Math 10 Review

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This review is a summary of the material studied in the course. It should only be considered as an addition to the book and to classes, and should not replace them.

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1 Applications of Integration

1.1 Area Between Curves

Suppose that the curve of a function $f(x)$ lies above the curve of another function $g(x)$ between two points a and b. Then the area confined between these two curves and the horizontal lines $x = a, x = b$ is

$$A = \int_a^b (f(x) - g(x))\, dx.$$

1.2 Volumes

1.2.1 Some Intuition

Recall, that for "nice" shapes, we have simple formulas to calculate volume:

- The volume of a rectangular box of dimensions a, b, c is given by $V = (a \times b) \times c$.
- The volume of a circular cylinder with radius r and height h is given by $V = \pi r^2 \times h$.

The recurring idea is that volume is given by *base area* multiplied by *height*. Thus, for any shape for which we may be required to compute the volume, we will attempt to cleverly approximate this shape by basic shapes.

The corresponding formula that we get is the following: suppose $A(x)$ is the area of a cross-section of our object as we progress along the x-axis. Then we may split the object into small "nice" shapes, with base $A(x)$ and height Δx. By taking the limit $\Delta x \longrightarrow 0$, we get the formula

$$V = \int_a^b A(x) dx,$$

if the object lies between the x-axis values of a and b.

1.2.2 Solids of Revolution

When dealing with some nice, symmetrical objects, we may notice that they are the result of rotating some curve all the way around some axis, to form a **solid of revolution**. For such an object, the formula for $A(x)$ is actually simple: the revolution means that at each cross-section, we get a circle of radius $f(x)$, so that

$$A(x) = \pi (f(x))^2.$$

1.3 Volumes by Cylindrical Shells

An alternative method for calculating the volume of a solid of revolution, is by summing spherical shells that are centered around the axis of revolution. Suppose the axis of revolution is the y-axis. Then each shell will be upright, the base varying with x, and the height depending on the curve, $f(x)$. The area of this shell will be

$$A(x) = 2\pi x\, f(x).$$

1.4 Summary: The Four (Actually Two) Different Methods to Find Volume

In this section we consider a function which may be presented as either $y = f(x)$ or $x = g(y)$, with bounds a, b and c, d respectively.

Example 1. As a simple example, consider the well known function $y = x^2$. Suppose our bounds on the x-axis are $a = 3$ and $b = 4$.

Our function $x = g(y)$ will be $x = \sqrt{y}$, and the bounds will be $c = 9$ and $d = 16$.

1.4.1 Rotating about the y-axis

I. Discs. When rotating about the y-axis, we get an object that is symmetric with respect to the y-axis. Thus, it would make sense to split it into discs "sitting" on each other. Thus the formula we get is

$$V = \int_c^d \pi(g(y))^2 dy.$$

II. Shells. If splitting into shells, the shells will be expanding as we advance along the x-axis, so that the resulting formula would be

$$V = \int_a^b 2\pi x\, f(x)\, dx.$$

1.4.2 Rotating about the x-axis

III. Discs. With similar considerations, we get

$$V = \int_a^b \pi(f(x))^2\, dx.$$

IV. Shells.

$$V = \int_c^d 2\pi y\, g(y)\, dy.$$

1.5 Average Value of a Function

The integral $\int_a^b f(x)dx$ measures the area under the curve $f(x)$ between the x values a and b. Now, recall that the area of a rectangle is given by *length* $\times$ *width*. Consider a rectangle whose base is $b - a$, laying on the x-axis between a and b. Now, suppose you want this rectangle to have the same area as under the curve $f(x)$ between a and b. What would have to be the length of the other side of the rectangle? It would have to be

$$\frac{1}{b-a}\int_a^b f(x)\, dx.$$

(Think about this). This number is called **the average value of $\boldsymbol{f(x)}$ between $\boldsymbol{a}$ and $\boldsymbol{b}$**, and is notated $\boldsymbol{f_{\mathbf{ave}}}$.

Theorem 2. (The Mean Value Theorem for Integrals) *If f is continuous on $[a, b]$, then there exists a number c between a and b, such that*

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\, dx.$$

If you draw an example of a continuous function f, and a rectangle as described above, you will see that all this theorem says, is that the upper side of the rectangle intersects the curve of the graph (at least) once.

2 Integration Methods

2.1 Integration by Parts

Recall the *product rule*, also known as *Leibniz's rule*,

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

Integrating this expression, we get

$$f(x)g(x) = \int f'(x)g(x)dx + \int f(x)g'(x)dx,$$

which we can rewrite as

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx.$$

If we now rename $u = f(x)$ and $v = g(x)$, then we have $du = f'(x)dx$ and $dv = g'(x)dx$ and we may write

$$\int u dv = uv - \int v du.$$

Example 3. Solve the integral $\int x\cos(x)dx$.

Solution. First let us attempt a "wrong" solution:

Let

$$\begin{array}{ll} u = \cos(x) & dv = x\,dx \\ du = -\sin(x)\,dx & v = \frac{1}{2}x^2 \end{array}$$

so that we get

$$\int x\cos(x)dx = \cos(x)\frac{1}{2}x^2 - \int \frac{1}{2}x^2(-\sin(x))\,dx.$$

So now we have to solve the integral $\int \frac{1}{2}x^2\sin(x)\,dx$, which is harder than the integral we first started with.

Thus, the "right" solution would be selecting

$$\begin{array}{ll} u = x & dv = \cos(x)dx \\ du = dx & v = \sin(x) \end{array}$$

to get

$$\begin{aligned} \int x\cos(x)dx &= x\sin(x) - \int \sin(x)dx \\ &= x\sin(x) + \cos(x) + C. \end{aligned}$$

2.2 Trigonometric Integrals

With some integrals involving the trigonometric functions, we utilize some of the identities that tie these functions with one another, and the fact that differentiating a trigonometric function gives us another trigonometric function (or sometimes two).

