Jeans's Theorem in Kinetic Theory

Student: Zhuo Min 'Harold' Lim
Supervisor: Jonathan Ben-Artzi

30 May, 2014

Contents

1 Introduction **1**
1.1 A Heuristic Derivation of the Vlasov-Poisson System 2
1.2 The Cauchy Problem for the Vlasov-Poisson System 3
1.3 Scope of the Essay 4
1.4 Acknowledgements 4

2 Jeans's Theorem **4**
2.1 Existence of Steady Solutions 4
2.2 A Rigorous Version of Jeans's Theorem 8
2.3 Failure of Jeans's Theorem for the Vlasov-Einstein System 10

3 Steady Solutions from Casimir Functionals **13**
3.1 The Energy-Casimir Method 13
3.2 An Early Stability Result of Camm Steady States 15
3.3 A Reduction Approach for Isotropic Steady States 18
3.4 An Unsuccessful Attempt at Anisotropic Steady States 24
3.5 More Recent Stability Results 34

A Rearrangements of Functions **35**

1 Introduction

The Vlasov-Poisson system is one of the fundamental systems in kinetic theory. It describes a very large, dilute collection of particles interacting with each other only via a Newtonian force; the effects of collisions or other body forces are assumed to be negligible. As such, it is widely used in such scientific disciplines as plasma physics or stellar dynamics.

The Vlasov-Poisson system consists of a Vlasov equation defining the evolution of a phase-space density function $f = f(t, x, v)$, with the forcing term being the negative gradient of the Newtonian potential of the spatial density function $\rho_{[f]}$. It reads

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \nabla U_{[f(t)]} \cdot \nabla_v f = 0$$

where, for $g = g(x, v) \in L^1(\mathbb{R}^6)$, $U_{[g]} : \mathbb{R}^3 \to \mathbb{R}$ satisfies Poisson's equation

$$\triangle U_{[g]} = 4\pi\gamma\rho_{[g]} \ , \quad \lim_{|x|\to\infty} U_{[g]}(x) = 0 \ .$$

with $\rho_{[g]} : \mathbb{R}^3 \to [0, \infty)$ denoting the spatial density function of g, i.e.

$$\rho_{[g]}(x) := \int_{\mathbb{R}^3} g(x, v) \, \mathrm{d}v \ .$$

Here, $\gamma \in \{\pm 1\}$ defines the nature of the force. The case $\gamma = +1$ describes an attractive force such as that of Newtonian gravitation; we henceforth refer to this as the *gravitational* case. The case $\gamma = -1$ describes a repulsive force such as Coulomb repulsion between like charges, and will henceforth be called the *plasma physics* case.

Various generalisations and alternatives of the Vlasov-Poisson system are possible, and some have been intensively studied as well. For example, it may be desired to study plasma comprising more than one species of particles, with differing charges, in which case one could consider a Vlasov-Poisson system comprising a Vlasov equation for each particle species, and a Poisson equation describing the interaction between different particle species. If collisions are not longer negligible, it may be desired to include a Boltzmann collision term in the system. Finally, there also exist other Vlasov-type system, such as the Vlasov-Maxwell system describing collisionless systems with electromagnetic interaction, or the Vlasov-Einstein system in case of a spacetime with nontrivial curvature. With the exception of Subsection 2.3, we do not consider these systems in this essay.

1.1 A Heuristic Derivation of the Vlasov-Poisson System

In this subsection we sketch a very formal and heuristic derivation of the Vlasov-Poisson system from Hamiltonian dynamics. The reader is warned that this subsection contains no rigorous mathematics, and is included only to motivate the study of the Vlasov-Poisson system. Indeed, the rigorous derivation of kinetic equations from particle dynamics is a difficult subject, in which there has been and still is active research.

Consider a system of $N \gg 1$ point particles of equal mass (which we may assume to be $\frac{1}{N}$, by making a suitable choice of units), interacting with each other via a potential ψ and also with an external potential ϕ. A state of the system can be described by a point $(X, V) \in \mathbb{R}^{3N} \times \mathbb{R}^{3N}$, where $X = (x_1, \ldots, x_N) \in (\mathbb{R}^3)^N$ are the positions of the particles and $V = (v_1, \ldots, v_N) \in (\mathbb{R}^3)^N$ are the velocities of the particles. The Hamiltonian of the system is

$$H(X, V) = \sum_{i=1}^{N} \frac{1}{2} |v_i|^2 + \frac{1}{2} \sum_{i \neq j} \psi(x_i - x_j) + \sum_{i=1}^{N} \phi(x_i) ,$$

and the system evolves according to Hamilton equations of motions

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{\partial H}{\partial v_i}(X, V) , \quad \frac{\mathrm{d}v_i}{\mathrm{d}t} = -\frac{\partial H}{\partial x_i}(X, V) .$$

Kinetic theory involves replacing Hamiltonian dynamics by time-evolution of a probability density function on phase space $\mathbb{R}^{6N}$. Specifically, the phase-space density function for N particles, $f_N(t) : \mathbb{R}^{6N} \to [0, \infty)$, should evolve according to

$$\frac{\partial f_N}{\partial t} + \sum_{i=1}^{N} \left(\frac{\partial H}{\partial v_i} \cdot \frac{\partial f_N}{\partial x_i} - \frac{\partial H}{\partial x_i} \cdot \frac{\partial f_N}{\partial v_i} \right) = 0 .$$

For our choice of Hamiltonian, this gives

$$\frac{\partial f_N}{\partial t} + \sum_{i=1}^{N} v_i \cdot \frac{\partial f_N}{\partial x_i} - \sum_{i=1}^{N} \nabla\phi(x_i) \cdot \frac{\partial f_N}{\partial v_i} - \sum_{i \neq j} \nabla\psi(x_i - x_j) \cdot \frac{\partial f_N}{\partial v_i} = 0 .$$

To get further, we now make the assumption of *chaos*: for very large N, the particles are almost independently distributed; we mention that this is the usual practice in kinetic theory, and that the rigorous derivation of the chaos property in kinetic equations from microscopic particle dynamics can be performed for many models, but is by no means straightforward. Thus we plug in an ansatz

$$f_N = f^{\otimes N} = f \otimes \cdots \otimes f$$

for some single-particle phase-space density function $f(t) : \mathbb{R}^6 \to [0, \infty)$. With this ansatz, taking the limit $N \to \infty$ in an appropriate way gives

$$\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial x} - \nabla\phi(x) \cdot \frac{\partial f}{\partial v} - \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \nabla\psi(x-y) f(y,w) \cdot \frac{\partial f}{\partial v}(x) \, \mathrm{d}w \, \mathrm{d}y = 0 \, .$$

In particular, if there is no external field, i.e. $\phi \equiv 0$, and ψ is the Newtonian potential

$$\psi := -\frac{\gamma}{|\cdot|} \, ,$$

then we recover the Vlasov-Poisson system.

1.2 The Cauchy Problem for the Vlasov-Poisson System

A starting point for the mathematical investigation of any evolutionary PDE or PDE system is the study of its Cauchy problem, i.e. issues of existence and uniqueness. The Vlasov-Poisson system is one of the few nonlinear systems of mathematical physics for which there is a reasonably complete theory for its Cauchy problem. Global existence results were obtained around 1989, by Pfaffelmoser [14] and independently by Lions and Perthame [12]. Pfaffelmoser's proof was later considerably simplified by Schaeffer [23]. A very general uniqueness result has also been obtained by Loeper [13] based on the theory of optimal transport.

Theorem 1.1 (Lions, Perthame). *Fix a choice of γ. Let $0 \le f_0 \in L^1 \cap L^\infty(\mathbb{R}^6)$ be such that there exists $s_0 > 3$ so that*

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |v|^s f_0(x,v) \, \mathrm{d}v \, \mathrm{d}x < +\infty \, , \quad \textit{for all } s \in [0, s_0) \, .$$

Then there exists a solution $f \in C^0([0,\infty), L^p(\mathbb{R}^6)) \cap L^\infty([0,\infty), L^\infty(\mathbb{R}^6))$ to the Vlasov-Poisson system with $f(0) = f_0$, and satisfying

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |v|^s f(t,x,v) \, \mathrm{d}v \, \mathrm{d}x < +\infty \, , \quad \textit{for all } s \in [0, s_0)$$

for any $T > 0$. □

The result of Schaeffer assumes an initial datum of class $C^1_{\mathrm{c}}(\mathbb{R}^6)$, but guarantees the existence of a global classical solution which at every time is also of class $C^1_{\mathrm{c}}(\mathbb{R}^6)$, and moreover provides an explicit quantitative bound on how quickly the velocity support can grow. More specifically, we have the following.

Theorem 1.2 (Schaeffer). *Fix a choice of γ. Any initial datum $0 \le f^0 \in C^1_{\mathrm{c}}(\mathbb{R}^6)$ will launch a global classical solution $f : [0,\infty) \times \mathbb{R}^6 \to [0,\infty)$ to the Vlasov-Poisson system such that $f(t) \in C^1_{\mathrm{c}}(\mathbb{R}^6)$ for every $t \ge 0$.*

Moreover, for any $p > \frac{33}{17}$, the solution f satisfies

$$1 + \sup\left\{ |v| \;\middle|\; \exists \tau \in [0,t], x \in \mathbb{R}^3 : f(\tau,x,v) > 0 \right\} \le C(f_0,p)\,(1+t)^p$$

where $C(f_0,p) > 0$ is some constant depending only on f_0 and p. □

The impressive result of Loeper asserts uniqueness in a very general class of solutions, namely the class of measure-valued solutions in the sense of distribution. In the following statement of Loeper's result, $(\mathcal{M}^+(\mathbb{R}^6), \mathrm{w}*)$ denotes the set of bounded positive measures on $\mathbb{R}^6$, endowed with the weak $\sigma(C^0_{\mathrm{b}}(\mathbb{R}^6), \mathcal{M}^+(\mathbb{R}^6))$ topology.

Theorem 1.3 (Loeper). *Fix a choice of γ. Given $f_0 \in \mathcal{M}^+(\mathbb{R}^6)$ and $T > 0$, there can exist at most one solution in $C([0,T), (\mathcal{M}^+(\mathbb{R}^6), \mathrm{w}*))$, starting at f_0, satisfying the Vlasov-Poisson system in the sense of distributions.* □

We omit the proofs of the above results as they are beyond the scope of this essay; the interested reader can find them in the papers cited above.

1.3 Scope of the Essay

In this essay we are concerned with steady (i.e. time-indepedent) solutions to the Vlasov-Poisson system, of finite mass and compact support, satisfying the functional form

$$f(x,v) = F\left(\frac{1}{2}|v|^2 + U_{[f]}(x)\ ,\ |x \wedge v|^2\right) \tag{1}$$

for some function $F = F(E,L) : \mathbb{R}^2 \to [0,\infty)$.

Our interest in this ansatz is due to the so-called "Jeans's theorem" which is often quoted in the astrophysics literature, as a starting point for any discussion. This is the statement that steady solutions to Vlasov-Poisson system can be expressed in terms of first integrals of motion, i.e. quantities which are conserved along the characteristic curves (X,V) to the Vlasov-Poisson system,

$$\dot{X} = V\ ,\quad \dot{V} = -\nabla U_{[f]}(X)\ .$$

Such claims possibly refer to a paper [7] by J. H. Jeans, pioneering the application of kinetic models to the study of stellar dynamics. There (see pp. 79ff) he made the assertion that the phase-space density function is really a function of these first integrals of motion, and then examined a few examples where the phase-space density function is a function of energy density $\frac{1}{2}|v|^2 + U_{[f]}(x)$ and angular momentum density $x \wedge v$. Although he mentioned the example of an "ellipsoidal universe" where another first integral was used instead of the angular momentum density, he dismissed it as artificial and of no physical interest. In short, he suggested that all physically relevant steady solutions to the Vlasov-Poisson system should be functions only of total energy density and angular momentum density.

The ansatz (1) is even more restrictive than the one considered by Jeans, since the phase-space density function f depends only on the magnitude of the angular momentum density, not on its direction. This suggests that our ansatz describes only particle systems with a very high degree of symmetry, e.g. spherically-symmetric solutions.

This essay is divided into two parts. In the first, we will discuss Jeans's theorem and provide a proof in the case of spherically-symmetric solutions; we will also outline a proof that the analogous statement does not hold for the Vlasov-Einstein system. In the second part, we will follow a framework, originally due to G. Rein, to identify some of these solutions as minimizers of certain energy-Casimir functionals. No result listed outside of Subsection 3.4 is due to original work by the author.

1.4 Acknowledgements

The author would like to thank his supervisor Dr. Jonathan Ben-Artzi for suggesting this mini-project, and for being a very helpful and encouraging mentor. This work was supported by a graduate research scholarship at St. John's College, Cambridge, and also by the UK Engineering and Physical Sciences Research Council (EPSRC) grant EP/H023348/1 for the University of Cambridge Centre for Doctoral Training, the Cambridge Centre for Analysis.

2 Jeans's Theorem

2.1 Existence of Steady Solutions

Before embarking on studying steady solutions having the functional form (1), it is natural to ask if steady solutions are known to exist in the first place. Energy estimates provide rather severe constraints on whether steady states can exist.

Definition 2.1. Fix a choice of γ. For $f \in L^1_+(\mathbb{R}^6)$, define

$$E_{\mathrm{kin}}(f) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1}{2}|v|^2 f(x,v)\,\mathrm{d}v\,\mathrm{d}x$$

$$E_{\mathrm{pot}}(f) := -\frac{\gamma}{8\pi}\int_{\mathbb{R}^3} \left|\nabla U_{[f]}(x)\right|^2 \mathrm{d}x\ .$$

We call them the *total kinetic energy* and *total potential energy* of f respectively. □

We now quote the following growth and decay estimates, whose proofs the reader can find in [20], Section 1.7.

Proposition 2.2. *For the plasma physics case $\gamma = -1$, any (global classical) solution f with $f(0) \in C^1_{\mathrm{c}}(\mathbb{R}^6)$ to the Vlasov-Poisson system satisfies, for $t > 0$, the estimates*

$$\left\|\nabla U_{[f(t)]}\right\|_{L^2} \leq C\,(1+t)^{-\frac{1}{2}}$$
$$\left\|\rho_{[f(t)]}\right\|_{L^{5/3}} \leq C\,(1+t)^{-\frac{3}{5}}$$

where $C > 0$ is some constant depending only on $f(0)$. □

Proposition 2.3. *For the gravitational case $\gamma = 1$, suppose f is a (global classical) solution to the Vlasov-Poisson system with $f(0) \in C^1_{\mathrm{c}}(\mathbb{R}^6)$ having positive total energy,*

$$E_{\mathrm{kin}}(f(0)) + E_{\mathrm{pot}}(f(0)) > 0\ .$$

Then

$$C_1 t^2 \leq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |x|^2 f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x \leq C_2 t^2$$

where $C_1, C_2 > 0$ are constants depending only on $f(0)$. □

Using the above estimates we deduce that steady solutions to the Vlasov-Poisson system, in the plasma physics case, are impossible in the absence of external fields or background radiation; and in the gravitational case they can exist only with negative total energy. Hence we make the following convention.

Convention 2.4. Henceforth, in the rest of this essay, unless explicitly stated otherwise, we will consider only the Vlasov-Poisson system in the gravitational case $\gamma = 1$. □

On the other hand, families of steady solutions, necessarily of negative total energy, have been constructed in the gravitational case. The paper [1], aside from providing a rigorous proof of Jeans's theorem, also provides a framework in which certain ansatzes can be shown to lead to steady states which are spherically symmetric, in the following sense.

Definition 2.5. We say a function $f : \mathbb{R}^6 \to \mathbb{R}$ is *spherically symmetric* if $f(Ax, Av) = f(x, v)$ for all rotations $A \in \mathrm{SO}(3)$. □

Notice that a spherically symmetric function is therefore a function only of the variables

$$r := |x|\ , \quad w := \frac{x \cdot v}{r}\ , \quad L := |x \wedge v|^2\ .$$

For simplicity we will ignore the set $\{r = 0\}$ or $\{L = 0\}$ in phase space, i.e. we assume the phase-space density function f is defined only on $\{(x, v) \in \mathbb{R}^6 \mid x \wedge v \neq 0\}$.

We make the following conventions.

Convention 2.6. In this subsection and the next, we consider a steady, spherically symmetric classical solution $0 \leq f = f(x, v) \in C^1(\mathbb{R}^6) \cap L^1(\mathbb{R}^6)$ to the Vlasov-Poisson system. We express such a solution as $f(x, v) = \varphi(r, w, L)$ on $(r, w, L) \in (0, \infty) \times \mathbb{R} \times (0, \infty)$.

We also write $U_{[\varphi]} = U_{[\varphi]}(r)$ for the solution of Poisson's equation

$$\frac{1}{r^2}\left(r^2 U'_{[\varphi]}(r)\right)' = 4\pi\rho_{[\varphi]}(r)\ , \quad \lim_{r\to\infty} U_{[\varphi]}(r) = 0\ ,$$

where $\rho_{[\varphi]} = \rho_{[\varphi]}(r)$ is

$$\rho_{[\varphi]}(r) = \frac{\pi}{r^2}\int_{-\infty}^{\infty}\mathrm{d}w\int_0^{\infty}\mathrm{d}L\ \varphi(r, w, L)\ .$$ □

We now return to the framework described in [1] for constructing spherically symmetric steady states. Suppose we postulate an ansatz satisfying the functional form (1), i.e. we postulate that there exists a "nice" function F such that

$$\varphi(r, w, L) = F\left(\frac{1}{2}|w|^2 + \frac{L}{2r^2} + U(r), L\right) .$$

Then the function $U_{[\varphi]}$ satisfies the differential equation

$$\frac{1}{r^2}\left(r^2 U'_{[\varphi]}(r)\right)' = \frac{4\pi^2}{r^2}\int_{-\infty}^{\infty} \mathrm{d}w \int_0^{\infty} \mathrm{d}L \; F\left(\frac{1}{2}|w|^2 + \frac{L}{2r^2} + U_{[\varphi]}(r), L\right) .$$

This motivates the more general study of the differential equation

$$\frac{1}{r^2}\left(r^2 U'(r)\right)' = h\left(r, U(r)\right) . \tag{$*$}$$

The main idea of the framework in [1] is to prove the existence of a solution U to $(*)$ given h, and to construct the corresponding steady solution φ such that $U_{[\varphi]} = U + \text{const}$ (the additive constant is necessary to ensure $U_{[\varphi]}(r) \to 0$ as $r \to \infty$. We state the relevant theorems below. We omit the rather technical proofs, for which we refer the reader to the paper [1].

Theorem 2.7. *Let $\alpha \in \mathbb{R}$ and let $h : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ be a measurable function such that*

(i) The function $s \mapsto sh(s, \alpha)$ is $L^1_{\mathrm{loc}}([0,\infty))$.

