ON RIGOROUS DERIVATION OF THE ENSKOG KINETIC EQUATION

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Outline

1. The Enskog kinetic equation
1.1. A brief chronology of the derivation of the Enskog equation in collisional kinetic theory

The Boltzmann kinetic equation with hard sphere collisions

$$
\begin{aligned}
\frac{\partial}{\partial t} f_1(t,q_1,p_1) = -\langle p_1, \frac{\partial}{\partial q_1}\rangle f_1(t,q_1,p_1) + \\
+ \int\limits_{\mathbb{R}^3\times\mathbb{S}^2_+} dp_2\, d\eta\, \langle \eta, (p_1-p_2)\rangle \big(f_1(t,q_1,p_1^*) f_1(t,q_1,p_2^*) - \\
- f_1(t,q_1,p_1) f_1(t,q_1,p_2)\big),
\end{aligned}
$$

$$
\begin{aligned}
p_1^* &= p_1 - \eta\, \langle \eta, (p_1-p_2)\rangle\,, \\
p_2^* &= p_2 + \eta\, \langle \eta, (p_1-p_2)\rangle\,.
\end{aligned}
$$

1922, *David Enskog*: The Enskog equation is introduced

$$\frac{\partial}{\partial t} f_1(t,q_1,p_1) = -\langle p_1, \frac{\partial}{\partial q_1}\rangle f_1(t,q_1,p_1) +$$

$$+\sigma^2 \int\limits_{\mathbb{R}^3\times\mathbb{S}^2_+} dp_2 d\eta \langle \eta, p_1 - p_2\rangle \Big(Y\big[f_1;\sigma\big] f_1(t,q_1,p_1^*) f_1(t,q_1-\sigma\eta,p_2^*) -$$

$$-Y\big[f_1;\sigma\big] f_1(t,q_1,p_1) f_1(t,q_1+\eta\sigma,p_2)\Big)$$

"In general, it is hard, or impossible, to justify Enskog equation in a somewhat rigorous fashion."

N. Bellomo, M. Lachowicz, J. Polewczak, G. Toscani, *Mathematical Topics in Nonlinear Kinetic Theory II: The Enskog Equation*, World Sci. Publ., **1991**.

1.2. Some references

- **V.I. Gerasimenko, D.Ya. Petrina.** Uspekhi Mat. Nauk, v.45, **No.3**, pp. 135–182, **1990**
- **H. Spohn**. *Large Scale Dynamics of Interacting Particles*. Springer, **1991**
- **C. Cercignani, R. Illner, M. Pulvirenti**. *The Mathematical Theory of Dilute Gases*. Springer, **1994**
- **C. Cercignani, V.I. Gerasimenko, D.Ya. Petrina.** *Many-Particle Dynamics and Kinetic Equations*. Springer, **2012**
- **I. Gallagher, L. Saint-Raymond, B. Texier**. *From Newton to Boltzmann: Hard Spheres and Short-range Potentials*. EMS Publ. House: Zurich Lect. in Adv. Math., **2014**

1.3. A hard sphere system: the BBGKY hierarchy

$$F(t) = \big(1, F_1(t,x_1), \ldots, F_s(t,x_1,\ldots,x_s), \ldots\big)$$

$$\frac{\partial}{\partial t}F_s(t) = \mathcal{L}_s^* F_s(t) + \epsilon^2 \sum_{i=1}^{s} \int_{\mathbb{R}^3\times\mathbb{R}^3} dx_{s+1}\mathcal{L}_{\mathrm{int}}^*(i,s+1)F_{s+1}(t),$$
$$F_s(t)_{|t=0} = F_s^{0,\epsilon}, \quad s \geq 1,$$

$$\mathcal{L}_s^* f_s \doteq -\sum_{i=1}^{s}\langle p_i, \frac{\partial}{\partial q_i}\rangle f_s + \sum_{j_1<j_2=1}^{s} \epsilon^2 \mathcal{L}_{\mathrm{int}}^*(j_1,j_2)f_s,$$
$$\mathcal{L}_{\mathrm{int}}^*(j_1,j_2)f_s \doteq \int_{\mathbb{S}_+^2} d\eta\langle\eta,(p_{j_1}-p_{j_2})\rangle\big(f_s(x_1,\ldots,q_{j_1},p_{j_1}^*,\ldots,$$
$$q_{j_2},p_{j_2}^*,\ldots,x_s)\delta(q_{j_1}-q_{j_2}+\epsilon\eta) - f_s(x_1,\ldots,x_s)\delta(q_{j_1}-q_{j_2}-\epsilon\eta)\big)$$

