The Vlasov-Poisson system for stellar dynamics in spaces of constant curvatures

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Outline

1.
2.
3.
4.

Outline

1. Derivation of the VP on curved spaces
2.
3.
4.

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2. Special distributions and reduced model
3.
4.

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3. Analysis of the reduced model: Well posedness, and Linear stability
4.

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2. Special distributions and reduced model
3. Analysis of the reduced model: Well posedness, and Linear stability
4. Sketch of the proofs

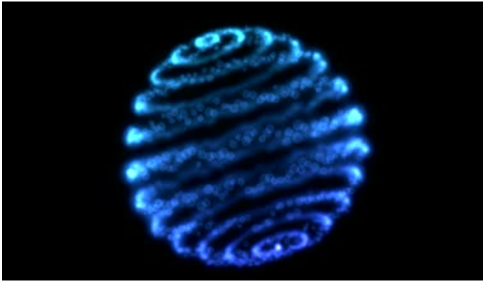

FIGURE: An abstract animation of colourful particles on a sphere.

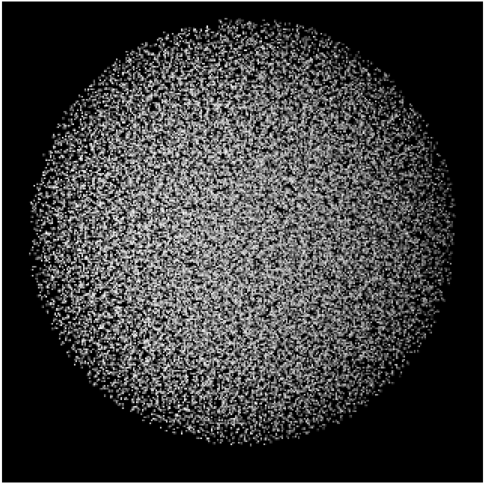

FIGURE: what happens if the number of particles becomes huge?

Introduction

The classical VP system models the density change of galaxies in a cluster of galaxies, stars in a galaxy, or particles in plasma.

$$\frac{\partial}{\partial t} f(t,\mathbf{x},\mathbf{v}) + \mathbf{v}\frac{\partial}{\partial \mathbf{x}} f(t,\mathbf{x},\mathbf{v}) + F(t,\mathbf{x})\frac{\partial}{\partial \mathbf{v}} f(t,\mathbf{x},\mathbf{v}) = 0,$$
$$F(t,\mathbf{x}) = -\int_{\mathbb{R}^3} \frac{\mathbf{x}-\mathbf{y}}{|\,\mathbf{x}-\mathbf{y}\,|^3}\rho(t,\mathbf{y})d\mathbf{y}, \quad \rho(t,\mathbf{x}) := \int_{\mathbb{R}^3} f(t,\mathbf{x},\mathbf{v})d\mathbf{v},$$

Goal: To broaden the scope of the VP system to spaces of non-zero constant Gaussian curvature within the framework of classical mechanics.
Reasons:

- To obtain a better understanding of the flat case by viewing the VP system as the limit of its counterpart in curved space when the curvature tends to zero.
- It might allow us to decide whether the universe is curved or not. Indeed, if a certain solution of the density function occurs only in, say, flat space but not in hyperbolic and elliptic space, and such behaviour is supported by astronomical evidence, then we could decide that universe is Euclidean.

Important fact: The derivation of the classical system for stellar dynamics uses the Newtonian equations of the gravitational N-body problem.
To generalize the VP system to elliptic and hyperbolic spaces, we have to employ a meaningful extension of the Newtonian equations of the N-body problem to spaces of constant Gaussian curvature.

- The idea of such an extension belonged to Bolyai '19 and Lobachevsky '49 in the 2-body case
- A suitable generalization of the classical Newtonian system was only recently obtained by Diacu '12-'14

Framework

Let $\mathbb{M}^2$ represent the unit 2-sphere,

$$\mathbb{S}^2 = \{(x_1, x_2, x_3) \,|\, x_1^2 + x_2^2 + x_3^2 = 1\} \hookrightarrow \mathbb{R}^3,$$

for positive curvature, and the unit hyperbolic 2-sphere (i.e. the upper sheet of the unit hyperboloid of two sheets),

$$\mathbb{H}^2 = \{(x_1, x_2, x_3) \,|\, x_1^2 + x_2^2 - x_3^2 = -1,\ x_3 > 0\} \hookrightarrow \mathbb{R}^{2,1},$$

for negative curvature. If κ denotes the Gaussian curvature, the signum function is defined as

$$\sigma = \begin{cases} +1, & \text{for} \quad \kappa \geq 0 \\ -1, & \text{for} \quad \kappa < 0. \end{cases}$$

Introduce the unified trigonometric functions:

$$\operatorname{sn} x := \begin{cases} \sin x, & \text{for} \quad \kappa > 0 \\ \sinh x, & \text{for} \quad \kappa < 0, \end{cases} \qquad \operatorname{csn} x := \begin{cases} \cos x, & \text{for} \quad \kappa > 0 \\ \cosh x, & \text{for} \quad \kappa < 0, \end{cases} \tag{1}$$

$$\operatorname{tn} x := \frac{\operatorname{sn} x}{\operatorname{csn} x}, \qquad \operatorname{ctn} x := \frac{\operatorname{csn} x}{\operatorname{sn} x},$$

the distance between $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ in $\mathbb{M}^2$ is $d(\mathbf{a}, \mathbf{b}) = \operatorname{csn}^{-1}(\sigma \mathbf{a} \cdot \mathbf{b})$, with scalar product