(ii) For all $(r_0, u_0) \in \{(0,\alpha)\} \cup (0,\infty) \times \mathbb{R}$, there exist $\delta > 0$ and a function

$$L_{[r_0,u_0]} : (r_0, r_0 + \delta) \to [0, \infty]$$

such that $(r - r_0)L_{[r_0,u_0]} \in L^1([r_0, r_0 + \delta])$, and

$$\left|h(r, u_1) - h(r, u_2)\right| \le L_{[r_0,u_0]}(r)\,|u_1 - u_2| \quad \forall\, r \in (r_0, r_0 + \delta), u_1, u_2 \in [u_0 - \delta, u_0 + \delta] .$$

(iii) There exists $0 \le H \in L^1_{\mathrm{loc}}((0,\infty))$ such that

$$|h(r, u)| \le H(r)\,(1 + |u|) \quad \forall\, r > 0, u \in \mathbb{R} .$$

Then $(*)$ has a unique solution $U : [0,\infty) \to \mathbb{R}$ with $U(0) = \alpha$. □

Theorem 2.8. *Let $\alpha, \beta \in \mathbb{R}$, let $r_* > 0$ and let $h : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ be a measurable function, such that*

(i) For all $(r_0, u_0) \in \{(0,\alpha)\} \cup (0,\infty) \times \mathbb{R}$, there exist $\delta > 0$ and a function

$$L_{[r_0,u_0]} : (r_0 - \delta, r_0 + \delta) \to [0, \infty]$$

such that $(r - r_0)L_{[r_0,u_0]} \in L^1([r_0 - \delta, r_0 + \delta])$, and

$$\left|h(r, u_1) - h(r, u_2)\right| \le L_{[r_0,u_0]}(r)\,|u_1 - u_2| \quad \forall\, r \in (r_0 - \delta, r_0 + \delta), u_1, u_2 \in [u_0 - \delta, u_0 + \delta] .$$

(ii) There exists $0 \le H \in L^1_{\mathrm{loc}}((0,\infty))$ such that

$$|h(r, u)| \le H(r)\,(1 + |u|) \quad \forall\, r > 0, u \in \mathbb{R} .$$

Then $(*)$ has a unique solution $U : (0,\infty) \to \mathbb{R}$ with $U(r_*) = \alpha, U'(r_*) = \beta$. □

These theorems allows us to reverse our previous heuristic argument. Specifically, suppose that $F_0 = F_0(E, L)$ is given, such that

$$h(r,u) := \frac{4\pi^2}{r^2}\int_{-\infty}^{\infty} \mathrm{d}w \int_0^{\infty} \mathrm{d}L\ F_0\left(\frac{1}{2}|w|^2 + \frac{L}{2r^2} + u, L\right)$$

satisfies the conditions of either Theorem 2.7 or Theorem 2.8. Having constructed the solution U to $(*)$, we may set

$$\varphi(r,w,L) := F_0\left(\frac{1}{2}|w|^2 + \frac{L}{2r^2} + U(r), L\right) .$$

Then $(*)$ simply says that U satisfies Poisson's equation

$$\frac{1}{r^2}\left(r^2 U'(r)\right)' = h(r, U(r)) = \frac{1}{4\pi}\rho_{[\varphi]}(r)$$

and therefore the potential energy density of φ satisfies

$$U_{[\varphi]}(r) = U(r) - U(\infty) .$$

Hence φ is indeed a solution to the time-independent Vlasov-Poisson system.

The following two families of solutions were constructed by this approach in [1].

Example 2.9. For $k > -1, l > -1, E_0 > 0$ such that $k + l + \frac{3}{2} > 0$, the ansatz

$$f_0(x,v) = \left(E_0 - \frac{1}{2}|v|^2 - U_{[f_0]}(x)\right)_+^k |x \wedge v|^{2l}$$

leads to a family of steady solutions to the Vlasov-Poisson system. These are known as *polytropic steady states*. For $k < 3l + \frac{7}{2}$, such steady states are known to have finite total mass and compact support. □

Example 2.10. For $k > -1, l > -1, E_0 > 0, \Gamma > 0$ such that $k + l + \frac{3}{2} > 0$, the ansatz

$$f_0(x,v) = \left(E_0 - \frac{1}{2}|v|^2 - U_{[f_0]}(x) - \Gamma\,|x \wedge v|^2\right)_+^k |x \wedge v|^{2l}$$

leads to a family of steady solutions to the Vlasov-Poisson system (see [1]). These are known as *Camm steady states*, after the paper [4] of G. L. Camm. For $k < 3l + \frac{7}{2}$ and Γ small enough, such steady states are known to have finite total mass and compact support. □

As a final note, we mention that steady solutions to the Vlasov-Poisson system, that are not spherically symmetric, are known to exist.

In [17], families of steady solutions, which are axially symmetric but are not spherically symmetric, are constructed by a perturbation about spherically symmetric solutions. More precisely, an ansatz of the form $f(x,v) = \phi(E_{[f]})\psi(\vartheta(x_1v_2 - x_2v_1))$, where ϕ, ψ are functions satisfying some given assumptions and ϑ is a small parameter, is shown to lead to a family of steady solutions. The solutions in this family are axially symmetric about the x_3-axis, and depend continuously on ϑ. Furthermore, the solutions with $\vartheta \neq 0$ are not spherically symmetric. These solutions are also shown to have finite mass and finite radius.

More recently, in the related work [25], steady solutions to the Vlasov-Poisson system, which are axially but not spherically symmetric, have been constructed by postulating an ansatz involving Jacobi's integral. These solutions describe rotating systems, and are also shown to have finite mass and finite radius.

We will make no use of these constructions in the sequel.

2.2 A Rigorous Version of Jeans's Theorem

As mentioned in the Introduction, the term "Jeans's theorem" is generally used to refer to claims that steady solutions to the Vlasov-Poisson system can be expressed as first integrals of the characteristic motion. A precise statement backed up by a rigorous proof does not seem to be available in the astrophysics literature except in certain trivial situations. For example, the version stated in the textbook [2] makes the very strong assumption that action-angle coordinates exist everywhere in phase space, from which Jeans's theorem is an immediate consequence (since we can simply write the phase-space density function in these coordinates).

To the author's knowledge, the first rigorous proof of Jeans's theorem was published by J. Batt, W. Faltenbacher and E. Horst [1] for steady spherically symmetric solutions to the Vlasov-Poisson system. Recently, G. Rein [21] obtained a proof of Jeans's theorem for steady spherically symmetric solutions to the Vlasov-MOND equation, a generalisation of the Vlasov-Poisson system based on M. Milgrom's theory of modified Newtonian dynamics (MOND).

The goal of this subsection is to present the proof of Rein, specialised to the Vlasov-Poisson system. We remind the reader that Convention 2.6 applies to this subsection.

In the (r, w, L) coordinates, the steady (time-indepedent) Vlasov-Poisson system is

$$w\frac{\partial \varphi}{\partial r} - \frac{\partial \Upsilon_{[\varphi]}}{\partial r}(r, L)\frac{\partial \varphi}{\partial w} = 0$$

where $\Upsilon_{[\varphi]} : (0, \infty) \times (0, \infty) \to \mathbb{R}$ denotes the effective potential energy density,

$$\Upsilon_{[\varphi]}(r, L) := \frac{L}{2r^2} + U_{[\varphi]}(r) \ .$$

Let us define

$$m_{[\varphi]}(r) := 4\pi \int_0^r s^2 \rho_{[\varphi]}(s) \ \mathrm{d}s \ .$$

Then we have an explicit formula for $U_{[\varphi]}$, namely

$$U_{[\varphi]}(r) = -\int_r^\infty \frac{m_{[\varphi]}(s)}{s^2} \ \mathrm{d}s \ .$$

Indeed, $U_{[\varphi]}$, given by this formula, satisfies Poisson's equation, and the condition $f \in L^1(\mathbb{R}^6)$ guarantees that $\rho_{[\varphi]}$ is integrable, so that $U_{[\varphi]}(r) \to 0$ as $r \to \infty$.

Lemma 2.11. *For any $L > 0$, the function $\Upsilon_{[\varphi]}(\cdot, L)$ has a unique global minimum $r_{\min}(L)$, and $\Upsilon_{[\varphi]}(\cdot, L)$ is strictly decreasing on $(0, r_{\min}(L))$ and strictly increasing on $(r_{\min}(L), \infty)$.*

Moreover $r_{\min}$ is a continuous, strictly increasing function of L.

Proof. Let

$$r_0 := \sup\left\{ r > 0 \;\middle|\; m(r) = 0 \right\} \ .$$

We find $(rm_{[\varphi]}(r))' = m_{[\varphi]}(r) + 4\pi\rho_{[\varphi]}(r)$, so $rm_{[\varphi]}(r)$ is zero on $(0, r_0)$ and strictly increasing on (r_0, ∞). Moreover, it is clear that $rm_{[\varphi]}(r) \to \infty$ as $r \to \infty$. Now,

$$\frac{\partial \Upsilon_{[\varphi]}}{\partial r}(r, L) = \frac{rm_{[\varphi]}(r) - L}{r^3} \ .$$

Thus there is exactly one solution $r_{\min}(L)$ to $\frac{\partial \Upsilon_{[\varphi]}}{\partial r}(\cdot, L) = 0$ in $(0, \infty)$. The behaviour of $\Upsilon_{[\varphi]}(\cdot, L)$ follows from noting that $\frac{\partial \Upsilon_{[\varphi]}}{\partial r}(\cdot, L)$ is negative on $(0, r_0)$ and positive on (r_0, ∞).

We have characterised $r_{\min}(L)$ as the unique solution of $rm_{[\varphi]}(r) = L$ in (r_0, ∞). Now $m_{[\varphi]}$ is continuous, so $rm_{[\varphi]}(r)$ gives a continuous increasing bijection, and hence a homeomorphism, $(r_0, \infty) \to (0, \infty)$. Its inverse $r_{\min}$ is therefore continuous and strictly increasing. $\square$

As an immediate consequence,

$$\Upsilon_{[\varphi],\min}(L) := \inf_{r\in\mathbb{R}} \Upsilon_{[\varphi]}(r,L) = \Upsilon_{[\varphi]}\left(r_{\min}(L),L\right)$$

defines a continuous function $(0,\infty)\to\mathbb{R}$, and

$$\Omega := \left\{(E,L)\in\mathbb{R}\times(0,\infty)\;\middle|\; E>\Upsilon_{[\varphi],\min}(L)\right\}$$

defines an open subset of $\mathbb{R}\times(0,\infty)$.

Lemma 2.12. *For any $L>0$,*

$$\lim_{r\to\infty}\Upsilon_{[\varphi]}(r,L)=\infty\ .$$

As such, $\Upsilon_{[\varphi]}(\cdot,L)$ defines a decreasing homeomorphism $(0,r_{\min}(L))\to(\Upsilon_{[\varphi],\min}(L),\infty)$.

Proof. Since

$$\Upsilon_{[\varphi]}(r,L)=\frac{1}{r^2}\left(\frac{L}{2}-r^2U_{[\varphi]}(r)\right)\ ,$$

it suffices to show that $r^2U_{[\varphi]}(r)\to 0$ as $r\to 0$.

If $U(r)$ remains bounded as $r\to 0$, we are done. Otherwise, by l'Hôpital's rule,

$$\lim_{r\to 0} r^2U_{[\varphi]}(r)=\lim_{r\to 0}\left(-\frac{1}{2r^{-3}}U'_{[\varphi]}(r)\right)=\lim_{r\to 0}\left(-\frac{r^3}{2}\frac{m(r)}{r^2}\right)=0$$

as required. □

We are finally ready to state and prove the following version of Jeans's theorem.

Theorem 2.13. *Suppose $f(x,v)=\varphi(r,w,L)$ is a steady, spherically symmetric classical solution to the Vlasov-Poisson system, such that φ is differentiable in the classical sense, and moreover $\rho_{[\varphi]}\in L^\infty_{\mathrm{loc}}((0,\infty))$.*

Then there exists a function $F:\Omega\to[0,\infty)$ such that

$$f(x,v)=F\left(\frac{1}{2}|v|^2+U_{[f]}(x),|x\wedge v|^2\right)\quad \textit{on}\ \left\{(x,v)\in\mathbb{R}^6\;\middle|\; x\wedge v\neq 0\right\}\ .$$

Proof. Define the total energy density by

$$E_{[\varphi]}(r,w,L):=\frac{1}{2}|w|^2+\Upsilon_{[\varphi]}(r,L)\ ,$$

which is numerically equal to $\frac{1}{2}|v|^2+U_{[f]}(x)$.

For any fixed $L>0$, the characteristic system

$$\dot{R}=W\ ,\quad \dot{W}=-\frac{\partial\Upsilon_{[\varphi]}}{\partial r}(R,L) \tag{$*$}$$

has well-defined orbits, because $\rho_{[\varphi]}$ being locally bounded guarantees that $\frac{\partial\Upsilon_{[\varphi]}}{\partial r}(\cdot,L)$ is locally Lipschitz. By direct calculation, we see that both $E_{[\varphi]}(\cdot,\cdot,L)$ and $\varphi(\cdot,\cdot,L)$ are constant on each orbit.

On the other hand, the monotonicity properties of $\Upsilon_{[\varphi]}$, given in Lemma 2.11, guarantees that every nonempty level set of $E_{[\varphi]}(\cdot,\cdot,L)$ is a single connected curve. Such a curve is, by definition, given by $\frac{1}{2}|w|^2+\Upsilon_{[\varphi]}(r,L)=\text{const}$. It is easy to see (by drawing these curves in the (r,w)-plane) that each such curve is an orbit of the characteristic system $(*)$.

By Lemma 2.12, we may define $\Upsilon_{[\varphi]}(\cdot,L)^{-1}:(0,r_{\min}(L))\to(\Upsilon_{[\varphi],\min}(L),\infty)$ to be the inverse function of $\Upsilon_{[\varphi]}(\cdot,L)$. The theorem is proved by simply setting

$$F(E,L):=\varphi\left(\left(\Upsilon_{[\varphi]}(\cdot,L)\right)^{-1}(E),0,L\right)$$

for $E>\Upsilon_{[\varphi],\min}(L)$, i.e. $(E,L)\in\Omega$. □

2.3 Failure of Jeans's Theorem for the Vlasov-Einstein System

In this subsection, which is somewhat tangential to the rest of the essay, we consider the Vlasov-Einstein system, a relativistic version of the gravitational Vlasov-Poisson system. The goal of this subsection is to outline the main idea of a construction, due to Schaeffer [24], of a class of counterexamples to the statement of Jeans's theorem for the Vlasov-Einstein system. Thus, Jeans's theorem does not hold for the Vlasov-Einstein system.

We warn the reader that the actual construction is very long and technical, and here we give a necessarily sketchy exposition of the main ideas of the construction. For the technical details we refer the reader to the original paper [24].

Assume an asymptotically-flat spacetime on which (t, x) coordinates, have been chosen. As before, a spherically symmetric function can be expressed in the variables

$$r := |x| \ , \quad w := \frac{x \cdot v}{r} \ , \quad L := |x \wedge v|^2 \ .$$

The Vlasov-Einstein system for a steady, spherically symmetric phase-space density function $f(x, v) = \varphi(r, w, L) \geq 0$ then reads

$$\frac{w}{\left(1 + w^2 + \frac{L}{r^2}\right)^{\frac{1}{2}}} \frac{\partial \varphi}{\partial r} - \left(1 + w^2 + \frac{L}{r^2}\right)^{\frac{1}{2}} \mu'_{[\varphi]}(r) \frac{\partial \varphi}{\partial w} = 0$$

with $\lambda_{[\varphi]}, \mu_{[\varphi]} : (0, \infty) \to \mathbb{R}$ being the space-time metric coefficients, given by

$$\mathrm{e}^{-2\lambda_{[\varphi]}} \left(2r\lambda'_{[\varphi]} - 1\right) + 1 = 8\pi r^2 \rho_{[\varphi]} \ , \qquad \lim_{r \to \infty} \lambda_{[\varphi]}(r) = 0 \ ,$$

$$\mathrm{e}^{-2\lambda_{[\varphi]}} \left(2r\mu'_{[\varphi]} + 1\right) - 1 = 8\pi r^2 p_{[\varphi]} \ , \qquad \lim_{r \to \infty} \mu_{[\varphi]}(r) = 0 \ ,$$

where

$$\rho_{[\varphi]}(r) := \frac{\pi}{r^2} \int_{-\infty}^{\infty} \mathrm{d}w \int_0^{\infty} \mathrm{d}L \ \varphi(r, w, L) \ ,$$

$$p_{[\varphi]}(r) := \frac{\pi}{r^2} \int_{-\infty}^{\infty} \mathrm{d}w \int_0^{\infty} \mathrm{d}L \ w^2 \varphi(r, w, L) \left(1 + w^2 + \frac{L}{r^2}\right)^{-2} \ .$$

The corresponding space time metric is

$$\mathrm{d}s^2 = -\mathrm{e}^{2\mu_{[\varphi]}(r)} \ \mathrm{d}t^2 + \mathrm{e}^{2\lambda_{[\varphi]}(r)} \ \mathrm{d}r^2 + r^2 \chi$$

where χ is the area 2-form of $\mathbb{S}^2$.

As before, L has the physical meaning of angular momentum. The generalisation of the total energy to the relativistic setting is the relativistic energy,

$$E_{[\varphi]}(r, w, L) := \mathrm{e}^{\mu_{[\varphi]}(r)} \sqrt{1 + w^2 + \frac{L}{r^2}} \ .$$

The statement of Jeans's theorem, in the context of the Vlasov-Einstein system, is that every steady, spherically symmetric solution to the Vlasov-Einstein system satisfies the functional form $\varphi(r, w, L) = F(E_{[\varphi]}(r, w, L), L)$ for some function F.

For convenience, we also define

$$m_{[\varphi]}(r) := \int_0^r 4\pi s^2 \rho_{[\varphi]}(s) \ \mathrm{d}s \ ,$$

Note that the equation for $\lambda_{[\varphi]}$ can be integrated immediately to give

$$\mathrm{e}^{-2\lambda_{[\varphi]}(r)} = 1 - 2\frac{m_{[\varphi]}(r)}{r} \ .$$

Substituting into the equation for $\mu_{[\varphi]}$ gives

$$\mu'_{[\varphi]}(r) = \frac{4\pi r p_{[\varphi]}(r) + r^{-2} m_{[\varphi]}(r)}{1 - 2r^{-1} m_{[\varphi]}(r)} \ .$$

Now we make the ansatz

$$\varphi(r, w, L) = D_0 \left(E_0 - \mathrm{e}^{\mu(r)} \sqrt{1 + w^2 + \frac{L}{r^2}} \right)_+^k (L - L_0)_+^l \tag{$*$}$$

with

$$D_0 > 0 \ , \quad E_0 > 0 \ , \quad L_0 > 0 \ , \quad k > 0 \ , \quad l > -\frac{1}{2} \tag{$*$.1}$$

and

$$\mu = \mu_{[\varphi]} - \text{const} \ . \tag{$*$.2}$$

If φ is such a solution, then μ solves the integro-differential equation

$$\mu'(r) = \frac{4\pi r p(r) + r^{-2} m(r)}{1 - 2r^{-1} m(r)} \tag{$\dagger$}$$

with

$$m(r) := 4\pi^2 D_0 \int_0^r \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}w \int_{L_0}^{\infty} \mathrm{d}L \, \left(E_0 - \mathrm{e}^{\mu(s)} \sqrt{1 + w^2 + \frac{L}{s^2}} \right)_+^k (L - L_0)^l \ , \tag{$\dagger$.1}$$

$$p(r) := \frac{\pi D_0}{r^2} \int_{-\infty}^{\infty} \mathrm{d}w \int_{L_0}^{\infty} \mathrm{d}L \, w^2 \left(E_0 - \mathrm{e}^{\mu(r)} \sqrt{1 + w^2 + \frac{L}{r^2}} \right)_+^k \frac{(L - L_0)^l}{\left(1 + w^2 + \frac{L}{r^2}\right)^{\frac{1}{2}}} \ . \tag{$\dagger$.2}$$

Remark 2.14. Suppose μ is a solution to ($\dagger$), ($\dagger$.1), ($\dagger$.2). Then ($\dagger$) implies that μ is a non-decreasing function. Therefore, we may define

$$r_0 := \left(\frac{L_0}{E_0^2 \mathrm{e}^{-2\mu(0)} - 1} \right)^{\frac{1}{2}} \ , \quad \text{provided } \mu(0) < \log(E_0) \ ,$$

so that $E_0 = \mathrm{e}^{\mu(0)} \sqrt{1 + L_0 r_0^{-2}}$. □

The key idea of Schaeffer's construction is the study of the integro-differential equation ($\dagger$) with ($\dagger$.1), ($\dagger$.2). The following lemma is the main result from such a study, and is crucial to Schaeffer's construction. However, its proof is very long and technical, so we shall omit it and refer the interested reader to the original paper [24].

Lemma 2.15. *Fix D_0, E_0, L_0, k, l as above, and let*

$$q := k + l + \frac{3}{2} \ .$$

Suppose μ solves ($\dagger$), ($\dagger$.1), ($\dagger$.2) *with initial condition satisfying*

$$\mu(0) < -\log(E_0) \ ,$$

so that r_0 may be defined in Remark 2.14.