A perturbative solution of the BBGKY hierarchy

$$F_s(t,x_1,\dots,x_s) = \sum_{n=0}^{\infty} \epsilon^{2n} \int_0^t dt_1 \dots \int_0^{t_{n-1}} dt_n \int_{(\mathbb{R}^3\times\mathbb{R}^3)^n} dx_{s+1}\dots dx_{s+n}$$
$$\times S_s^*(t-t_1) \sum_{i_1=1}^{s} \mathcal{L}_{\mathrm{int}}^*(i_1,s+1) S_{s+1}^*(t_1-t_2)\dots$$
$$\times S_{s+n-1}^*(t_n-t_n) \sum_{i_n=1}^{s+n-1} \mathcal{L}_{\mathrm{int}}^*(i_n,s+n) S_{s+n}^*(t_n) F_{s+n}^{0,\epsilon}, \quad s\geq 1,$$

$$(S_n^*(t)f_n)(x_1,\dots,x_n) \equiv S_n^*(t,1,\dots,n)f_n(x_1,\dots,x_n) \doteq$$
$$\doteq \begin{cases} f_n(X_1(-t,x_1,\dots,x_n),\dots,X_n(-t,x_1,\dots,x_n)), & \\ 0, & \mathrm{if}\,(q_1,\dots,q_n)\in\mathbb{W}_n \end{cases}$$

1.4. A non-perturbative solution of the BBGKY hierarchy

$$F_s(t,x_1,\ldots,x_s)=$$
$$=\sum_{n=0}^{\infty}\frac{1}{n!}\int_{(\mathbb{R}^3\times\mathbb{R}^3)^n}dx_{s+1}\ldots dx_{s+n}\mathfrak{A}_{1+n}(-t,\{Y\},X\setminus Y)F^{0,\epsilon}_{s+n},\quad s\geq 1,$$

We remark that nonperturbative solution of the BBGKY hierarchy is transformed to the form of the perturbation (iteration) solution as a result of applying of analogs of the Duhamel equation to cumulants of groups of operators.

V.I. Gerasimenko, T.V. Ryabukha, M.O. Stashenko
J. Phys. A: Math. Gen., **37**, 9861-9872, **2004**.

C. Cercignani, V.I. Gerasimenko, D.Ya. Petrina.
Many-Particle Dynamics and Kinetic Equations.
Springer, **2012**

The evolution operator $\mathfrak{A}_{1+n}(-t)$ is the $(n+1)th$-order cumulant of groups of operators defined by the expansion

$$\mathfrak{A}_{1+n}(-t,\{Y\},X\setminus Y)= \\ =\sum_{\mathrm{P}:\,(\{Y\},X\setminus Y)=\bigcup_i X_i}(-1)^{|\mathrm{P}|-1}(|\mathrm{P}|-1)!\prod_{X_i\subset\mathrm{P}}S^{*}_{|\theta(X_i)|}(t,\theta(X_i))$$

and the following notations are used: $\{Y\}$ is a set consisting of one element $Y\equiv(1,\ldots,s)$, i.e. $|\{Y\}|=1$, $\sum_{\mathrm{P}}$ is a sum over all possible partitions P of the set $(\{Y\},X\setminus Y)\equiv(\{Y\},s+1,\ldots,s+n)$ into $|\mathrm{P}|$ nonempty mutually disjoint subsets $X_i\in(\{Y\},X\setminus Y)$, the mapping θ is the declasterization mapping defined by the formula: $\theta(\{Y\},X\setminus Y)=X$. The simplest cumulants have the form

$$\mathfrak{A}_1(-t,\{Y\})=S^{*}_{s}(t,Y),$$
$$\mathfrak{A}_2(-t,\{Y\},s+1)=S^{*}_{s+1}(t,Y,s+1)-S^{*}_{s}(t,Y)S^{*}_{1}(t,s+1).$$

2. The main result
2.1. The generalized Enskog kinetic equation

Theorem

If the initial state

$$F^{(c)}(0) = \big(1, F_1^{0,\epsilon}(x_1), \ldots, \prod_{i=1}^{s} F_1^{0,\epsilon}(x_i)\mathcal{X}_{\mathbb{R}^{3s}\setminus\mathbb{W}_s^{\epsilon}}, \ldots\big),$$

then

$$F(t) = F(t \mid F_1(t)),$$

where $F(t) = \big(1, F_1(t, x_1), \ldots, F_s(t, x_1, \ldots, x_s), \ldots\big)$
$F(t \mid F_1(t)) = \big(1, F_1(t), F_2(t \mid F_1(t)), \ldots, F_s(t \mid F_1(t)), \ldots\big)$,
$\mathcal{X}_{\mathbb{R}^{3s}\setminus\mathbb{W}_s} \equiv \mathcal{X}_s(q_1, \ldots, q_s)$ is the Heaviside step function of allowed configurations $\mathbb{R}^{3s} \setminus \mathbb{W}_s$ of s hard spheres.

