Dispersive Equations

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Preface

These notes were written to accompany a course on dispersive equations taught jointly by J. Ben-Artzi and A. Shao during the Autumn term of 2015 at the Taught Course Centre, to PhD students at the universities of Bath, Bristol, Oxford and Warwick as well as Imperial College London.

The general topic of *Dispersive Equations* is meant to represent our two research interests, *Kinetic Theory* (J. Ben-Artzi) and *Nonlinear Wave Equations* (A. Shao). While the latter is a classic "dispersive" topic, we include the former here as well due to the dispersive nature of the Vlasov equation, which is a transport equation in phase space.

This course is 16 hours in total which leaves merely 8 hours for each topic, including introduction. The introduction includes a crash course on basic methods in ordinary and partial differential equations, including the Cauchy problem, existence and uniqueness of solutions, the method of characteristics, Picard iteration, the Fourier transform and Sobolev spaces.

These notes are by no means meant to be complete and should only be treated as an assertional reference. Please let us know if you find any typos or mistakes. The main books we used when preparing the course were:

- Introductory materials
  - L. C. Evans, *Partial Differential Equations (second edition)*, AMS, 2010
  - T. Tao, *Nonlinear Dispersive Equations: Local and Global Analysis*, CBMS-AMS, 2006
  - H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, 2011
- Kinetic Theory
  - F. Golse, *Lecture Notes (École polytechnique)*, `www.math.polytechnique.fr/~golse/M2/PolyKinetic.pdf`
  - R. T. Glassey, *The Cauchy Problem in Kinetic Theory*, SIAM, 1996
  - C. Mouhot, *Lecture Notes for Kinetic Theory Course (Cambridge)*, `https://cmouhot.wordpress.com/`

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– G. Rein, *Collisionless Kinetic Equations from Astrophysics – The Vlasov-Poisson System*, in Handbook of Differential Equations: Evolutionary Equations Volume 3, 2011

- Nonlinear Wave Equations
  - S. Selberg, *Lecture Notes for Nonlinear Wave Equations (Johns Hopkins)*, `http://www.math.ntnu.no/~sselberg/`
  - C. Sogge, *Lectures on Nonlinear Wave Equations*, International Press, 2006
  - L. Hörmander, *Lectures on Nonlinear Hyperbolic Differential Equations*, Springer-Verlag, 1997

Contents

1 ODEs and Connections to Evolution Equations

The main purpose of these notes is the study of two classes of nonlinear partial differential equations (PDEs) arising from physics: *wave equations*, and *transport equations* arising from kinetic theory (e.g., the *Vlasov-Poisson* and *Vlasov-Maxwell* equations). Both of these are subclasses of *evolution equations*, that is, PDEs that model a system evolving with respect to a "time" parameter.

When solving such evolution equations, the appropriate formulation of the problem is usually as an *initial value*, or *Cauchy*, *problem.* More specifically, we are given certain *initial data*, representing the state of the system at some initial time. The goal, then, is to "predict the future", that is, to find the solution of the PDE, which represents the behaviour of the system at all times.[1] Three classical examples of (linear) evolution equations are:

1. *Heat equation*: $\partial_t u - \Delta_x u = 0$, where the unknown is a function $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$. Here, the initial data is the value of u at $t = 0$, i.e., $u|_{t=0} = u_0 : \mathbb{R}^n \to \mathbb{R}$.

2. *Schrödinger equation*: $i\partial_t u + \Delta_x u = 0$, where the unknown is $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$. The initial data is again the value of u at $t = 0$, i.e., $u|_{t=0} = u_0 : \mathbb{R}^n \to \mathbb{C}$.

3. *Wave equation*: $-\partial_t^2 u + \Delta_x u = 0$, where the unknown is $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$. Here, we require initial values for both u and $\partial_t u$, i.e., $u|_{t=0} = u_0$ and $\partial_t u|_{t=0} = u_1$.

In all three examples, there is one "time" dimension, denoted by $t \in \mathbb{R}$, and n "space" dimensions, denoted by $x \in \mathbb{R}^n$. Moreover, Δ_x denotes the Laplacian in the spatial variables,

$$\Delta_x := \sum_{k=1}^{n} \partial^2_{x^k}.$$

For various technical reasons, the study of evolution equations can become quite complicated. Thus, it is beneficial to first look at some "model problems", which are technically simpler than our actual equations of interest, but still demonstrate many of the same fundamental features. A particularly useful model setting to consider is the theory of (first-order) ordinary differential equations (ODEs). The advantage of this is twofold: not only can many phenomena in evolutionary equations be demonstrated in the ODE setting, but also most readers will already have had some familiarity with ODEs.

Thus, in this section, we discuss various key aspects in the study of ODEs, and we highlight how these aspects are connected to the study of evolutionary PDE.[2]

1.1 Existence of Solutions

Throughout most of the upcoming discussion, we will consider the initial value problem for the following system of ODEs:

$$x' = y(t, x), \qquad x(t_0) = x_0. \tag{1.1}$$

Here, t is the independent variable, x is an $\mathbb{R}^n$-valued function to be solved, and the given function $y : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ defines the differential equation.[3] Recall the following:

[1] When solving on a finite domain, one requires in addition appropriate boundary conditions.
[2] A large portion of this chapter was inspired by the first chapter of T. Tao's monograph, [Tao2006].
[3] One can also restrict the domain of y to an open subset of $\mathbb{R} \times \mathbb{R}^n$, but we avoid this for simplicity.

Definition 1.1. *A differentiable function $x : I \to \mathbb{R}^n$, where I is a subinterval of $\mathbb{R}$ containing t_0, is a* solution *of* (1.1) *iff $x(t_0) = x_0$, and $x'(t) = y(t, x(t))$ for all $t \in I$.*

Such a solution x is called global *iff $I = \mathbb{R}$, and* local *otherwise.*

Abstractly, we can think of this as an evolution problem, with t in (1.1) functioning as the "time" parameter. A solution x of (1.1) can then be seen as a curve in the finite-dimensional space $\mathbb{R}^n$, parametrised by this time.

This perspective is also pertinent to evolution equations. For the sake of discussion, consider the ($(n+1)$-dimensional free) *Schrödinger equation*

$$i\partial_t u + \Delta_x u = 0 \tag{1.2}$$

where $u = u(t, x)$ is a complex-valued function of both time $t \in \mathbb{R}$ and space $x \in \mathbb{R}^n$. While the most apparent definition of a solution of (1.2) is as a sufficiently differentiable map $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$, one could alternatively think of u as mapping each time t to a function $u(t)$ of n space variables. In other words, analogous to the ODE situation, one can think of a solution $t \mapsto u(t)$ as a curve in some *infinite-dimensional* space H of functions $\mathbb{R}^n \to \mathbb{C}$.

Remark 1.2. In contrast to the finite-dimensional ODE setting, where $\mathbb{R}^n$ is essentially the only appropriate space to consider, there are different possibilities one can potentially take for the infinite-dimensional space H. The choice of an appropriate H in which solutions live is one of the many challenges in solving and understanding solutions of PDEs.[4]

As we shall see below, this viewpoint of evolutionary PDEs as ODEs in an infinite-dimensional space will prove to be immediately useful. For instance, we can solve many such nonlinear PDEs using essentially the same techniques (Picard iteration, contraction mapping theorem) as for ODEs; we briefly review this existence theory in this subsection. In the remaining subsections, we discuss several other important concepts in ODEs that have direct analogues in evolutionary PDEs; examples include unconditional uniqueness arguments, Duhamel's principle, and "bootstrap" arguments for treating nonlinear terms.

The first crucial ingredient in the existence theory for ODEs is expressing (1.1) as an equivalent *integral equation.* Formally, by integrating (1.1) in t, we obtain the relation

$$x(t) = x(t_0) + \int_{t_0}^{t} y(s, x(s)) ds. \tag{1.3}$$

Thus, in order to solve (1.1), it suffices to solve (1.3) instead.

Remark 1.3. One technical point to note is that one requires x to be differentiable to make sense of (1.1), while no such requirement is required for (1.3). However, for any x that satisfying the integral equation, the right-hand side of (1.3) automatically implies that x is differentiable. Thus, (1.1) and (1.3) are equivalent conditions.

On the other hand, for evolutionary PDEs, the analogous differential and integral equations will no longer be equivalent; in particular, the latter equation is often a strictly weaker requirement than the former. As a result, one must distinguish between solutions to the differential and integral equations, i.e., *classical* and *strong* solutions, respectively.

Next, we define the map Φ as follows: for an $\mathbb{R}^n$-valued curve x, we let $\Phi(x)$ be the $\mathbb{R}^n$-valued curve defined by the right-hand side of (1.3). From this viewpoint, solving (1.3)

[4]Common examples of H include $L^p(\mathbb{R}^n)$, as well as various Sobolev and Hölder spaces.

is equivalent to finding a *fixed point* of Φ, i.e., x such that

$$\Phi(x) = x. \tag{1.4}$$

To find such a fixed point, we resort to the following abstract theorem:

Theorem 1.4 (Contraction mapping theorem). *Let (X, d) be a nonempty complete metric space, and let $\Phi : X \to X$ be a* contraction, *i.e., there is some $c \in (0, 1)$ such that*

$$d(\Phi(x), \Phi(y)) \leq c \cdot d(x, y), \qquad x, y \in X. \tag{1.5}$$

Then, Φ has a unique fixed point in X.

Sketch of proof. Let x_0 be any element of X, and define the sequence (x_n) inductively by $x_{n+1} := \Phi(x_n)$. The contraction property (1.5) implies that (x_n) is a Cauchy sequence and hence has a limit x_∞. Since (1.5) also implies Φ is continuous, then

$$x_\infty = \lim_n x_{n+1} = \lim_n \Phi(x_n) = \Phi(x_\infty),$$

i.e., x_∞ is a fixed point of Φ.

For uniqueness, suppose $x, y \in X$ are fixed points of Φ. Then, (1.5) implies

$$d(x, y) = d(\Phi(x), \Phi(y)) \leq c \cdot d(x, y).$$

Since $c < 1$, it follows that $d(x, y) = 0$. $\square$

The strategy to solving (1.4) is to show that this Φ is indeed a contraction on the appropriate space. Then, Theorem 1.4 yields a fixed point of Φ, which is the solution of (1.1). The precise result is stated in the subsequent theorem:

Theorem 1.5 (Existence of solutions). *Consider the initial value problem* (1.1)*, and let $\Omega_{\mathcal{T},\mathcal{R}}$, where $\mathcal{T}, \mathcal{R} > 0$, be the following closed neighbourhood:*

$$\Omega_{\mathcal{T},\mathcal{R}} := \{(t, x) \mid |t - t_0| \leq \mathcal{T},\ |x| \leq 2\mathcal{R}\}.$$

Suppose also that the function y in (1.1) *satisfies the following:*

- *y is uniformly bounded on $\Omega_{\mathcal{T},\mathcal{R}}$—there exists $M > 0$ such that*

$$|y(t, x)| \leq M, \qquad (t, x) \in \Omega_{\mathcal{T},\mathcal{R}}. \tag{1.6}$$

- *y satisfies the following Lipschitz property on $\Omega_{\mathcal{T},\mathcal{R}}$—there exists $L > 0$ such that*

$$|y(t, x_1) - y(t, x_2)| \leq L|x_1 - x_2|, \qquad (t, x_1), (t, x_2) \in \Omega_{\mathcal{T},\mathcal{R}}. \tag{1.7}$$

Then, given any $x_0 \in \mathbb{R}^n$ with $|x_0| \leq \mathcal{R}$, the initial value problem (1.1) *has a solution $x : [t_0 - T, t_0 + T] \to \mathbb{R}^n$, for some $T \in (0, \mathcal{T})$ depending on $\mathcal{R}$, M, and L.*

Remark 1.6. Note the time T of existence of the solution in Theorem 1.5 depends on the size $|x_0|$ of the initial data. This is manifested in the dependence of T on $\mathcal{R}$.

Proof. Let $T > 0$, whose value will be determined later, and set $I = [t_0 - T, t_0 + T]$. Consider first the space $C(I; \mathbb{R}^n)$ of all continuous functions $x : I \to \mathbb{R}^n$, along with the sup-norm

$$\|x\| := \sup_{t \in I} |x(t)|.$$

In particular, the above forms a Banach space. Consider next the closed ball

$$X := \{x \in C(I;\mathbb{R}^n) \mid \|x\| \le 2\mathcal{R}\}.$$

Then, X, being closed, forms a complete metric space along with the induced metric

$$d(x_1, x_2) = \|x_2 - x_1\|.$$

Now, we define Φ to be the map arising from the integral equation (1.3):

$$\Phi : C(I;\mathbb{R}^n) \to C(I;\mathbb{R}^n), \qquad [\Phi(x)](t) := x_0 + \int_{t_0}^{t} y(s, x(s)) ds.$$

Then, in order to generate a fixed point of Φ, and hence a solution of (1.1), we must show:

1. Φ maps X into X.
2. $\Phi : X \to X$ is a contraction.

For the first point, given $x \in X$, we see from the definition of Φ that

$$\|\Phi(x)\| \le |x_0| + \int_I |y(s, x(s))| ds \le \mathcal{R} + 2MT.$$

Then, for small enough T (depending on M), we have that $\|\Phi(x)\| \le 2\mathcal{R}$, i.e., $\Phi(x) \in X$. This proves that Φ indeed maps X into X. For the second point, given $x_1, x_2 \in X$,

$$\begin{aligned}\|\Phi(x_2) - \Phi(x_1)\| &\le \int_I |y(s, x_2(s)) - y(s, x_1(s))| ds \\ &\le L \int_I |x_2(s) - x_1(s)| ds \\ &\le 2TL \|x_2 - x_1\|.\end{aligned}$$

Taking T small enough so that $TL < \frac{1}{4}$, then Φ is a contraction, completing the proof. □

Remark 1.7. Note that the proof the contraction mapping theorem is both relatively simple and more intuitive than the theorem statement itself. Thus, a more direct approach to proving Theorem 1.5 can be achieved by running the proof of the contraction mapping theorem directly for the Φ in (1.4), rather than applying the abstract Theorem 1.4. This process, which amounts to constructing an infinite sequence of closer and closer approximations to the desired solution of (1.1), is known as *Picard iteration*.

In the PDE setting, for instance for a nonlinear Schrödinger or wave equation such as

$$i\partial_t u + \Delta_x u = \pm |u|^2 u, \qquad -\partial_t^2 u + \Delta_x u = \pm |u|^2 u,$$

the main ideas for proving existence of solutions are largely the same as in Theorem 1.5. Indeed, the first step is to write the above differential equations as integral equations, which can again be cast as a fixed point problem. The goal then is once again to show that the map Φ obtained from the integral equation is a contraction.

In this setting, Φ now acts on a space of curves mapping into an infinite-dimensional space H of functions on $\mathbb{R}^n$. One of the main challenges, then, is to choose an appropriate space H with an appropriate norm so that Φ can be shown to be contraction. What H can be will of course depend very crucially on the PDE under consideration.

1.2 Uniqueness of Solutions

If an ODE models some system in the real world, then solving an initial value problem roughly amounts to "predicting the future", given the initial state of the system. However, this is not entirely accurate—although Theorem 1.5 states that solutions *exist* (for nice enough ODE), we have not discussed whether such solutions are *unique*. If multiple solutions were to exist, then the problem—at least in the way that we have stated it—is not deterministic, and we cannot "predict the future", so to speak.

The attentive reader may have noticed that the contraction mapping theorem, applied in the proof Theorem 1.5, guaranteed the existence of a unique fixed point. While this seems at first glance to take care of the uniqueness problem, there is, unfortunately, a hole in this chain of reasoning. Indeed, *the contraction mapping theorem only guarantees that the fixed point, or solution, is unique on the space* X, which is a closed ball of the larger Banach space $C([t_0-T, t_0+T]; \mathbb{R}^n)$. In particular, this does not rule out a different solution $z \in C([t_0 - T, t_0 + T]; \mathbb{R}^n)$ that grows large, i.e., that does not lie within the ball X.

The uniqueness obtained from the contraction mapping theorem is commonly referred to as *conditional uniqueness*—in the context of Theorem 1.5, we know that a solution x of (1.1) is unique as long as $\|x\| \leq 2\mathcal{R}$, that is, as x does not grow too large. The statement that we want, however, is *unconditional uniqueness*—that the solution x of (1.1) is in fact the only solution of all sizes, i.e., it is unique in all of $C([t_0 - T, t_0 + T]; \mathbb{R}^n)$.

Theorem 1.8 (Uniqueness of solutions). *Consider the initial value problem* (1.1). *Suppose also that y in* (1.1) *satisfies the following locally Lipschitz property: for each*

$$\Omega_{\mathcal{T},\mathcal{R}} := \{(t,x) \in \mathbb{R} \times \mathbb{R}^n \mid |t - t_0| \leq \mathcal{T},\ |x| \leq 2\mathcal{R}\}, \qquad \mathcal{T}, \mathcal{R} > 0,$$

there exists some $L_{\mathcal{T},\mathcal{R}} > 0$ such that

$$|y(t, x_1) - y(t, x_2)| \leq L_{\mathcal{T},\mathcal{R}} |x_1 - x_2|, \qquad (t, x_1), (t, x_2) \in \Omega_{\mathcal{T},\mathcal{R}}. \tag{1.8}$$

Let $T > 0$, and assume $x_1, x_2 \in C([t_0 - T, t_0 + T]; \mathbb{R}^n)$ are two solutions to (1.1). *Then, $x_1(t) = x_2(t)$ for all $t \in [t_0 - T, t_0 + T]$.*

Remark 1.9. Notice the Lipschitz condition in Theorem 1.8 is analogous to that in Theorem 1.5. The slight difference arises from the fact that here, one must assume y remains "nice", in the sense of Theorem 1.5, no matter how large the solutions x_i may get.

To obtain such an unconditional uniqueness statement, one generally requires another argument in addition to the proof of existence. For the most part, such uniqueness arguments are relatively simple, in that they use similar tools as the existence arguments.[5]

Theorem 1.8 can be proved in multiple ways. One of the most straightforward is via a linear estimate known as *Gronwall's inequality*. In fact, Gronwall's inequality is also an incredibly useful tool in the study of PDEs, for similar unconditional uniqueness arguments as well as other applications. The main idea derives from the method of integrating factors used in basic ODE theory, along with the observation that they are applicable to *inequalities* as well as to equations. We present some special cases here:

Theorem 1.10 (Gronwall inequality). *Let $x, C : [0, T] \to [0, \infty)$.*

[5] In some PDEs settings, unconditional uniqueness arguments can sometimes be much more nontrivial. Indeed, many such statements have only recently been proved, or even remain open.

1. Differential version: *Assume x is differentiable, and x satisfies*

$$x'(t) \leq C(t) \cdot x(t), \qquad t \in [0, T]. \tag{1.9}$$

Then, x also satisfies

$$x(t) \leq x(0) \cdot \exp \left[\int_0^t C(s) ds \right]. \tag{1.10}$$

2. Integral version: *Assume x is continuous, and x satisfies*

$$x(t) \leq A(t) + \int_0^t C(s) x(s) ds, \qquad t \in [0, T]. \tag{1.11}$$

for some nondecreasing $A : [0, T] \to [0, \infty)$. Then, x also satisfies

$$x(t) \leq A(t) \cdot \exp \left[\int_0^t C(s) ds \right]. \tag{1.12}$$

Proof. For the differential version, we multiply (1.9) by $\exp[-\int_0^t C(s)ds]$, which yields

$$\frac{d}{dt}[e^{-\int_0^t C(s)ds} x(t)] \leq 0, \qquad t \in [0, T].$$

Integrating the above from 0 to t results in (1.10).

For the integral version, we define

$$z(t) := \exp \left[-\int_0^t C(s) ds \right] \int_0^t C(s) x(s) ds.$$

Differentiating z, we see that

$$\begin{aligned} z'(t) &= C(t) \exp \left[-\int_0^t C(s) ds \right] \left[x(t) - \int_0^t C(s) x(s) ds \right] \\ &\leq A(t) C(t) \exp \left[-\int_0^t C(s) ds \right]. \end{aligned}$$

Since $z(0) = 0$ and A is nondecreasing, we have

$$\begin{aligned} z(t) &\leq \int_0^t A(s) C(s) \exp \left[-\int_0^s C(r) dr \right] ds \\ &\leq A(t) \int_0^t C(s) \exp \left[-\int_0^s C(r) dr \right] ds \\ &= A(t) - A(t) \exp \left[-\int_0^t C(s) \right]. \end{aligned}$$

Finally, by (1.11), we obtain, as desired,

$$x(t) \leq A(t) + \exp \left[\int_0^t C(s) ds \right] \cdot z(t) \leq A(t) \cdot \exp \left[\int_0^t C(s) ds \right]. \qquad \square$$

We now apply Gronwall's inequality to prove Theorem 1.8.

Proof of Theorem 1.8. Let us assume for convenience that $t_0 = 0$. Since both x_1 and x_2 are bounded on $[0, 0 + T]$, then (1.8) implies for any $t \in [0, T]$ that

$$|x_1(t) - x_2(t)| \leq \int_0^t |y(s, x_1(s)) - y(s, x_2(s))| ds \leq L \int_0^t |x_1(s) - x_2(s)| ds.$$

Applying the integral Gronwall's inequality, (1.12), with $x := |x_1 - x_2|$ and $A :\equiv 0$ yields

$$|x_1(t) - x_2(t)| \leq 0 \cdot \exp t = 0, \qquad 0 \leq t \leq T.$$

An analogous argument also shows that $x_1 = x_2$ on $[-T, 0]$. $\square$

Now that both existence and uniqueness have been established, one can discuss what is the largest time interval that a solution exists. The basic argument is as follows. First, Theorem 1.5 furnishes a (unique) solution x on $[t_0 - T, t_0 + T]$. Next, one can solve the same ODE, but with initial data given by x at times $t_0 \pm T$. Theorem 1.8 guarantees that these new solutions coincide with x wherever both are defined.

Thus, "patching" together these solutions yields a new solution x of (1.1) on a larger interval $[t_0 - T_1, t_0 + T_2]$. By iterating this process indefinitely, we obtain:

Corollary 1.11 (Maximal solutions). *Consider the initial value problem* (1.1)*, and suppose y satisfies the same hypotheses as in Theorem 1.8. Then, there exists a "maximal" interval (T_-, T_+) containing t_0, where $-\infty \leq T_- < T_+ \leq \infty$, such that:*

- *There exists a solution $x : (T_-, T_+) \to \mathbb{R}^n$ to* (1.1)*.*
- *x is the only solution to* (1.1) *on the interval (T_-, T_+).*
- *If $\tilde{x} : I \to \mathbb{R}^n$ is another solution of* (1.1)*, then $I \subseteq (T_-, T_+)$.*

We refer to x in Corollary 1.11 as the *maximal solution* of (1.1). In fact, one can say a bit more about the behaviour of the maximal solutions at the boundaries $T_\pm$.

Corollary 1.12 (Breakdown criterion). *Consider the initial value problem* (1.1)*, and suppose y satisfies the same hypotheses as in Theorem 1.8. Let $x : (T_-, T_+) \to \mathbb{R}^n$ be the maximal solution of* (1.1)*. Then, if $T_+ < \infty$, then*

$$\limsup_{t \nearrow T_+} |x(t)| = \infty. \tag{1.13}$$

An analogous result holds at T_-.

Proof. If (1.13) fails to hold, then $|x|$ is uniformly bounded near T_+. Since the time of existence in Theorem 1.5 depends only on the size of the initial data, then one can solve the ODE with initial data at a time $T_+ - \varepsilon$ for arbitrarily small $\varepsilon > 0$, but always for a fixed amount of time. By uniqueness, this allows us to push the solution past time T_+, contradicting that x is the maximal solution. ☐

Finally, we remark that in the setting of nonlinear evolution equations, one can often use Gronwall's inequality in a similar fashion as in Theorem 1.8 in order to show unconditional uniqueness. Then, the same argument behind Corollary 1.11 yields an analogous notion of maximal solutions for PDEs. Furthermore, for a large subclass of such PDEs—known as "subcritical"—one can establish that the time of existence of solutions depend only on the size of the initial data.[6] As a result, an analogue of Corollary 1.12 holds for these PDEs.

Later, we will formally demonstrate all these points for nonlinear wave equations.

1.3 Duhamel's Principle

We turn our attention to linear systems of ODE. Consider first the homogeneous case,

$$x' = y(t, x) = Ax, \qquad x(t_0) = x_0 \in \mathbb{R}^n, \tag{1.14}$$

where A is a constant $n \times n$ matrix.

[6] Here, "size" is measured by an appropriate norm on the infinite-dimensional space H of functions.

When $n = 1$, then A can be expressed as a constant $\lambda \in \mathbb{R}$. The resulting system $x' = \lambda x$ then has an explicit solution, $x(t) = e^{(t-t_0)\lambda} x_0$, which, depending on the sign of λ, either grows exponentially, decays exponentially, or stays constant.

In higher dimensions, one can still write the solution in the same manner

$$x(t) = e^{tA} x_0, \tag{1.15}$$

where e^{tA} is the matrix exponential, which, for instance, can be defined via a Taylor series:

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}.$$

In (1.15), the matrix e^{tA} is multiplied to x_0, represented as a column vector. The operator $x_0 \mapsto e^{tA} x_0$ is often called the *linear propagator* of the equation $x' = Ax$.

To better understand the solution $e^{tA} x_0$, one generally works with a basis of $\mathbb{R}^n$ which diagonalises A (or at least achieves Jordan normal form). In particular, by considering the eigenvalues of A, one can separate the solution curve x into individual directions which grow exponentially, decay exponentially, or oscillate.

Remark 1.13. In the more general case in which A also depends on t, one can still define linear propagators, though they may not be representable as matrix exponentials.

This reasoning extends almost directly to the PDE setting as well. Consider for instance the initial value problem for the free linear Schr̈dinger equation, which can be written as

$$\partial_t u = i\Delta_x u, \qquad u|_{t=0} = u_0 : \mathbb{R}^n \to \mathbb{C}.$$

Formally at least, thinking of $i\Delta_x$ as the (constant in time) linear operator on our infinite-dimensional space of functions, we can write the solution of the initial value problem in terms of a linear propagator,[7]

$$u(t, x) = (e^{it\Delta_x} u_0)(x).$$

Similarly, for a free linear transport equation,

$$\partial_t u + v \cdot \nabla_x u = 0, \qquad u|_{t=0} = u_0 : \mathbb{R}^n \to \mathbb{R},$$

where $v \in \mathbb{R}^n$, one can write a similar linear propagator,

$$u = e^{-t(v \cdot \nabla_x)} u_0,$$

although in this case the solution has a simpler explicit formula,

$$u(t, x) = u_0(x - tv).$$

Later, we will study the propagator for the wave equation in much greater detail.

Next, we consider inhomogeneous linear systems containing a forcing term,

$$x' = Ax + F, \qquad x(t_0) = x_0, \tag{1.16}$$

[7] There are multiple ways to make precise sense of the operators $e^{it\Delta_x}$. For example, this can be done using Fourier transforms, or through techniques from spectral theory.

where A is as before, and $F : \mathbb{R} \to \mathbb{R}^n$. To solve this system, one can apply the matrix analogue of the method of integrating factors from ODE theory. In particular, multiplying (1.16) by e^{-tA}, we can rewrite it as

$$(e^{-tA}x)' = e^{-tA}F.$$

Integrating the above with respect to t yields:

Proposition 1.14. *The solution to* (1.16) *is given by*

$$x(t) = e^{tA}x_0 + \int_{t_0}^{t} e^{(t-s)A}F(s)ds. \tag{1.17}$$

The first term $e^{tA}x_0$ in (1.17) is the solution to the homogeneous problem (1.14), while the other term represents the solution to the inhomogeneous equation with zero initial data. In the case that F is "small", one can think of (1.17) as the solution $e^{tA}x_0$ of the homogeneous problem plus a perturbative term. This should be contrasted with (1.3), which expresses the solution as a perturbation of the constant curve.

This viewpoint is especially pertinent for nonlinear equations. Consider the system

$$x' = -x + |x|x,$$

for (1.17) yields

$$x(t) = e^{-t}x_0 + \int_{t_0}^{t} e^{-(t-s)}[|x(s)|x(s)]ds.$$

Then, for small x or for small times $t - t_0$, the above indicates that x should behave like the linear equation, and that the nonlinear effects are perturbative.

Again, these ideas extend to PDE settings; the direct analogue of (1.17) in the PDE setting is often called *Duhamel's principle.* For example, a commonly studed family of nonlinear dispersive equations is the *nonlinear Schrödinger equation* (*NLS*), given by

$$i\partial_t u + \Delta_x u = \pm|u|^{p-1}u, \qquad p > 1.$$

Then, Duhamel's principle implies that

$$u(t) = e^{i(t-t_0)\Delta_x}u_0 \mp i\int_{t_0}^{t} e^{i(t-s)\Delta_x}[|u(s)|^{p-1}u(s)]ds.$$

In fact, in this case, the Picard iteration process is applied directly on the above formula, as it captures more effectively the qualitative properties of the solution.

As we will discuss in much more detail later, a similar Duhamel's formula exists for wave equations, and it has similar uses as for the NLS.

1.4 Continuity Arguments

The main part of these notes will deal with nonlinear PDE, for which solutions usually cannot be described by explicit equations. Thus, one must resort to other tools to capture various qualitative and quantative aspects of solutions.

One especially effective method is called the *continuity argument*, sometimes nicknamed "*bootstrapping*". The main step in this argument is to *assume what you want to prove* and then to *prove a strictly better version of what you assumed.* Until the process is properly explained, it seems suspiciously like circular reasoning. Moreover, because it is used so often in studying nonlinear evolution equations, continuity arguments often appear in research papers without comment or explanation, which can be confusing to many new readers.

Rather than discussing the most general possible result, let us consider as a somewhat general example the following "trivial" proposition:

Proposition 1.15. *Let $f : [0, T) \rightarrow [0, \infty)$ be continuous, where $0 < T \leq \infty$, and fix a constant $C > 0$. Suppose the following conditions hold:*

1. *$f(0) \leq C$.*
2. *If $f(t) \leq 4C$ for some $t > 0$, then in fact $f(t) \leq 2C$.*

Then, $f(t) \leq 4C$ (and hence $f(t) \leq 2C$) for all $t \in [0, T)$.

The intuition behind Proposition 1.15 is simple. Assumption (1) implies that f starts below $4C$. If f were to grow larger than $4C$, then it must cross the threshold $4C$ at some initial point t_0. But then, assumption (2) implies that f actually lies below $2C$, so that f could not have reached $4C$ at time t_0.

In applications of Proposition 1.15 (or some variant), the main problem is to show that assumption (2) holds. In other words, we assume what we want to prove ($f(t) \leq 4C$, called the *bootstrap assumption*), and we prove something strictly better ($f(t) \leq 2C$).

For completeness, let us give a more robust topological proof of Proposition 1.15:

Proof. Let $A := \{t \in [0, T) \mid f(s) \leq 4C \text{ for all } 0 \leq s \leq t\}$. Note that A is nonempty, since $0 \in A$, and that A is closed, since f is continuous. Now, if $t \in A$, then the second assumption implies $f(t) \leq 2C$, so that $t + \delta \in A$ for small enough $\delta > 0$. Thus, A is a nonempty, closed, and open subset of the connected set $[0, T)$, and hence $A = [0, T)$. □

In either the ODE or the PDE setting, one can think of $f(t)$ as representing the some notion of "size" of the solution up to time t. Of course, in general, one cannot compute explicitly how large $f(t)$ is. However, in order to better understand the behaviour of, or to further extend (say, using Corollary 1.12) the lifespan of, solutions, one often wishes to prove bounds on $f(t)$.[8] Continuity arguments, for instance via Proposition 1.15, provide a method for achieving precisely this goal, without requiring explicit formulas for the solution.