Then there exist positive constants D_1, D_2, D_3, D_4 independent of $\mu(0)$, such that, if it additionally holds that $\mu(0) < -D_1$, then there exists

$$R \in \left(r_0 + D_2 r_0^{1 + \frac{2}{q+1}}, r_0 + D_3 r_0^{1 + \frac{2}{q+1}} \right)$$

such that

$$\mathcal{E}'(R) > 0 \ , \quad \mathcal{E}(R) = E_0 \ , \quad \frac{m(R)}{R} < \frac{1}{2} - D_4 \ ,$$

where $\mathcal{E}$ is the function defined by

$$\mathcal{E}(r) := \mathrm{e}^{\mu(r)} \left(1 + \frac{L_0}{r^2}\right)^{\frac{1}{2}} \ . \qquad \square$$

Now, fix a solution $f(x,v) = \varphi(r,w,L)$ of the Vlasov-Einstein equation of the form $(*)$, $(*.1)$, $(*.2)$, such that

- The constant is chosen so that $\mu(0)$ is sufficiently negative, such that r_0, as defined in Remark 2.14, is small enough so that

  $$\frac{r_0 + D_3 r_0^{1+\frac{2}{q+1}}}{\sqrt{2D_4 L_0}} < 1 \ .$$

  Observe also that the definitions (†.1) and (†.2) give $m = m_{[\varphi]}$ and $p = p_{[\varphi]}$.

- With $R > 0$ chosen corresponding to μ in Lemma 2.15, $\varphi \equiv 0$ for $r \geq R$.

This is known to be possible; see [16]. Also, observe that, by definition of r_0, $\varphi \equiv 0$ for $r \leq r_0$. In summary,

$$\varphi(r,w,L) = 0 \qquad \text{for } r \leq r_0 \text{ or } r \geq R \ .$$

We therefore have

$$m_{[\varphi]}(r) = m_{[\varphi]}(R) \ , \quad \rho_{[\varphi]}(r) = p_{[\varphi]}(R) = 0 \qquad \text{for } r \geq R.$$

Since μ solves (†), we have

$$\mu'(r) = \frac{r^{-2} m_{[\varphi]}(R)}{1 - 2r^{-1} m_{[\varphi]}(R)} \qquad \text{for } r \geq R,$$

which may be integrated directly to give

$$\mathrm{e}^{\mu(r)} = \mathrm{e}^{\mu(R)} \left(\frac{1 - 2r^{-1} m_{[\varphi]}(R)}{1 - 2R^{-1} m_{[\varphi]}(R)}\right)^{\frac{1}{2}} \qquad \text{for } r \geq R.$$

By Lemma 2.15, we have

$$\begin{aligned}
\mathcal{E}(r) &= E_0 \left(\frac{1 - 2r^{-1} m_{[\varphi]}(R)}{1 - 2R^{-1} m_{[\varphi]}(R)}\right)^{\frac{1}{2}} \left(\frac{1 + L_0 r^{-2}}{1 + L_0 R^{-2}}\right)^{\frac{1}{2}} \\
&\leq E_0 \left(\frac{1 - 2r^{-1} m_{[\varphi]}(R)}{2D_4}\right)^{\frac{1}{2}} \left(\frac{1 + L_0 r^{-2}}{L_0 \left(r_0 + D_3 r_0^{1+\frac{2}{q+1}}\right)^{-2}}\right)^{\frac{1}{2}} \\
&= E_0 \frac{r_0 + D_3 r_0^{1+\frac{2}{q+1}}}{\sqrt{2D_4 L_0}} \left(1 - \frac{2m_{[\varphi]}(R)}{r}\right)^{\frac{1}{2}} \left(1 + \frac{L_0}{r^2}\right)^{\frac{1}{2}} \qquad \text{for } r \geq R.
\end{aligned}$$

By our choice of $\mu(0)$, so that r_0 is small, we have

$$\mathcal{E}(r) < E_0 \qquad \text{for } r \text{ sufficiently large.}$$

Schaeffer [24] asserts that it is possible to choose $r_1 \in (r_0, R), w_1 \in \mathbb{R}, L_1 \in (L_0, \infty)$ so that

$$\varphi(r_1, w_1, L_1) > 0 \ , \quad E_0 \frac{r_0 + D_3 r_0^{1+\frac{2}{q+1}}}{\sqrt{2D_4 L_0}} < \mathcal{E}(r_1) < E_0 = \mathcal{E}(R) \ .$$

By the intermediate value theorem, we may pick $r_2 > R$ such that

$$\mathcal{E}(r_2) = \mathcal{E}(r_1) ,$$

that is

$$e^{\mu(r_2)} \left(1 + \frac{L_0}{r_2^2}\right)^{\frac{1}{2}} = e^{\mu(r_1)} \left(1 + \frac{L_0}{r_1^2}\right)^{\frac{1}{2}} .$$

Since $L_1 > L_0$, the function $s \mapsto (1 + L_1 s^{-2})/(1 + L_0 s^{-2})$ is decreasing on $(0, \infty)$, so

$$\frac{1 + L_1 r_2^{-2}}{1 + L_0 r_2^{-2}} < \frac{1 + L_1 r_1^{-2}}{1 + L_0 r_1^{-2}} .$$

The preceding two displayed equations give

$$e^{\mu(r_2)} \left(1 + \frac{L_1}{r_2^2}\right)^{\frac{1}{2}} < e^{\mu(r_1)} \left(1 + \frac{L_1}{r_1^2}\right)^{\frac{1}{2}} .$$

Therefore we may choose $w_2 \in \mathbb{R}$, with $|w_2| > |w_1|$, so that

$$E_{[\varphi]}(r_2, w_2, L_1) = e^{\mu(r_2)} \left(1 + w_2^2 + \frac{L_1}{r_2^2}\right)^{\frac{1}{2}} = e^{\mu(r_1)} \left(1 + w_1^2 + \frac{L_1}{r_1^2}\right)^{\frac{1}{2}} = E_{[\varphi]}(r_1, w_1, L_1) .$$

Thus, with $L_2 := L_1$, we have $E_{[\varphi]}(r_2, w_2, L_2) = E_{[\varphi]}(r_1, w_1, L_1)$ and $L_2 = L_1$, but on the other hand $\varphi(r_2, w_2, L_2) = 0$ (since $r_2 > R$) and yet $\varphi(r_1, w_1, L_1) > 0$.

We conclude that Jeans's theorem fails to hold for this solution $f(x, v) = \varphi(r, w, L)$.

3 Steady Solutions from Casimir Functionals

Various families of steady, spherically symmetric solutions to the Vlasov-Poisson system, with finite mass and compact support, have been constructed by the application of Theorems 2.7 and 2.8 (see Subsection 2.2).

The present section is concerned with an alternative approach for the construction of such solutions, which involves the use of Casimir functionals. The advantage of this approach is that it provides a framework to investigate the issue of dynamical stability of the resulting solutions: given a time-dependent solution f to the Vlasov-Poisson system in the gravitational case, whose initial datum is close to a steady solution f_0, will $f(t)$ stay close to f_0? This and similar questions have important ramifications to the stability of models of stellar structure.

We remind the reader that throughout this section, Convention 2.4 will apply: unless otherwise stated, we only consider the Vlasov-Poisson system in the gravitational case. Moreover, throughout this section we will use the notation

$$E_{[f]}(x, v) := \frac{1}{2}|v|^2 + U_{[f]}(x)$$

3.1 The Energy-Casimir Method

The paper [6] contains a mostly heuristic discussion of a general abstract framework, the energy-Casimir method, for the investigation of nonlinear stability of stationary solutions to infinite-dimensional, degenerate Hamiltonian systems, of which many interesting physical systems are examples. Though this method cannot be applied verbatim to the Vlasov-Poisson system, it does inspire an alternative method which can be used to prove some interesting stability results for the Vlasov-Poisson system. We shall therefore give, in this subsection, an overview of the energy-Casimir method, and heuristically motivate how it can be used to demonstrate nonlinear stability.

Suppose the system under consideration has the equation of motion

$$\frac{\mathrm{d}u}{\mathrm{d}t} = A(u) \tag{$*$}$$

on some state space X, which should be a Banach space, and $A : \mathrm{D}(A) \to X$ is a nonlinear operator. Assume there exists a steady solution u_0 whose stability we wish to investigate. The principal steps are

1. Find an *energy functional*, i.e. a smooth function $\mathcal{H} : X \to \mathbb{R}$ which is conserved along flows of the system,
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(u(t)) = 0 \ .$$

2. Find a *Casimir functional*, i.e. a smooth function $\mathcal{C} : X \to \mathbb{R}$ which is also conserved along flows of the system, so that u_0 is a critical point of the *energy-Casimir functional* $\mathcal{H}_{\mathrm{C}} := \mathcal{H} + \mathcal{C}$, that is $\mathrm{D}\mathcal{H}_{\mathrm{C}}(u_0) = 0$.

3. Show that the quadratic part in the Taylor expansion of $\mathcal{H}_{\mathrm{C}}$ at u_0 is positive-definite, i.e.
$$\mathrm{D}^2\mathcal{H}_{\mathrm{C}}(u_0)\,(v, v) \geq C\|v\|_X^2 \qquad \text{for all } v \in X$$
for some constant $C > 0$.

The heuristic underlying the last step is the Taylor expansion

$$\mathcal{H}_{\mathrm{C}}(u) = \mathcal{H}_{\mathrm{C}}(u_0) + \frac{1}{2}\mathrm{D}^2\mathcal{H}_{\mathrm{C}}(u_0)\,(u - u_0, u - u_0) + \dots$$

so that the quadratic part is the main contribution to $\mathcal{H}_{\mathrm{C}}(u) - \mathcal{H}_{\mathrm{C}}(u_0)$ when $u - u_0$ is small. If the above steps could be carried out, then for a flow $u : [0, \infty) \to X$ of the above dynamical system $(*)$, we would have

$$\|u(t) - u_0\|_X^2 \leq C\left|\mathcal{H}_{\mathrm{C}}\left(u(0)\right) - \mathcal{H}_{\mathrm{C}}\left(u_0\right)\right|$$

for $\|u(0) - u_0\|_X$ sufficiently small; therefore the system $(*)$ is dynamically stable.

Let us illustrate why the above framework cannot be directly applied to the Vlasov-Poisson system. We proceed formally and heuristically, ignoring rigorous issues such as the choice of state space to work in. For the Vlasov-Poisson system, the natural energy functional is

$$\begin{aligned}\mathcal{H}(f) &:= E_{\mathrm{kin}}(f) + E_{\mathrm{pot}}(f)\\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{1}{2}|v|^2 f(x, v)\,\mathrm{d}v\,\mathrm{d}x - \frac{1}{8\pi}\int_{\mathbb{R}^3}\left|\nabla U_{[f]}(x)\right|^2\,\mathrm{d}x \ .\end{aligned}$$

Suppose we have a Casimir functional $\mathcal{C}$, and a steady state f_0 which is a critical point of the energy-Casimir functional $\mathcal{H}_{\mathrm{C}} := \mathcal{H} + \mathcal{C}$. Formally expanding, we have

$$\begin{aligned}\mathcal{H}_{\mathrm{C}}(f) - \mathcal{H}_{\mathrm{C}}(f_0) = &\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} E_{[f_0]}(x, v)\left(f(x, v) - f_0(x, v)\right)\,\mathrm{d}v\,\mathrm{d}x + \mathrm{D}\mathcal{C}(f_0)\,(f - f_0)\\ &- \frac{1}{8\pi}\int_{\mathbb{R}^3}\left|\nabla U_{[f]}(x) - \nabla U_{[f_0]}(x)\right|^2\,\mathrm{d}x + \frac{1}{2}\mathrm{D}^2\mathcal{C}(f_0)\,(f - f_0, f - f_0)\\ &+ \mathcal{O}\left(\|f - f_0\|^3\right)\end{aligned}$$

where

$$E_{[f]}(x, v) := \frac{1}{2}|v|^2 + U_{[f]}(x) \ .$$

Notice that the quadratic term is *not* necessarily positive-definite. Hence the above framework fails, and we need to modify our strategy.

Remark 3.1. Notice that the only reason for the lack of positive-definiteness, in the quadratic term in the above expansion, is our choice to work in the gravitational case. This is because the potential energy is negative in the gravitational case.

For the plasma physics case, the potential energy is positive and the above framework can be applied to certain situations where there are steady solutions, for example if a fixed ion background is present or if the problem is posed in a bounded domain rather than in the whole of $\mathbb{R}^3$. Some of these problems are treated in [15]. □

3.2 An Early Stability Result of Camm Steady States

The paper [5] is a pioneering work in the construction of steady, spherically symmetric solutions to the Vlasov-Poisson system, and the investigation of their dynamical stability, by minimising energy-Casimir functionals. The method used is reminiscent of the direct method of the calculus of variations.

The goal of this subsection is to give an overview of the proof that these minimisers are actually steady solutions to the Vlasov-Poisson equation, satisfying the functional form (1). We warn the reader that we will be employing notation that is different from that of the original paper, but aims to be consistent with the rest of this essay.

Guo and Rein considered the Casimir functional

$$\mathcal{C}(f) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \left[Q\left(f(x,v)|x\wedge v|^{2\lambda}\right)|x\wedge v|^{-2\lambda} + \Gamma f(x,v)\,|x\wedge v|^2 \right] \mathrm{d}v\,\mathrm{d}x$$

where $\Gamma \geq 0$ and $\lambda < 1$, and $Q \in C^1([0,\infty)) \cap C^2((0,\infty))$ is assumed to satisfy

(i) There exist $C_0 > 0, \kappa_0 \in (0, \frac{3}{2} - \lambda)$, such that

$$Q(h) \geq C_0 h^{\frac{\kappa_0+1}{\kappa_0}} .$$

(ii) There exist $C_1 > 0, \kappa_1 \in (0, \frac{3}{2} - \lambda)$, such that

$$Q(h) \leq C_1 h^{\frac{\kappa_1+1}{\kappa_1}} \quad \text{for sufficient small } h \geq 0 .$$

(iii) There exists $\kappa_0 \in (0, \frac{3}{2} - \lambda)$, such that

$$Q(ch) \geq c^{\frac{\kappa_0+1}{\kappa_0}} Q(h) \quad \text{for } c \in [0,1] .$$

(iv) $Q'(0) = 0$ and $Q'' > 0$.

In particular this means that Q' is an increasing homeomorphism $[0,\infty) \to [0,\infty)$, which is continuously differentiable on $(0,\infty)$.

For fixed $M > 0$, the associated energy-Casimir functional

$$\mathcal{H}_\mathrm{C} := \mathcal{H} + \mathcal{C} = E_\mathrm{kin} + E_\mathrm{pot} + \mathcal{C}$$

is to be minimised over

$$\mathcal{F}^M_\mathrm{sym} := \left\{ f \in L^1(\mathbb{R}^6) \,\middle|\, f \geq 0\,, \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(x,v)\,\mathrm{d}v\,\mathrm{d}x = M,\ f \text{ is spherically symmetric} \right\} .$$

For convenience, we will denote

$$h_M := \inf_{\mathcal{F}^M_\mathrm{sym}} \mathcal{H}_\mathrm{C} .$$

Remark 3.2. The assumptions above are motivated by the applicability to the consideration of Camm steady states. A typical Casimir functional of the above form would have

$$Q(h) = C_0 h^{\frac{\kappa_0+1}{\kappa_0}} + C_1 h^{\frac{\kappa_1+1}{\kappa_1}}$$

for $C_0 > 0, C_1 \geq 0$. Camm steady states would result from setting $C_1 = 0$. □

The first issue to verify is that $\mathcal{H}_{\mathrm{C}}$ does not take arbitrarily large negative values, i.e. that it is bounded below. The following result, whose proof by a scaling argument the reader can find in [5], Lemma 4, guarantees this.

Lemma 3.3. *Fix $M > 0$. Then for $\Gamma \geq 0$ sufficiently small, we have*

$$0 > h_M > -\infty \,. \qquad \square$$

With this reassuring fact, we can now pick a minimising subsequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}^M_{\mathrm{sym}}$. One then hopes, as in the direct method of the calculus of variations, to extract a convergent subsequence (with respect to some topology), and show that the limit is a minimiser of the energy-Casimir functional.

The natural function space to work in turns out to be

$$\mathfrak{F}^{\kappa_0,\lambda}_{\mathrm{sym}} := \left\{ f : \mathbb{R}^6 \to \mathbb{R} \;\middle|\; f \text{ measurable, spherically symmetric }, \|f\|_{\mathfrak{F}^{\kappa_0,\lambda}} < \infty \right\}$$

equipped with the norm

$$\|f\|_{\mathfrak{F}^{\kappa_0,\lambda}} := \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |f(x,v)|^{\frac{\kappa_0+1}{\kappa_0}} \, |x \wedge v|^{\frac{2\lambda}{\kappa_0}} \;\mathrm{d}v \,\mathrm{d}x \right)^{\frac{\kappa_0}{\kappa_0+1}} ,$$

which is simply the $L^{\frac{\kappa_0+1}{\kappa_0}}(|x\wedge v|^{\frac{2\lambda}{\kappa_0}} \;\mathrm{d}v \,\mathrm{d}x)$ norm. Under this norm, $\mathfrak{F}^{\kappa_0,\lambda}_{\mathrm{sym}}$ becomes a separable, reflexive Banach space. This choice of function space is motivated by the following estimate, whose proof the reader can find in [5], Lemma 1.

Lemma 3.4. *There exists a constant $C = C(M) > 0$ such that*

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |f(x,v)|^{\frac{\kappa_0+1}{\kappa_0}} \, |x \wedge v|^{\frac{2\lambda}{\kappa_0}} \;\mathrm{d}v \,\mathrm{d}x \leq C \left(1 + E_{\mathrm{kin}}(f) + \mathcal{C}(f)\right)$$

for every $f \in \mathcal{F}^M_{\mathrm{sym}}$. $\square$

Using Lemma 3.4 and some other estimates, it can be shown that a minimising sequence in $\mathcal{F}^M_{\mathrm{sym}}$ for $\mathcal{H}_{\mathrm{C}}$ must be a bounded sequence in $\mathfrak{F}^{\kappa_0,\lambda}_{\mathrm{sym}}$. Thus a subsequence, which is weakly convergent in $\mathfrak{F}^{\kappa_0,\lambda}_{\mathrm{sym}}$, can be extracted. The limit can then be shown to be a minimiser of the energy-Casimir functional $\mathcal{H}_{\mathrm{C}}$. Specifically, the following existence result is proved in [5], Theorem 1.

Theorem 3.5. *Let $M > 0$, and let $\Gamma \geq 0$ be sufficiently small so that Lemma 3.3 applies. Suppose $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}^M_{\mathrm{sym}}$ is a minimising sequence for $\mathcal{H}_{\mathrm{C}}$.*

Then there is a subsequence $\{f_{n(k)}\}_{k\in\mathbb{N}} \subseteq \{f_n\}_{n\in\mathbb{N}}$ which converges weakly in $\mathfrak{F}^{\kappa_0,\lambda}_{\mathrm{sym}}$ to a minimiser $f_0 \in \mathcal{F}^M_{\mathrm{sym}}$ of $\mathcal{H}_{\mathrm{C}}$. The corresponding spatial density $\rho_{[f_0]}$ has compact support, and $\nabla U_{[f_{n(k)}]} \to \nabla U_{[f_0]}$ strongly in $(L^2(\mathbb{R}^3))^3$. $\square$

We now arrive at the main result in this section: identifying minimisers of $\mathcal{H}_{\mathrm{C}}$ over $\mathfrak{F}^{\kappa_0,\lambda}_{\mathrm{sym}}$ as steady, spherically symmetric solutions to the Vlasov-Poisson system. Although we do not obtain an explicit formula for a minimiser f_0, we show that f_0 is of the functional form (1); therefore f_0 does indeed solve the Vlasov-Poisson system. The idea of the proof is almost identical to that used to derive Euler-Lagrange equations for minimisation problems in the calculus of variations.