2.2.1 Main identities

1. $\sin 2x = 2\sin x \cos x$	2. $\cos 2x = \cos^2 x - \sin^2 x$
3. $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$	4. $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
5. $\sin^2 x + \cos^2 x = 1$	
6. $\sec^2 x = 1 + \tan^2 x$	7. $\operatorname{cosec}^2 x = 1 + \cot^2 x$

2.2.2 Derivatives

8. $\sin' x = \cos x$	9. $\cos' x = -\sin x$
10. $\tan' x = \sec^2 x$	11. $\sec' x = \sec x \tan x$
12. $\cot' x = -\operatorname{cosec}^2 x$	13. $\operatorname{cosec}' x = -\operatorname{cosec} x \cot x$

2.2.3 Examples

Example 4. Solve the integral

$$\int \tan^5 x\, dx$$

Solution. We will solve this problem in two methods.

1.

$$\begin{aligned}
\int \tan^5 x\, dx &= \int \tan^5 x \frac{\sec^2 x}{\sec^2 x}\, dx \\
&= \int \frac{\tan^5 x}{1 + \tan^2 x} \sec^2 x\, dx \\
&= \int \frac{\tan^5 x}{1 + \tan^2 x}\, d(\tan x) \\
&= \int \frac{u^5}{1 + u^2}\, du.
\end{aligned}$$

This expression may be solved using the methods of the proceeding section.

2.

$$\begin{aligned}
\int \tan^5 x\, dx &= \int \tan^4 x \tan x \frac{\sec x}{\sec x}\, dx \\
&= \int \frac{\left(\tan^2 x\right)^2}{\sec x} \tan x \sec x\, dx \\
&= \int \frac{\left(\sec^2 x - 1\right)^2}{\sec x}\, d(\sec x) \\
&= \int \frac{\left(u^2 - 1\right)^2}{u}\, du \\
&= \int \frac{u^4 - 2u^2 + 1}{u}\, du \\
&= \int u^3\, du - 2\int u\, du + \int \frac{du}{u} \\
&= \frac{1}{4}u^4 - u^2 + \ln|u| + C \\
&= \frac{1}{4}\sec^4 x - \sec^2 x + \ln|\sec x| + C
\end{aligned}$$

Notice that this problem has an incorrect answer in the book.

2.3 Trigonometric Substitution

With some types of integrals, it actually makes the integration easier if we substitute x for a more complex function. Here's a list of substitutions: (look in the book for some restrictions on these)

Expression	Substitution	Identity Used	Derivative Used
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$1 - \sin^2\theta = \cos^2\theta$	$dx = a\cos\theta\, d\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$1 + \tan^2\theta = \sec^2\theta$	$dx = a\sec^2\theta\, d\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$\sec^2\theta - 1 = \tan^2\theta$	$dx = a\sec\theta\tan\theta\, d\theta$

Example 5. Solve the integral

$$\int \frac{x}{\sqrt{x^2+x+1}}\, dx.$$

Solution. We first complete the square in the denominator to achieve the desired form:

$$x^2 + x + 1 \;=\; (x+1/2)^2 + 3/4.$$

Substituting $u = x + 1/2$ we get

$$\int \frac{x}{\sqrt{x^2+x+1}}\, dx \;=\; \int \frac{u-1/2}{\sqrt{u^2+3/4}}\, du$$

We now substitute $u = \frac{\sqrt{3}}{2}\tan\theta$ to get

$$\begin{aligned}
\int \frac{u-1/2}{\sqrt{u^2+3/4}}\, du &= \int \frac{\frac{\sqrt{3}}{2}\tan\theta - 1/2}{\frac{\sqrt{3}}{2}\sec\theta}\, \frac{\sqrt{3}}{2}\sec^2\theta\, d\theta \\
&= \int \left(\frac{\sqrt{3}}{2}\tan\theta - \frac{1}{2}\right)\sec\theta\, d\theta \\
&= \frac{\sqrt{3}}{2}\int \tan\theta\sec\theta\, d\theta - \frac{1}{2}\int \sec\theta\, d\theta \\
&= \frac{\sqrt{3}}{2}\sec\theta - \frac{1}{2}\ln|\sec\theta + \tan\theta| + C \\
&= \frac{\sqrt{3}}{2}\sqrt{1+\frac{4}{3}u^2} - \frac{1}{2}\ln\left|\sqrt{1+\frac{4}{3}u^2} + \frac{2}{\sqrt{3}}u\right| + C \\
&= \frac{\sqrt{3}}{2}\sqrt{1+\frac{4}{3}\left(x+\frac{1}{2}\right)^2} - \frac{1}{2}\ln\left|\sqrt{1+\frac{4}{3}\left(x+\frac{1}{2}\right)^2} + \frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right| + C \\
&= \sqrt{\frac{3}{4}+\left(x+\frac{1}{2}\right)^2} - \frac{1}{2}\ln\left|\sqrt{1+\frac{4}{3}\left(x+\frac{1}{2}\right)^2} + \frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right| + C \\
&= \sqrt{x^2+x+1} - \frac{1}{2}\ln\left|\sqrt{\frac{4}{3}x^2+\frac{4}{3}x+\frac{4}{3}} + \frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right| + C \\
&= \sqrt{x^2+x+1} - \frac{1}{2}\ln\left|\frac{2}{\sqrt{3}}\sqrt{x^2+x+1} + \frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right| + C \\
&= \sqrt{x^2+x+1} - \frac{1}{2}\ln\left|\frac{2}{\sqrt{3}}\left(\sqrt{x^2+x+1} + \left(x+\frac{1}{2}\right)\right)\right| + C \\
&= \sqrt{x^2+x+1} - \frac{1}{2}\ln\left|\sqrt{x^2+x+1} + \left(x+\frac{1}{2}\right)\right| - \frac{1}{2}\ln\frac{2}{\sqrt{3}} + C \\
&= \sqrt{x^2+x+1} - \frac{1}{2}\ln\left|\sqrt{x^2+x+1} + \left(x+\frac{1}{2}\right)\right| + C,
\end{aligned}$$

the last step due to the fact that $\frac{1}{2}\ln\frac{2}{\sqrt{3}}$ is a constant that may be absorbed in C.

2.4 Method of Partial Fractions

When we are required to integrate a complicated fraction, that involves rather complicated polynomials (i.e. degrees of x. No sines, cosines, exponents, or any other such function) both in the numerator and in the denominator, we first attempt to simplify the expression as much as possible by performing **long division**.

Thus, a fraction $\frac{P(x)}{Q(x)}$ becomes $S(x)+\frac{R(x)}{Q(x)}$, where the degree of R is less than the degree of Q.