The marginal functionals of the state

$$F_s(t,x_1,\ldots,x_s \mid F_1(t)) \doteq \sum_{n=0}^{\infty}\frac{1}{n!}\int_{(\mathbb{R}^3\times\mathbb{R}^3)^n} dx_{s+1}\ldots dx_{s+n}\,\mathfrak{V}_{1+n}(t,\{Y\},X\setminus Y)\prod_{i=1}^{s+n}F_1(t,x_i),$$
$$s\geq 2,$$

Examples of the generating operators

$$\mathfrak{V}_1(t,\{Y\}) = \widehat{\mathfrak{A}}_1(t,\{Y\}) \doteq S_s^*(t,1,\ldots,s)\mathcal{X}_{\mathbb{R}^{3s}\setminus\mathbb{W}_s^{\epsilon}}\prod_{i=1}^{s}S_1^*(-t,i),$$

$$\mathfrak{V}_2(t,\{Y\},s+1) = \widehat{\mathfrak{A}}_2(t,\{Y\},s+1) - \widehat{\mathfrak{A}}_1(t,\{Y\})\sum_{i_1=1}^{s}\widehat{\mathfrak{A}}_2(t,i_1,s+1).$$

The generalized Enskog kinetic equation

$$F_1(t,x_1)=\sum_{n=0}^{\infty}\frac{1}{n!}\int_{(\mathbb{R}^3\times\mathbb{R}^3)^n}dx_2\ldots dx_{n+1}\mathfrak{A}_{1+n}(-t,$$

$$1,\ldots,n+1)\prod_{i=1}^{n+1}F_1^{0,\epsilon}(x_i)\mathcal{X}_{\mathbb{R}^{3(n+1)}\setminus\mathbb{W}^{\epsilon}_{n+1}}$$

$$\frac{\partial}{\partial t}F_1(t,x_1)=-\langle p_1,\frac{\partial}{\partial q_1}\rangle F_1(t,x_1)+\epsilon^2\int_{\mathbb{R}^3\times\mathbb{S}^2_+}dp_2d\eta\,\langle\eta,(p_1-p_2)\rangle\times$$

$$\times\big(F_2(t,q_1,p_1^*,q_1-\epsilon\eta,p_2^*\mid F_1(t))-F_2(t,x_1,q_1+\epsilon\eta,p_2\mid F_1(t))\big),$$

$$F_1(t,x_1)_{|t=0}=F_1^{0,\epsilon}(x_1).$$

V.I. Gerasimenko, I.V. Gapyak.
Kinet. Relat. Models, **5** (3) **2012**

2.2. The existence theorem

Theorem

A global in time solution of the Cauchy problem of the generalized Enskog equation is determined by the expansion

$$F_1(t,x_1)$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^3\times\mathbb{R}^3)^n} dx_2 \dots dx_{n+1}\, \mathfrak{A}_{1+n}(-t) \prod_{i=1}^{n+1} F_1^{0,\epsilon}(x_i) \mathcal{X}_{\mathbb{R}^{3(1+n)}\setminus\mathbb{W}_{1+n}}.$$

If $\|F_1^0\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)} < e^{-10}(1+e^{-9})^{-1}$*, then for* $F_1^0 \in L^1_0(\mathbb{R}^3\times\mathbb{R}^3)$ *it is a strong (classical) solution and for an arbitrary initial data* $F_1^0 \in L^1(\mathbb{R}^3\times\mathbb{R}^3)$ *it is a weak (generalized) solution.*

2.3. Some remarks

1. Bogolyubov kinetic equation
2. On the modified Enskog kinetic equation
3. Properties of Enskog collision integral

A. S. Trushechkin.
Kinet. Relat. Models, **7** (4) **2014**

3. Approaches to the derivation of the Enskog equation
3.1. The proof of a main result

$$F_s(t,x_1,\ldots,x_s)=\sum_{n=0}^{\infty}\frac{1}{n!}\int_{(\mathbb{R}^3\times\mathbb{R}^3)^n}dx_{s+1}\ldots$$

$$dx_{s+n}\mathfrak{A}_{1+n}(-t,\{Y\},X\setminus Y)\prod_{i=1}^{s+n}F_1^{0,\epsilon}(x_i)\mathcal{X}_{\mathbb{R}^{3(s+n)}\setminus\mathbb{W}_{s+n}},\quad s\geq 2$$