Theorem 3.6. *Let $f_0 \in \mathcal{F}^M_{\mathrm{sym}}$ be a minimiser of $\mathcal{H}_{\mathrm{C}}$ over $\mathcal{F}^M_{\mathrm{sym}}$. Then there exists $E_0 \in \mathbb{R}$ such that f_0 satisfies*

$$f_0(x,v) = \left[(Q')^{-1} \left(\left(E_0 - E_{[f_0]}(x,v) - \Gamma |x \wedge v|^2 \right)_+ \right) \right] |x \wedge v|^{-2\lambda} \quad \textit{for a.e. } (x,v) \in \mathbb{R}^6 \,.$$

Proof. Let $\varepsilon > 0$ be sufficiently small so that

$$\mathcal{S}_\varepsilon := \left\{ (x,v) \in \mathbb{R}^6 \;\middle|\; \varepsilon \leq f_0(x,v) \leq \frac{1}{\varepsilon} \right\}$$

has finite positive measure. Then let $\zeta : \mathbb{R}^6 \to \mathbb{R}$ be a bounded, measurable, spherically symmetric, compactly supported function, such that $\zeta \geq 0$ on $\mathbb{R}^6 \setminus \mathcal{S}_\varepsilon$. Set

$$\eta(x,v) := \zeta(x,v) - \frac{1}{|\mathcal{S}_\varepsilon|} \iint_{\mathcal{S}_\varepsilon} \zeta(y,w) \,\mathrm{d}w \,\mathrm{d}y \, \mathbb{1}_{\mathcal{S}_\varepsilon}(x,v) \,.$$

Note that $\eta \in L^\infty(\mathbb{R}^6)$ is spherically symmetric and has compact support.

For sufficiently small $\tau > 0$ we have $f_0 + \tau\eta \in \mathcal{F}^M_{\mathrm{sym}}$. As f_0 is a minimiser of $\mathcal{H}_\mathrm{C}$ over $\mathcal{F}^M_{\mathrm{sym}}$, we have

$$\begin{aligned}
0 &\leq \lim_{\tau\to 0^+} \frac{1}{\tau} \left(\mathcal{H}_\mathrm{C}\left(f_0 + \tau\eta\right) - \mathcal{H}_\mathrm{C}\left(f_0\right)\right) \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \left[E_{[f_0]}(x,v) + Q'\left(f_0(x,v)|x\wedge v|^{2\lambda}\right) + \Gamma|x\wedge v|^2\right] \eta(x,v) \,\mathrm{d}v \,\mathrm{d}x \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \left[E_{[f_0]}(x,v) + Q'\left(f_0(x,v)|x\wedge v|^{2\lambda}\right) + \Gamma|x\wedge v|^2\right] \zeta(x,v) \,\mathrm{d}v \,\mathrm{d}x \\
&\quad - \frac{1}{|\mathcal{S}_\varepsilon|} \iint_{\mathcal{S}_\varepsilon} \left[E_{[f_0]}(y,w) + Q'\left(f_0(y,w)|y\wedge w|^{2\lambda}\right) + \Gamma|y\wedge w|^2\right] \,\mathrm{d}y \,\mathrm{d}w \iint_{\mathcal{S}_\varepsilon} \zeta(x,v) \,\mathrm{d}v \,\mathrm{d}x \,.
\end{aligned}$$

In particular, since this holds for ζ vanishing outside $\mathcal{S}_\varepsilon$ but otherwise arbitrary (subject to being bounded and spherically symmetric) in $\mathcal{S}_\varepsilon$, we see that, for a.e. $(x,v) \in \mathcal{S}_\varepsilon$,

$$\begin{aligned}
&E_{[f_0]}(x,v) + Q'\left(f_0(x,v)|x\wedge v|^{2\lambda}\right) + \Gamma|x\wedge v|^2 \\
&\qquad = \frac{1}{|\mathcal{S}_\varepsilon|} \iint_{\mathcal{S}_\varepsilon} \left[E_{[f_0]}(y,w) + Q'\left(f_0(y,w)|y\wedge w|^{2\lambda}\right) + \Gamma|y\wedge w|^2\right] \,\mathrm{d}y \,\mathrm{d}w \,.
\end{aligned}$$

But the left-hand side of this equation does not depend on ε, so both sides are equivalent to a constant E_0 independent of ε, everywhere in $\mathcal{S}_\varepsilon$. Since $\varepsilon > 0$ could be chosen arbitrarily small, we deduce

$$E_{[f_0]}(x,v) + Q'\left(f_0(x,v)|x\wedge v|^{2\lambda}\right) + \Gamma|x\wedge v|^2 = E_0 \quad \text{for a.e. } (x,v) \in \{f_0 > 0\} \,.$$

On the other hand, taking ζ to vanish inside $\{f_0 > 0\}$ but otherwise arbitrary non-negative, bounded, spherically symmetric on $\{f_0 = 0\}$, we have that

$$E_{[f_0]}(x,v) + \Gamma|x\wedge v|^2 \geq E_0 \quad \text{for a.e. } (x,v) \in \{f_0 = 0\} \,.$$

The two preceding displayed equations prove the theorem. □

An advantage of constructing solutions using the above energy-Casimir method is that stability results can be proved for minimisers of the energy-Casimir functional. For this specific problem, we have the following dynamical stability result.

Theorem 3.7. *Let $f_0 \in \mathcal{F}^M_{\mathrm{sym}}$ be a minimiser of $\mathcal{H}_\mathrm{C}$ over $\mathcal{F}^M_{\mathrm{sym}}$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $f(0) \in C^1_\mathrm{c}(\mathbb{R}^6) \cap \mathcal{F}^M_{\mathrm{sym}}$ satisfies*

$$\mathcal{H}_\mathrm{C}(f(0)) - \mathcal{H}_\mathrm{C}(f_0) + \frac{1}{4\pi}\left\|\nabla U_{[f(0)]} - \nabla U_{[f_0]}\right\|^2_{L^2} < \delta \,,$$

the classical solution f to the Vlasov-Poisson system, launched by the initial data $f(0)$, satisfies

$$\mathcal{H}_\mathrm{C}(f(t)) - \mathcal{H}_\mathrm{C}(f_0) + \frac{1}{4\pi}\left\|\nabla U_{[f(t)]} - \nabla U_{[f_0]}\right\|^2_{L^2} < \varepsilon \quad \text{for all } t \geq 0 \,.$$

□

3.3 A Reduction Approach for Isotropic Steady States

The preceding stability result, Theorem 3.7, suffers from the defect that it caters only for the unphysical situation spherically symmetric perturbations about the minimiser f_0. Much work has been done in attempting to remove this restriction. The paper [18] presents an argument that proves dynamical stability, with respect to general perturbations, for *isotropic* steady states, i.e. steady states satisfying the functional form $f(x,v) = F(E_{[f]}(x,v))$ with no explicit dependence on $|x \wedge v|^2$.

An interesting alternative approach [19], also only for isotropic steady states, involves reducing the minimisation problem over phase-space densities f to a "reduced" minimisation problem over spatial densities ρ. A comprehensive exposition of the argument can be found in [20], whose presentation we follow. In this subsection we focus only on showing the equivalence of the two problems, demonstrating how isotropic steady states to the Vlasov-Poisson system can be recovered from the reduced problem, and stating the dynamical stability result.

The Casimir functional considered in [19] is

$$\mathcal{C}(f) := \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \Phi(f(x,v)) \, \mathrm{d}v \, \mathrm{d}x$$

where $\Phi \in C^1([0,\infty))$ is assumed to satisfy

(i) There exist $C_0 > 0, \kappa_0 \in (0, \frac{3}{2})$, such that

$$\Phi(h) \geq C_0 h^{\frac{\kappa_0+1}{\kappa_0}} \quad \text{for sufficiently large } h \geq 0 \, .$$

(ii) There exist $C_1 > 0, \kappa_1 \in (0, \frac{3}{2})$, such that

$$\Phi(h) \leq C_1 h^{\frac{\kappa_1+1}{\kappa_1}} \quad \text{for sufficiently small } h \geq 0 \, .$$

(iii) $\Phi(0) = \Phi'(0) = 0$, and Φ is strictly convex on $[0,\infty)$.
In particular Φ' is an increasing homeomorphism $[0,\infty) \to [0,\infty)$,

We also define $\Phi \equiv +\infty$ on $(-\infty, 0]$. This allows us to define the convex conjugate of Φ by

$$\Phi^*(\theta) := \sup_{h \in \mathbb{R}} \left(h\theta - \Phi(h)\right) \, .$$

Since Φ is strictly convex wherever it is finite, we have that Φ^* is differentiable on $\mathbb{R}$ (by [22], Theorem 26.3) and the inverse homeomorphism of $\Phi' : [0,\infty) \to [0,\infty)$ is given by $(\Phi')^{-1} = (\Phi^*)'|_{[0,\infty)}$ (by [22], Theorem 26.5). Since $\Phi^* \equiv 0$ on $(-\infty, 0]$, we see that $(\Phi^*)'$ is continuous on $\mathbb{R}$, hence $\Phi^* \in C^1(\mathbb{R})$.

Original Minimisation Problem. Minimise the energy-Casimir functional

$$\mathcal{H}_{\mathrm{C}} := E_{\mathrm{kin}} + E_{\mathrm{pot}} + \mathcal{C}$$

over

$$\mathcal{F}^M := \left\{ f \in L^1(\mathbb{R}^6) \;\middle|\; f \geq 0 \, , \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(x,v) \, \mathrm{d}v \, \mathrm{d}x = M \right\} \, .$$ □

Convention 3.8. In the following, for $\rho \in L^1(\mathbb{R}^3)$, we will write $U_{[\rho]}$ for the solution of

$$\triangle U_{[\rho]} = 4\pi\rho \, , \quad \lim_{|x|\to\infty} U_{[\rho]}(x) = 0 \, ,$$

and use the notation

$$E_{\mathrm{pot}}(\rho) := -\frac{1}{8\pi} \int_{\mathbb{R}^3} \left|\nabla U_{[\rho]}(x)\right|^2 \, \mathrm{d}x = \frac{1}{2} \int_{\mathbb{R}^3} U_{[\rho]}(x)\rho(x) \, \mathrm{d}x \, .$$

This abuse of notation should not cause any confusion. With this notation, we have, for example, $E_{\mathrm{pot}}(f) = E_{\mathrm{pot}}(\rho_{[f]})$ for any $f \in L^1(\mathbb{R}^6)$. □

To formulate the reduced problem, we define, for $s \geq 0$,

$$\mathcal{G}[s] := \left\{ g \in L^1(\mathbb{R}^3) \;\middle|\; g \geq 0 \,, \int_{\mathbb{R}^3} g(v) \,\mathrm{d}v = s \right\} \,,$$

$$\Psi(s) := \inf_{g \in \mathcal{G}[s]} \int_{\mathbb{R}^3} \left(\frac{1}{2}|v|^2 g(v) + \Phi(g(v)) \right) \,\mathrm{d}v \,.$$

It is immediate from the definition above that Ψ is convex and non-decreasing on $[0, \infty)$. For simplicity, we define $\Psi \equiv +\infty$ on $(-\infty, 0)$.

Reduced Minimisation Problem. Minimise the *reduced energy-Casimir functional*

$$\mathcal{H}_{\mathrm{r}}(\rho) := \int_{\mathbb{R}^3} \Psi(\rho(x)) \,\mathrm{d}x + E_{\mathrm{pot}}(\rho)$$

over

$$\mathcal{R}^M := \left\{ \rho \in L^1(\mathbb{R}^3) \;\middle|\; f \geq 0 \,, \int_{\mathbb{R}^3} \rho(x, v) \,\mathrm{d}x = M \right\} \,. \qquad \square$$

Our first goal in this subsection is to show that these two problems are equivalent. We first make the following straightforward observation.

Lemma 3.9. *For every $f \in \mathcal{F}^M$, we have*

$$\mathcal{H}_{\mathrm{C}}(f) \geq \mathcal{H}_{\mathrm{r}}\left(\rho_{[f]}\right) \,.$$

In particular,

$$\inf_{\mathcal{F}^M} \mathcal{H}_{\mathrm{C}} \geq \inf_{\mathcal{R}^M} \mathcal{H}_{\mathrm{r}} \,.$$

Proof. If $f \in \mathcal{F}^M$ then $\rho_{[f]} \in \mathcal{R}^M$, and

$$\begin{aligned} E_{\mathrm{kin}}(f) + \mathcal{C}(f) &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left(\frac{1}{2}|v|^2 f(x,v) + \Phi\left(f(x,v)\right) \right) \,\mathrm{d}v \,\mathrm{d}x \\ &\geq \int_{\mathbb{R}^3} \inf_{g \in \mathcal{G}[\rho_{[f]}(x)]} \left[\int_{\mathbb{R}^3} \left(\frac{1}{2}|v|^2 g(v) + \Phi\left(g(v)\right) \right) \,\mathrm{d}v \right] \mathrm{d}x = \int_{\mathbb{R}^3} \Psi\left(\rho_{[f]}(x)\right) \,\mathrm{d}x \,. \end{aligned}$$

Adding $E_{\mathrm{pot}}(f) = E_{\mathrm{pot}}(\rho_{[f]})$ to both sides gives the desired result. $\square$

We next derive a relation between the convex conjugates of Φ and Ψ.

Lemma 3.10. *The following relation holds:*

$$\Psi^*(\theta) = \int_{\mathbb{R}^3} \Phi^*\left(\theta - \frac{1}{2}|v|^2\right) \,\mathrm{d}v \,.$$

Proof. From the definitions we have

$$\begin{aligned} \Psi^*(\theta) &= \sup_{s \geq 0} \left[\theta s - \inf_{g \in \mathcal{G}[s]} \int_{\mathbb{R}^3} \left(\frac{1}{2}|v|^2 g(v) + \Phi\left(g(v)\right) \right) \mathrm{d}v \right] \\ &= \sup_{s \geq 0} \sup_{g \in \mathcal{G}[s]} \left[\theta s - \int_{\mathbb{R}^3} \left(\frac{1}{2}|v|^2 g(v) + \Phi\left(g(v)\right) \right) \mathrm{d}v \right] \\ &= \sup_{0 \leq g \in L^1(\mathbb{R}^3)} \int_{\mathbb{R}^3} \left(\left(\theta - \frac{1}{2}|v|^2 \right) g(v) - \Phi\left(g(v)\right) \right) \mathrm{d}v \\ &\leq \int_{\mathbb{R}^3} \sup_{z \geq 0} \left[\left(\theta - \frac{1}{2}|v|^2 \right) z - \Phi\left(z\right) \right] \mathrm{d}v \\ &= \int_{\mathbb{R}^3} \Phi^*\left(\theta - \frac{1}{2}|v|^2\right) \mathrm{d}v \,. \end{aligned}$$

Therefore it suffices to prove the reverse inequality.

For $\theta \leq 0$, note that the left-hand side is non-negative, while the right-hand side is zero. For $\theta > 0$, set

$$g_1(v) := \begin{cases} (\Phi')^{-1}\left(\theta - \frac{1}{2}|v|^2\right) & , \quad \text{if } |v| < \sqrt{2\theta} \\ 0 & , \quad \text{if } |v| \geq \sqrt{2\theta} \end{cases} .$$

Then $0 \leq g_1 \in L^1(\mathbb{R}^3)$, and

$$\sup_{z \geq 0}\left[\left(\theta - \frac{1}{2}|v|^2\right) z - \Phi(z)\right] = \left(\theta - \frac{1}{2}|v|^2\right) g_1(v) - \Phi(g_1(v)) \ ,$$

so a direct integration yields

$$\begin{aligned}\int_{\mathbb{R}^3} \sup_{z \geq 0}\left[\left(\theta - \frac{1}{2}|v|^2\right) z - \Phi(z)\right] \mathrm{d}v &= \int_{\mathbb{R}^3}\left(\left(\theta - \frac{1}{2}|v|^2\right) g_1(v) - \Phi(g_1(v))\right) \mathrm{d}v \\ &\leq \sup_{0 \leq g \in L^1(\mathbb{R}^3)} \int_{\mathbb{R}^3}\left(\left(\theta - \frac{1}{2}|v|^2\right) g(v) - \Phi(g(v))\right) \mathrm{d}v\end{aligned}$$

which is the desired reverse inequality. □

Lemma 3.11. *$\Psi^* \in C^1(\mathbb{R})$, $\Psi \in C^1([0,\infty))$ and Ψ is strictly convex. In particular Ψ' defines an increasing homeomorphism $[0,\infty) \to [0,\infty)$, with inverse $(\Psi')^{-1} = (\Psi^*)'|_{[0,\infty)}$.*

Proof. Differentiating the formula in Lemma 3.10, we get

$$(\Psi^*)'(\theta) = \int_{\mathbb{R}^3} (\Phi^*)'\left(\theta - \frac{1}{2}|v|^2\right) \mathrm{d}v \ .$$

From this we see that Ψ^* is continuously differentiable and $(\Phi^*)'$ is strictly increasing.

Since Ψ is convex lower-semicontinuous, we have $\Psi = \Psi^{**}$ (by [3], Theorem 4.2.1). Thus, by [22], Theorems 26.3 and 26.5 again, we have that $\Psi \in C^1([0,\infty))$ and $(\Psi^*)' = (\Psi')^{-1}$. Hence Ψ' is strictly increasing, so Ψ is strictly convex. □

We are finally ready to prove the following theorem, which shows that the two minimisation problems stated above are equivalent, in the sense that equality holds in Lemma 3.9, i.e.

$$\inf_{\mathcal{F}^M} \mathcal{H}_\mathrm{C} = \inf_{\mathrm{R}^M} \mathcal{H}_\mathrm{r} \ ,$$

and minimisers of $\mathcal{H}_\mathrm{C}$ over $\mathcal{F}^M$ can be constructed from minimisers of $\mathcal{H}_\mathrm{r}$ over $\mathcal{R}^M$.

Theorem 3.12. *Let $\rho_0 \in \mathcal{R}^M$ be a minimiser of $\mathcal{H}_\mathrm{r}$ over $\mathcal{R}^M$. Then there exists $E_0 \in \mathbb{R}$ such that ρ_0 satisfies*

$$\rho_0(x) = (\Psi^*)'\left(E_0 - U_{[\rho_0]}(x)\right) \quad \textit{for a.e. } x \in \mathbb{R}^3 \ .$$

Moreover, defining

$$f_0(x,v) := (\Phi^*)'\left(E_0 - \frac{1}{2}|v|^2 - U_{[\rho_0]}(x)\right) \ ,$$

we have $\rho_0 = \rho_{[f_0]}$ and $\mathcal{H}_\mathrm{C}(f_0) = \mathcal{H}_\mathrm{r}(\rho_0)$. Consequently, f_0 is a minimiser of $\mathcal{H}_\mathrm{C}$ over $\mathcal{F}^M$.

Proof. The first part is proved in an identical manner to Theorem 3.6. Let $\varepsilon > 0$ be sufficiently small so that

$$\mathcal{S}_\varepsilon := \left\{x \in \mathbb{R}^3 \;\middle|\; \varepsilon \leq \rho_0(x) \leq \frac{1}{\varepsilon}\right\}$$

has finite positive measure. Then let $\zeta : \mathbb{R}^3 \to \mathbb{R}$ be a bounded, measurable, compactly supported function, such that $\zeta \geq 0$ on $\mathbb{R}^3 \setminus \mathcal{S}_\varepsilon$.