To see bootstrapping in action, let us consider two (ODE) examples:

Example 1.16. Let $n = 1$, and consider the nonlinear system

$$x'(t) = \frac{|x(t)|^2}{1 + t^2}, \qquad x(0) = x_0 \in \mathbb{R}. \tag{1.18}$$

We wish to show the following: *If $|x_0|$ is sufficiently small, then the solution to (1.18) is global, i.e., $x(t)$ is defined for all $t \in \mathbb{R}$. Furthermore, x is everywhere uniformly small.*

By the breakdown criterion, Corollary 1.12, it suffices to prove the appropriate uniform bound for x, since this implies x can be further extended.[9] For this, let

$$f(t) = \sup_{0 \leq s \leq t} |x(s)|.$$

[8] In fact, in many PDE settings, estimates for this $f(t)$ are often essential to *solving* the equation itself.

[9] More precisely, one lets (T_-, T_+) be the maximal interval of definition for x, and one applies the continuity argument to show that x is small on this domain. Corollary 1.12 then implies that $T_\pm = \pm\infty$.

Moreover, let $\varepsilon > 0$ be a small constant, to be determined later, and suppose $|x_0| \leq \varepsilon$.

For the continuity argument, let us impose the bootstrap assumption

$$f(T) = \sup_{0 \leq t \leq T} |x(t)| \leq 4\varepsilon, \qquad T > 0.$$

Then, for any $0 \leq t \leq T$, we have, from (1.3) and the bootstrap assumption,

$$|x(t)| \leq |x_0| + \int_0^t \frac{|x(s)|^2}{1+s^2} ds \leq \varepsilon + 16\varepsilon^2 \int_0^t \frac{1}{1+s^2} ds.$$

Since $t \mapsto (1+t^2)^{-1}$ is integrable on $[0, \infty)$, then

$$|x(t)| \leq \varepsilon + C\varepsilon^2,$$

and as long as ε is sufficiently small (with respect to C), we have $|x(t)| \leq 2\varepsilon$, and hence $f(T) \leq 2\varepsilon$ (a strictly better result). Proposition 1.15 now implies $|x(t)| \leq 2\varepsilon$ for all $t \geq 0$.

An analogous argument proves the same bound for $t \leq 0$, hence x is small for all times.

Example 1.17. Let x be a solution of (1.1), and suppose x satisfies

$$|x| \leq A + B|x|^p, \qquad A, B > 0, \quad 0 < p < 1.$$

We wish to show that x *is uniformly bounded* and that x *is a global solution*.

Again, by Corollary 1.12, we need only show a uniform bound for x. Let

$$f(t) = \sup_{0 \leq s \leq t} |x(s)|,$$

and assume $f(T) \leq 4C$ for some sufficiently large C. Then, by our bootstrap assumption,

$$|x(t)| \leq A + B(4C)^p \leq A + 4^p BC^p, \qquad 0 \leq t \leq T$$

which for C large enough implies $f(T) \leq 2C$.

By Proposition 1.15, the above shows that x is uniformly bounded for all positive times. An analogous argument controls negative times as well.

The above examples give simple analogues for how continuity arguments can be used to study nonlinear PDE. Such arguments will be essential later once we consider global existence questions for nonlinear wave equations.

2 PDEs and Kinetic Theory

In the previous section some basic properties of PDEs (and how to solve them as infinite dimensional ODEs) were discussed along with some examples of prototypical evolution equations. In this section we shall discuss additional evolution equations, namely *transport equations*, methods for converting them into infinitely many ODEs, and we'll recall the definitions and properties of Fourier transforms and Sobolev spaces. We start with some motivation from physics.

2.1 Introduction to Kinetic Theory

Kinetic theory is concerned with the statistical description of many-particle systems (gasses, plasmas, galaxies) on a *mesoscopic scale.* [We typically denote the number of particles by N, thought to be large: $N \gg 1$]. This is an intermediate scale, complementing the two well-known scales:

- *Microscopic scale.* This is the naïve Newtonian description where one keeps track of each particle, and the evolution is due to all binary interactions. Already for $N = 3$ this description becomes highly nontrivial (the three body problem), and this is certainly true when considering realistic systems with $N \sim 10^{23}$ particles (Avogadro number).
- *Macroscopic scale.* This a description on the level of observables, taking into consideration conservation laws (mass, momentum, energy). One thus obtains a hydrodynamic description, with equations such as Euler's equations or the Navier-Stokes equations.

In contrast, in the **mesoscopic scale** the system is described by a *probability distribution function* (pdf), so that one does not care about each individual particle, but rather the *statistics* of all particles. More precisely, we introduce a function

$$f = f(t, x, p)$$

which measures the *density* of particles that at time $t \geq 0$ are located at the point $x \in \mathbb{R}^n$ and have momentum $p \in \mathbb{R}^n$ (n is the dimension, and is typically 1, 2 or 3). The pdf f is not an observable, but its moments in the momentum variable are. Let us mention the first two:

$$\begin{aligned}
\rho(t, x) &= \int_{\mathbb{R}^n} f(t, x, p)\,\mathrm{d}p = \text{particle density}, \\
u(t, x) &= \frac{1}{\rho(t, x)} \int_{\mathbb{R}^n} f(t, x, p) p\,\mathrm{d}p = \text{mean velocity}.
\end{aligned}$$

In these lectures we shall always take the spatial domain (i.e. the x variable) to be unbounded, although one could always restrict the domain and impose appropriate boundary conditions. The momentum domain is almost always taken to be unbounded, as there is no *a priori* reason why particle momenta must remain bounded.

2.1.1 The Vlasov equation

Informally speaking, Liouville's equation asserts that the full time derivative of f (also known as the *material derivative*) is nonzero only if there are collisions between particles. That is, as long as collisions are negligible, the pdf f is *transported* and the chain rule gives:

$$0 = \frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \dot{x} \cdot \nabla_x f + \dot{p} \cdot \nabla_p f$$

which is called the *Vlasov equation* and is written (applying Newton's second law $\dot{p} = \mathbf{F}$):

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + \mathbf{F}[f] \cdot \nabla_p f = 0 \tag{2.1}$$

where $v = v(p)$ is the velocity and $\mathbf{F}$ is a driving force and depends on the physics of the problem at hand. We use the convention that

$$\nabla_x = (\partial_{x_1}, \ldots, \partial_{x_n}) \quad \text{and} \quad \nabla_p = (\partial_{p_1}, \ldots, \partial_{p_n}).$$

To simplify presentation, we take all constants that typically appear (such as the mass of particles, or the speed of light which will be relevant in the relativistic case) to be 1. Hence we have

$$\begin{aligned} \text{classical case} \qquad & v = p \\ \text{relativistic case} \qquad & v = \frac{p}{\sqrt{1+|p|^2}}. \end{aligned}$$

2.1.2 The Vlasov-Poisson system

Consider first the case of a plasma, i.e. a gas of ions or electrons. In the electrostatic case where all interactions are due to the instantaneous electric fields generated by the particles, the force $\mathbf{F} = \mathbf{E}$ is simply the electric field generated by the bulk. The system is hence closed by coupling *Poisson's equation* to the Vlasov equation (2.1):

$$\nabla \cdot \mathbf{E} = \rho. \tag{2.2}$$

Note that the resulting interaction is repulsive (this makes sense: all charged particles have the same charge, and are thus repelling one another). We note that this system can also be modified to be *attractive* by simply flipping the sign of $\mathbf{F}$. This results in a model for *galactic dynamics*. This can be summarised as follows:

$$\mathbf{F} = \gamma \mathbf{E}, \qquad \gamma = \begin{cases} +1 & \text{plasma problems} \\ -1 & \text{galactic dynamics} \end{cases} \tag{2.3}$$

2.1.3 The Vlasov-Maxwell system

In the plasma case, one can consider a more accurate model in which all electro*magnetic* interactions are taken into account, hence coupling the Vlasov equation (2.1) to *Maxwell's equations*

$$\nabla \cdot \mathbf{E} = \rho, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B} = \mathbf{j} + \frac{\partial \mathbf{E}}{\partial t}, \tag{2.4}$$

through the *Lorentz force*

$$\mathbf{F} = \mathbf{E} + v \times \mathbf{B} \tag{2.5}$$

where $\mathbf{B}(t, x)$ is the magnetic field, and

$$\mathbf{j}(t, x) = \int_{\mathbb{R}^n} f(t, x, p) v \, \mathrm{d}p = \text{current density}.$$

It is well known that Maxwell's equations are hyperbolic (i.e. have the form of a wave equation). This can be seen, for instance, by taking a time derivative. Alternatively, letting ϕ and $\mathbf{A}$ be the electric and magnetic potentials satisfying

$$\nabla \phi = -\mathbf{E} \qquad \text{and} \qquad \nabla \times \mathbf{A} = \mathbf{B}$$

and imposing the Lorenz gauge condition $\nabla\cdot\mathbf{A}+\frac{\partial\phi}{\partial t}=0$ Maxwell's equations are transformed into the system

$$\frac{\partial^2}{\partial t^2}\phi-\Delta\phi=\rho,$$
$$\frac{\partial^2}{\partial t^2}\mathbf{A}-\Delta\mathbf{A}=\mathbf{j}.$$

2.2 Linear Transport Equations: The Method of Characteristics

As we have seen before, Liouville's equation implies that the pdf f is transported by the Vlasov equation (in the case where collisions are negligible). In this section our goal is to study first order (linear) PDEs. Letting the unknown be $u=u(t,y):\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$, we consider equations of the form

$$\frac{\partial u}{\partial t}(t,y)+w(t,y)\cdot\nabla_y u(t,y)=0 \tag{2.6}$$

where $w:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ is given, and ask whether given some initial data

$$u(0,y)=u_0(y) \tag{2.7}$$

solutions exist, and, if so, are they unique?[10] In what follows, this question is converted into a question on infinite dimensional ODEs. Intuitively, instead of thinking of (2.6) as transporting the initial data u_0 (this is also known as the **Eulerian viewpoint**), we write down an ODE for each "particle" (this is also known as the **Lagrangian viewpoint**).

Definition 2.1 (Characteristics). *A* characteristic *for* (2.6) *is a function $X\in C^1(I;\mathbb{R}^n)$ where $I\subset\mathbb{R}$ is an interval (that is, $X:I\to\mathbb{R}^n$ and $I\ni s\mapsto X(s)\in\mathbb{R}^n$ is C^1) satisfying*

$$\dot{X}(s)=w(s,X(s)).$$

Theorem 2.2 (Existence, uniqueness and regularity of characteristics). *Assume that $w(t,y)$ satisfies the two following conditions:*

$$\text{(H1)}:\ w\in C([0,T]\times\mathbb{R}^n;\mathbb{R}^n)\ \textit{and}\ D_y w\in C([0,T]\times\mathbb{R}^n;\mathrm{M}_n(\mathbb{R}^n)).$$

$$\text{(H2)}:\ \exists c>0\ \textit{such that}\ |w(t,y)|\le c(1+|y|)\ \textit{for all}\ (t,y)\in[0,T]\times\mathbb{R}^n.$$

Then the following holds:

1. *For all $(t,y)\in[0,T]\times\mathbb{R}^n$ there exists a unique characteristic $X(s;t,y)$[11] defined on $[0,T]$ such that $X(t;t,y)=y$. Moreover, $X\in C^1([0,T]\times[0,T]\times\mathbb{R}^n;\mathbb{R}^n)$.*

2. *For all $(s,t,y)\in[0,T]\times[0,T]\times\mathbb{R}^n$ and all $j=1,\dots,n$ the mixed partial derivatives $\partial_{x_j}\partial_s X$ and $\partial_s\partial_{x_j}X$ exist and are equal, and are $C([0,T]\times[0,T]\times\mathbb{R}^n;\mathbb{R}^n)$.*

Remark 2.3. We use the convention that a gradient of a scalar function is denoted ∇ (or ∇_y to make clear that the gradient is with respect to $y=(y_1,\dots,y_n)$) and the matrix of partial derivatives of a vector-valued function is denoted D (or D_y).

[10] Note that this is a *linear* problem, while the Vlasov-Poisson and Vlasov-Maxwell equations are *nonlinear* (the nonlinear term is the forcing term $\mathbf{F}[f]\cdot\nabla_p f$).
[11] X is thought of as a function of s, with t and y being parameters.

Proof. Part (1). The assumption (H1) ensures that there exists a unique local solution to the initial value problem

$$\dot{z}(s) = w(s, z(s)), \qquad z(t) = y, \qquad t \in [0,T], y \in \mathbb{R}^n$$

due to *Picard's theorem*. To show that this solution can be extended to $[0,T]$ write $X(s;t,y) = y + \int_t^s w(\zeta, X(\zeta;t,y))\,\mathrm{d}\zeta$ and show that blowup of $|X(s;t,y)|$ cannot occur in $[0,T]$ due to (H2) and *Gronwall's lemma*.

Part (2). This follows from the continuous dependence of solutions to ODEs on parameters and initial data. $\square$

Theorem 2.4 (Properties of the characteristics). *Assume that $w(t,y)$ satisfies (H1) and (H2) and let X be as defined in Definition 2.1. Then:*

1. *For all $t_1, t_2, t_3 \in [0,T]$ and $y \in \mathbb{R}^n$, $X(t_1;t_2,X(t_2;t_3,y)) = X(t_1;t_3,y)$.*
2. *For $(s,t) \in [0,T] \times [0,T]$, $\mathbb{R}^n \ni y \mapsto X(s;t,y)$ is a C^1-diffeomorphism on $\mathbb{R}^n$ (denoted $X(s;t,\cdot)$) with inverse $X(t;s,\cdot)$. Furthermore, $X(s;t,)$ is orientation preserving.*
3. *Define $J(s;t,y) := \det(D_y X(s;t,y))$. Then J solves*
$$\begin{cases} \dot{J}(s;t,y) = \nabla_y \cdot w(s, X(s;t,y))J(s;t,y), & y \in \mathbb{R}^n,\ s,t \in [0,T], \\ J(t;t,y) = 1. \end{cases}$$
4. *If $\nabla_y \cdot w(t,y) = 0$ for all $y \in \mathbb{R}^n, t \in [0,T]$ then $X(s;t,\cdot)$ leaves the Lebesgue measure on $\mathbb{R}^n$ invariant, i.e.*
$$\int_{\mathbb{R}^n} \psi(X(s;t,y))\,\mathrm{d}y = \int_{\mathbb{R}^n} \psi(y)\,\mathrm{d}y, \qquad \forall s,t \in [0,T],\ \forall \psi \in C_0(\mathbb{R}^n). \tag{2.8}$$

Proof. Part (1). This is a simple consequence of the uniqueness of solutions.

Part (2). Due to the smooth dependence on initial data $X \in C^1([0,T] \times [0,T] \times \mathbb{R}^n)$. In particular $y \mapsto X(s;t,y)$ and $y \mapsto X(t;s,y)$ are C^1. Due to Part (1), $X(t;s,\cdot) \circ X(s;t,\cdot) = X(s;t,\cdot) \circ X(t;s,\cdot) = \mathrm{Id}_{\mathbb{R}^n}$ hence $X(s;t,\cdot)$ is one-to-one and onto with inverse $X(t;s,\cdot)$. The fact that they are both orientation preserving relies on the next part (in particular, it relies on the fact that $J(s;t,y) > 0$ for all $s,t \in [0,T], y \in \mathbb{R}^n$).

Part (3). Recall *Jacobi's formula*:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\det A(t)) = \det(A)\,\mathrm{trace}\left(A^{-1}(t)\frac{\mathrm{d}}{\mathrm{d}t}(A(t))\right).$$

which implies that

$$\dot{J}(s;t,y) = J(s;t,y)\,\mathrm{trace}(D_y X(s;t,y)^{-1}\partial_s D_y X(s;t,y)).$$

In addition, due to Theorem 2.2 we know that $\partial_s D_y X = D_y \partial_s X \in C([0,T] \times [0,T] \times \mathbb{R}^n; \mathrm{M}_n(\mathbb{R}^n))$, hence we have

$$\begin{aligned}
\dot{J}(s;t,y) &= J(s;t,y)\,\mathrm{trace}(D_y X(s;t,y)^{-1} D_y \partial_s X(s;t,y)) \\
&= J(s;t,y)\,\mathrm{trace}(D_y X(s;t,y)^{-1} D_y(w(s,X(s;t,y)))) \\
&= J(s;t,y)\,\mathrm{trace}(D_y X(s;t,y)^{-1}(D_y w)(s,X(s;t,y))D_y X(s;t,y)) \\
&= J(s;t,y)\,\mathrm{trace}((D_y w)(s,X(s;t,y))) \\
&= J(s;t,y)\nabla_y \cdot w(s,X(s;t,y)).
\end{aligned}$$

Since $X(t;t,y) = y$ we have $D_y X(t;t,y) = I_{\mathbb{R}^n}$ so that $J(t;t,y) = 1$.

Part (4). From the previous part we know that $J(s;t,y) = \exp\left(\int_t^s \nabla_y \cdot w(\tau, X(\tau;t,y))\mathrm{d}\tau\right) > 0$. If $\nabla_y \cdot w \equiv 0$ then $J \equiv 1$. $\square$

Theorem 2.5 (Solution to the transport equation). *Let $w(t,y)$ satisfy (H1) and (H2) and let $u_0 \in C^1(\mathbb{R}^n)$. Then the Cauchy problem*

$$\begin{cases} \partial_t u(t,y) + w(t,y) \cdot \nabla_y u(t,y) = 0, \quad y \in \mathbb{R}^n, t \in (0,T) \\ u(0,y) = u_0(y) \end{cases}$$

has a unique solution $u \in C^1([0,T] \times \mathbb{R}^n)$ which is given by the formula

$$u(t,y) = u_0(X(0;t,y)), \qquad \forall t \in [0,T], y \in \mathbb{R}^n. \tag{2.9}$$

Finally, if u_0 is compactly supported then so is $u(t,\cdot)$.

Proof. **Uniqueness.** Uniqueness follows from the uniqueness of characteristics (the PDE is reduced to an ODE along the characteristic), as follows. Since the map $t \mapsto X(t;0,z)$ is C^1 and $u \in C^1$ (we show this in the next part), so is the map $t \mapsto u(t, X(t;0,z))$. Applying the chain rule one easily sees that

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t, X(t;0,z)) = 0.$$

Hence $t \mapsto u(t, X(t;0,z))$ is constant on $[0,T]$ and $u(t, X(t;0,z)) = u(0, X(0;0,z)) = u_0(z)$. Letting $y = X(t;0,z)$ we have $z = X(0;t,y)$ so that

$$u(t,y) = u_0(X(0;t,y)), \qquad \forall (t,y) \in [0,T] \times \mathbb{R}^n.$$

Existence. a) The formula (2.9) defines a C^1 function. This is clear due to Theorem 2.2, since u is defined as the composition of the C^1 maps u_0 and $(t,y) \mapsto X(0;t,y)$. This function obviously satisfies $u|_{t=0} = u_0$.

b) We need to verify that (2.9) defines a solution to the transport equation. That is, we need to show that

$$\left(\partial_t + w(t,y) \cdot \nabla_y\right) u_0(X(0;t,y)) = 0.$$

By the chain rule, the left hand side is

$$\left(\partial_t + w(t,y) \cdot \nabla_y\right) u_0(X(0;t,y)) = \nabla u_0(X(0;t,y)) \cdot \left(\partial_t + w(t,y) \cdot \nabla_y\right) X(0;t,y)$$

and we claim that

$$\left(\partial_t + w(t,y) \cdot \nabla_y\right) X(0;t,y) = 0, \qquad \forall t \in [0,T], y \in \mathbb{R}^n.$$

In fact, let us show that the following stronger identity holds:

$$\left(\partial_t + w(t,y) \cdot \nabla_y\right) X(s;t,y) = 0, \qquad \forall s,t \in [0,T], y \in \mathbb{R}^n.$$

Recall the identity $X(t_1;t_2,X(t_2;t_3,y)) = X(t_1;t_3,y)$ from Theorem 2.4. Differentiating with respect to t_2 one obtains[12]

$$\begin{aligned} &\partial_t X(t_1;t_2,X(t_2;t_3,y)) + D_y X(t_1;t_2,X(t_2;t_3,y)) \cdot \dot{X}(t_2;t_3,y) \\ &\quad = \partial_t X(t_1;t_2,X(t_2;t_3,y)) + D_y X(t_1;t_2,X(t_2;t_3,y)) \cdot w(t_2, X(t_2;t_1,y)) = 0. \end{aligned}$$

[12]We denote $\dot{X}$ the partial derivative of X with respect to the first variable, and $\partial_t X$ the partial derivative of X with respect to the parameter t (the second "variable"), and $D_y X$ the matrix of partial derivatives with respect to the second parameter (third "variable").

Letting $t_2 = t_3 = t$ and $t_1 = s$ this identity becomes

$$\begin{aligned}&\partial_t X(s; t, X(t; t, y)) + D_y X(s; t, X(t; t, y)) \cdot \dot{X}(s; t, y) \\ &\quad = \partial_t X(s; t, y) + D_y X(s; t, y) \cdot \dot{X}(s; t, y) = 0\end{aligned}$$

which proves the claim.

Compact support. This is a simple consequence of the formula

$$\operatorname{supp}(u(t, \cdot)) = \operatorname{supp}(u_0(X(0; t, \cdot))) = X(0; t, \operatorname{supp}(u_0))$$

and the fact that characteristics remain bounded in $[0, T]$. □

2.3 The Fourier Transform

The Fourier transform is an essential tool when studying PDEs for many reasons. Notably, it relates *differentiation* to *multiplication*, which is much easier to analyse in many cases. There are many ways to define the Fourier transform (they usually differ from one another by where one inserts the factor of 2π). We choose a definition that renders the transform an *isometry.*

Definition 2.6. *Given $u \in L^1(\mathbb{R}^n)$ we define its Fourier transform $\mathcal{F}u$ to be*

$$(\mathcal{F}u)(\xi) = \hat{u}(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x) \, \mathrm{d}x, \quad \xi \in \mathbb{R}^n. \tag{2.10}$$

Given $v \in L^1(\mathbb{R}^n)$ we define the inverse Fourier transform[13] *to be*

$$(\mathcal{F}^{-1}v)(x) = \check{v}(x) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} v(\xi) \, \mathrm{d}\xi, \quad x \in \mathbb{R}^n. \tag{2.11}$$

Lemma 2.7. *The Fourier transform and its inverse embed $L^1(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$.*

Theorem 2.8 (Riemann-Lebesgue Lemma). *If $u \in L^1(\mathbb{R}^n)$ then $\hat{u}$ is continuous and $|\hat{u}(\xi)| \to 0$ as $|\xi| \to \infty$.*

Theorem 2.9 (Plancherel). *The Fourier transform can be defined as an operator on $L^2(\mathbb{R}^n)$, and $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a unitary isomorphism.*

Proposition 2.10 (Properties of the Fourier transform). *Some of the important properties of $\mathcal{F}$ are:*

1. $\mathcal{F}[u(\cdot - x_0)] = e^{-ix_0\cdot\xi}\mathcal{F}u$
2. $\mathcal{F}[u(\cdot/\lambda)] = |\lambda|^n(\mathcal{F}u)(\lambda\cdot)$
3. $\mathcal{F}[u * v] = \mathcal{F}u\mathcal{F}v$
4. $\mathcal{F}[\partial_{x_j}u] = -i\xi_j\mathcal{F}u$

[13] Strictly speaking, one cannot *define* the inverse transform; rather, one has to *show* that the inverse of $\mathcal{F}$ is indeed given by (2.11). However, this requires a detailed discussion of domains and ranges of the transform, a topic which we do not attempt to cover here.

2.4 Sobolev Spaces

Sobolev spaces are an essential tool in the analysis of PDEs. It is well known that working with L^2 spaces is highly beneficial due to their Hilbert space structure. Sobolev spaces are meant to adapt this to the study of PDEs: these are functional spaces (with a Hilbert space structure) that include functions that are L^2, as well as some of their derivatives.

Definition 2.11 (The Sobolev spaces $H^k(\mathbb{R}^n)$). *Let $u \in L^2(\mathbb{R}^n)$. We say that $u \in H^k(\mathbb{R}^n)$ (where $k \in \mathbb{N}$), if $\partial^\alpha u \in L^2(\mathbb{R}^n)$,[14] for any $|\alpha| \leq k$.[15] The norm on $H^k(\mathbb{R}^n)$ is given by*

$$\|u\|^2_{H^k(\mathbb{R}^n)} := \sum_{|\alpha|\leq k} \|\partial^\alpha u\|^2_{L^2(\mathbb{R}^n)}.$$

When there is no room for confusion, we may replace $\|\cdot\|_{H^k(\mathbb{R}^n)}$ by $\|\cdot\|_{H^k}$ or by $\|\cdot\|_k$.

In other words, $H^k(\mathbb{R}^n)$ is the space of all functions $u \in L^2(\mathbb{R}^n)$ such that all possible partial derivatives of u (including mixed partial derivatives), up to (and including) order k, are square integrable.

In Proposition 2.10 we saw how the Fourier transform relates differentiation to multiplication: ∂_{x_j} becomes $-i\xi_j$. This provides a tool for defining the Sobolev spaces H^k in a more efficient way. Moreover, this will allow us to replace the discrete parameter k by a *continuous* parameter, typically denoted s, and hence providing us with a far richer family of spaces that include functions that have "fractional derivatives" that are square integrable.

Theorem 2.12 (Definition of $H^k(\mathbb{R}^n)$ in terms of the Fourier transform). *$u \in H^k(\mathbb{R}^n)$ if and only if $(1+|\xi|^2)^{k/2}\hat{u} \in L^2(\mathbb{R}^n)$ and the norm $\|u\|^2_k$ defined above is equivalent to the norm*

$$\int_{\mathbb{R}^n} (1+|\xi|^2)^k |\hat{u}(\xi)|^2 \, \mathrm{d}\xi.$$

In this definition there is no apparent reason why k must be discrete, and indeed we may extend this definition:

Definition 2.13 (Sobolev spaces $H^s(\mathbb{R}^n)$ with continuous parameter). *We define the Sobolev space $H^s(\mathbb{R}^n)$, where $s \geq 0$,[16] to be the space of functions $u \in L^2(\mathbb{R}^n)$ such that $(1+|\xi|^2)^{s/2}\hat{u} \in L^2(\mathbb{R}^n)$. The associated norm $\|u\|^2_s$ is defined as*

$$\|u\|^2_s := \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{u}(\xi)|^2 \, \mathrm{d}\xi.$$

Theorem 2.14 (Hilbert space structure). *$H^s(\mathbb{R}^d)$ is a Hilbert space with inner product*

$$(u,v)_s := \int_{\mathbb{R}^n} (1+|\xi|^2)^s \hat{u}(\xi)\overline{\hat{v}(\xi)} \, \mathrm{d}\xi.$$

Remark 2.15 (Inequalities and embeddings). Later in this course (as the need arises) we shall get a glimpse into a vast field of inequalities and embeddings, relating different

[14] Partial derivatives of u are defined *in the sense of distributions* or *weakly*, meaning that they are defined by integrating by parts against test functions.

[15] α is a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_j \geq 0$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and $\partial^\alpha = \partial^{\alpha_1}_{x_1} \cdots \partial^{\alpha_n}_{x_n}$

[16] We can consider also values $s < 0$, however in that case u is not an element of L^2, rather u must be taken to be a *tempered distribution*.

spaces (Sobolev spaces, L^p spaces, Hölder spaces ...) to one another. For instance, it is often desirable to know whether a function belonging to H^s has certain continuity or boundedness properties. Furthermore, if we knew that H^s were *compactly* embedded in some C^k space, say, then we could conclude that a bounded sequence of elements in H^k has a convergent subsequence in C^k.

3 The Vlasov-Poisson System: Local Existence and Uniqueness

In this section we demonstrate local existence of classical solutions to the Vlasov-Poisson system of equations. This will involve obtaining some *a priori* estimates and an iteration scheme. *A priori* estimates are an essential tool in the analysis of PDEs, and in particular for establishing existence of solutions. We follow [Rein2007].

3.1 Classical Solutions to Vlasov-Poisson: A Rigorous Definition

We start by precisely stating the meaning of a *classical solution.*[17] We recall that the Vlasov-Poisson system is the following system of equations for the unknown $f(t,x,p) : \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}_+$:

$$\frac{\partial f}{\partial t}(t,x,p) + p \cdot \nabla_x f(t,x,p) + \gamma \mathbf{E}_f(t,x) \cdot \nabla_p f(t,x,p) = 0, \tag{3.1}$$

$$\nabla \cdot \mathbf{E}_f(t,x) = \rho_f(t,x) = \int_{\mathbb{R}^3} f(t,x,p)\,\mathrm{d}p, \tag{3.2}$$

where $\gamma = +1$ for plasma problems and $\gamma = -1$ for galactic dynamics. We indicate quantities that depend upon f by a corresponding subscript. Alternatively, one could write this system with the field replaced by its potential. Since the potential is only determined up to a constant, one imposes an additional restriction (for instance decay at ∞):

$$\frac{\partial f}{\partial t}(t,x,p) + p \cdot \nabla_x f(t,x,p) - \gamma \nabla \phi_f(t,x) \cdot \nabla_p f(t,x,p) = 0, \tag{3.3}$$

$$-\Delta \phi_f(t,x) = \rho_f(t,x) = \int_{\mathbb{R}^3} f(t,x,p)\,\mathrm{d}p, \qquad \lim_{|x|\to\infty} \phi_f(t,x) = 0. \tag{3.4}$$

Definition 3.1 (Classical Solution). *A function $f : I \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}_+$ is a classical solution of the Vlasov-Poisson system on the interval $I \subset \mathbb{R}$ if:*

- $f \in C^1(I \times \mathbb{R}^3 \times \mathbb{R}^3)$
- *ρ_f and ϕ_f are well-defined, and belong to $C^1(I \times \mathbb{R}^3)$. Moreover, ϕ_f is twice continuously differentiable with respect to x.*
- *For every compact subinterval $J \subset I$, $\mathbf{E}_f = -\nabla \phi_f$ is bounded on $J \times \mathbb{R}^3$.*

Finally, obviously one requires that f satisfy (3.3) *and* (3.4) *on $I \times \mathbb{R}^3 \times \mathbb{R}^3$ and correspondingly that ρ_f and ϕ_f satisfy* (3.3) *and* (3.4) *on $I \times \mathbb{R}^3$.*

Theorem 3.2 (Local Existence of Classical Solutions). *Let $f_0(x,p) \in C_0^1(\mathbb{R}^3 \times \mathbb{R}^3)$ with $f_0 \geq 0$ be given. Then there exists a unique classical solution $f(t,x,p)$ for the system* (3.3)-(3.4) *on some interval $[0,T)$ with $T > 0$ and $f(0,\cdot,\cdot) = f_0$.*

Furthermore, for all $t \in [0,T)$ the function $f(t,\cdot,\cdot)$ is compactly supported and nonnegative.

Finally, we have the following breakdown criterion*: if $T > 0$ is chosen to be maximal, and if*

$$\sup_{\substack{(x,p)\in \operatorname{supp} f(t,\cdot,\cdot) \\ t\in[0,T)}} |p| < \infty$$

[17] We shall specialise to the three dimensional classical case.

or

$$\sup_{\substack{x\in\mathbb{R}^3\\ t\in[0,T)}} \rho_f(t,x) < \infty$$

then the solution is global ($T=\infty$).

Remark 3.3. The last part tells us how breakdown of solutions occurs: *both* momenta *and* the particle density must become unbounded. To show global existence (later in the course) one would have to establish *a priori* bounds on these quantities.

Remark 3.4. The assumption that f_0 is compactly supported can be relaxed, to include initial data that decays "sufficiently fast" at infinity (this was done, e.g. by [Horst1981]).

