Linear instability, exponential trichotomy and invariant manifold for Hamiltonian PDEs

Zhiwu Lin and Chongchun Zeng
Georgia Institute of Technology

Recent Progress on Collisionless models, London

September 7, 2015

Linear Hamiltonian system

Consider a linear Hamiltonian system

$$\partial_t u = JLu, \ \ u \in X,$$

where X is a Hilbert space. We assume that:
(H1) The operator $L : X \to X^*$ generates a bounded bilinear symmetric form $\langle L\cdot, \cdot\rangle$ on X. There exists a decomposition $X = N \oplus \ker L \oplus P$ satisfying that $\langle L\cdot, \cdot\rangle|_N < 0, \ \dim N = n^-(L)$ (the dimension of maximally negative subspace of X), and there exists $\delta > 0$ such that

$$\langle Lu, u\rangle \geq \delta \|u\|_X^2 \ , \text{ for any } u \in P.$$

(H2) $J : X^* \to X$ is a skew-adjoint operator, $\ker L \subset D(J)$.
For the consideration of complex eigenvalues, we consider the complexified space $X + iX$ of X and extend J, L. For convenience, we still use $X, \ J, \ L$ below.

Notations

$n^-(L|_S)$: the number of negative modes of $\langle L\cdot,\cdot\rangle$ restricted to a subspace $S \subset X$.

k_r : the total algebraic multiplicities of $\lambda \in \sigma(JL)$ with $\lambda > 0$.

k_c : the total algebraic multiplicities of $\lambda \in \sigma(JL)$ with $\mathrm{Re}\,\lambda, \mathrm{Im}\,\lambda > 0$.

$k_0^- = n^-\left(L|_{g\ker(JL)}\right)$, where

$g\ker(JL) = \left\{u \in X \mid (JL)^k u = 0, \text{ for some } k\right\}$.

For $0 \neq i\mu \in \sigma(JL)$, let $E_{i\mu}$ be the generalized eigenspace,

$k^-(i\mu) = n^-\left(L|_{E_{i\mu}}\right)$; $k_i^- = \sum_{0\neq\mu\in\mathbf{R}^+} k^-(i\mu)$.

An index formula

Theorem (Lin & Zeng, 2015) Under the assumptions (H1)-(H2), the eigenvalues of JL are symmetric to both real and imaginary axis and

$$k_r + 2k_c + 2k_i^- + k_0^- = n^-(L).$$

Corollary:
(i) If $k_0^- \geq n^-(L)$, then (LH) is spectrally stable.
(ii) If $n^-(L) - k_0^-$ is odd, then there exists a positive eigenvalue of (LH). In particular, if $n^-(L) - k_0^- = 1$, then $k_r = 1$ and $k_c = k_i^- = 0$.

Notations

For any eigenvalue $i\mu$ $(0 \neq \mu \in \mathbf{R})$ of JL, let $1 \leq k_1 < \cdots < k_l$ be the lengths of Jordan chains in $E_{i\mu}$. Suppose there are l_j Jordan chains of length k_j. When k_j is odd, let $\left\{ v_j^i \right\}_{1 \leq i \leq l_j}$ be the $(k_j + 1)/2$-th element (i.e middle element) of each chain. Define the $l_j \times l_j$ Hermitian matrix $M_j = \left(\left\langle L v_j^{i_1}, v_j^{i_2} \right\rangle \right)$, $1 \leq i_1, i_2 \leq l_j$. Let $n_j^- (\mu) = n^- (M_j)$. For zero eigenvalue, consider the projection of the chain spaces in $g \ker (JL)$ to $(\ker L)^{\perp}$. Define $n_j^- (0)$ as above for the projected chains. The numbers $n_j^- (\mu)$, $n_j^- (0)$ are independent of the choice of basis for Jordan chains.

Formula-1

Proposition 1. For any $\mu \in \mathbf{R}$,

$$k^{-}\left(i\mu\right) = \sum_{k_j \text{ even}} \frac{l_j k_j}{2} + \sum_{k_j \text{ odd}} \left[\frac{l_j \left(k_j - 1\right)}{2} + n_j^{-}\left(\mu\right) \right].$$

It follows from this formula that for any imaginary eigenvalue (including origin), the number and length of Jordan chains are uniformly bounded by $n^{-}\left(L\right)$.