For sufficiently small $\tau > 0$ we have that

$$\rho_\tau := \rho_0 + \tau \left(\zeta - \frac{1}{|\mathcal{S}_\varepsilon|} \int_{\mathcal{S}_\varepsilon} \zeta(y)\, \mathrm{d}y\, \mathbb{1}_{\mathcal{S}_\varepsilon} \right)$$

belongs to $\mathcal{R}^M$. As ρ_0 is a minimiser of $\mathcal{H}_\mathrm{r}$ over $\mathcal{R}^M$, we have

$$\begin{aligned} 0 &\le \lim_{\tau \to 0^+} \frac{1}{\tau} \left(\mathcal{H}_\mathrm{r}(\rho_\tau) - \mathcal{H}_\mathrm{r}(\rho_0) \right) \\ &= \int_{\mathbb{R}^3} \left[U_{[\rho_0]}(x) + \Psi'(\rho_0(x)) \right] \left[\zeta(x) - \frac{1}{|\mathcal{S}_\varepsilon|} \int_{\mathcal{S}_\varepsilon} \zeta(y)\, \mathrm{d}y\, \mathbb{1}_{\mathcal{S}_\varepsilon}(x) \right] \mathrm{d}x \\ &= \int_{\mathbb{R}^3} \left[U_{[\rho_0]}(x) + \Psi'(\rho_0(x)) - \frac{1}{|\mathcal{S}_\varepsilon|} \int_{\mathcal{S}_\varepsilon} \left[U_{[\rho_0]}(y) + \Psi'(\rho_0(y)) \right] \mathrm{d}y\, \mathbb{1}_{\mathcal{S}_\varepsilon}(x) \right] \zeta(x)\, \mathrm{d}x\ . \end{aligned}$$

In particular, since this holds for ζ vanishing outside $\mathcal{S}_\varepsilon$ but otherwise an arbitrary bounded measurable function in $\mathcal{S}_\varepsilon$, we see that, for a.e. $x \in \mathcal{S}_\varepsilon$,

$$U_{[\rho_0]}(x) + \Psi'(\rho_0(x)) = \frac{1}{|\mathcal{S}_\varepsilon|} \int_{\mathcal{S}_\varepsilon} \left[U_{[\rho_0]}(y) + \Psi'(\rho_0(y)) \right] \mathrm{d}y\ .$$

But the left-hand side of this equation does not depend on ε, so both sides are equivalent to a constant E_0 independent of ε, everywhere in $\mathcal{S}_\varepsilon$. Since $\varepsilon > 0$ could be chosen arbitrarily small, we deduce

$$U_{[\rho_0]}(x) + \Psi'(\rho_0(x)) = E_0 \quad \text{for a.e. } x \in \{\rho_0 > 0\}\ .$$

On the other hand, taking ζ to vanish inside $\{\rho_0 > 0\}$ but otherwise arbitrary bounded non-negative on $\{\rho_0 = 0\}$, we have that

$$U_{[\rho_0]}(x) + \Psi'(\rho_0(x)) \ge E_0 \quad \text{for a.e. } x \in \{\rho_0 = 0\}\ .$$

Recalling from Lemma 3.11 that $(\Psi')^{-1} = (\Psi^*)'$ on $[0, \infty)$, and that $(\Phi^*)' \equiv 0$ on $(-\infty, 0)$, we see that the two preceding displayed equations prove the first part of the theorem.

Now let f_0 be defined as in the statement of the theorem. By Lemma 3.10,

$$\begin{aligned} \rho_0(x) &= (\Psi^*)' \left(E_0 - U_{[\rho_0]}(x) \right) \\ &= \int_{\mathbb{R}^3} (\Phi^*)' \left(E_0 - \frac{1}{2}|v|^2 - U_{[\rho_0]}(x) \right) \mathrm{d}v = \int_{\mathbb{R}^3} f_0(x, v)\, \mathrm{d}v\ , \end{aligned}$$

that is $\rho_{[f_0]} = \rho_0$. Thus, $U_{[f_0]} = U_{[\rho_0]}$ also.

Next observe the formulae

$$\begin{cases} \Phi^*(\theta) = \theta(\Phi')^{-1}(\theta) - \Phi\left((\Phi')^{-1}(\theta) \right) \\ \Psi^*(\theta) = \theta(\Psi')^{-1}(\theta) - \Psi\left((\Psi')^{-1}(\theta) \right) \end{cases} \quad \text{for } \theta \ge 0\ .$$

Using these, and Lemma 3.10, we find

$$\begin{aligned}
\int_{\mathbb{R}^3} \Psi\left(\rho_0(x)\right) \, \mathrm{d}x &= \int_{\{\rho_0>0\}} \Psi\left((\Psi')^{-1}\left(E_0 - U_{[f_0]}(x)\right)\right) \, \mathrm{d}x \\
&= \int_{\{\rho_0>0\}} \left[\left(E_0 - U_{[f_0]}(x)\right)(\Psi^*)'\left(E_0 - U_{[f_0]}(x)\right) - \Psi^*\left(E_0 - U_{[f_0]}(x)\right)\right] \, \mathrm{d}x \\
&= \int_{\{\rho_0>0\}} \int_{\mathbb{R}^3} \Bigg[\left(E_0 - U_{[f_0]}(x)\right)(\Phi^*)'\left(E_0 - \frac{1}{2}|v|^2 - U_{[f_0]}(x)\right) \\
&\qquad - \Phi^*\left(E_0 - \frac{1}{2}|v|^2 - U_{[f_0]}(x)\right)\Bigg] \, \mathrm{d}v \, \mathrm{d}x \\
&= \int_{\{\rho_0>0\}} \int_{\mathbb{R}^3} \Bigg[\frac{1}{2}|v|^2(\Phi^*)'\left(E_0 - \frac{1}{2}|v|^2 - U_{[f_0]}(x)\right) \\
&\qquad + \Phi\left((\Phi')^{-1}\left(E_0 - \frac{1}{2}|v|^2 - U_{[f_0]}(x)\right)\right)\Bigg] \, \mathrm{d}v \, \mathrm{d}x \\
&= \int_{\{\rho_0>0\}} \int_{\mathbb{R}^3} \left[\frac{1}{2}|v|^2 f_0(x,v) + \Phi\left(f_0(x,v)\right)\right] \, \mathrm{d}v \, \mathrm{d}x \\
&= E_{\mathrm{kin}}\left(f_0\right) + \mathcal{C}\left(f_0\right) \; .
\end{aligned}$$

Adding $E_{\mathrm{pot}}(f_0)$ to both sides completes the proof of the second part of the theorem. □

We have therefore proven the equivalence of the two minimisation problems. It remains to show that the reduced minimisation problem admits minimisers in $\mathcal{R}^M$. This is guaranteed by the following theorem, whose proof we shall omit; the reader can find it in the papers we have cited above.

Theorem 3.13. *For any $M > 0$, the reduced energy-Casimir functional $\mathcal{H}_{\mathrm{r}}$ is bounded from below on $\mathcal{R}^M$.*

If $\{\rho_j\}_{j\in\mathbb{N}} \subset \mathcal{R}^M$ is a minimising sequence for $\mathcal{H}_{\mathrm{r}}$ over $\mathcal{R}^M$, then $\{\rho_j\}_{j\in\mathbb{N}}$ is bounded in $L^{\frac{\nu_0+1}{\nu_0}}(\mathbb{R}^3)$, where $\nu_0 := \kappa_0 + \frac{3}{2}$. Moreover there exists a subsequence $\{\rho_j\}_{j\in\mathbb{N}}$ (not relabelled), and a sequence $\{z_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^3$, such that

$$\begin{aligned}
\rho_j\left(\cdot + z_j\right) &\rightharpoonup \rho_0 \quad \textit{weakly in } L^{\frac{\nu_0+1}{\nu_0}}(\mathbb{R}^3) \; , \\
\nabla U_{[\rho_j]}\left(\cdot + z_j\right) &\to \nabla U_{[\rho_0]} \quad \textit{strongly in } \left(L^2(\mathbb{R}^3)\right)^3 \; ,
\end{aligned}$$

where $\rho_0 \in \mathcal{R}^M$ is a minimiser of $\mathcal{H}_{\mathrm{r}}$ over $\mathcal{R}^M$. □

We shall focus only on one aspect of this theorem, namely that weak convergence in $L^{\frac{\nu_0+1}{\nu_0}}(\mathbb{R}^3)$ implies strong convergence in $(L^2(\mathbb{R}^3))^3$ of the Newtonian force. For this, we will need the following inequality, whose proof the reader can find in [11], Theorem 4.2.

Proposition 3.14. *Let $p, q, r \geq 1$ be such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$. Then there exists a constant $C = C(d,p,q,r) > 0$ such that*

$$\left|\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) g(x-y) h(y) \, \mathrm{d}x \, \mathrm{d}y\right| \leq C \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)} \|h\|_{L^r(\mathbb{R}^d)}$$

for every $f \in L^p(\mathbb{R}^d), g \in L^q(\mathbb{R}^d), h \in L^r(\mathbb{R}^d)$. □

Proposition 3.15. *Let $1 < \nu < 5$, and suppose $\{\rho_j\}_{j\in\mathbb{N}} \subset L^{\frac{\nu+1}{\nu}}(\mathbb{R}^3)$ is such that*

$$\begin{aligned}
&\rho_j \rightharpoonup \rho_0 \quad \textit{weakly in } L^{\frac{\nu+1}{\nu}}(\mathbb{R}^3) \; , \\
&\forall\, \varepsilon > 0 \;\exists\, R > 0 \;:\; \limsup_{j\to\infty} \int_{\mathbb{R}^3 \setminus R\mathbb{B}^3} |\rho_j(x)| \, \mathrm{d}x < \varepsilon \; .
\end{aligned}$$

Then $\nabla U_{[\rho_j]} \to \nabla U_{[\rho_0]}$ strongly in $(L^2(\mathbb{R}^3))^3$.

Proof. For any $r > 0$, Hölder's inequality gives

$$\int_{r\mathbb{B}^3} |\rho(x)| \, \mathrm{d}x \le \left|r\mathbb{B}^3\right|^{\frac{1}{\nu+1}} \left(\int_{\mathbb{R}^3} |\rho(x)|^{\frac{\nu+1}{\nu}} \, \mathrm{d}x \right)^{\frac{\nu}{\nu+1}} \le \left|r\mathbb{B}^3\right|^{\frac{1}{\nu+1}} \|\rho\|_{L^{\frac{\nu+1}{\nu}}(\mathbb{R}^3)} \; .$$

Therefore the hypotheses guarantee that $\{\rho_j\}_{j\in\mathbb{N}}$ is bounded in $L^1(\mathbb{R}^3)$, and weak convergence guarantees that $\rho_0 \in L^1(\mathbb{R}^3)$ as well.

The sequence $\sigma_j := \rho_j - \rho_0$ converges weakly to 0 in $L^{\frac{\nu+1}{\nu}}(\mathbb{R}^3)$, and is bounded in $L^1(\mathbb{R}^3)$. We need to show that $\nabla U_{[\sigma_j]} \to 0$ strongly in $L^2(\mathbb{R}^3)$, which is equivalent to showing

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\sigma_j(x)\sigma_j(y)}{|x-y|} \, \mathrm{d}x \, \mathrm{d}y \to 0 \; . \tag{$*$}$$

For convenience, denote

$$P := \sup_{j\in\mathbb{N}} \|\sigma_j\|_{L^1(\mathbb{R}^3)} \; , \quad Q := \sup_{j\in\mathbb{N}} \|\sigma_j\|_{L^{\frac{\nu+1)}{\nu}}(\mathbb{R}^3} \; .$$

Let $\delta > 0$ be small and $R > 0$ large, to be chosen later. We partition the domain of integration, $\{(x,y) \in \mathbb{R}^3 \times \mathbb{R}^3\}$, into

$$\begin{aligned} A_1 &:= \{|x-y| \ge \delta\} \cap (\{|x| \ge R\} \cup \{|y| \ge R\}) \; , \\ A_2 &:= \{|x-y| \ge \delta\} \cap \{|x| < R\} \cap \{|y| < R\} \; , \\ A_3 &:= \{|x-y| < \delta\} \; . \end{aligned}$$

We will estimate the integral $(*)$ over each of these sets.

The integral over A_1 is estimated by

$$\iint_{A_1} \frac{|\sigma_j(x)||\sigma_j(y)|}{|x-y|} \, \mathrm{d}x \, \mathrm{d}y \le \frac{2}{\delta} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3 \setminus R\mathbb{B}^3} |\sigma_j(x)||\sigma_j(y)| \, \mathrm{d}x \, \mathrm{d}y \le \frac{2P}{\delta} \int_{\mathbb{R}^3 \setminus R\mathbb{B}^3} |\sigma_j(y)| \, \mathrm{d}y \; .$$

We next consider the integral over A_2. Let

$$h_j(x) := \mathbb{1}_{R\mathbb{B}^3}(x) \int_{\mathbb{R}^3} \frac{\mathbb{1}_{\mathbb{R}^3 \setminus \delta\mathbb{B}^3}(x-y)}{|x-y|} \mathbb{1}_{R\mathbb{B}^3}(y) \sigma_j(y) \, \mathrm{d}y \; .$$

For any fixed $x \in \mathbb{R}^3$, note that

$$\frac{\mathbb{1}_{\mathbb{R}^3 \setminus \delta\mathbb{B}^3}(x-\cdot)}{|x-\cdot|} \mathbb{1}_{R\mathbb{B}^3} \in L^{\nu+1}(\mathbb{R}^3)$$

defines a continuous linear functional on $L^{\frac{\nu+1}{\nu}}(\mathbb{R}^3)$. Since $\sigma_j \rightharpoonup 0$ in $L^{\frac{\nu+1}{\nu}}(\mathbb{R}^3)$, we see that $h_j \to 0$ pointwise on $\mathbb{R}^3 \setminus \{0\}$. On the other hand, $|h_j| \le \frac{P}{\delta} \mathbb{1}_{R\mathbb{B}^3}$, so the dominated convergence theorem guarantees $h_j \to 0$ in $L^{\nu+1}(\mathbb{R}^3)$. Hence,

$$\iint_{A_2} \frac{\sigma_j(x)\sigma_j(y)}{|x-y|} \, \mathrm{d}x \, \mathrm{d}y = \int_{\mathbb{R}^3} h_j(x)\sigma_j(x) \, \mathrm{d}x$$

tends to 0 as $j \to \infty$.

The integral over A_3 is estimated using Proposition 3.14 to obtain

$$\iint_{A_3} \frac{|\sigma_j(x)||\sigma_j(y)|}{|x-y|} \, \mathrm{d}x \, \mathrm{d}y \le C(\nu) \|\sigma_j\|^2_{L^{\frac{\nu+1}{\nu}}(\mathbb{R}^3)} \left\| \frac{\mathbb{1}_{\delta\mathbb{B}^3}}{|\cdot|} \right\|_{L^{\frac{\nu+1}{2}}(\mathbb{R}^3)} \le C(\nu) Q^2 \delta^{\frac{5-\nu}{\nu+1}} \; .$$

We collect the above estimates to conclude

$$\begin{aligned} \left| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\sigma_j(x)\sigma_j(y)}{|x-y|} \, \mathrm{d}x \, \mathrm{d}y \right| \le{}& C(\nu) Q^2 \delta^{\frac{5-\nu}{\nu+1}} + \frac{2P}{\delta} \int_{\mathbb{R}^3 \setminus R\mathbb{B}^3} |\sigma_j(y)| \, \mathrm{d}y \\ &+ \left| \iint_{A_2} \frac{\sigma_j(x)\sigma_j(y)}{|x-y|} \, \mathrm{d}x \, \mathrm{d}y \right| \; . \end{aligned}$$

Now, given $\varepsilon > 0$, choose $\delta > 0$ small enough, and then choose $R > 0$ large enough, so that

$$C(\nu)Q^2\delta^{\frac{5-\nu}{\nu+1}} < \frac{\varepsilon}{2} \quad \text{and} \quad \frac{2P}{\delta}\limsup_{j\to\infty}\int_{\mathbb{R}^3\setminus R\mathbb{B}^3} |\sigma_j(y)|\, \mathrm{d}y < \frac{\varepsilon}{2} \, .$$

Then, since the integral over A_2 vanish in the limit $j \to \infty$, we obtain

$$\limsup_{j\to\infty}\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\sigma_j(x)\sigma_j(y)}{|x-y|}\, \mathrm{d}x\, \mathrm{d}y\right| < \varepsilon \, .$$

Since $\varepsilon > 0$ was arbitrary, we have proven the desired result $(*)$. □

In the next subsection, we will prove a similar result in the next subsection, using a slightly different technique.

We end this subsection with the statement of a dynamical stability result, whose proof we shall omit; the reader can find the proof in the papers we have cited above.

Theorem 3.16. *Let $f_0 \in \mathcal{F}^M$ be a minimiser of $\mathcal{H}_{\mathrm{C}}$ over $\mathcal{F}^M$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $f(0) \in C^1_{\mathrm{c}}(\mathbb{R}^6) \cap \mathcal{F}^M$ satisfies*

$$\mathcal{H}_{\mathrm{C}}(f(0)) - \mathcal{H}_{\mathrm{C}}(f_0) + \frac{1}{4\pi}\left\|\nabla U_{[f(0)]} - \nabla U_{[f_0]}\right\|^2_{L^2} < \delta \, ,$$

the classical solution f to the Vlasov-Poisson system, launched by the initial data $f(0)$, satisfies the condition that for any $t \geq 0$, there exists $z_t \in \mathbb{R}^3$, such that

$$\mathcal{H}_{\mathrm{C}}(f(t, \cdot + z_t)) - \mathcal{H}_{\mathrm{C}}(f_0) + \frac{1}{4\pi}\left\|\nabla U_{[f(t)]}(\cdot + z_t) - \nabla U_{[f_0]}\right\|^2_{L^2} < \varepsilon \, .$$ □

3.4 An Unsuccessful Attempt at Anisotropic Steady States

Inspired by the previous results, the author has attempted, as part his research mini-project in the first year of his doctoral studies, to construct anisotropic steady states (i.e. steady states satisfying the functional form (1) with explicit dependence on the angular momentum density $|x \wedge v|^2$) by identifying them as minimisers of certain energy-Casimir functionals. The Casimir part is taken to be

$$\mathcal{C}(f) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \Phi\left(f(x,v)\, , |x\wedge v|^2\right)\, \mathrm{d}v\, \mathrm{d}x$$

where Φ should satisfy certain assumptions.

Unfortunately the attempt has yet to be successful, and this subsection is a record of the results that the author has managed to obtain. The reader is advised that the results in this subsection do not in any way represent a complete theory, nor do they lead up to any interesting theorem.

In the sequel, we will work with the function spaces

$$\mathfrak{F}^{\kappa,\lambda} := L^{\frac{\kappa+1}{\kappa}}\left(\mathbb{R}^6, \ |x\wedge v|^{\frac{2\lambda}{\kappa}}\, \mathrm{d}\mathcal{L}^3(x)\mathrm{d}\mathcal{L}^3(v)\right) \quad \text{for } \kappa, \lambda > 0 \, ,$$

and

$$\mathfrak{R}^{\nu,\lambda} := L^{\frac{\nu+1}{\nu}}\left(\mathbb{R}^3, \ |x|^{\frac{2\lambda}{\nu}}\, \mathrm{d}\mathcal{L}^3(x)\right) \quad \text{for } \nu, \lambda > 0 \, .$$

The assumptions on $\Phi : [0,\infty) \times [0,\infty) \to \mathbb{R}$ used by the author are as follows.

(i) There exist $C_0 > 0, \kappa_0 > 0, \lambda_0 \in (0,1)$, such that

$$\nu_0 := \frac{3 - 2\kappa_0 - 2\lambda_0}{2} \quad \text{satisfies } 1 < \nu_0 < 3 \ \text{ and } \ \nu_0 + 4\lambda_0 < 5 \, ,$$

and

$$\Phi(h, L) \geq C_0 h^{\frac{\kappa_0+1}{\kappa_0}} L^{\frac{\lambda_0}{\kappa_0}} \, .$$

(ii) There exist $C_1 > 0, \kappa_1 > 1, \lambda_1 \in (-\frac{3}{2}\kappa_1, \frac{3}{2} - \kappa_1)$, such that

$$\Phi(h, L) \leq C_1 h^{\frac{\kappa_1+1}{\kappa_1}} L^{\frac{\lambda_1}{\kappa_1}} \quad \text{for all sufficiently small } h \geq 0.$$

(iii) $\Phi(ch, L) \geq c^3 \Phi(h, L)$ for all $c \in [0, 1]$.

(iv) Φ should satisfy some strict convexity assumption with respect to h.

As before, for $M > 0$, denote

$$\mathcal{F}^M := \left\{ f \in \mathfrak{F}^{\kappa_0, \lambda_0} \;\middle|\; f \geq 0 \;,\; \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(x, v)\, \mathrm{d}v\, \mathrm{d}x = M \right\}$$

and

$$h_M := \inf_{\mathcal{F}^M} \mathcal{H}_\mathrm{C} \;.$$

These assumptions are similar to the ones imposed in the paper [5] by Guo and Rein. The original hope of the author was to prove similar results to those in that paper. The author's approach, however, is more similar to that employed in Rein's review paper [20] to solve the minimisation problem referred to in that paper as "Version 2 of the variational problem".