`This is the same process as writing` $\frac{11}{4}=2+\frac{3}{4}$.

Then, we **factor** the denominator into the simplest possible products (for example, $x^2-1=(x-1)(x+1)$).

`This is the same as writing` $4=2\times 2$.

Last, we use the method of **partial fractions**, in order to separate the various terms in the denominator.

`This part is the one that has no real equivalent with simple numbers.`

Here are the main cases when using partial fractions:

2.4.1 The denominator $Q(x)$ is a product of distinct linear factors.

If $Q(x)$ can be decomposed into $Q(x)=(a_1x+b_1)(a_2x+b_2)\cdots(a_kx+b_k)$, then we write

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_1}+\frac{A_2}{a_2x+b_2}+\cdots+\frac{A_k}{a_kx+b_k} \tag{1}$$

2.4.2 $Q(x)$ is a product of linear factors, some of which are repeated.

If $Q(x)$ is decomposed similarly to the above decomposition, but with some of the factors appearing with higher powers, then in (1) we get extra terms with higher powers. For example, if the factor a_1x+b_1 appears with the power r, then we need to add the terms

$$\frac{A_{12}}{(a_1x+b_1)^2}+\frac{A_{13}}{(a_1x+b_1)^3}+\cdots+\frac{A_{1r}}{(a_1x+b_1)^r}$$

to equation (1).

2.4.3 $Q(x)$ is a product of irreducible quadratic factors, none of which is repeated.

This case is similar to the first case, with the sole difference that instead of having a number A_i in the numerator, we now write A_ix+B_i.

2.4.4 $Q(x)$ contains a repeated quadratic factor.

Same as with the second case: we simply add terms with degrees up to (and including) the degree of the factor.

2.5 Improper Integrals

As we know, definite integrals measure the area under a curve. But what if the curve tends to infinity (vertical asymptote)? What if the interval which we are dealing with is infinite? Those are called **improper integrals**.

2.5.1 L'Hôpital's Rule

Before dealing with improper integrals, we remind ourselves of an important theorem that has to do with limits, **l'Hôpital's rule**:

Let a be either some number, or $\pm\infty$. Assume two functions $f(x)$, $g(x)$ both tend to 0 (or $\pm\infty$) when x tends to a. Then we have the result:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

2.5.2 Improper Integrals of Type I

The first type of improper integrals deals with integrals where one (or two) of the bounds are infinite. These are integrals of the form

$$\int_a^\infty f(x)dx \quad \text{or}$$
$$\int_{-\infty}^b f(x)dx \quad \text{or}$$
$$\int_{-\infty}^\infty f(x)dx.$$

The solution to these integrals is actually not complicated: for example, say we wanted to solve the second of these three examples; we'd first solve the integral

$$\int_{-t}^b f(x)dx.$$

Next, we let $t\longrightarrow\infty$, so that our final answer is given by

$$\lim_{t\to\infty}\int_{-t}^b f(x)dx.$$

2.5.3 Improper Integrals of Type II

The second type of improper integrals deals with functions that have some point of "blow up" around which we attemp to integrate. The method is similar to the previous one:

Say we look for the integral

$$\int_a^b f(x)dx$$

and suppose the function has a blow up at the point b (i.e. as we approach b the function tends to ∞ or $-\infty$). Then the integral is given by

$$\lim_{t\uparrow b}\int_a^t f(x)dx,$$

where the notation $t\uparrow b$ means that t approaches b from below: t is a smaller number than b, but tends to it and gets nearer and nearer to b.

2.6 Arc Length

Up until now we were busy calculating area under curves, given by a straight-forward integral. Now we investigate the question of calculating the length of a curve.

The formula for the length of a curve $f(x)$ between two points a and b is given by

$$L = \int_a^b \sqrt{1+(f'(x))^2}dx.$$

2.7 Area of a Surface of Revolution

See (?)

3 Sequences and Series

3.1 Sequences

As the name implies, “sequences” are an infinite number of numbers written one after the other, with importance to the order in which they are written. We usually write the sequence as

$$a_0, a_1, a_2, \ldots, a_n, \ldots$$

or, in short,

$$\{a_n\}$$

where each a_n stands for some number.

For instance, consider the sequence

$$3, -7, 5492, 45, -78, \ldots$$

Then $a_0 = 3, a_1 = -7, a_2 = 5492, a_3 = 45$ and $a_4 - 78$.

Usually when dealing with sequences, the next step would be to try to say something about the series as a whole. The most common would be “does the sequence have a limit?”.

When we look at the sequence given above, we can't really come up with a number a_5 and then a_6, a_7, etc. that will make more sense than others. That is the case, because the sequence has no apparent rule by which it abides.

An example for a sequence with a rule, would be

$$1, 2, 3, 4, 5, \ldots$$

or

$$1, 4, 9, 16, 25, \ldots$$

or

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$$

where the general elements are $a_n = n$, $a_n = n^2$ and $a_n = 1/n$ respectively.

3.2 Series

A series is the (infinite) sum of a sequence. If our sequence is $\{a_n\}$, then the corresponding series will be given by

$$\sum_{n=0}^{\infty} a_n.$$

The main question we ask regarding series, is convergence: does this infinite sum equal some number, or is it infinity (or negative infinity)?

To determine this question, we define the N^{th} partial sum:

$$\begin{aligned} S_N &= \sum_{n=0}^{N} a_n \\ &= a_0 + a_1 + a_2 + \cdots + a_{N-1} + a_N. \end{aligned}$$

Definition 6. *The series $\sum_{n=0}^{\infty} a_n$ is said to* ***converge*** *if the limit $\lim_{N\to\infty} S_N$ exists and is finite. Otherwise, we say that the series* ***diverges****.*

Example 7. If our series is $\sum_{n=0}^{\infty} 1$, then obviously the series diverges, since the N^{th} partial sum is

$$\begin{aligned} S_N &= \sum_{n=0}^{N} 1 \\ &= 1+1+1+\cdots+1 \\ &= N+1 \end{aligned}$$

which tends to infinity as $N \to \infty$.

Example 8. The series $\sum_{n=1}^{\infty} 1/n^2 = 1 + 1/2 + 1/4 + \cdots$ does converge, to the number 2. Try to figure out why. *Hint: look up the formula for a geometric series.*

In the next subsections, we will be exploring various methods of determining convergence/divergence of series.

