Stability of a collisionless plasma in a solid torus

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Joint work with Walter Strauss

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- The dynamics of ions / electrons: $f^{\pm}(t,x,v) \geq 0$ solves Vlasov

$$f_t^{\pm} + \hat{v} \cdot \nabla_x f^{\pm} \pm \Big(E + \hat{v} \times B \Big) \cdot \nabla_v f^{\pm} = 0$$

on $\Omega \times \mathbb{R}^3$ (ignore collisions), with the fields E, B solving the Maxwell equations. In term of potentials (Coulomb gauge $\nabla \cdot A = 0$):

$$-\Delta \phi = \rho, \qquad \partial_t^2 A - \Delta A = j - \partial_t \nabla \phi.$$

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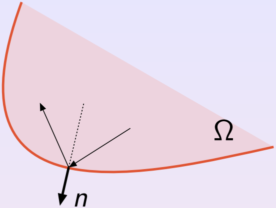
• Relativistic VM: particle velocity $\hat{v} = \frac{v}{\langle v \rangle}$, with $\langle v \rangle = \sqrt{1 + |v|^2}$.

• Wellposedness is a big open problem (Ω with or without a boundary)! I focus on the issue of the stability theory of equilibria.



When Ω has a boundary, we impose

- Specular boundary condition on $f^{\pm}$:

$$f^{\pm}(t,x,v) = f^{\pm}(t,x,v-2(v\cdot n)n)$$

for $n \cdot v < 0$ and $x \in \partial\Omega$.

- Perfect conductor boundary conditions on E, B:

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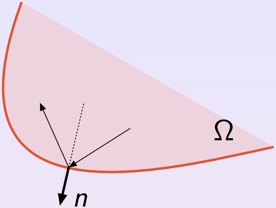

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• Specular BC assures casimirs' invariants: $\frac{d}{dt}\iint_{\Omega\times\mathbb{R}^3}\Phi(f^{\pm})\,dxdv = 0$.

• Perfect conductor BC (in fact, $(E\times B)\cdot n{=}0$) assures the conservation of energy:

$$\iint_{\Omega\times\mathbb{R}^3}\langle v\rangle(f^+ + f^-)\,dvdx + \frac{1}{2}\int_\Omega\Big(|E|^2+|B|^2\Big)\,dx = E_0.$$

- Equilibria: infinitely many (e.g., Rein '92), including

$$f^+ = \mu^+(e^+), \qquad f^- = \mu^-(e^-)$$

for arbitrary $\mu^\pm(\cdot)$, where $e^\pm := \langle v \rangle \pm \phi(x)$ denote the particle energy (invariant), together with Maxwell equations: $j = 0, B = 0$, and

$$-\Delta\phi = \int_{\mathbb{R}^3} (\mu^+ - \mu^-)\, dv, \qquad \phi_{|\partial\Omega} = \mathrm{const}.$$

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- In a domain with symmetry: additional conservations. Equilibria:

$$f^+ = \mu^+(e^+, p^+), \qquad f^- = \mu^-(e^-, p^-).$$

  - Guo '99: Ω is x_3-translation invariant. $p^\pm := v_3 \pm A_3(x_1, x_2)$
  - Guo'99, Lin-Strauss '07-08: Ω is rotational invariant. $p^\pm := r(v_\theta \pm A_\theta(r, z))$
  - Maxwell becomes a semi-linear elliptic system for (ϕ, A_3).

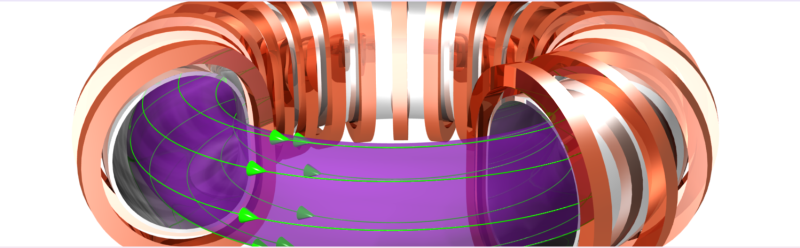

Figure : *Illustrated is a tokamak! Figure credit: internet.*

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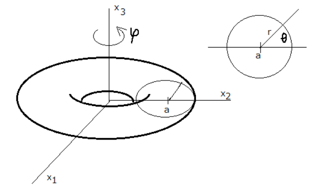
$\Omega =$ 

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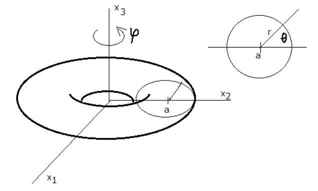
$\Omega =$ 

Figure : *Shown are toroidal coordinates* (r, θ, φ). *Set* $\beta := a + r\cos\theta$.