Formula-2

Proposition 2: Define $S_1 = \ker(JL) \cap (\ker L)^{\perp}$. Let $\ker L \cap R(JL) = \{\xi_1, \cdots, \xi_m\}$ and $\{\eta_1, \cdots, \eta_m\} \subset (\ker(JL))^{\perp}$ be such that $JL\eta_i = \xi_i$, $1 \leq i \leq m$. Define $S_2 = span\{\eta_1, \cdots, \eta_m\}$. Denote n_1 to be the dimension of maximally nonpositive subspace of $S_1 \oplus S_2$ for $\langle L\cdot, \cdot\rangle$. Then: (i) $k_0^- \geq n_1$ and (ii) If $\langle L\cdot, \cdot\rangle|_{S_1 \oplus S_2}$ is non-degenerate, then $k_0^- = n_1$.

Comments

- Various index theorems have been studied by Mackay (1986) for finite-d case, by Kapitula, Kevrekidis, Sandstede, Pelinovsky, Chugunova, Stefnov, Bronski, Johnson, ... for infinite-d case (2000 to now). Mostly, under the assumptions that J is invertible and D or $\langle L\cdot, \cdot\rangle|_{S_1 \oplus S_2}$ is non-degenerate.

Comments

- Various index theorems have been studied by Mackay (1986) for finite-d case, by Kapitula, Kevrekidis, Sandstede, Pelinovsky, Chugunova, Stefnov, Bronski, Johnson, ... for infinite-d case (2000 to now). Mostly, under the assumptions that J is invertible and D or $\langle L\cdot,\cdot\rangle\,|_{S_1\oplus S_2}$ is non-degenerate.
- Here, we allow J to be an arbitrary unbounded skew-adjoint operator, even with $\dim\ker J=\infty$.

Comments

- Various index theorems have been studied by Mackay (1986) for finite-d case, by Kapitula, Kevrekidis, Sandstede, Pelinovsky, Chugunova, Stefnov, Bronski, Johnson, ... for infinite-d case (2000 to now). Mostly, under the assumptions that J is invertible and D or $\langle L\cdot, \cdot\rangle \,|_{S_1 \oplus S_2}$ is non-degenerate.
- Here, we allow J to be an arbitrary unbounded skew-adjoint operator, even with $\dim \ker J = \infty$.
- The explicit formula for k_i^- is new even for the finite dimensional case.

Exponential trichotomy of semigroup

Theorem Under assumptions (H1)-(H2), we have

$$X = E^u \oplus E^c \oplus E^s,$$

satisfying: i) E^u, E^s and E^c are invariant under e^{tJL}. ii) $\exists\ M > 0,\ \lambda_u > 0$, such that

$$\begin{aligned} \left| e^{tJL}|_{E^s} \right|_X &\leq Me^{-\lambda_u t}, \quad \forall\ t \geq 0, \\ |e^{tJL_c}|_{E^u}|_X &\leq Me^{\lambda_u t}, \quad \forall\ t \leq 0. \end{aligned}$$

and

$$|e^{tJL_c}|_{E^c}|_X \leq M(1 + |t|^{k_0}),\ \forall\ t \in \mathbf{R}.$$

where

$$k_0 = \max\left\{2k_i^-, 2k_0^-, 1\right\} \leq \max\left\{2\left(n^-\left(L\right) - k_r - 2k_c\right), 1\right\}.$$

Exponential trichotomy (continue)

For $k \geq 1$, define the space $X^k \subset X$ to be

$$X^k = \{u \in X \mid (JL)^n u \in X, \ n = 1, \cdots, k.\}$$

and

$$\|u\|_{X^k} = \|u\|_X + \|JLu\|_X + \cdots + \left\|(JL)^k u\right\|_X .$$

Assume $E^{u,s} \subset X^k$, then the exponential trichotomy holds true for X^k with

$$X^k = E^u \oplus E^c_k \oplus E^s, \ E^c_k = E^c \cap X^k$$

Remarks

- The exponential dichotomy (trichotomy) of the semigroup e^{tJL} is the first step to construct invariant manifolds for the nonlinear problem. It does not follow from the spectral gap of $\sigma(JL)$ even though $\sigma_{\text{ess}}(JL) \subset i\mathbf{R}$, due to the issue of spectral mapping $\sigma\left(e^{tJL}\right) \supsetneqq e^{t\sigma(JL)}$.