3.3 The Integral Test and Estimates of Sums

The first, most intuitive method of determining convergence/divergence of a series, is plotting a function that will represent the series. To do this, we construct the function $f(x)$ as follows:

Suppose our series is given by $\sum_{n=0}^{\infty} a_n$.

At all non-negative integers (i.e., at 0, 1, 2, 3,...) we mark dots corresponding to the values of the sequence a_n: at 0 we mark a dot at a_0, at 1 we mark a dot at a_1, and so on.

Then, we connect the dots with a smooth line.

Finally we try to calculate the area under this function. If the area is finite, the series converges. If the area is infinite, the series diverges.

Now read the theorem in the book, and make sure you see why this description corresponds to the theorem presented there.

3.4 The Comparison Tests

Another very intuitive method for determining convergence/divergence, is comparing to known series. Suppose $\sum_{n=0}^{\infty} a_n$ is the series in question, and suppose that there is another series $\sum_{n=0}^{\infty} b_n$ for which we know convergence/divergence. Then we have the following (obvious rules):

1. If $a_n \leq b_n$ and $\sum_{n=0}^{\infty} b_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.
2. If $a_n \geq b_n$ and $\sum_{n=0}^{\infty} b_n$ diverges, then $\sum_{n=0}^{\infty} a_n$ diverges.

Warning 9. In this case, we require all series to be positive! (i.e. all a_n and all b_n are ≥ 0)

3.5 Alternating Series

Theorem 10. *Let $\{b_n\}$ be a non-negative sequence, i.e. $b_n \geq 0$. Assume that the sequence is decreasing, i.e. $b_{n+1} \leq b_n$ and that it tends to 0, i.e. $\lim_{n\to\infty} b_n = 0$.*

Then the series

$$\sum_{n=0}^{\infty} (-1)^n b_n = b_0 - b_1 + b_2 - b_3 + b_4 - + \cdots$$

converges.

This theorem makes dealing with series with alternating signs *much* simpler. It is important to make sure that the series in question is, indeed, an alternating series! This might by tricky. Here are two examples, one of an actual alternating series, the other of a fake alternating series.

Example 11. (Alternating harmonic series)

The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - + \cdots$ is convergent, since it is an alternating series that satisfies the conditions of Theorem 10.

Notice, that without the " $-$ " signs, the series becomes the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ which diverges!

Example 12. (Fake alternating series)

The series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin n}{n} = \frac{\sin 1}{1} - \frac{\sin 2}{2} + \frac{\sin 3}{3} - \frac{\sin 4}{4} + \frac{\sin 5}{5} - + \cdots$$

is *not* an alternating series, because the sine function may take negative values. This means, that our sequence $\{b_n\}$, which is, in this case $\left\{\frac{\sin n}{n}\right\}$ is not necessarily a non-negative sequence.

Thus Theorem 10 does not apply, and we cannot deduce if this series converges or diverges.

3.6 Absolute Convergence and the Ratio and Root Tests

3.6.1 Absolute Convergence

Definition 13. *We say that a series $\sum a_n$ is* ***absolutely convergent****, if the series of absolute values $\sum |a_n|$ is convergent. In other words, a series is absolutely convergent if it converges even when we replace any " $-$ " sign with a " $+$ " sign.*

Replacing all the $-$'s with $+$'s is sort of a "worst case scenario": instead of allowing the series to go up or down as we sum, we allow the series to only go up. The risk of diverging (reaching ∞) is greater than it was before. This motivates the following definition, and the theorem the proceeds it:

Definition 14. *A series $\sum a_n$ is called* ***conditionally convergent*** *if it is convergent, but not absolutely convergent.*

Theorem 15. *If a series is absolutely convergent, it is also convergent.*

Strategy for determining if an alternating series is convergent:

1. Make sure the series is really an alternating series! Recall Example 12 of the fake alternating series.
2. If it is alternating, change all $-$ signs to $+$ signs. If this series converges, you're done, since it means that the series is absolutely convergent, and thus also convergent.
3. If the series in not absolutely convergent (like in Example 11), check if the conditions in Theorem 10 are satisfied. If they are, you just found a conditionally convergent sequence.

3.6.2 The Ratio Test

Theorem 16. *Let $\{a_n\}$ be a sequence. Define*

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

Then:

1. *If* $\boldsymbol{L < 1}$ *then the series* $\sum \boldsymbol{a_n}$ ***converges absolutely*** *(and so also converges).*
2. *If* $\boldsymbol{L > 1}$ ***or*** $\boldsymbol{L = \infty}$ *then the series* $\sum \boldsymbol{a_n}$ ***diverges****.*
3. *If* $\boldsymbol{L = 1}$*, we cannot determine whether the series converges or diverges.*

3.6.3 The Root Test

Theorem 17. *Let* $\{a_n\}$ *be a sequence. Define*

$$L \;=\; \lim_{n\to\infty} \sqrt[n]{a_n}.$$

Then:

1. *If* $\boldsymbol{L < 1}$ *then the series* $\sum \boldsymbol{a_n}$ ***converges absolutely*** *(and so also converges).*
2. *If* $\boldsymbol{L > 1}$ ***or*** $\boldsymbol{L = \infty}$ *then the series* $\sum \boldsymbol{a_n}$ ***diverges****.*
3. *If* $\boldsymbol{L = 1}$*, we cannot determine whether the series converges or diverges.*

3.7 Power Series

A power series is a series that in addition to having numbers being added up (those a_n's we're used to by now), also has powers of x.

Thus, instead of having the series

$$\sum_{n=0}^{\infty} a_n \;=\; a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + \cdots + a_n + \cdots$$

we have the series

$$\sum_{n=0}^{\infty} a_n x^n \;=\; a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + a_4\,x^4 + a_5\,x^5 + \cdots + a_n\,x^n + \cdots$$

More generally, we could "shift" the series by a factor b, to get the series

$$\sum_{n=0}^{\infty} a_n (x-b)^n \;=\; a_0 + a_1\,(x-b) + a_2\,(x-b)^2 + a_3\,(x-b)^3 + a_4\,(x-b)^4 + \cdots + a_n\,(x-b)^n + \cdots$$

However, we will usually deal with the simpler case, with the base point $b = 0$.