- Particle energy and angular momentum invariants:

$$e^{\pm}(x, v) := \langle v \rangle \pm \phi^0(r, \theta), \qquad p^{\pm}(x, v) := \beta(v_\varphi \pm A^0_\varphi(r, \theta)).$$

- Equilibria: $f^{\pm} = \mu^{\pm}(e^{\pm}, p^{\pm})$. Elliptic problem for scalar potentials:

$$-\Delta\phi^0 = \rho^0, \qquad \Big(-\Delta + \frac{1}{\beta^2}\Big) A^0_\varphi = j^0_\varphi, \qquad (\phi, \beta A_\varphi)_{|\partial\Omega} = \mathrm{const}.$$

- Quick theorem: if $\|\mu^{\pm}\|_{\mathrm{Lip}_\omega} \ll 1$, equilibria exist: $\phi^0, A^0_\varphi \in C^{2+\alpha}(\Omega)$.

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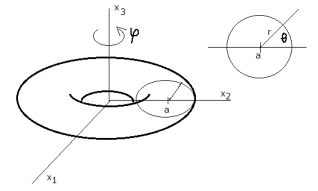
- Objective: a linear stability theory for "monotonic" equilibria ($\mu^{\pm}_e < 0$)

$$E^0 = -\nabla\phi^0 = \partial_r\phi^0 e_r + \frac{1}{r}\partial_\theta\phi^0 e_\theta \quad \in \quad \mathrm{span}\{e_r, e_\theta\}$$

$$B^0_\varphi = 0, \qquad B^0 = \nabla \times A^0_\varphi e_\varphi \quad \in \quad \mathrm{span}\{e_r, e_\theta\}.$$



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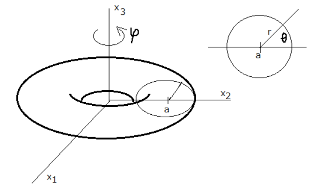
- Boundary conditions:

$$E^0_{\theta\,|\partial\Omega} = 0, \qquad B^0_{r\,|\partial\Omega} = 0.$$



Particle trajectories (for ion particle):

$$\dot{X} = \hat{V}, \qquad \dot{V} = E^0 + \hat{V} \times B^0, \qquad (X, V)_{|_{t=0}} = (x, v),$$

following the rule of specular condition, when it hits $\partial\Omega$.

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Proof.

The map $\Phi_{t(\cdot)}(\cdot)$ is σ-measure preserving on $\partial\Omega \times \mathbb{R}^3$. The Poincaré recurrence theorem asserts that

$$Z := \Big\{(x, v) \in \partial\Omega \times \mathbb{R}^3 \ : \ \sum_{k \geq 0} t_k(\Phi_{t_k(x,v)}(x, v)) < \infty\Big\}$$

must have σ-measure zero! $\square$

Linearization under the toroidal symmetry ($f = f^+$):

$$\begin{aligned} D_t f &= -(E + \hat{v} \times B) \cdot \nabla_v \mu(e, p) \\ &= -\mu_e \hat{v} \cdot E - \beta \mu_p e_\varphi \cdot (E + \hat{v} \times (\nabla \times A^0_\varphi e_\varphi)) \\ &= -\mu_e \hat{v} \cdot E + \beta \mu_p \partial_t A_\varphi + \mu_p \hat{v} \cdot \nabla(\beta A_\varphi). \end{aligned}$$

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Lemma (Energy conservation)

Set $F := f - \beta \mu_p A_\varphi$. The energy functional

$$\mathcal{I}(f^\pm, E, B) := \sum_\pm \iint \Big[\frac{1}{|\mu_e^\pm|} |F^\pm|^2 - \beta \hat{v}_\varphi \mu_p^\pm |A_\varphi|^2 \Big] + \int \Big[|E|^2 + |B|^2 \Big]$$

is independent of time. Stable equilibria: $\mu_p = 0$ and $\mu_e < 0$.

Expanding B, a sufficient condition for stability:

$$\int_\Omega |B|^2 \, \mathrm{d}x = \int_\Omega \Big[|\nabla A_\varphi|^2 + \frac{1}{\beta^2}|A_\varphi|^2 + |B_\varphi|^2 \Big] \, \mathrm{d}x \geq \iint \beta \hat{v}_\varphi \mu_p^\pm |A_\varphi|^2$$

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or equivalently (similar to Guo '99),

$$\mathcal{L}_{\mathrm{Guo}} := (-\Delta + \frac{1}{\beta^2}) - \int \beta \hat{v}_\varphi \mu_p^\pm \, dv \geq 0.$$

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Theorem (N.-Strauss, ARMA 2014)

- *Stable equilibria:* $p\mu_p^\pm \leq 0$ *and* $\|A_\varphi^0\|_{L^\infty} \ll 1$.
- *Stable equilibria:* $|\mu_p^\pm| \ll 1$.