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \sigma a_3 b_3, \quad \text{and} \quad ||\mathbf{a}|| = |a_1^2 + a_2^2 + \sigma a_3^2|^{1/2}.$$

Equations of motion

For $(\mathbf{x},\mathbf{v}) \in T\mathbb{M}^2$, $f = f(t,\mathbf{x},\mathbf{v})$ is the phase space density, $\rho = \int_{T_x\mathbb{M}^2} f(t,\mathbf{x},\mathbf{v})\, dv$ the spatial density, and $U = U(\mathbf{x})$ the gravitational potential function.

Proposition

In extrinsic coordinates having the origin at the centre of $\mathbb{M}^2$, the equations of motion of a particle $(\mathbf{x},\mathbf{v}) \in T\mathbb{M}^2$, under the effect of a potential function $U : \mathbb{M}^2 \to \mathbb{R}$, are

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{v}, \\ \dot{\mathbf{v}} = \nabla_{\mathbf{x}} U(\mathbf{x}) - \sigma(\mathbf{v}\cdot\mathbf{v})\mathbf{x}. \end{cases} \tag{2}$$

Idea Variational approach from constrained Lagrangian mechanics: the motion that minimize the integral of Lagrangian

$$L(\mathbf{x},\mathbf{v}) = T(\mathbf{x},\mathbf{v}) - V(t,\mathbf{x}) = \frac{1}{2}\sigma(\mathbf{v}\cdot\mathbf{v})(\mathbf{x}\cdot\mathbf{x}) + U(\mathbf{x})$$

under the constraint $\mathbf{x}\cdot\mathbf{x} = \sigma$. Euler-Lagrange system:

$$\frac{d}{dt}(\partial_{\mathbf{v}} L(\mathbf{x},\mathbf{v})) - \partial_{\mathbf{x}} L(\mathbf{x},\mathbf{v}) - \lambda\, \partial_{\mathbf{x}} g(\mathbf{x}) = 0, \tag{3}$$

gives $\dot{v} - |v|^2 x - \nabla_x U = \lambda x$. Then Euler's formula and the constraint show that $\lambda = -2v\cdot v$, and therefore $\dot{\mathbf{v}} = \nabla_{\mathbf{x}} U(\mathbf{x}) - \sigma(\mathbf{v}\cdot\mathbf{v})\mathbf{x},$.

Remark

- *In local coordinates, the equations of motion for a particle with position* $\mathbf{q} = \alpha \mathbf{e}_\alpha + \theta \mathbf{e}_\theta \in \mathbb{M}^2$ *and velocity* $\mathbf{p} = \omega_\alpha \mathbf{e}_\alpha + \operatorname{sn}\alpha\, \omega_\theta \mathbf{e}_\theta \in \mathbf{T}\mathbb{M}^2$*, under the effect of a potential function* $\tilde{U} : \mathbb{M}^2 \to \mathbb{R}$ *are given by the system*

$$\begin{cases} \dot{\alpha} = \omega_\alpha, \\ \dot{\theta} = \omega_\theta, \\ \dot{\omega}_\alpha = \partial_\alpha \tilde{U}(\alpha, \theta) + \omega_\theta^2 \operatorname{sn}\alpha \operatorname{csn}\alpha, \\ \dot{\omega}_\theta = \dfrac{1}{\operatorname{sn}^2\alpha} \partial_\theta \tilde{U}(\alpha, \theta) - 2\omega_\alpha \omega_\theta \operatorname{ctn}\alpha. \end{cases} \tag{4}$$

- *In extrinsic coordinates* $(\mathbf{x}, \mathbf{v}) \in T\mathbb{M}^2$*, the equations of motion for a particle of mass 1 moving on* $\mathbb{M}^2$ *can be written in Hamiltonian form as*

$$\begin{cases} \dot{\mathbf{x}} = \partial_{\mathbf{v}} H \\ \dot{\mathbf{v}} = -\partial_{\mathbf{x}} H, \end{cases}$$

where $H(\mathbf{x}, \mathbf{v}) = \dfrac{1}{2}\sigma(\mathbf{v} \cdot \mathbf{v})(\mathbf{x} \cdot \mathbf{x}) - U(\mathbf{x})$ *is the Hamiltonian function.*

Gravitational potential and force

Proposition (Cohl '11& '12)

The gravitational potential function U at a point $\mathbf{x} \in \mathbb{M}^2$ due to a spatial mass distribution $\rho = \rho(t, \mathbf{x})$ is given by

$$U(t, \mathbf{x}) = \frac{1}{4\pi} \iint_{\mathbb{M}^2} \rho(t, \mathbf{y}) G(\mathbf{x}, \mathbf{y}) d\sigma(\mathbf{y}) \tag{5}$$

with an interaction potential

$$G(\mathbf{x}, \mathbf{y}) = -\frac{1}{2\pi} \log \operatorname{ctn} \frac{d(\mathbf{x}, \mathbf{y})}{2}.$$

