A uniqueness result for the Vlasov-Poisson system

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The Vlasov-Poisson system

Aim of the talk: study uniqueness issues for the Vlasov-Poisson system in dimension $n = 2$ or $n = 3$

$$\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0 \quad \text{on} \quad \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n,$$

where

$$f(t,x,v) \geq 0, \quad (x,v) \in \mathbb{R}^n \times \mathbb{R}^n \quad \text{density of particles or stars,}$$

$$\rho(t,x) = \int f(t,x,v)\, dv \quad \text{macroscopic density,}$$

$$E(t,x) = \gamma \int \frac{x-y}{|x-y|^n} \rho(t,y)\, dy \quad \text{force field,} \quad \gamma = \pm 1.$$

Hence $\quad E = \gamma \nabla \phi, \quad \Delta \phi = c(n)\rho.$

Previous existence results

- Arsenev 75: Global existence of weak solutions $f \in L^\infty(L^1 \cap L^\infty)$ with finite energy.
- DiPerna & Lions 88: Global existence of renormalized solutions $f \in L^\infty(L^1)$ with finite entropy and energy.

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- Lions & Perthame 91, Pallard 11: Global existence of weak solutions $f \in L^\infty(L^1 \cap L^\infty)$ with finite moments:

$$\text{if} \quad \iint |v|^m f(0)\, dx\, dv < +\infty \quad \text{for some } m > 3$$

$$\text{then} \quad \forall T > 0, \quad \sup_{t\in[0,T]} \iint |v|^m f(t)\, dx\, dv < +\infty.$$

Moreover if $m > 6$ then
  - $E \in L^\infty([0,T], L^\infty)$,
  - the solution is **unique** under an additionnal assumption on $f(0)$ (see later).

Previous uniqueness results

- Bardos & Degond 85: Global existence and uniqueness of classical solutions with small initial data.
- Pfaffelmoser 92, Schaeffer 91: Global existence and uniqueness of classical solutions with **compact velocity support**.

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- Loeper 06: Uniqueness among weak solutions $f \in C(\mathcal{M}_+ - w^*)$ with **bounded macroscopic density**

$$\forall T > 0, \qquad \sup_{t\in[0,T]} \|\rho(t)\|_{L^\infty} < +\infty.$$

In fact $\rho \in L^\infty([0,T],L^\infty)$ implies

- $E \in L^\infty([0,T],L^\infty)$ so the weak formulation makes sense if f is only assumed to be a measure,
- E is almost-Lipschitz:

$$|E(t,x)-E(t,y)| \leq \left(\|\rho(t)\|_{L^1} + \|\rho(t)\|_{L^\infty}\right)|x-y|(1+|\ln|x-y||).$$

Main result

Theorem 1

Let $f_0 \in \mathcal{M}_+(\mathbb{R}^n \times \mathbb{R}^n)$ be a nonnegative bounded measure. For all $T > 0$, there exists at most one weak solution $f \in C([0,T], \mathcal{M}_+(\mathbb{R}^n \times \mathbb{R}^n) - w^*)$ of the Vlasov-Poisson system on $[0,T]$ with $f(0) = f_0$ such that

$$\sup_{[0,T]} \sup_{p \geq 1} \frac{\|\rho(t)\|_{L^p}}{p} < +\infty.$$

Remark

Since $\rho \in L^\infty([0,T], L^1 \cap L^p)$ for $p > n$ we have $E \in L^\infty([0,T], L^\infty)$ so the weak formulation makes sense.

Monokinetic densities

Theorem 1 applies to measure solutions of the form

$$f(t,x,v) = \rho(t,x)\delta_{v=u(t,x)},$$

with ρ satisfying the uniqueness criterion and (ρ, u) a solution of the **Euler-Poisson system**

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0 \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \rho E = 0 \\ E = \gamma \nabla \Delta^{-1} \rho. \end{cases}$$

An example for $n = 2$

Consider the interaction of a **bounded density** $f(t)$ of light particles with a **heavy particle** of opposite charge $\xi(t)$.

Caprino, Marchioro, M & Pulvirenti 12: There exists a global solution such that

$$\forall t \geq 0, \quad \rho(t,x) \leq C(1 + \ln_{-} |x - \xi(t)|).$$

Then ρ satisfies the uniqueness condition.

Link with the Euler equation

Consider the two-dimensional incompressible Euler equations

$$\partial_t \omega + u \cdot \nabla \omega = 0 \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^2,$$
$$u(t,x) = \frac{1}{2\pi} \int \frac{(x-y)^\perp}{|x-y|^2} \omega(t,y)\, dy,$$

where $\omega : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ is the vorticity.