Remarks

- The exponential dichotomy (trichotomy) of the semigroup e^{tJL} is the first step to construct invariant manifolds for the nonlinear problem. It does not follow from the spectral gap of $\sigma(JL)$ even though $\sigma_{\text{ess}}(JL) \subset i\mathbf{R}$, due to the issue of spectral mapping $\sigma\left(e^{tJL}\right) \underset{\neq}{\supset} e^{t\sigma(JL)}$.
- We prove the exponential trichotomy by using the invariance of $\langle L\cdot, \cdot\rangle$ under e^{tJL} and the counting of negative index of L.

Remarks

- The exponential dichotomy (trichotomy) of the semigroup e^{tJL} is the first step to construct invariant manifolds for the nonlinear problem. It does not follow from the spectral gap of $\sigma(JL)$ even though $\sigma_{\text{ess}}(JL) \subset i\mathbf{R}$, due to the issue of spectral mapping $\sigma\left(e^{tJL}\right) \underset{\neq}{\supseteq} e^{t\sigma(JL)}$.
- We prove the exponential trichotomy by using the invariance of $\langle L\cdot, \cdot\rangle$ under e^{tJL} and the counting of negative index of L.
- Define $E^c = \{u \in X \mid \langle Lu, v\rangle = 0, \ \text{ for any } v \in E^u \oplus E^s\}$. A key step of our proof is to construct a decomposition $E^c = X_- \oplus X_+$ such that: $\dim X_- < \infty$, X_- is invariant under e^{tJL}, $L|_{X_+} \geq c_0 > 0$ and $X_+ \oplus \ker L$ is invariant under e^{tJL}.

Remarks

- The exponential dichotomy (trichotomy) of the semigroup e^{tJL} is the first step to construct invariant manifolds for the nonlinear problem. It does not follow from the spectral gap of $\sigma(JL)$ even though $\sigma_{\text{ess}}(JL) \subset i\mathbf{R}$, due to the issue of spectral mapping $\sigma\left(e^{tJL}\right) \underset{\neq}{\supset} e^{t\sigma(JL)}$.
- We prove the exponential trichotomy by using the invariance of $\langle L\cdot, \cdot\rangle$ under e^{tJL} and the counting of negative index of L.
- Define $E^c = \{u \in X \mid \langle Lu, v\rangle = 0, \quad \text{for any } v \in E^u \oplus E^s\}$. A key step of our proof is to construct a decomposition $E^c = X_- \oplus X_+$ such that: $\dim X_- < \infty$, X_- is invariant under e^{tJL}, $L|_{X_+} \geq c_0 > 0$ and $X_+ \oplus \ker L$ is invariant under e^{tJL}.
- The exponential dichotomy can fail if there is no spectral gap for L, for example, the Rayleigh-Taylor instability for 2D incompressible stratified flows.

BBM

- $$\partial_t u + \partial_x u + \partial_x f(u) - \partial_{txx} u = 0$$

BBM

- $$\partial_t u + \partial_x u + \partial_x f(u) - \partial_{txx} u = 0$$
- travelling waves $u(x,t) = u_c(x - ct)$ satisfying
$$-u_c'' + \left(1 - \frac{1}{c}\right) u_c - \frac{1}{c} f(u_c) = 0.$$

BBM

- $$\partial_t u + \partial_x u + \partial_x f(u) - \partial_{txx} u = 0$$
- travelling waves $u(x,t) = u_c(x - ct)$ satisfying
$$-u_c'' + \left(1 - \frac{1}{c}\right) u_c - \frac{1}{c} f(u_c) = 0.$$
- The linearized equation $\partial_t u = JLu$, where
$$J = c\partial_x (1 - \partial_{xx})^{-1}, \; L = -\partial_{xx} + \left(1 - \frac{1}{c}\right) - \frac{1}{c} f'(u_c).$$

We have $\ker L = \{u_{c,x}\}$ and $n^-(L) = 1$.