The corresponding gravitational force is

$$F = \nabla_{\mathbf{x}} U(t, \mathbf{x}) = \frac{1}{2\pi} \int_{\mathbb{M}^2} \frac{\mathbf{y} - \sigma(\mathbf{x} \cdot \mathbf{y})\mathbf{x}}{1 - (\mathbf{x} \cdot \mathbf{y})^2} \rho(t, \mathbf{y}) d\mathbf{y}. \tag{6}$$

Remark

Noticing that $d(\mathbf{x}, \mathbf{y}) = \operatorname{csn}^{-1}(\sigma \mathbf{x} \cdot \mathbf{y})$ *and* $\operatorname{ctn} \frac{d}{2} = \operatorname{tn} d + \frac{1}{\operatorname{sn} d}$*, we get*

$$\operatorname{ctn} \frac{d(\mathbf{x}, \mathbf{y})}{2} = \sqrt{\frac{\sigma \mathbf{x} \cdot \mathbf{y} + 1}{\sigma - \mathbf{x} \cdot \mathbf{y}}}, \quad \textit{and} \quad U(t, \mathbf{x}) = \frac{1}{4\pi} \iint_{\mathbb{M}^2} \rho(t, \mathbf{y}) \log\left(\frac{\sigma \mathbf{x} \cdot \mathbf{y} + 1}{\sigma - \mathbf{x} \cdot \mathbf{y}}\right) d\mathbf{y} \tag{7}$$

Corollary

In local coordinates at a point $\mathbf{q} = \alpha \mathbf{e}_\alpha + \theta \mathbf{e}_\theta \in \mathbb{M}^2$*, the gravitational potential function* $\tilde{U} = \tilde{U}(t, \alpha, \theta) := U(t, \mathbf{x}(\mathbf{q}))$ *due to a spatial mass distribution* $\tilde{\rho} = \tilde{\rho}(t, \alpha, \theta) := \rho(t, \mathbf{x}(\mathbf{q}))$ *is given by*

$$\tilde{U}(t, \alpha, \theta) = \frac{1}{4\pi} \iint_{\mathbb{M}^2} \tilde{\rho}(t, \alpha', \theta') \log \left[\frac{\sigma \mathbf{x}(\alpha, \theta) \cdot \mathbf{y}(\alpha', \theta') + 1}{\sigma - \mathbf{x}(\alpha, \theta) \cdot \mathbf{y}(\alpha', \theta')} \right] \operatorname{sn} \alpha' \, d\alpha' d\theta', \tag{8}$$

where

$$\mathbf{x}(\alpha, \theta) \cdot \mathbf{y}(\alpha', \theta') = \operatorname{sn} \alpha \cos \theta \operatorname{sn} \alpha' \cos \theta' + \operatorname{sn} \alpha \sin \theta \operatorname{sn} \alpha' \sin \theta' + \sigma \operatorname{csn} \alpha \operatorname{csn} \alpha'.$$

Remark

From the proposition, it becomes evident that $U(x)$ *is a homogeneous function of degree 0. Euler's formula for homogeneous functions yields*

$$\mathbf{x} \cdot \nabla_{\mathbf{x}} U(t, \mathbf{x}) = 0. \tag{9}$$

The physical interpretation of (9) *is that, in an extrinsic coordinate system having the origin at the centre of* $\mathbb{M}^2$*, the gravitational force acting on each particle is always orthogonal to its position vector* $\mathbf{x}$*. Therefore we can conclude that if particles are initially located on* $\mathbb{M}^2$ *with velocities tangent to the manifold, then they will remain on* $\mathbb{M}^2$ *for all time.*

Example of a point mass

Let a mass distribution on $\mathbb{S}^2$ be given by

$$\rho(\alpha, \theta) := \frac{\delta(\alpha) \otimes \delta(\theta)}{\sin \alpha},$$

i.e. a point mass located at the north pole of the sphere, as shown in Figure 3.

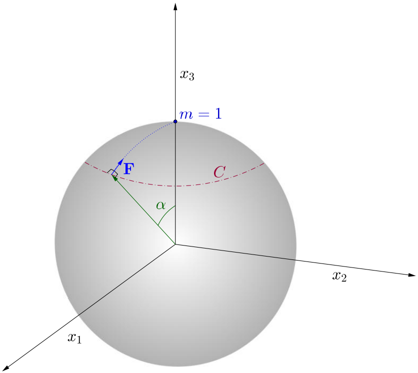

FIGURE: Illustration of a point mass located at the north pole of $\mathbb{S}^2$.

