ERRATUM TO: "APPROXIMATIONS OF STRONGLY CONTINUOUS FAMILIES OF UNBOUNDED SELF-ADJOINT OPERATORS"

JONATHAN BEN-ARTZI AND THOMAS HOLDING

ABSTRACT. A gap in the proof of the original article is fixed. As a result, the formulation of the main theorem is modified accordingly.

Acknowledgement: The authors are grateful to Nathanaël Berestycki and Martin Hairer for pointing out the gap in the original proof.

DATE: May 8, 2017.

1. INTRODUCTION

The original article [BAH16] dealt with finite-dimensional symmetric approximations of families of self-adjoint operators of the form

$$\mathcal{M}^\lambda = \mathcal{A}^\lambda + \mathcal{K}^\lambda = \begin{bmatrix} -\Delta + \alpha(\lambda) & 0 \\ 0 & \Delta - \alpha(\lambda) \end{bmatrix} + \begin{bmatrix} \mathcal{K}^\lambda_{++} & \mathcal{K}^\lambda_{+-} \\ \mathcal{K}^\lambda_{-+} & \mathcal{K}^\lambda_{--} \end{bmatrix}, \quad \lambda \in [0,1] \tag{1.1}$$

acting in an appropriate subspace of $L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$, where $\{\mathcal{K}^\lambda\}_{\lambda\in[0,1]}$ is a bounded, symmetric and strongly continuous family and $\alpha(\lambda) > \alpha > 1$ is continuous. The spectrum of $\mathcal{M}^\lambda$ was discretised by adding a potential, leading us to define

$$\mathcal{M}^\lambda_\varepsilon = \mathcal{A}^\lambda + \mathcal{K}^\lambda + \varepsilon \mathcal{W}^\lambda \tag{1.2}$$

which is assumed to have a compact resolvent for all $\varepsilon > 0$ (the precise details are omitted in this note). Finally, an $n \times n$ matrix $\widetilde{\mathcal{M}}^\lambda_{\varepsilon,n}$ was defined by restricting $\mathcal{M}^\lambda_\varepsilon$ to a subspace spanned by n eigenfunctions of $\mathcal{A}^\lambda + \varepsilon\mathcal{W}^\lambda$ (chosen in an appropriate way). The main result – Theorem 3 – asserted that $\widetilde{\mathcal{M}}^\lambda_{\varepsilon,n}$ recover the spectrum of $\mathcal{M}^\lambda$ in $(-1,1)$ and moreover that they converge uniformly in λ to the spectrum of $\mathcal{M}^\lambda$ on compact subsets of $(-1,1)$ as $\varepsilon \to 0$ and $n \to \infty$.

The purpose of this erratum is to correct this statement which may fail due to the possible appearance of eigenvalues entering $(-1,1)$ at the boundary (in other words, we lack upper-semicontinuity).

The possible failure of the original statement stems from a gap in the proof: while the theorem treats the convergence of spectra in the *open* interval $(-1,1)$, the crucial compactness result meant to show upper-semicontinuity (Proposition 18) deals with the *closed* interval $[-1,1]$. We solve this problem by considering (roughly speaking) a *coarser topology.* The approach of the original article was to think of the spectrum as a subset of the real line and measure distance according to the Hausdorff distance

$$d_H(X,Y) := \max\left(\sup_{y\in Y}\inf_{x\in X}|x-y|, \sup_{x\in X}\inf_{y\in Y}|x-y|\right), \qquad X, Y \subset \mathbb{R}.$$

Instead, we think of the spectrum as a *measure* (counting multiplicities) and we assess convergence in terms of *weak convergence of measures.* We recall that a sequence of finite Borel measures (on some measure space $\mathcal{X}$) μ_n is said to converge to a measure μ weakly