BBM

- $$\partial_t u + \partial_x u + \partial_x f(u) - \partial_{txx} u = 0$$
- travelling waves $u(x,t) = u_c(x - ct)$ satisfying
$$-u_c'' + \left(1 - \frac{1}{c}\right) u_c - \frac{1}{c} f(u_c) = 0.$$
- The linearized equation $\partial_t u = JLu$, where
$$J = c\partial_x (1 - \partial_{xx})^{-1}, \; L = -\partial_{xx} + \left(1 - \frac{1}{c}\right) - \frac{1}{c} f'(u_c).$$

  We have $\ker L = \{u_{c,x}\}$ and $n^-(L) = 1$.
- The operator J is not invertible, $0 \in \sigma_{\text{ess}}(J)$.

BBM

- $$\partial_t u + \partial_x u + \partial_x f(u) - \partial_{txx} u = 0$$
- travelling waves $u(x,t) = u_c(x - ct)$ satisfying
$$-u_c'' + \left(1 - \frac{1}{c}\right) u_c - \frac{1}{c} f(u_c) = 0.$$
- The linearized equation $\partial_t u = JLu$, where
$$J = c\partial_x (1 - \partial_{xx})^{-1},\ L = -\partial_{xx} + \left(1 - \frac{1}{c}\right) - \frac{1}{c} f'(u_c).$$
We have $\ker L = \{u_{c,x}\}$ and $n^-(L) = 1$.
- The operator J is not invertible, $0 \in \sigma_{\text{ess}}(J)$.
- Linear instability is equivalent to $dP/dc < 0$, where $P(c) = \int \left(u_c^2 + u_{c,x}^2\right) dx$.

Invariant manifolds

Theorem (L, Zeng) Assume $f \in C^k$ $(k \geq 1)$, consider BBM in the traveling frame of $u_c(x - ct)$.
1) There exists an unstable manifold M^u in H^k such that: M^u is locally invariant in time; If $u(0) \in M^u$, then $u(t) \to u_c(x)$ exponentially fast when $t \to -\infty$; M^u is a C^k graph over the (one dimensional) unstable eigenspace.
2) The stable manifold M^s is obtained by reversing the time.
3) There exists a central manifold M^c in the energy space H^1, near the translation orbit $S = \{u_c(x - \theta), \theta \in \mathbf{R}\}$, such that:

continue-central manifold

- M^c is a C^k manifold, of codimension 2 in H^1.

continue-central manifold

- M^c is a C^k manifold, of codimension 2 in H^1.
- (Orbital stability and uniqueness of M^c) For any small neighborhood U of S, there is a smaller neighborhood V of S, such that every solution starting from $V \cap M^c$ remains in $U \cap M^c$ for all $t \in \mathbf{R}$. This implies that the center manifold is locally unique. Here, to get the orbital stability on center manifold, we use the property $L|_{E^c \cap (\ker L)^{\perp}} \geq \delta_0 > 0$.

continue-central manifold

- M^c is a C^k manifold, of codimension 2 in H^1.
- (Orbital stability and uniqueness of M^c) For any small neighborhood U of S, there is a smaller neighborhood V of S, such that every solution starting from $V \cap M^c$ remains in $U \cap M^c$ for all $t \in \mathbf{R}$. This implies that the center manifold is locally unique. Here, to get the orbital stability on center manifold, we use the property $L|_{E^c \cap (\ker L)^\perp} \geq \delta_0 > 0$.
- $\exists$ a neighborhood U of S such that every solution starting from $U \setminus M^c$ exits U in finite time.

2D Euler

- 2D vorticity equation $\omega_t + \vec{v} \cdot \nabla\omega = 0$ in a bounded domain $\Omega \subset \mathbf{R}^2$, where ω is the vorticity and $\vec{v} = cur^{-1}\omega$ is the velocity.

2D Euler

- 2D vorticity equation $\omega_t + \vec{v} \cdot \nabla \omega = 0$ in a bounded domain $\Omega \subset \mathbf{R}^2$, where ω is the vorticity and $\vec{v} = cur^{-1}\omega$ is the velocity.
- Steady flows: if $-\Delta \psi_0 = F(\psi_0)$, $\psi_0 = 0$ on $\partial\Omega$ and Ω is simply connected. Then $\vec{v}_0 = \nabla^{\perp}\psi_0$ is a steady flow.