Yudovich 63: Global existence and uniqueness if $\omega_0 \in L^1 \cap L^\infty \rightsquigarrow \omega \in L^\infty(L^1 \cap L^\infty)$.

Yudovich 95: extension to the case $\|\omega_0\|_{L^p} \leq C \ln p, \quad \forall p > 1 \rightsquigarrow$ the same bound holds for $\|\omega(t)\|_{L^p} = \|\omega(0)\|_{L^p}$.
Allows for initial vorticities like $|\omega_0(x)| \leq C \ln|\ln x|$.

How to guarantee the uniqueness criterion ?

- $\|\rho(0)\|_{L^p} \leq Cp$ does not imply that $\|\rho(t)\|_{L^p} \leq C_1 p$ for $t > 0$.
  $\rightsquigarrow$ we need more information on $f(0)$.

- In the context of Lions & Perthame's solutions with finite moments of order $m > 6$, the **boundedness of** $\rho(t)$ for $t > 0$ is ensured by the condition

$$\forall R > 0, \quad \forall T > 0,$$

$$\sup_{x\in\mathbb{R}^n} \sup_{t\in[0,T]} \int \sup_{|y-x|\leq RT^2, |w-v|\leq RT} f(0, y - vt, w)\, dv < +\infty.$$

True in particular if $f(0)$ has compact velocity support.

Theorem 2

Let $f_0 \in L^1 \cap L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ be nonnegative and such that

$$\iint_{\mathbb{R}^n \times \mathbb{R}^n} |v|^m f_0(x, v)\, dx\, dv < +\infty$$

for some $m > 2$ if $n = 2$, and $m > 6$ if $n = 3$.

Let $T > 0$ and let f be a corresponding weak solution provided by Lions & Perthame's result. If f_0 satisfies moreover

$$\forall k \geq 1, \quad \iint_{\mathbb{R}^n \times \mathbb{R}^n} |v|^k f_0(x, v)\, dx\, dv \leq (C_0 k)^{\frac{k}{n}},$$

for some constant C_0, then f satisfies the uniqueness condition of Theorem 1.

A few examples of such initial data

- $f_0 \in L^1 \cap L^\infty$ with compact velocity support.
- For any $h_0 \in L^1 \cap L^\infty \cap L^\infty_v(L^1_x)$, for any $p \geq 0$,

$$f_0(x,v) = e^{-|v|^n} |v|^p h_0(x,v).$$

Then $\rho_0 \in L^\infty$.

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- There exists f_0 satisfying the assumptions of Theorem 2 and such that

$$\rho_0(x) = \omega_n \ln_- |x|, \quad \text{thus } \rho_0 \notin L^\infty.$$

Perturbations of stationary states

Let $n = 2$ and consider the **gravitational case** $\gamma = -1$:

$$E(t,x) = -\nabla\Phi(t,x) = -\int \frac{x-y}{|x-y|^2}\rho(t,y)\,dy,$$

$$\Phi(t,x) = \int \ln|x-y|\rho(t,y)\,dy.$$

Dolbeault-Fernandez-Sanchez 06: There exists a steady state $\overline{f} \in L^1(\mathbb{R}^2 \times \mathbb{R}^2)$, with finite energy, of the form

$$\overline{f}(x,v) = \varphi\left(\frac{|v|^2}{2} + \Phi(x)\right),$$

where φ is nonincreasing, unbounded, and $\mathrm{supp}(\varphi) \subset]-\infty, M]$. Moreover $\overline{\rho} = \int \overline{f}\,dv$ is radially symmetric, compactly supported in $B(0,1)$, bounded, and

$$\Phi(x) = \ln|x| \int_{|y|\le|x|} \overline{\rho}(|y|)\,dy + \int_{|y|>|x|} \ln|y|\rho(|y|)\,dy.$$

Perturbations of steady states (2)

- Let $K > 0$ and $h_0 \in L^1 \cap L^\infty$. Let $\overline{f}$ be of the previous form with the single asssumption that $\overline{\rho}$ is radially symmetric and compactly supported. Then

$$f_0 = \overline{f}\chi_{\{\overline{f} \leq K\}} h_0$$

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- Same conclusion for initial data like

$$f_0(x, v) = \varphi\left(|v|^2 + \Phi(x) + a(x, v)\right),$$

with $a \geq 0$, $\rho_0 = \int f_0 \, dv$ has compact support in $B \subset \mathbb{R}^n$, and with

$$\forall p \geq 1, \quad \int_B (M - \Phi(x))_+^p \, dx \leq (C_0 p)^{\frac{2p}{n}},$$

for some constant C_0.

