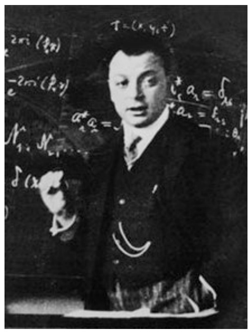
"Recent Progress in Collisionless Models," Imperial College 8 Sept 2015

Phase mixing vs. nonlinear advection in drift-kinetic plasma turbulence

Alex Schekochihin

w i t h

Joseph Parker, Paul Dellar (Oxford Maths),
Edmund Highcock (Oxford Th. Phys.),
Bill Dorland (Maryland), Greg Hammett (Princeton)



with thanks to the Wolfgang Pauli Institute, Vienna,
where much of this work was conceived and done
(and where maths meet physics as if they'd always been together)

arXiv:1508.05988

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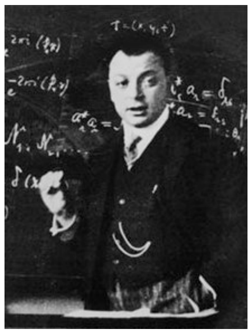

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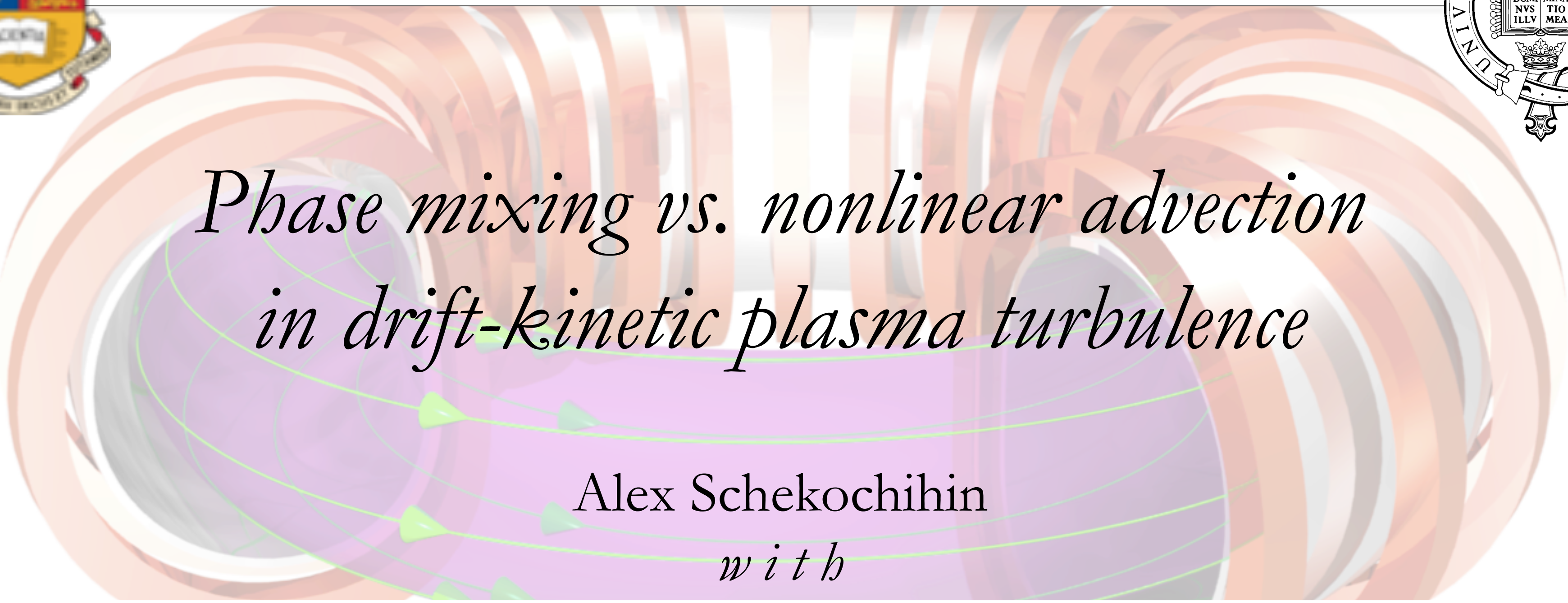

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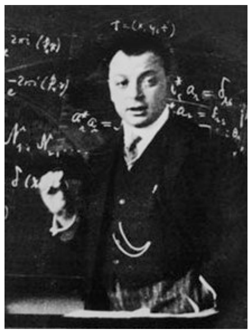

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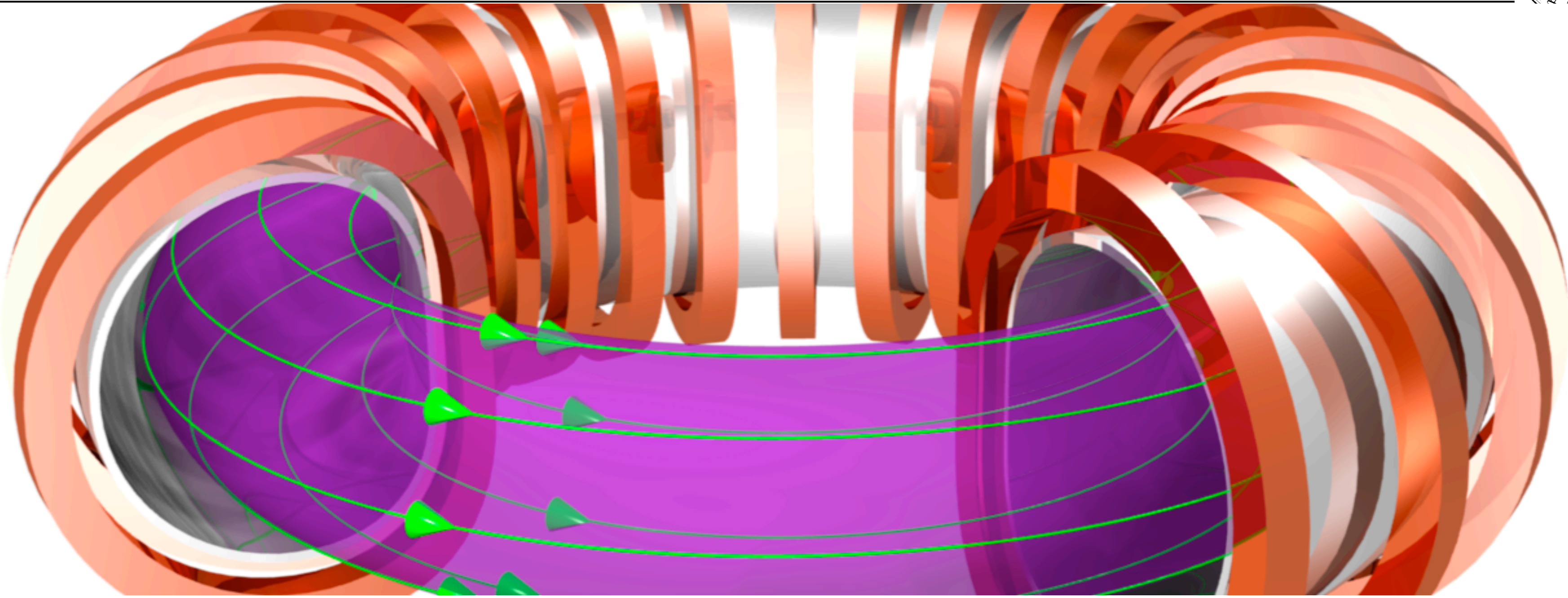

I will assume you really want to know
what sort of problems fusion physicists think about

(actually, what I will give you is my take on what's interesting, to me, right now;
__disclaimer:__ I am not really a hard-core fusion physicist,
most of them actually think of much more practical things…)

A Prototypical Kinetic Problem

Plasma near Maxwellian equilibrium: $f = F_0 + \delta f$

Strong (uniform) magnetic field: $\omega \ll \Omega_i,\ k_\parallel \ll k_\perp$

Electrostatic: $\delta\mathbf{E} = -\nabla\phi,\ \delta\mathbf{B} = 0$

Long wavelength: $k_\perp \rho_i \ll 1$

$\Downarrow$

Drift-kinetic equation for $g(t, \mathbf{r}, v_\parallel) = \dfrac{1}{n_{0i}} \displaystyle\int d^2\mathbf{v}_\perp \delta f_i$

$$\boxed{\frac{\partial g}{\partial t} + v_\parallel \nabla_\parallel (g + \underbrace{\varphi F_0}_{\text{parallel el. field}}) + \underbrace{\mathbf{u}_\perp \cdot \nabla_\perp g}_{\mathbf{E}\text{x}\mathbf{B}\text{ drift nonlinearity}} = \underbrace{C[g]}_{\text{coll. phase space cutoff}} + \underbrace{\chi}_{\text{energy injection}}}$$

$$\mathbf{u}_\perp = \frac{\rho_i v_{\rm th}}{2} \hat{\mathbf{z}} \times \nabla_\perp \varphi$$

Boltzmann electrons: $\varphi \equiv \dfrac{Ze\phi}{T_i} = \alpha \displaystyle\int dv_\parallel g, \quad \alpha = \dfrac{ZT_e}{T_i}$

ITG/drift-wave problem:

$$\chi = -\mathbf{u}_\perp \cdot \nabla F_0 = -\frac{\rho_i v_{\rm th}}{2} \frac{\partial \varphi}{\partial y} \left[\frac{1}{L_n} + \left(\frac{v_\parallel^2}{v_{\rm th}^2} - \frac{1}{2} \right) \frac{1}{L_T} \right]$$

A Prototypical Kinetic Problem

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$$\boxed{\frac{\partial g}{\partial t} + v_\parallel \nabla_\parallel (g + \underbrace{\varphi F_0}_{\substack{\text{parallel}\\ \text{el. field}}}) + \underbrace{\mathbf{u}_\perp \cdot \nabla_\perp g}_{\substack{\mathbf{E}\text{x}\mathbf{B}\text{ drift}\\ \text{nonlinearity}}} = \underbrace{C[g]}_{\substack{\text{coll.}\\ \text{phase}\\ \text{space}\\ \text{cutoff}}} + \underbrace{\chi}_{\substack{\text{energy}\\ \text{injection}}}}$$

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ITG/drift-wave problem:

$$\chi = -\mathbf{u}_\perp \cdot \nabla F_0 = -\frac{\rho_i v_{\text{th}}}{2} \frac{\partial \varphi}{\partial y} \left[-\frac{1}{n_i}\frac{dn_i}{dx} - \left(\frac{v_\parallel^2}{v_{\text{th}}^2} - \frac{1}{2} \right) \frac{1}{T_i}\frac{dT_i}{dx} \right]$$

The Plasma Turbulence Problem

Energy injected into perturbations can be thermalised:

EITHER by phase mixing (=Landau damping), producing fine scales in $v_\parallel$ and thus making $C[g]$ finite even if the collisionality is small:

$$C[g] \sim \nu\, v_{\rm th}^2 \frac{\partial^2 g}{\partial v_\parallel^2} \sim \omega g \quad \text{if} \quad \frac{\delta v_\parallel}{v_{\rm th}} \sim \left(\frac{\nu}{\omega}\right)^{1/2}$$

$$\boxed{\frac{\partial g}{\partial t} + \underbrace{v_\parallel \nabla_\parallel (g}_{\text{phase mixing}} + \varphi F_0) + \underbrace{\mathbf{u}_\perp \cdot \nabla_\perp g}_{\text{turbulent mixing}} = C[g] + \chi}$$

AND/OR by turbulent mixing, producing fine scales in real space, eventually accessing various dissipation mechanisms at $k_\perp \rho_i \lesssim 1$ (which are an interesting but separate story, for another talk)

So what does the system choose to do?

The Plasma Turbulence Problem

"Idle" theory questions:

- Which thermalisation route does the system favour?

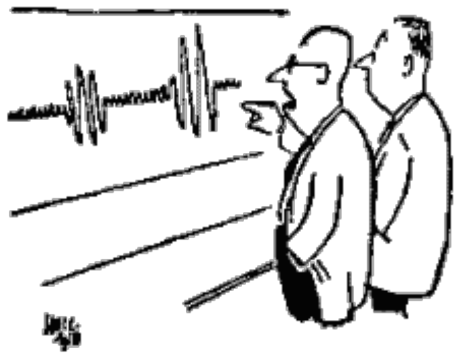
- Therefore, what is the structure of turbulence at scales between injection and dissipation (in phase space, so φ, g vs. $k_\perp, k_\|, v_\perp$)

'Which here is the "+" and which is the "–"?'

"Pragmatic" modeling questions:

- At what rate is the injected energy removed to small scales?

- Therefore, what is the typical amplitude of the fluctuations? (get that by balancing injection rate with removal rate)



- Therefore, what is the typical "turbulent diffusivity" relaxing large-scale gradients?

$$D_T \sim \underbrace{\langle u_\perp^2 \rangle}_{\text{amplitude}} \underbrace{\tau_c}_{\text{correlation time}}$$

Free Energy

"Energy" in δf kinetics is in fact the free energy of the fluctuations:

$$\mathcal{F} = -\sum_s T_s \delta S_s = -\sum_s T_s \delta \int d^3\mathbf{v} \langle f_s \ln f_s \rangle = \sum_s \int d^3\mathbf{v} \frac{T_s \langle \delta f_s^2 \rangle}{2F_{0s}} = n_i T_i W$$

where $W = \int dv_\| \frac{\langle g^2 \rangle}{2F_0} + \frac{\langle \varphi^2 \rangle}{2\alpha}$ is conserved by our equations

$$\frac{\partial g}{\partial t} + v_\| \nabla_\| (g + \varphi F_0) + \mathbf{u}_\perp \cdot \nabla_\perp g = C[g] + \chi$$

$$\varphi \equiv \frac{Ze\phi}{T_i} = \alpha \int dv_\| g, \quad \alpha = \frac{ZT_e}{T_i}$$

$$\boxed{\frac{dW}{dt} = \int dv_\| \frac{\langle g\chi \rangle}{F_0} + \int dv_\| \frac{\langle gC[g] \rangle}{F_0}}$$

↑ injection (instabilities, forcing...)