1st way: Gauss's law in two dimensions says that the gravitational flux at any point is proportional to the mass enclosed by a Gaussian curve passing through the point, i.e.

$$\int \mathbf{F} \cdot \mathbf{n} = km, \tag{10}$$

where $\mathbf{F}$ denotes the gravitational field, m is the enclosed mass, and k is a constant.
If we choose a Gaussian curve C as pictured in Figure 3, the spherical symmetry of the mass results in significant simplification of (10). Since the field generated by the mass intersects the Gaussian curve perpendicularly within $\mathbb{S}^2$ and the magnitude of the field is the same at each point on the curve, our equation reduces to

$$2\pi \sin\alpha |\mathbf{F}| = k.$$

Solving for $\mathbf{F}$ yields

$$\mathbf{F} = -\frac{k}{2\pi \sin\alpha} \mathbf{e}_\alpha. \tag{11}$$

2nd way: Use (8) to calculate the gravitational potential function and its gradient for the point mass.

$$\begin{aligned}
\tilde{U}(t,\alpha,\theta) &= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{\delta(\alpha')\delta(\theta')}{\sin\alpha'}\log\left[\frac{\mathbf{x}(\alpha,\theta)\cdot\mathbf{y}(\alpha',\theta')+1}{\mathbf{x}(\alpha,\theta)\cdot\mathbf{y}(\alpha',\theta')-1}\right]\sin\alpha' d\alpha' d\theta' \\
&= \frac{1}{4\pi}\log\left[\frac{\mathbf{x}(\alpha,\theta)\cdot\mathbf{y}(0,\theta')+1}{\mathbf{x}(\alpha,\theta)\cdot\mathbf{y}(0,\theta')-1}\right] = \frac{1}{4\pi}\log\left(\frac{\cos\alpha+1}{\cos\alpha-1}\right) \\
&= \frac{1}{4\pi}\log\left[\cot\left(\frac{\alpha}{2}\right)\right], \qquad (12)
\end{aligned}$$

then use (12) to take the gradient and obtain

$$\nabla_{\mathbf{q}}\tilde{U}(t,\alpha,\theta) = \frac{1}{4\pi}\partial_\alpha\left(\log\cot\frac{\alpha}{2}\right)\mathbf{e}_\alpha = -\frac{1}{4\pi\sin\alpha}\,\mathbf{e}_\alpha,$$

which agrees with (11) for $k = 1/2$.

VE on spaces of constant curvature

Liouville's theorem: the equation that governs the motion of a continuous particle distribution with no collisions is given (in extrinsic coordinates) by

$$\partial_t f(t,\mathbf{x},\mathbf{v}) + \mathbf{v}\cdot\nabla_{\mathbf{x}} f(t,\mathbf{x},\mathbf{v}) + [\nabla_{\mathbf{x}} U(t,\mathbf{x}) - \sigma(\mathbf{v}\cdot\mathbf{v})\mathbf{x}]\cdot\nabla_{\mathbf{v}} f(t,\mathbf{x},\mathbf{v}) = 0, \tag{13}$$

with $(\mathbf{x}\cdot\mathbf{x}) = \sigma$ and $\mathbf{x}\cdot\mathbf{v} = 0$. Then,
the *gravitational Vlasov-Poisson system in spaces of constant curvature*, or for short the *curved gravitational Vlasov-Poisson system*,

$$\begin{cases} \partial_t f + \mathbf{v}\cdot\nabla_{\mathbf{x}} f + (\nabla_{\mathbf{x}} U - \sigma(\mathbf{v}\cdot\mathbf{v})\mathbf{x})\cdot\nabla_{\mathbf{v}} f = 0, \\ U(t,\mathbf{x}) = \dfrac{1}{4\pi}\displaystyle\iint_{\mathbb{M}^2} \rho(t,\mathbf{y}) \log\left(\dfrac{\sigma\mathbf{x}\cdot\mathbf{y}+1}{\sigma-\mathbf{x}\cdot\mathbf{y}}\right) d\mathbf{y}, \\ \rho(t,\mathbf{x}) = \displaystyle\int_{T_{\mathbf{x}}(\mathbb{M}^2)} f(t,\mathbf{x},\mathbf{v})\,d\mathbf{v}. \end{cases} \tag{14}$$

In local coordinates, VP system on $\mathbb{M}^2$ takes the form

$$\begin{cases} \partial_t \tilde f + \omega_\alpha \partial_\alpha \tilde f + \omega_\theta \partial_\theta \tilde f + (\partial_\alpha \tilde U + \omega_\theta^2 \operatorname{sn}\alpha \operatorname{csn}\alpha)\partial_{\omega_\alpha} \tilde f \\ \qquad\qquad + \left(\dfrac{1}{\operatorname{sn}^2\alpha}\partial_\theta \tilde U - 2\omega_\alpha\omega_\theta \operatorname{ctn}\alpha\right)\partial_{\omega_\theta}\tilde f = 0 \\ \tilde U(t,\alpha,\theta) = \dfrac{1}{4\pi}\displaystyle\iint_{\mathbb{M}^2} \tilde\rho(t,\alpha',\theta') \log\left[\dfrac{\sigma\mathbf{x}(\alpha,\theta)\cdot\mathbf{y}(\alpha',\theta')+1}{\sigma\mathbf{x}(\alpha,\theta)\cdot\mathbf{y}(\alpha',\theta')-1}\right] \operatorname{sn}\alpha\, d\alpha' d\theta \\ \tilde\rho(t,\alpha,\theta) = \displaystyle\int_{T_{\mathbf{x}}(\mathbb{M}^2)} \tilde f(t,\alpha,\theta) \operatorname{sn}\alpha\, d\omega_\alpha d\omega_\theta. \end{cases} \tag{15}$$

VE on spaces of constant curvature Cnt.