2D Euler

- 2D vorticity equation $\omega_t + \vec{v} \cdot \nabla \omega = 0$ in a bounded domain $\Omega \subset \mathbf{R}^2$, where ω is the vorticity and $\vec{v} = cur^{-1}\omega$ is the velocity.
- Steady flows: if $-\Delta \psi_0 = F(\psi_0)$, $\psi_0 = 0$ on $\partial\Omega$ and Ω is simply connected. Then $\vec{v}_0 = \nabla^{\perp}\psi_0$ is a steady flow.
- Assume $F'(\psi_0) > 0$. Linearized equation can be written as $\partial_t \omega = JL\omega$, where

$$J = F'(\psi_0)\,\vec{v}_0 \cdot \nabla, \quad L = \frac{1}{F'(\psi_0)} - (-\Delta)^{-1},$$

and

$$X = \left\{ \omega \mid \int\int_{\Omega} \frac{\omega^2}{F'(\psi_0)} dx < \infty \right\}.$$

continue

- Let $A = -\Delta - F'(\psi_0)(I - P)$, where P is the L^2 projection to $\ker J$. Note that $P\phi$ is the phase space average of ϕ on the phase space trajectory.

continue

- Let $A = -\Delta - F'(\psi_0)(I - P)$, where P is the L^2 projection to $\ker J$. Note that $P\phi$ is the phase space average of ϕ on the phase space trajectory.
- Assume $\ker A = \{0\}$. Then $L|_{R(J)}$ is non-degenerate and $n^-\left(L|_{R(J)}\right) = n^-(A)$.

continue

- Let $A = -\Delta - F'(\psi_0)(I - P)$, where P is the L^2 projection to $\ker J$. Note that $P\phi$ is the phase space average of ϕ on the phase space trajectory.
- Assume $\ker A = \{0\}$. Then $L|_{R(J)}$ is non-degenerate and $n^-\left(L|_{R(J)}\right) = n^-(A)$.
- The index formula becomes $k_r + 2k_c + 2k_i^- = n^-(A)$.

continue

- Let $A = -\Delta - F'(\psi_0)(I - P)$, where P is the L^2 projection to $\ker J$. Note that $P\phi$ is the phase space average of ϕ on the phase space trajectory.
- Assume $\ker A = \{0\}$. Then $L|_{R(J)}$ is non-degenerate and $n^-\left(L|_{R(J)}\right) = n^-(A)$.
- The index formula becomes $k_r + 2k_c + 2k_i^- = n^-(A)$.
- It was proved in (Lin, CMP, 03) that the oddness of $n^-(A)$ implies the linear instability, for general $F \in C^1$.

continue

- Let $A = -\Delta - F'(\psi_0)(I - P)$, where P is the L^2 projection to $\ker J$. Note that $P\phi$ is the phase space average of ϕ on the phase space trajectory.
- Assume $\ker A = \{0\}$. Then $L|_{R(J)}$ is non-degenerate and $n^-\left(L|_{R(J)}\right) = n^-(A)$.
- The index formula becomes $k_r + 2k_c + 2k_i^- = n^-(A)$.
- It was proved in (Lin, CMP, 03) that the oddness of $n^-(A)$ implies the linear instability, for general $F \in C^1$.
- When $\ker A \neq \{0\}$, define $D = (\langle Lw_i, w_j \rangle)$ with $w_i \in R(J)$ and $JLw_i \in \ker L$, then

$$k_r + 2k_c + 2k_i^- = n^-(A) - n^-(D),$$

if D is non-degenerate.

A second order system

Consider the second order psystem

$$A\partial_{tt}u + Lu = 0, \tag{1}$$

with the following assumption (H'): L satisfies (H1) in X; $A > 0$ is self-adjoint, and L and A are real operators. Let $v = \sqrt{A}\partial_t u$, then it becomes

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = J\tilde{L}\begin{pmatrix} u \\ v \end{pmatrix}, \tag{2}$$

where

$$J = \begin{pmatrix} 0 & A^{-\frac{1}{2}} \\ -A^{-\frac{1}{2}} & 0 \end{pmatrix}, \ \tilde{L} = \begin{pmatrix} L & 0 \\ 0 & I \end{pmatrix}.$$