The first step is to show that $h_M > -\infty$ for all $M > 0$, so that the minimisation problem makes sense. The following three propositions show that $0 > h_M > -\infty$ for all $M > 0$. We have indicated where and how the assumptions made above are used.

Proposition 3.17. *Assume*

$$\Phi(ch, L) \geq c^3 \Phi(h, L) \quad \textit{for all } c \in [0, 1] \;.$$

Then $h_{cM} \geq c^3 h_M$ *for* $c \in (0, 1)$.

Proof. We employ a scaling argument. Fix $c \in (0, 1)$, and let $f \in \mathcal{F}^M$. Put

$$\overline{f}(x, v) := cf\left(cx, c^{-1}v\right) \;.$$

Then $\overline{f} \in \mathcal{F}^{cM}$ while $E_\mathrm{kin}(\overline{f}) = c^3 E_\mathrm{kin}(f)$ and $E_\mathrm{pot}(\overline{f}) = c^3 E_\mathrm{pot}(f)$. Thus,

$$\begin{aligned}
\mathcal{H}_\mathrm{C}(\overline{f}) &= c^3 \left(E_\mathrm{kin}(f) + E_\mathrm{pot}(f)\right) + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \Phi\left(cf(x, v)\;, |x \wedge v|^2\right)\, \mathrm{d}v\, \mathrm{d}x \\
&\geq c^3 \left(E_\mathrm{kin}(f) + E_\mathrm{pot}(f) + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \Phi\left(f(x, v)\;, |x \wedge v|^2\right)\, \mathrm{d}v\, \mathrm{d}x\right) \\
&\geq c^3 h_M \;.
\end{aligned}$$

Since any $g \in \mathcal{F}^{cM}$ can be constructed as $\overline{f}$ for some $f \in \mathcal{F}^M$, taking the infimum over the left-hand side proves our assertion. □

Proposition 3.18. *Assume there exist* $C_1 > 0, \kappa_1 > 1, \lambda_1 \in (-\frac{3}{2}\kappa_1, \frac{3}{2} - \kappa_1)$*, such that*

$$\Phi(h, L) \leq C_1 h^{\frac{\kappa_1+1}{\kappa_1}} L^{\frac{\lambda_1}{\kappa_1}} \qquad \textit{for all sufficiently small } h > 0 \;.$$

Then $h_M < 0$ *for all sufficiently small* $M > 0$.

Proof. Since $3 - 2\lambda_1 - \kappa_1 > \kappa_1 > 0$, we may pick $\alpha \in \mathbb{R}$ such that

$$-2 < \alpha < -1 - \frac{\kappa_1}{3 - 2\lambda_1 - \kappa_1} \;.$$

We may then pick $\beta \in \mathbb{R}$ such that

$$0 < \beta < \min\left(\alpha + 2 \;,\; \frac{(3 - 2\lambda_1 - \kappa_1)|\alpha| - (3 - 2\lambda_1)}{\kappa_1 - 1}\right) \;.$$

Let $\varepsilon > 0$ be sufficiently small. Consider the phase-space density function

$$f(x,v) := \mathbb{1}_{[0,\varepsilon^\alpha)}(|x|)\ \mathbb{1}_{[0,\varepsilon)}(|v|)\ \left(\frac{4\pi}{3}\right)^{-2} \varepsilon^{\beta-3\alpha-3} \ .$$

which has total mass $M = \varepsilon^\beta$. Clearly,

$$E_{\mathrm{kin}}(f) \le \frac{1}{2}\varepsilon^{\beta+2} \ .$$

We calculate

$$\rho_{[f]}(x) = \mathbb{1}_{[0,\varepsilon^\alpha)}(|x|)\ \left(\frac{4\pi}{3}\right)^{-1} \varepsilon^{\beta-3\alpha} \ .$$

Thus,

$$E_{\mathrm{pot}}(f) = -\frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\rho_{[f]}(x)\rho_{[f]}(y)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y \le -\frac{1}{2}\left(\varepsilon^{\beta-3\alpha}\right)^2 \left(2\varepsilon^\alpha\right)^{-1} \left(\varepsilon^\alpha\right)^6 = -\frac{1}{4}\varepsilon^{2\beta-\alpha}$$

Since ε was chosen sufficiently small and $\beta - 3\alpha - 3 > 0$, the hypothesis gives

$$\begin{aligned}
\mathcal{C}(f) &= \int_{\varepsilon^\alpha\mathbb{B}^3} \mathrm{d}x \int_{\varepsilon\mathbb{B}^3} \mathrm{d}v\ \Phi\left(\left(\frac{4\pi}{3}\right)^{-2}\varepsilon^{\beta-3\alpha-3}\ ,\ |x\wedge v|^2\right) \\
&\le C_1 \left(\frac{4\pi}{3}\right)^{-2\frac{\kappa_1+1}{\kappa_1}} \left(\varepsilon^{\beta-3\alpha-3}\right)^{\frac{\kappa_1+1}{\kappa_1}} \int_{\varepsilon^\alpha\mathbb{B}^3}\mathrm{d}x \int_{\varepsilon\mathbb{B}^3}\mathrm{d}v\ |x\wedge v|^{\frac{2\lambda_1}{\kappa_1}} \\
&= C\left(C_1\right)\left(\varepsilon^{\beta-3\alpha-3}\right)^{\frac{\kappa_1+1}{\kappa_1}} \int_0^{\varepsilon^\alpha}\mathrm{d}|x| \int_0^{\varepsilon}\mathrm{d}|v| \int_0^{\pi}\mathrm{d}\vartheta\ |x|^{2+\frac{2\lambda_1}{\kappa_1}}|v|^{2+\frac{2\lambda_1}{\kappa_1}}\left(\sin(\vartheta)\right)^{1+\frac{2\lambda_1}{\kappa_1}} \\
&= C(\kappa_1,\lambda_1,C_1)\left(\varepsilon^{\beta-3\alpha-3}\right)^{\frac{\kappa_1+1}{\kappa_1}} \varepsilon^{(1+\alpha)(3+\frac{2\lambda_1}{\kappa_1})} \\
&= C(\kappa_1,\lambda_1,C_1)\varepsilon^{\beta\frac{\kappa_1+1}{\kappa_1}-\frac{2\lambda_1-3}{\kappa_1}(\alpha+1)} \ .
\end{aligned}$$

The condition $\lambda_1 > -\frac{3}{2}\kappa_1$ guarantees that the above integrals converge.

Putting the above estimates together, we have

$$\mathcal{H}_{\mathrm{C}}(f) \le \frac{1}{2}\varepsilon^{\beta+2} - \frac{1}{4}\varepsilon^{2\beta-\alpha} + C(\kappa_1,\lambda_1,C_2)\varepsilon^{\beta\frac{\kappa_1+1}{\kappa_1}-\frac{2\lambda_1-3}{\kappa_1}(\alpha+1)} \ . \tag{$*$}$$

Now,

$$\beta < \alpha+2 \quad \text{implies} \quad 2\beta-\alpha < \beta+2 \ ,$$

$$\beta < \frac{(3-2\lambda_1-\kappa_1)|\alpha|-(3-2\lambda_1)}{\kappa_1-1} \quad \text{implies} \quad 2\beta-\alpha < \beta\frac{\kappa_1+1}{\kappa_1} - \frac{2\lambda_1-3}{\kappa_1}(\alpha+1) \ .$$

Hence, if $\varepsilon > 0$ is taken sufficiently small, the negative term in $(*)$ dominates the positive terms, so that $\mathcal{H}_{\mathrm{C}}(f) < 0$. Thus $h_M < 0$ for $M > 0$ sufficiently small. □

It turns out that the various terms in $\mathcal{H}_{\mathrm{C}}$ can be estimated by the $\mathfrak{F}^{\kappa_0,\lambda_0}$ norm, and the $\mathfrak{R}^{\nu_0,\lambda_0}$ norm of the corresponding spatial density. The specific estimates are provided in the next few lemmas.

Lemma 3.19. *Assume $\kappa_0 > 0$ and $\lambda_0 \in (0,1)$. Set*

$$\nu_0 := \frac{3+2\kappa_0-2\lambda_0}{2} \ .$$

Then for any $f \in \mathfrak{F}^{\kappa_0,\lambda_0}$, we have

$$\left\|\rho_{[f]}\right\|_{\mathfrak{R}^{\nu_0,\lambda_0}} \le C(\kappa_0,\lambda_0)E_{\mathrm{kin}}(|f|)^{\frac{3-2\lambda_0}{2(\nu_0+1)}}\|f\|_{\mathfrak{F}^{\kappa_0,\lambda_0}}^{\frac{\kappa_0+1}{\nu_0+1}} \ .$$

Proof. In this proof, C always denotes generic positive constants, whose values may change from line to line, but depending only on κ_0, λ_0.

For any $S > 0$, we have

$$
\begin{aligned}
|\rho_{[f]}(x)| &\leq \int_{\{|v|\leq S\}} |f(x,v)|\, |x\wedge v|^{\frac{2\lambda_0}{\kappa_0+1}} |x\wedge v|^{-\frac{2\lambda_0}{\kappa_0+1}} \,\mathrm{d}v + \int_{\{|v|>S\}} |f(x,v)|\,\mathrm{d}v \\
&\leq \left(\int_{\mathbb{R}^3} |f(x,v)|^{\frac{\kappa_0+1}{\kappa_0}} |x\wedge v|^{\frac{2\lambda_0}{\kappa_0}} \,\mathrm{d}v\right)^{\frac{\kappa_0}{\kappa_0+1}} \left(\int_{\{|v|\leq S\}} |x\wedge v|^{-2\lambda_0}\,\mathrm{d}v\right)^{\frac{1}{\kappa_0+1}} \\
&\quad + \frac{1}{S^2}\int_{\mathbb{R}^3} |f(x,v)||v|^2 \mathrm{d}v \\
&= C|x|^{-\frac{2\lambda_0}{\kappa_0+1}} \left(\int_{\mathbb{R}^3} |f(x,v)|^{\frac{\kappa_0+1}{\kappa_0}} |x\wedge v|^{\frac{2\lambda_0}{\kappa_0}} \,\mathrm{d}v\right)^{\frac{\kappa_0}{\kappa_0+1}} S^{\frac{3-2\lambda_0}{\kappa_0+1}} \\
&\quad + \frac{1}{S^2}\int_{\mathbb{R}^3} |f(x,v)||v|^2 \mathrm{d}v
\end{aligned}
$$

where the condition $\lambda_0 < 1$ guarantees

$$
C = \left(2\pi \int_0^1 s^{2-2\lambda_0}\mathrm{d}s \int_0^\pi (\sin(\vartheta))^{1-2\lambda_0}\,\mathrm{d}\vartheta\right)^{\frac{1}{\kappa_0+1}} < \infty
$$

We optimise the preceding estimate by setting

$$
S^{\frac{5+2\kappa_0-2\lambda_0}{\kappa_0+1}} = C|x|^{\frac{2\lambda_0}{\kappa_0+1}} \left(\int_{\mathbb{R}^3} |f(x,v)|^{\frac{\kappa_0+1}{\kappa_0}} |x\wedge v|^{\frac{2\lambda_0}{\kappa_0}} \,\mathrm{d}v\right)^{-\frac{\kappa_0}{\kappa_0+1}} \left(\int_{\mathbb{R}^3} |f(x,v)||v|^2\mathrm{d}v\right) ,
$$

which gives

$$
|\rho_{[f]}(x)| \leq C|x|^{-\frac{2\lambda_0}{\nu_0+1}} \left(\int_{\mathbb{R}^3} |f(x,v)|^{\frac{\kappa_0+1}{\kappa_0}} |x\wedge v|^{\frac{2\lambda_0}{\kappa_0}} \,\mathrm{d}v\right)^{\frac{\kappa_0}{\nu_0+1}} \left(\int_{\mathbb{R}^3} |f(x,v)||v|^2\mathrm{d}v\right)^{\frac{3-2\lambda_0}{2(\nu_0+1)}} .
$$

Therefore, using Hölder's inequality,

$$
\begin{aligned}
&\int_{\mathbb{R}^3} |\rho_{[f]}(x)|^{\frac{\nu_0+1}{\nu_0}} |x|^{\frac{2\lambda_0}{\nu_0}}\,\mathrm{d}x \\
&\quad\leq C\int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} |f(x,v)|^{\frac{\kappa_0+1}{\kappa_0}} |x\wedge v|^{\frac{2\lambda_0}{\kappa_0}} \,\mathrm{d}v\right)^{\frac{\kappa_0}{\nu_0}} \left(\int_{\mathbb{R}^3} |f(x,v)||v|^2\mathrm{d}v\right)^{\frac{3-2\lambda_0}{2\nu_0}} \mathrm{d}x \\
&\quad\leq C\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |f(x,v)|^{\frac{\kappa_0+1}{\kappa_0}} |x\wedge v|^{\frac{2\lambda_0}{\kappa_0}} \,\mathrm{d}v\,\mathrm{d}x\right)^{\frac{\kappa_0}{\nu_0}} \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |f(x,v)||v|^2\mathrm{d}v\,\mathrm{d}x\right)^{\frac{3-2\lambda_0}{2\nu_0}} \\
&\quad= CE_{\mathrm{kin}}(|f|)^{\frac{3-2\lambda_0}{2\nu_0}} \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |f(x,v)|^{\frac{\kappa_0+1}{\kappa_0}} |x\wedge v|^{\frac{2\lambda_0}{\kappa_0}} \,\mathrm{d}v\,\mathrm{d}x\right)^{\frac{\kappa_0}{\nu_0}}
\end{aligned}
$$

which proves the result. □

In order to estimate the potential energy term, we will need to relate the usual L^p norms with the $\mathfrak{R}^{\nu_0,\lambda_0}$ norm. In general these two norms are not equivalent on spaces of functions defined over $\mathbb{R}^3$. However, when we restrict to bounded neighbourhoods of the origin, we have the following relations.

Lemma 3.20. *Let $1 \leq p < \infty$ and $s > 0$.*

(i) There is an inclusion

$$
L^p_{\mathrm{loc}}\left(\mathbb{R}^d, \mathcal{L}^d\right) \hookrightarrow L^p_{\mathrm{loc}}\left(\mathbb{R}^d,\ |\cdot|^s\mathcal{L}^d\right)
$$

which is continuous as a linear map between Frechét spaces: For $R > 0$,

$$
\|\cdot\|_{L^p(R\mathbb{B}^d,|\cdot|^s\mathcal{L}^d)} \leq R^{\frac{s}{p}} \|\cdot\|_{L^p(R\mathbb{B}^d,\mathcal{L}^d)} \,.
$$

(ii) If $1 \le q < p$ and $\frac{qs}{p-q} < d$, then there is an inclusion

$$L^p_{\mathrm{loc}}\left(\mathbb{R}^d,\ |\cdot|^s \mathcal{L}^d\right) \hookrightarrow L^q_{\mathrm{loc}}\left(\mathbb{R}^d, \mathcal{L}^d\right)$$

which is continuous as a linear map between Fréchet spaces: For $R > 0$,

$$\|\cdot\|_{L^q(R\mathbb{B}^d,\mathcal{L}^d)} \le \left(\left|\mathbb{S}^{d-1}\right| \frac{p-q}{(p-q)d-qs}\right)^{\frac{p-q}{pq}} R^{\frac{(p-q)d-qs}{pq}} \|\cdot\|_{L^p(R\mathbb{B}^d,|\cdot|^s\mathcal{L}^d)} .$$

Proof. The first statement is obvious from

$$\|f\|^p_{L^p(R\mathbb{B}^d,|\cdot|^s\mathcal{L}^d)} = \int_{R\mathbb{B}^d} |f(x)|^p |x|^s \,\mathrm{d}x \le R^s \int_{R\mathbb{B}^d} |f(x)|^p \,\mathrm{d}x = R^s \|f\|^p_{L^p(R\mathbb{B}^d,\mathcal{L}^d)} .$$

The second statement is an easy exercise in Hölder's inequality:

$$\begin{aligned}
\|f\|^q_{L^q(R\mathbb{B}^d,\mathcal{L}^d)} &= \int_{R\mathbb{B}^d} |x|^{-\frac{sq}{p}} \left(|f(x)|^p|x|^s\right)^{\frac{q}{p}} \,\mathrm{d}x \\
&\le \left(\int_{R\mathbb{B}^d} |x|^{-\frac{sq}{p-q}} \,\mathrm{d}x\right)^{\frac{p-q}{p}} \left(\int_{R\mathbb{B}^d} |f(x)|^p|x|^s \,\mathrm{d}x\right)^{\frac{q}{p}} \\
&= \left(\left|\mathbb{S}^{d-1}\right| \left(d - \frac{sq}{p-q}\right)^{-1} R^{d-\frac{sq}{p-q}}\right)^{\frac{p-q}{p}} \|f\|^q_{L^p(R\mathbb{B}^d,|\cdot|^s\mathcal{L}^d)} .
\end{aligned}$$

This completes the proof. □

Lemma 3.21. *Let $1 < \nu < 3$ and $0 < \lambda < 1$ be such that $\nu + 4\lambda < 5$. Then there exists a constant $\tau = \tau(\nu, \lambda) \in (0, 1)$ such that any $\rho \in L^1(\mathbb{R}^3) \cap \mathfrak{R}^{\nu,\lambda}$ satisfies*

$$\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\rho(x)\rho(y)}{|x-y|} \,\mathrm{d}y \,\mathrm{d}x\right| \le C\left(\nu, \lambda, \|\rho\|_{L^1(\mathbb{R}^3)}\right) \|\rho\|^{\frac{(\nu+1)}{\nu}\tau}_{\mathfrak{R}^{\nu,\lambda}} + C\left(\|\rho\|_{L^1(\mathbb{R}^3)}\right) .$$

The constants can be chosen to be non-decreasing in $\|\rho\|_{L^1(\mathbb{R}^3)}$.

Proof. Without loss of generality we may assume that $\rho \ge 0$. We partition the domain of integration, $\{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3\}$, into

$$\begin{aligned}
A_1 &:= \{|x-y| \ge 1\} \cap \left(\{\rho(x) \le 1\} \cup \{\rho(y) \le 1\}\right) , \\
A_2 &:= \{|x-y| \ge 1\} \cap \{\rho(x) > 1\} \cap \{\rho(y) > 1\} , \\
A_3 &:= \{|x-y| < 1\} ,
\end{aligned}$$

and estimate the integral over each of these regions.

The integrals over A_1 and A_3 are easily estimated:

$$\begin{aligned}
\iint_{A_1} \frac{\rho(x)\rho(y)}{|x-y|} \,\mathrm{d}y \,\mathrm{d}x &\le 2\int_{\mathbb{R}^3} \rho(y) \int_{\mathbb{B}^3} \frac{\mathrm{d}x}{|x-y|} \,\mathrm{d}y = 4\pi\|\rho\|_{L^1(\mathbb{R}^3)} , \\
\iint_{A_3} \frac{\rho(x)\rho(y)}{|x-y|} \,\mathrm{d}y \,\mathrm{d}x &\le \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \rho(x)\rho(y) \,\mathrm{d}y \,\mathrm{d}x = \|\rho\|^2_{L^1(\mathbb{R}^3)} .
\end{aligned}$$

We now turn to the integral over A_2. Fix $\alpha, \sigma \in \mathbb{R}$ such that

$$\frac{6}{5} < \alpha < \frac{3(\nu+1)}{3\nu+2\lambda} < \sigma < \frac{\nu+1}{\nu}$$

and

$$\tau := \frac{2\nu}{\nu+1}\frac{\sigma}{\sigma-1}\frac{\alpha-1}{\alpha} < 1 .$$

We claim that this is possible. Indeed, the hypothesis $\nu + 4\lambda < 5$ guarantees $\frac{6}{5} < \frac{3(\nu+1)}{3\nu+2\lambda}$, so there exist α, σ in the ranges specified. Moreover, $\frac{\sigma}{\sigma-1}\frac{\alpha-1}{\alpha}$ decreases when σ increases or α decreases, and so τ can be made slightly larger than $\frac{\nu}{3}$, by choosing α slightly larger than $\frac{6}{5}$ and σ slightly smaller than $\frac{\nu+1}{\nu}$.