($\mu_n \rightharpoonup \mu$) if $\int_{\mathcal{X}} f \,\mathrm{d}\mu_n \to \int_{\mathcal{X}} f \,\mathrm{d}\mu$ for any f that is bounded and continuous. The space of finite positive Borel measures equipped with the topology of weak convergence is metrisable, for example with the bounded Lipschitz distance

$$d_{BL}(\mu,\nu) := \sup_{\|\varphi\|_{\mathrm{Lip}}\le 1, |\varphi|\le 1} \int \varphi \,\mathrm{d}(\mu-\nu).$$

2. Reformulating the main theorem

In the original article we studied continuity properties (in the sense of the Hausdorff distance) of the two set-valued maps

$$\Sigma : [0,1]\times[0,\varepsilon^*] \to (\text{closed subsets of } (-1,1), d_H)$$
$$\Sigma(\lambda,\varepsilon) = (-1,1)\cap \mathrm{sp}(\mathcal{M}_\varepsilon^\lambda)$$

and

$$\Sigma_\varepsilon : [0,1]\times\mathbb{N} \to (\text{closed subsets of } (-1,1), d_H)$$
$$\Sigma_\varepsilon(\lambda,n) = (-1,1)\cap \mathrm{sp}(\widetilde{\mathcal{M}}_{\varepsilon,n}^\lambda).$$

Instead, for $\lambda\in[0,1]$, $\varepsilon\ge 0$ and $n\in\mathbb{N}$ we define the measures (where we *always* take multiplicities into account!)

$$\nu_{\lambda,\varepsilon} = \sum_{x\in \mathrm{sp}_{\mathrm{pp}}(\mathcal{M}_\varepsilon^\lambda)\setminus \mathrm{sp}_{\mathrm{ess}}(\mathcal{M}_\varepsilon^\lambda)} \delta_x$$

and for any $\varepsilon>0$ the measures

$$\widetilde{\nu}_{\lambda,\varepsilon,n} = \sum_{x\in\mathrm{sp}(\widetilde{\mathcal{M}}_{\varepsilon,n}^\lambda)} \delta_x,$$

where δ_x is the standard Dirac delta function centred at x. Consider a cutoff function φ_η satisfying

$$\varphi_\eta(x) = \begin{cases} 1 & x\in[-1,1] \\ 0 & x\in\mathbb{R}\setminus(-1-\eta,1+\eta) \end{cases}, \quad \varphi_\eta\in C(\mathbb{R},[0,1]), \quad \eta\in(0,\alpha). \tag{$*$}$$

Finally, define the measures

$$\mu_{\lambda,\varepsilon}^\eta = \varphi_\eta \nu_{\lambda,\varepsilon}$$

and

$$\widetilde{\mu}_{\lambda,\varepsilon,n}^\eta = \varphi_\eta \widetilde{\nu}_{\lambda,\varepsilon,n}.$$

The main theorem may now be restated as:

Theorem 2.1. *The mappings* $[0,1]\times[0,\infty)\ni(\lambda,\varepsilon)\mapsto\mu_{\lambda,\varepsilon}^\eta$ *and* $[0,1]\ni\lambda\mapsto\widetilde{\mu}_{\lambda,\varepsilon,n}^\eta$ *(here* $\varepsilon>0$*) are weakly continuous and as* $n\to\infty$*,* $d_{BL}(\widetilde{\mu}_{\lambda,\varepsilon,n}^\eta,\mu_{\lambda,\varepsilon}^\eta)\to 0$ *uniformly in* $\lambda\in[0,1]$*.*

Remark 2.2. Note that the above statement does not depend on the particular choice of cutoff function φ_η, as long as the requirements in $(*)$ are satisfied.