↑ dissipation (collisions)

Kruskal & Oberman 1958
Bernstein 1958
Fowler 1963, 68
Krommes & Hu 1994
Krommes 1999
Sugama et al. 1996
Hallatschek 2004
Howes et al. 2006
Candy & Waltz 2006
Schekochihin et al. 2007-09
Scott 2010
Banon, Teaca, Hatch, Morel, Jenko et al. 2011-14
Plunk et al 2012
Abel et al. 2013
Kunz et al. 2015
…

Landau Damping = Phase Mixing

Landau damping/phase mixing is the transfer of free energy from φ to g via refinement of velocity-space structure of the perturbed distribution

$$W = \int dv_\parallel \frac{\langle g^2 \rangle}{2F_0} + \frac{\langle \varphi^2 \rangle}{2\alpha}$$ is conserved by our equations

$$\frac{\partial g}{\partial t} + v_\parallel \nabla_\parallel (g + \varphi F_0) + \mathbf{u}_\perp \cdot \nabla_\perp g = C[g] + \chi$$

$$\varphi \equiv \frac{Ze\phi}{T_i} = \alpha \int dv_\parallel g, \quad \alpha = \frac{ZT_e}{T_i}$$

$$\boxed{\frac{dW}{dt} = \underbrace{\int dv_\parallel \frac{\langle g\chi \rangle}{F_0}}_{\text{injection (instabilities, forcing...)}} + \underbrace{\int dv_\parallel \frac{\langle gC[g] \rangle}{F_0}}_{\text{dissipation (collisions)}}}$$

Hammett, Perkins, Dorland, Beer, Smith, Snyder 1990-2001:

development of Landau fluid models based on understanding of phase mixing as energy removal into phase space and eventual collisional thermalisation

Life in Hermite Space

$$H_m(x) = (-1)^m e^{x^2} \frac{d^m}{dx^m} e^{-x^2} \quad \Rightarrow \quad g_m = \int dv_\parallel \frac{H_m(v_\parallel/v_{\rm th})}{\sqrt{2^m m!}} g(v_\parallel)$$

$$H_0 = 1 \quad \Rightarrow \quad g_0 = \frac{\delta n}{n} = \frac{\varphi}{\alpha}$$

$$H_1 = 2x \quad \Rightarrow \quad g_1 = \sqrt{2}\frac{u_\parallel}{v_{\rm th}}$$

$$H_2 = 4\left(x^2 - \frac{1}{2}\right) \quad \Rightarrow \quad g_2 = \frac{1}{\sqrt{2}}\frac{\delta T_\parallel}{T}, \quad \text{etc.}$$

"fluid" moments

$m = 0, 1, 2$

$$\frac{\partial g}{\partial t} + v_\parallel \nabla_\parallel (g + \varphi F_0) + \mathbf{u}_\perp \cdot \nabla_\perp g = C[g] + \chi$$

$$\varphi \equiv \frac{Ze\phi}{T_i} = \alpha \int dv_\parallel g, \quad \alpha = \frac{ZT_e}{T_i}$$

$$\chi = -\mathbf{u}_\perp \cdot \nabla F_0 = -\frac{\rho_i v_{\rm th}}{2}\frac{\partial \varphi}{\partial y}\left[\frac{1}{L_n} + \left(\frac{v_\parallel^2}{v_{\rm th}^2} - \frac{1}{2}\right)\frac{1}{L_T}\right]$$

H_0 H_2 energy injection only at low m

Life in Hermite Space

$$\underline{\underline{\frac{\partial}{\partial t}\frac{\varphi}{\alpha}}} + \underline{\underline{v_{\rm th}\nabla_\parallel \frac{u_\parallel}{v_{\rm th}}}} = -\frac{v_{\rm th}}{2L_n}\rho_i\frac{\partial\varphi}{\partial y}$$

$$\left(\underline{\underline{\frac{\partial}{\partial t}}} + \mathbf{u}_\perp\cdot\nabla_\perp\right)\frac{u_\parallel}{v_{\rm th}} + v_{\rm th}\nabla_\parallel\left(\underline{\underline{\frac{1}{2}\frac{\delta T_\parallel}{T}}} + \frac{1+\alpha}{\alpha}\varphi\right) = 0$$

$$\left(\underline{\underline{\frac{\partial}{\partial t}}} + \mathbf{u}_\perp\cdot\nabla_\perp\right)\frac{\delta T_\parallel}{T} + v_{\rm th}\nabla_\parallel\left(\sqrt{3}\,g_3 + 2\frac{u_\parallel}{v_{\rm th}}\right) = \underline{\underline{-\frac{v_{\rm th}}{2L_T}\rho_i\frac{\partial\varphi}{\partial y}}}$$

Standard "fluid" ITG equations (e.g., Cowley et al. 1992)

$$\omega^3 \approx \frac{\alpha}{2}(k_\parallel v_{\rm th})^2\omega_{*T}$$

$$\omega_{*T} = \frac{k_y\rho_i v_{\rm th}}{2L_T}$$

From *m*=3 on, the equations are universal:

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_\perp\cdot\nabla_\perp\right)g_m + v_{\rm th}\nabla_\parallel\left(\sqrt{\frac{m+1}{2}}\,g_{m+1} + \sqrt{\frac{m}{2}}\,g_{m-1}\right) = -\nu m g_m$$

all moments are advected by the same velocity

$$\mathbf{u}_\perp = \frac{\rho_i v_{\rm th}}{2}\hat{\mathbf{z}}\times\nabla_\perp\varphi$$

higher moments couple to lower ones, so even though free energy is injected at low *m*, it gets to large *m*

at large enough *m*, free energy is removed by collisions

NB: we use the LB operator

$$C[g] = \nu\frac{\partial}{\partial v_\parallel}\left(\frac{1}{2}\frac{\partial}{\partial v_\parallel} + v_\parallel\right)g$$

Life in Hermite Space

$$\frac{\partial}{\partial t}\frac{\varphi}{\alpha} + v_{\rm th}\nabla_\parallel \frac{u_\parallel}{v_{\rm th}} = -\frac{v_{\rm th}}{2L_n}\rho_i\frac{\partial \varphi}{\partial y}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_\perp\cdot\nabla_\perp\right)\frac{u_\parallel}{v_{\rm th}} + v_{\rm th}\nabla_\parallel\left(\frac{1}{2}\frac{\delta T_\parallel}{T} + \frac{1+\alpha}{\alpha}\varphi\right) = 0$$

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Standard "fluid"
ITG equations
(e.g., Cowley et al. 1992)

The free energy is (via Parseval's theorem for H_m's)

$$W = \sum_{m=3}^{\infty}\frac{\langle g_m^2\rangle}{2} + \frac{1}{4}\frac{\langle \delta T_\parallel^2\rangle}{T^2} + \frac{\langle u_\parallel^2\rangle}{v_{\rm th}^2} + \frac{1+\alpha}{2\alpha^2}\langle\varphi^2\rangle$$

Life in Hermite Space

$$\frac{\partial}{\partial t}\frac{\varphi}{\alpha} + v_{\rm th}\nabla_\parallel \frac{u_\parallel}{v_{\rm th}} = -\frac{v_{\rm th}}{2L_n}\rho_i\frac{\partial\varphi}{\partial y}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_\perp\cdot\nabla_\perp\right)\frac{u_\parallel}{v_{\rm th}} + v_{\rm th}\nabla_\parallel\left(\frac{1}{2}\frac{\delta T_\parallel}{T} + \frac{1+\alpha}{\alpha}\varphi\right) = 0$$

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Standard "fluid" ITG equations (e.g., Cowley et al. 1992)

The free energy is (via Parseval's theorem for H_m's)

Landau damping/phase mixing is the transfer of free energy From low moments ($\varphi, u_\parallel, \delta T_\parallel$) into higher ones ($g_{m\geq 3}$).

$$W = \sum_{m=3}^{\infty}\frac{\langle g_m^2\rangle}{2} + \frac{1}{4}\frac{\langle\delta T_\parallel^2\rangle}{T^2} + \frac{\langle u_\parallel^2\rangle}{v_{\rm th}^2} + \frac{1+\alpha}{2\alpha^2}\langle\varphi^2\rangle$$

Turbulence (in the usual sense) is the mixing of $\varphi, u_\parallel, \delta T_\parallel$ by $\mathbf{u}_\perp$ transferring their energy to small scales (large $k_\perp$).

“Fluid” Turbulence Theory

$$\frac{\partial}{\partial t}\frac{\varphi}{\alpha} + v_{\rm th}\nabla_\parallel \frac{u_\parallel}{v_{\rm th}} = -\frac{v_{\rm th}}{2L_n}\rho_i\frac{\partial \varphi}{\partial y}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_\perp\cdot\nabla_\perp\right)\frac{u_\parallel}{v_{\rm th}} + v_{\rm th}\nabla_\parallel\left(\frac{1}{2}\frac{\delta T_\parallel}{T} + \frac{1+\alpha}{\alpha}\varphi\right) = 0$$

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Standard “fluid” ITG equations (e.g., Cowley et al. 1992)

Let us construct a turbulence theory for ITG ignoring coupling to phase space…

$$W = \cancel{\sum_{m=3}^{\infty}\frac{\langle g_m^2\rangle}{2}} + \frac{1}{4}\frac{\langle \delta T_\parallel^2\rangle}{T^2} + \frac{\langle u_\parallel^2\rangle}{v_{\rm th}^2} + \frac{1+\alpha}{2\alpha^2}\langle\varphi^2\rangle$$

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[Barnes, Parra & AAS PRL **107**, 115003 (2011)]

“Fluid” Turbulence Theory: Outer Scale

$$\frac{\partial}{\partial t}\frac{\varphi}{\alpha} + v_{\rm th}\nabla_\parallel \frac{u_\parallel}{v_{\rm th}} = -\frac{v_{\rm th}}{2L_n}\rho_i\frac{\partial\varphi}{\partial y}$$

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Standard “fluid” ITG equations (e.g., Cowley et al. 1992)

Let us construct a turbulence theory for ITG ignoring coupling to phase space…

Where is energy injected?

- “Linear balance”: $k_{\parallel 0}v_{\rm th} \sim \omega_{*T} = k_{y0}\rho_i\frac{v_{\rm th}}{L_T}$
  (for ITG injection to work)
- Largest possible scale: $k_{\parallel 0} \sim \frac{1}{L_\parallel}\left(=\frac{1}{qR}\right)$
- Isotropy: $k_{x0} \sim k_{y0} \sim k_{\perp 0}$
  ($k_{x0} \sim S_{\rm ZF}k_{y0}\tau_c \sim k_{y0}$ if $S_{\rm ZF} \sim \tau_c^{-1}$)
  zonal flow shear

$$\Rightarrow \boxed{k_{\perp 0}\rho_i \sim \frac{L_T}{L_\parallel}}$$

[Barnes, Parra & AAS PRL **107**, 115003 (2011)]

“Fluid” Turbulence Theory: Outer Scale

$$\frac{\partial}{\partial t}\frac{\varphi}{\alpha} + v_{\rm th}\nabla_\parallel \frac{u_\parallel}{v_{\rm th}} = -\frac{v_{\rm th}}{2L_n}\rho_i\frac{\partial\varphi}{\partial y}$$

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Standard “fluid” ITG equations (e.g., Cowley et al. 1992)

Let us construct a turbulence theory for ITG ignoring coupling to phase space…

At what rate is energy removed from this scale?

$$\omega_{*T} \sim k_{\perp 0}\rho_i\frac{v_{\rm th}}{L_T} \sim k_{\perp 0}u_{\perp 0} \sim \rho_i v_{\rm th}k_{\perp 0}^2\varphi_0$$

injection

nonlinear removal to smaller scales, $k_\perp > k_{\perp 0}$

$$\Downarrow$$

$$\varphi_0 \sim \frac{1}{k_{\perp 0}L_T}$$

$$\boxed{k_{\perp 0}\rho_i \sim \frac{L_T}{L_\parallel}}$$

$$\Rightarrow \boxed{\varphi_0 \sim \frac{\rho_i L_\parallel}{L_T^2}}$$

[Barnes, Parra & AAS PRL **107**, 115003 (2011)]

“Fluid” Turbulence Theory: Outer Scale

$$\frac{\partial}{\partial t}\frac{\varphi}{\alpha} + v_{\rm th}\nabla_\parallel \frac{u_\parallel}{v_{\rm th}} = -\frac{v_{\rm th}}{2L_n}\rho_i\frac{\partial\varphi}{\partial y}$$

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$$\left(\frac{\partial}{\partial t} + \mathbf{u}_\perp\cdot\nabla_\perp\right)\frac{\delta T_\parallel}{T} + v_{\rm th}\nabla_\parallel\left(\cancel{\sqrt{8}g_3} + 2\frac{u_\parallel}{v_{\rm th}}\right) = -\frac{v_{\rm th}}{2L_T}\rho_i\frac{\partial\varphi}{\partial y}$$

Standard “fluid” ITG equations (e.g., Cowley et al. 1992)

Let us construct a turbulence theory for ITG ignoring coupling to phase space…

What is the turbulent diffusivity?

$$D_T \sim u_{\perp 0}^2\tau_c \sim \frac{u_{\perp 0}}{k_{\perp 0}} \sim \rho_i v_{\rm th}\varphi_0 \sim \frac{\rho_i^2 v_{\rm th}}{L_\parallel}\left(\frac{L_\parallel}{L_T}\right)^2$$

$$\boxed{k_{\perp 0}\rho_i \sim \frac{L_T}{L_\parallel}}$$

$$\boxed{\varphi_0 \sim \frac{\rho_i L_\parallel}{L_T^2}}$$

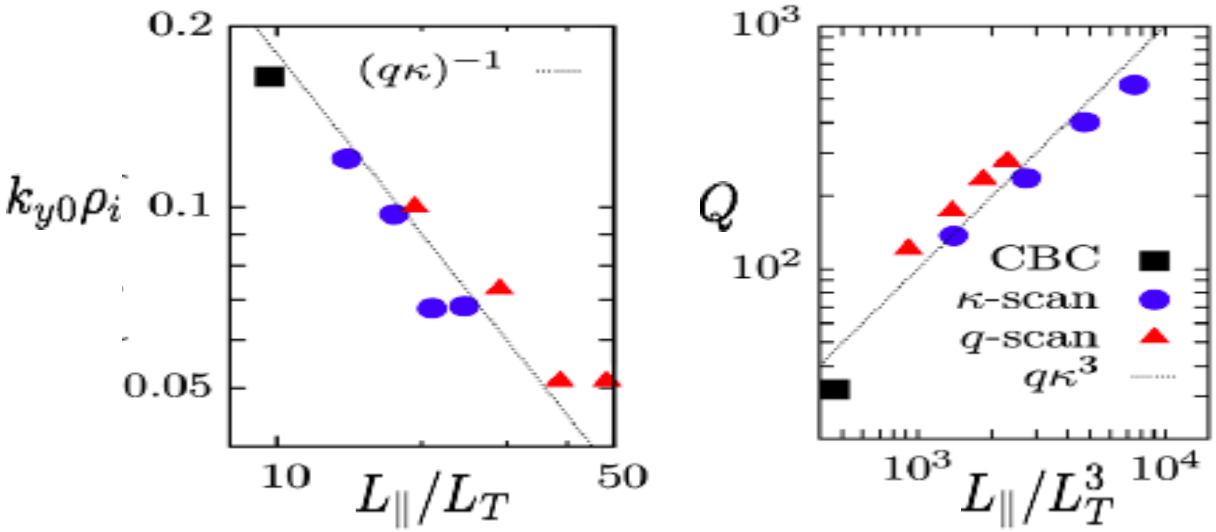
and so the heat flux is

$$\boxed{Q \sim \frac{nD_T T}{L_T} \sim \underbrace{\frac{n\rho_i^2 v_{\rm th}}{L_\parallel^2}}_{\text{gyro-Bohm}}\underbrace{\left(\frac{L_\parallel}{L_T}\right)^3}_{\text{stiff}}}$$

More formally, $Q = \langle u_x \delta T_\parallel\rangle$ and you want to bound it from above by some function of L_T.