Proof of Theorem 1: uniqueness if $\|\rho(t)\|_{L^p} \leq Cp$

Lagrangian formulation

- If $\rho \in L^\infty([0,T], L^p)$ for $p > n$ then $E \in L^\infty([0,T], L^\infty)$ and $\nabla E \in L^\infty([0,T], L^p)$.
- DiPerna & Lions theory $\rightsquigarrow$ there exists a unique Lagrangian flow $\Psi = (X, V)$ i.e. for a.e. $(x,v) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$\begin{cases} \dot{X}(t,x,v) = V(t,x,v), & X(0,x,v) = x \\ \dot{V}(t,x,v) = E(t, X(t,x,v)), & V(0,x,v) = v. \end{cases}$$

and moreover

$$\forall t \in [0,T], \quad f(t) = \Psi(t)_\# f_0.$$

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Sobolev embedding

- Calderón-Zygmund inequality $\rightsquigarrow$ $\|\nabla E\|_{L^p} \leq Cp\|\rho\|_{L^p}$.
- Sobolev embedding $\rightsquigarrow$ $|E(x) - E(y)| \leq Cp\|\rho\|_{L^p}|x-y|^{1-n/p}$.
- We use a finer version:

$$\int \left| \frac{x-z}{|x-z|^n} - \frac{y-z}{|y-z|^n} \right| \rho(z)\, dz \leq C\, p\, (\|\rho\|_{L^1} + \|\rho\|_{L^p})|x-y|^{1-n/p}.$$

If $\rho \in L^\infty$,

$$\int \left| \frac{x-z}{|x-z|^n} - \frac{y-z}{|y-z|^n} \right| \rho(z)\, dz$$
$$\leq C(\|\rho\|_{L^1} + \|\rho\|_{L^\infty})|x-y|(1 + \ln_- |x-y|).$$

Proof of Theorem 1: uniqueness if $\|\rho(t)\|_{L^p} \leq Cp$

A distance functional

Given two solutions ρ_1 and ρ_2, introduce

$$\mathcal{D}(t) = \iint |X_1(t,x,v) - X_2(t,x,v)| f_0(x,v)\, dx\, dv.$$

Proposition

For $t \in [0,T]$, we have

$$\mathcal{D}(t) \leq Cp\left(1 + \|\rho_1\|_{L^\infty(L^p)} + \|\rho_2\|_{L^\infty(L^p)}\right) \int_0^t \int_0^s \mathcal{D}(\tau)^{1-n/p}\, d\tau\, ds.$$

Comparison with other settings

- For 2D Euler, setting $\mathcal{D}(t) = \int |X_1(t,x) - X_2(t,x)||\omega_0(x)|\,dx$, where $\dot{X}_i(t,x) = u_i(t, X_i(t,x))$ one gets

$$\mathcal{D}(t) \leq Cp\left(1 + \|\omega_0\|_{L^\infty(L^p)}\right)\int_0^t \mathcal{D}(s)^{1-n/p}\,ds.$$

- For Vlasov-Poisson, Loeper considers the W_2-like distance:

$$\mathcal{D}(t) = \left(\iint (|X_1(t) - X_2(t)|^2 + |V_1(t) - V_2(t)|^2) f_0(x,v)dxdv\right)^{1/2}$$
$$\geq \left(\inf_{T_\# \rho_1 = \rho_2} \iint |T(x) - x|^2 \rho_1(x)\,dx\right)^{1/2} = W_2(\rho_1, \rho_2).$$

Optimal transportation methods yield the estimate

$$\mathcal{D}(t) \leq C(1 + \|\rho_1\|_{L^\infty(L^\infty)} + \|\rho_2\|_{L^\infty(L^\infty)})\int_0^t \mathcal{D}(s)|\ln \mathcal{D}(s)|\,ds.$$

Proof of Theorem 1 with the Proposition

Set

$$\mathcal{F}(t) = \int_0^t \int_0^s \mathcal{D}(\tau)^{1-n/p}\, d\tau\, ds$$

so that

$$\mathcal{F}'' \leq Cp^2 \mathcal{F}^{1-n/p}$$

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Letting $p \to \infty$ we obtain $\mathcal{D} = 0$ on $[0, 1/C] \rightsquigarrow \mathcal{D} = 0$ on $[0, T]$.

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Remark

The inequality $\mathcal{G}' \leq Cp^2 \mathcal{G}^{1-n/p}$ yields $\mathcal{G} \leq (Cpt)^p$.