Define $\beta > 0$ by

$$\frac{2}{\alpha} + \frac{1}{\beta} := 2 \ .$$

The condition $\frac{6}{5} < \alpha < \frac{\nu+1}{\nu} < 2$ then guarantees $1 < \beta < 3$.

Let ρ° be the symmetric decreasing rearrangement of $\rho \mathbb{1}_{\{\rho \geq 1\}}$. By Markov's inequality, ρ° is supported in $C\|\rho\|_{L^1(\mathbb{R}^3)}\overline{\mathbb{B}^3}$ for some constant C independent of ν, λ, ρ. Moreover, we have, from Corollary A.5, that

$$\|\rho^\circ\|_{L^1(\mathbb{R}^3)} = \|\rho\|_{L^1(\mathbb{R}^3)} \ , \quad \|\rho^\circ\|_{\mathfrak{R}^{\nu,\lambda}} \leq \|\rho\|_{\mathfrak{R}^{\nu,\lambda}} \ .$$

Using Hölder's inequality and Lemma 3.20,

$$\begin{aligned} \|\rho^\circ\|^\alpha_{L^\alpha(\mathbb{R}^3)} &\leq \|\rho^\circ\|^{\frac{\sigma-\alpha}{\sigma-1}}_{L^1(\mathbb{R}^3)} \|\rho^\circ\|^{\frac{\alpha-1}{\sigma-1}\sigma}_{L^\sigma(\mathbb{R}^3)} \\ &\leq \|\rho\|^{\frac{\sigma-\alpha}{\sigma-1}}_{L^1(\mathbb{R}^3)} \left[C\left(\nu, \lambda, \|\rho\|_{L^1(\mathbb{R}^3)}\right) \|\rho^\circ\|_{\mathfrak{R}^{\nu,\lambda}} \right]^{\frac{\sigma}{\sigma-1}(\alpha-1)} \\ &= C\left(\nu, \lambda, \|\rho\|_{L^1(\mathbb{R}^3)}\right) \|\rho^\circ\|^{\frac{\sigma}{\sigma-1}(\alpha-1)}_{\mathfrak{R}^{\nu,\lambda}} \end{aligned}$$

where the constant $C = C(\nu, \lambda, \|\rho\|_{L^1(\mathbb{R}^3)})$ can be chosen to be of the form $C(\nu, \lambda)\|\rho\|^\theta_{L^1(\mathbb{R}^3)}$ for some $\theta > 0$. Using Riesz's rearrangement inequality (Theorem A.6) and Proposition 3.14, we have

$$\begin{aligned} \iint_{A_2} \frac{\rho(x)\rho(y)}{|x-y|} \,\mathrm{d}y \,\mathrm{d}x &\leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \rho^\circ(x) \frac{\mathbb{1}_{\mathbb{B}^3}(x-y)}{|x-y|} \rho^\circ(y) \,\mathrm{d}y \,\mathrm{d}x \\ &\leq \left\| \frac{\mathbb{1}_{\mathbb{B}^3}}{|\cdot|} \right\|_{L^\beta(\mathbb{R}^3)} \|\rho^\circ\|^2_{L^\alpha(\mathbb{R}^3)} \\ &\leq C\left(\nu, \lambda, \|\rho\|_{L^1(\mathbb{R}^3)}\right) \|\rho^\circ\|^{2\frac{\sigma}{\sigma-1}\frac{\alpha-1}{\alpha}}_{\mathfrak{R}^{\nu,\lambda}} \\ &\leq C\left(\nu, \lambda, \|\rho\|_{L^1(\mathbb{R}^3)}\right) \|\rho\|^{\frac{(\nu+1)}{\nu}\tau}_{\mathfrak{R}^{\nu,\lambda}} \ . \end{aligned}$$

This completes the proof. ☐

Lemma 3.22. *If $\alpha, \beta \geq 0$ satisfy $\alpha + \beta < 1$, then there exist $a \in (\alpha, 1), b \in (\beta, 1), C > 0$, such that*

$$A^\alpha B^\beta \leq \frac{\alpha}{a} A^a + \frac{\beta}{b} B^b + C \qquad \textit{for all } A, B \in [0, \infty) \ .$$

Proof. Pick $p, q > 1$ such that $\alpha < \frac{1}{p}$, $\beta < \frac{1}{q}$ and $\frac{1}{p} + \frac{1}{q} < 1$. The usual Young's inequality then gives

$$A^\alpha B^\beta \leq \frac{1}{p} A^{p\alpha} + \frac{1}{q} B^{q\beta} + \left(1 - \frac{1}{p} - \frac{1}{q}\right) \qquad \text{for all } A, B \in [0, \infty) \ .$$

So the lemma follows from setting $a := p\alpha$ and $b := q\beta$. ☐

With the preceding lemmas in place, we are now ready to prove an estimate which shows that $h_M > -\infty$ for $M > 0$. In other words, our minimisation problem makes sense.

Proposition 3.23. *Assume $\kappa_0 > 0$ and $\lambda_0 \in (0, 1)$ are such that $\nu_0 := \frac{3+2\kappa_0-2\lambda_0}{2}$ satisfies $1 < \nu_0 < 3$ and $\nu_0 + 4\lambda_0 < 5$. Assume also*

$$\Phi(h, L) \geq C_0 h^{\frac{\kappa_0+1}{\kappa_0}} L^{\frac{\lambda_0}{\kappa_0}} \ .$$

Then there exist $a = a(\kappa_0, \lambda_0) \in (0,1), b = b(\kappa_0, \lambda_0) \in (0,1)$ such that for any fixed $M > 0$, there exist constants $C = C(\kappa_0, \lambda_0, C_0, M)$ which satisfies the property that whenever $f \in \mathcal{F}^M$ has $E_{\mathrm{kin}}(f) < \infty$, it holds that

$$\mathcal{H}_{\mathrm{C}}(f) \geq C\left[E_{\mathrm{kin}}(f) - CE_{\mathrm{kin}}(f)^a\right] + C\left[\|f\|_{\mathfrak{F}^{\kappa_0,\lambda_0}}^{\frac{\kappa_0+1}{\kappa_0}} - C\left(\|f\|_{\mathfrak{F}^{\kappa_0,\lambda_0}}^{\frac{\kappa_0+1}{\kappa_0}}\right)^b\right] - C\,.$$

In particular, $h_M > -\infty$.

Proof. In this proof, C always denotes generic positive constants, which may have different values, but depending only on $\kappa_0, \lambda_0, C_0, M$.

Pick τ for (ν_0, λ_0) as in Lemma 3.21. Let $f \in \mathcal{F}^M$. Then, using Lemmas 3.21 and 3.19,

$$\begin{aligned}
\mathcal{H}_{\mathrm{C}}(f) &= E_{\mathrm{kin}}(f) + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \Phi\left(f(x,v),\ |x \wedge v|^2\right)\, \mathrm{d}v\, \mathrm{d}x - |E_{\mathrm{pot}}(f)| \\
&\geq E_{\mathrm{kin}}(f) + C_0\|f\|_{\mathfrak{F}^{\kappa_0,\lambda_0}}^{\frac{\kappa_0+1}{\kappa_0}} - C\left\|\rho_{[f]}\right\|_{\mathfrak{R}^{\nu_0,\lambda_0}}^{\frac{(\nu_0+1)}{\nu_0}\tau} - C \\
&\geq E_{\mathrm{kin}}(f) + C_0\|f\|_{\mathfrak{F}^{\kappa_0,\lambda_0}}^{\frac{\kappa_0+1}{\kappa_0}} - C\left[E_{\mathrm{kin}}(f)^{\frac{3-2\lambda_0}{2(\nu_0+1)}}\|f\|_{\mathfrak{F}^{\kappa_0,\lambda_0}}^{\frac{\kappa_0+1}{\nu_0+1}}\right]^{\frac{(\nu_0+1)}{\nu_0}\tau} - C \\
&= E_{\mathrm{kin}}(f) + C_2\|f\|_{\mathfrak{F}^{\kappa_0,\lambda_0}}^{\frac{\kappa_0+1}{\kappa_0}} - CE_{\mathrm{kin}}(f)^{\frac{3-2\lambda_0}{2\nu_0}\tau}\left(\|f\|_{\mathfrak{F}^{\kappa_0,\lambda_0}}^{\frac{\kappa_0+1}{\kappa_0}}\right)^{\frac{\kappa_0}{\nu_0}\tau} - C\,.
\end{aligned}$$

Since $\frac{3-2\lambda_0}{2\nu_0}\tau + \frac{\kappa_0}{\nu_0}\tau = \tau < 1$, we may use Lemma 3.22 to split the middle term as

$$E_{\mathrm{kin}}(f)^{\frac{3-2\lambda_0}{2\nu_0}\tau}\left(\|f\|_{\mathfrak{F}^{\kappa_0,\lambda_0}}^{\frac{\kappa_0+1}{\kappa_0}}\right)^{\frac{\kappa_0}{\nu_0}\tau} \leq CE_{\mathrm{kin}}(f)^a + C\left(\|f\|_{\mathfrak{F}^{\kappa_0,\lambda_0}}^{\frac{\kappa_0+1}{\kappa_0}}\right)^b + C$$

with $a > \frac{3-2\lambda_0}{2\nu_0}\tau$ and $b > \frac{\kappa_0}{\nu_0}\tau$, but $a + b < 1$. Hence,

$$\mathcal{H}_{\mathrm{C}}(f) \geq C\left[E_{\mathrm{kin}}(f) - CE_{\mathrm{kin}}(f)^a\right] + C\left[\|f\|_{\mathfrak{F}^{\kappa_0,\lambda_0}}^{\frac{\kappa_0+1}{\kappa_0}} - C\left(\|f\|_{\mathfrak{F}^{\kappa_0,\lambda_0}}^{\frac{\kappa_0+1}{\kappa_0}}\right)^b\right] - C$$

where the right-hand side is bounded below. Since $f \in \mathcal{F}^M$ was arbitrary, we are done. □

Corollary 3.24. *Let assumptions (i)–(iii) hold for Φ. Then*

$$0 > h_M > -\infty \quad \text{for any } M > 0\,.$$

Moreover, if $\{f_j\}_{j\in\mathbb{N}} \subset \mathcal{F}^M$ is a minimising sequence for $\mathcal{H}_{\mathrm{C}}$ over $\mathcal{F}^M$, then

- *The sequences $\{E_{\mathrm{kin}}(f_j)\}_{j\in\mathbb{N}}, \{|E_{\mathrm{pot}}(f_j)|\}_{j\in\mathbb{N}}, \{\mathcal{C}(f_j)\}_{j\in\mathbb{N}}$ are bounded.*
- *$\{f_j\}_{j\in\mathbb{N}}$ is a bounded sequence in $\mathfrak{F}^{\kappa_0,\lambda_0}$.*
- *$\{\rho_{[f_j]}\}_{j\in\mathbb{N}}$ is a bounded sequence in $\mathfrak{R}^{\nu_0,\lambda_0}$.* □

With these boundedness properties for a minimising sequence, it is now possible to prove the following compactness result, which is an analogue of Proposition 3.15. Notice, however, that since we now have control over the $\mathfrak{R}^{\nu_0,\lambda_0}$ instead of a Lebesgue space norm, one of the integrals has to be estimated using a different technique.

Proposition 3.25. *Let $\nu > 1$ and $0 < \lambda < 1$ satisfy $\nu + 4\lambda < 5$. Let $\{\rho_j\}_{j\in\mathbb{N}} \cup \{\rho_0\} \subset \mathfrak{R}^{\nu,\lambda}$ be such that*

$$\rho_j \rightharpoonup \rho_0 \text{ weakly in } \mathfrak{R}^{\nu,\lambda}\,,$$

$$\forall\, \varepsilon > 0\ \exists\, R > 0\ :\ \limsup_{j\to\infty} \int_{\mathbb{R}^3\setminus R\mathbb{B}^3} |\rho_j(x)|\, \mathrm{d}x < \varepsilon\,,$$

Then $\nabla U_{[\rho_j]} \to \nabla U_{[\rho_0]}$ strongly in $(L^2(\mathbb{R}^3))^3$.

Proof. For any $r > 0$, Hölder's inequality gives

$$\int_{r\mathbb{B}^3} |\rho(x)| \, \mathrm{d}x \le \left(\int_{r\mathbb{B}^3} |x|^{-2\lambda} \, \mathrm{d}x\right)^{\frac{1}{\nu+1}} \left(\int_{r\mathbb{B}^3} |\rho(x)|^{\frac{\nu+1}{\nu}} |x|^{\frac{2\lambda}{\nu}} \mathrm{d}x\right)^{\frac{\nu}{\nu+1}} \le C(\nu,\lambda) \|\rho\|_{\mathfrak{R}^{\nu,\lambda}} \, .$$

Therefore the hypotheses guarantee that $\{\rho_j\}_{j\in\mathbb{N}}$ is bounded in $L^1(\mathbb{R}^3)$, and weak convergence guarantees that $\rho_0 \in L^1(\mathbb{R}^3)$ as well.

The sequence $\sigma_j := \rho_j - \rho_0$ converges weakly to 0 in $\mathfrak{R}^{\nu,\lambda}$, and is bounded in $L^1(\mathbb{R}^3)$. We need to show $\nabla U_{[\sigma_j]} \to 0$ strongly in $(L^2(\mathbb{R}^3))^3$, which is equivalent to showing

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\sigma_j(x)\sigma_j(y)}{|x-y|} \, \mathrm{d}y \, \mathrm{d}x \to 0 \, . \tag{$*$}$$

For convenience, denote

$$P := \sup_{j\in\mathbb{N}} \|\sigma_j\|_{L^1(\mathbb{R}^3)} \ , \quad Q := \sup_{j\in\mathbb{N}} \|\sigma_j\|_{\mathfrak{R}^{\nu,\lambda}} \ .$$

In addition, fix some

$$\alpha \in \left(\frac{6}{5} \, , \, \frac{3(\nu+1)}{3\nu+2\lambda}\right) \, .$$

This is possible since the hypothesis $\nu + 4\lambda < 5$ guarantees $\frac{6}{5} < \frac{3(\nu+1)}{3\nu+2\lambda}$. Define $\beta > 0$ by

$$\frac{2}{\alpha} + \frac{1}{\beta} := 2 \, .$$

The condition $\frac{6}{5} < \alpha < \frac{\nu+1}{\nu} < 2$ then guarantees $1 < \beta < 3$.

Let $\delta > 0$ be small and $R > 0$ be large, to be chosen later. We partition the domain of integration, $\{(x,y) \in \mathbb{R}^3 \times \mathbb{R}^3\}$, into

$$\begin{aligned}
A_1 &:= \{|x-y| \ge \delta\} \cap \big(\{|x| \ge R\} \cup \{|y| \ge R\}\big) \ , \\
A_2 &:= \{|x-y| \ge \delta\} \cap \{|x| < R\} \cap \{|y| < R\} \ , \\
B_{j,1} &:= \{|x-y| < \delta\} \cap (\{|\sigma_j(x)| \le P\} \cup \{|\sigma_j(y)| \le P\}) \ , \\
B_{j,2} &:= \{|x-y| < \delta\} \cap \{|\sigma_j(x)| > P\} \cap \{|\sigma_j(y)| > P\} \ .
\end{aligned}$$

We will estimate the integral $(*)$ over each of these sets.

The integral over A_1 is estimated by

$$\iint_{A_1} \frac{|\sigma_j(x)|\,|\sigma_j(y)|}{|x-y|} \, \mathrm{d}x \, \mathrm{d}y \le \frac{2}{\delta} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3 \setminus R\mathbb{B}^3} |\sigma_j(x)|\,|\sigma_j(y)| \, \mathrm{d}x \, \mathrm{d}y \le \frac{2P}{\delta} \int_{\mathbb{R}^3 \setminus R\mathbb{B}^3} |\sigma_j(y)| \, \mathrm{d}y \, .$$

We next consider the integral over A_2. Let

$$\begin{aligned}
h_j(x) &:= |x|^{-\frac{2\lambda}{\nu}} \mathbb{1}_{R\mathbb{B}^3}(x) \int_{\mathbb{R}^3} \frac{\mathbb{1}_{\mathbb{R}^3\setminus\delta\mathbb{B}^3}(x-y)}{|x-y|} \mathbb{1}_{R\mathbb{B}^3}(y) \sigma_j(y) \, \mathrm{d}y \\
&= |x|^{-\frac{2\lambda}{\nu}} \mathbb{1}_{R\mathbb{B}^3}(x) \int_{\mathbb{R}^3} \left(\frac{\mathbb{1}_{\mathbb{R}^3\setminus\delta\mathbb{B}^3}(x-y)}{|x-y|} |y|^{-\frac{2\lambda}{\nu}} \mathbb{1}_{R\mathbb{B}^3}(y) \, \sigma_j(y)\right) |y|^{\frac{2\lambda}{\nu}} \mathrm{d}y \, .
\end{aligned}$$

For any $x \in \mathbb{R}^3$, a straightforward estimate yields

$$\frac{\mathbb{1}_{\mathbb{R}^3\setminus\delta\mathbb{B}^3}(x - \cdot\,)}{|x - \cdot\,|} |\cdot|^{-\frac{2\lambda}{\nu}} \mathbb{1}_{R\mathbb{B}^3} \in L^{\nu+1}\left(\mathbb{R}^3, |\cdot|^{\frac{2\lambda}{\nu}} \mathcal{L}^3\right) = \left(\mathfrak{R}^{\nu,\lambda}\right)' \, .$$

Since $\sigma_j \rightharpoonup 0$ in $\mathfrak{R}^{\nu,\lambda}$, we see that $h_j \to 0$ pointwise on $\mathbb{R}^3 \setminus \{0\}$. On the other hand, we have the uniform bound $|h_j| \le \frac{P}{\delta} |\cdot|^{-\frac{2\lambda}{\nu}} \mathbb{1}_{R\mathbb{B}^3}$, so the dominated convergence theorem guarantees that $h_j \to 0$ in $\mathfrak{R}^{\nu,\lambda}$. Thus

$$\iint_{A_2} \frac{\sigma_j(x)\sigma_j(y)}{|x-y|} \, \mathrm{d}x \, \mathrm{d}y = \int_{\mathbb{R}^3} h_j(x) \sigma_j(x) |x|^{\frac{2\lambda}{\nu}} \mathrm{d}x$$

tends to 0 as $j \to \infty$.