[Barnes, Parra & AAS PRL **107**, 115003 (2011)]

"Fluid" Turbulence Theory: Outer Scale



Let us construct a turbulence theory for ITG ignoring coupling to phase space…

These scalings basically work.

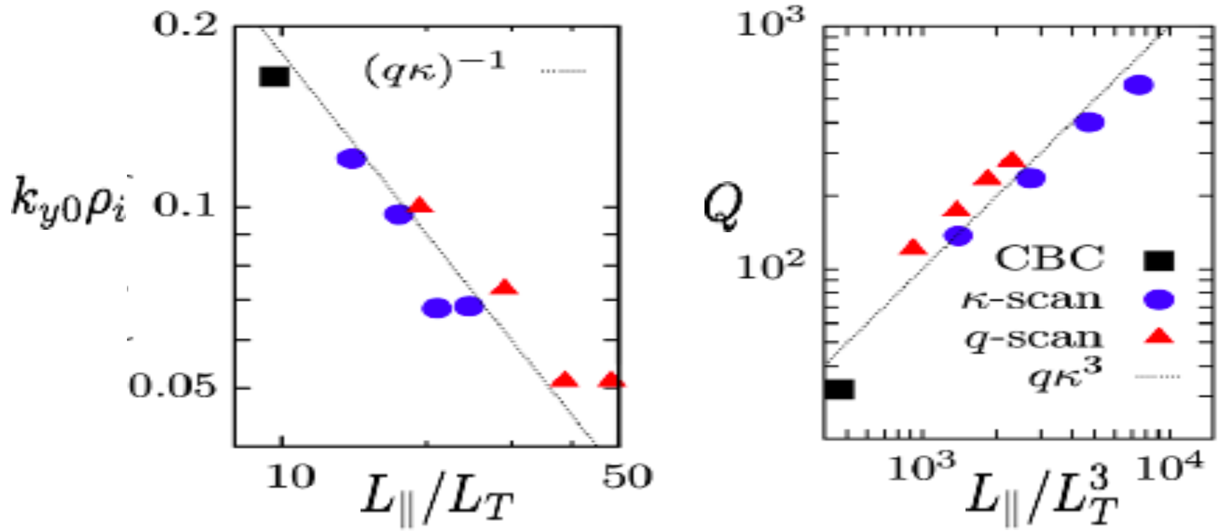
$$\boxed{k_{\perp 0}\rho_i \sim \frac{L_T}{L_\parallel}}$$

$$\boxed{\varphi_0 \sim \frac{\rho_i L_\parallel}{L_T^2}}$$

$$\boxed{Q \sim \frac{n D_T T}{L_T} \sim \underbrace{\frac{n\rho_i^2 v_{\rm th}}{L_\parallel^2}}_{\text{gyro-Bohm}} \underbrace{\left(\frac{L_\parallel}{L_T}\right)^3}_{\text{stiff}}}$$

[Barnes, Parra & AAS PRL **107**, 115003 (2011)]

"Fluid" Turbulence Theory: Inertial Range



Let us construct a turbulence theory for ITG ignoring coupling to phase space…

These scalings basically work.

So what? All this says is that leakage rate into phase space, $\sim k_{\parallel 0} v_{\rm th}$,
is at most same order as $k_{\perp 0} u_{\perp 0}$, as we indeed assumed!

A more sensitive (if less interesting to modelers) question is how free energy cascades to smaller scales…

[Barnes, Parra & AAS PRL **107**, 115003 (2011)]

"Fluid" Turbulence Theory: Inertial Range

Kolmogorov-style argument: constant flux of free energy to small scales

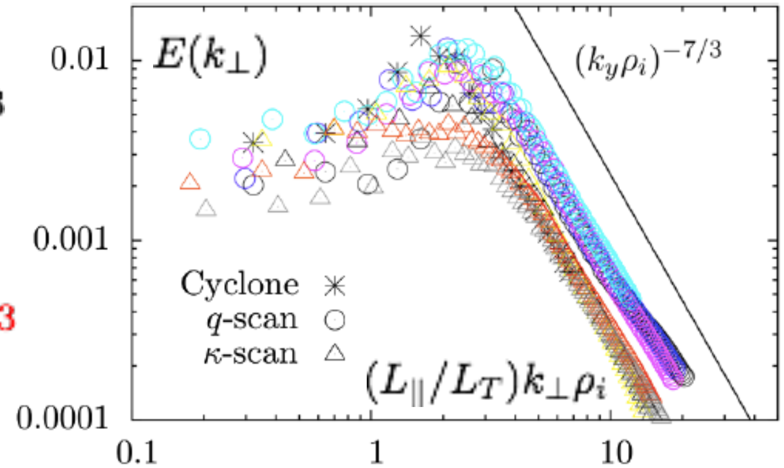
NB: assuming no damping, i.e., energy stays in "fluid" (m=0, 1, 2) moments.

Then, at $k_\perp > k_{\perp 0}$,

$$\frac{\varphi^2}{\tau_c} \sim k_\perp u_\perp \varphi^2 \sim k_\perp^2 \varphi^3 = \text{const} \;\Rightarrow\; \varphi \propto k_\perp^{-2/3}$$

The "1D spectrum":

$$E(k_\perp) = 2\pi k_\perp \int dk_\parallel \langle |\varphi_{\mathbf{k}}|^2 \rangle \sim \frac{\varphi^2}{k_\perp} \propto k_\perp^{-7/3}$$



$$W = \cancel{\sum_{m=3}^{\infty} \frac{\langle g_m^2 \rangle}{2}} + \frac{1}{4}\frac{\langle \delta T_\parallel^2 \rangle}{T^2} + \frac{\langle u_\parallel^2 \rangle}{v_{\rm th}^2} + \frac{1+\alpha}{2\alpha^2}\langle \varphi^2 \rangle$$

[Barnes, Parra & AAS PRL **107**, 115003 (2011)]

"Fluid" Turbulence Theory: Inertial Range

Kolmogorov-style argument: constant flux of free energy to small scales

NB: assuming no damping, i.e., energy stays in "fluid" (m=0, 1, 2) moments.

Then, at $k_\perp > k_{\perp 0}$,

$$\frac{\varphi^2}{\tau_c} \sim k_\perp u_\perp \varphi^2 \sim k_\perp^2 \varphi^3 = \mathrm{const} \;\Rightarrow\; \varphi \propto k_\perp^{-2/3}$$

The "1D spectrum":

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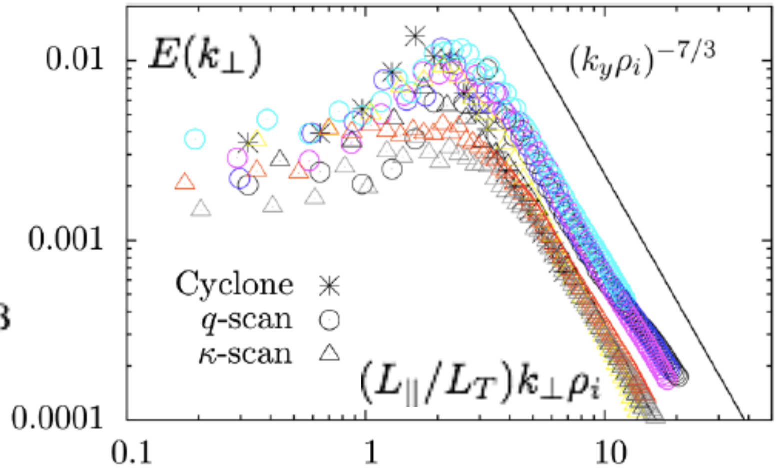
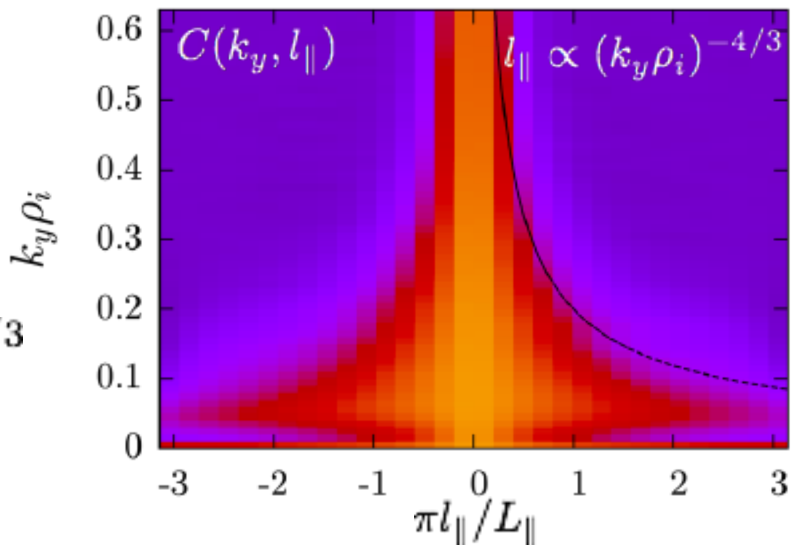


Critical balance: by causality, turbulence cannot stay correlated at parallel scales larger than those over which linear communication happens faster than nonlinear decorrelation:
so, no correlation if

$$k_\parallel v_{\rm th} < k_\perp u_\perp \propto k_\perp^{4/3} \;\Rightarrow\; k_\parallel L_\parallel < \left(\frac{k_\perp}{k_{\perp 0}}\right)^{4/3}$$



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That this works suggests there is no phase mixing in the inertial range

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That this works suggests the notional phase mixing rate $\sim k_\parallel v_{\mathrm{th}}$ is nevertheless same order as $k_\perp u_\perp$ at all scales.

So why is there no exponential cutoff of the spectrum?

[Barnes, Parra & AAS PRL **107**, 115003 (2011)]

Hermite "Cascade"

Let us go back to our kinetic equation and now ask how transfer of free energy to high *m*'s occurs <u>linearly</u>:

$$\left(\frac{\partial}{\partial t} + \cancel{\mathbf{u}_\perp \cdot \nabla_\perp}\right) g_m + v_{\rm th} \nabla_\| \left(\sqrt{\frac{m+1}{2}}\, g_{m+1} + \sqrt{\frac{m}{2}}\, g_{m-1}\right) = -\nu m g_m$$

In Fourier space: $\nabla_\| \to i k_\|, \quad \tilde{g}_m(k_\|) = (i\, \mathrm{sgn} k_\|)^m g_m(k_\|)$

$$\frac{\partial \tilde{g}_m}{\partial t} + \frac{|k_\||v_{\rm th}}{\sqrt{2}} \left(\sqrt{m+1}\, \tilde{g}_{m+1} - \sqrt{m}\, \tilde{g}_{m-1}\right) = -\nu m \tilde{g}_m$$

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$\uparrow$

this looks like a derivative: indeed,

$$= \sqrt{m}\left(\sqrt{1+\frac{1}{m}}\,\tilde{g}_{m+1} - \tilde{g}_{m-1}\right)$$

$$\approx \sqrt{m}\left(\tilde{g}_m + \frac{1}{2m} + \frac{\partial \tilde{g}_m}{\partial m} - \tilde{g}_m + \frac{\partial \tilde{g}_m}{\partial m}\right)$$

$$= 2m^{1/4}\frac{\partial}{\partial m}m^{1/4}\tilde{g}_m$$

this propagates perturbations towards higher *m*

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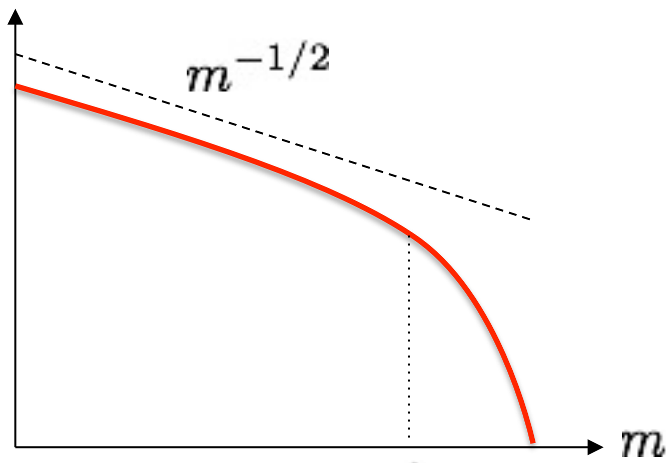
Hermite spectrum $C_m = \frac{1}{2}\langle |g_m|^2\rangle$ satisfies

$$\cancel{\frac{\partial C_m}{\partial t}} + |k_\parallel| v_{\rm th}\frac{\partial}{\partial m}\sqrt{2m}\, C_m = -2\nu m C_m$$

stationary

Hermite flux,
constant for $m \ll m_c$.

$$\boxed{C_m \propto \frac{1}{\sqrt{m}}\, e^{-(m/m_c)^{3/2}}}$$



$$m_c = \left(\frac{3|k_\parallel| v_{\rm th}}{2\sqrt{2}\,\nu}\right)^{2/3}$$

[Zocco & AAS PoP **18**, 102309 (2011)]

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So this is what Landau damping looks like in a system with some persistent energy source at low m [see detailed tutorial in Kanekar et al. JPP **81**, 305810104 (2015)]