3.7.1 Radius of convergence

Since the series is now a function of x, we may ask for which values of x the series converges. Usually, the most efficient way is to use the *ratio test*: we want only x's that satisfy

$$\lim_{n\to\infty} \left| \frac{a_{n+1}\,x^{n+1}}{a_n\,x^n} \right| \;<\; 1.$$

Notice that this condition translates into the condition

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \;<\; \frac{1}{|x|}$$

if $x \neq 0$.

Intuitively it is clear that the smaller x is, the greater the chance for the series to converge (you should also see that it follows from the last formula above). In fact, we have the following theorem:

Theorem 18. *The power series $\sum_{n=0}^{\infty} a_n x^n$ satisfies one (and only one) of the following:*

1. *The series converges only when $x=0$.*
2. *The series converges for all x.*
3. *There is a positive number R such that the series converges if $-R<x<R$ and diverges if $x<-R$ or $x>R$.*

3.7.2 Representation of functions as power series

Consider the following:

$$\begin{aligned}(1-x)\left(1+x+x^2+\cdots+x^n\right) &= 1-x+x-x^2+x^2-+\cdots-x^n+x^n-x^{n+1}\\ &= 1-x^{n+1}\end{aligned}$$

So that, by dividing both sides by $1-x$ we get

$$1+x+x^2+\cdots+x^n = \frac{1-x^{n+1}}{1-x}.$$

Now, if $|x|<1$, then $x^n\to 0$ as $n\to\infty$, so that, taking this limit, we get

$$\sum_{n=0}^{\infty} x^n = \underbrace{1+x+x^2+x^3\cdots}_{\text{infinite series}} = \frac{1}{1-x}. \tag{2}$$

Equation (2) is very important. It allows us to find power series representation to many functions. Please look at the examples in the book.

3.7.3 Taylor series

Equation (2) showed us that we can represent the function $\frac{1}{1-x}$, as well as its derivatives and integrals, in a power series form. Now we develop a method to represent almost any function we encounter in this form.

Let $f(x)$ be some function, and suppose it is "nice" enough so that we can take as many derivatives of it as we want (i.e. it doesn't have any jumps for example). Suppose we could write it in a power series form:

$$f(x) = a_0+a_1x+a_2x^2+\cdots+a_nx^n+\cdots \tag{3}$$

Let us try to find out what the coefficients $a_0, a_1, a_2, \ldots$ will be:
Plug in $x=0$ to both sides of (3). We get

$$\begin{aligned}f(0) &= a_0+a_1 0+a_2 0^2+\cdots\\ &= a_0.\end{aligned}$$

Now, take a derivative of (3), to get

$$f'(x) = a_1+2a_2x+3a_3x^2+\cdots+n\,a_nx^{n-1}+\cdots \tag{4}$$

and again plug in $x=0$:

$$\begin{aligned}f'(0) &= a_1+2a_2 0+3a_3 0^2+\cdots\\ &= a_1.\end{aligned}$$

Repeat this process a few more times, to get:

$$\begin{aligned} f''(x) &= 2a_2 + 3\cdot 2\cdot a_3 x + 4\cdot 3\cdot a_4 x^2 + 5\cdot 4\cdot a_5 x^3 + \cdots + n\cdot(n-1)\cdot a_n x^{n-2} + \cdots \\ f'''(x) &= 3\cdot 2\cdot a_3 + 4\cdot 3\cdot 2\cdot a_4 x + 5\cdot 4\cdot 3\cdot a_5 x^2 + \cdots + n\cdot(n-1)\cdot(n-2)\cdot a_n x^{n-3} + \cdots \end{aligned}$$

and

$$\begin{aligned} f''(0) &= 2a_2 \\ f'''(0) &= 3\cdot 2\cdot a_3. \end{aligned}$$

In general, we find that the n^{th} derivative of f at 0 is given by

$$\begin{aligned} f^{(n)}(0) &= n\cdot(n-1)\cdot(n-2)\cdots 3\cdot 2\cdot a_n \\ &= n!\cdot a_n. \end{aligned} \tag{5}$$

Dividing both sides of (5) by $n!$, we have

$$a_n = \frac{f^{(n)}(0)}{n!}. \tag{6}$$

Plugging (6) into (3), we see that we get

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \tag{7}$$

This is called the ***Maclaurin Series*** of f. We can write it in short form, as

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n, \tag{8}$$

with the agreement that $0! = 1$.

Now, our choice of the point 0 to always plug into $f(x)$ above was convenient, but the procedure may be modified slightly, to allow the choice of any point. Please see the book for more details. This will give us the general ***Taylor series***

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n. \tag{9}$$

4 Differential Equations

A *differential equation* with variable x, is an equation involving x, functions of x, and their derivatives. We will only deal with simple equations of **first** and **second** *order*.

Definition 19. *A* ***first order differential equation*** *is an equation of the form*

$$(y' =) \quad \frac{dy}{dx} = F(x,y). \tag{10}$$

If, in addition, we are given that

$$y(a) = b \tag{11}$$

then we call the entire problem an ***initial value problem****.*

Definition 20. *A* ***second order differential equation*** *is an equation of the form*

$$(y'' =) \quad \frac{d^2y}{dx^2} = G(x,y,y'). \tag{12}$$

The corresponding initial value problem is defined in a similar fashion as was above, but note that a second order equation must be accompanied by two initial conditions, for example

$$\begin{aligned} y(a) &= b \\ y'(a) &= c. \end{aligned}$$

In general, for one to be able to find a specific solution for a differential equation, it must be accompanied by as many "pieces of information" as its order.

4.1 First Order Equations

4.1.1 Separable Equations

When we encounter equations of the form

$$\frac{dy}{dx} = F(x,y) = f(x)\cdot g(y), \tag{13}$$

a trick that we may use is to rewrite (13) in the form

$$\frac{dy}{g(y)} = f(x)\,dx \tag{14}$$

and integrate both sides separately, to get

$$\int \frac{dy}{g(y)} = \int f(x)\,dx. \tag{15}$$

Remark 21. We have to make sure that we are not performing an "illegal operation" by dividing by 0 when writing (14): It is important to be aware of the values of y for which $g(y) = 0$.