The examples of kinetic cluster expansions

$$\mathfrak{A}_1(-t,\{Y\})\mathcal{X}_{\mathbb{R}^{3s}\setminus\mathbb{W}_s}\prod_{i=1}^{s}\mathfrak{A}_1(-t,i)\doteq\widehat{\mathfrak{A}}_1(t,\{Y\})=\mathfrak{V}_1(t,\{Y\}),$$

$$\widehat{\mathfrak{A}}_2(t,\{Y\},s+1)=\mathfrak{V}_2(t,\{Y\},s+1)+\mathfrak{V}_1(t,\{Y\})\sum_{i_1=1}^{s}\widehat{\mathfrak{A}}_2(t,i_1,s+1),$$

Kinetic cluster expansions

$$\mathfrak{A}_{1+n}(-t,\{Y\},X\setminus Y)\mathfrak{I}_{s+n}(X)$$

$$=\sum_{k_1=0}^{n}\frac{n!}{(n-k_1)!k_1!}\mathfrak{V}_{1+n-k_1}(t,\{Y\},s+1,\ldots,s+n-k_1)$$

$$\times\sum_{k_2=0}^{k_1}\frac{k_1!}{k_2!(k_1-k_2)!}\ldots\sum_{k_{n-k_1+s}=0}^{k_{n-k_1+s-1}}\frac{k_{n-k_1+s-1}!}{k_{n-k_1+s}!(k_{n-k_1+s-1}-k_{n-k_1+s})!}$$

$$\times\prod_{i=1}^{s+n-k_1}\mathfrak{A}_{1+k_{n-k_1+s+1-i}-k_{n-k_1+s+2-i}}(-t,i,s+n-k_1+1+k_{s+n-k_1+2-i},\ldots$$

$$\ldots,s+n-k_1+k_{s+n-k_1+1-i})\mathfrak{I}_{1+k_{n-k_1+s+1-i}-k_{n-k_1+s+2-i}}(i,$$

$$s+n-k_1+1+k_{s+n-k_1+2-i},\ldots,s+n-k_1+k_{s+n-k_1+1-i}),\quad n\geq 0,$$

$$F_s\big(t,x_1,\ldots,x_s \mid F_1(t)\big)$$
$$\doteq \sum_{n=0}^{\infty}\frac{1}{n!}\int_{(\mathbb{R}^3\times\mathbb{R}^3)^n} dx_{s+1}\ldots dx_{s+n}\,\mathfrak{V}_{1+n}(t,\{Y\},X\setminus Y)\prod_{i=1}^{s+n}F_1(t,x_i),$$

where

$$F_1(t,x_1)$$
$$=\sum_{n=0}^{\infty}\frac{1}{n!}\int_{(\mathbb{R}^3\times\mathbb{R}^3)^n} dx_2\ldots dx_{n+1}\,\mathfrak{A}_{1+n}(-t)\prod_{i=1}^{n+1}F_1^{0,\epsilon}(x_i)\mathcal{X}_{\mathbb{R}^{3(1+n)}\setminus\mathbb{W}_{1+n}},$$

3.2. The dual BBGKY hierarchy

$$(B(t),F(0)) = (B(0),F(t)) =$$
$$= \sum_{s=0}^{\infty}\frac{1}{s!}\int_{(\mathbb{R}^3\times\mathbb{R}^3)^s} dx_1\ldots dx_s\, B_s^{0,\epsilon}(x_1,\ldots,x_s)\, F_s(t,x_1,\ldots,x_s)$$

$$B_s(t,x_1,\ldots,x_s) = \sum_{n=0}^{s}\frac{1}{n!}\sum_{j_1\neq\ldots\neq j_n=1}^{s}\mathfrak{A}_{1+n}\big(t,\{Y\setminus Z\},Z\big)\,B_{s-n}^{0,\epsilon}(x_1,$$
$$\ldots x_{j_1-1},x_{j_1+1},\ldots,x_{j_n-1},x_{j_n+1},\ldots,x_s),\quad s\geq 1$$

$$\mathfrak{A}_{1+n}(t,\{Y\setminus Z\},Z) \doteq \sum_{\mathrm{P}:\,(\{Y\setminus Z\},Z)=\bigcup_i X_i}(-1)^{|\mathrm{P}|-1}(|\mathrm{P}|-1)!\prod_{X_i\subset \mathrm{P}} S_{|\theta(X_i)|}(t,\theta(X_i)),$$
$$Y\equiv(1,\ldots,s),\quad Z\equiv(j_1,\ldots,j_n),\quad \theta(\{Y\setminus Z\},Z) = Y$$