Remark 2.3. From the results of the original paper we know that the following hold:

- *Upper-semicontinuity:* If $(\lambda_m,\varepsilon_m)\to(\lambda_\infty,\varepsilon_\infty)$, $[-1-\alpha,1+\alpha]\ni\sigma_m\to\sigma_\infty$ and $\mathcal{M}_{\varepsilon_m}^{\lambda_m}u_m = \sigma_m u_m$ where $\|u_m\|=1$ then u_m has a subsequence converging strongly to some $u_\infty\neq 0$ and $\mathcal{M}_{\varepsilon_\infty}^{\lambda_\infty}u_\infty = \sigma_\infty u_\infty$. That is, we have upper-semicontinuity of the spectrum on the closed interval $[-1-\alpha,1+\alpha]$: eigenvalues of $\mathcal{M}_{\varepsilon_m}^{\lambda_m}$ converge to eigenvalues of $\mathcal{M}_{\varepsilon_\infty}^{\lambda_\infty}$.

- *Lower-semicontinuity:* The spectrum is lower-semicontinuous under strong resolvent perturbations. This implies that near each eigenvalue of $\mathcal{M}^{\lambda_\infty}_{\varepsilon_\infty}$ there is an eigenvalue of $\mathcal{M}^{\lambda_m}_{\varepsilon_m}$.

Proof. We split the proof into three parts, denoted **I, II, III**.

I. Claim: along any sequence $(\lambda_m, \varepsilon_m) \to (\lambda_\infty, \varepsilon_\infty)$ **it holds that** $\mu^\eta_{\lambda_m,\varepsilon_m} \rightharpoonup \mu^\eta_{\lambda_\infty,\varepsilon_\infty}$ **as** $m \to \infty$**.** Indeed, we have to show that for any bounded continuous function f it holds that, as $m \to \infty$,

$$\int f \, \mathrm{d}\mu^\eta_{\lambda_m,\varepsilon_m} = \sum_{y \in \mathrm{sp}(\mathcal{M}^{\lambda_m}_{\varepsilon_m})} \varphi_\eta(y) f(y) \to \sum_{y \in \mathrm{sp}(\mathcal{M}^{\lambda_\infty}_{\varepsilon_\infty})} \varphi_\eta(y) f(y) = \int f \, \mathrm{d}\mu^\eta_{\lambda_\infty,\varepsilon_\infty} \tag{2.1}$$

where (as before) multiplicity is taken into account in the summations. Without loss of generality we assume that $f \geq 0$. We know that the spectrum of $\mathcal{M}^{\lambda_\infty}_{\varepsilon_\infty}$ inside the support of φ_η is discrete, consisting of a finite number of eigenvalues, each of finite multiplicity. Let them be $\sigma_1, \dots, \sigma_M$ of respective multiplicities $N_1, \dots, N_M$. We split the proof of (2.1) into two steps.

I1. Claim: $\liminf_m \sum_{y \in \mathrm{sp}(\mathcal{M}^{\lambda_m}_{\varepsilon_m})} \varphi_\eta(y) f(y) \geq \sum_{y \in \mathrm{sp}(\mathcal{M}^{\lambda_\infty}_{\varepsilon_\infty})} \varphi_\eta(y) f(y)$**.**

By the strong resolvent convergence of $\mathcal{M}^{\lambda_m}_{\varepsilon_m}$ to $\mathcal{M}^{\lambda_\infty}_{\varepsilon_\infty}$ we know that for any $\delta > 0$ small enough there are only finitely many ms for which $\mathcal{M}^{\lambda_m}_{\varepsilon_m}$ does not have, for each $i = 1, \dots, M$, at least N_i eigenvalues (counting multiplicity!) within δ of σ_i. Thus, by the continuity and non-negativity of $\varphi_\eta f$, for any $\varepsilon' > 0$, we may choose δ small enough so that

$$\sum_{y \in \mathrm{sp}(\mathcal{M}^{\lambda_m}_{\varepsilon_m})} \varphi_\eta(y) f(y) \geq \sum_{y \in \mathrm{sp}(\mathcal{M}^{\lambda_\infty}_{\varepsilon_\infty})} \varphi_\eta(y) f(y) - \varepsilon'$$

for all but finitely many ms, which completes I1.