Geometric mechanic approach: Marsden-Weinstein '85
Define the Hamiltonian $H(q,p) = \frac{1}{2}|p|^2 + U(q) : T^*\mathbb{M}^2 \to \mathbb{R}$. Then

Conservation of a distribution function $f(t,q,p)$ along the phase-space trajectories of H i.e. $\frac{d}{dt} f = 0$.

Using the canonical bracket $\{\cdot,\cdot\}$, it can be rewritten as

$$\partial_t f = \{f, H\}.$$

We obtain VP if moreover the potential U solves the Poisson's equation

$$-\Delta_{\mathbb{M}^2} U = \rho(q) := \int_{T_p\mathbb{M}^2} f(t,q,p)\, dp.$$

Mean field approximation

Suppose we have N particles on $\mathbb{M}^2 \hookrightarrow \mathbb{R}^3$ (or $\mathbb{R}^{2,1}$) satisfying

$$\ddot{\mathbf{x}}_i = -c(N)\sum_j \nabla_{\mathbb{M}^2} G(\mathbf{x}_i, \mathbf{x}_j) + \sigma |\mathbf{v}|^2 \mathbf{x}_i.$$

Introduce the empirical measure:

$$\mu^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{(x_i, v_i)},$$

Proposition

If $\nabla_{\mathbb{M}^2} G$ is <u>uniformly continuous</u>, $\mu^N_{|t=0} \rightharpoonup \mu_0$ and $c(N)N \to \gamma > 0$. Then $\mu^N \rightharpoonup \mu^t$ solving

$$\frac{\partial \mu}{\partial t} + v \cdot \nabla_{\mathbb{M}^2}\mu + \left(F(t,x) - |v|^2 x\right)\cdot \nabla_v \mu = 0, \quad F = -\gamma \int_{T_y\mathbb{M}^2} \nabla_{\mathbb{M}^2} G(x,y)\mu(dvdy)$$

If $\mu(dvdy) = f(t,x,v)dvd\sigma(y)$, then we recover VP ($\gamma = 1$) on $\mathbb{M}^d$

$$\begin{cases} \dfrac{\partial f}{\partial t} + v\cdot \nabla_{\mathbb{M}^2} f + \left(F(t,x) - |v|^2 x\right)\cdot \nabla_v f = 0 \\ F(t,x) = -\displaystyle\int_{\mathbb{M}^2} \nabla_{\mathbb{M}^2} G(x,y)\rho(t,y)d\sigma(y) \\ \rho(t,x) = \displaystyle\int_V f(t,x,v)dv \end{cases} \tag{16}$$

Initial data along a geodesic

A great circle is the hyperbola obtained by intersecting the upper sheet of the hyperboloid of two sheets with a plane through the origin of the extrinsic coordinate system.
WLG, we can choose the geodesic $\mathcal{G}$

$$G := \begin{cases} \{(x_1, x_2, x_3) \,|\, x_1^2 + x_2^2 = 1,\, x_3 = 0\}\,, & \text{for } \mathbb{M}^2 = \mathbb{S}^2 \\ \{(x_1, x_2, x_3) \,|\, x_2^2 - x_3^2 = -1,\, x_1 = 0\}\,, & \text{for } \mathbb{M}^2 = \mathbb{H}^2, \end{cases}$$

Consider a configuration of particles satisfying:

1. the position of each particle is always in the set $\mathcal{G}$ where
2. the velocity of each particle is always in the $\mathbf{e}_\theta$ (resp. $\mathbf{e}_\alpha$)-direction for $\mathbb{S}^2$ (resp. $\mathbb{M}^2$),

Since the gravitational force on each particle is directed along the geodesic, if the particles are initially aligned on $\mathcal{G}$ with initial velocities along that geodesic, then they remain on $\mathcal{G}$ for all time. With these conditions, the phase space distribution takes the form

$$f(t, \mathbf{q}, \mathbf{p}) = \begin{cases} \dfrac{\tilde{f}(t, \theta, \omega_\theta)}{\sin^2 \alpha} \delta\left(\alpha - \dfrac{\pi}{2}\right) \otimes \delta\left(\omega_\alpha\right) \\ \dfrac{\tilde{f}(t, \theta, \omega_\theta)}{\sinh^2 \alpha} \delta\left(\theta - \dfrac{\pi}{2}\right) \otimes \delta\left(\omega_\alpha\right). \end{cases}$$

Then

$$\rho(t, \alpha, \theta) = \begin{cases} \delta_{(\alpha - \frac{\pi}{2})} \otimes \frac{\tilde{\rho}(t,\theta)}{\sin^2 \alpha} \\ \delta_{(\alpha - \frac{\pi}{2})} \otimes \frac{\tilde{\rho}(t,\theta)}{\sinh^2 \theta} \end{cases}$$