3.2 *A Priori* Estimates

3.2.1 The Free Transport Equation

We start with the basic *free transport equation* which models the force-free transport of particles in the classical case. Letting $f = f(t,x,p)$ with $t \geq 0$, $(x,p) \in \mathbb{R}^n \times \mathbb{R}^n$ and $p = \dot{x}$, the initial value problem is

$$\partial_t f + p\cdot\nabla_x f = 0, \qquad f(0,\cdot,\cdot) = f_0. \tag{3.5}$$

We already know that there exists a unique solution for this problem on $[0,\infty)$ (in fact on $(-\infty,\infty)$). Moreover, in this simple case the solution can be written explicitly (the characteristics are trivially $(\dot{X},\dot{V}) = (V,0)$)[18]

$$f(t,x,p) = f_0(x-pt,p)$$

and models particles that move freely (and therefore linearly) without any forces whatsoever acting on them.

Proposition 3.5 (Dispersion). *Let f be the solution to* (3.5) *and assume that $f_0 \in L^\infty(\mathbb{R}^n\times\mathbb{R}^n)\cap L^1(\mathbb{R}^n\times\mathbb{R}^n)$. Then:*[19]

$$\operatorname{ess\,sup}_{x\in\mathbb{R}^n}\int_{\mathbb{R}^n}|f(t,x,p)|\,\mathrm{d}p \leq \frac{1}{t^n}\int_{\mathbb{R}^n}\operatorname{ess\,sup}_{q\in\mathbb{R}^n}|f_0(y,q)|\,\mathrm{d}y.$$

In the kinetic case where $f\geq 0$ the density $\rho_f = \int f\,\mathrm{d}p$ decays:

$$\|\rho_f(t,\cdot)\|_\infty \leq \frac{c}{t^n}.$$

3.2.2 The Linear Transport Equation

Now let us consider the linear transport equation

$$\begin{cases} \partial_t u(t,y) + w(t,y)\cdot\nabla_y u(t,y) = 0, & y\in\mathbb{R}^n, t\in(0,T)\\ u(0,y) = u_0(y) \end{cases} \tag{3.6}$$

where $w(t,y) : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$ is given and satisfies, as before,

[18] We use V for the momentum variable characteristic since P will be used later for a different purpose.

[19] For brevity, this is often written as: $\|f(t,\cdot,\cdot)\|_{L^\infty_x(L^1_p)} \leq t^{-n}\|f_0\|_{L^1_x(L^\infty_p)}$

(H1) : $w \in C([0,T] \times \mathbb{R}^n; \mathbb{R}^n)$ and $D_y w \in C([0,T] \times \mathbb{R}^n; \mathrm{M}_n(\mathbb{R}^n))$.

(H2) : $\exists c > 0$ such that $|w(t,y)| \leq c(1+|y|)$ for all $(t,y) \in [0,T] \times \mathbb{R}^n$.

Comparing with Vlasov-Poisson, we have:

$$y = (x,p) \in \mathbb{R}^6, \qquad w(t,y) = (p, \gamma \mathbf{E}), \qquad \nabla_y = \nabla_{(x,p)},$$

so that $w \cdot \nabla_y = p \cdot \nabla_x + \gamma \mathbf{E} \cdot \nabla_p$. Of course, the Vlasov-Poisson system is nonlinear (and non-local[20]) since the force depends on f itself. However, it is a common strategy to "forget" this, and imagine that the force is given (then, for instance, *a priori* estimates for the linear transport equation such as Theorem 3.6 below can be used for Vlasov-Poisson). Notice that in any case, the Vlasov flow is divergence-free:

$$\nabla_y \cdot w = \nabla_x \cdot p + \gamma \nabla_p \cdot \mathbf{E} = 0. \tag{3.7}$$

Theorem 3.6 (Properties of the Linear Transport Equation). *Assume that $w(t,y)$ satisfies (H1) and (H2) and that $\nabla_y \cdot w = 0$. Let $u_0 \in C_0^1(\mathbb{R}^n)$. Then the solution u to* (3.6) *satisfies:*

1. *$u(t,\cdot)$ is compactly supported.*
2. *If $u_0 \geq 0$ then $u(t,\cdot) \geq 0$.*
3. *For all $p \in [1,\infty]$, $\|u(t,\cdot)\|_{L^p(\mathbb{R}^n)} = \|u_0\|_{L^p(\mathbb{R}^n)}$.*[21]
4. *For any $\Phi \in C^1(\mathbb{R};\mathbb{R})$ with $\Phi(0) = 0$ we have*

$$\int_{\mathbb{R}^n} \Phi(u(t,y))\,\mathrm{d}y = \int_{\mathbb{R}^n} \Phi(u_0(y))\,\mathrm{d}y, \qquad \forall t \in [0,T]. \tag{3.8}$$

Proof. The first property was already proven in Theorem 2.5, and the second property is an easy consequence of the representation (2.9) of the solution to the linear transport equation using the characteristics. Let us prove (3.8) first (note that this resembles (2.8)). Notice that if u solves the transport equation then so does $\Phi(u)$. This is easily verified by an application of the chain rule. Hence $\Phi(u)$ satisfies the transport equation with initial condition $\Phi(u_0)$. Integrating the transport equation in y we get:

$$\begin{aligned}
0 &= \int_{\mathbb{R}^n} \left(\partial_t \Phi(u(t,y)) + w(t,y) \cdot \nabla \Phi(u(t,y))\right) \mathrm{d}y \\
&= \int_{\mathbb{R}^n} \partial_t \Phi(u(t,y))\mathrm{d}y + \int_{\mathbb{R}^n} w(t,y) \cdot \nabla \Phi(u(t,y))\,\mathrm{d}y \\
&= \partial_t \left(\int_{\mathbb{R}^n} \Phi(u(t,y))\mathrm{d}y\right) + \int_{\mathbb{R}^n} \nabla \cdot \left(w(t,y)\Phi(u(t,y))\right) \mathrm{d}y \\
&= \partial_t \left(\int_{\mathbb{R}^n} \Phi(u(t,y))\,\mathrm{d}y\right),
\end{aligned}$$

where in the third equality we used the fact that $\nabla \cdot w = 0$. This proves Part 4. By letting $\Phi(u) = |u|^p$ for $p \in (1,\infty)$, this also proves conservation of these L^p norms. Note that for $p = 1$ this won't work, as $\Phi(u) = |u|$ isn't C^1. However, in this case we can prove for a smoothed version of $\Phi(u) = |u|$ (i.e. we smooth the singularity at 0) and let the smoothing parameter tend to 0. The details are omitted here.

The fact that the L^∞ norm is conserved is evident from the representation (2.9) and since $u_0 \in C^1$. If u is less smooth then in general the L^∞ norm may decrease. ☐

[20] This means that the evolution depends on the system as a whole.

[21] One can also consider less smooth initial data in which case this is only true for $p \in [1,\infty)$, and for $p = \infty$ one has $\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \leq \|u_0\|_{L^\infty(\mathbb{R}^n)}$.

3.2.3 Poisson's Equation

Due to the nature of the nonlinearity in the Vlasov-Poisson system having the form $\nabla\phi_f(t,x)\cdot \nabla_p f(t,x,p)$ we want to obtain some *a priori* estimates on $\nabla\phi_f(t,x)$. We have the following:

Proposition 3.7 (Properties of Solutions to Poisson's Equation). *Given $\rho(x) \in C_0^1(\mathbb{R}^3)$ we define*

$$\phi_\rho(x) := \int_{\mathbb{R}^3} \frac{\rho(y)}{|x-y|}\mathrm{d}y.$$

Then:

1. *ϕ_ρ is the unique solution in $C^2(\mathbb{R}^3)$ of $-\Delta\phi = \rho$ with $\lim_{|x|\to\infty}\phi(x) = 0$.*[22]
2. *The force is given by*

$$\nabla\phi_\rho(x) = -\int \frac{x-y}{|x-y|^3}\rho(y)\,\mathrm{d}y$$

and we have the decay properties as $|x| \to \infty$

$$\phi_\rho(x) = O(|x|^{-1}) \quad \text{and} \quad \nabla\phi_\rho(x) = O(|x|^{-2}).$$

3. *For any $p \in [1,3)$*

$$\|\nabla\phi_\rho\|_\infty \leq c_p\|\rho\|_p^{p/3}\|\rho\|_\infty^{1-p/3} \qquad (c_p \text{ only depends on } p).$$

4. *For any $p \in [1,3)$, $R > 0$ and $d \in (0,R]$, $\exists c > 0$ independent of ρ, R, d, s.t.*

$$\|D^2\phi_\rho\|_\infty \leq c\left(\frac{\|\rho\|_1}{R^3} + d\|\nabla\rho\|_\infty + (1+\ln(R/d))\|\rho\|_\infty\right),$$
$$\|D^2\phi_\rho\|_\infty \leq c(1+\|\rho\|_\infty)(1+\ln_+\|\nabla\rho\|_\infty) + c\|\rho\|_1.$$

3.3 Sketch of Proof of Local Existence and Uniqueness

The proof of local existence is "standard" in the sense that it follows the ideas outlined in Section 1. However, this does not mean that the proof is easy. This result is due to [Batt1977] and [Ukai1978]. We remind that the system we want to solve is

$$\partial_t f(t,x,p) + p\cdot\nabla_x f(t,x,p) - \gamma\nabla\phi_f(t,x)\cdot\nabla_p f(t,x,p) = 0,$$
$$-\Delta\phi_f(t,x) = \rho_f(t,x) = \int_{\mathbb{R}^3} f(t,x,p)\,\mathrm{d}p, \qquad \lim_{|x|\to\infty}\phi_f(t,x) = 0.$$

with initial data

$$f(0,\cdot,\cdot) = f_0 \in C_0^1(\mathbb{R}^6).$$

The proof shall follow the following iterative scheme:

Step 0. Set $f^0(t,x,p) = f_0(x,p)$ and define $\phi_{f^0}(t,x)$.

[22] There should be a factor of 4π in Poisson's equation which we omit in accordance with our convention.

Step 1. Let $f^1(t, x, p)$ be the solution to the linear transport equation

$$\partial_t f^1(t, x, p) + p \cdot \nabla_x f^1(t, x, p) - \gamma \nabla \phi_{f^0}(t, x) \cdot \nabla_p f^1(t, x, p) = 0.$$

Define $\phi_{f^1}(t, x)$ which is used to obtain $f^2(t, x, p)$.

... and so on ...

Step N. Let $f^N(t, x, p)$ be the solution to the linear transport equation

$$\partial_t f^N(t, x, p) + p \cdot \nabla_x f^N(t, x, p) - \gamma \nabla \phi_{f^{N-1}}(t, x) \cdot \nabla_p f^N(t, x, p) = 0.$$

Step ∞. Show that as $N \to \infty$, f^N has a C^1 limit, and that this limit satisfies the Vlasov-Poisson system on some time interval $[0, T)$.

Key Ingredients of the Proof. From our study of linear transport equations, we know that the most important ingredient is estimating the vector-field. This amounts to estimating $\nabla \phi_f$ (we drop the superscript N). Define the (crucial!) quantity for $t \in [0, T)$:

$$P(t) := \sup_{(x,p) \in \operatorname{supp} f(t,\cdot,\cdot)} |p|. \tag{3.9}$$

From Theorem 3.6 we know that the norms $\|f(t, \cdot, \cdot)\|_p$ are constant for all $p \in [1, \infty]$; hence

$$\|\rho_f(t, \cdot)\|_\infty = \sup_{x \in \mathbb{R}^3} \int_{\mathbb{R}^3} f(t, x, p) \, \mathrm{d}p \leq \|f(t, \cdot, \cdot)\|_\infty P^3(t) = cP^3(t)$$

and

$$\|\rho_f(t, \cdot)\|_1 = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(t, x, p) \, \mathrm{d}p \, \mathrm{d}x = c$$

where c is a constant that may change from line to line. Hence by Proposition 3.7

$$\|\nabla \phi_f(t, \cdot)\|_\infty \leq c \|\rho_f(t, \cdot)\|_1^{1/3} \|\rho_f(t, \cdot)\|_\infty^{2/3} \leq cP^2(t).$$

The problem therefore reduces to controlling $P(t)$, the maximal momentum. Momentum growth can only happen due to the forcing term in the Vlasov equation: $\nabla \phi_f$. As we have just seen, $\nabla \phi_f$ is controlled by $P^2(t)$, and so we have a typical *Gronwall inequality*:

$$P(t) \leq P(0) + c \int_0^t P^2(s) \, \mathrm{d}s.$$

To complete the proof one would have to repeat a similar analysis for derivatives (since we need to show convergence in the C^1-norm). For those, we will show that

$$\|\nabla \rho_f(t, \cdot)\|_\infty \leq c \qquad \text{and} \qquad \|D^2 \phi_f(t, \cdot)\|_\infty \leq c.$$

3.4 Detailed Proof of Local Existence and Uniqueness

3.4.1 Defining Approximate Solutions

We initiate the problem by iteratively defining a sequence of solutions $(f^N, \rho^N = \rho^{f^N}, \phi^N = \phi^{f^N})$ to an approximated Vlasov-Poisson.

Set

$$f^0(t,x,p) = f_0(x,p)$$

and define

$$\rho^0(t,x) = \int_{\mathbb{R}^3} f^0(t,x,p)\,\mathrm{d}p \quad \text{and} \quad \phi^0(t,x) = \int_{\mathbb{R}^3} \frac{\rho^0(t,z)}{|x-z|}\mathrm{d}z.$$

Suppose that $(f^{N-1}, \rho^{N-1}, \phi^{N-1})$ have been defined and define f^N to be the solution to the linear transport equation

$$\begin{cases} \partial_t f^N(t,x,p) + p\cdot\nabla_x f^N(t,x,p) - \gamma\nabla\phi^{N-1}(t,x)\cdot\nabla_p f^N(t,x,p) = 0, \\ f^N(0,\cdot,\cdot) = f_0. \end{cases}$$

Hence one can write

$$\begin{aligned} f^N(t,x,p) &= f_0(X^{N-1}(0;t,x,p), V^{N-1}(0;t,x,p)), \\ \rho^N(t,x) &= \int_{\mathbb{R}^3} f^N(t,x,p)\,\mathrm{d}p, \\ \phi^N(t,x) &= \int_{\mathbb{R}^3} \frac{\rho^N(t,z)}{|x-z|}\mathrm{d}z. \end{aligned}$$

Goal: show that $\exists T > 0$ such that $\lim_{N\to\infty} f^N$ exists in $C^1([0,T)\times\mathbb{R}^6)$, $\lim_{N\to\infty}(\rho^N,\phi^N)$ exists in $C^1([0,T)\times\mathbb{R}^3)$, that the limits satisfy the Vlasov-Poisson system (uniquely), and that the continuation criterion holds.

3.4.2 The Iterates are Well-Defined

This is a simple proof by induction and is omitted here. One can show that the following holds:

$$\begin{aligned} &f^N(t,x,p) \in C^1([0,\infty)\times\mathbb{R}^6), \\ &\rho^N(t,x) \in C^1([0,\infty)\times\mathbb{R}^3), \\ &\nabla\phi^N(t,x) \in C^1([0,\infty)\times\mathbb{R}^3). \end{aligned}$$

Define $R_0 > 0$ and $P_0 > 0$ to be such that

$$\operatorname{supp} f^0 \subset \{|x| < R_0\} \cap \{|p| < P_0\}$$

and

$$\begin{aligned} P_0(t) &= P_0 \\ P_N(t) &= \sup_{\substack{(x,p)\in\operatorname{supp} f^0(t,\cdot,\cdot) \\ s\in[0,t]}} |V^{N-1}(s;0,x,p)| \end{aligned}$$

Then the following holds:

$$\operatorname{supp} f^N \subset \left\{|x| < R_0 + \int_0^t P_N(s)\,\mathrm{d}s\right\} \cap \left\{|p| < P_N(t)\right\}$$
$$\operatorname{supp} \rho^N \subset \left\{|x| < R_0 + \int_0^t P_N(s)\,\mathrm{d}s\right\}$$
$$\|f^N(t)\|_1 = \|\rho^N(t)\|_1 = \|f^0\|_1$$
$$\|f^N(t)\|_\infty = \|f^0\|_\infty$$
$$\|\rho^N(t)\|_\infty \le c\|f^0\|_\infty P_N^3(t)$$
$$\|\nabla\phi^N(t)\|_\infty \le C(f^0)P_N^2(t)$$

where by Proposition 3.7

$$C(f^0) = \|f^0\|_1^{1/3}\|f^0\|_\infty^{2/3} \tag{3.10}$$

up to some multiplicative constant.

3.4.3 A Uniform Bound for the Maximal Momentum

Intuitively, we know that for each N, the particle acceleration is given by $\nabla\phi^N(t)$ for which we have the bound $C(f^0)P_N^2(t)$. This suggests that all momenta can be uniformly bounded as follows. Let $\delta > 0$ and $P : [0,\delta) \to (0,\infty)$ be such that P is the maximal solution of

$$P(t) = P_0 + C(f^0)\int_0^t P^2(s)\,\mathrm{d}s \qquad \text{i.e. } P(t) = \frac{P_0}{1 - P_0C(f^0)t} \text{ and } \delta = \frac{1}{P_0C(f^0)}.$$

Claim. We have the uniform bound:

$$P_N(t) \le P(t), \qquad \forall N \ge 0, t \in [0,\delta).$$

Assuming this for the moment, we immediately have for all $N \ge 0$ and $t \in [0,\delta)$:

$$\|\rho^N(t,\cdot)\|_\infty \le c\|f^0\|_\infty P^3(t),$$
$$\|\nabla\phi^N(t,\cdot)\|_\infty \le C(f^0)P^2(t).$$

Proof of claim. The claim is clearly true for $N = 0$. Hence we assume it is true for N and prove for $N+1$. For any $0 \le s \le t < \delta$ and $(x,p) \in \operatorname{supp} f^0$:

$$\begin{aligned}|V^N(s;0,x,p)| &\le |p| + \int_0^s \|\nabla\phi^N(\tau,\cdot)\|_\infty \mathrm{d}\tau\\ &\le P_0 + C(f^0)\int_0^s P_N^2(\tau)\,\mathrm{d}\tau\\ &\le P_0 + C(f^0)\int_0^t P^2(\tau)\,\mathrm{d}\tau = P(t).\end{aligned}$$

3.4.4 A Uniform Bound for $\nabla\rho^N$ and $D^2\phi^N$

For C^1 convergence we need we need uniform convergence of derivatives of ρ^N and $\nabla\phi^N$ on subintervals of $[0,\delta)$. Hence we let $\delta_0 \in (0,\delta)$ and claim the following:

Claim. $\exists c = c(f^0,\delta_0) > 0$ such that

$$\|\nabla\rho^N(t,\cdot)\|_\infty + \|D^2\phi^N(t,\cdot)\|_\infty \le c, \qquad \forall t \in [0,\delta_0],\ N \ge 0.$$

To show this, we first claim:

Sub-Claim 1. We can estimate

$$\|\nabla\rho^{N+1}(t,\cdot)\|_\infty \le c\exp\left[\int_0^t \|D^2\phi^N(\tau,\cdot)\|_\infty \mathrm{d}\tau\right], \qquad 0\le t\le \delta_0.$$

Assuming this for the moment, we prove the claim:

Proof of Claim. Recall the estimate on $\|D^2\phi^N(t,\cdot)\|_\infty$ from Proposition 3.7:

$$\|D^2\phi^{N+1}(t,\cdot)\|_\infty \le c(1+\|\rho^{N+1}(t,\cdot)\|_\infty)(1+\ln_+\|\nabla\rho^{N+1}(t,\cdot)\|_\infty)+c\|\rho^{N+1}(t,\cdot)\|_1.$$

Using Sub-Claim 1 together with the estimate for $\|\rho^N(t,\cdot)\|_\infty$ that we obtained in the previous step, we have:

$$\begin{aligned}\|D^2\phi^{N+1}(t,\cdot)\|_\infty &\le c(1+\|\rho^{N+1}(t,\cdot)\|_\infty)(1+\ln_+\|\nabla\rho^{N+1}(t,\cdot)\|_\infty)+c\|\rho^{N+1}(t,\cdot)\|_1\\ &\le c(1+\|f^0\|_\infty P^3(t))\left(1+\ln_+\exp\left[\int_0^t\|D^2\phi^N(\tau,\cdot)\|_\infty\mathrm{d}\tau\right]\right)+c\\ &\le c\left(1+\int_0^t\|D^2\phi^N(\tau,\cdot)\|_\infty\mathrm{d}\tau\right).\end{aligned}$$

Hence by induction

$$\|D^2\phi^N(t,\cdot)\|_\infty\le ce^{ct}, \qquad \forall t\in[0,\delta_0], N\ge 0$$

(here we assumed that c is so large that $\|D^2\phi^0(t,\cdot)\|_\infty\le c$). □

Now we are left with proving the sub-claim:

Proof of Sub-Claim 1. We first note that

$$\rho^{N+1}(t,x)=\int_{\mathbb{R}^3} f^{N+1}(t,x,p)\,\mathrm{d}p=\int_{\mathbb{R}^3} f_0(X^N(0;t,x,p),V^N(0;t,x,p))\,\mathrm{d}p.$$

Hence

$$\begin{aligned}|\nabla\rho^{N+1}(t,x)| &\le \int_{|p|\le P(t)}\left|\nabla_x\left(f_0(X^N(0;t,x,p),V^N(0;t,x,p))\right)\right|\,\mathrm{d}p\\ &\le c\left(\|\nabla_x X^N(0;t,\cdot,\cdot)\|_\infty+\|\nabla_x V^N(0;t,\cdot,\cdot)\|_\infty\right).\end{aligned}$$

From Sub-Claim 2 below, we know that

$$\begin{aligned}&|\nabla_x X^N(s;t,x,p)|+|\nabla_x V^N(s;t,x,p)|\le\\ &\qquad 1+\int_s^t\left(1+\|D^2\phi^N(\tau,\cdot)\|_\infty\right)\left(|\nabla_x X^N(\tau,t,x,p)|+|\nabla_x V^N(\tau,t,x,p)|\right)\,\mathrm{d}\tau.\end{aligned}$$

So *Gronwall's inequality* leads to

$$|\nabla_x X^N(s;t,x,p)|+|\nabla_x V^N(s;t,x,p)|\le\exp\left[\int_0^t\left(1+\|D^2\phi^N(\tau,\cdot)\|_\infty\right)\,\mathrm{d}\tau\right]$$

which completes the proof of the sub-claim. □

Sub-Claim 2. We claim that for all $(x,p)\in\mathbb{R}^6$ and $s,t\in[0,\delta_0]$,

$$\begin{aligned}&|\nabla_x X^N(s;t,x,p)|+|\nabla_x V^N(s;t,x,p)|\le\\ &\qquad 1+\int_s^t\left(1+\|D^2\phi^N(\tau,\cdot)\|_\infty\right)\left(|\nabla_x X^N(\tau,t,x,p)|+|\nabla_x V^N(\tau,t,x,p)|\right)\,\mathrm{d}\tau.\end{aligned}$$

Proof of Sub-Claim 2. Exercise. □

3.4.5 The Sequence $\{f^N\}$ Has a Limit

Since $f^N(t,x,p) = f_0(X^{N-1}(0;t,x,p), V^{N-1}(0;t,x,p))$ we have

$$\begin{aligned}
&|f^{N+1}(t,x,p) - f^N(t,x,p)| \\
&\qquad = |f_0(X^N(0;t,x,p), V^N(0;t,x,p)) - f_0(X^{N-1}(0;t,x,p), V^{N-1}(0;t,x,p))| \\
&\qquad \le c\left(|X^N(0;t,x,p) - X^{N-1}(0;t,x,p)| + |V^N(0;t,x,p) - V^{N-1}(0;t,x,p)|\right).
\end{aligned}$$

Claim. The following estimate holds:

$$\begin{aligned}
&|X^N(s;t,x,p) - X^{N-1}(s;t,x,p)| + |V^N(s;t,x,p) - V^{N-1}(s;t,x,p)| \\
&\qquad \le c\int_0^t \|f^N(\tau,\cdot,\cdot) - f^{N-1}(\tau,\cdot,\cdot)\|_\infty \,\mathrm{d}\tau.
\end{aligned}$$

Assuming this claim, we then have

$$\|f^{N+1}(t,\cdot,\cdot) - f^N(t,\cdot,\cdot)\|_\infty \le c\int_0^t \|f^N(\tau,\cdot,\cdot) - f^{N-1}(\tau,\cdot,\cdot)\|_\infty \,\mathrm{d}\tau$$

which yields

$$\|f^{N+1}(t,\cdot,\cdot) - f^N(t,\cdot,\cdot)\|_\infty \le ct^N(N!)^{-1}, \qquad n \ge 0, t \in [0,\delta_0].$$

Hence the sequence $\{f^N\}$ is uniformly Cauchy and converges uniformly on $[0,\delta_0]\times\mathbb{R}^6$ to some function $f \in C([0,\delta_0]\times\mathbb{R}^6)$.

3.4.6 Properties of the Limit

ρ^N and ϕ^N also converge uniformly:

$$\rho^N \to \rho_f,\ \phi^N \to \phi_f, \qquad \text{uniformly on } [0,\delta_0]\times\mathbb{R}^3.$$

The support of f satisfies:

$$\operatorname{supp} f \subset \left\{|x| < R_0 + \int_0^t P(s)\,\mathrm{d}s\right\} \cap \left\{|p| < P(t)\right\}.$$

The estimates on on Poisson's equation (Proposition 3.7) lead to (exercise):

$$\phi_f, \nabla\phi_f, D^2\phi_f \in C([0,\delta_0]\times\mathbb{R}^3).$$

This implies that

$$(X^N, V^N) \to (X,V) \in C^1([0,\delta_0]\times[0,\delta_0]\times\mathbb{R}^6;\mathbb{R}^6)$$

and this is the flow due to the limiting field $\nabla\phi_f$. Hence

$$f(t,x,p) = \lim_{N\to\infty} f_0(X^N(0;t,x,p), V^N(0;t,x,p)) = f_0(X(0;t,x,p), V(0;t,x,p))$$

and $f \in C^1([0,\delta_0]\times\mathbb{R}^6)$.

3.4.7 Uniqueness

Uniqueness is a simple consequence of *Gronwall's inequality.*

3.4.8 Proof of the Continuation Criterion

This is a proof by contradiction. Assume that the maximal time interval on which the solution f can defined is $[0,T)$, with $T < \infty$, but that neither $\operatorname{supp}|p|$ nor $\|\rho_f\|_\infty$ blowup as $t \nearrow T$. Take some $T_0 = T - \varepsilon$ (ε to be chosen sufficiently small) and restart the problem from T_0, showing that it is possible to go beyond T.

4 The Vlasov-Poisson System: Global Existence and Uniqueness

In this section we plan to show that the local existence result – Theorem 3.2 – can in fact be extended indefinitely, that is the maximal time of existence is $T = +\infty$. In the proof of the local result, we saw that the fields and their derivatives were all bounded by powers of the crucial quantity $P(t)$ – the momentum of the "fastest" particle at time t. Hence, as long as this quantity can be controlled, the solution can be continued. Previously, the optimal estimate on $P(t)$ we could obtain was related to the maximal solution of

$$P(t) = P_0 + C(f^0) \int_0^t P^2(s)\,\mathrm{d}s,$$

that is:

$$P(t) = \frac{P_0}{1 - P_0 C(f^0) t} \text{ and } \delta = \frac{1}{P_0 C(f^0)}.$$

Improving this bound shall require some more *a priori* estimates.

4.1 *A Priori* Estimates

In Section 3.2 we obtained some preliminary basic *a priori* estimates for the Vlasov-Poisson system

$$\frac{\partial f}{\partial t}(t,x,p) + p \cdot \nabla_x f(t,x,p) - \gamma \nabla \phi_f(t,x) \cdot \nabla_p f(t,x,p) = 0, \tag{4.1}$$

$$-\Delta \phi_f(t,x) = \rho_f(t,x) = \int_{\mathbb{R}^3} f(t,x,p)\,\mathrm{d}p, \qquad \lim_{|x|\to\infty} \phi_f(t,x) = 0. \tag{4.2}$$

with initial data

$$f(0,\cdot,\cdot) = f_0 \in C_0^1(\mathbb{R}^6),$$

where $\gamma = \pm 1$ differentiates between repulsive $(+1)$ and attractive (-1) dynamics. Let us recall these basic estimates. First, we identified the conservation of all L^p norms, $p \in [1,\infty]$:

$$\|f(t,\cdot,\cdot)\|_p = \|f_0\|_p$$

as long as the solution exists. Then we recalled estimates related to the solution of Poisson's equation $-\Delta\phi = \rho$ in $\mathbb{R}^3$: For any $p \in [1,3)$

$$\|\nabla \phi_\rho\|_\infty \le c_p \|\rho\|_p^{p/3} \|\rho\|_\infty^{1-p/3} \qquad (c_p \text{ only depends on } p),$$

and for any $p \in [1,3)$, $R > 0$ and $d \in (0,R]$, $\exists c > 0$ independent of ρ, R, d, s.t.

$$\|D^2\phi_\rho\|_\infty \le c\left(\frac{\|\rho\|_1}{R^3} + d\|\nabla\rho\|_\infty + (1 + \ln(R/d))\|\rho\|_\infty\right),$$

$$\|D^2\phi_\rho\|_\infty \le c(1 + \|\rho\|_\infty)(1 + \ln_+ \|\nabla\rho\|_\infty) + c\|\rho\|_1.$$

These estimates are insufficient if one wants to bound momenta. For that we need to use the conservative nature of the Vlasov-Poisson system. More precisely, in what follows not only will we show that one can define an energy and that it is conserved, we will also show that we can bound the *kinetic* energy.

Lemma 4.1 (Continuity Equation). *Let $T > 0$ be the time of local existence of a solution f to the Vlasov-Poisson system (as given in Theorem 3.2). Let $t \in [0, T)$ and define the (vector-valued)* ***flux:***

$$j_f(t,x) = \int_{\mathbb{R}^3} pf(t,x,p)\,\mathrm{d}p.$$

Then the following (continuity) equation holds for every $x \in \mathbb{R}^3$:

$$\partial_t \rho_f + \nabla_x \cdot j_f = 0. \tag{4.3}$$

Proof. Integrate the Vlasov equation in p and eliminate the last term due to the divergence theorem. □

Definition 4.2 (Energy). *For a solution $f(t,x,p)$ of the Vlasov-Poisson system we define its* ***kinetic energy***

$$\mathcal{E}_{kin}(t) := \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{1}{2}|p|^2 f(t,x,p)\,\mathrm{d}x\,\mathrm{d}p,$$

its ***potential energy***

$$\mathcal{E}_{pot}(t) := \frac{\gamma}{2}\int_{\mathbb{R}^3} |\nabla\phi_f(t,x)|^2\,\mathrm{d}x = \frac{\gamma}{2}\int_{\mathbb{R}^3} \rho_f(t,x)\phi_f(t,x)\,\mathrm{d}x$$

and ***total energy***

$$\mathcal{E}(t) := \mathcal{E}_{kin}(t) + \mathcal{E}_{pot}(t).$$

Remark 4.3 (Attractive vs Repulsive Dynamics). Notice that the kinetic energy is a positive. However, the potential energy is only positive in the repulsive (plasma) regime; in the attractive (galactic) regime, the *potential energy is negative*. Hence in the **repulsive case**

$$\mathcal{E}(t) = \mathcal{E}_{kin}(t) + \mathcal{E}_{pot}(t), \qquad \mathcal{E}_{kin}(t), \mathcal{E}_{pot}(t) \geq 0,$$

since $\gamma = +1$, and in the **attractive case**

$$\mathcal{E}(t) = \mathcal{E}_{kin}(t) - |\mathcal{E}_{pot}(t)|, \qquad \mathcal{E}_{kin}(t) \geq 0, \mathcal{E}_{pot}(t) \leq 0,$$

since $\gamma = -1$. Assuming for the moment that the total energy is conserved (we will show this later), we immediately obtain a uniform bound for the potential and kinetic energies in the *repulsive* case. However, in the *attractive* case there's no *a priori* reason why both energies cannot blow up, while their sum remains constant. We will show that this does not happen.