Continue

The energy space for (2) is $\tilde{X} = X \times L^2$ and

$$n^{-}\left(\tilde{L}|_{\tilde{X}}\right) = n^{-}\left(L|_{X}\right).$$

Theorem For the second order system (1), assume (H'). Then: i) $k_r = n^{-}\left(L|_X\right)$ and $L|_X \geq 0$ implies linear stability. ii) the linear exponential trichotomy holds true for (1) in the space

$$\bar{X} = \left\{(u, \partial_t u) \mid \left(u, \sqrt{A}\partial_t u\right) \in \tilde{X}\right\}.$$

1.5D Vlasov-Maxwell

The $1\frac{1}{2}$D Vlasov Maxwell system for electrons with a constant ion background n_0 is

$$\partial_t f + v_1 \partial_x f - (E_1 + v_2 B)\,\partial_{v_1} f - (E_2 - v_1 B)\,\partial_{v_2} f = 0$$

$$\partial_t E_1 = -j_1 = \int v_1 f \; dv, \;\; \partial_t B = -\partial_x E_2$$

$$\partial_t E_2 + \partial_x B = -j_2 = \int v_2 f \; dv$$

with the constraint

$$\partial_x E_1 = n_0 - \int f dv.$$

Steady states

Consider the $P-$periodic equilibrium $f^0 = \mu(e, p)$,
$E_1^0 = -\partial_x \phi^0$, $E_2^0 = 0$, $B^0 = \partial_x \psi^0$, where (ϕ^0, ψ^0) satisfy the ODE system

$$\partial_x^2 \phi^0 = n_0 - \int \mu(e, p) dv, \quad \partial_x^2 \psi^0 = \int v_2 \mu(e, p) dv$$

with the electron energy and the angular momentum defined by

$$e = \frac{1}{2} |v|^2 - \phi^0(x), \quad p = v_2 - \psi^0(x).$$

We assume

$$\mu \geq 0, \quad \mu \in C^1, \quad \mu_e \equiv \frac{\partial \mu}{\partial e} < 0$$

Linearized equation

The linearized evolution equations are

$$(\partial_t + D)f = \mu_e v_1 E_1 - \mu_p v_1 B + (\mu_e v_2 + \mu_p)E_2,$$

where D is the transport operator associated with the steady fields,

$$\begin{aligned} D &= v_1\partial_x - \left(E_1^0 + v_2 B^0\right)\partial_{v_1} + v_1 B^0 \partial_{v_2} \\ &= v_1\partial_x + \partial_x\phi^0\ \partial_{v_1} + \partial_x\psi^0\ (v_1\partial_{v_2} - v_2\partial_{v_1}), \end{aligned}$$

together with

$$\begin{aligned} \partial_x E_1 &= -\int f dv,\ \partial_t E_1 = \int v_1 f dv, \\ \partial_t E_2 + \partial_x B &= \int v_2 f dv,\ \partial_t B + \partial_x E_2 = 0. \end{aligned}$$

Second order form

Split f into its even and odd parts in the variable v_1:

$$f = f_{ev} + f_{od}, \quad \text{where } f_{ev}(x, v_1, v_2) = \frac{1}{2}\{f(x, v_1, v_2) + f(x, -v_1, v_2)\}.$$

Then the linearized VM system can be written as a second order system for $u = (f_{od}, E_2)$:

$$\begin{aligned}
(\partial_t^2 - D^2)\left(\frac{f_{od}}{|\mu_e|}\right) - D(E_2 v_2) + \left(\int v_1 f_{od}\, dv\right) v_1 &= 0, \\
\partial_t^2 E_2 - \partial_x^2 E_2 + \int v_2 D f_{od}\ dv - \int v_2 \partial_{v_2} f^0 dv\ E_2 &= 0.
\end{aligned}$$

Define the functional

$$\begin{aligned}
W(f_{od}, E_2) &= \iint \frac{1}{|\mu_e|} |Df_{od} - \mu_e v_2 E_2|^2\, dvdx + \int |\partial_x E_2|^2\, dx \\
&\quad + \int \left|\int v_1 f_{od}\, dv\right|^2 dx - \iint \mu_p v_2 |E_2|^2\, dvdx \\
&= (Lu, u).
\end{aligned}$$