It will dissipate collisionally all the energy that is injected, at the rate $\sim |k_\parallel| v_{\rm th}$, independent of collisionality (because the m spectrum is shallow):

$$\frac{dW}{dt} = (\text{injection}) - \sum_{k_\parallel} \sum_m \underbrace{2\nu m C_m(k_\parallel)}_{\sim \nu \int_0^{m_c} dm \sqrt{m} \sim \nu m_c^{3/2} \sim |k_\parallel| v_{\rm th}}$$

NB: W diverges as $\sim \nu^{-1/3}$

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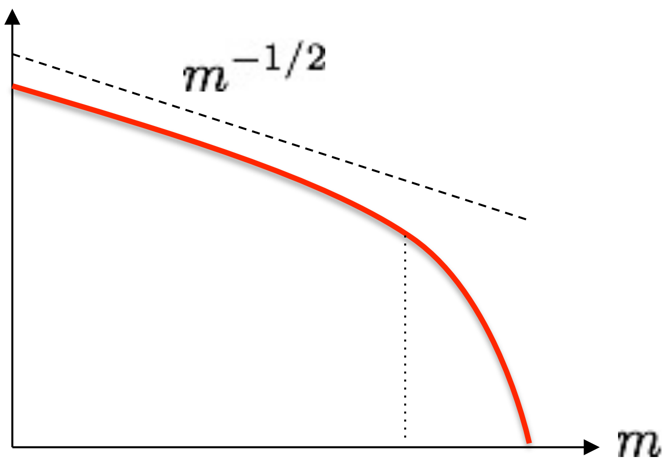

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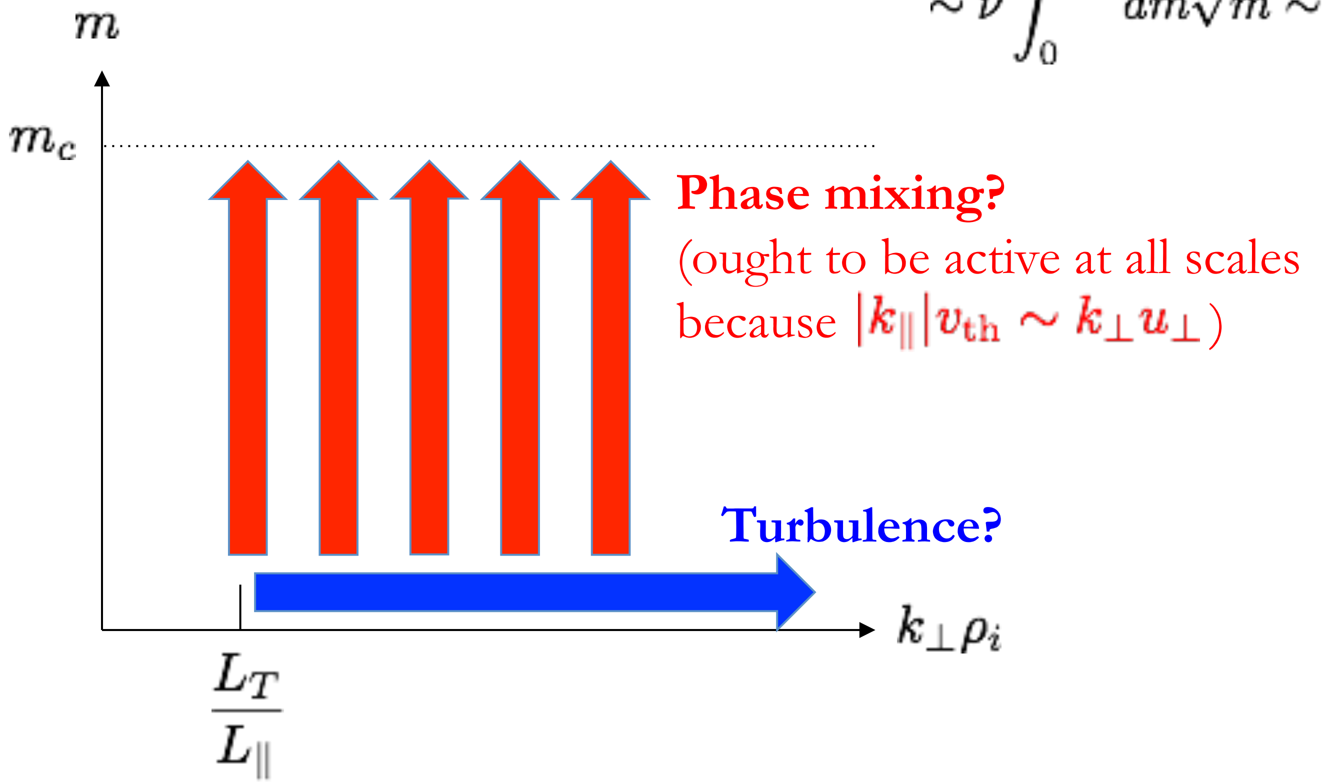

"Un-phase-mixing"

The crucial step that gave us robust phase mixing was assuming continuity in *m* space:

$$\frac{\partial \tilde{g}_m}{\partial t} + \frac{|k_\parallel| v_{\rm th}}{\sqrt{2}} \left(\sqrt{m+1}\,\tilde{g}_{m+1} - \sqrt{m}\,\tilde{g}_{m-1}\right) = -\nu m \tilde{g}_m$$

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[AAS et al., arXiv:1508.05988]

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For $1 \ll m \ll m_c$, to lowest order,

$$\sqrt{m+1}\,\tilde{g}_{m+1} - \sqrt{m}\,\tilde{g}_{m-1} = 0 \;\Rightarrow\; \tilde{g}_{m+1} \approx \tilde{g}_{m-1}$$

This allows two solutions: $\tilde{g}_{m+1} \approx \pm \tilde{g}_m$, so either $\tilde{g}_m$ or $(-1)^m \tilde{g}_m$ is continuous.

This can be encoded in the following decomposition:

$$\boxed{\tilde{g}_m = \tilde{g}_m^+ + (-1)^m \tilde{g}_m^-}$$

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[AAS et al., arXiv:1508.05988]

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(phase mixing)

propagates
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(un-phase-mixing!)

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$\tilde{g}_m^+$: propagates from low to high *m* (phase mixing)

$\tilde{g}_m^-$: propagates from high to low *m* (un-phase-mixing!)

Linearly, none of this happens because there are no energy sources at high *m* and to satisfy

$$\tilde{g}_{m\to\infty} \to 0,$$

we must have

$$\tilde{g}_m^- = 0$$

Time to bring back nonlinearity…

[AAS et al., arXiv:1508.05988]

"Un-phase-mixing"

Restore nonlinearity:

$$\left(\frac{\partial g_m}{\partial t}\right)_{\rm nl} = -[\mathbf{u}_\perp \cdot \nabla_\perp g_m](k_\parallel) = -\sum_{p_\parallel + q_\parallel = k_\parallel} \mathbf{u}_\perp(p_\parallel)\cdot\nabla_\perp g_m(q_\parallel)$$

For $\tilde{g}_m = (i\,\mathrm{sgn} k_\parallel)^m g_m$, the nonlinearity becomes

$$\left(\frac{\partial \tilde{g}_m}{\partial t}\right)_{\rm nl} = -(i\,\mathrm{sgn}\,k_\parallel)^m[\mathbf{u}_\perp \cdot \nabla_\perp g_m](k_\parallel) = -\sum_{p_\parallel + q_\parallel = k_\parallel} \mathbf{u}_\perp(p_\parallel)\cdot\nabla_\perp \frac{(i\,\mathrm{sgn}\,k_\parallel)^m}{(i\,\mathrm{sgn}\,q_\parallel)^m}\tilde{g}_m(q_\parallel)$$

And for the '+' and '–' modes (add/subtract the above for *m* and *m*+1),

$$\left(\frac{\partial \tilde{g}_m^\pm}{\partial t}\right)_{\rm nl} = -\sum_{p_\parallel + q_\parallel = k_\parallel} \mathbf{u}_\perp(p_\parallel)\cdot\nabla_\perp \left[\underset{\uparrow}{\delta^+_{k_\parallel,q_\parallel}}\tilde{g}_m^\pm(q_\parallel) + \underset{\uparrow}{\delta^-_{k_\parallel,q_\parallel}}\tilde{g}_m^\mp(q_\parallel)\right]$$

1 if $k_\parallel$ and $q_\parallel$ have same sign
0 otherwise

1 if $k_\parallel$ and $q_\parallel$ have opposite sign
0 otherwise

[AAS et al., arXiv:1508.05988]

Plasma Echo

$$\frac{\partial \tilde{g}_m^\pm}{\partial t} \pm \sqrt{2}|k_\parallel| v_{\rm th} m^{1/4} \frac{\partial}{\partial m} m^{1/4} \tilde{g}_m^\pm = -\nu m \tilde{g}_m^\pm$$

$$- \sum_{p_\parallel + q_\parallel = k_\parallel} \mathbf{u}_\perp(p_\parallel) \cdot \nabla_\perp \left[\delta^+_{k_\parallel, q_\parallel} \tilde{g}_m^\pm(q_\parallel) + \delta^-_{k_\parallel, q_\parallel} \tilde{g}_m^\mp(q_\parallel) \right]$$

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$\uparrow$ '+' and '–' modes couple!

$\tilde{g}_m^- = 0$ is no longer a solution.

Free energy can come back from phase space!

[AAS et al., arXiv:1508.05988]

[cf. Hammett et al. 1993]

Plasma Echo

A very compact form of the Hermite-space equation is achieved by defining

$$f = m^{1/4} \begin{cases} \tilde{g}_m^+ & \text{for } k_\parallel \geq 0, \\ \tilde{g}_m^- & \text{for } k_\parallel < 0 \end{cases} \quad \text{and} \quad s = \sqrt{m}$$

This is a bit like a Fourier transform, with $iv_\parallel \sim \partial_s$

$$\boxed{\frac{\partial f}{\partial t} + \frac{k_\parallel v_{\rm th}}{\sqrt{2}} \frac{\partial f}{\partial s} + \nu s^2 f = -\sum_{p_\parallel} \mathbf{u}_\perp(p_\parallel) \cdot \nabla_\perp f(k_\parallel - p_\parallel)}$$

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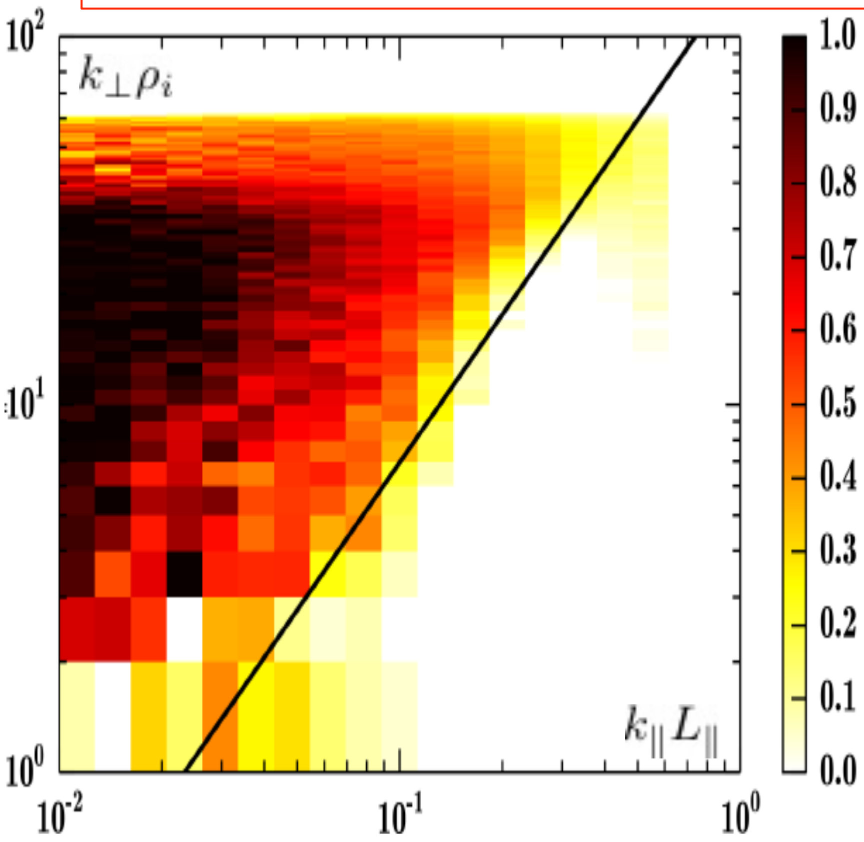