Proof of the Proposition

Since $\ddot{X}_i = E_i(t, X_i)$,

$$\mathcal{D}(t) \le \int_0^t \int_0^s \iint |E_1(\tau, X_1(\tau)) - E_2(\tau, X_2(\tau))| f_0(x, v)\, dx\, dv\, d\tau\, ds$$

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&\leq \int_0^t \int_0^s \iint |E_1(\tau, X_1(\tau)) - E_1(\tau, X_2(\tau)| f_0(x,v)\, dx\, dv\, d\tau\, ds \\
&+ \int_0^t \int_0^s \iint |E_1(\tau, X_2(\tau)) - E_2(\tau, X_2(\tau)| f_0(x,v)\, dx\, dv\, d\tau\, ds
\end{aligned}
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$$\leq \int_0^t \int_0^s \iint |E_1(\tau, X_1(\tau)) - E_1(\tau, X_2(\tau)| f_0(x, v)\, dx\, dv\, d\tau\, ds$$
$$+ \int_0^t \int_0^s \iint |E_1(\tau, X_2(\tau)) - E_2(\tau, X_2(\tau)| f_0(x, v)\, dx\, dv\, d\tau\, ds$$
$$\leq Cp(1 + \|\rho_1\|_{L^\infty(L^p)}) \int_0^t \int_0^s \mathcal{D}(\tau)^{1-n/p}\, d\tau\, ds$$
$$+ \int_0^t \int_0^s \iint |E_1(\tau, x) - E_2(\tau, x)| f_2(\tau, x, v)\, dx\, dv\, d\tau\, ds.$$

Proof of the Proposition (2)

Since $f_i(\tau) = (X_i, V_i)(\tau)_{\#} f_0$, we get:

$$\int |E_1(\tau,x) - E_2(\tau,x)| \rho_2(\tau,x)\, dx$$

$$\leq \int \left(\iint \left| \frac{x - X_1(\tau,y,w)}{|x - X_1(\tau,y,w)|^n} - \frac{x - X_2(\tau,y,w)}{|x - X_2(\tau,y,w)|^n} \right| f_0(y,w) \right) \rho_2(\tau,x)$$

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&= \iint f_0(y,w) \left(\int \left| \frac{x - X_1(\tau,y,w)}{|x - X_1(\tau,y,w)|^n} - \frac{x - X_2(\tau,y,w)}{|x - X_2(\tau,y,w)|^n} \right| \rho_2(\tau,x) \right)
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&\leq Cp(1 + \|\rho_2\|_{L^\infty(L^p)}) \\
&\qquad \times \iint |X_1(\tau,y,w) - X_2(\tau,y,w)|^{1-n/p} f_0(y,w)\, dy\, dw.
\end{aligned}
$$

Proof of Theorem 2

Set $M_k = \iint |v|^k f \, dx \, dv$. We have the interpolation inequality:

$$\|\rho\|_{L^{\frac{k+n}{n}}} \leq C \|f\|_{L^\infty}^{\frac{k}{k+n}} M_k^{\frac{n}{k+n}}.$$

Proposition

Assume that $M_k(0) \leq (C_0 k)^{k/n}$. Then there exists $C_1 > C_0$ such that

$$\sup_{t \in [0,T]} M_k(t) \leq (C_1 k)^{\frac{k+n}{k}}.$$

Proof.

Gronwall estimate on $M_k(t)$, using that

$$\frac{d}{dt} M_k(t) \leq k \int |E(t,x)| |v|^{k-1} f(t,x,v) \, dx \, dv.$$

Further extensions: Orlicz spaces for the density (with Thomas Holding)

Definition (Luxemburg norm)

Define the Luxemburg norm of a function f as

$$\|f\|_{L_\Psi} = \inf\{\lambda > 0 : \int_{\mathbb{R}^n} \Psi(|f(x)|/\lambda)\, dx < 1\}.$$

Lemme (Holding)

Let $\Psi(\tau) = \exp(\tau) - \tau - 1$. For $\rho \in L^1 \cap L_\Psi$,

$$\int_{\mathbb{R}^n} \left| \frac{x-z}{|x-z|^n} - \frac{y-z}{|y-z|^n} \right| |\rho(z)|\, dz \leq C(\|\rho\|_{L^1} + \|\rho\|_{L_\Psi})|x-y| \ln^2(|x-y|).$$

Consequence: if

$$\sup_{t\in[0,T]} \|\rho_i(t)\|_{L_\Psi} < +\infty$$

we obtain

$$\mathcal{F}''(t) \leq C\mathcal{F}(t)\ln^2 \mathcal{F}(t), \quad \forall t \in [0, T],$$

and we conclude as before that uniqueness holds.

Further extensions (in progress):

- Other Orlicz spaces.
- Stability estimate "à la Dobrushin" for the Monge-Kantorovitch distance between two solutions: control $W_1(f_1(t), f_2(t))$ in terms of $W_1(f_1(0), f_2(0))$.