I2. Claim: $\limsup_m \sum_{y \in \mathrm{sp}(\mathcal{M}^{\lambda_m}_{\varepsilon_m})} \varphi_\eta(y) f(y) \leq \sum_{y \in \mathrm{sp}(\mathcal{M}^{\lambda_\infty}_{\varepsilon_\infty})} \varphi_\eta(y) f(y)$**.**

We first claim that for all but finitely many ms we have

$$\#\left(\mathrm{sp}(\mathcal{M}^{\lambda_m}_{\varepsilon_m}) \cap [-\eta - 1, 1 + \eta]\right) \leq \#\left(\mathrm{sp}(\mathcal{M}^{\lambda_\infty}_{\varepsilon_\infty}) \cap [-\eta - 1, 1 + \eta]\right) =: M',$$

counting multiplicities. Indeed, suppose not. Then there would exist a subsequence (for which we abuse notation and still denote by m) for which $\mathcal{M}^{\lambda_m}_{\varepsilon_m}$ has (at least) $M'+1$ distinct eigenvalues (counting multiplicity). Say $\sigma_{m,1}, \dots, \sigma_{m,M'+1}$ with normalised eigenfunctions $u_{m,1}, \dots, u_{m,M'+1}$. By compactness of $[-\eta - 1, 1 + \eta]^{M'+1}$ we may pass to a subsequence (again we retain the index m) on which $\sigma_{m,i} \to \sigma_{\infty,i}$ for each $i = 1, \dots, M' + 1$ and some $\sigma_{\infty,i}$s. By the upper-semicontinuity result we may pass to successive subsequences to obtain a final subsequence (still denoted m) for which additionally $u_{m,i} \to u_{\infty,i}$ strongly for each i where $u_{\infty,i}$ is a normalised eigenfunction of $\mathcal{M}^{\lambda_\infty}_{\varepsilon_\infty}$ with eigenvalue $\sigma_{\infty,i}$. Moreover, as all the operators involved are self-adjoint, for each m the eigenfunctions $\{u_{m,i}\}_{i=1}^{M'+1}$ form an orthonormal system, and as orthonormality is preserved by strong limits, this holds also for $\{u_{\infty,i}\}_{i=1}^{M'+1}$. But this implies that $\mathcal{M}^{\lambda_\infty}_{\varepsilon_\infty}$ has at least $M' + 1$ eigenvalues in $[-\eta - 1, 1 + \eta]$, a contradiction, proving the claim.

We can now complete the proof of I2. Suppose that the claimed bound fails, then there would exist $\varepsilon' > 0$ and a subsequence (still denoted m) for which

$$\sum_{y \in \mathrm{sp}(\mathcal{M}^{\lambda_m}_{\varepsilon_m})} \varphi_\eta(y) f(y) \geq \sum_{y \in \mathrm{sp}(\mathcal{M}^{\lambda_\infty}_{\varepsilon_\infty})} \varphi_\eta(y) f(y) + \varepsilon'$$

for each m. Let $M_m = \#\left(\mathrm{sp}(\mathcal{M}^{\lambda_m}_{\varepsilon_m}) \cap [-\eta - 1, 1 + \eta]\right)$. Then by the previous claim we know that for all but finitely many ms we have $M_m \leq M'$. Thus some number $M'' \in \{1, \dots, M'\}$ is equal to infinitely many of the M_ms. We pass to this subsequence (still denoted m) so that $M_m = M''$ for every m. Let these eigenvalues be $\{\sigma_{m,i}\}_{i=1}^{M''}$. As in the proof of the claim above, after passing to another subsequence we have $\sigma_{m,i} \to \sigma_{\infty,i}$ for each i where $\{\sigma_{\infty,i}\}_{i=1}^{M''}$ are distinct (counting multiplicity) eigenvalues of $\mathcal{M}_{\varepsilon_\infty}^{\lambda_\infty}$. Hence, by continuity and non-negativity of $f\varphi_\eta$, we have