... and calculating the corresponding potential

$$U(t,\alpha,\theta) = \begin{cases} \frac{1}{4\pi}\int_{-\pi}^{\pi} \tilde{\rho}(t,\theta')\log\left[\frac{1+\sin\alpha\cos(\theta-\theta')}{1-\sin\alpha\cos(\theta-\theta')}\right] d\theta' \\ \\ \frac{1}{4\pi}\int_{-\pi}^{\pi} \tilde{\rho}(t,\theta')\log\left[\frac{1+\cosh\theta\cosh\theta'-\sinh\theta\sinh(\theta')\cos(\alpha-\alpha')}{-1-\sinh\theta\sinh\theta'\cos(\alpha-\alpha')+\cosh\theta\cosh(\theta')}\right] d\theta'. \end{cases}$$

... and after calculating the force, the gravitational Vlasov-Poisson system in spaces of constant curvature from (15) becomes

$$\begin{cases} \partial_t \tilde{f} + \omega\partial_\theta \tilde{f} + \partial_\theta \tilde{U}\,\partial_\omega \tilde{f} = 0, \\ \tilde{U}(t,\theta) = \dfrac{1}{2\pi}\displaystyle\int_I \rho(t,\theta')\log\left|\mathrm{ctn}\left(\frac{\theta-\theta'}{2}\right)\right| d\theta', \\ \tilde{\rho}(t,\theta) = \displaystyle\int_{\mathbb{R}} \tilde{f}(t,\theta,\omega)d\omega, \end{cases} \tag{17}$$

which can be written in the form

$$\begin{cases} \frac{\partial \tilde{f}}{\partial t} + \omega\frac{\partial \tilde{f}}{\partial \theta} + (W * \frac{\partial}{\partial_\theta}\rho)\frac{\partial \tilde{f}}{\partial \omega} = 0 \\ \rho(t,\theta) = \int_{\mathbb{R}} \tilde{f}(t,\theta,\omega)d\omega. \\ W(\theta) = \frac{1}{2\pi}\log\left|\mathrm{ctn}\left(\frac{\theta}{2}\right)\right| \end{cases} \tag{18}$$

Remark

- *Conserved quantities: total mass, mechanical energy and Casimirs.*
- *The reduced system* (18) *is more singular than the classical 1D VP system where the* $F = \rho * B$, *and* $B(\theta) = \frac{1}{2} - \frac{\theta}{2\pi}$.
- *The system is different from the HMF model.*
- *The mathematical challenges are similar to those for VDPE, also called gyrokinetic model, (important in analysis of plasma fusion) derived from VE by Ghendrih, Hauray and Nouri '09 (also derived from NLS by Grenier '95, Jin-Levermore-McLaughlin '99).*

Well-posedness

For all $a \in \mathbb{N}$, and $s \geq 1$, define $D^{a,s}$ on the cone of polynomials P with positive coefficients as

$$D^{a,s}(\lambda^n) = \lambda^{n-a} \left(\frac{n!}{(n-a)!} \right)^s, \quad n \in \mathbb{N}.$$

Gevrey class G^s

$$G^s := \left\{ f : \exists M_f > 0 \quad \text{such that} \quad \forall k \in \mathbb{N}, \quad |\partial_x^k g| \leq M_f^k (k!)^s \right\}$$

For any function $f(t,x,v)$ and positive real numbers (λ_0, K), define

$$f_{k,l} = \partial_x^k \partial_v^l f, \quad |f(t)|_\lambda := \sum_{k+l\geq 0} \frac{\lambda^{k+l}}{(k!l!)^s} |f_{k,l}|_{L^\infty_{x,v}}, \quad |f(t)|_{\lambda,a} := D^{a,s}|f(t)|_\lambda,$$

$$\lambda(t) = \lambda_0 - (1+K)t, \quad H_f(t) := \sum_{a\geq 0} \frac{|f|_{\lambda(t),a}(t)}{(a!)^{2s}}, \quad \tilde{H}_f(t) := \sum_{a\geq 0} a^{2s}(a+1)^{s-1} \frac{|f|_{\lambda(t),a}(t)}{(a!)^{2s}}.$$

Note that all the norms above can be applied to functions depending on just one variable x or v by setting $k=0$ or $l=0$.

Well-posedness

Let $f_i(x,v)$ be the initial condition of (18) and satisfies

$$h_0 := H_{f_i}(0) < \infty. \tag{19}$$

For all $M > h_0 > 0$, define $\tilde{M} = \frac{1}{18^s C_w} \ln \frac{M}{h_0}$, where $C_w = \|W\|_{L^1}$. For all $T > 0$, define the space $\mathcal{H}_{T,M}$ by

$$\mathcal{H}_{T,M} = \{f(t,x,v) : \sup_{t\in[0,T]} H_f(t) \leq M, \quad \& \quad \int_0^T \tilde{H}_f(t)dt \leq \tilde{M}\},$$

endowed with the norm

$$\|f\|_T = \sup_{t\in[0,T]} H_f(t) + \int_0^T \tilde{H}_f(t)dt.$$

Now we can state the local existence result.

Theorem

Let $s \geq 1$, $f_i \in G^s$ be the IC of (18) satisfying (19), then there exist time $T > 0$ and a constant $M > h_0$ such that (18) has a unique solution f on $[0,T]$ and $f \in \mathcal{H}_{T,M} \subset L^\infty(G^s)$.