Next, to analyze the role of $|F|^2$ and $|\nabla\phi|^2$ in the energy, we write

$$D_t F = -\mu_e \hat{v} \cdot E = \mu_e \hat{v} \cdot (\nabla\phi + \partial_t A)$$

The linearization now reads

$$\partial_t(F - \mu_e \hat{v} \cdot A) + D(F - \mu_e \phi) = 0. \tag{1}$$

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Classical idea: minimizing the energy functional under the orthogonality constraint (e.g., Lin '04, Lin-Strauss '07). By (1), minimizer (f, ϕ) satisfies

$$F - \mu_e \phi \quad \in \quad \ker(D).$$

- Hence, $F = \mu_e (1 - \mathbb{P})\phi + \mu_e \mathbb{P}(\hat{v} \cdot A)$, with $\mathbb{P} = \mathbb{P}_{\ker(D)}$.

At the minimizer: $F = \mu_e(1-\mathbb{P})\phi + \mu_e\mathbb{P}(\hat{v}\cdot A)$, we compute

$$-\Delta\phi = \int F\,dv + \int \beta\mu_p A_\varphi\,dv \quad\Leftrightarrow\quad \phi = -(\mathcal{A}_1^0)^{-1}(\mathcal{B}^0)^* A_\varphi$$

$$\iint \frac{1}{|\mu_e|}|F|^2 = \|(1-\mathbb{P})\phi\|^2_{L^2_{|\mu_e|}} + \|\mathbb{P}(\hat{v}\cdot A)\|^2_{L^2_{|\mu_e|}}$$

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Theorem (Lin-Strauss '07 (no boundary); N.-Strauss, ARMA 2014)

Sufficient condition for stability: $\mathcal{I}(f^\pm, E, B) \ge 0$, *or equivalently,*

$$\mathcal{L}_{\mathrm{LinStr}} := \mathcal{L}_{\mathrm{Guo}} - \mathcal{B}^0(\mathcal{A}_1^0)^{-1}(\mathcal{B}^0)^* - \int \hat{v}_\varphi\mu_e\mathbb{P}(\hat{v}_\varphi(\cdot))\,dv \ge 0$$

Last two terms are nonnegative (and no contribution from boundary).

Much striking and delicate: $\mathcal{L}_{LinStr} \geq 0$ is also necessary for stability (see also our very next talk!)

Theorem (Lin-Strauss '08 (no boundary); N.-Strauss, ARMA 2014)

Let Ω be a solid torus. Under toroidally symmetric perturbations

- *Equilibrium is linearly stable if and only if*

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- *Stable equilibria: $|\mu_p^\pm| \ll 1$.*
- *Unstable equilibria: $\mu^+(e,p) = \mu^-(e,-p)$ and $p\mu_p^-(e,p) \geq c_0 p^2 \nu(e)$, with $c_0 \gg 1$.*

- A nonlinear stability theory?
  - Guo '99: sufficiency of $\mathcal{L}_{\mathrm{Guo}} \geq 0$ for nonlinear stability (under x_3-translation or rotation symmetry; only study for 3D case). Should also apply to the torus!
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- Linear implies nonlinear instability.
  - A series of Guo-Strauss '95-'00 for homogenous and weakly inhomogenous equilibria, and periodic BGK waves.
  - Lin '04 (for VP) and Lin-Strauss' 07 (for $1\frac{1}{2}DVM$): $\mathcal{L}_{LinStr} \geq 0$ is also necessary for nonlinear stability.
  - Open for the torus or even 3D cases without a boundary!

Let us sign off with the following nonlinear instability of 3D RVM!

Theorem (Instability in the classical limit; Han-Kwan & N., 2015)

Let $\mu(v)$ be smooth unstable equilibrium (in the sense of Penrose). Let $c \gg 1$ be the speed of light. There are smooth solutions (f^c, E^c, B^c) to RVM so that

$$\|\langle v\rangle^m (f^c_{|_{t=0}} - \mu)\|_{H^s} \le c^{-N} \to 0,$$

but at time $t_c \approx \log c$,

$$\liminf_{c\to\infty} \left\| f^c_{|_{t=t_c}} - \mu \right\|_{H^{-s'}} > 0,$$

for arbitrary fixed m, s, s', N.

Classical limit of VM $\to$ VP: Asano-Ukai '86, Degond '86, and Schaeffer '86, within a finite interval of time.