$$\sum_{y\in\mathrm{sp}(\mathcal{M}_{\varepsilon_\infty}^{\lambda_\infty})} \varphi_\eta(y)f(y) \geq \sum_{i=1}^{M''} \varphi_\eta(\sigma_{\infty,i})f(\sigma_{\infty,i})$$
$$= \lim_{m\to\infty} \sum_{i=1}^{M''} \varphi_\eta(\sigma_{m,i})f(\sigma_{m,i}) \geq \sum_{y\in\mathrm{sp}(\mathcal{M}_{\varepsilon_\infty}^{\lambda_\infty})} \varphi_\eta(y)f(y) + \varepsilon'$$

where the limit is on the subsequence we obtained. This is a contradiction which completes I2, and the weak convergence $\mu^\eta_{\lambda_m,\varepsilon_m} \rightharpoonup \mu^\eta_{\lambda_\infty,\varepsilon_\infty}$ follows.

II. Claim: for any $\varepsilon > 0$ and $n \in \mathbb{N}$ fixed and along any sequence $\lambda_m \to \lambda_\infty$ it holds that $\widetilde{\eta}_{\lambda_m,\varepsilon,n} \rightharpoonup \widetilde{\eta}_{\lambda_\infty,\varepsilon,n}$. This may be shown either by the same proof as in I, or we may simply note that the operators involved are finite dimensional matrices whose coefficients vary continuously in λ.

III. Claim: for any fixed $\varepsilon > 0$ we have $d_{BL}(\widetilde{\mu}^\eta_{\lambda,\varepsilon,n}, \mu^\eta_{\lambda,\varepsilon}) \to 0$ uniformly in $\lambda \in [0,1]$ as $n \to \infty$. The convergence $\widetilde{\mu}^\eta_{\lambda_n,\varepsilon,n} \rightharpoonup \mu^\eta_{\lambda_\infty,\varepsilon}$ along any sequence $\lambda_n \to \lambda_\infty$ follows from the same proof as in I. Uniform convergence follows from the compactness of $[0,1]$. Indeed, suppose that this uniform convergence does not hold. Then there would exist $\delta > 0$ such that, for infinitely many ns it holds that $d_{BL}(\widetilde{\mu}^\eta_{\lambda_n,\varepsilon,n}, \mu^\eta_{\lambda_n,\varepsilon}) > \delta$ for some $\lambda_n \in [0,1]$. Extract a subsequence (we abuse notation and retain the index n) for which $\lambda_n \to \lambda_\infty \in [0,1]$. From I we know that for all but finitely many ns we must have $d_{BL}(\mu^\eta_{\lambda_n,\varepsilon}, \mu^\eta_{\lambda_\infty,\varepsilon}) < \delta/2$. Therefore, by the triangle inequality

$$d_{BL}(\widetilde{\mu}^\eta_{\lambda_n,\varepsilon,n}, \mu^\eta_{\lambda_\infty,\varepsilon}) \geq \left| d_{BL}(\widetilde{\mu}^\eta_{\lambda_n,\varepsilon,n}, \mu^\eta_{\lambda_n,\varepsilon}) - d_{BL}(\mu^\eta_{\lambda_n,\varepsilon}, \mu^\eta_{\lambda_\infty,\varepsilon}) \right| > \delta/2$$

for infinitely many ns, a contradiction to the weak convergence $\widetilde{\mu}^\eta_{\lambda_n,\varepsilon,n} \rightharpoonup \mu^\eta_{\lambda_\infty,\varepsilon}$. $\square$

References

[BAH16] Jonathan Ben-Artzi and Thomas Holding. Approximations of Strongly Continuous Families of Unbounded Self-Adjoint Operators. *Commun. Math. Phys.*, 345(2):615–630, 2016.

School of Mathematics, Cardiff University, Cardiff CF24 4AG, Wales, United Kingdom.
E-mail address: Ben-ArtziJ@cardiff.ac.uk

Mathematics Institute, University of Warwick, Gibbet Hill Rd, Coventry CV4 7AL, United Kingdom.
E-mail address: T.Holding@warwick.ac.uk