Linear stability

Consider

$$\begin{cases} \dfrac{\partial h}{\partial t} + \omega \dfrac{\partial h}{\partial \theta} + F(t,\theta)\dfrac{\partial f^0}{\partial \omega} = 0 \\ F = W * \dfrac{\partial}{\partial \theta}\rho, \quad W(\theta) = \dfrac{1}{2\pi} \log |\mathrm{ctn}\,(\dfrac{\theta}{2})| \\ \rho = \displaystyle\int_{\mathbb{R}} h(t,\theta,\omega) d\omega. \end{cases} \tag{20}$$

Theorem

For (20) on $\mathbb{S}^2$, if $f^0(\omega)$ and $h_0(\theta,\omega)$ are both analytic functions, $(f^0)'(\omega) = O(\frac{1}{|\omega|})$ for large $|\omega|$, and under Penrose stability condition,

$$\forall \omega \in \mathbb{R}, \quad (f^0)'(\omega) = 0 \Longrightarrow p.v. \int_{-\infty}^{\infty} \frac{(f^0)'(v)}{v-\omega} dv > -1, \tag{21}$$

(20) is linearly stable i.e. there are positive constants δ and C st.

$$\|\rho(t,\theta) - \int_{\mathbb{R}} \int_0^{2\pi} h_0(\theta,\omega) d\theta d\omega\|_{C^r(\mathbb{T})} \leq Ce^{-\delta t}, \quad \text{and} \quad \|F(t,\theta)\|_{C^r(\mathbb{T})} \leq Ce^{-\delta t},$$

where $\|u\|_{C^r(\mathbb{T})} = \max_{0\leq n \leq r, \alpha \in \mathbb{T}} |\partial_\theta^n u(\theta)|$ and $r \in \mathbb{N}^+$.

Theorem

For (20) on $\mathbb{H}^2$, if $f^0(\omega)$ and $h_0(\alpha,\omega)$ are both analytic functions, , $(f^0)'(\omega) = O(\frac{1}{|\omega|})$ for large $|\omega|$, and under Penrose stability condition,

$$\forall \omega \in \mathbb{R}, \quad (f^0)'(\omega) = 0 \Longrightarrow \left(p.v. \int_{-\infty}^{\infty} \frac{(f^0)'(v)}{v-\omega} dv \right) > -\frac{4}{\pi}, \tag{22}$$

Then the stationary solution is stable and there exist constants $\lambda', C > 0$, dependent on the initial data, such that for large values of t we have

$$\|\rho(t,\cdot)\|_{\mathcal{F}^{\lambda'}} \leq \frac{2C}{\lambda' t} \quad \text{and} \quad \|F(t,\cdot)\|_{\mathcal{F}^{\lambda'}} \leq \frac{C}{\lambda' t}.$$

Recall that

$$\|f\|_{\mathcal{F}^{\lambda}} := \int_{\mathbb{R}} e^{\lambda|\xi|} |\hat{f}(\xi)| d\xi.$$

Remark

- *The algebraic decay rate in the hyperbolic case is due to the low frequencies: Glassey-Schaeffer for VP on real line*
- *VDPE: similar results were obtained by Jabin & Nouri '11 for the LWP and Bardos & Nouri' 13 for the linear stability*
- *VP ...Degond'86 and for Nonlinear Landau Damping: Villani-Mouhot '10, Bedrossian-Masmoudi-Mouhot '14 ...*
- *Remark about illposedness*

Proof of Theorem 1

Fixed point: Given a density $\rho \in \mathcal{H}_{T,M}$, let g solve

$$\begin{cases} \partial_t g + v\partial_x g + (W * \partial_x \rho)(\partial_v g + \gamma g) = 0, \\ g(0,x,v) = g_i(x,v). \end{cases} \tag{23}$$

with $\gamma := \alpha'/\alpha$ and $\alpha(v) : \mathbb{R} \to (0,\infty)$ satisfying

$$\int \alpha dv \leq 1, \quad \text{and} \quad \tilde{\gamma} := \sum_{a\geq 0} \frac{a^{2s}}{(a!)^{2s}} \sum_{k\geq a} \frac{\lambda_0^{k-a}}{((k-a)!)^s} |\partial_v^k \gamma|_{L_v^\infty} < 1.$$

Then, define a map Φ by setting $\sigma = \phi(\rho) := \int \alpha g dv$.