Proposition 4.4 (Conservation of Total Energy). *The total energy $\mathcal{E}(t)$ is conserved.*

Proof. Multiply the Vlasov equation by $\frac{|p|^2}{2}$ and integrate in (x,p) to obtain

$$\begin{aligned}
0 &= \iint_{\mathbb{R}^6} \frac{|p|^2}{2}\partial_t f(t,x,p) + \iint_{\mathbb{R}^6} \frac{|p|^2}{2} p\cdot\nabla_x f(t,x,p) - \gamma\iint_{\mathbb{R}^6} \frac{|p|^2}{2}\nabla_x\phi_f(t,x)\cdot\nabla_p f(t,x,p) \\
&= \dot{\mathcal{E}}_{kin}(t) + \iint_{\mathbb{R}^6} \nabla_x \cdot \frac{|p|^2}{2} pf(t,x,p) + \gamma\iint_{\mathbb{R}^6} \nabla_x\phi_f(t,x)\cdot pf(t,x,p) \\
&= \dot{\mathcal{E}}_{kin}(t) + 0 - \gamma\int_{\mathbb{R}^3} \phi_f(t,x)\nabla_x\cdot j_f(t,x) \\
&= \dot{\mathcal{E}}_{kin}(t) + \gamma\int_{\mathbb{R}^3} \phi_f(t,x)\partial_t\rho_f(t,x) \\
&= \dot{\mathcal{E}}_{kin}(t) + \frac{\gamma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \phi_f(t,x)\rho_f(t,x) = \dot{\mathcal{E}}_{kin}(t) + \dot{\mathcal{E}}_{pot}(t) = \dot{\mathcal{E}}(t).
\end{aligned}$$

We used the continuity equation going from the second to the third line (exercise: verify the transition from the fourth to the fifth line) $\square$

Proposition 4.5 (Bounds on the Energy). *Let $f(t,x,p)$ be a classical solution of Vlasov-Poisson on $[0,T)$. Then for all $t \in [0,T)$:*

$$\mathcal{E}_{kin}(t), |\mathcal{E}_{pot}(t)|, \|\rho_f(t,\cdot)\|_{L^{5/3}(\mathbb{R}^3)} \leq C$$

where $C = C(\|f_0\|_\infty, \|f_0\|_1, \mathcal{E}_{kin}(0))$.

Proving Proposition 4.5 will require the following lemma (which we prove later):

Lemma 4.6 (Moment Estimates). *Let $g = g(x,p) : \mathbb{R}^6 \to \mathbb{R}_+$ be measurable and define*

$$m_k(g)(x) := \int_{\mathbb{R}^3} |p|^k g(x,p)\,\mathrm{d}p$$
$$M_k(g) := \iint_{\mathbb{R}^3\times\mathbb{R}^3} |p|^k g(x,p)\,\mathrm{d}p\,\mathrm{d}x$$

Let $p, p^* \in [1,\infty]$ satisfy $\frac{1}{p} + \frac{1}{p^*} = 1$, let $0 \leq k' \leq k < \infty$ and define

$$r = \frac{k + 3/p^*}{k' + 3/p^* + (k-k')/p}.$$

Then

$$\|m_{k'}(g)\|_r \leq c\|g\|_p^{\frac{k-k'}{k+3/p^*}} M_k(g)^{\frac{k'+3/p^*}{k+3/p^*}}$$

whenever the above quantities are finite, and $c = c(k,k',p) > 0$.

Proof of Proposition 4.5. Due to the conservation of total energy, the bounds on $\mathcal{E}_{kin}(t)$ and $\mathcal{E}_{pot}(t)$ in the repulsive case ($\gamma = +1$) are trivial. In the attractive case we use the Hardy-Littlewood-Sobolev inequality and Lemma 4.6 with $k = 2, k' = 0, p = 9/7, r = 6/5$ to obtain:

$$\begin{aligned}
|\mathcal{E}_{pot}(t)| &= \frac{1}{2}\int_{\mathbb{R}^3} \rho_f(t,x)\phi_f(t,x)\,\mathrm{d}x \\
&= \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_f(t,x)\rho_f(t,y)}{|x-y|}\,\mathrm{d}y\,\mathrm{d}x \\
&\leq c\|\rho_f(t,\cdot)\|_{6/5}^2 \\
&\leq c\|f(t,\cdot,\cdot)\|_{9/7}^{3/2}\mathcal{E}_{kin}^{1/2}(t).
\end{aligned}$$

Now we use the conservation of energy to bound $\mathcal{E}_{kin}(t)$:

$$\begin{aligned}
\text{const} = \mathcal{E}(t) &= \mathcal{E}_{kin}(t) + \mathcal{E}_{pot}(t) \\
&\geq \mathcal{E}_{kin}(t) - c\|f(t,\cdot,\cdot)\|_{9/7}^{3/2}\mathcal{E}_{kin}^{1/2}(t)
\end{aligned}$$

and since $c\|f(t,\cdot,\cdot)\|_{9/7}^{3/2}$ is some constant, the desired bound is achieved. For the bound on ρ, we use Lemma 4.6 with $k = 2, k' = 0, p = \infty, r = 5/3$ to obtain:

$$\|\rho_f(t,\cdot)\|_{5/3} \leq c\|f(t,\cdot,\cdot)\|_\infty^{2/5}\mathcal{E}_{kin}^{3/5}(t).$$

$\square$

Proof of Lemma 4.6. This will be included in these notes in the future. For now, this is left as an exercise (you can also find the proof in textbooks). $\square$

4.2 Remarks on Global Existence

Let us see if we can improve the Gronwall inequality stemming from the equation

$$P(t) = P(0) + C(f^0)\int_0^t P^2(s)\,\mathrm{d}s.$$

We know that $\|\nabla\phi_f\|_\infty \le c_p\|\rho_f\|_p^{p/3}\|\rho_f\|_\infty^{1-p/3}$, and so with $p = 5/3$ we obtain

$$\|\nabla\phi_f\|_\infty \le c\underbrace{\|\rho_f\|_{5/3}^{5/9}}_{\le\text{const}}\underbrace{\|\rho_f\|_\infty^{4/9}}_{\le(P^3(t))^{4/9}} \le cP^{4/3}(t).$$

Hence we obtain the improved bound

$$P(t) = P(0) + \int_0^t \|\nabla\phi_f(s,\cdot)\|_\infty\,\mathrm{d}s \le P(0) + c\int_0^t P^{4/3}(s)\,\mathrm{d}s.$$

This is better, but not good enough: the solution to this equation still has finite-time blowup. The exponent 4/3 comes from the estimate $\|\nabla\phi_f\|_\infty \le c_p\|\rho_f\|_p^{p/3}\|\rho_f\|_\infty^{1-p/3}$, and to lower it to an exponent that is ≤ 1 we would need the term $\|\rho_f\|_\infty$ to appear with a power no more than 1/3, hence $p = 2$. To summarise, using Lemma 4.6 we have

$$\begin{aligned}\|\nabla\phi_f\|_\infty &\le c\|\rho_f\|_2^{2/3}\|\rho_f\|_\infty^{1/3}\\ &\le c\|\rho_f\|_2^{2/3}P(t)\\ &\le c\|f\|_\infty^{1/2}\underbrace{\left(\iint_{\mathbb{R}^3\times\mathbb{R}^3}|p|^3 f(t,x,p)\,\mathrm{d}p\,\mathrm{d}x\right)^{1/2}}_{M_3^{1/2}(f)}P(t)\end{aligned}$$

which leads to:

Proposition 4.7. *The breakdown criterion of Theorem 3.2 can be modified: as long as $\|\rho_f\|_2$ or $M_3(f)$ remain bounded, the solution does not blowup.*

4.3 Proof of Global Existence and Uniqueness

As of the typing of these notes there are two main approaches for proving global existence and uniqueness of solutions. The first, which we shall pursue, involves a detailed analysis of the trajectory of one particle thereby obtaining *a priori* bounds for the maximal momentum $P(t)$. This approach is due to Pfaffelmoser [Pfaffelmoser1992], later improved somewhat by Schaeffer [Schaeffer1991] (Pfaffelmoser's paper took a long time to publish, and was therefore published after Schaeffer's paper) . The other approach, due to Lions and Perthame [Lions1991] follows the idea of Proposition 4.7 by obtaining *a priori* estimates for higher moments of f. The main difference between these two approaches is in that the second approach does not require the initial datum f^0 to be compactly supported, while the first does. This makes the second approach somewhat more physically relevant.

As we have already seen (see Proposition 3.7 for instance) momentum growth is due to the field $\mathbf{E}_f = -\nabla\phi_f$ which is given by the relation

$$\nabla\phi_f(t,x) = -\int\frac{x-y}{|x-y|^3}\rho_f(t,y)\,\mathrm{d}y = -\iint\frac{x-y}{|x-y|^3}f(t,y,p)\,\mathrm{d}p\,\mathrm{d}y.$$

This leads to the estimate

$$|\nabla\phi_f(t,x)| \le \iint\frac{f(t,y,p)}{|x-y|^2}\,\mathrm{d}p\,\mathrm{d}y.$$

As in Section 3 we denote $T > 0$ to be the maximal time of local existence and uniqueness of the Vlasov-Poisson system, and for $s, t \in [0, T)$ the characteristics of the Vlasov-Poisson system $(X(s;t,x,p), V(s;t,x,p))$ are solutions to:

$$\dot{X}(s;t,x,p) = V(s;t,x,p)$$
$$\dot{V}(s;t,x,p) = -\gamma \nabla \phi_f(t, X(s;t,x,p))$$

with initial conditions

$$X(t;t,x,p) = x \qquad V(t;t,x,p) = p$$

so that

$$f(t,x,p) = f_0(X(0;t,x,p), V(0;t,x,p)).$$

We now follow the approach of Pfaffelmoser, and *fix a characteristic* $(\widetilde{X}(s), \widetilde{V}(s))$ corresponding to one particle (we don't care here about dependence upon the other parameters) with $(\widetilde{X}(0), \widetilde{V}(0)) \in \operatorname{supp} f_0$. We want to study the increase in momentum of this particle, satisfying the simple estimate

$$|\widetilde{V}(t) - \widetilde{V}(t-\Delta)| \le \int_{t-\Delta}^{t} |\nabla \phi_f(s, \widetilde{X}(s))| \, \mathrm{d}s \le \int_{t-\Delta}^{t} \iint \frac{f(s,y,w)}{|\widetilde{X}(s) - y|^2} \, \mathrm{d}w \, \mathrm{d}y \, \mathrm{d}s. \tag{4.4}$$

This shall be the main task in the proof of the following theorem:

Theorem 4.8 (Global Existence of Classical Solutions). *Let $f_0(x,p) \in C_0^1(\mathbb{R}^3 \times \mathbb{R}^3)$ with $f_0 \ge 0$ be given. Then there exists a unique classical global solution $f(t,x,p)$ for the system* (4.1)-(4.2) *with $f(0,\cdot,\cdot) = f_0$.*

Proof. We first recall (see Theorem 2.4) that the mapping $(X(s;t,\cdot,\cdot), V(s;t,\cdot,\cdot)) : \mathbb{R}^6 \to \mathbb{R}^6$ is orientation and measure preserving. Hence letting

$$y = X(s;t,x,p), \qquad w = V(s;t,x,p),$$
$$x = X(t;s,y,w), \qquad p = V(t;s,y,w),$$

and using the fact that f is constant along characteristics, the estimate (4.4) can be rewritten as

$$|\widetilde{V}(t) - \widetilde{V}(t-\Delta)| \le \int_{t-\Delta}^{t} \iint \frac{f(t,x,p)}{|\widetilde{X}(s) - X(s;t,x,p)|^2} \, \mathrm{d}p \, \mathrm{d}x \, \mathrm{d}s. \tag{4.5}$$

The goal in this proof will be to estimate this integral. We shall break up the domain of integration into three parts, *the good*, *the bad* and *the ugly*. Fix some parameters (to be made precise later) $q, r > 0$ and define:

$$M_g := \left\{ (s,x,p) \in [t-\Delta, t] \times \mathbb{R}^6 \ : \ |p| \le q \ \vee \ |p - \widetilde{V}(t)| \le q \right\}$$

$$M_b := \Big\{ (s,x,p) \in [t-\Delta, t] \times \mathbb{R}^6 \ : \ |p| > q \ \wedge \ |p - \widetilde{V}(t)| > q \ \wedge$$
$$\wedge \left[|X(s;t,x,p) - \widetilde{X}(s)| \le r|p|^{-3} \ \vee \ |X(s;t,x,p) - \widetilde{X}(s)| \le r|p - \widetilde{V}(t)|^{-3} \right] \Big\}$$

$$M_u := \Big\{ (s,x,p) \in [t-\Delta, t] \times \mathbb{R}^6 \ : \ |p| > q \ \wedge \ |p - \widetilde{V}(t)| > q \ \wedge$$
$$\wedge \ |X(s;t,x,p) - \widetilde{X}(s)| > r|p|^{-3} \ \wedge \ |X(s;t,x,p) - \widetilde{X}(s)| > r|p - \widetilde{V}(t)|^{-3} \Big\}.$$

The *good set* is the set on which either momenta are small, or momenta relative to the chosen particle are small.

The *bad set* is the set on which both the momenta and relative momenta are large, but the positions are close to the chosen particle.

The *ugly set* is the set on which momenta and relative momenta are large, and particles are far away from the chosen particle.

To proceed, we first redefine $P(t)$ (the maximal momentum at time t, originally defined in (3.9)) so that it is a monotonically increasing function:

$$P(t) := \sup_{\substack{(x,p)\in \operatorname{supp} f(s,\cdot,\cdot) \\ s\in[0,t]}} |p|. \tag{4.6}$$

Choice of Δ. We choose Δ to be sufficiently small so that momenta don't change too much between $t-\Delta$ and t. Recalling the estimate $\|\nabla\phi_f(t,\cdot)\|_\infty \le cP^{4/3}(t)$, we define

$$\Delta := \min\left\{t, \frac{q}{4cP^{4/3}(t)}\right\} \tag{4.7}$$

so that for all $s \in [t-\Delta, t]$, $(x,p) \in \mathbb{R}^6$,

$$|V(s;t,x,p) - p| \le \Delta c P^{4/3}(t) \le \frac{1}{4}q. \tag{4.8}$$

The Good Set. We will show that

$$\iiint_{M_g} \frac{f(t,x,p)}{|\widetilde{X}(s) - X(s;t,x,p)|^2}\,\mathrm{d}p\,\mathrm{d}x\,\mathrm{d}s \le cq^{4/3}\Delta. \tag{4.9}$$

The Bad Set. We will show that

$$\iiint_{M_b} \frac{f(t,x,p)}{|\widetilde{X}(s) - X(s;t,x,p)|^2}\,\mathrm{d}p\,\mathrm{d}x\,\mathrm{d}s \le cr\ln\left(\frac{4P(t)}{q}\right)\Delta. \tag{4.10}$$

The Ugly Set. We will show that

$$\iiint_{M_u} \frac{f(t,x,p)}{|\widetilde{X}(s) - X(s;t,x,p)|^2}\,\mathrm{d}p\,\mathrm{d}x\,\mathrm{d}s \le \frac{c}{r}. \tag{4.11}$$

Estimating the Integral. Before proceeding to prove (4.9),(4.10) and (4.11) we first add these estimates up to see what we get. They give:

$$\begin{aligned}
|\widetilde{V}(t) - \widetilde{V}(t-\Delta)| &\le c\left(q^{4/3} + r\ln\left(\frac{4P(t)}{q}\right) + \frac{1}{r\Delta}\right)\Delta \\
&\le c\left(q^{4/3} + r\ln\left(\frac{4P(t)}{q}\right) + \frac{1}{r}\max\left\{\frac{1}{t}, \frac{4cP^{4/3}(t)}{q}\right\}\right)\Delta.
\end{aligned}$$

Now we want to optimise the choice of q and r. Without loss of generality, there exists t for which $P(t) > 1$ (otherwise we replace $P(t)$ by $P(t)+1$). Recall also that $P(t)$ is now defined to be monotonically increasing. Hence, defining

$$q = P^{4/11}(t) \qquad r = P^{16/33}(t),$$

we have that $q \le P(t)$. Moreover, if the maximal time of existence $T < \infty$, then we know that $P(t) \uparrow \infty$ as $T \uparrow \infty$. Hence there exists $T^* \in (0,T)$ such that $\frac{1}{t} \le \frac{4cP^{4/3}(t)}{q} = 4cP^{32/33}(t)$ for all $t \in [T^*, T)$. Therefore, for all $t \in [T^*, T)$

$$|\widetilde{V}(t) - \widetilde{V}(t-\Delta)| \le cP^{16/33}(t)\ln P(t)\Delta$$

so that for any $\varepsilon > 0$ there exists some $c = c(\varepsilon) > 0$ such that

$$|\widetilde{V}(t) - \widetilde{V}(t-\Delta)| \le cP^{16/33+\varepsilon}(t)\Delta. \tag{4.12}$$

Now, notice that $\Delta(t)$ is monotonically decreasing on (T^*, T). Fix some $t \in (T^*, T)$, and define

$$\begin{aligned} t_0 &= t \\ t_{i+1} &= t_i - \Delta(t_i). \end{aligned}$$

Since $\Delta(t)$ is monotonically decreasing, there exists some k such that

$$t_k < T^* \le t_{k-1} < t_{k-2} < \cdots < t_1 < t_0 = t.$$

Hence from (4.12) we get

$$\begin{aligned} |\widetilde{V}(t) - \widetilde{V}(t_k)| &\le \sum_{i=1}^{k} |\widetilde{V}(t_{i-1}) - \widetilde{V}(t_i)| \\ &\le cP^{16/33+\varepsilon}(t) \sum_{i=1}^{k} (t_{i-1} - t_i) \\ &\le cP^{16/33+\varepsilon}(t)t. \end{aligned}$$

However this implies (due to the very definition of $P(t)$) that $P(t) \le P(t_k) + cP^{16/33+\varepsilon}(t)t$ so that for any $\delta > 0$ there exists some $c = c(\delta) > 0$ such that

$$P(t) \le c(1+t)^{33/17+\delta}, \qquad \forall t \in [0, T)$$

which implies that $T = \infty$ (see Theorem 3.2) and finishes the proof. Hence we are only left with verifying the estimates (4.9),(4.10) and (4.11).

The Good: Proof of (4.9)**.** We want to show that on the *good set*, the following holds:

$$\iiint_{M_g} \frac{f(t,x,p)}{|\widetilde{X}(s) - X(s;t,x,p)|^2} \,\mathrm{d}p\,\mathrm{d}x\,\mathrm{d}s \le cq^{4/3}\Delta.$$

Recall (4.8): $|V(s;t,x,p) - p| \le \frac{1}{4}q$. Then if $(s,x,p) \in M_g$, in the original coordinates (s,y,w) the following holds:

$$|w| < 2q \vee |w - \widetilde{V}(s)| < 2q.$$

Hence we get

$$\begin{aligned} &\iiint_{M_g} \frac{f(t,x,p)}{|\widetilde{X}(s) - X(s;t,x,p)|^2} \,\mathrm{d}p\,\mathrm{d}x\,\mathrm{d}s \\ &\le \int_{t-\Delta}^{t} \int_{\mathbb{R}^3} \int_{|w|<2q \,\vee\, |w-\widetilde{V}(s)|<2q} \frac{f(s,y,w)}{|\widetilde{X}(s) - y|^2} \,\mathrm{d}w\,\mathrm{d}y\,\mathrm{d}s \\ &= \int_{t-\Delta}^{t} \int_{\mathbb{R}^3} \frac{\tilde{\rho}(s,y)}{|\widetilde{X}(s) - y|^2} \,\mathrm{d}y\,\mathrm{d}s \qquad \left[\text{where } \tilde{\rho}(s,y) := \int_{|w|<2q \,\vee\, |w-\widetilde{V}(s)|<2q} f(s,y,w)\,\mathrm{d}w\right] \\ &\le c \int_{t-\Delta}^{t} \underbrace{\|\tilde{\rho}(s,\cdot)\|_{5/3}^{5/9}}_{\le \text{const}} \underbrace{\|\tilde{\rho}(s,\cdot)\|_{\infty}^{4/9}}_{\le (cq^3)^{4/9}} \,\mathrm{d}s \qquad \text{[by Propositions 3.7 and 4.5]} \\ &\le cq^{4/3}\Delta. \end{aligned}$$

The Bad: Proof of (4.10)**.** We want to show that on the *bad set*, the following holds:

$$\iiint_{M_b} \frac{f(t,x,p)}{|\widetilde{X}(s) - X(s;t,x,p)|^2} \,\mathrm{d}p\,\mathrm{d}x\,\mathrm{d}s \le cr \ln\left(\frac{4P(t)}{q}\right)\Delta.$$

Recall again (4.8): $|V(s;t,x,p) - p| \le \frac{1}{4}q$. If $(s,x,p) \in M_b$, in the original coordinates (s,y,w) the following holds:

$$\begin{aligned} &\frac{1}{2}q < |w| < 2|p| \wedge \frac{1}{2}q < |w - \widetilde{V}(s)| < 2|p - \widetilde{V}(t)| \wedge \\ &\qquad \wedge \left[|y - \widetilde{X}(s)| \le 8r|w|^{-3} \vee |y - \widetilde{X}(s)| \le 8r|w - \widetilde{V}(s)|^{-3}\right]. \end{aligned}$$

Now, we also use the fact that for $s \in [0,t]$ and $w \in \operatorname{supp} f(s,y,\cdot)$ it holds that $|w| \le P(t)$, $|w - \widetilde{V}(s)| \le 2P(t)$. Using also the conservation of L^∞ norms, we estimate

$$\begin{aligned} &\iiint_{M_b} \frac{f(t,x,p)}{|\widetilde{X}(s) - X(s;t,x,p)|^2} \,\mathrm{d}p\,\mathrm{d}x\,\mathrm{d}s \\ &\le \int_{t-\Delta}^{t} \int_{\frac{1}{2}q<|w|<2P(t)} \int_{|y-\widetilde{X}(s)|<8r|w|^{-3}} \frac{f(s,y,w)}{|\widetilde{X}(s) - y|^2} \,\mathrm{d}y\,\mathrm{d}w\,\mathrm{d}s \\ &\quad + \int_{t-\Delta}^{t} \int_{\frac{1}{2}q<|w-\widetilde{V}(s)|<2P(t)} \int_{|y-\widetilde{X}(s)|<8r|w-\widetilde{V}(s)|^{-3}} \frac{f(s,y,w)}{|\widetilde{X}(s) - y|^2} \,\mathrm{d}y\,\mathrm{d}w\,\mathrm{d}s \\ &\le c\int_{t-\Delta}^{t} \int_{\frac{1}{2}q<|w|<2P(t)} 4\pi \cdot 8r|w|^{-3} \,\mathrm{d}w\,\mathrm{d}s \\ &\quad + c\int_{t-\Delta}^{t} \int_{\frac{1}{2}q<|w-\widetilde{V}(s)|<2P(t)} 4\pi \cdot 8r|w - \widetilde{V}(s)|^{-3} \,\mathrm{d}w\,\mathrm{d}s \\ &\le cr\ln\left(\frac{4P(t)}{q}\right)\Delta \end{aligned}$$

where we have integrated in y and in w by changing to spherical coordinates.

The Ugly: Proof of (4.11)**.** In estimating the integral over the ugly set we shall, for the first time, use some smoothing properties of the time integral, rather than simply estimate it by Δ. Recall, that we want to show that

$$\iiint_{M_u} \frac{f(t,x,p)}{|\widetilde{X}(s) - X(s;t,x,p)|^2} \,\mathrm{d}p\,\mathrm{d}x\,\mathrm{d}s \le \frac{c}{r}$$

where M_u is defined as

$$\begin{aligned} M_u := \Big\{ &(s,x,p) \in [t-\Delta, t] \times \mathbb{R}^6 \;:\; |p| > q \wedge |p - \widetilde{V}(t)| > q \wedge \\ &\wedge |X(s;t,x,p) - \widetilde{X}(s)| > r|p|^{-3} \wedge |X(s;t,x,p) - \widetilde{X}(s)| > r|p - \widetilde{V}(t)|^{-3} \Big\}. \end{aligned}$$

First, we want a lower bound for the distance between the particles, which we denote, for $s \in [t-\Delta, t]$,

$$D(s) := X(s;t,x,p) - \widetilde{X}(s).$$

Note that $D : [t-\Delta, t] \to \mathbb{R}^3$. We claim that for all $s \in [t-\Delta, t]$ and $(x,p) \in \mathbb{R}^3 \times \mathbb{R}^3$ with $|p - \widetilde{V}(t)| > q$ the following lower bound holds:

$$|D(s)| \ge \frac{1}{4}|p - \widetilde{V}(t)||s - s_0|. \tag{4.13}$$

We prove this by comparing to a linear approximation, defined as follows. Let $s_0 \in [t-\Delta, t]$ be such that $|D(s)|$ attains a minimum there and let

$$\overline{D}(s) := D(s_0) + \dot{D}(s_0)(s - s_0)$$

be the tangent (i.e. linear approximation) to $D(s)$ at s_0. Hence D and $\overline{D}$ and their first derivatives agree at s_0. As for the second derivative, we have

$$|\ddot{D}(s) - \ddot{\overline{D}}(s)| = |\dot{V}(s;t,x,p) - \dot{\widetilde{V}}(s)| \leq 2\|\nabla\phi_f(s,\cdot)\|_\infty \leq cP^{4/3}(t).$$

Therefore, a simple Taylor expansion gives

$$\begin{aligned}|D(s) - \overline{D}(s)| &\leq cP^{4/3}(t)(s-s_0)^2 \\ &\leq cP^{4/3}(t)\Delta|s-s_0| \\ &\leq \frac{1}{4}q|s-s_0| \\ &< \frac{1}{4}|p - \widetilde{V}(t)||s-s_0|. \end{aligned} \tag{4.14}$$

Next, let us show that

$$|\overline{D}(s)|^2 \geq \frac{1}{4}|p - \widetilde{V}(t)|^2|s-s_0|^2. \tag{4.15}$$

Indeed, observe that

$$\begin{aligned}|p - \widetilde{V}(t)| &\leq |p - V(s_0;t,x,p)| + |\widetilde{V}(s_0) - \widetilde{V}(t)| + |V(s_0;t,x,p) - \widetilde{V}(s_0)| \\ &\leq \frac{1}{2}q + |V(s_0;t,x,p) - \widetilde{V}(s_0)|\end{aligned}$$

so that

$$|\dot{D}(s_0)| = |V(s_0;t,x,p) - \widetilde{V}(s_0)| \geq |p - \widetilde{V}(t)| - \frac{1}{2}q > \frac{1}{2}|p - \widetilde{V}(t)|.$$

By the definition of s_0 we have that $(s-s_0)D(s_0)\cdot\dot{D}(s_0) \geq 0$ for all $s \in [t-\Delta, t]$, which is enough to prove (4.15). Combining (4.14) and (4.15) we have (4.13).

Now define functions $\sigma_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, 2$, as

$$\sigma_1(\xi) := \begin{cases} \xi^{-2} & \xi > r|p|^{-3} \\ (r|p|^{-3})^{-2} & \xi \leq r|p|^{-3} \end{cases}$$

$$\sigma_2(\xi) := \begin{cases} \xi^{-2} & \xi > r|p - \widetilde{V}(t)|^{-3} \\ (r|p - \widetilde{V}(t)|^{-3})^{-2} & \xi \leq r|p - \widetilde{V}(t)|^{-3}. \end{cases}$$

Using (4.13), the definition of M_u and the fact that both σ_i are monotonically decreasing, we have the estimate (here $\mathbb{1}_U$ denoted the characteristic function of the set U)

$$\frac{\mathbb{1}_{M_u}(s,x,p)}{|D(s)|^2} \leq \sigma_i(|D(s)|) \leq \sigma_i\left(\frac{1}{4}|p - \widetilde{V}(t)||s-s_0|\right)$$

for $i = 1, 2$ and $s \in [t-\Delta, t]$. This allows us to estimate the integral over M_u by first integrating in time:

$$\begin{aligned}\int_{t-\Delta}^{t} \frac{\mathbb{1}_{M_u}(s,x,p)}{|D(s)|^2}\,\mathrm{d}s &\leq 8|p - \widetilde{V}(t)|^{-1}\int_0^\infty \sigma_i(\xi)\,\mathrm{d}\xi \\ &= 16|p - \widetilde{V}(t)|^{-1}\begin{cases} r^{-1}|p|^3 & i = 1 \\ r^{-1}|p - |\widetilde{V}(t)|^3 & i = 2\end{cases}\end{aligned}$$

which in turn implies (by taking the minimum of the right hand side)

$$\int_{t-\Delta}^{t} \frac{\mathbb{1}_{M_u}(s,x,p)}{|D(s)|^2}\,\mathrm{d}s \le 16r^{-1}|p|^2.$$

Therefore we are finally left with

$$\begin{aligned}\iiint_{M_u} \frac{f(t,x,p)}{|\widetilde{X}(s) - X(s;t,x,p)|^2}\,\mathrm{d}p\,\mathrm{d}x\,\mathrm{d}s &\le \iint_{\mathbb{R}^6} f(t,x,p) \int_{t-\Delta}^{t} \frac{\mathbb{1}_{M_u}(s,x,p)}{|D(s)|^2}\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}p \\ &\le \frac{c}{r} \iint_{\mathbb{R}^6} |p|^2 f(t,x,p)\,\mathrm{d}x\,\mathrm{d}p \\ &\le \frac{c}{r}\end{aligned}$$

since the kinetic energy is bounded. This concludes the proof. □

5 Linear Wave Equations

One of the main topics of this course is the study of *wave equations*. Throughout, we use the term "wave equations" to describe a broad class of PDEs, both linear and nonlinear, whose principal part consists of the wave operator,

$$\Box := -\partial_t^2 + \Delta_x := -\partial_t^2 + \sum_{k=1}^{n} \partial_{x^k}^2. \tag{5.1}$$

Historically, wave equations arose in the study of vibrating strings. Since then, wave equations have played fundamental roles in both mathematics and physics. For instance:

- Wave equations serve as prototypical examples of a wider class of PDEs known as *hyperbolic PDEs*. Solutions of hyperbolic PDEs share a number of fundamental properties, such as finite speed of propagation. Thus, understanding the basic wave equation is an important step in studying hyperbolic PDEs in greater generality.

- Wave (and, more generally, hyperbolic) equations can be found in many fundamental equations of physics, including the Maxwell equations of electromagnetism and the Einstein field equations of general relativity. Therefore, in order to better grasp these physical theories, one must understand phenomena arising from wave equations.

The study of nonlinear wave equations in general is very difficult, with many important questions still left unanswered. However, research efforts over the past few decades have proved vastly fruitful in situations where the nonlinear theory as a perturbation of the linear theory. These refer to settings in which solutions of the nonlinear wave equation behave very closely to solutions of the linearised equation. In particular, the nonlinear theory we will eventually consider in these notes will be perturbative in this sense.

Thus, before tackling nonlinear wave equations, one must first understand the theory of *linear wave equations*. This is the main topic of this chapter. More specifically, we will discuss the *initial value*, or *Cauchy*, problem for both of the following:

1. *Homogeneous* linear wave equation: we look for a solution $\phi : \mathbb{R}_t \times \mathbb{R}_x^n \to \mathbb{R}$ of

$$\Box\phi = 0, \qquad \phi|_{t=0} = \phi_0, \quad \partial_t\phi|_{t=0} = \phi_1, \tag{5.2}$$

   where $\phi_0, \phi_1 : \mathbb{R}^n \to \mathbb{R}$ comprise the *initial data*.