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…And it does indeed do it! Here are some plots of the relative Hermite flux

$$\frac{C_m^+ - C_m^-}{C_m^+ + C_m^-}$$

from a drift-kinetic slab ITG simulation by **J. Parker & E. Highcock**

Phase Mixing vs. Turbulence

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The perturbation at low *s*, $f(s \sim 1) \sim \varphi$, and at some fixed $k_\perp$ and $k_\parallel$, will propagate to higher *s* along the characteristic:

$$s \sim k_\parallel v_{\rm th} t,$$

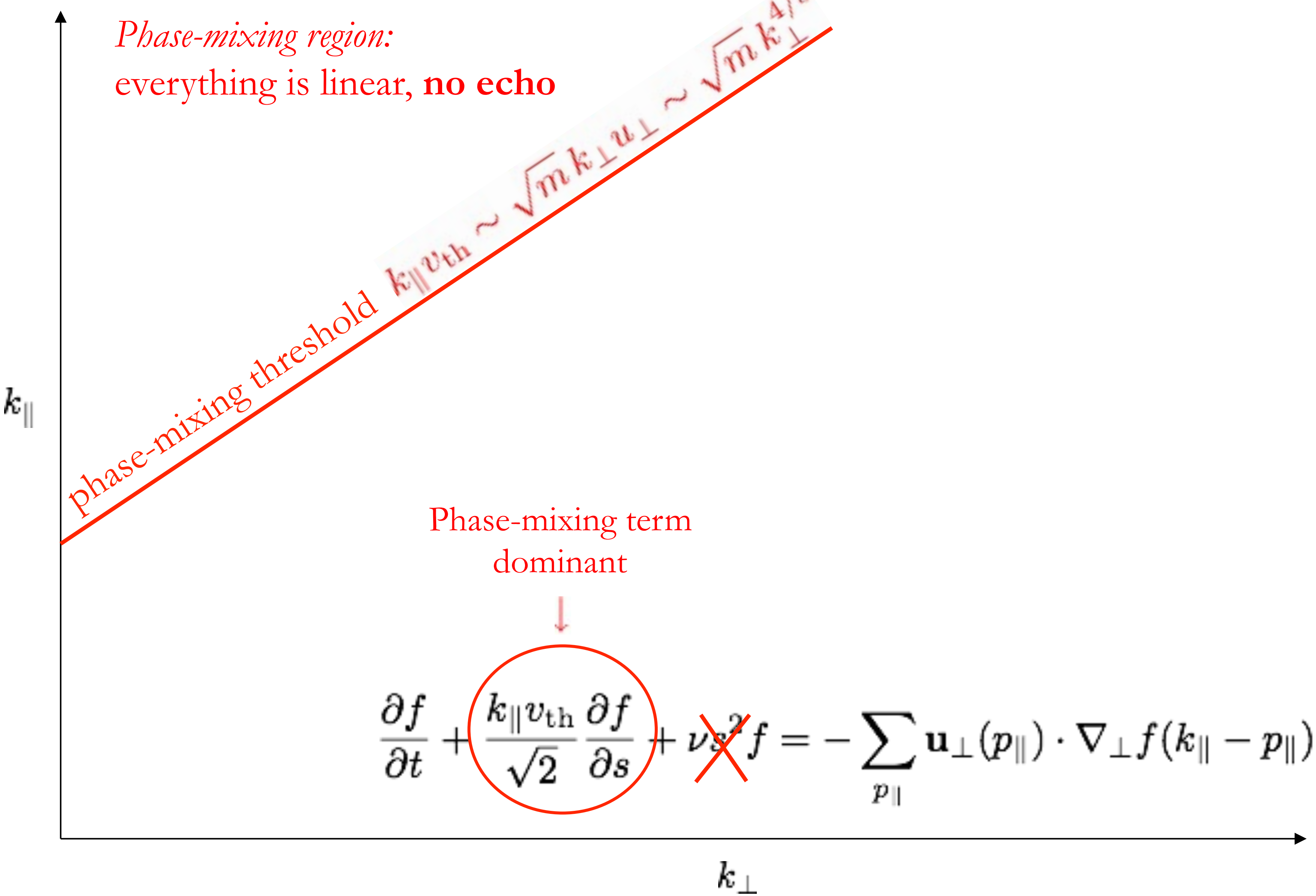
until it is swept by nonlinear advection ($\mathbf{u}_\perp$) to higher $k_\perp$ in one nonlinear time,

$$t \sim (k_\perp u_\perp)^{-1} \propto k_\perp^{-4/3}.$$

Thus, $f(s) \sim \varphi$ for $s \lesssim \dfrac{k_\parallel v_{\rm th}}{k_\perp u_\perp}$, or, equivalently, for the phase-space spectrum:

$$\boxed{E_m(k_\perp, k_\parallel) = 2\pi k_\perp \langle |g_m|^2 \rangle \sim \frac{E_\varphi(k_\perp, k_\parallel)}{\sqrt{m}} \quad \text{for } k_\parallel \gtrsim \frac{k_\perp u_\perp}{v_{\rm th}} \sqrt{m} \propto k_\perp^{4/3} \sqrt{m}}$$

Phase-Space Spectra



[AAS et al., arXiv:1508.05988]

Phase-Space Spectra

Phase-mixing region:
everything is linear, **no echo**

$$E_m^+ \sim \frac{k_\perp^3 k_\parallel^{-5}}{\sqrt{m}}$$

phase-mixing threshold $k_\parallel v_{\rm th} \sim \sqrt{m}\, k_\perp u_\perp \sim \sqrt{m}\, k_\perp^{4/3}$

✓ By pure kinematics of correlation functions, in 2D,

$$E_m \sim \text{const}\, k_\perp^3 + \dots \quad \text{as } k_\perp \to 0$$

✓ Parallel exponent fixed by **matching** at the phase-mixing threshold

$k_\parallel$

Phase-mixing term dominant

↓

$$\frac{\partial f}{\partial t} + \frac{k_\parallel v_{\rm th}}{\sqrt{2}}\frac{\partial f}{\partial s} + \cancel{\nu s^2} f = -\sum_{p_\parallel} \mathbf{u}_\perp(p_\parallel)\cdot\nabla_\perp f(k_\parallel - p_\parallel)$$

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[AAS et al., arXiv:1508.05988]

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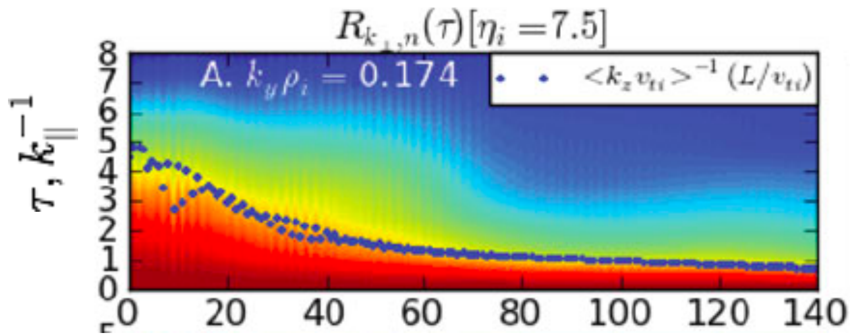
phase-mixing threshold $k_\parallel v_{\rm th} \sim \sqrt{m}\, k_\perp u_\perp \sim \sqrt{m}\, k_\perp^{4/3}$

$k_\parallel$

✓ By pure kinematics of correlation functions, in 2D,

$$E_m \sim \text{const}\, k_\perp^3 + \dots \quad \text{as } k_\perp \to 0$$

✓ Parallel exponent fixed by **matching** at the phase-mixing threshold



"phase-space critical balance"
found numerically
by Hatch et al. JPP **80**, 531 (2014)

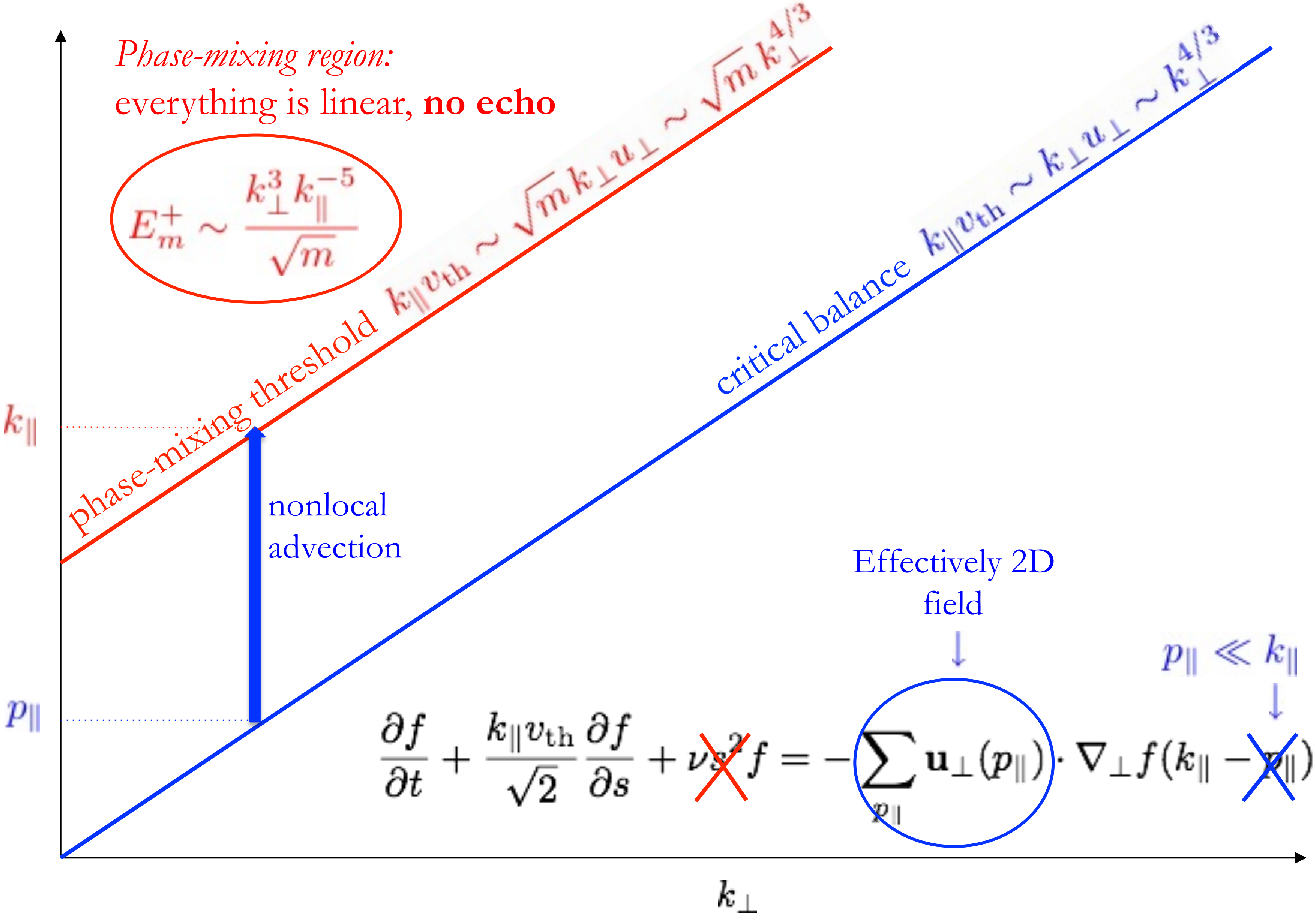
Phase-mixing term dominant
↓

$$\frac{\partial f}{\partial t} + \frac{k_\parallel v_{\rm th}}{\sqrt{2}}\frac{\partial f}{\partial s} + \nu \cancel{s^2} f = -\sum_{p_\parallel} \mathbf{u}_\perp(p_\parallel)\cdot\nabla_\perp f(k_\parallel - p_\parallel)$$

$k_\perp$

[AAS et al., arXiv:1508.05988]

Phase-Space Spectra



[AAS et al., arXiv:1508.05988]

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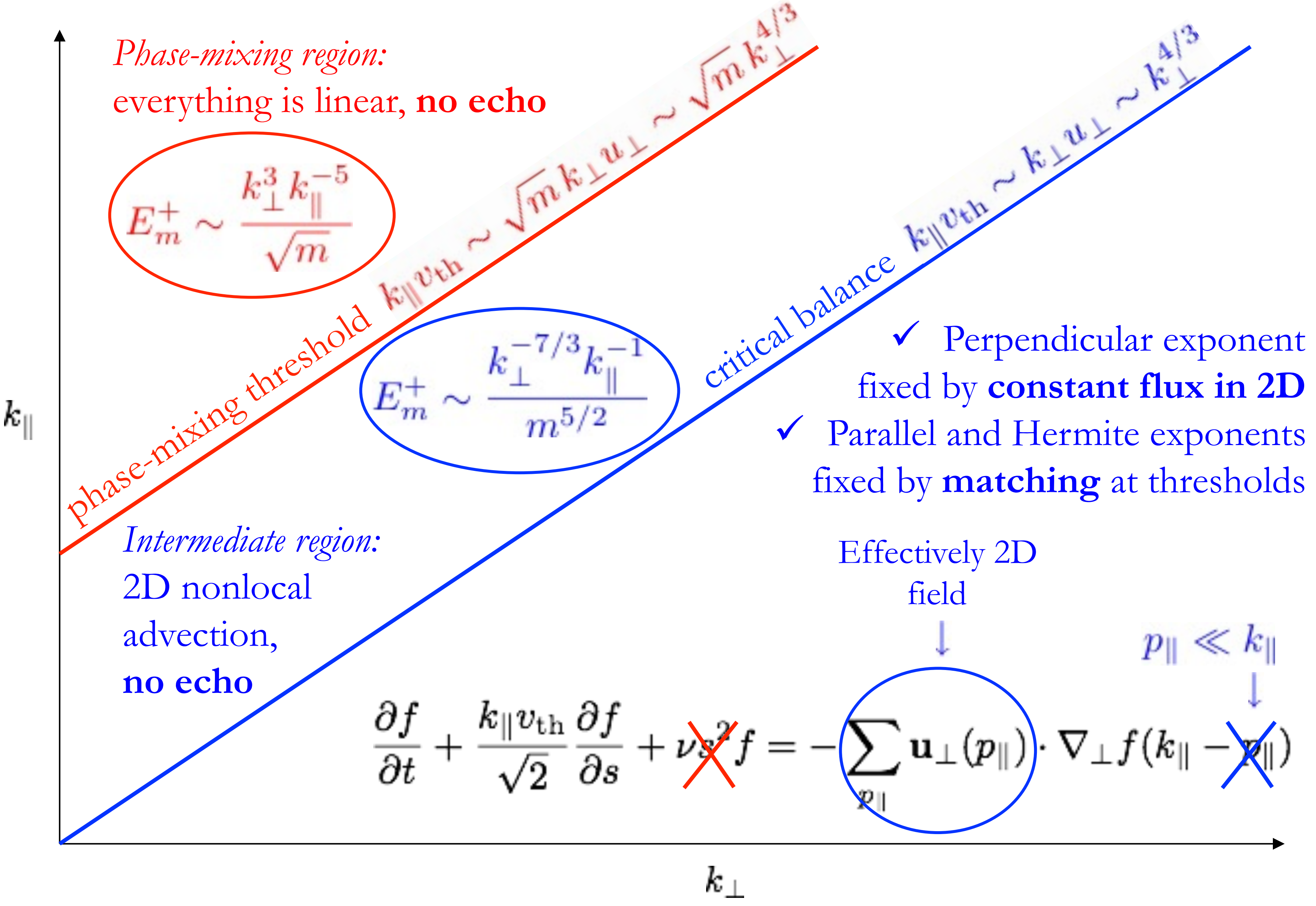

[AAS et al., arXiv:1508.05988]

