} = \frac{d+2}{d}$ so that the (Φ,p)-WPI in this case is

$$\|u\|^2_{L^2} \le C\mathcal{E}(u)^{1/p}\Phi(u)^{1-1/p} = C\mathcal{E}(u)^{\frac{d}{d+2}}\Phi(u)^{\frac{2}{d+2}} = C\left(\|\nabla u\|^2_{L^2}\right)^{\frac{d}{d+2}}\left(\|u\|^2_{L^1}\right)^{\frac{2}{d+2}}.$$

This demonstrates how our methodology gives a general framework for many known important inequalities. We shall see this again in the next example as well, see Remark 4.5.

4.2. **The fractional Laplacian.** For our final example, we turn to the fractional Laplacian – a nonlocal operator which has received significant interest in recent years.

Example 4.4. Consider now the operator $H = (-\Delta)^p$, $p\in(0,1)$, acting in $\mathcal{H} = L^2(\mathbb{R}^d)$. In analogy to (4.1), the resolution of the identity of H satisfies

$$(E(\lambda)u,v)_{\mathcal{H}} = \int_{|\xi|^{2p}\le\lambda}\widehat{u}(\xi)\overline{\widehat{v}(\xi)}\,\mathrm{d}\xi.$$

Following the arguments that lead to (4.5), and letting $\mathcal{Y} = \mathcal{X} = L^1(\mathbb{R}^d)$ as before, we now obtain

$$\psi(\lambda) = \frac{1}{2p}\left|\mathbb{S}^{d-1}\right|\lambda^{\frac{d}{2p}-1}.$$

Now, from [6] we know that $\|u(t,\cdot)\|_{L^1} \leq \|u_0\|_{L^1}$ and as such, much like the previous example, we conclude that

$$\begin{aligned}\mathrm{Var}(u(t,\cdot)) &\leq \left(\mathrm{Var}(u_0)^{-\frac{2p}{d}} + C\frac{2p}{d}\left(\frac{d}{2p|\mathbb{S}^{d-1}|}\right)^{\frac{2p}{d}} \|u_0\|_{L^1}^{-\frac{4p}{d}} t\right)^{-\frac{d}{2p}} \\ &\leq C\left(\frac{d}{2p}\right)^{\frac{d}{2p}-1} |\mathbb{S}^{d-1}| \|u_0\|_{L^1}^2 t^{-\frac{d}{2p}}\end{aligned}$$

and hence the asymptotic decay rate is

$$\mathrm{Var}(u(t,\cdot)) = O(t^{-\frac{d}{2p}}), \qquad \text{as } t \to \infty.$$

Remark 4.5. The functional inequality we obtain in this example is the Nash-type inequality for the fractional Laplacian:

$$\|u\|_{L^2}^2 \leq C\left(\| (-\Delta)^{\frac{p}{2}} u\|_{L^2}^2\right)^{\frac{d}{d+2p}} \left(\|u\|_{L^1}^2\right)^{\frac{2p}{d+2p}}.$$

5. Additional results and lemmas

In this last section we prove two auxiliary lemmas that were used above.

Lemma 5.1 (**$L^{2,s}$ norm of solutions to the heat equation).** *Let $u(t,x)$ be a solution to the heat equation in $\mathbb{R}^d$ with initial condition $u(0,x) = u_0(x)$. Let $s \in (0,1]$. Then*

$$\|u(t,\cdot)\|_{L^{2,s}}^2 \leq \|u_0\|_{L^{2,s}}^2 + 2s(2s-2+d)\|u_0\|_{L^2}^2 t.$$

Proof. This lemma can be proved either directly or using Fourier methods. Here we provide the direct proof. Differentiating the expression $\|u(t,\cdot)\|_{L^{2,s}}^2 = \int_{\mathbb{R}^d} u(t,x)^2(1+|x|^2)^s \,\mathrm{d}x$ in time (using the fact that solutions to the heat equation are smooth) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t,\cdot)\|_{L^{2,s}}^2 = 2\int_{\mathbb{R}^d} u(t,x)\Delta u(t,x)(1+|x|^2)^s \,\mathrm{d}x.$$

Integration by parts yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t,\cdot)\|_{L^{2,s}}^2 &= -2\int_{\mathbb{R}^d} \nabla u(t,x)\cdot\left(\nabla u(t,x)(1+|x|^2)^s + 2s(1+|x|^2)^{s-1}xu(t,x)\right)\mathrm{d}x \\ &\leq -4s\underbrace{\int_{\mathbb{R}^d} \nabla u(t,x)\cdot(1+|x|^2)^{s-1}xu(t,x)\,\mathrm{d}x}_{I(t)}.\end{aligned}$$

Integrating $I(t)$ by parts again gives us

$$\begin{aligned}I(t) &= \int_{\mathbb{R}^d}\sum_{j=1}^d x_j\nabla_j u(t,x)(1+|x|^2)^{s-1}u(t,x)\,\mathrm{d}x \\ &= -d\int_{\mathbb{R}^d} u(t,x)^2(1+|x|^2)^{s-1}\mathrm{d}x - I(t) - 2(s-1)\int_{\mathbb{R}^d} u(t,x)^2|x|^2(1+|x|^2)^{s-2}\,\mathrm{d}x\end{aligned}$$

so that

$$I(t) = -\left(s-1+\frac{d}{2}\right)\|u(t,\cdot)\|_{L^{2,s-1}(\mathbb{R}^d)}^2 + (s-1)\,\|u(t,\cdot)\|_{L^{2,s-2}(\mathbb{R}^d)}^2.$$

Returning to $\frac{\mathrm{d}}{\mathrm{d}t}\|u(t,\cdot)\|^2_{L^{2,s}}$, using the fact that $s>0$ but $s-1\leq 0$, we conclude that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t,\cdot)\|^2_{L^{2,s}} &\leq 4s\left(s-1+\frac{d}{2}\right)\|u(t,\cdot)\|^2_{L^{2,s-1}} - 4s\,(s-1)\,\|u(t,\cdot)\|^2_{L^{2,s-2}} \\ &\leq 2s\,(2s-2+d)\,\|u(t,\cdot)\|^2_{L^2}\end{aligned}$$

which completes the proof. □

Lemma 5.2 (L^1 norm of solutions to the heat equation). *Let $u(t,x)$ be a solution to the heat equation in $\mathbb{R}^d$ with initial condition $u(0,x)=u_0(x)$. Then*

$$\|u(t,\cdot)\|_{L^1}\leq\|u_0\|_{L^1}.$$

Proof. This follows from the invariance of the measure and the order preserving property of the heat process, in light of the Crandall-Tartar Lemma [8]. □

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