2. *Inhomogeneous* linear wave equation: we look for a solution $\phi : \mathbb{R}_t \times \mathbb{R}_x^n \to \mathbb{R}$ of

$$\Box\phi = F, \qquad \phi|_{t=0} = \phi_0, \quad \partial_t\phi|_{t=0} = \phi_1, \tag{5.3}$$

   where ϕ_0, ϕ_1 are as before, and $F : \mathbb{R}_t \times \mathbb{R}_x^n \to \mathbb{R}$ is the forcing term.

Even the linear equations (5.2) and (5.3) have a rich theory. There are multiple viewpoints that one may adopt when studying these equations, which, broadly speaking, one can separate into "physical space" and "Fourier space" methods. As we shall see, different methods will prove useful in extracting different properties of solutions. In this chapter, we focus on this diversity of methods, and we compare and contrast the types of information that can be gleaned from each of these methods.

For convenience, we will adopt the following notations throughout this chapter:

- We write $A \lesssim_{c_1,\dots,c_m} B$ to mean $A \leq CB$ for some constant depending on $c_1, \dots, c_m$. When no constants c_k are given, the constant C is presumed to be universal.

- Unless otherwise specified, we assume various spaces of functions are over $\mathbb{R}^n$—for instance, $L^p := L^p(\mathbb{R}^n)$ and $H^s := H^s(\mathbb{R}^n)$. Similar conventions hold for the corresponding norms: $\| \cdot \|_{L^p} := \| \cdot \|_{L^p(\mathbb{R}^n)}$ and $\| \cdot \|_{H^s} := \| \cdot \|_{H^s(\mathbb{R}^n)}$.
- We let $\mathcal{S} := \mathcal{S}(\mathbb{R}^n)$ denote the Schwartz space of smooth, rapidly decreasing functions:

$$\mathcal{S} = \{ f \in C^\infty(\mathbb{R}^n) \mid |\nabla_x^k f(x)| \leq (1 + |x|)^{-N} \text{ for all } k, N \geq 0 \}.$$

- Given $x_0 \in \mathbb{R}^n$ and $R > 0$, we define the ball and sphere about x_0 of radius R in $\mathbb{R}^n$:

$$B_{x_0}(R) := \{ x \in \mathbb{R}^n \mid |x - x_0| < R \}, \qquad S_{x_0}(R) := \{ x \in \mathbb{R}^n \mid |x - x_0| = R \}.$$

  In particular, we view the unit sphere $\mathbb{S}^{n-1}$ as the embedded sphere $S_0(1) \subseteq \mathbb{R}^n$.

To be precise, unless otherwise specified, integrals over $S_{x_0}(R)$ will be with respect to the volume forms induced from $\mathbb{R}^n$. Sometimes, this volume form is denoted by $d\sigma$.

5.1 Physical Space Formulas

The first method we discuss is to derive explicit formulas for the solutions of the homogeneous wave equation (5.2) in physical space. By "physical space" formulas, we mean formulas for the solution ϕ itself, as functions of the Cartesian coordinates t and x (as opposed to formulas for the Fourier transform, spatial or spacetime, of ϕ).

Explicit formulas for ϕ are nice in that they provide very direct information. For instance, these immediately imply the all-important finite speed of propagation properties of waves (as well as the strong Huygens principle for odd n). Moreover, from a more careful examination of these equations, one can also derive various asymptotic decay properties of waves.

On the other hand, what is not at all apparent from these formulas are the L^2-based properties of waves, i.e., conservation of energy and other energy-type estimates. These are not only fundamental properties, but they are also robust, in that they carry over to the analysis of nonlinear waves as well as waves on other backgrounds besides $\mathbb{R} \times \mathbb{R}^n$. In particular, methods which are heavily reliant on explicit formulas for (5.2) will likely fail to be useful in other more general settings of interest. As a result of this, we give only an abridged treatment of this method for completeness; the reader is referred to Section 2.4 of [Evan2002] for more detailed developments.

Another unfortunate feature of these explicit formulas is that they differ largely depending on the dimension n. As a result, we will have to treat different dimensions separately.

5.1.1 $n = 1$: D'Alembert's Formula

In this case, one can solve (5.2) using a simple change of variables. More specifically, rather than t and x, we consider instead the *null coordinates*,

$$u = t - x, \qquad v = t + x.$$

In these coordinates, the wave equation can expanded as

$$\partial_u \partial_v \phi = -\Box \phi = 0, \tag{5.4}$$

One can then integrate (5.4) directly. This yields

$$\partial_v \phi = G(v), \qquad \phi = \int_{-u}^{v} G(s) ds + H(u), \qquad \partial_u \phi = G(-u) + H'(u), \tag{5.5}$$

for some functions G and H (we integrate G from $-u$ here for later convenience). Both G and H in (5.4) can then be determined by the initial data (note that $x = -u = v$ when $t = 0$, and that $\partial_t = \partial_u + \partial_v$); a brief computation then yields

$$G(s) = \frac{1}{2}\phi_1(s) + \frac{1}{2}\phi_0'(s), \qquad H(s) = \phi_0(-s). \tag{5.6}$$

Combining (5.5) and (5.6), we arrive at *d'Alembert's formula*:

Theorem 5.1 (D'Alembert's formula). *Consider the problem* (5.2)*, with* $n = 1$*. If* $\phi_0 \in C^2(\mathbb{R})$ *and* $\phi_1 \in C^1(\mathbb{R})$*, then the function* $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$*, defined*

$$\phi(t,x) = \frac{1}{2}[\phi_0(x-t) + \phi_0(x+t)] + \frac{1}{2}\int_{x-t}^{x+t} \phi_1(\xi) d\xi, \tag{5.7}$$

is C^2 *and solves* (5.2)*.*

Proof. By a direct computation, one can verify that ϕ in (5.7) indeed solves (5.2). $\square$

Remark 5.2. Note the discussion preceding the statement of Theorem 5.1 serves as a proof of uniqueness for solutions of (5.2). Indeed, any solution of (5.2) that is regular enough so that the above manipulations are well-defined must in fact be the function (5.7). The same observation holds for the formulas in higher dimensions discussed below.

5.1.2 Odd $n > 1$: Method of Spherical Means

Unfortunately, in higher dimensions, one cannot concoct a similar change of variables to obtain an equation that can be integrated directly. However, there is a trick which reduces the problem, at least for odd spatial dimensions, to that of the previous case $n = 1$.

Roughly, the main idea is to write the wave equation in spherical coordinates,

$$-\partial_t^2\phi + \partial_r^2\phi + \frac{n-1}{r}\partial_r\phi + r^{-2}\not{\Delta}\phi = 0, \tag{5.8}$$

with $\not{\Delta}$ denoting the Laplace operator on the $(n-1)$-dimensional unit sphere $\mathbb{S}^{n-1}$. Now, if we integrate (5.8) over a sphere about the origin at a fixed time (i.e., over a level set of (t,r)), then the divergence theorem eliminates the spherical Laplacian. The resulting equation for these spherical averages of ϕ is then very close to the $(1+1)$-dimensional wave equation, for which we can solve using d'Alembert's formula. Moreover, this process can be repeated for spheres centered around any point $x \in \mathbb{R}^n$, not just the origin.

Because the main step of the process is this averaging of ϕ over spheres, this trick is usually referred to as the *method of spherical means.* Here, we briefly summarise the process in the case $n = 3$, for which the formulas remain relatively simple. We only state without derivation the result for higher (odd) dimensions.

For any point $(t,x) \in \mathbb{R} \times \mathbb{R}^3$ and $r \in \mathbb{R}$, we define

$$M(t,x,r) := \frac{1}{4\pi}\int_{\mathbb{S}^2} \phi(t, x+ry) d\sigma(y),$$

where $d\sigma$ denotes the surface measure of the unit sphere $\mathbb{S}^2$. In particular, $M(t,x,r)$ represents the mean of ϕ over a sphere about (t,x) of radius $|r|$. Furthermore, one computes (noting that the spherical integral kills the spherical Laplacian) that

$$-\partial_t^2(rM) + \partial_r^2(rM) = 0,$$

Thus, rM satisfies the (1+1)-dimensional wave equation in t and r (for any fixed $x \in \mathbb{R}^3$), hence rM can be expressed explicitly using d'Alembert's formula. Also, by definition,

$$\phi(t, x) := \lim_{r \to 0} M(t, x, r).$$

Combining all the above, we arrive at *Kirchhoff's formula*:

Theorem 5.3 (Kirchhoff's formula). *Consider the problem* (5.2)*, with* $n = 3$*. Assuming* $\phi_0 \in C^3(\mathbb{R}^3)$ *and* $\phi_1 \in C^2(\mathbb{R}^3)$*, then the function* $\phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$*, defined*

$$\begin{aligned} \phi(t, x) &= \frac{1}{4\pi} \int_{\mathbb{S}^2} [\phi_0(x + ty) + ty \cdot \nabla_x \phi_0(x + ty) + t \cdot \phi_1(x + ty)] d\sigma(y) \\ &= \frac{1}{4\pi t^2} \int_{S_x(|t|)} [\phi_0(y) + (y - x) \cdot \nabla_y \phi_0(y) + t \cdot \phi_1(y)] d\sigma(y). \end{aligned} \tag{5.9}$$

is C^2 *and solves* (5.2)*.*

Again, this can be proved by directly verifying that (5.9) satisfies (5.2).

For odd $n > 3$, an explicit solution can be derived by a similar process via spherical means. For brevity, we merely state the result here:

Theorem 5.4. *Consider the problem* (5.2)*, with* $n > 1$ *being odd. If* $\phi_0 \in C^{(n+3)/2}(\mathbb{R}^3)$ *and* $\phi_1 \in C^{(n+1)/2}(\mathbb{R}^3)$*, then the function* $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$*, defined*

$$\phi(t, x) = \frac{1}{\gamma_n} \partial_t (t^{-1} \partial_t)^{\frac{n-3}{2}} \left[t^{-1} \int_{S_x(|t|)} \phi_0 \right] + \frac{1}{\gamma_n} (t^{-1} \partial_t)^{\frac{n-3}{2}} \left[t^{-1} \int_{S_x(|t|)} \phi_1 \right], \tag{5.10}$$

is C^2 *and solves* (5.2)*, where* $\gamma_n := [1 \cdot 3 \cdot \dots \cdot (n - 2)] \cdot |\mathbb{S}^{n-1}|$*.*

5.1.3 Even n: Method of Descent

For even dimensions, the main idea is to convert this to an odd-dimensional problem by adding a "dummy" variable $x_{n+1} \in \mathbb{R}$, with both ϕ_0 and ϕ_1 independent of this x_{n+1}. Thinking of this as an $(n + 1)$-dimensional problem, we can now apply the previous results of Theorems 5.3 and 5.4. This is known as the *method of descent.*

Again, we summarise this process only for $n = 2$, for which the formulas are relatively simple. We now add a dummy variable $x^3 \in \mathbb{R}$, and we define

$$\tilde{\phi}_0(t, x, x^3) := \phi_0(t, x), \qquad \tilde{\phi}_1(t, x, x^3) := \phi_1(t, x).$$

Letting $x' = (x, x^3)$, then by (5.9), we see that the solution to (5.2), in the case $n = 3$ with initial data $\tilde{\phi}_0$ and $\tilde{\phi}_1$, is given by

$$\tilde{\phi}(t, x') = \frac{1}{4\pi} \int_{\mathbb{S}^2} [\tilde{\phi}_0(x' + ty') + ty' \cdot \nabla_{x'} \tilde{\phi}_0(x' + ty') + t \cdot \tilde{\phi}_1(x' + ty')] d\sigma(y'). \tag{5.11}$$

Now, the integrals over the hemispheres $x^3 > 0$ and $x^3 < 0$ of $\mathbb{S}^2$ can be written as weighted integrals of the unit disk in $\mathbb{R}^2$. Indeed, one can compute that for any $f : \mathbb{S}^2 \to \mathbb{R}$,

$$\int_{\mathbb{S}^2 \cap \{x^3 > 0\}} f(y') d\sigma(y') = \int_{B_0(1)} \frac{f(y^1, y^2, \sqrt{1 - |y|^2})}{\sqrt{1 - |y|^2}} dy, \tag{5.12}$$

$$\int_{\mathbb{S}^2 \cap \{x^3 < 0\}} f(y') d\sigma(y') = \int_{B_0(1)} \frac{f(y^1, y^2, -\sqrt{1 - |y|^2})}{\sqrt{1 - |y|^2}} dy. \tag{5.13}$$

Thus, combining (5.11) and (5.12), and recalling the definitions of $\tilde{\phi}_0$ and $\tilde{\phi}_1$, we obtain

$$\tilde{\phi}(t, x') = \frac{1}{2\pi} \int_{B_0(1)} \frac{\phi_0(x + ty) + ty \cdot \nabla_x \phi_0(x + ty) + t \cdot \phi_1(x + ty)}{\sqrt{1 - |y|^2}} dy. \tag{5.14}$$

Note that $\tilde{\phi}$ is in fact independent of x^3. Thus, defining

$$\phi(t, x) := \tilde{\phi}(t, x'),$$

it follows that ϕ solves (5.2), for $n = 2$ and for initial data ϕ_0 and ϕ_1:

Theorem 5.5 (Method of descent). *Consider the problem* (5.2)*, in the case $n = 2$. If $\phi_0 \in C^3(\mathbb{R}^2)$ and $\phi_1 \in C^2(\mathbb{R}^2)$, then the function $\phi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$, defined*

$$\begin{aligned} \phi(t, x) &= \frac{1}{2\pi} \int_{B_0(1)} \frac{\phi_0(x + ty) + ty \cdot \nabla_x \phi_0(x + ty) + t \cdot \phi_1(x + ty)}{\sqrt{1 - |y|^2}} dy & (5.15) \\ &= \frac{1}{2\pi t} \int_{B_x(|t|)} \frac{\phi_0(y) + (y - x) \cdot \nabla_y \phi_0(y) + t \cdot \phi_1(y)}{\sqrt{t^2 - |y - x|^2}} dy, & (5.16) \end{aligned}$$

is C^2 and solves (5.2).

Again, one can verify that (5.15) is a solution of (5.2) through direct computation. Finally, we state without proof the formulas for the solution of (5.2) in all even dimensions:

Theorem 5.6. *Consider the problem* (5.2)*, with n being even. If $\phi_0 \in C^{(n+4)/2}(\mathbb{R}^n)$ and $\phi_1 \in C^{(n+2)/2}(\mathbb{R}^n)$, then the function $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, defined*

$$\begin{aligned} \phi(t, x) &= \frac{1}{\gamma_n} \partial_t (t^{-1} \partial_t)^{\frac{n-2}{2}} \left[\int_{B_x(|t|)} \frac{\phi_0(y)}{\sqrt{t^2 - |y - x|^2}} dy \right] \\ &\qquad + \frac{1}{\gamma_n} (t^{-1} \partial_t)^{\frac{n-2}{2}} \left[\int_{B_x(|t|)} \frac{\phi_1(y)}{\sqrt{t^2 - |y - x|^2}} dy \right], \end{aligned} \tag{5.17}$$

is C^2 and solves (5.2)*, and where $\gamma_n := (2 \cdot 4 \cdot \dots \cdot n) \cdot |B_0(1)|$.*

5.1.4 Finite Speed of Propagation

A fundamental property of waves that can be immediately seen from the physical space formulas (5.7), (5.10), and (5.17) is *finite speed of propagation.* If one alters the initial data ϕ_0, ϕ_1 in a small region, then that change travels in the solution ϕ at a finite speed, so that ϕ will not change at any point "far away" from where ϕ_0 and ϕ_1 were changed.

To be more illustrative, suppose we first consider trivial initial data, $\phi_0 = \phi_1 \equiv 0$. Then, the solution ϕ to (5.2) is simply the zero function, $\phi(t, x) = 0$. Next, suppose we alter ϕ_0 and ϕ_1, so that they are now nonzero on the unit ball $|x| < 1$. Then, applying the appropriate physical space formula, we see that at time $t = 1$, the solution $\phi(1, x)$ can be nonzero only when $|x| < 2$; this is because $\phi(1, x)$ is expressed as an integral of ϕ_0 and ϕ_1 on a ball or sphere of radius 1 about x. More generally, $\phi(t, x)$, at a time t, can be nonzero only when $|x| < 1 + |t|$. Thus, any change to ϕ_0 and ϕ_1 propagates at most at finite speed 1.

We give more precise statements of this below:

Theorem 5.7 (Finite speed of propagation). *Suppose ϕ is a solution of* (5.2)*, with ϕ_0 and ϕ_1 satisfying the assumptions of Theorem 5.1, 5.4, or 5.6, depending on n. In addition, we fix a point $x_0 \in \mathbb{R}^n$ and a radius $R > 0$.*

- *If the supports of ϕ_0 and ϕ_1 are contained in the ball $\overline{B_{x_0}(R)}$, then for any $t \in \mathbb{R}$, the support of $\phi(t, \cdot)$ is contained in the ball $\overline{B_{x_0}(R + |t|)}$.*
- *If ϕ_0 and ϕ_1 vanish on $\overline{B_{x_0}(R)}$, then ϕ vanishes on the cone*

$$C = \{(t, x) \in (-R, R) \times \mathbb{R}^n \mid |x - x_0| \leq R - |t|\}.$$

Proof. These can be directly observed using (5.7), (5.10), and (5.17). □

If n is odd, then we have an even stronger property. Indeed, from (5.9) and (5.10), we see that at time $t > 0$, the formula for $\phi(t, x)$ is expressed as an integral of ϕ_0 and ϕ_1 on a *sphere* of radius t about x. In other words, a change in ϕ_0 and ϕ_1 at a point $x_0 \in \mathbb{R}^n$ propagates entirely along the cone $\mathcal{C} = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n \mid |t| = |x - x_0|\}$. This is known in physics literature as the *strong Huygens principle*.

Theorem 5.8 (Strong Huygens principle). *Suppose ϕ solves* (5.2)*, with n odd. Then, if the first $(n-1)/2$ derivatives of ϕ_0 and the first $(n-3)/2$ derivatives of ϕ_1 vanish on the sphere $S_{x_0}(|t|)$, where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, then $\phi(t, x_0) = 0$.*

5.2 Fourier Space Formulas

We now discuss representation formulas for the solutions of (5.2) in Fourier space, that is, for the spatial Fourier transform of ϕ. In contrast to the physical space formulas in the previous subsection, the Fourier space formulas have the same format in all dimensions. This is an advantage in that one can use these formulas in the same way independently of dimension. On the other hand, this also means the Fourier representation hides many of the important qualititative differences among the formulas (5.7), (5.9), (5.10), (5.15), (5.17).

Suppose now that ϕ_0 and ϕ_1 are “nice enough” functions such that their Fourier transforms $\hat{\phi}_i : \mathbb{R}^n \to \mathbb{C}$ exist, and that after making “reasonable” transformations to $\hat{\phi}_0$ and $\hat{\phi}_1$, their inverse Fourier transforms also still exist.[23] Suppose ϕ solves (5.2), and let $\hat{\phi}$ denote its spatial Fourier transform, i.e., the Fourier transform of ϕ of only the x-variables:

$$\hat{\phi} : \mathbb{R}_t \times \mathbb{R}^n_\xi \to \mathbb{C}, \qquad \hat{\phi}(t, \xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \phi(t, x) dx.$$

In particular, taking the spatial Fourier transform of (5.2) yields

$$-\partial_t^2 \hat{\phi}(t, \xi) - |\xi|^2 \hat{\phi}(t, \xi) \equiv 0, \qquad \hat{\phi}|_{t=0} = \hat{\phi}_0, \qquad \partial_t \hat{\phi}|_{t=0} = \hat{\phi}_1.$$

For each $\xi \in \mathbb{R}^n$, the above is a second-order ODE in t, which can be solved explicitly:

$$\hat{\phi}(t, \xi) = \cos(t|\xi|) \hat{\phi}_0(\xi) + \frac{\sin(t|\xi|)}{|\xi|} \hat{\phi}_1(\xi). \tag{5.18}$$

This is the general representation formula for ϕ in Fourier space. Taking an inverse Fourier transform of (5.18) (assuming it exists) yields a formula for the solution ϕ itself. In particular, this inverse Fourier transform exists when both ϕ_0 and ϕ_1 lie in L^2.

For concise notation, one usually denotes this formula for ϕ via the operators

$$f \mapsto \cos(t\sqrt{-\Delta}) f = \mathcal{F}^{-1}[\cos(t|\xi|)\mathcal{F}f], \qquad f \mapsto \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} f = \mathcal{F}^{-1}\left[\frac{\sin(t|\xi|)}{|\xi|}\mathcal{F}f\right],$$

corresponding to multiplication by $\cos(t|\xi|)$ and $|\xi|^{-1}\sin(t|\xi|)$ in Fourier space.[24] Thus, from (5.18) and the above considerations, we obtain:

[23] To sidestep various technical issues, we avoid the topic of distributional solutions.

[24] There are spectral-theoretic justifications for such notations, but we will not discuss these here.

Theorem 5.9. *Consider the problem* (5.2)*, for general* n*, and suppose* $\phi_0, \phi_1 \in L^2$*. Then, the solution* ϕ *to* (5.2) *can be expressed as*

$$\phi(t) = \cos(t\sqrt{-\Delta})\phi_0 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}\phi_1. \tag{5.19}$$

Remark 5.10. Note that in general, the right-hand side of (5.19) may not be twice differentiable in the classical sense. Thus, one must address what is meant by (5.19) being a "solution". While there are multiple ways to characterise such "weak solutions", we note here that (5.19) does solve (5.2) in the sense of distributions.

As mentioned before, the physical and Fourier space formulas highlight rather different aspects of waves. For instance, the finite speed of propagation properties that were immediate from the physical space formulas cannot be readily seen from (5.19). On the other hand, it is easy to obtain L^2-type estimates for ϕ from (5.19) via Plancherel's theorem, while such estimates are not at all apparent from the physical space formulas:

Theorem 5.11. *Suppose* $\phi_0 \in H^{s+1}$ *and* $\phi_1 \in H^s$ *for some* $s \geq 0$*. Then, for any* $t \in \mathbb{R}$*, the solution* ϕ *to* (5.2) *satisfies the following estimate:*

$$\|\nabla_x \phi(t)\|_{H^s} + \|\partial_t \phi(t)\|_{H^s} \lesssim \|\nabla_x \phi_0\|_{H^s} + \|\phi_1\|_{H^s}. \tag{5.20}$$

Proof. This follows from Plancherel's theorem and the definition of H^s-norms:[25]

$$\begin{aligned} \|\nabla_x \phi(t)\|_{H^s} &\leq \| \cos(t|\xi|) \cdot (1+|\xi|^2)^{\frac{s}{2}} \cdot \xi \phi_0\|_{L^2} + \| \sin(t|\xi|) \cdot |\xi|^{-1} \xi \cdot (1+|\xi|^2)^{\frac{s}{2}} \cdot \phi_1 \|_{H^s} \\ &\leq \|(1+|\xi|^2)^{\frac{s}{2}} \cdot \xi \phi_0\|_{L^2} + \|(1+|\xi|^2)^{\frac{s}{2}} \cdot \phi_1\|_{H^s} \\ &= \|\nabla_x \phi_0\|_{H^s} + \|\phi_1\|_{H^s}. \end{aligned}$$

Similarly, for $\partial_t \phi$, we have

$$\begin{aligned} \|\partial_t \phi(t)\|_{H^s} &\leq \| \sin(t|\xi|) \cdot (1+|\xi|^2)^{\frac{s}{2}} \cdot |\xi| \phi_0\|_{L^2} + \| \cos(t|\xi|) \cdot (1+|\xi|^2)^{\frac{s}{2}} \cdot \phi_1 \|_{H^s} \\ &\leq \|\nabla_x \phi_0\|_{H^s} + \|\phi_1\|_{H^s}. \end{aligned}$$

□

In particular, the H^s-regularity of (first derivatives of) solutions of (5.2) is propagated. In other words, as long as the hypotheses of Theorem 5.11 are satisfied, the curves $t \mapsto \nabla_x \phi(t)$ and $t \mapsto \partial_t \phi(t)$ lie in the (infinite-dimensional) space H^s.

Remark 5.12. From either the fundamental theorem of calculus or from Fourier space estimates, one can show that under the hypotheses of Theorem 5.11,

$$\|\phi(t)\|_{H^s} \lesssim 1 + |t|, \qquad t \in \mathbb{R}.$$

Moreover, this is nearly optimal, one can construct solutions ϕ of (5.2) such that $\|\phi(t)\|_{L^2}$ grows faster than $|t|^{1-\varepsilon}$ for any $\varepsilon > 0$; see Section 4.5 in [Selb2001].

Remark 5.13. Alternatively, one can rewrite the Fourier space formula (5.18) as

$$\hat{\phi}(t,\xi) = \frac{1}{2} e^{it|\xi|} [\hat{\phi}_0(\xi) - i|\xi|^{-1} \hat{\phi}_1(\xi)] + \frac{1}{2} e^{-it|\xi|} [\hat{\phi}_0(\xi) + i|\xi|^{-1} \hat{\phi}_1(\xi)].$$

This leads to the *half-wave decomposition* of ϕ:

$$\phi(t) = \frac{1}{2} e^{it\sqrt{-\Delta_x}} \phi_- + \frac{1}{2} e^{-it\sqrt{-\Delta_x}} \phi_+, \qquad \phi_\pm := \phi_0 \pm i(-\Delta_x)^{-\frac{1}{2}} \phi_1. \tag{5.21}$$

[25] Here, for generality and for convenience, we use the Fourier definition of H^s-norms.

In particular, this recasts (5.2) as solving the following first-order *half-wave equations*:

$$\partial_t \beta \pm \sqrt{-\Delta_x}\beta = 0, \qquad \beta|_{t=0} = \beta_0. \tag{5.22}$$

This half-wave representation is often better suited for harmonic analysis techniques.

5.3 Duhamel's Principle

We now turn our attention to the general inhomogeneous wave equation, (5.3). As was mentioned in Chapter 1, one could construct a solution of inhomogeneous linear ODE from a solution of the corresponding homogeneous ODE via Duhamel's principle, and this idea extends to PDEs as well. However, Duhamel's principle, as discussed, applied only to first-order equations, while the wave equation is of course second-order.

The trick for overcoming this issue to adopt $\psi := \partial_t \phi$ as a second unknown. By considering (ϕ, ψ) as the unknowns, then (5.3) can be written as a first-order system:

$$\partial_t \begin{bmatrix} \phi \\ \psi \end{bmatrix} = \begin{bmatrix} \psi \\ -\Delta_x \phi - F \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\Delta_x & 0 \end{bmatrix} \begin{bmatrix} \phi \\ \psi \end{bmatrix} - \begin{bmatrix} 0 \\ F \end{bmatrix}, \qquad \begin{bmatrix} \phi \\ \psi \end{bmatrix}\Bigg|_{t=0} = \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix}. \tag{5.23}$$

Remark 5.14. Perhaps some clarification is needed for the term "first-order". In (5.23), we adopt the ODE-inspired perspective of treating the solution (ϕ, ψ) to the wave equation as a curve in an infinite-dimensional space of (pairs of) functions $\mathbb{R}^n \to \mathbb{R}$. In this viewpoint, the Laplacian Δ_x is thought of as a linear operator on this infinite-dimensional space.

Let L denote the *linear propagator* for the above system (5.23), that is,[26]

$$L(t) \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} := \begin{bmatrix} \tilde{\phi}(t) \\ \tilde{\psi}(t) \end{bmatrix}, \qquad t \in \mathbb{R}$$

where $\tilde{\phi}$ and $\tilde{\psi}$ solve (5.23), *with* $F \equiv 0$. Then, from (5.19) and its t-dervative, we have

$$L(t) \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} = \begin{bmatrix} \cos(t\sqrt{-\Delta})\phi_0 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}\phi_1 \\ \sin(t\sqrt{-\Delta})\sqrt{-\Delta}\phi_0 + \cos(t\sqrt{-\Delta})\phi_1 \end{bmatrix}. \tag{5.24}$$

Formally, one can repeat the derivation of Proposition 1.14, assuming the integrating factors process extends to our setting. This yields that the solution (ϕ, ψ) of (5.23) satisfies

$$\begin{bmatrix} \phi \\ \psi \end{bmatrix} = L(t) \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} - \int_0^t L(t-s) \begin{bmatrix} 0 \\ F(s) \end{bmatrix} ds.$$

Finally, recalling the formula (5.24) for $L(t)$ and restricting our attention only to the first component (the solution ϕ of the original wave equation), we obtain:

Theorem 5.15 (Duhamel's principle (Fourier)). *Consider the problem* (5.3). *Let* $\phi_0, \phi_1 \in L^2$, *and assume* $F \in L^\infty(\mathbb{R}_t; L^2(\mathbb{R}^n_x))$. *Then, the solution* ϕ *can be written as*

$$\phi(t) = \cos(t\sqrt{-\Delta})\phi_0 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}\phi_1 - \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(s) ds. \tag{5.25}$$

Furthermore, if $\nabla_x \phi_0 \in H^s$, $\phi_1 \in H^s$, *and* $F \in L^\infty(\mathbb{R}; H^s)$, *then for any* $t \in \mathbb{R}$,

$$\|\nabla_x \phi(t)\|_{H^s} + \|\partial_t \phi(t)\|_{H^s} \lesssim \|\nabla_x \phi_0\|_{H^s} + \|\phi_1\|_{H^s} + \left| \int_0^t \|F(\tau)\|_{H^s} d\tau \right|. \tag{5.26}$$

[26]For now, one simply assume that $\phi_0, \phi_1 \in \mathcal{S}$ for convenience.

Proof. Again, recalling the definitions of the operators in (5.25) in Fourier space, one can differentiate (5.25) and check directly that it satisfies (5.3). The estimate (5.26) follows from Plancherel's theorem in the same manner as (5.20). □

One can also derive analogues of Duhamel's principle using the physical space formulas. Consider, for concreteness, the case $n = 3$ (the other dimensions can be handled analogously). Let L_ϕ denote the projection of the linear propagator to its first (ϕ-)component. Then, assuming ϕ_0 and ϕ_1 to be sufficiently differentiable, $L_1(t)(\phi_0, \phi_1)$ is given by the right-hand side of (5.9). Duhamel's formula then yields the following:

Theorem 5.16 (Duhamel's principle (physical)). *Consider the problem* (5.3)*, with* $n = 3$*. If* $\phi_0 \in C^3(\mathbb{R}^3)$*,* $\phi_1 \in C^2(\mathbb{R}^3)$*, and* $F \in C^2(\mathbb{R} \times \mathbb{R}^3)$*, then the function*

$$\begin{aligned} \phi(t,x) = {} & \frac{1}{4\pi t^2} \int_{S_x(|t|)} [\phi_0(y) + (y - x) \cdot \nabla_y \phi_0(y) + t \cdot \phi_1(y)] d\sigma(y) \\ & + \frac{1}{4\pi} \int_0^t \int_{S_x(|s|)} \frac{F(t-s,y)}{s} d\sigma(y) ds \end{aligned} \tag{5.27}$$

is C^2 *and satisfies* (5.3)*. Moreover, analogous formulas hold in other dimensions.*

5.4 The Energy Identity

By the "energy" of a wave ϕ, one generally refers to L^2-type norms for first derivatives of ϕ. Recall that (5.26) already provides such an energy estimate.[27] Below, we will show how physical space methods can be used to achieve more precise energy *identities*.

Furthermore, physical space methods will also allow us to *localise* energy identities and estimates within spacetime cones. This property is closely connected to finite speed of propagation and hence is far less apparent from Fourier space techniques.