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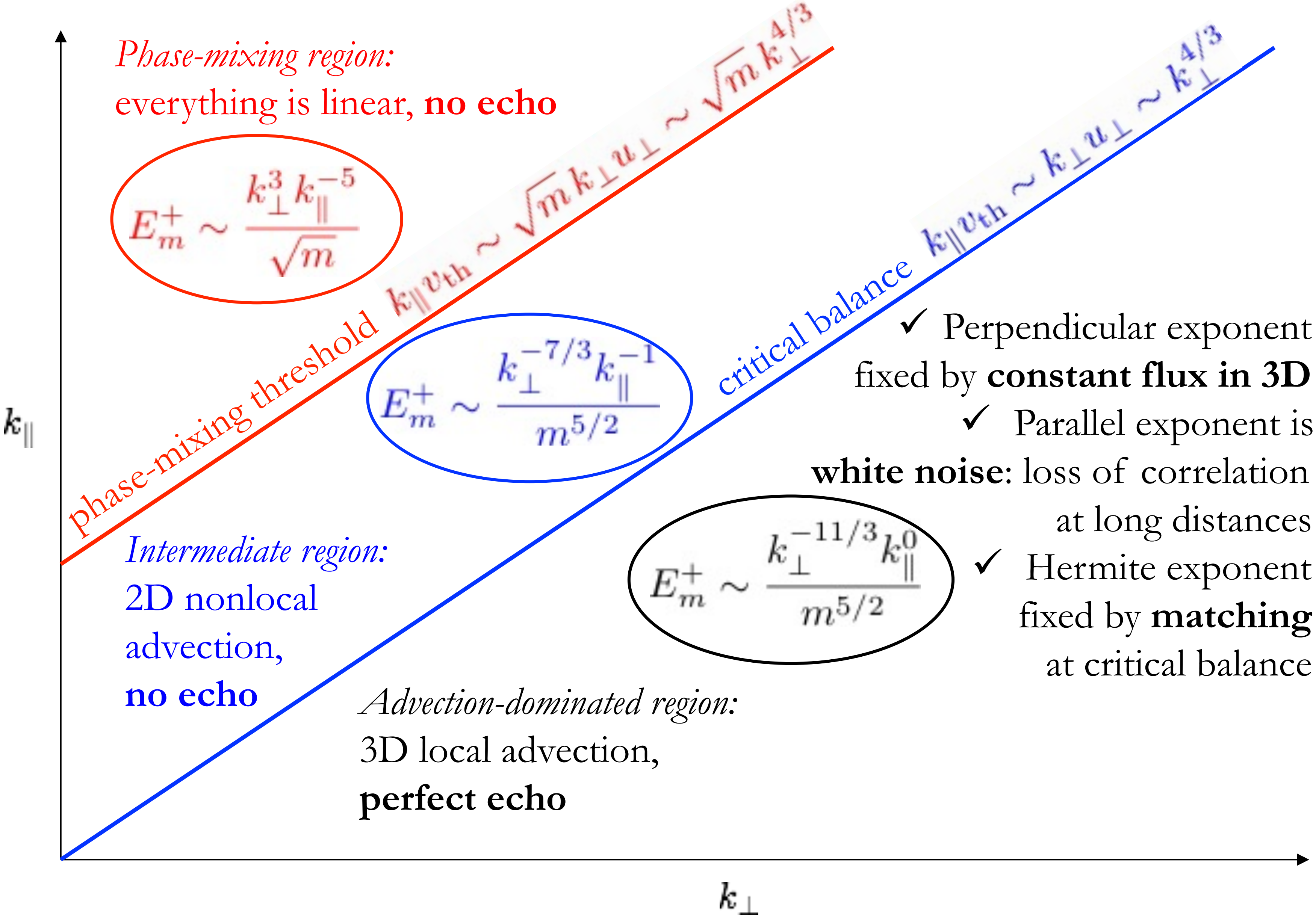

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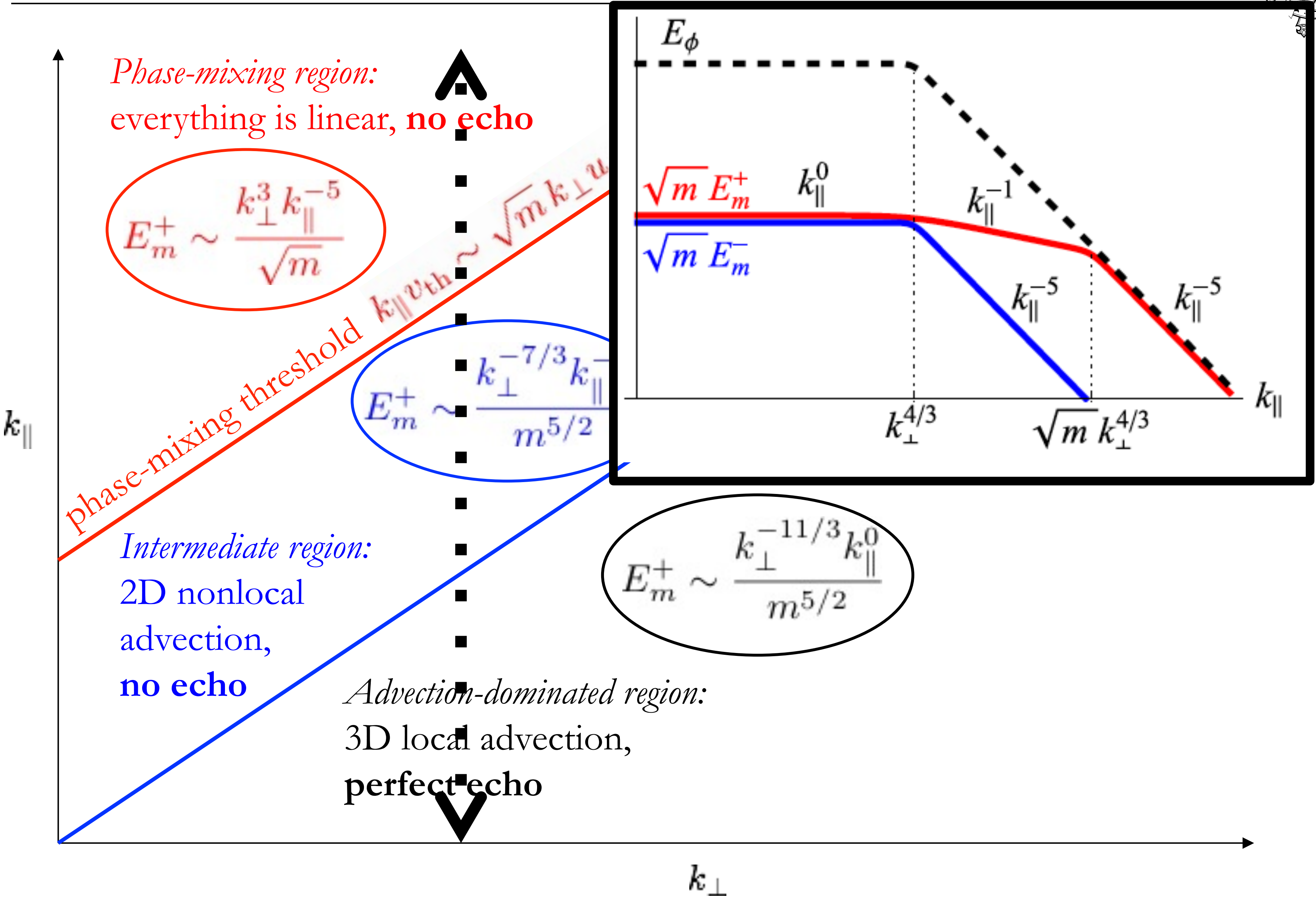

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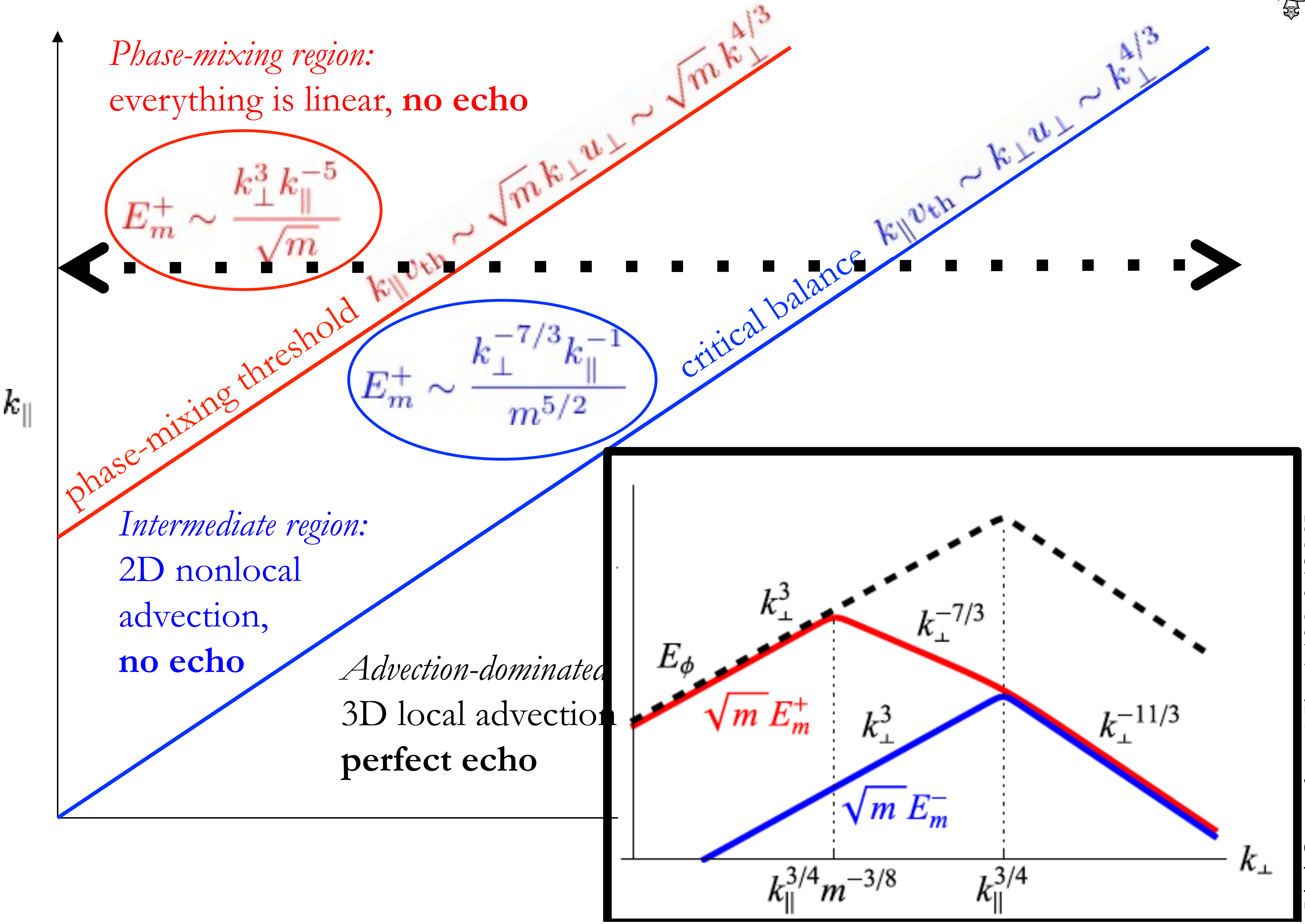

[AAS et al., arXiv:1508.05988]

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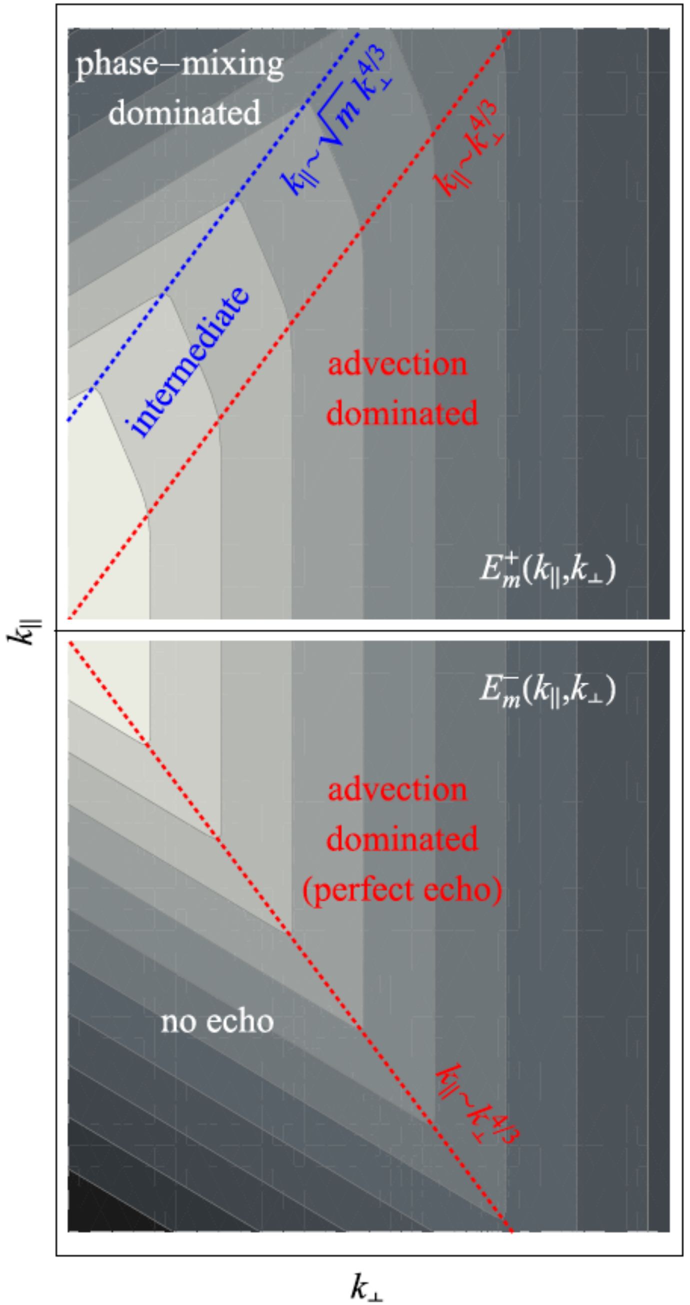