ESTIMATES:

- LEMMA
  For a given $M > 0$ and $\lambda_0 > 0$, take $\gamma(v)$ as above, and set $\tilde{M} = \frac{1}{18^s C_w + 2C_\gamma} \ln \frac{M}{h_0}$. There exists K such that for any ρ satisfying

$$\sup_{0 \le t \le T} H_\rho(t) \le M \quad \text{and} \quad \int_t^T \tilde{H}_\rho(t) dt \le \tilde{M},$$

  the solution to (24) is in $g \in \mathcal{H}_{T,M}$ and enjoys

$$\partial_t H_g \le (\lambda_0 - K + CC_\gamma C_w H_\rho)\tilde{H}_g + CC_\gamma C_w H_g \tilde{H}_\rho.$$

- LEMMA
  There exist K, T and M ($\lambda(t) = \lambda_0 - (1 + K)t$) such that for any $\sigma \in \mathcal{H}_{T,M}$, the macroscopic density ρ associated to the solution to (23) satisfies

$$g \in \mathcal{H}_{T,M}.$$

Very sketchy proof of Lemma 1

Observations:

- Using Leibnitz rule in Gevrey spase, we make a suitable choice of the weight function γ so that
$$H_{\gamma f} \leq C_\gamma H_f \quad \text{and} \quad \tilde{H}_{\gamma f} \leq C_\gamma \tilde{H}_f.$$
- Differentiating the equation gives
$$\begin{aligned} & \partial_t(g_{k,l}) + v\partial_x(g_{k,l}) + (W * \partial_x\rho)\partial_v(g_{k,l}) \\ = \quad & -lg_{k+1,l-1} + \sum_{m=0}^{k-1} C_k^m (W * \partial_x^{k-m+1}\rho)(g_{m,l+1}) + \sum_{m=0}^{k} C_k^m (W * \partial_x^{k-m+1}\rho)(\gamma g)_{m,l}. \end{aligned} \tag{24}$$

- Lemma

Given σ, for any solution of (23) in $\mathcal{H}_{T,M}$, it holds that

$$\partial_t |g|_{\lambda,a} \lesssim_s a^s |g|_{\lambda,a} + \lambda |g|_{\lambda,a+1} + C_W D_\lambda^a (\|\sigma\|_{\lambda,1} |g|_{\lambda,1} + \|\sigma\|_{\lambda,1} |\gamma|_\lambda |g|_\lambda).$$

Proof of Theorem 2

Observations:

- $\frac{d}{dt}\hat{\rho}(t,0) = 0$
- Taking Fourier transform both in θ, we have (knowing that $\widehat{W} \sim \frac{1}{|k|}$)

$$\hat{\rho}(t,k) \quad = \quad \tilde{h}_0(k,kt) + \int_0^t K(t-\tau)\hat{\rho}(\tau,k)dt, \tag{25}$$

where

$$K(t,k) = 0 \quad k \text{ is even, and} \quad K(t,k) = |k|t\hat{f}^0(kt) \quad \text{if} \quad k \text{ is odd.}$$

- Lemma

Assume that ϕ is a solution of the Volterra equation

$$\phi(t) = a(t) + \int_0^t K(t-\tau)\phi(\tau)d\tau,$$

where a, the kernel K and its Laplace transform $K^L(\xi)$ satisfy

(I) $|K(t)| \leq C_0 e^{-\lambda_0 t}$;
(II) $|K^L(\xi) - 1| \geq \kappa > 0$ *for* $0 \leq Re\xi \leq \Lambda$;
(III) $|a(t)| \leq \alpha e^{-\lambda t}$.

Then for any $\lambda' \leq \min(\lambda, \lambda_0, \Lambda)$, $C = \alpha + \frac{C_0\alpha}{2\sqrt{(\lambda_0-\lambda')(\lambda-\lambda')}}$, we have

$$|\phi(t)| \leq Ce^{-\lambda' t}.$$

- The Laplace transform $K^L(\xi)$ of K in time at $\xi = (\lambda - i\omega)k$:

$$K^L(\xi) \quad = \quad \frac{-1}{|k|}\int_{\mathbb{R}} \frac{(f^0)'(v)}{i\lambda + (v-\omega)}dv \to \frac{1}{k}\left(p.v\int_{-\infty}^{\infty} \frac{(f^0)'(v)}{v-\omega}dv - i\pi(f^0)'(\omega)\right)$$

Proof of Theorem 2

Observations:

- In Fourier transform, we have (knowing that $\widehat{W(\xi)} \sim \frac{\tanh(\frac{\xi\pi}{2})}{2\xi}$)

$$\hat{\rho}(t,\xi) \quad = \quad \tilde{h}_0(k,\xi t) + \int_0^t K(t-\tau)\hat{\rho}(\tau,\xi)dt, \tag{26}$$

where

$$K(t,\xi) = -\xi t/2 \tanh(\pi\xi/2)\hat{f}^0(\xi t).$$

From previous Lemma, we have

$$|\hat{\rho}(t,\xi)| \leq Ce^{-\lambda'|\xi|t},$$

and

$$\begin{aligned} \widehat{F}(t,\xi) &= i\xi\widehat{W}\hat{\rho} \\ &= \frac{i}{2}\tanh(\frac{\xi\pi}{2})\hat{\rho}. \end{aligned}$$

- Computing the Laplace transform $K^L(\zeta,\xi)$ at $\zeta = (\lambda - i\omega)\xi$, we ge

$$K^L(\zeta,\xi) = -\frac{1}{2\xi}\tanh(\frac{\xi\pi}{2})\int_{\mathbb{R}} \frac{(f^0)'(v)}{i\lambda + (v-\omega)}dv.$$

and use $\frac{2\xi}{\tanh(\frac{\xi\pi}{2})} \geq \frac{4}{\pi}$.

Thank you

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