To state the general local energy identity, we need a few more notations. Given $x_0 \in \mathbb{R}^n$:

- We denote by $\partial_{r(x_0)}$ the radial derivative centred about x_0,

$$\partial_{r(x_0)} := \frac{x - x_0}{|x - x_0|} \cdot \nabla_x .$$

- We denote by $\nabla\!\!\!/_{y(x_0)}$ the angular gradients on the spheres $S_{x_0}(R)$, $R > 0$.

With the above in mind, the local energy identity can now be stated as follows:

Theorem 5.17 (Local energy identity). *Let* ϕ *be a* C^2*-solution of* (5.3)*, and suppose* F *is continuous. Given* $x_0 \in \mathbb{R}^n$ *and* $R > 0$*, we define the* local energy *of* ϕ *by*

$$\mathcal{E}_{\phi,x_0,R}(t) := \frac{1}{2} \int_{B_{x_0}(R-t)} [|\partial_t \phi(t,x)|^2 + |\nabla_x \phi(t,x)|^2] dx, \qquad 0 \leq t < R, \tag{5.28}$$

Then, for any $0 \leq t_1 < t_2 < R$*, the following* local energy identity *holds:*

$$\mathcal{E}_{\phi,x_0,R}(t_2) + \mathcal{F}_{\phi,x_0,R}(t_1,t_2) = \mathcal{E}_{\phi,x_0,R}(t_1) - \int_{t_1}^{t_2} \int_{B_{x_0}(R-t)} F(t,x) \partial_t \phi(t,x) dx dt, \tag{5.29}$$

where $\mathcal{F}_{\phi,x_0,R}$ *is the corresponding* local energy flux*,*

$$\mathcal{F}_{\phi,x_0,R}(t_1,t_2) = \frac{1}{2} \int_{t_1}^{t_2} \int_{S_{x_0}(R-t)} [|(\partial_t - \partial_{r(x_0)})\phi(t,y)|^2 + |\nabla\!\!\!/_{y(x_0)} \phi(t,y)|^2] d\sigma(y) dt. \tag{5.30}$$

[27] Taking $s > 0$ in (5.26) yields higher-order energy estimates.

Proof. Writing the integral in $\mathcal{E}_{\phi,x_0,R}(t)$ in polar coordinates as

$$\mathcal{E}_{\phi,x_0,R}(t) := \frac{1}{2}\int_0^{R-t}\int_{S_{x_0}(r)}[|\partial_t\phi(t,y)|^2 + |\nabla_x\phi(t,y)|^2]d\sigma(y)dr,$$

we derive that

$$\begin{aligned}\mathcal{E}'_{\phi,x_0,R}(t) &= \int_{B_{x_0}(R-t)}[\partial_t^2\phi(t,x)\partial_t\phi(t,x) + \partial_t\nabla_x\phi(t,x)\cdot\nabla_x\phi(t,x)]dx \\ &\qquad - \frac{1}{2}\int_{S_{x_0}(R-t)}[|\partial_t\phi(t,y)|^2 + |\nabla_x\phi(t,y)|^2]d\sigma(y) \\ &= I_1 + I_2.\end{aligned}$$

The term I_1 can be further expanded using the wave equation (5.2):

$$\begin{aligned}I_1 &= \int_{B_{x_0}(R-t)}[-F(t,x)\partial_t\phi(t,x) + \Delta_x\phi(t,x)\partial_t\phi(t,x) + \partial_t\nabla_x\phi(t,x)\cdot\nabla_x\phi(t,x)]dx \\ &= I_{1,1} + I_{1,2} + I_{1,3}.\end{aligned}$$

Integrating $I_{1,2}$ by parts yields

$$I_{1,2} = -I_{1,3} + \int_{S_{x_0}(R-t)}\partial_{r(x_0)}\phi(t,x)\partial_t\phi(t,x).$$

Thus, combining the above results in the identity

$$\begin{aligned}\mathcal{E}'_{\phi,x_0,R}(t) &= -\frac{1}{2}\int_{S_{x_0}(R-t)}[|\partial_t\phi(t,y)|^2 + |\nabla_x\phi(t,y)|^2 - 2\partial_{r(x_0)}\phi(t,x)\partial_t\phi(t,x)]d\sigma(y) \\ &\qquad - \int_{B_{x_0}(R-t)}F(t,x)\partial_t\phi(t,x)dx.\end{aligned}$$

Recall now that the gradient of ϕ can be decomposed into its radial and angular parts, with respect to x_0. In terms of lengths, we have

$$|\nabla_x\phi|^2 = |\partial_{r(x_0)}\phi|^2 + |\not\nabla_{x(x_0)}\phi|^2.$$

As a result of some algebra, our identity for $\mathcal{E}'_{\phi,x_0,R}$ becomes

$$\begin{aligned}\mathcal{E}'_{\phi,x_0,R}(t) &= -\frac{1}{2}\int_{S_{x_0}(R-t)}[|\partial_t\phi(t,y) - \partial_{r(x_0)}\phi(t,y)|^2 + |\not\nabla_{x(x_0)}\phi(t,y)|^2]d\sigma(y) \\ &\qquad - \int_{B_{x_0}(R-t)}F(t,x)\partial_t\phi(t,x)dx.\end{aligned}$$

Integrating the above in t from t_1 to t_2 results in (5.29). □

Remark 5.18. Since $\tilde{\phi}(t) := \phi(-t)$ also satsifies a wave equation, then under the assumptions of Theorem 5.17, an analogous result holds for negative times $-R < t \leq 0$.

In the homogeneous case, one can use to Theorem 5.17 to almost immediately recover finite speed of propagation (though not the strong Huygens principle):

Corollary 5.19 (Finite speed of propagation). *Suppose ϕ is a C^2-solution of* (5.2), *and let $x_0 \in \mathbb{R}^n$ and $R > 0$. If ϕ_0 and ϕ_1 vanish on $\overline{B_{x_0}(R)}$, then ϕ vanishes on*

$$C = \{(t,x) \in (-R,R)\times\mathbb{R}^n \mid |x - x_0| \leq R - |t|\}.$$

Proof. Noting that the flux (5.30) is always nonnegative, then (5.29) implies

$$\mathcal{E}_{\phi,x_0,R}(t) \leq \mathcal{E}_{\phi,x_0,R}(0) = 0, \qquad 0 \leq t < R,$$

where the initial energy vanishes due to our assumptions. Since these $\mathcal{E}_{\phi,x_0,R}(t)$ vanish, then both $\partial_t \phi$ and $\nabla_x \phi$ vanish on $C^+ := \{(t,x) \in C \mid t \geq 0\}$. Since $\phi(0)$ vanishes on $B_{x_0}(R)$, then the fundamental theorem of calculus implies ϕ also vanishes on C^+. A similar conclusion can be reached for negative times using time symmmetry. $\square$

Thus far, we have constructed, via both physical and Fourier space methods, solutions to (5.2) and (5.3). However, we have devoted only cursory attention to is the uniqueness of solutions to (5.2). Suppose ϕ and $\tilde{\phi}$ both solve (5.3) (with the same F, ϕ_0, ϕ_1). Then, $\phi - \tilde{\phi}$ solves (5.2), with zero initial data. Thus, applying Corollary 5.19 with various x_0 and R yields that $\phi = \tilde{\phi}$. As a result, we have shown:

Corollary 5.20 (Uniqueness). *If ϕ and $\tilde{\phi}$ are C^2-solutions to* (5.3)*, then $\phi = \tilde{\phi}$.*

Remark 5.21. In fact, Holmgren's theorem implies that solutions to (5.2) (and also to (5.3) for real-analytic F) are unique in the much larger class of distributions.[28]

5.4.1 Global Energy Identities

One can also use physical space methods to derive *global* energy bounds similar to (5.26), as long as there is sufficiently fast decay in spatial directions. This approach has an additional advantage in that one obtains an energy identity rather than just an estimate.

Theorem 5.22 (Energy identity). *Let ϕ be a C^2-solution of* (5.3)*, and suppose for any $t \in \mathbb{R}$ that $\nabla_x \phi(t)$, $\partial_t \phi(t)$, and $F(t)$ decay rapidly.*[29] *Define the* energy *of ϕ by*

$$\mathcal{E}_\phi(t) := \frac{1}{2} \int_{\mathbb{R}^n} [|\partial_t \phi(t,x)|^2 + |\nabla_x \phi(t,x)|^2] dx, \qquad t \in \mathbb{R}. \tag{5.31}$$

Then, for any $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$, the following energy identity *holds:*

$$\mathcal{E}_\phi(t_2) = \mathcal{E}_\phi(t_1) - \int_{t_1}^{t_2} \int_{\mathbb{R}^n} F \partial_t \phi \cdot dx dt \tag{5.32}$$

Proof. This follows by taking $R \nearrow \infty$ in (5.29) and by noticing that the local energy flux (5.30) vanishes in this limit due to our decay assumptions. $\square$

Remark 5.23. From (5.32), one can recover the energy estimate (5.26) with $s = 0$. Note that one also obtains higher-order energy identities from (5.32), since the wave operator commutes with $\langle -\Delta_x \rangle^s := (1 + |-\Delta_x|)^{\frac{s}{2}}$, and hence[30]

$$\square(\langle -\Delta_x \rangle^s \phi) = \langle -\Delta_x \rangle^s F.$$

Moreover, in the homogeneous case, one captures an even stronger statement:

[28] Holmgren's theorem is a classical result which states that for any linear PDE with analytic coefficients, solutions of the noncharacteristic Cauchy problem are unique in the class of distributions.

[29] More specifically, $|\partial_t \phi(t)| + |\nabla_x \phi(t)| + |F(t)| \lesssim (1 + |x|)^{-N}$ for any $N > 0$. Note that such assumptions for ϕ are not unreasonable, since Corollary 5.19 implies this holds for compactly supported initial data.

[30] Of course, one must assume additional differentiability for ϕ if $s > 0$.

Corollary 5.24 (Conservation of energy). *Let ϕ be a C^2-solution of* (5.2)*, and suppose for any $t \in \mathbb{R}$ that $\nabla_x \phi(t)$, $\partial_t \phi(t)$, and $F(t)$ decay rapidly. Then, for any $t \in \mathbb{R}$,*

$$\mathcal{E}_\phi(t) = \mathcal{E}_\phi(0) = \frac{1}{2} \int_{\mathbb{R}^n} [|\phi_1(x)|^2 + |\nabla_x \phi_0(x)|^2] dx. \tag{5.33}$$

5.4.2 Some Remarks on Regularity

Thus far, our energy identities have required that ϕ is at least C^2, which from our (physical space) representation formulas may necessitate even more regularity for ϕ_0 and ϕ_1. One can hence ask whether these results still apply when ϕ is less smooth.

In fact, one can recover this local energy theory (and, by extension, finite speed of propagation and uniqueness) for rough solutions ϕ of (5.2), arising from initial data $\phi_0 \in H^1_{\mathrm{loc}}$ and $\phi_1 \in L^2_{\mathrm{loc}}$. (For solutions of (5.3), one also requires some integrability assumptions for F.) In general, this is done by approximating ϕ_0 and ϕ_1 by smooth functions and applying the existing theory to the solutions arising from the regularised data. Then, by a limiting argument, one can transfer properties for the regularised solutions to ϕ itself.

The remainder of these notes will deal mostly with highly regular functions, for which all the methods we developed will apply. As a result, we avoid discussing these regularity issues here, as they can be rather technical and can obscure many of the main ideas.

5.5 Dispersion of Free Waves

While we have shown energy conservation for solutions ϕ of the homogeneous wave equation (5.2), we have not yet discussed how solutions decay in time. On one hand, the total energy of $\phi(t)$, given by the L^2-norms of $\partial_t \phi(t)$ and $\nabla_x \phi(t)$, does not change in t. However, what happens over large times is that the wave will propagate further outward (though at a finite speed), and the profile of $\phi(t)$ disperses over a larger area in space. Correspondingly, the magnitude of $|\phi(t)|$ will become smaller as the profile spreads out further.

A pertinent question is to ask what is the generic rate of decay of $|\phi(t)|$ as $|t| \nearrow \infty$. The main result, which is often referred to as a *dispersive estimate*, is the following:

Theorem 5.25 (Dispersive estimate). *Suppose ϕ solves* (5.2)*, with $\phi_0, \phi_1 \in \mathcal{S}$. Then,*

$$\|\phi(t)\|_{L^\infty} \leq C t^{-\frac{n-1}{2}}, \tag{5.34}$$

where the constant C depends on various properties of ϕ_0 and ϕ_1.

Remark 5.26. The representation formulas (5.7), (5.10), and (5.17) demonstrate that (5.34) is false if ϕ_0 and ϕ_1 do not decay sufficiently quickly.

One traditional method to establish Theorem 5.25 is by using harmonic analysis methods. Recalling the half-wave decomposition (5.21), Theorem 5.25 reduces to proving dispersion estimates for the half-wave propagators $e^{\pm it\sqrt{-\Delta_x}}$. These can be shown to be closely connected to decay properties for the Fourier transform of the surface measure of $\mathbb{S}^{n-1}$.

There also exist more recent physical space methods for deriving (5.34). Very roughly, these are based primarily on establishing weighted integral estimates for $\partial_t \phi$ and $\partial_x \phi$ over certain *spacetime* regions. While these methods require more regularity from ϕ_0 and ϕ_1, they have the additional advantage of being applicable to wave equations on backgrounds that are not $\mathbb{R} \times \mathbb{R}^n$; see the lecture notes of M. Dafermos and I. Rodnianski, [Dafe2013].

6 The Vlasov-Maxwell System: Conditional Global Existence

The proof of existence and uniqueness for the Vlasov-Poisson system relied heavily on the elliptic structure of Poisson's equation in order to obtain bounds for the momentum of the "fastest" particle, $P(t)$. These tools are not available to us here, and, indeed, an *a priori* bound on $P(t)$ is still an open problem.[31]

6.1 The Glassey-Strauss Theorem

We recall that the Vlasov-Maxwell system describes the evolution of a pdf $f(t,x,p) : \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}_+$ due to electromagnetic forces. It reads

$$\frac{\partial f}{\partial t}(t,x,p) + v \cdot \nabla_x f(t,x,p) + (\mathbf{E}_f(t,x) + v \times \mathbf{B}_f(t,x)) \cdot \nabla_p f(t,x,p) = 0, \tag{6.1}$$

$$\nabla \cdot \mathbf{E}_f = \rho_f, \quad \nabla \cdot \mathbf{B}_f = 0, \quad \nabla \times \mathbf{E}_f = -\frac{\partial \mathbf{B}_f}{\partial t}, \quad \nabla \times \mathbf{B}_f = \mathbf{j}_f + \frac{\partial \mathbf{E}_f}{\partial t}, \tag{6.2}$$

where

$$\rho_f(t,x) = \int_{\mathbb{R}^n} f(t,x,p)\,\mathrm{d}p = \text{particle density},$$

$$\mathbf{j}_f(t,x) = \int_{\mathbb{R}^n} f(t,x,p) v\,\mathrm{d}p = \text{current density},$$

and

$$\begin{aligned} &\text{classical case} && v = p, \\ &\text{relativistic case} && v = \frac{p}{\sqrt{1+|p|^2}}. \end{aligned}$$

The most famous existence and uniqueness results – due to Glassey and Strauss [Glassey1986] – is *conditional* in the sense that it requires momenta to be bounded in time though such a condition is not *a priori* known to hold. In this section we shall sketch the proof.

Theorem 6.1 (Conditional Existence of Classical Solutions (Relativistic Case)). *Let $f_0(x,p) \in C_0^1(\mathbb{R}^3 \times \mathbb{R}^3)$ with $f_0 \geq 0$, and let $\mathbf{E}(0,x) = \mathbf{E}_0(x)$ and $\mathbf{B}(0,x) = \mathbf{B}_0(x)$ be such that $\nabla \cdot \mathbf{E}_0 = \rho_{f_0}$ and $\nabla \cdot \mathbf{B}_0 = 0$. Assume that there exists a function $\beta(t)$ such that for all x, $f(t,x,p) = 0$ for $|p| > \beta(t)$. Then there exists a unique classical global solution $f(t,x,p)$ for the system* (6.1)-(6.2) *with $f(0,\cdot,\cdot) = f_0$ in the* relativistic case.

Remark 6.2. Note that the electromagnetic fields satisfy wave equations which require additional initial data (see Section 5). Hence the initial data $(\mathbf{E}_0(x), \mathbf{B}_0(x))$ must be complemented by another pair $(\mathbf{E}_1(x), \mathbf{B}_1(x))$. However these (dynamic) conditions are not reflected in the (static) problem at time $t = 0$.

6.2 Sketch of Proof of Conditional Existence and Uniqueness

6.2.1 Defining Approximate Solutions

The construction of approximate solutions follows the same ideas as in the proof for Vlasov-Poisson. However, now Poisson's (elliptic) equation is replaced by Maxwell's (hyperbolic) equations, and hence defining the vector field is more involved.

[31] Such bounds exists in some special cases, for instance if assuming the initial data is "small" in some sense [Glassey1987a], or if the system possesses some symmetries [Glassey1990, Glassey1997].

Set

$$f^0(t,x,p) = f_0(x,p), \quad \mathbf{E}^0(t,x) = \mathbf{E}_0(x), \quad \text{and} \quad \mathbf{B}^0(t,x) = \mathbf{B}_0(x),$$

and define

$$\rho^0(t,x) = \int_{\mathbb{R}^3} f^0(t,x,p)\,\mathrm{d}p \quad \text{and} \quad \mathbf{j}^0(t,x) = \int_{\mathbb{R}^3} f^0(t,x,p)v\,\mathrm{d}p.$$

Suppose that $(f^{N-1}, \mathbf{E}^{N-1}, \mathbf{B}^{N-1})$ have been defined and define f^N to be the solution to the linear transport equation

$$\begin{cases} \partial_t f^N(t,x,p) + p \cdot \nabla_x f^N(t,x,p) + \big(\mathbf{E}^{N-1}(t,x) + v \times \mathbf{B}^{N-1}(t,x)\big) \cdot \nabla_p f^N(t,x,p) = 0, \\ f^N(0,\cdot,\cdot) = f_0. \end{cases}$$

Hence one can write

$$\begin{aligned} &f^N(t,x,p) = f_0(X^{N-1}(0;t,x,p), V^{N-1}(0;t,x,p)), \\ &\rho^N(t,x) = \int_{\mathbb{R}^3} f^N(t,x,p)\,\mathrm{d}p \quad \text{and} \quad \mathbf{j}^N(t,x) = \int_{\mathbb{R}^3} f^N(t,x,p)v\,\mathrm{d}p, \end{aligned}$$

and we can define $\mathbf{E}^N$ and $\mathbf{B}^N$ to be the solutions of

$$\begin{aligned} -\Box\mathbf{E}^N &= (\partial_t^2 - \Delta_x)\mathbf{E}^N = -\nabla_x \rho^N - \partial_t \mathbf{j}^N, \\ -\Box\mathbf{B}^N &= (\partial_t^2 - \Delta_x)\mathbf{B}^N = \nabla_x \times \mathbf{j}^N. \end{aligned}$$

with initial data $(\mathbf{E}_0, \mathbf{E}_1)$ and $(\mathbf{B}_0, \mathbf{B}_1)$, respectively.

> **Goal:** show that under the assumption that an upper bound $\beta(t)$ to momenta exists $\lim_{N\to\infty} f^N$ exists in $C^1([0,T)\times\mathbb{R}^6)$, $\lim_{N\to\infty}(\mathbf{E}^N, \mathbf{B}^N)$ exists in $C^1([0,T)\times\mathbb{R}^3)$, that the limits satisfy the relativistic Vlasov-Maxwell system (uniquely).

6.2.2 Representation of the Fields and Their Derivatives

In the scheme defined above, the vector field governing the evolution of f^N depends on the fields $\mathbf{E}^{N-1}$ and $\mathbf{B}^{N-1}$ which depend upon f^{N-1} through Maxwell's equations in a complicated way. We must ensure that there is no loss in regularity, i.e. that f^N has the same regularity of f^{N-1}.

New Basis. To this end, it is convenient to work in coordinates that respect the symmetries of the Vlasov equation (free transport) and of Maxwell's equations (the light cone). That is, we wish to replace the partial derivatives ∂_t and ∂_{x_i} by suitably chosen directional derivatives. Fix a point $x \in \mathbb{R}^3$. A signal arriving at x at time t from a different point $y \in \mathbb{R}^3$ would have had to leave y at time $t - |x-y|$ (we have taken the speed of light to be 1). Hence we define:

$$\begin{aligned} S &:= \partial_t + v \cdot \nabla_x \\ T_i f &:= \partial_{y_i}[f(t - |x-y|, y, p)], \quad i = 1,2,3. \end{aligned}$$

Inverting, one has the representation:

$$\partial_t = \frac{S - v \cdot T}{1 + v \cdot \omega}$$
$$\partial_{x_i} = T_i + \frac{\omega_i}{1 + v \cdot \omega}(S - v \cdot T) = \frac{\omega_i S}{1 + v \cdot \omega} + \sum_{j=1}^{3} \left(\delta_{ij} - \frac{\omega_i v_j}{1 + v \cdot \omega} \right) T_j.$$

where

$$\omega = \frac{y - x}{|y - x|}.$$

Representation of the Fields. We use the new coordinates to express the fields. We drop the N superscripts for brevity here, and in all the subsequent arguments that do not require an explicit identification of the iterates.

Proposition 6.3. *Under the hypotheses of Theorem 6.1 the fields admit the following representation:*

$$\mathbf{E}(t,x) = \mathbf{E}_0(t,x) + \mathbf{E}_T(t,x) + \mathbf{E}_S(t,x) \quad \textit{and} \quad \mathbf{B}(t,x) = \mathbf{B}_0(t,x) + \mathbf{B}_T(t,x) + \mathbf{B}_S(t,x)$$

where

$$\mathbf{E}_T^i(t,x) = -\int_{|y-x|\le t} \int_{\mathbb{R}^3} \frac{(\omega_i + v_i)(1 - |v|^2)}{(1 + v \cdot \omega)^2} f(t - |y-x|, y, p) \,\mathrm{d}p \,\frac{\mathrm{d}y}{|y-x|^2},$$
$$\mathbf{E}_S^i(t,x) = -\int_{|y-x|\le t} \int_{\mathbb{R}^3} \frac{\omega_i + v_i}{1 + v \cdot \omega} (Sf)(t - |y-x|, y, p) \,\mathrm{d}p \,\frac{\mathrm{d}y}{|y-x|}$$

and similar expressions for $\mathbf{B}$.

Proof. Let us show this for $\mathbf{E}$. We know that

$$-\Box \mathbf{E}^i = -\partial_{x_i}\rho - \partial_t \mathbf{j}^i = -\int_{\mathbb{R}^3} (\partial_{x_i} f + v \partial_t f) \,\mathrm{d}p.$$

We express the operator $\partial_{x_i} + v\partial_t$ appearing in the integrand on the right hand side as

$$\partial_{x_i} + v\partial_t = \frac{(\omega_i + v_i)S}{1 + v \cdot \omega} + \sum_{j=1}^{3} \left(\delta_{ij} - \frac{(\omega_i + v_i) v_j}{1 + v \cdot \omega} \right) T_j.$$

Now one proceeds by applying *Duhamel's principle* (Theorem 5.16) to the resulting inhomogeneous wave equation for $\mathbf{E}^i$ and integrating by parts (the full details can be found in [Glassey1996]). The proof for $\mathbf{B}$ is analogous. $\square$

Representation of the Derivatives of the Fields. We have expressions analogous to the one obtained in Proposition 6.3:

Proposition 6.4. *Under the hypotheses of Theorem 6.1 the partial derivatives of the fields admit the following representation, for* $i, k = 1, 2, 3$*:*

$$\partial_k \mathbf{E}^i = (\partial_k \mathbf{E}^i)_0 + \int_{r \le t} \int_{\mathbb{R}^3} a(\omega, v) f \,\mathrm{d}p \,\frac{\mathrm{d}y}{r^3} + \int_{|\omega|=1} \int_{\mathbb{R}^3} d(\omega, v) f(t,x,p) \,\mathrm{d}\omega \,\mathrm{d}p$$
$$+ \int_{r \le t} \int_{\mathbb{R}^3} b(\omega, v) Sf \,\mathrm{d}p \,\frac{\mathrm{d}y}{r^2} + \int_{r \le t} c(\omega, v) S^2 f \,\mathrm{d}p \,\frac{\mathrm{d}y}{r}$$

where f, Sf, S^f without explicit arguments are evaluated at $(t-|x-y|, y, p)$ and $r=|y-x|$. The functions a, b, c, d are smooth except at $1+v\cdot\omega=0$, have algebraic singularities at such points, and $\int_{|\omega|=1} a(\omega, v)\,\mathrm{d}\omega = 0$. Therefore the first integral converges if f is sufficiently smooth. Similar expressions exist for the derivatives of $\mathbf{B}$.

Proof. This is obtained from applying $\frac{\partial}{\partial x_k}$ to the expressions obtained in Proposition 6.3. For the long computation involved we refer to [Glassey1996]. For simplicity and future reference we shall write the expression for $\partial_k \mathbf{E}^i$ as:

$$\partial_k \mathbf{E}^i = \partial_k \mathbf{E}^i_0 + \partial_k \mathbf{E}^i_{TT} - \partial_k \mathbf{E}^i_{TS} + \partial_k \mathbf{E}^i_{ST} - \partial_k \mathbf{E}^i_{SS}$$

□

6.2.3 The Iterates are Well-Defined

Lemma 6.5. *If $f^N \in C^2([0,T)\times\mathbb{R}^6)$ then $\mathbf{E}^N, \mathbf{B}^N \in C^2([0,T)\times\mathbb{R}^3)$.*

Proof. Recall that $\mathbf{E}^N$ and $\mathbf{B}^N$ satisfy the wave equations

$$\begin{aligned} -\Box\mathbf{E}^N &= -\nabla_x\rho^N - \partial_t\mathbf{j}^N, \\ -\Box\mathbf{B}^N &= \nabla_x \times \mathbf{j}^N. \end{aligned}$$

If $f^N \in C^2$ then the right hand sides of these equations are in C^1 and hence so are the fields. To show that they are in fact C^2 we need to employ the representation results and proceed by induction. Recall that

$$\mathbf{E}^N(t,x) = \mathbf{E}_0(t,x) + \mathbf{E}^N_T(t,x) + \mathbf{E}^N_S(t,x) \quad \text{and} \quad \mathbf{B}^N(t,x) = \mathbf{B}_0(t,x) + \mathbf{B}^N_T(t,x) + \mathbf{B}^N_S(t,x).$$

The data terms $\mathbf{E}_0(t,x)$ and $\mathbf{B}_0(t,x)$ are C^2, so we only need to analyse the other terms. Take for instance the expression

$$\mathbf{E}^N_S(t,x) = -\int_{|y-x|\le t}\int_{\mathbb{R}^3} \frac{\omega+v}{1+v\cdot\omega}(Sf^N)(t-|y-x|, y, p)\,\mathrm{d}p\,\frac{\mathrm{d}y}{|y-x|}.$$

Notice that

$$Sf^N = -\nabla_p \cdot \left[(\mathbf{E}^{N-1} + v\times\mathbf{B}^{N-1})f^N\right]$$

which allows us to integrate by parts in p and use the induction hypothesis that $\mathbf{E}^{N-1}$ and $\mathbf{B}^{N-1}$ are C^2. A similar argument can be employed for $\mathbf{E}^N_T(t,x)$. Hence $\mathbf{E}^N$ is C^2 and the same holds for $\mathbf{B}^N$. □

6.2.4 A Uniform Bound for the Particle Density

Proposition 6.6. *The particle density satisfies the bound:*

$$\|f(t,\cdot,\cdot)\|_{C^1} \le c + c_T\int_0^t \left[1 + \|\mathbf{E}(\tau,\cdot)\|_{C^1} + \|\mathbf{B}(\tau,\cdot)\|_{C^1}\right]\|f(\tau,\cdot,\cdot)\|_{C^1}\,\mathrm{d}\tau \tag{6.3}$$

for all $t\in[0,T]$.

Proof. Let $D \in \{\partial_{x_j}\}_{j=1}^3$ and denote $\mathbf{F} = \mathbf{E}_f(t,x) + v\times\mathbf{B}_f(t,x)$. Then

$$(\partial_t + v\cdot\nabla_x + \mathbf{F}\cdot\nabla_p)(Df) = -D\mathbf{F}\cdot\nabla_p f.$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}s} Df(s, X(s;t,x,p), V(s;t,x,p)) = -D\mathbf{F} \cdot \nabla_p f(s, X(s;t,x,p), V(s;t,x,p))$$

which leads to the estimate

$$\begin{aligned} |Df(t,x,p)| \leq{}& |Df(0, X(0;t,x,p), V(0;t,x,p))| \\ &+ \int_0^t |D\mathbf{F} \cdot \nabla_p f(s, X(s;t,x,p), V(s;t,x,p))| \, \mathrm{d}s. \end{aligned}$$

A similar bound can be obtained for $D \in \{\partial_{p_j}\}_{j=1}^3$. The assertion follows from these two bounds. $\square$

6.2.5 A Uniform Bound for the Fields

Proposition 6.7. *The fields admit the uniform bound*

$$\|\mathbf{E}(t,\cdot)\|_{C^0} + \|\mathbf{B}(t,\cdot)\|_{C^0} \leq c_T, \qquad \forall t \in [0,T]. \tag{6.4}$$

Proof. We omit this proof which is technical (though it is at the heart of the proof). Here, the assumption on the existence of a bound $\beta(t)$ of momenta is used crucially. $\square$

6.2.6 A Uniform Bound for the Gradients of the Fields

Proposition 6.8. *Let* $\log^* s = \begin{cases} s & s \leq 1, \\ 1 + \ln s & s \geq 1. \end{cases}$ *Then the gradients of the fields admit the uniform bound*

$$\|\mathbf{E}(t,\cdot)\|_{C^1} + \|\mathbf{B}(t,\cdot)\|_{C^1} \leq c_T \left[1 + \log^*\left(\sup_{\tau \leq t} \|f(\tau,\cdot,\cdot)\|_{C^1}\right)\right], \qquad \forall t \in [0,T]. \tag{6.5}$$

Proof. We omit this proof which is technical (though it is at the heart of the proof). Here, the assumption on the existence of a bound $\beta(t)$ of momenta is used crucially. $\square$

6.2.7 The Limit $\lim_{N\to\infty}(f^N, \mathbf{E}^N, \mathbf{B}^N)$ and its Properties

From the bounds on the fields and their gradients, (6.4) and (6.5) respectively, uniform bounds for the iterates follow. Similarly, we obtain uniform bounds for the particle density. Together with compactness, this shows that the sequences admit limits. However some of the elements of this proof are also lengthy and omitted here.

6.2.8 Uniqueness

For uniqueness we only require the expressions obtained in Proposition 6.3. This is also omitted here.

7 Nonlinear Wave Equations: Classical Existence and Uniqueness

From our previous discussions, we have built some considerable intuition behind how solutions of linear wave equations behave. In addition, we have studied numerous viewpoints and tools for analysing waves. From here on, we now turn our attention to nonlinear wave equations, i.e., partial differential equations of the form

$$\Box\phi = \mathcal{N}(t, x, \phi, \partial\phi),$$

where $\partial\phi$ denotes the *spacetime* gradient of ϕ, and where $\mathcal{N}(t, x, \phi, \partial\phi)$ is some nonlinear function of ϕ and $\partial\phi$. As mentioned before, our understanding of linear waves will be useful, as we work in perturbative settings in which nonlinear waves behave like linear ones.