Theorem

If the initial state

$$F^{(c)}(0)=\big(1,F_1^{0,\epsilon}(x_1),\ldots,\prod_{i=1}^{s}F_1^{0,\epsilon}(x_i)\mathcal{X}_{\mathbb{R}^{3s}\setminus\mathbb{W}_s^{\epsilon}},\ldots\big),$$

then the equality holds

$$\big(B(t),F^{(c)}(0)\big)=\big(B(0),F(t\mid F_1(t))\big),$$

where the sequence of marginal functionals of the state

$$F(t\mid F_1(t))=\big(1,F_1(t),F_2(t\mid F_1(t)),\ldots,F_s(t\mid F_1(t)),\ldots\big)$$

was defined above.

In case of initial data specified by additive-type marginal observables, i.e. $B^{(1)}(0)=(0,b_1,0,\ldots)$, the equality holds

$$\big(B^{(1)}(t),F^{(c)}(0)\big)=\int_{\mathbb{R}^3\times\mathbb{R}^3}dx_1\,b_1(x_1)F_1(t,x_1),$$

where the one-particle marginal distribution function $F_1(t)$ is the solution of the generalized Enskog kinetic equation.

If initial data specified by the s-ary marginal observable ($s\geq 2$), i.e. $B^{(s)}(0)=(0,\ldots,0,b_s,0,\ldots)$, the equality holds

$$\big(B^{(s)}(t),F^{(c)}(0)\big)=\frac{1}{s!}\int_{(\mathbb{R}^3\times\mathbb{R}^3)^s}dx_1\ldots dx_s\,b_s(x_1,\ldots,x_s)F_s(t\mid F_1(t)),$$

where $F_s(t\mid F_1(t))$ is the marginal functionals of the state.

3.3. The Enskog kinetic equation with initial correlations

Let initial states

$$F^{(cc)}(0)=\big(1,F_1^{0,\epsilon}(x_1),\ldots,\prod_{i=1}^{n}F_1^{0,\epsilon}(x_i)g_n,\ldots\big),$$

where $g_n\equiv g_n(x_1,\ldots,x_n)$ is bounded function characterizing the correlation of the initial states, then the equality holds

$$\big(B(t),F^{(cc)}(0)\big)=\big(B(0),F(t\mid F_1(t))\big),$$

where $F(t\mid F_1(t))=(F_1(t,x_1),\ldots,F_s(t,x_1,\ldots,x_s\mid F_1(t)),\ldots)$

$$F_s\big(t, x_1, \ldots, x_s \mid F_1(t)\big) \doteq$$
$$\doteq \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^3\times\mathbb{R}^3)^n} dx_{s+1} \ldots dx_{s+n}\, \mathfrak{B}_{1+n}(t, \{Y\}, X \setminus Y) \prod_{i=1}^{s+n} F_1(t, x_i),$$

Examples of generating operators

$$\mathfrak{B}_1(t, \{Y\}) = \widehat{\mathfrak{A}}_1(t, \{Y\}) \doteq S_s^*(t, Y) g_s(x_1, \ldots, x_s) \prod_{i=1}^{s} S_1^*(-t, i),$$

$$\mathfrak{B}_2(t, \{Y\}, s+1) = \widehat{\mathfrak{A}}_2(t, \{Y\}, s+1) - \widehat{\mathfrak{A}}_1(t, \{Y\}) \sum_{i_1=1}^{s} \widehat{\mathfrak{A}}_2(t, i_1, s+1).$$

4. The Boltzmann – Grad scaling behavior of the Enskog equation
4.1. The Boltzmann–Grad limit theorem

Theorem

If initial one-particle marginal distribution function $F_1^{0,\epsilon}$ *satisfies the condition:* $|F_1^{0,\epsilon}(x_1)| \leq c e^{-\frac{\beta}{2} p_1^2}$ *and there exists its limit:* $\mathrm{w-}\lim_{\epsilon\rightarrow 0}(\epsilon^2 F_1^{0,\epsilon}(x_1) - f_1^0(x_1)) = 0$, *then for finite time interval the Boltzmann–Grad limit of a solution of the Cauchy problem of the generalized Enskog equation exists in the same sense*

$$\mathrm{w-}\lim_{\epsilon\rightarrow 0} \big(\epsilon^2 F_1(t,x_1) - f_1(t,x_1)\big) = 0,$$

where the limit one-particle distribution function is represented by the uniformly convergent on arbitrary compact set series expansion which is a weak solution of the Cauchy problem of the Boltzmann kinetic equation with hard sphere collisions.