When solving (15) it is **crucial** to remember the constant we get after integrating. This constant will remain unknown, unless we are given an initial condition. This is true in general, not only for this type of equation: **if you solved a differential equation and no initial condition was given, your answer will be y as a function of x, but an unknown constant C will appear. Only if an initial condition is given will you be able to determine the value of C!**

Example 22. Solve the initial value problem

$$\begin{aligned} y^2\frac{dy}{dx} &= x^2 + 2x + 1 \\ y(1) &= 2. \end{aligned}$$

Solution. We first solve the differential equation, and then use the initial condition.

1. In this case, simple multiplication by dx is enough, to get

$$\begin{aligned} y^2 dy &= \left(x^2+2x+1\right)dx \\ \int y^2 dy &= \int \left(x^2+2x+1\right)dx \\ \frac{1}{3}y^3 &= \frac{1}{3}x^3+x^2+x+C \\ y^3 &= x^3+3x^2+3x+C. \end{aligned}$$

   It is enough to leave the solution as is. Note that "C" remained as is in the last transition, since at this stage it is an unknown constant, and, thus, "C" and "$3C$" are the same for us.

2. Now we apply the initial value: we plug in $x=1$ and $y=2$ to the solution, to get

$$\begin{aligned} 2^3 &= 1^3+3\times 1^2+3\times 1+C \\ 8 &= 7+C \\ 1 &= C. \end{aligned}$$

3. Finally, our solution to the initial value problem is

$$y^3 = x^3+3x^2+3x+1.$$

4.1.2 Linear First Order Equations

Definition 23. *A **linear first order equation** is a first order differential equation of the form*

$$\frac{dy}{dx}+P(x)\cdot y = Q(x), \tag{16}$$

with $P(x)$ and $Q(x)$ some functions of x.

To solve such an equation, we introduce the **integrating factor**

$$\rho(x) = e^{\int P(x)dx}.$$

This is a trick that we use, since notice what we get now:

$$\begin{aligned} [y\cdot\rho(x)]' &= y'\cdot\rho(x)+y\cdot\rho'(x) \\ &= y'\cdot e^{\int P(x)dx}+y\cdot\left(e^{\int P(x)dx}\right)' \\ &= y'\cdot e^{\int P(x)dx}+y\cdot P(x)\cdot e^{\int P(x)dx} \\ &= \left\{\frac{dy}{dx}+y\cdot P(x)\right\}\cdot e^{\int P(x)dx} \\ &= \left\{\frac{dy}{dx}+y\cdot P(x)\right\}\cdot\rho(x) \end{aligned}$$

which is exactly the left hand side of (16), multiplied by the integrating factor. Plugging this into (16), we have

$$[y\cdot\rho(x)]' = Q(x)\cdot\rho(x). \tag{17}$$

The solution now follows easily: integrating, we get

$$y\cdot\rho(x) = \int Q(x)\cdot\rho(x)\,dx,$$

so that

$$y(x) = \frac{1}{\rho(x)} \int Q(x) \cdot \rho(x)\, dx. \tag{18}$$

Again: **do not forget (1) to make sure that $\boldsymbol{\rho(x) \neq 0}$ when dividing, and (2) the constant $\boldsymbol{C}$ you get when integrating in (18)!**

If we have an initial value, the same procedure as above applies.

Example 24. Solve the initial value problem

$$\begin{aligned}(1+x)\frac{dy}{dx} + y &= \cos x \\ y(0) &= 1.\end{aligned} \tag{19}$$

Solution.

1. We first want to bring the equation to the form we have in (16). To do this, we divide by $1+x$, and remember that this means that we assume $x \neq -1$!

$$\frac{dy}{dx} + \frac{1}{1+x} \cdot y = \frac{\cos x}{1+x}.$$

2. We identify $P(x) = \frac{1}{1+\mathrm{x}}$, so that we have

$$\begin{aligned}\rho(x) &= \exp\left[\int \frac{1}{1+x} dx\right] \\ &= \exp[\ln|1+x|] \\ &= |1+x|.\end{aligned}$$

We now notice that our initial value is given at $x = 0$, so that our solution will be going through $1 + 0 = 1 > 0$, so that we may lose the absolute value, as it makes no difference. Notice, also, that since we assumed $x \neq -1$, the solution will always have the property that $1 + x > 0$.

Using (18), we have

$$\begin{aligned}y(x) &= \frac{1}{1+x} \int \frac{\cos x}{1+x} \cdot (1+x)\, dx \\ &= \frac{1}{1+x} \int \cos x\, dx \\ &= \frac{1}{1+x}(\sin x + C).\end{aligned}$$

3. Applying the initial condition, we get

$$\begin{aligned}1 &= \frac{1}{1+0}(\sin 0 + C) \\ &= C.\end{aligned}$$

4. Finally, our solution to the initial value problem is

$$y(x) = \frac{1}{1+x}(\sin x + 1).$$

Exercise 1. Plug this answer into (19) and make sure this is indeed a solution, that also satisfies the initial value.

4.1.3 Population Models

Population models are a specific type of linear first order differential equation, that model population growth (or decrease) for bacteria in a petri dish, to name one example. These are pretty basic models, which can be made much more complex.

The general equation is of the form

$$\frac{dP}{dt} = kP(M-P) \tag{20}$$

where $P(t)$ is the *population size*, M is the *carrying capacity*, i.e. the “optimal” population size, and k is a constant number that measures how “well” the population can grow/decrease.

Treatment of such an equation is by separation of variables.

4.2 Second Order Equations

4.2.1 What to expect when dealing with second order equations?

Second order equations, in general, have the form

$$y'' = G(x, y, y').$$

The pendulum is a familiar physical object whose motion is described by a second order equation.

Similar to first order equations, we have a few ground rules:

1. Any solution to a second order linear, homogeneous equation must have the form
$$y(x) = C_1 y_1(x) + C_2 y_2(x). \tag{21}$$
You should always end up having two constant coefficients, C_1 and C_2.
2. To become an initial value problem, a second order equation must have two conditions, for example
$$\begin{aligned} y(a) &= b \\ y'(a) &= c. \end{aligned} \tag{22}$$
In this case, C_1 and C_2 can be calculated (you should get two equations with C_1, C_2 the unknowns).