Of course, there are countless possibilities for the nonlinearity $\mathcal{N}$, even within the subset of equations relevant to physics. However, in order to achieve some mathematical understanding of nonlinear waves, it is convenient to restrict our attention to model nonlinearities, in particular those which "scale consistently". Of particular interest to mathematicians are the following classes of nonlinear wave equations:

1. *NLW*: $\mathcal{N}(t, x, \phi, \partial\phi)$ is some power of ϕ, that is,[32]
$$\Box\phi = \pm|\phi|^{p-1}\phi, \qquad p > 1. \tag{7.1}$$
2. *dNLW*: $\mathcal{N}(t, x, \phi, \partial\phi)$ is some power of $\partial\phi$,
$$\Box\phi = (\partial\phi)^p, \qquad p > 1, \tag{7.2}$$
where $(\partial\phi)^p$ refers to some quantity that involves p-th powers of $\partial\phi$.

In order to keep our discussions concrete, we restrict our attention in this chapter to derivative nonlinear waves (dNLW), in the specific case $p = 2$. In particular, we consider the case when $\mathcal{N}(\phi, \partial\phi)$ is a quadratic form in $\partial\phi$:[33]

$$\mathcal{N}(t, x, \phi, \partial\phi) := Q(\partial\phi, \partial\phi).$$

In the remaining chapters, we restrict our stuides to the following initial value problem,

$$\Box\phi = Q(\partial\phi, \partial\phi), \qquad \phi|_{t=0} = \phi_0 : \mathbb{R}^n \to \mathbb{R}, \qquad \partial_t\phi|_{t=0} = \phi_1 : \mathbb{R}^n \to \mathbb{R}. \tag{7.3}$$

In this chapter, we show that (7.3) has unique local-in-time solutions. In the subsequent chapter, we explore the existence of global solutions, in particular for small initial data.

7.1 The ODE Perspective

From here on, it will be useful to view (7.3) as an analogue of the ODE setting—that ϕ is a curve, parametrised by the time t, taking values in some infinite-dimensional space X of real-valued functions on $\mathbb{R}^n$. Similar to ODEs, this is captured by considering ϕ as an element of the space $C(I; X)$ of continuous X-valued functions, where I is some interval

[32] For various reasons related to the qualitative behaviours of solutions, the "+" case in (7.1) is called *defocusing*, while the "−" case is called *focusing*.

[33] To avoid technical issues, we avoid settings in which $\mathcal{N}$ fails to be smooth.

containing the initial time $t_0 := 0$. In the ODE setting, X is simply the finite-dimensional space $\mathbb{R}^d$, with d the number of unknowns. On the other hand, in the PDE setting, one has far more freedom (and pitfalls) in the choice of the space X.

The upshot of this ambiguity is that one must choose X carefully. In particular, since $\phi(t)$ must lie in X for each time t, then X must be chosen such that its properties are propagated by the evolution of (7.3). Furthermore, we wish to apply the contraction mapping theorem (Theorem 1.4), hence it follows that X must necessarily be complete.

Recall that for linear wave equations, one has energy-type estimates in the Sobolev spaces $H^s(\mathbb{R}^n)$; see (5.26). Since one expects nonlinear waves to approximate linear waves in our setting, then one can reasonably hope that taking $X = H^s(\mathbb{R}^n)$ is sufficient to solve (7.3), at least for some values of s. Recall also that it was sometimes useful to consider both ϕ and $\partial_t\phi$ as unknowns in an equivalent first-order system,

$$\phi \in C(I; X), \qquad \partial_t\phi \in C(I; X').$$

From (5.26), we see it is reasonable to guess $X := H^{s+1}$ and $X' = H^s$. Returning to the viewpoint of a single second-order equation, the above can then be consolidated as

$$\phi \in C^0(I; H^{s+1}) \cap C^1(I; H^s).$$

The main objective of this chapter is to show that the above intuition can be validated, at least over a sufficiently small time interval. Over small times, one expects that the nonlinearity does not yet have a chance to significantly affect the dynamics, hence this seems reasonable. For large times, the nonlinearity could potentially play a significant role, in which case the above reasoning would collapse.

7.1.1 Strong and Classical Solutions

Before stating the main result, one must first discuss in more detail what is meant by a "solution" of (7.3). Recall that in the ODE theory, one works not with the differential equation (1.1) itself, but rather with an equivalent integral equation (1.3). For analogous reasons, one wishes to do the same in our current setting. In particular, this converts (7.3) into a fixed point problem, which one then solves by generating a contraction mapping.

The direct analogue of (1.3) would be the following first-order system:

$$\partial_t \begin{bmatrix} \phi(t) \\ \partial_t\phi(t) \end{bmatrix} = \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} + \int_0^t \begin{bmatrix} \partial_t\phi(s) \\ -\Delta_x\phi(s) - Q(\partial\phi, \partial\phi)(s) \end{bmatrix} ds. \tag{7.4}$$

However, this description immediate runs into problems, most notably with the desire to propagate the H^{s+1}-property for all $\phi(t)$. Indeed, suppose the pairs $(\phi(s), \partial_t\phi(s))$, where $0 \leq s < t$, are presumed to lie in $H^{s+1} \times H^s$. Then, the $\Delta_x\phi(s)$'s live in H^{s-1}, hence (7.4) implies that $(\phi(t), \partial_t\phi(t))$ only lie in the (strictly larger) space $H^s \times H^{s-1}$. Consequently, (7.4) is incompatible with the H^s-propagation that we desire.

The resolution to this issue is to use a more opportunistic integral equation that is more compatible with the H^s-propagation, namely, the representation which yielded the energy estimates (5.26) in the first place. For our specific setting, the idea is to use Duhamel's formula, (5.25), but with F replaced by our current nonlinearity:[34]

$$\phi(t) = \cos(t\sqrt{-\Delta})\phi_0 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}\phi_1 - \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}}[Q(\partial\phi, \partial\phi)](s)ds. \tag{7.5}$$

[34] In comparison to the ODE setting, this is the analogue of (1.17).

Since the operators $\cos(t\sqrt{\Delta})$ and $\sin(t\sqrt{-\Delta})$ do preserve H^s-regularity, (7.5) seems to address the shortcoming inherent in (7.4). The remaining issue is the nonlinearity $Q(\partial\phi, \partial\phi)$ and whether this multiplication also "preserves" H^s-regularity in a similar fashion. This is the main new technical content of this section (which we unfortunately will not have time to cover in detail, as it involves a fair bit of harmonic analysis).

As had been mentioned before, the PDE setting differs from the ODE setting in that our integral description (7.5) is no longer equivalent to (7.3). In particular (ignoring for now the contribution from the nonlinearity), (7.5) makes sense for functions ϕ which are not (classically) differentiable enough for (7.3) to make sense. As a result of this, we make the following definition generalising the notion of solution.

Definition 7.1. *Let $s \geq 0$. We say that $\phi \in C^0(I; H^{s+1}) \cap C^1(I; H^s)$ is a* strong solution *(in $H^{s+1} \times H^s$) of* (7.3) *iff ϕ satisfies* (7.5) *for all $t \in I$.*

One must of course demonstrate that this definition is sensible. First, one can easily show that any $H^{s+1} \times H^s$-strong solution is also a $H^{s'+1} \times H^{s'}$-strong solution for any $0 \leq s' \leq s$, hence the notion of strong solutions is compatible among all H^s-spaces. Furthermore, from distribution theory, one can also show that any strong solution of (7.3) which is C^2 is also a solution of (7.3) in the classical sense. In this way, strong solutions are a direct generalisation of classical solutions. For brevity, we omit the details of these derivations.

7.2 Local Existence and Uniqueness

With the preceding discussion in mind, we are now prepared to give a precise statement of our local existence and uniqueness theorem for (7.3):

Theorem 7.2 (Local existence and uniqueness). *Let $s > n/2$, and suppose*

$$\phi_0 \in H^{s+1}(\mathbb{R}^n), \qquad \phi_1 \in H^s(\mathbb{R}^n). \tag{7.6}$$

Then, there exists $T > 0$, depending only on n and $\|\phi_0\|_{H^{s+1}} + \|\phi_1\|_{H^s}$, such that the initial value problem (7.3) *has a unique strong solution*

$$\phi \in C^0([-T, T]; H^{s+1}(\mathbb{R}^n)) \cap C^1([-T, T]; H^s(\mathbb{R}^n)). \tag{7.7}$$

In the remainder of this subsection, we prove Theorem 7.2.[35] Note throughout that the basic argument mirrors that of ODEs in Section 1. However, the technical steps are further complicated here, since many estimates that were previously trivial in the ODE setting now rely on various properties and estimates for H^s-spaces.

7.2.1 Proof of Theorem 7.2: Analytical Tools

Before engaging in the proof, let us first discuss some of the main tools used within:

- Existence is again achieved by treating it as a fixed point problem for an integral equation (though one uses the Duhamel representation rather than the direct integral equation). This fixed point is then found via the contraction mapping theorem.[36]

[35] Much of this discussion will be heavily based on the contents of [Selb2001, Ch. 5].

[36] Alternately, this can be done via Picard iteration.

- As before, the contraction mapping theorem only achieves conditional uniqueness—i.e., uniqueness within a closed ball of the space of interest. As such, one needs an additional, though similar, argument to obtain the full uniqueness statement.

The main contrast with the ODE setting is the estimates one obtains for the sizes of the solution and the initial data. In the ODE setting, this involves measuring the lengths of finite-dimensional vectors, a relatively simple task. However, in the PDE setting, this involves measuring H^s-norms. As such, an additional layer of technicalities is required in order to understand and apply the toolbox of available estimates for these norms.

These H^s-estimates can be divided into two basic categories. The first are linear estimates, referring to estimates that are used to treat linear wave equations. For the current setting, this refers specifically to the energy estimate (5.26) from our previous discussions, which will be used to treat the main, non-perturbative part of the solution.

The remaining category contains nonlinear estimates, which, in the context of Theorem 7.2, refers to H^s-estimates for the nonlinearity $Q(\partial\phi, \partial\phi)$. Technically speaking, this is the main novel component of the proof that has not been encountered before. To be more specific, we will use the following classical estimate:

Theorem 7.3 (Product estimate). *If $f, g \in H^\sigma$, where $\sigma > n/2$, then $fg \in H^\sigma$, and*

$$\|fg\|_{H^\sigma} \lesssim \|f\|_{H^\sigma} \|g\|_{H^\sigma}. \tag{7.8}$$

Theorem 7.3 is in fact a direct consequence of the following estimates:

Theorem 7.4. *The following estimates hold:*

1. Product estimate*: If $\sigma \geq 0$ and $f, g \in L^\infty \cap H^\sigma$, then $fg \in H^\sigma$, and*

$$\|fg\|_{H^\sigma} \lesssim \|f\|_{L^\infty} \|g\|_{H^\sigma} + \|f\|_{H^\sigma} \|g\|_{L^\infty}. \tag{7.9}$$

2. Sobolev embedding*: If $f \in H^\sigma$ and $\sigma > n/2$, then $f \in L^\infty$, and*

$$\|f\|_{L^\infty} \lesssim \|f\|_{H^\sigma}. \tag{7.10}$$

While Sobolev embedding is a much more general topic, the special case (7.10) has a simple and concise proof. The main step is to write f in terms of its Fourier transform:

$$|f(x)| \simeq \left| \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{f}(\xi) d\xi \right| \lesssim \left[\int_{\mathbb{R}^n} (1 + |\xi|^2)^{-\sigma} d\xi \right]^{\frac{1}{2}} \left[\int_{\mathbb{R}^n} (1 + |\xi|^2)^{\sigma} |\hat{f}(\xi)|^2 d\xi \right]^{\frac{1}{2}}.$$

The first integral on the right-hand side is finite since $\sigma > n/2$, while the second integral is precisely the H^σ-norm. Since the above holds for all $x \in \mathbb{R}^n$, then (7.10) is proved.

The product estimate (7.9) is considerably more involved, and unfortunatly only a brief and basic discussion of the ideas can be presented here. For detailed proofs, the reader is referred to [Selb2001, Ch. 5, 6] and [Tao2006, App. A].

First, one should note that the case $\sigma = 0$ is trivial by Hölder's inequality, and that the case $\sigma = 1$ then follows from the Leibniz rule:

$$\|\nabla_x (fg)\|_{L^2} \leq \|f \cdot \nabla_x g\|_{L^2} + \|\nabla_x f \cdot g\|_{L^2} = \|f\|_{L^\infty} \|\nabla_x g\|_{L^2} + \|\nabla_x f\|_{L^2} \|g\|_{L^\infty}.$$

For more general differential operators $\mathcal{D}$, in particular $\mathcal{D} := (1 - \Delta)^{\sigma/2}$, one no longer has the Leibniz rule $\mathcal{D}(fg) = f \cdot \mathcal{D}g + \mathcal{D}f \cdot g$, hence the above simple argument no longer works. In spite of this, one can still recover something “close to the Leibniz rule”, and this observation can be exploited to arrive at (7.9). However, capturing this property requires some creative use of Fourier transforms and harmonic analysis.

The rough idea is to understand how different frequences of f and g interact with each other in the product. For this, one applies what is called a *Littlewood-Paley decomposition*, in which a function h is decomposed into “frequency bands”, i.e., components $P_k h$ whose Fourier transform is supported on the annulus $|\xi| \simeq 2^k$.[37] In particular, this decomposition is applied to both f and g, as well as the product fg. Then, each component $\mathcal{D}P_m(P_k f \cdot P_l g)$ can be treated separately, depending on the relative sizes of k, l, m. If done carefully enough, one can sum the estimates for each such component and recover (7.9).

Remark 7.5. Product estimates such as (7.9) form the beginning of a more general area of study known as *paradifferential calculus.*

7.2.2 Proof of Theorem 7.2: Existence

For convenience, we adopt the abbreviation

$$\mathcal{X} := C^0([-T,T]; H^{s+1}) \cap C^1([-T,T]; H^s),$$

and we denote the corresponding norm for $\mathcal{X}$ by

$$\|u\|_{\mathcal{X}} = \sup_{0 \leq t \leq T} [\|u(t)\|_{H^{s+1}} + \|\partial_t u(t)\|_{H^s}].$$

Furthermore, let $A > 0$ (to be chosen later), and consider the closed ball

$$Y = \{\phi \in \mathcal{X} \mid \|\phi\|_{\mathcal{X}} \leq A\}.$$

Since $\mathcal{X}$ is a Banach space, Y (with the metric induced by the $\mathcal{X}$-norm) forms a closed metric space. Consider now the map $\Psi : \mathcal{X} \to \mathcal{X}$ given by

$$[\Psi(\phi)](t) := \cos(t\sqrt{-\Delta})\phi_0 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}\phi_1 - \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}}[Q(\partial\phi, \partial\phi)](s)ds. \quad (7.11)$$

Then, finding a strong solution of (7.3) is equivalent to finding a fixed point of Ψ.

Similar to the ODE setting, we accomplish this by showing that for large enough A and small enough T, depending on n and the size of the initial data, we have:

1. Ψ maps Y into itself.

2. $\Psi : Y \to Y$ is a contraction.

It suffices to prove (1) and (2), since then the contraction mapping theorem furnishes a fixed point for $\Psi \in Y$, which would complete the proof for existence in Theorem 7.2.

For (1), suppose $\phi \in Y$. By definition, $\Psi(\phi)$ satisfies the wave equation

$$\Box\Psi(\phi) = Q(\partial\phi, \partial\phi), \qquad \Psi(\phi)|_{t=0} = \phi_0, \qquad \partial_t\Psi(\phi)|_{t=0} = \phi_1,$$

[37] Essentially, applying a derivative to $P_k h$ is like multiplying by a constant 2^k in Fourier space.

at least in the Duhamel formula sense of (7.11).[38] As a result, one can apply the linear estimate (5.26), with $F := Q(\partial\phi, \partial\phi)$, in order to obtain[39]

$$\begin{aligned}
&\|\nabla_x[\Psi(\phi)](t)\|_{H^s} + \|\partial_t[\Psi(\phi)](t)\|_{H^s} \qquad (7.12)\\
&\quad \leq C\left[\|\nabla_x\phi_0\|_{H^s} + \|\phi_1\|_{H^s} + \left|\int_0^t \||\partial\phi(\tau)|^2\|_{H^s} d\tau\right|\right]\\
&\quad \leq C\left[\|\phi_0\|_{H^{s+1}} + \|\phi_1\|_{H^s} + CT \sup_{\tau\in[-T,T]} \||\partial\phi(\tau)|^2\|_{H^s}\right],
\end{aligned}$$

for each $t \in [-T, T]$. Here, we use C to denote a positive constant that can change from line to line. Applying (7.8) to the nonlinear term in (7.12), we see for each t that

$$\begin{aligned}
&\|\nabla_x\Psi(\phi)(t)\|_{H^s} + \|\partial_t\Psi(\phi)(t)\|_{H^s} \qquad (7.13)\\
&\quad \leq C\left[\|\phi_0\|_{H^{s+1}} + \|\phi_1\|_{H^s} + T \sup_{\tau\in[-T,T]} \|\partial\phi(\tau)\|^2_{H^s}\right]\\
&\quad \leq C[\|\phi_0\|_{H^{s+1}} + \|\phi_1\|_{H^s} + TA^2],
\end{aligned}$$

where in the last step, we recalled that $\phi \in Y$. Furthermore, from the fundamental theorem of calculus, we can control $\Psi(\phi)$ by $\partial_t\Psi(\phi)$: we have for each t that

$$\begin{aligned}
\|\Psi(\phi)(t)\|_{L^2} &\leq \|\Phi(\phi)(0)\|_{L^2} + T \sup_{\tau\in[-T,T]} \|\partial_t\Psi(\phi)(\tau)\|_{L^2} \qquad (7.14)\\
&\leq C(1+T)[\|\phi_0\|_{H^{s+1}} + \|\phi_1\|_{H^s} + TA^2].
\end{aligned}$$

Since the size $M := \|\phi_0\|_{H^{s+1}} + \|\phi_1\|_{H^s}$ is fixed, then by choosing A to be much larger than M and T to be sufficiently small, we see from (7.13) and (7.14) that

$$\|\Psi(\phi)(t)\|_{H^{s+1}} + \|\partial_t(\Psi)(t)\|_{H^s} \leq A, \qquad t \in [-T, T]. \qquad (7.15)$$

Taking a supremum in t, it follows that $\|\Psi(\phi)\|_{\mathcal{X}} \leq A$, and hence Ψ indeed maps any $\phi \in Y$ into Y. This completes the proof of property (1).

The proof of property (2) uses the same set of tools. Supposing $\phi, \tilde{\phi} \in Y$, then the difference $\Psi(\phi) - \Psi(\tilde{\phi})$ satisfies, again in the Duhamel sense, the wave equation

$$\Box[\Psi(\phi) - \Psi(\tilde{\phi})] = Q(\partial\phi, \partial\phi) - Q(\partial\tilde{\phi}, \partial\tilde{\phi}) = Q(\partial(\phi - \tilde{\phi}), \partial\phi) + Q(\partial\tilde{\phi}, \partial(\phi - \tilde{\phi})),$$

with zero initial data. Thus, applying (5.26) and then (7.8) to this quantity yields

$$\begin{aligned}
&\|\nabla_x\Psi(\phi) - \Psi(\tilde{\phi})](t)\|_{H^s} + \|\partial_t\Psi(\phi) - \Psi(\tilde{\phi})](t)\|_{H^s} \qquad (7.16)\\
&\quad \leq CT \sup_{\tau\in[-T,T]} [\||\partial\phi||\partial(\phi - \tilde{\phi})|(\tau)\|_{H^s} + \||\partial\tilde{\phi}||\partial(\phi - \tilde{\phi})|(\tau)\|_{H^s}]\\
&\quad \leq CT \sup_{\tau\in[-T,T]} [\|\partial\phi(\tau)\|_{H^s} + \|\partial\tilde{\phi}(\tau)\|_{H^s}]\|\partial(\phi - \tilde{\phi})(\tau)\|_{H^s}\\
&\quad \leq CTA\|\phi - \tilde{\phi}\|_{\mathcal{X}}.
\end{aligned}$$

Another application of the fundamental theorem of calculus then yields

$$\|[\Psi(\phi) - \Psi(\tilde{\phi})](t)\|_{L^2} \leq T \sup_{\tau\in[-T,T]} \|\partial_t[\Psi(\phi) - \Psi(\tilde{\phi})](\tau)\|_{L^2} \leq CT^2A\|\phi - \tilde{\phi}\|_{\mathcal{X}}. \quad (7.17)$$

[38]The above also holds in the classical sense when ϕ is sufficiently smooth.

[39]The main point is that the proof of (5.26) uses only the Duhamel representation of waves.

Combining (7.16) and (7.17) and taking a supremum over $t \in [-T, T]$, we see that

$$\|\Psi(\phi) - \Psi(\tilde{\phi})\|_{\mathcal{X}} \leq CT(1+T)A\|\phi - \tilde{\phi}\|_{\mathcal{X}}.$$

Taking T to be sufficiently small yields

$$\|\Psi(\phi) - \Psi(\tilde{\phi})\|_{\mathcal{X}} \leq \frac{1}{2}\|\phi - \tilde{\phi}\|_{\mathcal{X}},$$

and hence $\Psi : Y \to Y$ is indeed a contraction. This concludes the proof of (2).

7.2.3 Proof of Theorem 7.2: Uniqueness

Suppose $\phi, \tilde{\phi} \in \mathcal{X}$ are solutions of (7.3). Then, $\psi := \phi - \tilde{\phi}$ solves, in the Duhamel sense,

$$\Box\psi = Q(\partial(\phi - \tilde{\phi}), \partial\phi) + Q(\partial\tilde{\phi}, \partial(\phi - \tilde{\phi})), \qquad \psi|_{t=0} = \partial_t\psi|_{t=0} = 0.$$

For convenience, we consider only $t \geq 0$; the $t < 0$ case is proved in an analogous manner. Similar to the preceding proof of existence, we apply (5.26) and (7.8) to obtain

$$\begin{aligned} &\|\nabla_x\psi(t)\|_{H^s} + \|\partial_t\psi(t)\|_{H^s} \\ &\quad\lesssim \int_0^t [\||\partial\phi||\partial(\phi - \tilde{\phi})|(\tau)\|_{H^s} + \||\partial\tilde{\phi}||\partial(\phi - \tilde{\phi})|(\tau)\|_{H^s}]d\tau \\ &\quad\lesssim \int_0^t [\|\partial\phi(\tau)\|_{H^s} + \|\partial\tilde{\phi}(\tau)\|_{H^s}]\|\partial\psi(\tau)\|_{H^s}d\tau. \end{aligned} \tag{7.18}$$

Since $[-T, T]$ is compact, by continuity and the definition if $\mathcal{X}$, the quantities

$$\|\partial\phi(t)\|_{H^s} + \|\partial\tilde{\phi}(t)\|_{H^s}, \qquad t \in [-T, T]$$

are uniformly bounded, and it follows that

$$\|\nabla_x\psi(t)\|_{H^s} + \|\partial_t\psi(t)\|_{H^s} \lesssim \int_0^t [\|\nabla_x\psi(\tau)\|_{H^s} + \|\partial_t\psi(\tau)\|_{H^s}]d\tau. \tag{7.19}$$

An application of the Gronwall inequality (1.12) yields that

$$\|\nabla_x\psi(t)\|_{H^s} + \|\partial_t\psi(t)\|_{H^s} = 0, \qquad t \in [0, T]. \tag{7.20}$$

Another application of the fundamental theorem of calculus then yields

$$\|\psi(t)\|_{L^2} = 0, \qquad t \in [0, T], \tag{7.21}$$

completing the proof of uniqueness.

7.3 Additional Comments

Finally, we address some additional issues related to Theorem 7.2.

7.3.1 Unconditional Uniqueness

Note that unconditional uniqueness in Theorem 7.2 is only a minor issue, as it is addressed using essentially the same tools as the proof of existence. However, there do exist other settings in which unconditional uniqueness becomes nontrivial, or even an open problem. [40]

[40] For instance, low-regularity existence results that use Strichartz-type estimates would obtain uniqueness only in a "Strichartz subspace" of $\mathcal{X}$. Establishing uniqueness in all of $\mathcal{X}$ requires new arguments.

7.3.2 Maximal Solutions

Another similarity with the ODE setting is that the time of existence in Theorem 7.2 again depends only on the *size* of the initial data. One consequence of this is that the proofs of Corollaries 1.11 and 1.12 can be directly carried over to the current setting. Thus, we have:

Corollary 7.6 (Maximal solutions). *Assume the hypotheses of Theorem 7.2. Then, there is a "maximal" interval* (T_-, T_+)*, where* $-\infty \leq T_- < 0 < T_+ \leq \infty$*, such that:*

- *There exists a strong solution* $\phi \in C^0((T_-, T_+); H^{s+1}) \cap C^1((T_-, T_+); H^s)$ *to* (7.3).
- ϕ *is the only strong solution to* (7.3) *on the interval* (T_-, T_+).
- *If* $\tilde{\phi} \in C^0(I; H^{s+1}) \cap C^1(I; H^s)$ *is another solution of* (7.3)*, then* $I \subseteq (T_-, T_+)$.

Furthermore, if $T_+ < \infty$*, then*

$$\limsup_{t \nearrow T_+} [\|\phi(t)\|_{H^{s+1}} + \|\partial_t \phi\|_{H^s}] = \infty. \tag{7.22}$$

An analogous statement holds for T_-.

One can also view (7.22) as a continuation criterion for (7.3): if the left-hand side of (7.22) is instead finite, then one can extend the solution further in time beyond T_+. As a result of this, it is useful to establish results that replace (7.22) by some other quantity that can be more easily checked. One such classical result is the following:

Corollary 7.7 (Breakdown criterion). *Assume the hypotheses of Theorem 7.2, and let* $\phi : (T_-, T_+) \times \mathbb{R}^n \to \mathbb{R}$ *be the maximal solution to* (7.3)*. If* $T_+ < \infty$*, then*

$$\|(\partial_t \phi, \nabla_x \phi)\|_{L^\infty([0,T_+) \times \mathbb{R}^n)} = \infty. \tag{7.23}$$

Furthermore, an analogous statement holds for T_-.

Proof. Suppose instead that

$$\|(\partial_t \phi, \nabla_x \phi)\|_{L^\infty([0,T_+) \times \mathbb{R}^n)} < \infty. \tag{7.24}$$

Similar to the proof of Theorem 7.2, we apply (5.26) to (7.3), with $F := Q(\partial \phi, \partial \phi)$, along with the fundamental theorem of calculus for $\|\phi(t)\|_{L^2}$ to obtain the bound

$$\|\phi(t)\|_{H^{s+1}} + \|\partial_t \phi(t)\|_{H^s} \lesssim (1+T) \left[\|\phi_0\|_{H^{s+1}} + \|\phi_1\|_{H^s} + \int_0^t \||\partial \phi(\tau)|^2\|_{H^s} d\tau \right]$$

for each $0 \leq t < T_+$. Applying (7.9) and then (7.24) yields

$$\begin{aligned} &\|\phi(t)\|_{H^{s+1}} + \|\partial_t \phi(t)\|_{H^s} \\ &\quad \lesssim (1+T) \left[\|\phi_0\|_{H^{s+1}} + \|\phi_1\|_{H^s} + \int_0^t \|\partial \phi(\tau)\|_{L^\infty} \|\partial \phi(\tau)\|_{H^s} d\tau \right] \\ &\quad \lesssim (1+T) \left\{ \|\phi_0\|_{H^{s+1}} + \|\phi_1\|_{H^s} + \int_0^t [\|\phi(\tau)\|_{H^{s+1}} + \|\partial_t \phi(\tau)\|_{H^s}] d\tau \right\}. \end{aligned}$$

By the Gronwall inequality (1.12), one uniformly bounds $\|\phi(t)\|_{H^{s+1}} + \|\partial_t \phi(t)\|_{H^s}$ for every $t \in [0, T_+)$. Corollary 7.6 then implies that $T_+ = \infty$. $\square$

7.3.3 Finite Speed of Propagation

We revisit the finite speed of propagation property for homogeneous linear wave equations, Corollary 5.19. This property extends to many nonlinear waves, including (7.3).

Corollary 7.8 (Local uniqueness). *Let* $\phi, \tilde{\phi} \in C^2([-T,T] \times \mathbb{R}^n)$ *solve*

$$\Box\phi = Q(\partial\phi, \partial\phi), \qquad \phi|_{t=0} = \phi_0, \qquad \partial_t\phi|_{t=0} = \phi_1,$$
$$\Box\tilde{\phi} = Q(\partial\tilde{\phi}, \partial\tilde{\phi}), \qquad \tilde{\phi}|_{t=0} = \tilde{\phi}_0, \qquad \partial_t\tilde{\phi}|_{t=0} = \tilde{\phi}_1.$$

Moreover, fix $x_0 \in \mathbb{R}^n$ *and* $0 < R \leq T$*, and suppose* ϕ_i *and* $\tilde{\phi}_i$*, where* $i = 0, 1$*, are identical on* $\overline{B_{x_0}(R)}$*. Then,* ϕ *and* $\tilde{\phi}$ *are identical in the region*

$$\mathcal{C} = \{(t,x) \in [-R,R] \times \mathbb{R}^n \mid |x - x_0| \leq R - |t|\}.$$

Remark 7.9. In fact, both the local energy theory and Corollary 7.8 hold in the H^s-setting of this chapter. For brevity, we avoid proving this here.

Proof. Let $\psi := \phi - \tilde{\phi}$, which solves

$$\Box\psi = F := Q(\partial\psi, \partial\phi) + Q(\partial\tilde{\phi}, \partial\psi), \qquad \psi|_{t=0} = \psi_0, \qquad \partial_t\psi|_{t=0} = \psi_1,$$

with ψ_0 and ψ_1 vanishing on $\overline{B_{x_0}(R)}$. We then see from (5.29), with the above F, that

$$\begin{aligned} \mathcal{E}_{\psi,x_0,R}(t) &\leq \mathcal{E}_{\psi,x_0,R}(0) + C\int_0^t \int_{B_{x_0}(R-\tau)} (|\partial\phi(\tau)| + |\partial\tilde{\phi}(\tau)|)|\partial\psi(\tau)|^2 d\tau \\ &\leq C[\|\partial\phi\|_{L^\infty(\mathcal{C})} + \|\partial\tilde{\phi}\|_{L^\infty(\mathcal{C})}] \int_0^t \mathcal{E}_{\psi,x_0,R}(\tau) d\tau. \end{aligned} \tag{7.25}$$

for any $0 \leq t < R$, where $\mathcal{E}_{\psi,x_0,R}(t)$ is as defined in (5.28).

By compactness, the L^∞-norms in (7.25) are finite, hence (1.12) implies $\mathcal{E}_{\psi,x_0,R}(t) = 0$ for all $0 \leq t < R$. An analogous result can also be shown to hold for negative times $-R < t \leq 0$. By the fundamental theorem of calculus, ψ vanishes on all of $\mathcal{C}$. $\Box$

Remark 7.10. If $R > T$ in Corollary 7.8, the conclusion still holds in the truncated cone

$$\mathcal{C}_T = \{(t,x) \in [-T,T] \times \mathbb{R}^n \mid |x - x_0| \leq R - |t|\}.$$

The proof is a slight modification of the above.

7.3.4 Lower Regularity

For Theorem 7.2 and all the discussions above, we required that the initial data for (7.3) lie in $H^{s+1} \times H^s$ for $s > n/2$. One can then ask is whether there must still exist local solutions for less regular initial data, with $s \leq n/2$. Note that to find such solutions, one would require new ingredients in the proof, since one can no longer rely on Theorem 7.3.