Continue

Here, the symmetric operator L is defined by

$$L\begin{pmatrix} f_{od} \\ E_2 \end{pmatrix} = \begin{pmatrix} -D^2\left(\frac{f_{od}}{|\mu_e|}\right) - D(E_2 v_2) + \left(\int v_1 f_{od}\, dv\right) v_1 \\ -\partial_x^2 E_2 + \int v_2 D f_{od}\ dv - \int v_2 \partial_{v_2} f^0 dv\ E_2 \end{pmatrix}.$$

Define the positive operator

$$A\begin{pmatrix} f_{od} \\ E_2 \end{pmatrix} = \begin{pmatrix} \frac{f_{od}}{|\mu_e|} \\ E_2 \end{pmatrix}.$$

Then the linearized system becomes

$$A\partial_t^2 \begin{pmatrix} f_{od} \\ E_2 \end{pmatrix} + L\begin{pmatrix} f_{od} \\ E_2 \end{pmatrix} = 0.$$

Continue

The energy space X for $u = (f_{od}, E_2)$ is the Hilbert space defined by the norm

$$\begin{aligned}\|u\|_X^2 &= \iint \frac{1}{|\mu_e|} \left|Df_{od} - \mu_e v_2 E_2\right|^2 dvdx \\ &+ \int \left|\iint v_1 f_{od} dv\right|^2 dx + \int \left(|\partial_x E_2|^2 + |E_2|^2\right) dx.\end{aligned}$$

Lemma The operator L satisfies the assumption (H1) on X and $n^-(L|_X) = n^-(\mathcal{L}^0)$.

Continue

Here,

$$\mathcal{L}^0 = (\mathcal{B}^0)^*(\mathcal{A}_1^0)^{-1}\mathcal{B}^0 + \mathcal{A}_2^0 : H_P^2 \to L_P^2,$$

and $\mathcal{A}_1^0, \mathcal{A}_2^0,\ \mathcal{B}^0$ are defined by

$$\mathcal{A}_1^0 h = -\partial_x^2 h - \left(\int \mu_e dv\right) h + \int \mu_e\ \mathcal{P}h\ dv,$$

$$\mathcal{A}_2^0 h = -\partial_x^2 h - \left(\int v_2 \mu_p dv\right) h - \int \mu_e v_2 \mathcal{P}(\hat{v}_2 h)\ dv,$$

$$\mathcal{B}^0 h = -\int \mu_e\,(I - \mathcal{P})\,(v_2 h)\, dv.$$

Stability criterion and semigroup estimate

Theorem i) $k_r = n^-\left(\mathcal{L}^0\right)$ and $\mathcal{L}^0 \geq 0$ implies linear stability; ii) The exponential trichotomy is proved for linearized VM.

Remarks:

1. The linear stability criterion recovers the results of Lin & Strauss (06, 07) which used the method of trajectory integration and continuity argument. The similar results for 3D cylindrically symmetric Vlasov-Maxwell system can also be obtained by the second order equation.
2. The semigroup estimates can be proved for 1.5D RVM by using the compact perturbation theory of semigroups. But it strongly used the operator splitting techniques of Glassey & Strauss and does not work for nonrelativistic VM.
3. For VM, the semigroup is first proved for the 2nd order system and then go back to the original 1st order system.

Summary

- An index theorem and the linear exponential trichotomy is proved for general linear Hamiltonian PDEs with an energy functional bounded from below and with finitely many negative dimensions .

Summary

- An index theorem and the linear exponential trichotomy is proved for general linear Hamiltonian PDEs with an energy functional bounded from below and with finitely many negative dimensions .
- The framework is also used for some other problems in fluids and plasmas, including 2D Euler equation and Vlasov-Maxwell systems for collisionless plasmas.

Summary

- An index theorem and the linear exponential trichotomy is proved for general linear Hamiltonian PDEs with an energy functional bounded from below and with finitely many negative dimensions .
- The framework is also used for some other problems in fluids and plasmas, including 2D Euler equation and Vlasov-Maxwell systems for collisionless plasmas.
- One remaining challenge is to develop a stability theory for cases when the energy functional is unbounded from both below and above (water waves, regularized Boussinesq, 2D Euler with general F etc.). There exists some approaches to get sufficient conditions for linear instability for many of these cases, however it is more difficult to get sharp linear stability criterion and prove exponential trichotomy of linearized equations.