4.2.2 Linear Second Order Equations

Definition 25. *A* ***linear, second order*** *equation, is an equation of the form*

$$A(x)\,y'' + B(x)\,y' + C(x)\,y = D(x). \tag{23}$$

Definition 26. *A linear, second order* ***homogeneous*** *equation, has the form*

$$A(x)\,y'' + B(x)\,y' + C(x)\,y = 0. \tag{24}$$

The linear second order equations we deal with in this course are homogeneous and have constant coefficients. They are of the form

$$a\,y'' + b\,y' + c\,y = 0. \tag{25}$$

To solve, we consider the quadratic equation

$$a\,r^2 + b\,r + c = 0.$$

We solve for r, and get the two roots r_1 and r_2. From those we deduce the solution for (25). We have three cases:

I. If $r_1 \neq r_2$ and both are real, then (25) has solution
$$y(x) = C_1\,e^{r_1 x} + C_2\,e^{r_2 x}.$$

II. If $r_1 = r_2$, then (25) has solution

$$y(x) \;=\; C_1 e^{r_1 x} + C_2\, x\, e^{r_1 x}.$$

III. If $r_1 \neq r_2$ are complex numbers, then they must have the form $r_{1,2} = p \pm q\, i$ and (25) has solution

$$y(x) \;=\; e^{px}(C_1 \cos qx + C_2 \sin qx).$$

As usual, if initial values are given, they will be of the form shown in (22), and, when plugged into the solution, we should get a system of two equations for C_1, C_2 which ought to solve them.

4.2.3 Mechanical Vibrations

Mechanical vibrations (like a coil or a pendulum) are characterized by slightly more complex equations, though not essentially more.

The most general form of such equations that we will deal with is

$$m\, x'' + c\, x' + k\, x \;=\; F_0 \cos \omega t. \tag{26}$$

Here, if we consider a coil attached horizontally to some object and thus moving it back and forth, m represents the object's mass, c represents the friction (damping) between the object and the surface upon which it is moving, k represents how "strong" the coil is and $F_0 \cos \omega t$ is an external forcing (i.e. you are holding the object and forcing it to move in a certain way).

Note 27. It is common notation, when dealing with this type of equation, to have x as a function of t, rather than y as a function of x. Don't get confused by this!

We solve equation (26) in three stages:

I. We solve the homogeneous equation

$$m\, x'' + c\, x' + k\, x \;=\; 0 \tag{27}$$

as in the previous section. We get a solution

$$x_c(t) \;=\; C_1 x_1(t) + C_2 x_2(t).$$

II. We solve the non-homogeneous equation (26) by taking the function

$$x_p(t) \;=\; A \cos \omega t + B \sin \omega t$$

and plugging it into (26). We solve for A and for B by comparing coefficients in front of the sines and the cosines.

III. The general solution of (26) will be

$$x(t) \;=\; x_c(t) + x_p(t).$$

Example 28. Solve the initial value problem

$$x'' + 4x' + 5x \;=\; 40 \cos 3t$$

$$\begin{aligned} x(0) &= 0 \\ x'(0) &= 0 \end{aligned}$$

Solution.

I. We start by solving the homogeneous equation

$$x'' + 4x' + 5x \;=\; 0.$$

We consider the quadratic equation

$$r^2 + 4r + 5 = 0,$$

which has roots

$$\begin{aligned} r_{1,2} &= \frac{-4 \pm \sqrt{16 - 4 \cdot 5}}{2} \\ &= -2 \pm i. \end{aligned}$$

Thus, our solution to the homogeneous equation is

$$x_c(t) = e^{-2t}(C_1 \cos t + C_2 \sin t).$$

II. Now we solve the non-homogeneous problem, by attempting $x_p = A\cos 3t + B \sin 3t$.
First we calculate x_p' and x_p'':

$$\begin{aligned} x_p' &= -3A \sin 3t + 3B \cos 3t \\ x_p'' &= -9A \cos 3t - 9B \sin 3t \end{aligned}$$

So that we get:

$$\begin{aligned} x_p'' + 4x_p' + 5x_p &= 40 \cos 3t \\ -9A \cos 3t - 9B \sin 3t + 4(-3A \sin 3t + 3B \cos 3t) + 5(A \cos 3t + B \sin 3t) &= 40 \cos 3t \\ (-9A + 12B + 5A)\cos 3t + (-9B - 12A + 5B)\sin 3t &= 40 \cos 3t \end{aligned}$$

So that, by comparing coefficients, we have

$$\begin{aligned} -4A + 12B &= 40 \\ -4B - 12A &= 0 \end{aligned}$$

which imply that $B = -3A$ (by the second relation), and plugging this into the first relation

$$\begin{aligned} -4A + 12(-3A) &= 40 \\ -40A &= 40 \\ A &= -1 \\ B &= 3. \end{aligned}$$

III. Thus, our solution is

$$x(t) = e^{-2t}(C_1 \cos t + C_2 \sin t) - \cos 3t + 3 \sin 3t.$$

Next we find C_1 and C_2 by applying the initial conditions:
First we calculate

$$x'(t) = -2e^{-2t}(C_1 \cos t + C_2 \sin t) + e^{-2t}(-C_1 \sin t + C_2 \cos t) + 3\sin 3t + 9 \cos 3t.$$

and now

$$\begin{aligned} 0 &= x(0) \\ &= C_1 - 1 \end{aligned}$$

$$\begin{aligned} 0 &= x'(0) \\ &= -2C_1 + C_2 + 9 \\ &= C_2 + 7 \end{aligned}$$

So we have $C_1 = 1, C_2 = -7$, and our final answer can be written

$$x(t) = e^{-2t}(\cos t - 7 \sin t) - \cos 3t + 3 \sin 3t.$$

4.3 Series Solutions to Differential Equations

The method of series solution to differential equations is a general method, that can help us solve many types of equations.

When solving in this method, we have two main approaches:

4.3.1 If we are asked to "find $a_0, a_1, \ldots, a_4$"

In the title, the choice of "4" is of course arbitrary, but by that, the meaning is that these are problem types in which you are asked to find the first few coefficients of y.