Thus, any results that push s down to and below $n/2$ would require some mechanism to make up for the lack of spatial derivatives. The key observation is the time integral on the right-hand side of (7.5), which one can interpret as an antiderivative with respect to t. Note the wave equation $\partial_t^2 u = \Delta u$ can be interpreted as begin able to trade space derivatives for time derivatives, and vice versa. As a result, one can roughly think of being able to convert this antiderivative in t into antiderivatives in x. The effect is manifested in a class of estimates for wave equation known as *Strichartz estimates.*

Using such Strichartz estimates, one can reduce the regularity required in Theorem 7.2 to $s > n/2 - 1/2$ whenever $n \geq 3$.[41] For expositions on Strichartz estimates for wave equations, see [Selb2001,Tao2006]. To further reduce the needed regularity in (7.3), one requires instead *bilinear estimates* for the wave equation. In many cases, if Q in (7.3) has particularly favourable structure,[42] then one can further push down the required regularity. Lastly, for sufficiently small s, local existence of solutions to (7.3) is false; see, e.g., [Selb2001, Ch. 9].

[41] When $n = 2$, Strichartz estimates reduce the required regularity to $s > n/2 - 1/4$.

[42] In particular, if Q is a null form; see the upcoming chapter.

8 Nonlinear Wave Equations: Vector Field Methods, Global and Long-time Existence

Thus far, we have, through Theorem 7.2, established local existence and uniqueness for the quadratic derivative nonlinear wave equation (7.3). One consequence of this result is the notion of maximal solution (see Corollary 7.6), as well as a basic understanding of what must happen if such a solution breaks down in finite time (see (7.22) and Corollary 7.7).

What is not yet clear, however, is whether there actually exist solutions that break down in finite time. Unfortunately, explicit "blow-up" solutions can be easily constructed. For example, consider the following special case of (7.3):

$$\Box\phi = (\partial_t\phi)^2, \qquad \phi|_{t=0} = \phi_0, \qquad \partial_t\phi|_{t=0} = \phi_1. \tag{8.1}$$

If we assume that ϕ depends only on t, and we set $y := \partial_t$, then (8.1) becomes

$$y' = -y^2.$$

This is now an ODE that can be solved explicitly:[43]

$$\partial_t\phi(t) = y(t) = \frac{1}{t + \frac{1}{C}}, \qquad \phi_1 = y(0) = C, \qquad C \in \mathbb{R} \setminus \{0\}.$$

In particular, if $C < 0$, then $\partial_t\phi$ (and also ϕ) blows up at finite time $T_+ = 1/C$.

One can object to the above "counterexample" because the initial data ϕ_1, a constant function, fails to lie in $H^s(\mathbb{R}^n)$. However, this shortcoming can be addressed using the local uniqueness property of Corollary 7.8 (and the remark following its proof). Indeed, suppose one alters ϕ_1 so that it remains a negative constant function on a large enough ball $B_0(R)$, but then smoothly transitions to the zero function outside a larger ball $B_0(R+1)$. Then, finite speed of propagation implies that within the cone

$$\mathcal{C} := \{(t,x) \mid |x| \leq R - |t|\},$$

the new solution ϕ is identical to ODE solution. As a result, as long as R is large enough, this new ϕ will see the same blowup behaviour that was constructed in the ODE setting.

On the other hand, one can still ask whether global existence may hold for *sufficiently small* initial data, for which the linear behaviour is expected to dominate for long times. The main result of this chapter is an affirmative answer for sufficiently high dimensions:

Theorem 8.1 (Small-data global and long-time existence, [?]). *Consider the initial value problem*

$$\Box\phi = Q(\partial\phi, \partial\phi), \qquad \phi|_{t=0} = \varepsilon\phi_0, \qquad \partial_t\phi|_{t=0} = \varepsilon\phi_1, \tag{8.2}$$

where $\varepsilon > 0$, and where the profiles of the initial data satisfy $\phi_0, \phi_1 \in \mathcal{S}(\mathbb{R}^n)$. Suppose in addition that ε is sufficiently small, with respect to n, ϕ_0, and ϕ_1:

- *If $n \geq 4$, then* (8.2) *has a unique global solution.*
- *Otherwise, letting $|T_\pm|$ be as in Corollary 7.6, the maximal solution ϕ to* (8.2) *satisfies*
  - *If $n = 3$, then $|T_\pm| \geq e^{C\varepsilon^{-1}}$.*

[43] There is also the trivial solution $y \equiv 0$.

– *If $n = 2$, then $|T_\pm| \geq C\varepsilon^{-2}$.*

– *If $n = 1$, then $|T_\pm| \geq C\varepsilon^{-1}$.*

Here, the constants C depend on the profiles ϕ_i.

The remainder of this chapter is dedicated to the proof of Theorem 8.1. To keep the exposition brief, we omit some of the more computational and technical elements of the proof; for more detailed treatments, as well as generalisations of Theorem 8.1, the reader is referred to [Selb2001, Ch. 7] or [**?**, **?**].

Before this, a few preliminary remarks on the theorem are in order.

Remark 8.2. Since the ϕ_i are assumed to lie in $\mathcal{S}(\mathbb{R}^n)$, the initial data $\varepsilon\phi_i$ lie in every H^s-space. As a result, all the machinery from the local theory applies, and one can speak of maximal solutions of (8.2). Furthermore, since these solution curves lie in every H^s-space, it follows that the maximal solution ϕ is actually a smooth classical solution of (8.2).

Remark 8.3. The uniqueness arguments from Theorem 7.2 also carry over to the current setting. Thus, we only need to concern ourselves with existence here.

Remark 8.4. Note that although small-data global existence is not proved for low dimensions $n < 4$ in Theorem 8.1, one does obtain weaker *long-time existence* results, in the form of lower bounds on the timespan $T_\pm$ of solutions.

8.1 Preliminary Ideas

From now on, we let $\phi : (T_-, T_+) \times \mathbb{R}^n \to \mathbb{R}$ be the maximal solution to (8.2), as obtained from Theorem 7.2 and Corollary 7.6. To prove Theorem 8.1, we must hence show $|T_\pm| = \infty$. Moreover, we focus on showing $T_+ = \infty$, since negative times can be handled analogously.

Recall from the previous chapter that the local theory behind (8.2) revolves around energy-type estimates of the form

$$\mathcal{E}_0(t) \lesssim \mathcal{E}(0) + \int_0^t [\mathcal{E}(\tau)]^2 d\tau, \qquad t \in [0, T_+), \tag{8.3}$$

where the "energy" $\mathcal{E}(t)$ is given in terms of H^s-norms:

$$\mathcal{E}(t) := \|\phi(t)\|_{H^{s+1}} + \|\partial_t \phi(t)\|_{H^s}, \qquad s > \frac{n}{2}. \tag{8.4}$$

In particular, both local existence and uniqueness followed from this type of estimate.

A major guiding intuition was that whenever t is small, the nonlinear $\mathcal{E}^2$-integral in (8.3) will not interfere appreciably with the linear evolution. However, since this intuition breaks down whenever t is large, (8.3) is not enough to ensure $\mathcal{E}(t)$ does not blow up at a finite time. Thus, our local theory, based around (8.3), cannot be sufficient to derive global existence.

Suppose on the other hand that we have a stronger "energy estimate",

$$\mathcal{E}(t) \lesssim \mathcal{E}(0) + \int_0^t \frac{[\mathcal{E}(\tau)]^2}{(1+\tau)^p} d\tau, \qquad p > 0, \tag{8.5}$$

where $\mathcal{E}(t)$ now denotes some alternate "energy quantity". In other words, suppose the nonlinear estimate comes with an additional decay in time. If $p > 1$, and hence $(1 + \tau)^{-p}$ is integrable on $[0, \infty)$, then the largeness of t is no longer the devastating obstruction it once was. In this case, the smallness of $\mathcal{E}(t)$ itself is sufficient to show that the nonlinear evolution is dominated by the linear evolution, regardless of the size of t.

Indeed, using the integrability of $(1+\tau)^{-p}$ results in the estimate

$$\sup_{0\leq\tau\leq t} \mathcal{E}(\tau) \leq C\mathcal{E}(0) + C_p \left[\sup_{0\leq\tau\leq t} \mathcal{E}(\tau) \right]^2$$

Using a continuity argument, as described in Section 1.4, one can then uniformly bound $\mathcal{E}(t)$ for all $t \in [0, T_+)$. (In fact, this was essentially demonstrated by the computations in Example 1.16.) This propagation property for the modified energy for all times is the main ingredient to improving from local to global existence when $n \geq 4$.

On the other hand, when $n \leq 3$, the power p that one can obtain will be small enough such that $(1+\tau)^{-p}$ is no longer integrable. In this case, one can no longer obtain global existence, but one can still estimate how large t can be before the nonlinear evolution can dominate. Indeed, the nonlinear effects become non-negligible whenever the integral

$$\int_0^t (1+\tau)^{-p} d\tau$$

becomes large.[44] In fact, this consideration is directly responsible for the lower bounds on the times of existence $|T_\pm|$ in Theorem 8.1 when $n \leq 3$.

In light of the above, the pressing questions are then the following:

1. What is this modified energy quantity $\mathcal{E}(t)$?
2. How does one obtain this improved energy estimate (8.5) for $\mathcal{E}(t)$?

8.2 The Invariant Vector Fields

Recall that the unmodified energy is obtained by taking s derivatives of $\partial\phi$ and measuring the L^2-norm. These derivatives ∂_t and ∇_x are handy in particular because they commute with the wave operator. In fact, one can view the H^s-energy estimate for $\partial\phi$ as the L^2-energy estimate applied to both $\partial\phi$ and "$\partial\nabla_x^s\phi$".

With this in mind, it makes sense to enlarge our set of derivatives to other operators that commute with $\Box$. We do this by defining the following set of vector fields on $\mathbb{R}^{n+1}$:

- *Translations*: The Cartesian coordinate vector fields

$$\partial_0 := \partial_t, \partial_1 := \partial_{x^1}, \ldots, \partial_n := \partial_{x^n}, \tag{8.6}$$

  which generate the spacetime translations of $\mathbb{R} \times \mathbb{R}^n$.[45]

- *Spatial rotations*: The vector fields,

$$\Omega_{ij} := x^j\partial_i - x^i\partial_j, \qquad 1 \leq i < j \leq n, \tag{8.7}$$

  which generate spatial rotations on each level set of t.

- *Lorentz boosts*: The vector fields,

$$\Omega_{0j} := x^j\partial_t + t\partial_j, \qquad 1 \leq i \leq n, \tag{8.8}$$

  which generate *Lorentz boosts* on $\mathbb{R} \times \mathbb{R}^n$.

[44] Whenever this integral is not large, one can still bound $\mathcal{E}(t)$ via the above continuity argument.

[45] More specifically, transport along the integral curves of the ∂_α's are precisely translations in $\mathbb{R} \times \mathbb{R}^n$.

- *Scaling/dilation*: The vector field

$$S := t\partial_t + \sum_{i=1}^{n} x^i \partial_i, \tag{8.9}$$

  which generates the (spacetime) dilations on $\mathbb{R} \times \mathbb{R}^n$.

Note that (8.6)-(8.9) define exactly

$$\gamma_n := (n+1) + \frac{n(n-1)}{2} + n + 1 = \frac{(n+2)(n+1)}{2} + 1$$

independent vector fields. For future notational convenience, we order these vector fields in some arbitrary manner, and we label them as $\Gamma_1, \dots, \Gamma_{\gamma_n}$.

The main algebraic properties of the Γ_a's are given in the following lemma:

Lemma 8.5. *The scaling vector field satisfies*

$$[\Box, S] := \Box S - S\Box = 2\Box, \tag{8.10}$$

while any other such vector field $\Gamma_a \neq S$ satisfies

$$[\Box, \Gamma_a] := \Box\Gamma_a - \Gamma_a\Box = 0. \tag{8.11}$$

Furthermore, for any Γ_b and Cartesian derivative ∂_α, we have

$$[\partial_\alpha, \Gamma_b] = \sum_{\beta=0}^{n} c^\beta_{\alpha b} \partial_\beta, \qquad c^\beta_{\alpha b} \in \mathbb{R}. \tag{8.12}$$

Proof. These identities can be verified through direct computation. In particular, (8.12) is an consequence of the fact that for any such Γ_a, its coefficients (expressed in Cartesian coordinates) are always either constant or one of the Cartesian coordinate functions. $\Box$

We will use multi-index notation to denote successive applications of various such Γ_a's. More specifically, given a multi-index $I = (I_1, \dots, I_d)$, where $1 \leq I_i \leq \gamma_n$, we define

$$\Gamma^I = \Gamma_{I_1}\Gamma_{I_2}\dots\Gamma_{I_d}. \tag{8.13}$$

Note that since the Γ_a's generally do not commute with each other, the ordering of the coefficients in such a multi-index I carries nontrivial information.

8.2.1 Geometric Ideas

The key intutions behind the vector fields (8.6)-(8.9) are actually geometric in nature. To fully appreciate these ideas, one must invoke some basic notions from differential geometry. We give a brief summary of these observations here.

Remark 8.6. In the context of Theorem 8.1, the properties we will need are the identities in Lemma 8.5, which can be computed without reference to any geometric discussions. Thus, the intuitions discussed here are not essential to the proof of Theorem 8.1.

Recall that *Minkowksi spacetime* can be described as the manifold $(\mathbb{R} \times \mathbb{R}^n, m)$, where m is the *Minkowski metric*, i.e., the symmetric covariant 2-tensor on $\mathbb{R}^{n+1}$ given by

$$m := -(dt)^2 + (dx^1)^2 + (dx^n)^2.$$

In particular, the Minkowski metric differs from the Euclidean metric on $\mathbb{R}^{1+n}$,

$$e := (dt)^2 + (dx^1)^2 + (dx^n)^2,$$

only by a reversal of sign in the t-component. However, this change in sign makes Minkowski geometry radically different from the more familiar Euclidean geometry.

Remark 8.7. Minkowsi spacetime is the setting for Einstein's theory of *special relativity*.

Furthermore, the wave operator $\Box$ is intrinsic to Minkowski spacetime. Indeed, $\Box$ is the Laplace-Beltrami operator associated with $(\mathbb{R}^{n+1}, m)$,

$$\Box = m^{ij} D^2_{ij} = -\partial_t^2 + \sum_{i=1}^{n} \partial_i^2,$$

where D is the covariant derivative with respect to m. As a result, one would expect that any symmetry of Minkowski spacetime would behave well with respect to $\Box$.

Observe next that any vector field Γ_a given by (8.6)-(8.8) is a *Killing vector field* on Minkowski spacetime, that is, Γ_a *generates symmetries of Minkowski spacetime.* In differential geometric terms, this is given by the condition $\mathcal{L}_{\Gamma_a} m = 0$, where $\mathcal{L}$ denotes the Lie derivative. In other words, transport along the integral curves of Γ_a does not change the Minkowski metric, hence such a transformation yields a symmetry of Minkowski spacetime. Since $\Box$ arises entirely from Minkowski geometry, transporting along Γ_a also preserves the wave operator. This is the main geometric intuition behind (8.11).

Remark 8.8. In fact, the vector fields (8.6)-(8.8) generate the Lie algebra of all Killing vector fields on Minkowski spacetime.

On the other hand, the scaling vector field S in (8.9) is *not* a Killing vector field and hence does not generate a symmetry of Minkowski spacetime. However, S is a *conformal Killing vector field*, that is, S generates a conformal symmetry of Minkowski spacetime. As this is not a full symmetry, S will not commute with $\Box$, but the conformal symmetry property ensures that this commutator is relatively simple; see (8.10).

8.3 The Modified Energy

Because the vector fields Γ_a commute so well with $\Box$, see (8.10) and (8.11), then $\Gamma_a \phi$ also satisfies a "nice" nonlinear wave equation:

$$\Box \Gamma_a \phi = \Gamma_a \Box u + c \Box u = \Gamma_a \partial u \cdot \partial u + c(\partial u)^2, \qquad c \in \mathbb{R}. \tag{8.14}$$

As a result, one can also apply energy estimates to control $\partial \Gamma_a \phi$ in terms of the initial data. Moreover, the same observations hold for any number of Γ_a's applied to ϕ—for any multi-index $I = (I_1, \dots, I_d)$, with $1 \leq I_i \leq \gamma_n$, one has

$$|\Box \Gamma^I \phi| \leq |\Gamma^I \Box \phi| + |[\Box, \Gamma^I]\phi| \lesssim \sum_{|J| \leq |I|} |\Gamma^J \Box \phi|, \tag{8.15}$$

where the sum is over all multi-indices $J = (J_1, \dots, J_m)$ with length $|J| = m \leq d = |I|$. Applying (8.2) and then (8.12) to the right-hand side of (8.15), we see that

$$|\Box \Gamma^I \phi| \lesssim \sum_{|J|+|K| \leq |I|} |\Gamma^J \partial \phi||\Gamma^K \partial \phi| \lesssim \sum_{|J|+|K| \leq |I|} |\partial \Gamma^J \phi||\partial \Gamma^K \phi|. \tag{8.16}$$

This leads us to define the following modified energy quantity for ϕ:[46]

$$\mathcal{E}(t) = \sum_{|I| \leq n+4} \|\partial \Gamma^I \phi(t)\|_{L^2}. \tag{8.17}$$

We now wish to show that this satisfies an improved energy estimate (8.5).

Applying the linear estimate (5.26) to $\Gamma^I \phi$, with $s = 0$, yields

$$\|\partial \Gamma^I \phi(t)\|_{L^2} \lesssim \|\partial \Gamma^I \phi(0)\|_{L^2} + \int_0^t \|\Box \Gamma^I \phi(\tau)\|_{L^2} d\tau.$$

Summing the above over $|I| \leq n+4$ and applying (8.16) yields

$$\mathcal{E}(t) \lesssim \mathcal{E}(0) + \sum_{|J|+|K| \leq n+4} \int_0^t \| |\partial \Gamma^J \phi(\tau)| |\partial \Gamma^K \phi(\tau)| \|_{L^2} d\tau. \tag{8.18}$$

Now, since $|J|+|K|$ on the right-hand side of (8.18) is at most $n+4$, we can assume without loss of generality that $|J| \leq n/2 + 2$. Using Hölder's inequality results in the bound

$$\begin{aligned} \mathcal{E}(t) &\lesssim \mathcal{E}(0) + \int_0^t \sum_{|J| \leq \frac{n}{2}+2} \|\partial \Gamma^J \phi(\tau)\|_{L^\infty} \sum_{|K| \leq n+4} \|\partial \Gamma^K \phi(\tau)\|_{L^2} d\tau \\ &\lesssim \mathcal{E}(0) + \int_0^t \sum_{|J| \leq \frac{n}{2}+2} \|\partial \Gamma^J \phi(\tau)\|_{L^\infty} \cdot \mathcal{E}(\tau) \cdot d\tau. \end{aligned} \tag{8.19}$$

8.3.1 Sobolev Bounds with Decay

Previously, we controlled L^∞-norms of ϕ by H^s-energies by applying the Sobolev inequality (7.10). In our setting, this results in the crude bound

$$\|\phi(t)\|_{L^\infty} \lesssim \sum_{k \leq \frac{n}{2}+1} \|\partial^k \phi(t)\|_{L^2} \lesssim \sum_{|I| \leq \frac{n}{2}+1} \|\Gamma^I \phi(t)\|_{L^2}. \tag{8.20}$$

Note we are losing a large amount of information here, since we are considering all the vector fields Γ_a, not just the ∂_α's. By leveraging the fact that many of the Γ_a's have growing weights, one sees the possibility of an improvement to (8.20), with additional weights on the left-hand side that grow. In fact, there does exist such an estimate, which is known as the *Klainerman-Sobolev*, or *global Sobolev*, *inequality*:

Theorem 8.9 (Klainerman-Sobolev inequality). *Let $v \in C^\infty([0,\infty) \times \mathbb{R}^n)$ such that $v(t) \in \mathcal{S}(\mathbb{R}^n)$ for any $t \geq 0$. Then, the following estimate holds for each $t \geq 0$ and $x \in \mathbb{R}^n$:*

$$(1 + t + |x|)^{\frac{n-1}{2}} (1 + |t - |x||)^{\frac{1}{2}} |v(t,x)| \lesssim \sum_{|I| \leq \frac{n}{2}+1} \|\Gamma^I v(t)\|_{L^2}. \tag{8.21}$$

Roughly, the main idea behind the proof of (8.21) is to write ∂_α as linear combinations of the Γ_a's, which introduce decaying weights. This can be expressed in multiple ways, with each resulting in different weights in time and space. One then applies standard Sobolev inequalities (either on $\mathbb{R}^n$ or on $\mathbb{S}^{n-1}$) and uses the aforementioned algebraic relations to pick up decaying weights. Moreover, depending on the relative sizes of t and $|x|$, one can choose the specific relations and estimates to maximise the decay in the weight. For details, the reader is referred to either [Selb2001, Ch. 7] or [**?**, **?**].

[46] Recall again that $\partial := (\partial_t, \nabla_x)$ denotes the spacetime gradient.

Remark 8.10. In the context of Theorem 8.1, the Klainerman-Sobolev estimate suggests decay for ϕ in both t and $|x|$. Furthermore, the weight on the left-hand side of (8.21) indicates that ϕ will decay a half-power *faster* away from the cone $t = |x|$. For our current problem, though, we will not need to consider the decay in $|x|$ or in $|t - |x||$.

In particular, when we apply Theorem 8.9 to (8.19), we obtain

$$\mathcal{E}(t) \lesssim \mathcal{E}(0) + \int_0^t (1+\tau)^{-\frac{n-1}{2}} \sum_{|J|+|K| \leq n+4} \|\Gamma^K \partial \Gamma^J \phi(\tau)\|_{L^2} \cdot \mathcal{E}(\tau) \cdot ds. \tag{8.22}$$

Commuting Γ_a's and ∂_α's using (8.12) yields the following bound:

Lemma 8.11. *For any* $0 \leq t < T_+$,

$$\mathcal{E}(t) \lesssim \mathcal{E}(0) + \int_0^t \frac{[\mathcal{E}(\tau)]^2}{(1+\tau)^{\frac{n-1}{2}}} d\tau. \tag{8.23}$$

Remark 8.12. Note that one must prescribe a high enough number of derivatives in the definition of $\mathcal{E}(t)$, so that after applying (8.21) to the L^∞-factor in (8.19), the resulting L^2-norms are still controlled by $\mathcal{E}(t)$. This is the rationale behind our choice $n + 4$.

8.4 Completion of the Proof

We now apply (8.23) to complete the proof of Theorem 8.1. The main step is the following:

Lemma 8.13. *Assume* ε *in* (8.2) *is sufficiently small. Then:*

- *If* $n = 4$, *then* $\mathcal{E}(t) \lesssim \mathcal{E}(0)$ *for all* $0 \leq t < T_+$.
- *If* $n = 3$, *then* $\mathcal{E}(t) \lesssim \mathcal{E}(0)$ *for all* $0 \leq t < \min(T_+, e^{C\varepsilon^{-1}})$.
- *If* $n = 2$, *then* $\mathcal{E}(t) \lesssim \mathcal{E}(0)$ *for all* $0 \leq t < \min(T_+, C\varepsilon^{-2})$.
- *If* $n = 1$, *then* $\mathcal{E}(t) \lesssim \mathcal{E}(0)$ *for all* $0 \leq t < \min(T_+, C\varepsilon^{-1})$.

Here, C *is a constant depending on* ϕ_0 *and* ϕ_1.

Let us first assume Lemma 8.13 has been established. Applying the standard Sobolev embedding (7.10), we can uniformly bound the spacetime gradient of ϕ:

$$\|\partial \phi(t)\|_{L^\infty} \lesssim \sum_{|I| \lesssim \frac{n}{2}+1} \|\partial \Gamma^I \phi(t)\|_{L^2} \lesssim \mathcal{E}(t). \tag{8.24}$$

When $n \geq 4$, combining Lemma 8.13 and (8.24) results in a uniform bound on $\partial\phi$ on all of $[0, T_+) \times \mathbb{R}^n$. By Corollary 7.7, it follows that $T_+ = \infty$, as desired.

Consider now the case $n = 3$, and suppose $T_+ \leq e^{C\varepsilon^{-1}}$. Again, by Lemma 8.13 and (8.24), one can bound $\partial\phi$ uniformly on $[0, T_+) \times \mathbb{R}^n$. Corollary 7.7 then implies $T_+ = \infty$, resulting in a contradiction. Thus, we conclude $T_+ \geq e^{C\varepsilon^{-1}}$, as desired.

The remaining cases $n < 3$ can be proved in the same manner as for $n = 3$. Thus, to complete the proof of Theorem 8.1, it remains only to prove Lemma 8.13.

8.4.1 The Bootstrap Argument

As mentioned before, the proof of Lemma 8.13 revolves around a continuity argument.[47] For this, we first fix positive constants A and B such that $\mathcal{E}(0) := \varepsilon B \ll \varepsilon A$. Given a time $t \geq 0$, we make the following bootstrap assumption: [48]

$\mathbf{BS}(t)$: $\mathcal{E}(t') \leq 2A\varepsilon$ for all $0 \leq t' \leq t$.

The goal then is to derive a strictly better version of $\mathbf{BS}(t)$.

Suppose first that $n \geq 4$, so that $(1+\tau)^{-\frac{n-1}{2}}$ is integrable on all of $[0, \infty)$. Then, applying (8.23) and the bootstrap assumption $\mathbf{BS}(t)$, we obtain, for any $0 \leq t' \leq t$,

$$\begin{aligned} \mathcal{E}(t') &\leq C \cdot \mathcal{E}(0) + C \int_0^{t'} \frac{[\mathcal{E}(\tau)]^2}{(1+\tau)^{\frac{n-1}{2}}} d\tau \\ &\leq \varepsilon CB + 4\varepsilon^2 CA^2 \int_0^{\infty} \frac{1}{(1+\tau)^{\frac{n-1}{2}}} d\tau \\ &\leq \varepsilon CB + \varepsilon^2 C' A^2, \end{aligned} \tag{8.25}$$

where $C' > 0$ is another constant. Note in particular that if ε is sufficiently small, then (8.25) implies a strictly better version of $\mathbf{BS}(t)$:

$$\mathcal{E}(t') \leq \varepsilon A, \qquad 0 \leq t' \leq t.$$

This implies the desired uniform bound for $\mathcal{E}(t)$ and proves Lemma 8.13 when $n \geq 4$.

Consider next the case $n = 3$. The main idea is that the above bootstrap argument still applies as long as t is not too large. More specifically, assuming $\mathbf{BS}(t)$ as before, one sees that as long as $t' \leq t \leq e^{C\varepsilon^{-1}}$, the following estimate still holds:

$$\begin{aligned} \mathcal{E}(t') &\leq \varepsilon C' B + 4\varepsilon^2 C' A^2 \int_0^{e^{C\varepsilon^{-1}}} \frac{1}{1+\tau} d\tau \\ &\leq \varepsilon C' B + \varepsilon A \cdot CC'' A. \end{aligned} \tag{8.26}$$

Letting C be small, then one once again obtains a strictly improved version of $\mathbf{BS}(t)$,

$$\mathcal{E}(t') \leq \varepsilon A, \qquad 0 \leq t' \leq t,$$

as long as $t \leq e^{C\varepsilon^{-1}}$ *for the above* C. A continuity argument (which can be localised to a finite interval) then implies that $\mathcal{E}(t)$ is uniformly bounded for all times $0 \leq t \leq e^{C\varepsilon^{-1}}$.

The proofs of the remaining cases $n < 3$ resemble that of $n = 3$, hence we omit the details here. This completes the proof of Theorem 8.1.

8.5 Additional Remarks

We conclude this chapter with some additional remarks on variants of Theorem 8.1.

8.5.1 Higher-Order Nonlinearities

Theorem 8.1 can be extended to higher-order derivative nonlinearities $\mathcal{N}(\phi, \partial\phi) \approx (\partial\phi)^p$ for $p > 2$. Consider, for concreteness, the cubic derivative nonlinear wave equation

$$\Box\phi = U(\partial\phi, \partial\phi, \partial\phi), \qquad \phi|_{t=0} = \varepsilon\phi_0, \quad \partial_t\phi|_{t=0} = \varepsilon\phi_1, \tag{8.27}$$

[47] For background on continuity arguments, see Section 1.4 and in particular Example 1.16.

[48] Note that the constants A and B depend on the profiles ϕ_0 and ϕ_1.

where U is some trilinear form. Since ϕ is presumed small, then the cubic nonlinearity $(\partial\phi)^3$ should be even smaller than the previous $(\partial\phi)^2$. As a result, one can expect improved small-data global existence results for (8.27).[49]

To be more specific, if we rerun the proof of Theorem 8.1, with $\mathcal{E}$ the modified energy, then the nonlinear term contains two L^∞-factors. This results in the estimate

$$\mathcal{E}(t) \lesssim \mathcal{E}(0) + \int_0^t \frac{[\mathcal{E}(\tau)]^2}{(1+\tau)^{n-1}} d\tau.$$

Since $(1+\tau)^{-(n-1)}$ is integrable when $n \geq 3$, small-data global existence holds for (8.27) whenever $n \geq 3$. Moreover, when $n < 3$, one can again obtain lower bounds on $|T_\pm|$.

This reasoning extends to even higher-order nonlinearities. For instance, for quartic derivative nonlinear wave equations, small-data global existence holds whenever $n \geq 2$.

8.5.2 The Null Condition

Returning to the quadratic case (8.2), small-data global existence now fails for $n = 3$. For example, when $Q(\partial\phi, \partial\phi) = (\partial_t\phi)^2$, every solution with smooth, compactly supported data blows up in finite time; see [**?**]. However, one can still ask whether small-data global existence holds for quadratic nonlinearities containing some special structure.

The key observation is that for such nonlinear waves, not all derivatives of ϕ decay at the same rate; in fact, there are "good" directions which decay better the usual $(1+t)^{-\frac{n-1}{2}}$-rate. For instance, this can be seen in the extra weight $(1+|t-|x||)^{\frac{1}{2}}$ in the Klainerman-Sobolev inequality, (8.21).[50] As a result, one could possibly expect improved results when Q has the special algebraic property that every term contains at least one "good" component of $\partial\phi$.

The formal expression of this algebraic criterion is known as the *null condition* and was first discovered by Klainerman and Christodoulou; see [**?**, **?**]. For this, one first defines the *fundamental null forms*:

$$\begin{aligned} Q_0(\partial f, \partial g) &= -\partial_t f \partial_t g + \sum_{i=1}^n \partial_i f \partial_i g, \\ Q_{\alpha\beta}(\partial f, \partial g) &= \partial_\alpha f \partial_\beta g - \partial_\beta f \partial_\alpha g. \end{aligned} \tag{8.28}$$

Then, the null condition is simply that Q is a linear combination of the above forms:

Theorem 8.14. *Let $n = 3$, and suppose Q in* (8.2) *satisfies the above* null condition.[51] *Then, for sufficiently small $\varepsilon > 0$, the solution to* (8.2) *is global.*

We conclude by demonstrating Theorem 8.14 via an example:

Example 8.15. Consider the initial value problem

$$\Box\phi = -Q_0(\partial\phi, \partial\phi), \qquad \phi|_{t=0} = \varepsilon\phi_0, \quad \partial_t\phi|_{t=0} = \varepsilon\phi_1, \tag{8.29}$$

and let $v := e^\phi$. A direct computation shows that v must formally satisfy

$$\Box v = 0, \qquad v|_{t=0} = e^{\phi_0}, \qquad \partial_t v|_{t=0} = \phi_1 e^{\phi_0},$$

which by Theorem 5.3 has a global solution.

[49] The local existence theory of the previous chapter also extends directly to (8.27).
[50] In particular, the proof of Theorem 8.1 did not take advantage of this extra decay.
[51] Note however that $Q_{\alpha\beta}$ can only appear in systems of wave equations.

One can now recover the solution ϕ for (8.29) by reversing the change of variables, $\phi = \log v$. In particular, this solution ϕ exists as long as $v > 0$. A direct computation using (5.9) shows that this indeed holds as long as ϕ_0 and ϕ_1 are small.

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Abstract: We discuss some of the key ideas of Perelman's proof of Poincar\'e's conjecture via the Hamilton program of using the Ricci flow, from the perspective of the modern theory of nonlinear partial differential equations.

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