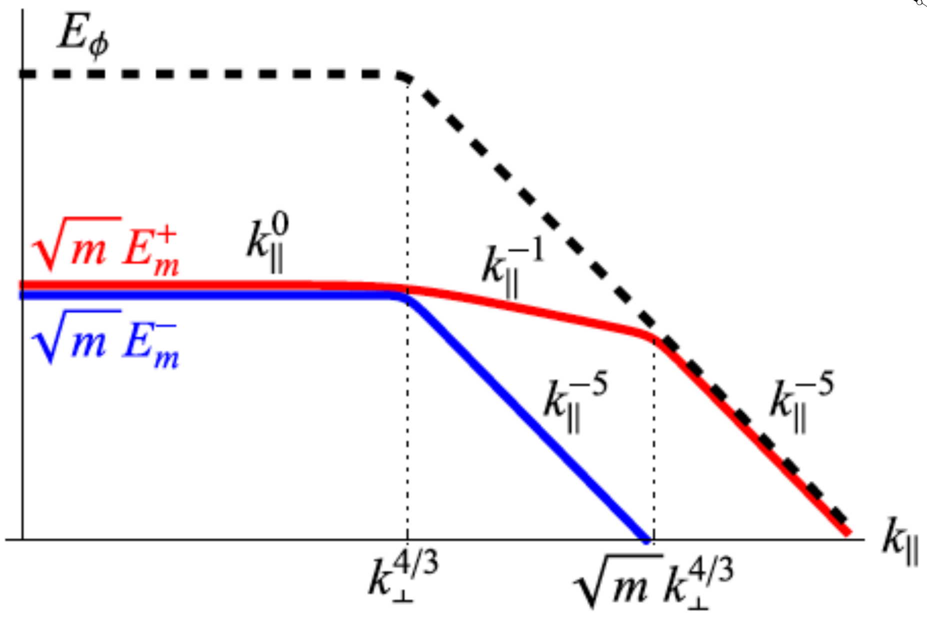

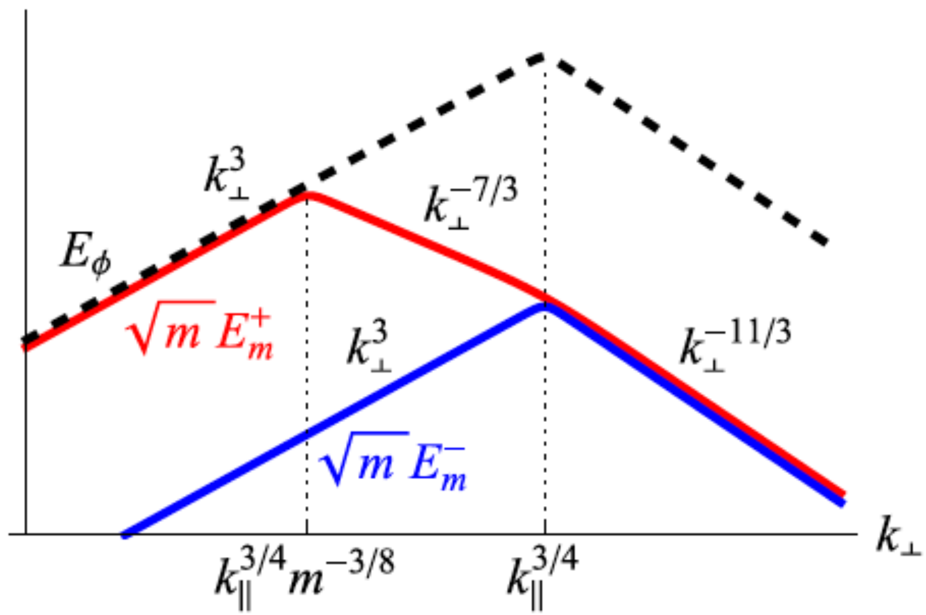

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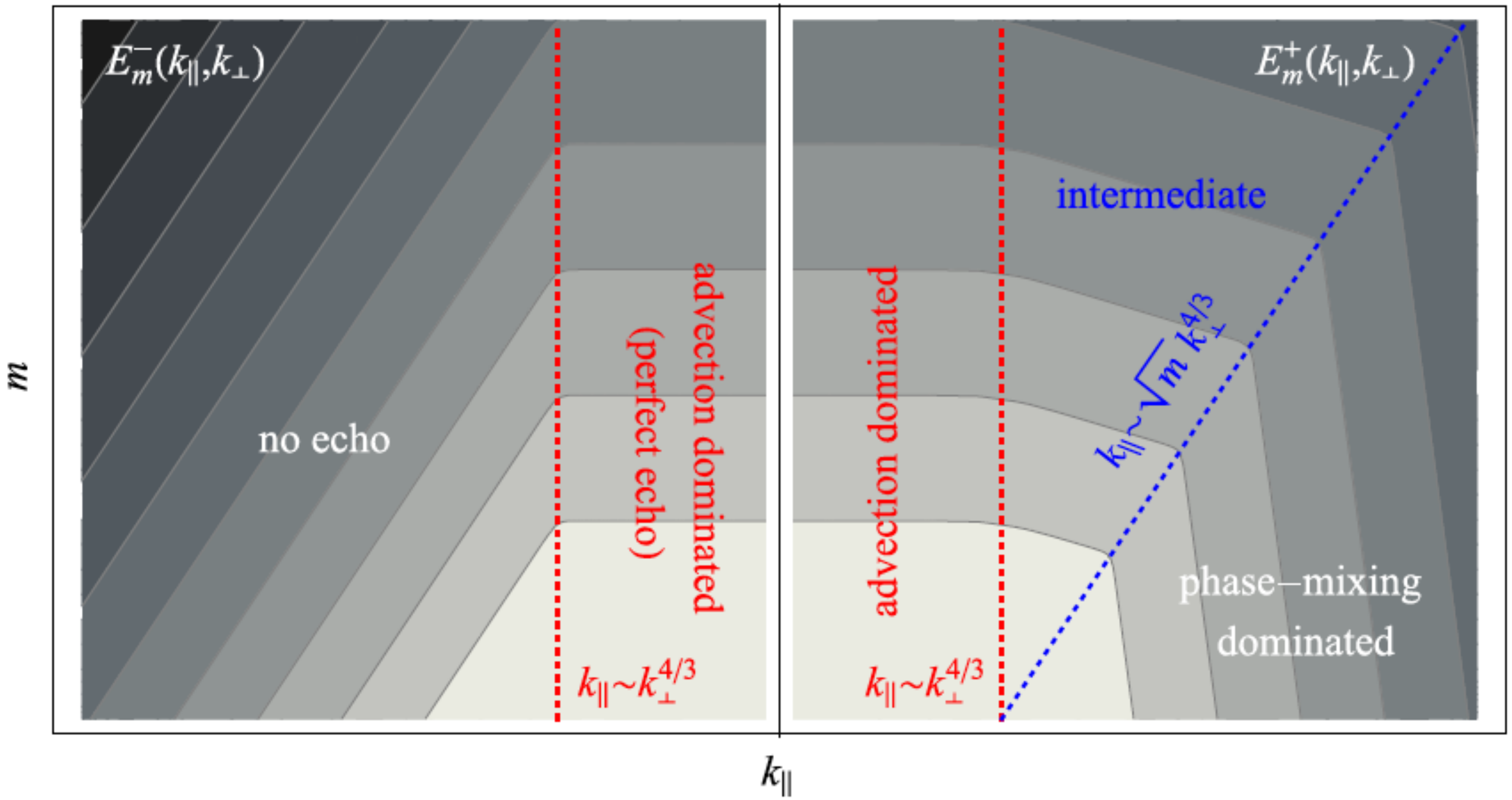

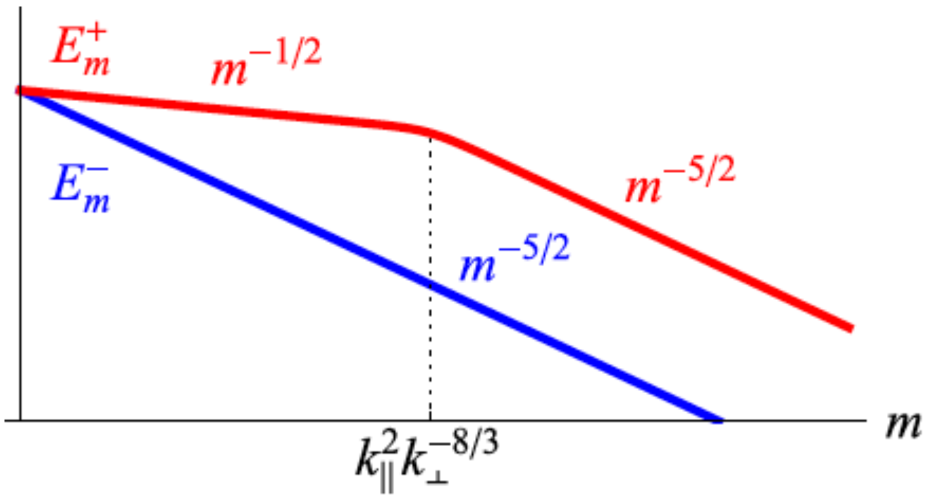

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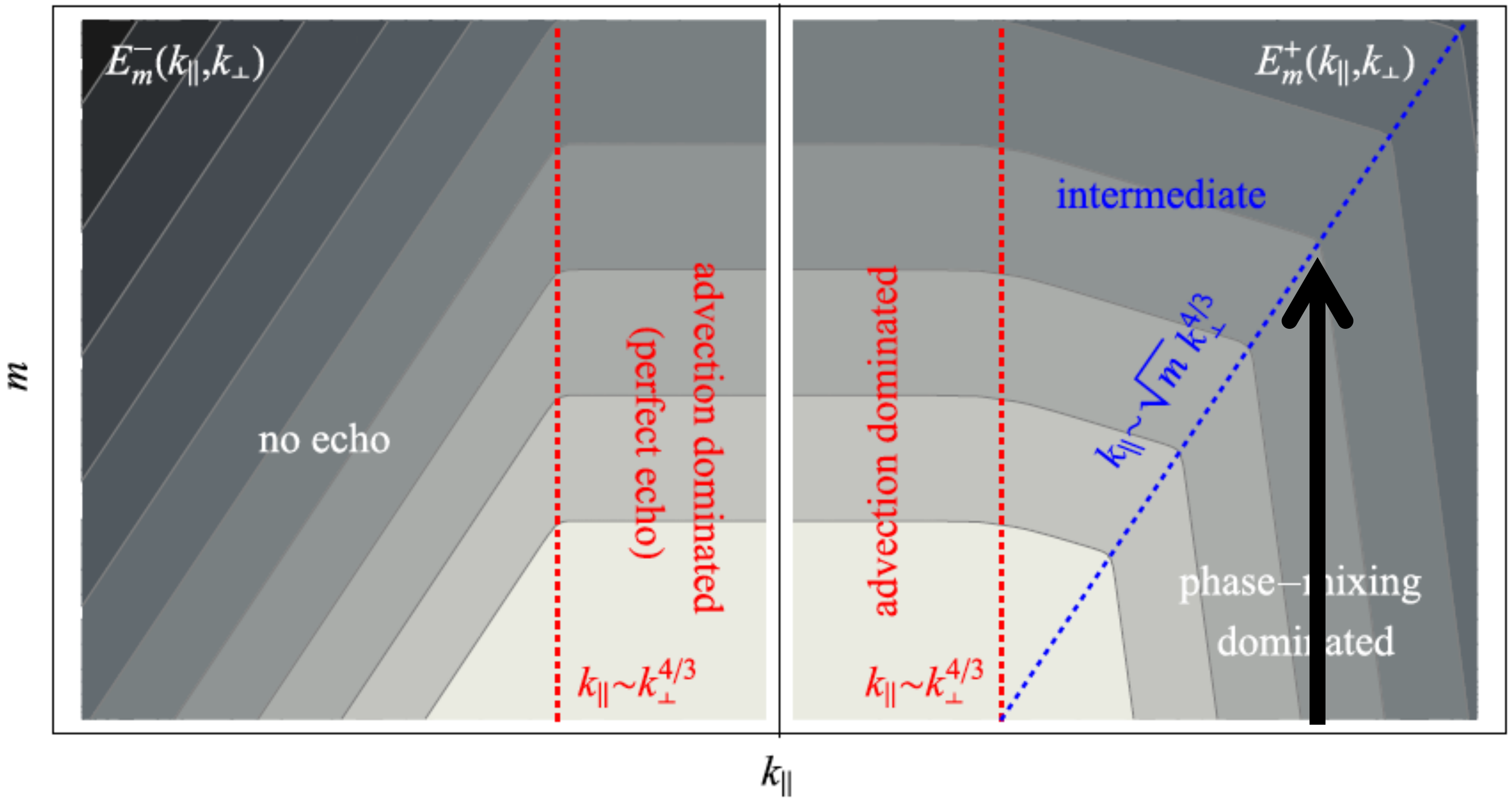
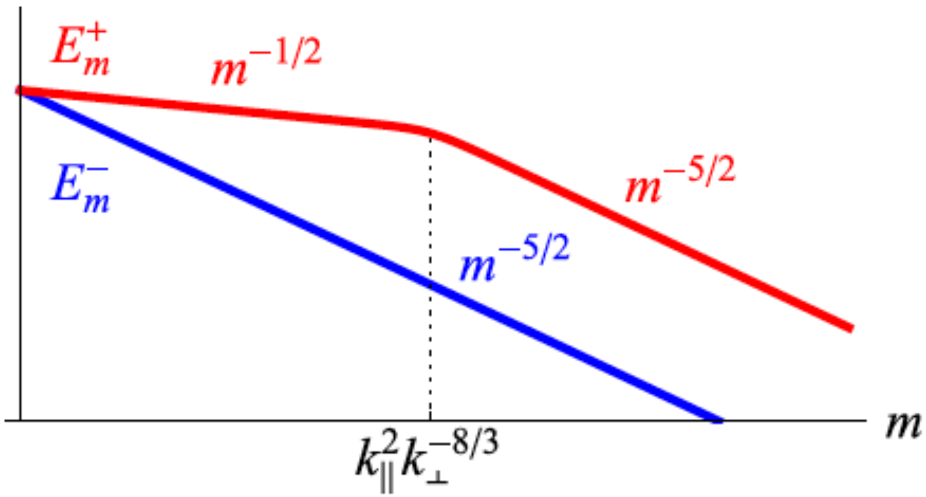
Energy Flows





[AAS et al., arXiv:1508.05988]