4.2. A Boltzmann-Grad scaling limit of marginal functionals of the state

The Boltzmann–Grad limit of marginal functionals of the state exists in the same sense

$$\mathrm{w}-\lim_{\epsilon\rightarrow 0}\big(\epsilon^{2s}F_s\big(t,x_1,\ldots,x_s\mid F_1(t)\big)-\prod_{j=1}^{s}f_1(t,x_j)\big)=0,$$

Thus, in the Boltzmann–Grad limit for the marginal functionals of the state the chaos condition holds.

In order to establish this fact we prove that for generating operators such equalities in weak sense hold

$$\mathrm{w}-\lim_{\epsilon\rightarrow 0}\big(\mathfrak{V}_1(t,\{Y\})f_s-If_s\big)=0,$$
$$\mathrm{w}-\lim_{\epsilon\rightarrow 0}\epsilon^{-2n}\mathfrak{V}_{1+n}(t,\{Y\},X\setminus Y)f_{s+n}=0,\quad n\geq 1,$$

4.3. The Boltzmann-Grad scaling limit of the Enskog equation with initial correlations

The Boltzmann-type kinetic equation with initial correlations

$$f^{(cc)} = \big(1, f_1^0(x_1), g_2(x_1,x_2)\prod_{i=1}^{2} f_1^0(x_i), \ldots, g_n(x_1,\ldots,x_n)\prod_{i=1}^{n} f_1^0(x_i), \ldots\big).$$

$$\frac{\partial}{\partial t} f_1(t,x_1) = -\langle p_1, \frac{\partial}{\partial q_1}\rangle f_1(t,x_1) +$$

$$\int\limits_{\mathbb{R}^3\times\mathbb{S}^2_+} dp_2\, d\eta\, \langle \eta, (p_1-p_2)\rangle \Big(g_2(q_1 - p_1^* t, p_1^*, q_2 - p_2^* t, p_2^*) f_1(t,q_1,p_1^*) \times$$

$$\times f_1(t,q_1,p_2^*) - g_2(q_1 - p_1 t, p_1, q_2 - p_2 t, p_2) f_1(t,x_1) f_1(t,q_1,p_2)\Big),$$

$$f_1(t,x_1)|_{t=0} = f_1^0(x_1).$$

The propagation of initial correlations in the Boltzmann-Grad limit

The Boltzmann-Grad scaling limit of the marginal functional of the state exists in the same sense

$$\mathrm{w-}\lim_{\epsilon\rightarrow 0}\big(\epsilon^{2s}F_s(t,x_1,\ldots,x_s\mid F_1(t))-$$
$$-g_s(q_1-tp_1,p_1;\ldots;q_s-tp_s,p_s)\prod_{j=1}^{s}f_1(t,x_j)\big)=0,$$

where the limiting one-particle distribution function f_1 is the solution of the Boltzmann-type kinetic equation with initial correlations.

Thus, in the Boltzmann–Grad limit the initial correlations are saved.

5. Outlook

5.1. Some open mathematical problems

$*$ High-order corrections to scaling asymptotics

$*$ Collisional kinetic theory of the Fokker – Planck equation
I.V. Gapyak and V.I. Gerasimenko.
Reports of NAS of Ukraine **12** 2014

$*$ The creation and the propagation of correlations in the Enskog gas
V.I. Gerasimenko, D.O. Polishchuk.
Math. Meth. Appl. Sci. **36** (16) 2013

5.2. Some applications

a. Granular gases
M.S. Borovchenkova and V.I. Gerasimenko.
J. Phys. A: Math. Theor. **47** (3) (2014) 035001

b. Kinetic equations of systems in condensed states

c. Active soft matter: living organism (biokinetics)
Yu.Yu. Fedchun and V.I. Gerasimenko.
In: Semigroups of Operators – Theory and Applications.
Series: Springer Proceedings in Mathematics and Statistics,
113, pp. 165–182. Springer, 2015

Et cetera...