We write the first few elements in the Taylor expansion of y (a good idea is to write two more than what is asked for)

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots$$

Now we look back at the equation, and if it contains y' and y'', we write them out

$$\begin{aligned} y' &= a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + \cdots \\ y'' &= 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4 + \cdots \end{aligned}$$

Generally, we now have two steps:

I. If initial values are given, plug them into the above expressions to get (usually...) a_0 and a_1.

II. Use this result, and plug the above expressions into the given equation to find $a_2, a_3, \ldots$

4.3.2 If we are asked to find the full series expansion

In this case, we write y in the same way, i.e. a Taylor expansion, but we write a general summation formula, instead of writing individual elements:

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ y' &= \sum_{n=1}^{\infty} n\, a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n \\ y'' &= \sum_{n=2}^{\infty} n\,(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n \end{aligned}$$

We plug these expressions back into the equation, and compare coefficients in front of x^j.

5 Polar Coordinates and Parametric Curves

When dealing with problems in the plane (i.e. a problem that has x and y coordinates), it is sometimes easier to describe coordinates in terms of angle and distance, rather than the usual rectangular x and y coordinates.

For example, imagine an ant that has to direct ant traffic in the entrance to the ant colony. It is standing on the surface tracking incoming ant traffic from the outside. If it wanted to pin-point an incoming ant, what would be best: to describe the ant's location in x, y coordinates (answer: no), or to describe the ant's location in terms of its angle compared to some object, and its distance along the line pointing in that direction?

5.1 Polar Coordinates

The transitions that are used when discussing polar coordinates, are:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ r &= \sqrt{x^2+y^2} \\ \theta &= \arctan\frac{y}{x} \end{aligned}$$

where the last holds only if $x \neq 0$.

Note, that in polar coordinates, a point may be described in many ways:

i. If we take $\theta + 2\pi$ instead of θ we get the exact same direction in the plane, and so the point described will be the same. This is true, in fact, if we add (or subtract) any multiple of 2π to θ.

ii. Draw a small sketch to convince yourself also that

$$(r,\theta) = (-r, \theta+\pi).$$

Thus, we will usually only discuss angles θ that take values in $0 \leq \theta < 2\pi$; however, we do allow negative r values.

Also note, that for $r = 0$, any θ value will give the same point - the origin.

5.2 Area Computations in Polar Coordinates

In rectangular coordinates we know that given a function $f(x)$, if we want to calculate the area under its curve, say between some point a and some other point b, we need to take its integral

$$\int_a^b f(x)\,dx.$$

For a function defined in polar coordinates the concept is the same, with one difference: when measuring area from one base point about which we turn around (like the ant...), there is distortion. Thus we get that area calculation in polar coordinates, say of a function $g(\theta)$ between θ values of α and β, is given by

$$\int_\alpha^\beta \frac{1}{2}[g(\theta)]^2\,d\theta.$$

5.2.1 The Area Between Two Curves

The area between two curves $g(\theta)$ and $h(\theta)$, where g is the outer curve, will simply be given by

$$\int_\alpha^\beta \left\{\frac{1}{2}[g(\theta)]^2 - \frac{1}{2}[h(\theta)]^2\right\} d\theta.$$

If we need to determine the domain of integration, i.e. we need to determine α and β, we will usually be asked to do so between two points where the curves meet. Thus we find those points by equating

$$g(\theta) = h(\theta).$$

5.3 Parametric Curves

Suppose you know that there is a ball rolling on a plane. Apply the usual x and y axes to this plane. First imagine that the sun is setting at the edge of the y axis, so that the ball is casting its shadow on the x axis. Now imagine that the sun is setting at the edge of the x axis, so that the ball is casting its shadow on the y axis.

Let the first shadow cast, the one cast on the x axis, be some function $f(t)$, i.e. as time changes, the location of the shadow changes, but it is always on the x axis. Similarly, let the shadow cast on the y axis be some function $g(t)$.

It now should be evident that the ball's location at time t is simply given by $(f(t), g(t))$. As time changes, the ball's location changes, and thus forms a curve. This is called a ***parametric curve***, with parameter t.

5.3.1 Sketching a Parametric Curve

To sketch a parametric curve, we have these crucial steps:

I. Recall that $r^2 = x^2 + y^2 = f(t)^2 + g(t)^2$. Often, considering $f(t)^2 + g(t)^2$ will give us a nice, clean expression. This expression will be no other than $r(t)^2$, i.e. the square of the curve's distance from the origin at time t.

II. Check critical points: if it is clear that $f(t) = 0$ for some t, check what is the value of $g(t)$ for that t and vice versa. Also, if there are other values of t for which $f(t), g(t)$ are easy to compute - do it!

III. **Always make sure to compute the location of *at least* three points explicitly!**

These steps should suffice to get a rough idea of how the curve looks like.

5.3.2 The Line Tangent to a Parametric Curve

The slope of the line tangent to the parametric curve $(f(t), g(t))$ can be easily found using the chain rule. The formula we get is

$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$$

If we also want the second derivative, it is given by:

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \\ &= \frac{\frac{d}{dt}\left(\frac{g'(t)}{f'(t)}\right)}{f'(t)} \end{aligned}$$

5.3.3 Parametric Curves in Polar Coordinates

Note that many times parametric curves that are polar (i.e. "revolving" about the origin) can be written in the form

$$\begin{aligned} x(\theta) &= f(\theta)\cos\theta \\ y(\theta) &= f(\theta)\sin\theta. \end{aligned}$$

Treatment of this form is similar to the above.

5.4 Integral Computations with Parametric Curves

5.4.1 Some Basic Integrals

Recall the following important integrals:

1. Area under a curve
$$A = \int_a^b y\,dx.$$
2. Volume of revolution about the x-axis
$$V_x = \int_a^b \pi y^2 dx.$$
3. Volume of revolution about the y-axis
$$V_y = \int_a^b 2\pi x y\,dx.$$
4. Arc length of a curve
$$\begin{aligned} s &= \int_0^s ds \\ &= \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx. \end{aligned}$$
5. Area of surface of revolution about the x-axis
$$S_x = \int_{x=a}^b 2\pi y\,ds.$$
6. Area of surface of revolution about the y-axis
$$S_y = \int_{x=a}^b 2\pi x\,ds.$$

5.4.2 Applying These Integrals to Parametric Curves

Given a parametric curve $x = f(t)$, $y = g(t)$, we substitute the following expressions into the above integrals:

$$\begin{aligned} x &= f(t) \\ dx &= f'(t)\,dt \\ y &= g(t) \\ dy &= g'(t)\,dt \\ ds &= \sqrt{[f'(t)]^2+[g'(t)]^2}\,dt \end{aligned}$$