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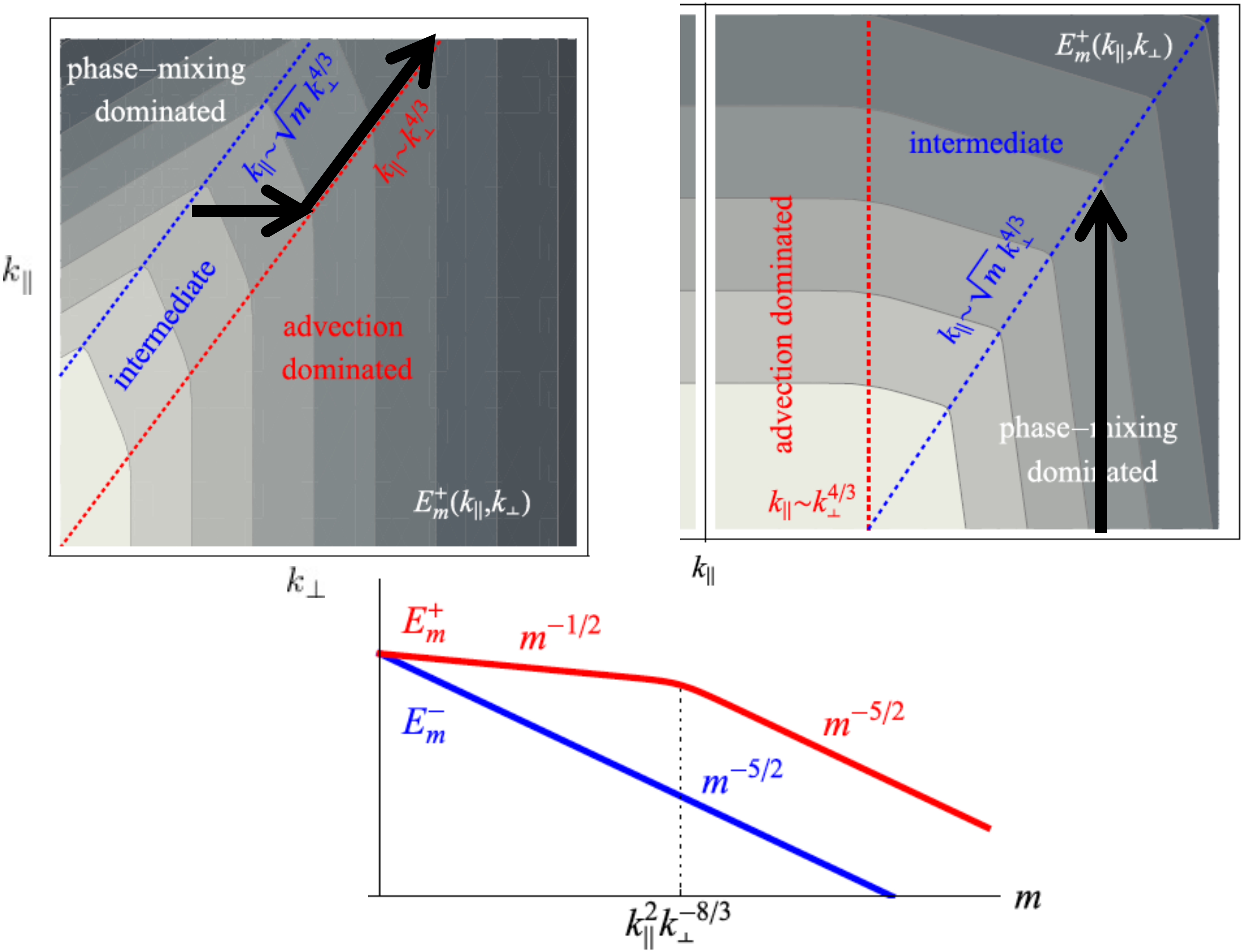


[AAS et al., arXiv:1508.05988]

Conclusions

✓ At $k_\parallel \gtrsim k_\perp^{4/3}\sqrt{m}$, linear phase mixing dominates, $E_m \propto \frac{1}{\sqrt{m}}$,
but there is very little energy $(\sim k_\parallel^{-5})$

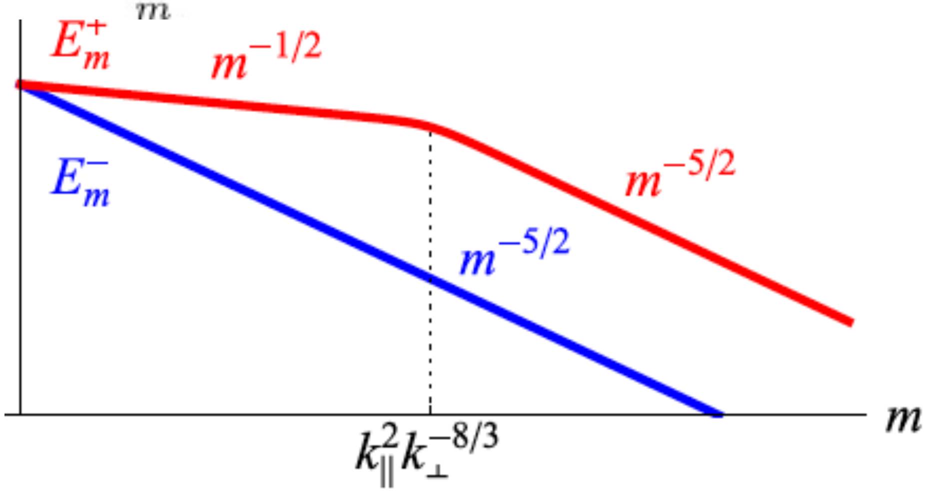
✓ At $k_\parallel \lesssim k_\perp^{4/3}\sqrt{m}$, nonlinear mixing (turbulence) dominates, $E_m \propto \frac{1}{m^{5/2}}$,
most energy is there, but collisional dissipation $\to 0$ as $\nu \to 0$;
total free energy stored in phase space is finite and independent of collisionality

$$\boxed{\sum_m E_m \to \text{const}, \quad \nu \sum_m m E_m \to 0 \text{ as } \nu \to +0}$$

<u>*This means spatial mixing ("turbulence") wins over phase mixing*</u>

In contrast, in the linear problem,

$$\sum_m E_m \sim \nu^{-1/3}, \quad \nu \sum_m m E_m \to \text{const as } \nu \to +0$$



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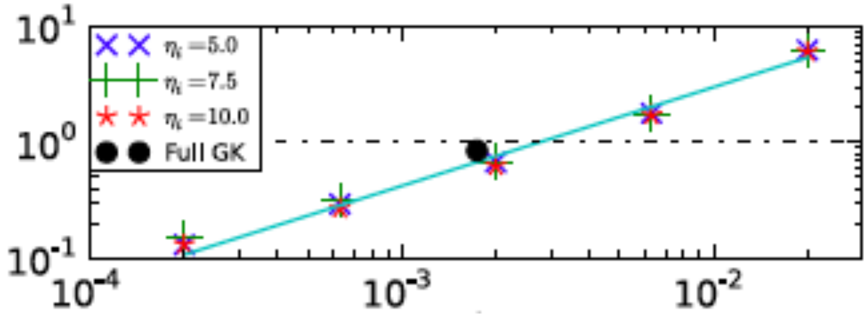
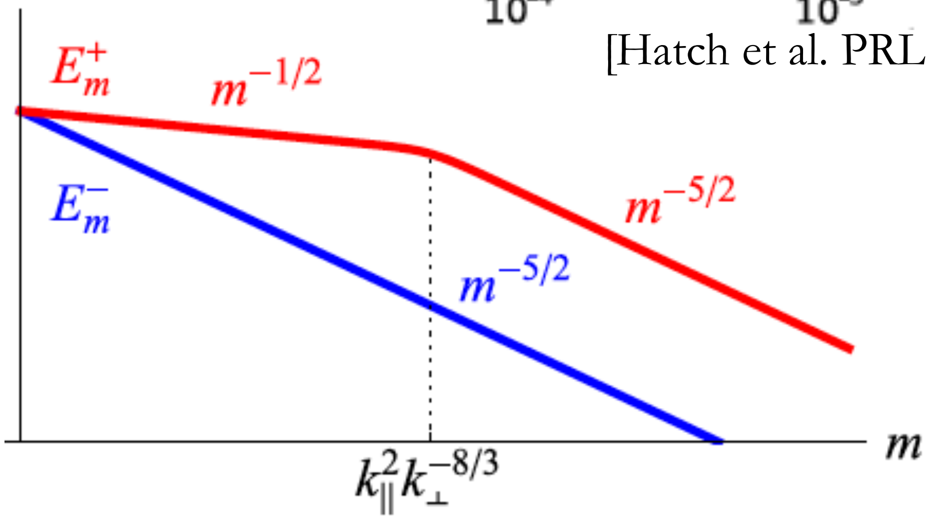


[Hatch et al. PRL **111**, 175001 (2013)]



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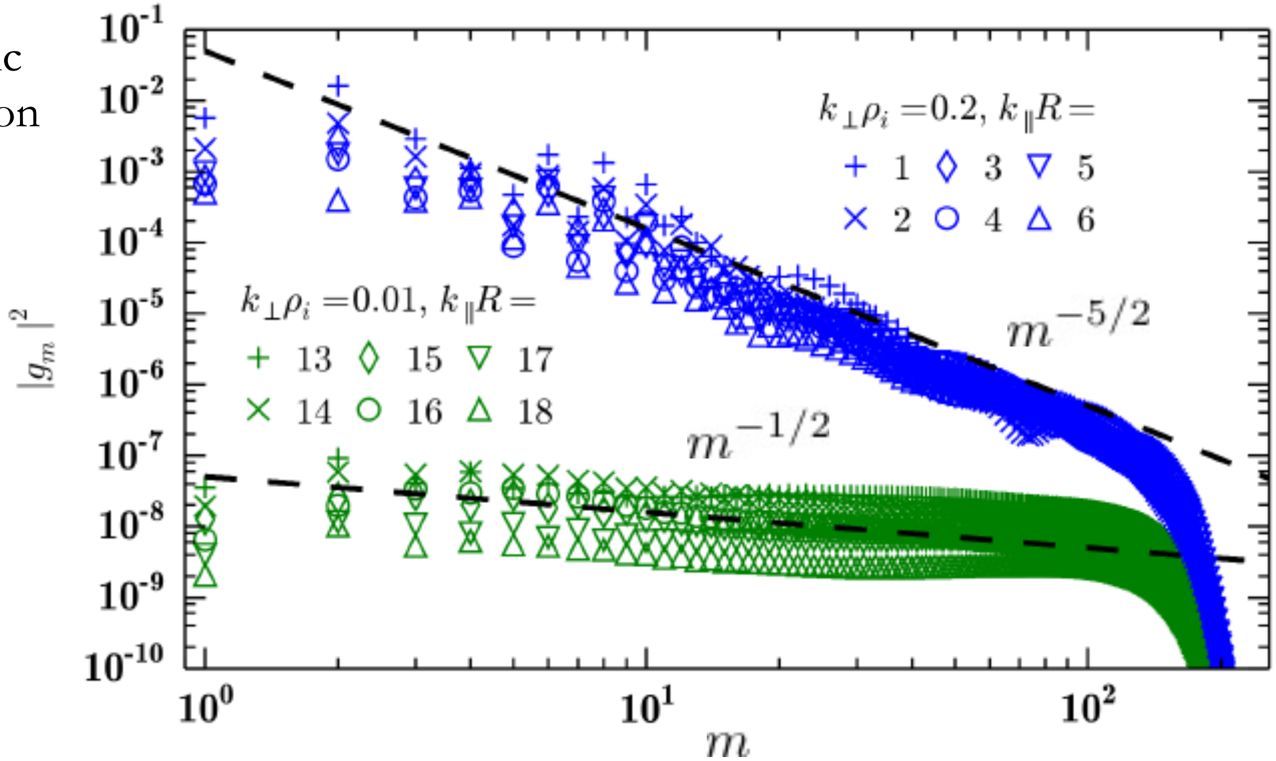
from a drift-kinetic slab ITG simulation by **J. Parker & E. Highcock:**



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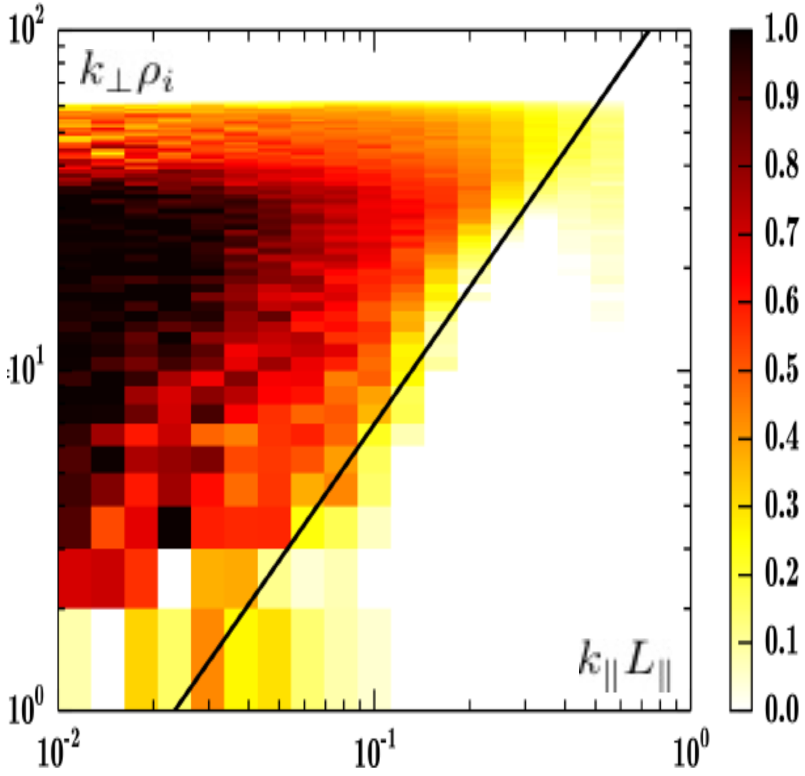

✓ Return echo flux cancels phase-mixing flux at $k_\| \lesssim k_\perp^{4/3}$
(below critical balance);
turbulent cascade of low Hermite moments is effectively fluid…
we might say that, for the purposes of free-energy accounting in turbulence,
"*Landau damping is suppressed*"

[AAS et al., arXiv:1508.05988]

NB

All of the arguments presented above rely on the approximation of

$$m \gg 1 \text{ and, indeed, } m^{1/4} \gg 1\,,$$

i.e., truly asymptotically small collisionality (= a lot of velocity-space structure).

In reality (experimental and certainly numerical), the collisionality or effective collisionality (in codes) is rarely truly small. When it is moderate and only relatively little Hermite space is available to the free energy, processes that require such space – most notably the echo flux – are likely to be less pronounced.
This probably accounts for how well Landau fluid closures have tended to capture quantitative behaviour of turbulence in tokamaks.

So perhaps (perhaps!) the scenario is

$\nu \gg \omega$ – collisional system, fluid

$\nu \lesssim \omega$ – weakly collisional system, "Landau-fluid"

$\nu \ll \omega$ – "collisionless" system, like fluid again?

In any event, this is quite a cute problem to think about…

[AAS et al., arXiv:1508.05988]