The integral over $B_{j,1}$ is estimated by

$$\iint_{B_{j,1}} \frac{|\sigma_j(x)|\,|\sigma_j(y)|}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y \le 2P \int_{\mathbb{R}^3} |\sigma_j(y)| \int_{\delta\mathbb{B}^3} \frac{\mathrm{d}x}{|x-y|}\,\mathrm{d}y = CP^2\delta^2 \,.$$

We now turn to the rather tricky task of estimating the integral over $B_{j,2}$. Let σ_j° be the symmetric decreasing rearrangement of $|\sigma_j|\mathbb{1}_{\{|\sigma_j|>P\}}$. Since Markov's inequality gives $\mathcal{L}^3\{|\sigma_j| > P\} \le C$, we see that each σ_j° is supported in $C\overline{\mathbb{B}^3}$. Since $\alpha < \frac{3\nu+3}{3\nu+2\lambda}$, Lemma 3.20 applies to give

$$\left\|\sigma_j^\circ\right\|_{L^\alpha(\mathcal{L}^3)} \le C(\nu,\lambda,\alpha)\left\|\sigma_j^\circ\right\|_{\mathfrak{R}^{\nu,\lambda}} \,.$$

Moreover, by Corollary A.5,

$$\left\|\sigma_j^\circ\right\|_{\mathfrak{R}^{\nu,\lambda}} \le \|\sigma_j\|_{\mathfrak{R}^{\nu,\lambda}} \le Q \,.$$

Using Riesz's rearrangement inequality (Theorem A.6) and Proposition 3.14, we obtain the following estimate for the integral $(*)$ over $B_{j,2}$.

$$\begin{aligned}\iint_{B_{j,2}} \frac{|\sigma_j(x)|\,|\sigma_j(y)|}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y &\le \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \sigma_j^\circ(x)\frac{\mathbb{1}_{\delta\mathbb{B}^3}(x-y)}{|x-y|}\sigma_j^\circ(y)\,\mathrm{d}x\,\mathrm{d}y \\ &\le C(\alpha)\left\|\frac{\mathbb{1}_{\delta\mathbb{B}^3}}{|\cdot|}\right\|_{L^\beta(\mathbb{R}^3)}\left\|\sigma_j^\circ\right\|^2_{L^\alpha(\mathbb{R}^3)} \\ &\le C(\nu,\lambda,\alpha)Q^2\delta^{\frac{3-\beta}{\beta}} \,.\end{aligned}$$

Finally, we collect our estimates above to conclude

$$\begin{aligned}\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\sigma_j(x)\sigma_j(y)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y\right| &\le CP^2\delta^2 + C(\nu,\lambda,\alpha)Q^2\delta^{\frac{3-\beta}{\beta}} \\ &\quad + \frac{2P}{\delta}\int_{\mathbb{R}^3\setminus R\mathbb{B}^3} |\sigma_j(y)|\,\mathrm{d}y + \left|\iint_{A_2} \frac{\sigma_j(x)\sigma_j(y)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y\right|\end{aligned}$$

Now, given $\varepsilon > 0$, choose $\delta > 0$ small enough so that

$$CP^2\delta^2 + C(\nu,\lambda,\alpha)Q^2\delta^{\frac{3-\beta}{\beta}} < \frac{\varepsilon}{2} \,.$$

Then choose $R > 0$ large enough so that

$$\frac{2P}{\delta}\limsup_{j\to\infty}\int_{\mathbb{R}^3\setminus R\mathbb{B}^3} |\sigma_j(y)|\,\mathrm{d}y < \frac{\varepsilon}{2} \,.$$

Since the integral over A_2 tends to 0 as $j \to \infty$, we find

$$\limsup_{j\to\infty}\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\sigma_j(x)\sigma_j(y)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y\right| < \frac{\varepsilon}{2} \,.$$

Since $\varepsilon > 0$ was arbitrary, we have proved the desired result $(*)$. □

The last result the author managed to prove says that, along a minimising sequence for $\mathcal{H}_{\mathrm{C}}$ over $\mathcal{F}^M$, some minimal mass must remain within a ball of sufficiently large radius. Specifically, we have the following result. The proof uses a similar strategy to that employed in the proof of the preceding Proposition 3.25.

Proposition 3.26. *Let assumptions (i)–(iii) hold for* Φ.

Let $\{f_j\}_{j\in\mathbb{N}}$ *be a minimising sequence for* $\mathcal{H}_{\mathrm{C}}$ *over* $\mathcal{F}^M$. *Then there exist* $m_0 > 0, R_0 > 0$ *and a sequence* $\{z_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^3$, *such that*

$$\int_{z_j+R_0\mathbb{B}^3} \rho_{[f_j]}(x)\,\mathrm{d}x \ge m_0 \quad \textit{for sufficiently large } j \in \mathbb{N} \,.$$

Proof. Recalling from Corollary 3.24 that $h_M < 0$, we may, after discarding the first few terms of the minimising sequence $\{f_j\}_{j\in\mathbb{N}}$ if necessary, assume without loss of generality that

$$\frac{h_M}{2} > \mathcal{H}_{\mathrm{C}}(f_j) \quad \text{for all } j \in \mathbb{N} \, .$$

Since also $\{\rho_{[f_j]}\}_{j\in\mathbb{N}}$ is bounded in $\mathfrak{R}^{\nu_0,\lambda_0}$, we may let

$$P := \sup_{j\in\mathbb{N}} \left\|\rho_{[f_j]}\right\|_{\mathfrak{R}^{\nu_0,\lambda_0}} \, .$$

Fix some

$$\alpha \in \left(\frac{6}{5} \, , \, \frac{3(\nu_0+1)}{3\nu_0+2\lambda_0}\right) \, .$$

This is possible since $\nu_0 + 4\lambda_0 < 5$ guarantees $\frac{6}{5} < \frac{3(\nu_0+1)}{3\nu_0+2\lambda_0}$. Define $\beta > 0$ by

$$\frac{2}{\alpha} + \frac{1}{\beta} := 2 \, .$$

The condition $\frac{6}{5} < \alpha < \frac{\nu_0+1}{\nu_0} < 2$ then guarantees $1 < \beta < 3$.

Let $R > 1$ be large, to be chosen later. We partition $\{(x,y) \in \mathbb{R}^3 \times \mathbb{R}^3\}$ into

$$A_1 := \{|x-y| \geq R\} \, , \quad A_2 := \left\{R^{-1} \leq |x-y| \leq R\right\} \, ,$$

$$B_{j,1} := \left\{|x-y| < R^{-1}\right\} \cap \left(\left\{\rho_{[f_j]}(x) \leq 1\right\} \cup \left\{\rho_{[f_j]}(y) \leq 1\right\}\right) \, ,$$
$$B_{j,2} := \left\{|x-y| < R^{-1}\right\} \cap \left\{\rho_{[f_j]}(x) > 1\right\} \cup \left\{\rho_{[f_j]}(y) > 1\right\} \, .$$

We have the estimates

$$\iint_{A_1} \frac{\rho_{[f_j]}(x)\rho_{[f_j]}(y)}{|x-y|} \, \mathrm{d}x \, \mathrm{d}y \leq \frac{1}{R} \iint_{A_1} \rho_{[f_j]}(x)\rho_{[f_j]}(y) \, \mathrm{d}x \, \mathrm{d}y = \frac{M^2}{R} \, ,$$

and

$$\begin{aligned} \iint_{A_2} \frac{\rho_{[f_j]}(x)\rho_{[f_j]}(y)}{|x-y|} \, \mathrm{d}x \, \mathrm{d}y &\leq R \int_{\mathbb{R}^3} \int_{y+R\mathbb{B}^3} \rho_{[f_j]}(x)\rho_{[f_j]}(y) \, \mathrm{d}x \, \mathrm{d}y \\ &\leq MR \sup_{y\in\mathbb{R}^3} \int_{y+R\mathbb{B}^3} \rho_{[f_j]}(x) \, \mathrm{d}x \, , \end{aligned}$$

and

$$\iint_{B_{j,1}} \frac{\rho_{[f_j]}(x)\rho_{[f_j]}(y)}{|x-y|} \, \mathrm{d}x \, \mathrm{d}y \leq 2 \int_{\mathbb{R}^3} \rho_{[f_j]}(y) \int_{y+R^{-1}\mathbb{B}^3} \frac{\mathrm{d}x}{|x-y|} \mathrm{d}y = \frac{4\pi M}{R} \, .$$

Let ρ_j° be the symmetric decreasing rearrangement of $\rho_{[f_j]} \mathbb{1}_{\{\rho_{[f_j]}>1\}}$. By Markov's inequality, each ρ_j° is supported in $CP\overline{\mathbb{B}^3}$ for some $C > 0$ independent of j. Using Lemma 3.20 and Corollary A.5, we have

$$\left\|\rho_j^\circ\right\|_{L^\alpha(\mathbb{R}^3)} \leq C(\nu_0,\lambda_0,\alpha,P) \left\|\rho_j^\circ\right\|_{\mathfrak{R}^{\nu_0,\lambda_0}} \leq C(\nu_0,\lambda_0,\alpha,P) \left\|\rho_{[f_j]}\right\|_{\mathfrak{R}^{\nu_0,\lambda_0}} \leq C(\nu_0,\lambda_0,\alpha,P) \, .$$

Therefore, by Riesz's rearrangement inequality (Theorem A.6) and Proposition 3.14,

$$\begin{aligned} \iint_{B_{j,2}} \frac{\rho_{[f_j]}(x)\rho_{[f_j]}(y)}{|x-y|} \, \mathrm{d}x \, \mathrm{d}y &\leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \rho_j^\circ(x) \frac{\mathbb{1}_{R^{-1}\mathbb{B}^3}(x-y)}{|x-y|} \rho_j^\circ(y) \, \mathrm{d}x \, \mathrm{d}y \\ &\leq C(\alpha) \left\| \frac{\mathbb{1}_{R^{-1}\mathbb{B}^3}}{|\cdot|} \right\|_{L^\beta(\mathbb{R}^3)} \left\|\rho_j^\circ\right\|_{L^\alpha(\mathbb{R}^3)}^2 \\ &\leq C(\nu_0,\lambda_0,\alpha,P) R^{-\frac{3-\beta}{\beta}} \, . \end{aligned}$$

Combining the estimates above, we find

$$\begin{aligned}-E_{\text{pot}}(f_j) &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\rho[f_j](x)\rho_{[f_j]}(y)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y \\ &\le \frac{C(M)}{R} + C(\nu_0,\lambda_0,\alpha,P)R^{-\frac{3-\beta}{\beta}} + MR\sup_{y\in\mathbb{R}^3}\int_{y+R\mathbb{B}^3}\rho_{[f_j]}(x)\,\mathrm{d}x\;.\end{aligned}$$

Since $\frac{h_M}{2} > E_{\text{pot}}(f_j)$ for all j, we have

$$\frac{h_M}{2} > -\frac{C(M)}{R} - C(\nu_0,\lambda_0,\alpha,P)R^{-\frac{3-\beta}{\beta}} - MR\sup_{y\in\mathbb{R}^3}\int_{y+R\mathbb{B}^3} rho_{[f_j]}(x)\,\mathrm{d}x\;,$$

which on rearranging gives

$$\sup_{y\in\mathbb{R}^3}\int_{y+R\mathbb{B}^3} rho_{[f_j]}(x)\,\mathrm{d}x > \frac{1}{MR}\left[\frac{|h_M|}{2} - \frac{C(M)}{R} - C(\nu_0,\lambda_0,\alpha,P)R^{-\frac{3-\beta}{\beta}}\right]\;.$$

Taking $R = R_0 > 1$ sufficiently large, so that the right-hand side is strictly positive, proves the assertion. □

One might hope that using Proposition 3.26, a minimising sequence could, after translation, be made to satisfy the hypothesis of the compactness result, Proposition 3.25. This is, in fact, the strategy employed in [20], pages 70–72.

Unfortunately, this turns out not to work here: for our minimisation problem, a minimising sequence would not remain minimising after translation, so one cannot employ the direct method to construct minimisers. It might be possible to salvage the situation by showing that the sequence $\{z_j\}_{j\in\mathbb{N}}$ in Proposition 3.26 can be chosen to be bounded, but the author has not been successful in doing so.

3.5 More Recent Stability Results

The proofs of the stability results presented earlier in this section relied on the fact that the Casimir functional is conserved along flows of the Vlasov-Poisson system (this means that $\mathcal{C}(f(t)) = \mathcal{C}(f(0))$ for a time-dependent solution f to the Vlasov-Poisson system) in each of the situations considered. This holds true when considering radial perturbations about anisotropic, spherically symmetric steady states (Subsection 3.2) or arbitrary perturbations about isotropic steady states (Subsection 3.3).

Toward the end of this mini-project, the author was introduced to a series of papers by Lemou, Méhats and Raphaël proving stability results by instead exploiting monotonicity of the Hamiltonian $\mathcal{H}$ under generalised notions of symmetric rearrangement. Their methods apply to large classes of steady states, including those not yet treated by the energy-Casimir methods; one such model is the so-called King model, with

$$f(x,v) = \left(\exp\left(E_0 - E_{[f]}(x,v)\right) - 1\right)_+\;.$$

One of their results involves the stability of anisotropic, spherically symmetric models under radial perturbations, as stated in the following result. The full proof can be found in [9], while a summary of the method of proof can be found in [8].

Theorem 3.27. *Let $f_0(x,v)$ be a continuous, non-negative, compactly supported steady solution to the Vlasov-Poisson system, satisfying the functional form*

$$f_0(x,v) = F\left(E_{[f_0]}(x,v), |x\wedge v|^2\right)\;.$$

Assume that there exists $E_0 < 0$ such that $\{F > 0\} \subseteq (-\infty, E_0)\times[0,\infty)$,0 F is C^1 on $\{F > 0\}$, and

$$\frac{\partial F}{\partial E} < 0 \quad \text{on } \{F > 0\}\;.$$

Then f_0 is stable under radial pertubations in the following sense: For all $L > 0, \varepsilon > 0$, there exists $\delta > 0$ such that whenever $f(0) \in C^1_c(\mathbb{R}^6)$ is spherically symmetric with

$$\|f(0) - f_0\|_{L^1(\mathbb{R}^6)} \leq \delta \ , \quad \|f(0)\|_{L^\infty(\mathbb{R}^6)} \leq \|f_0\|_{L^\infty(\mathbb{R}^6)} + L \quad \text{ and } |\mathcal{H}(f(0)) - \mathcal{H}(f_0)| \leq \delta \ ,$$

the corresponding global solution f satisfies

$$\left\| \left(1 + |v|^2\right) (f(t) - f_0) \right\|_{L^1(\mathbb{R}^6)} \leq \varepsilon \ . \qquad \square$$

The strategy of proof was later adapted to obtain the following stability result for isotropic steady states under arbitrary perturbations. Its full proof can be found in [10].

Theorem 3.28. *Let $f_0(x, v)$ be a continuous, non-negative, compactly supported steady solution to the Vlasov-Poisson system, satisfying the functional form*

$$f_0(x, v) = F\left(E_{[f_0]}(x, v)\right) \ .$$

Assume that there exists $E_0 < 0$ such that $\{F > 0\} \subseteq (-\infty, E_0)$, F is C^1 on $\{F > 0\}$, and

$$F' < 0 \quad \text{on } \{F > 0\} \ .$$

Then f_0 is stable in the following sense: For all $L > 0, \varepsilon > 0$, there exists $\delta > 0$ such that whenever $0 \leq f(0) \in L^1(\mathbb{R}^6) \cap L^\infty(\mathbb{R}^6)$ with $|v|^2 f(0) \in L^1(\mathbb{R}^6)$ is such that

$$\|f(0) - f_0\|_{L^1(\mathbb{R}^6)} \leq \delta \ , \quad \|f(0)\|_{L^\infty(\mathbb{R}^6)} \leq \|f_0\|_{L^\infty(\mathbb{R}^6)} + L \quad \text{ and } \mathcal{H}(f(0)) \leq \mathcal{H}(f_0) + \delta \ ,$$

there is a translation shift $z = z(t)$ such that the corresponding global solution f satisfies

$$\left\| \left(1 + |v|^2\right) (f(t, x, v) - f_0(x - z(t), v)) \right\|_{L^1(\mathbb{R}^6)} \leq \varepsilon \ . \qquad \square$$

A Rearrangements of Functions

The preceding portions of this essay have made heavy use of rearrangements of functions. In this short appendix, we remind the reader of the basic notions and results. Our treatment is based on [11], Chapter 3.

Throughout this appendix, $\mathcal{L}^d$ denotes the Lebesgue measure on $\mathbb{R}^d$.

Definition A.1. If $A \subset \mathbb{R}^d$ is a Borel set with $\mathcal{L}^d(A) < \infty$, its *symmetric rearrangement* is

$$A^\circ := \left(\frac{\mathcal{L}^d(A)}{\mathcal{L}^d(\mathbb{B}^d)} \right)^{\frac{1}{d}} \mathbb{B}^d \ ,$$

i.e. the open ball centred at the origin with the same Lebesgue measure as A. $\square$

Definition A.2. We say a Borel measurable function $f : \mathbb{R}^d \to \mathbb{C}$ *vanishes at infinity* if

$$\mathcal{L}^d\{|f| > \lambda\} < \infty \quad \text{ for every } \lambda > 0 \ .$$

For a function $f : \mathbb{R}^d \to \mathbb{C}$ which vanishes at infinity, we define its *symmetric decreasing rearrangement* to be the function $f^\circ : \mathbb{R}^d \to [0, \infty)$ given by

$$f^\circ(x) := \int_0^\infty \mathbb{1}_{(|x|^d, \infty)} \left(\frac{\mathcal{L}^d\{|f| > \lambda\}}{\mathcal{L}^d(\mathbb{B}^d)} \right) \mathrm{d}\lambda \ . \qquad \square$$

Remark A.3. Observe that

$$x \in \{|f| > \lambda\}^\circ \iff |x|^d < \frac{\mathcal{L}^d\{|f| > \lambda\}}{\mathcal{L}^d(\mathbb{B}^d)} \ .$$

Therefore the symmetric decreasing rearrangement of f satisfies

$$f^\circ(x) = \int_0^\infty \mathbb{1}_{\{|f| > \lambda\}^\circ}(x) \, \mathrm{d}\lambda \ .$$

This is, in fact, the definition provided in [11], Chapter 3. The equivalent Definition A.2 is, however, easier to work with. $\square$

The symmetric decreasing rearrangement enjoys the following nice properties, whose proofs, when not given, are straightforward from Definition A.2.

(i) $f^\circ(x) < \infty$ for every $x \in \mathbb{R}^d \setminus \{0\}$ (since f vanishes at infinity).
Moreover, $f^\circ(0) < \infty$ if and only if $f \in L^\infty(\mathbb{R}^d)$.

(ii) f° is radially symmetric and non-increasing.

(iii) f° is a lower-semicontinuous function.

Proof. If $\{x_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^d$ converges to $x \in \mathbb{R}^d$, then

$$\mathbb{1}_{(|x|^d,\infty)}\left(\frac{\mathcal{L}^d\{|f|>\lambda\}}{\mathcal{L}^d(\mathbb{B}^d)}\right) \le \liminf_{k\to\infty} \mathbb{1}_{(|x_k|^d,\infty)}\left(\frac{\mathcal{L}^d\{|f|>\lambda\}}{\mathcal{L}^d(\mathbb{B}^d)}\right) \quad \text{for every } \lambda > 0 .$$

Therefore, Fatou's lemma gives

$$\int_0^\infty \mathbb{1}_{(|x|^d,\infty)}\left(\frac{\mathcal{L}^d\{|f|>\lambda\}}{\mathcal{L}^d(\mathbb{B}^d)}\right) \mathrm{d}\lambda \le \liminf_{k\to\infty} \int_0^\infty \mathbb{1}_{(|x_k|^d,\infty)}\left(\frac{\mathcal{L}^d\{|f|>\lambda\}}{\mathcal{L}^d(\mathbb{B}^d)}\right) \mathrm{d}\lambda$$

which proves the claim. □

(iv) $\{f^\circ > \lambda\} = \{|f| > \lambda\}^\circ$.

Proof. We have

$$\begin{aligned} f^\circ(x) > t &\iff \exists k \in \mathbb{N} \,:\, \mathbb{1}_{(|x|^d,\infty)}\left(\frac{\mathcal{L}^d\left\{|f| > t + \frac{1}{k}\right\}}{\mathcal{L}^d(\mathbb{B}^d)}\right) = 1 \\ &\iff x \in \bigcup_{k\in\mathbb{N}} \left[\left(\frac{\mathcal{L}^d\left\{|f| > t + \frac{1}{k}\right\}}{\mathcal{L}^d(\mathbb{B}^d)}\right)^{\frac{1}{d}} \mathbb{B}^d\right] \end{aligned}$$

and observe this last set is

$$\begin{aligned} \bigcup_{k\in\mathbb{N}} \left[\left(\frac{\mathcal{L}^d\left\{|f| > t + \frac{1}{k}\right\}}{\mathcal{L}^d(\mathbb{B}^d)}\right)^{\frac{1}{d}} \mathbb{B}^d\right] &= \left[\lim_{k\to\infty}\left(\frac{\mathcal{L}^d\left\{|f| > t + \frac{1}{k}\right\}}{\mathcal{L}^d(\mathbb{B}^d)}\right)^{\frac{1}{d}}\right] \mathbb{B}^d \\ &= \left(\frac{\mathcal{L}^d\left\{|f| > t\right\}}{\mathcal{L}^d(\mathbb{B}^d)}\right)^{\frac{1}{d}} \mathbb{B}^d \\ &= \{|f| > t\}^\circ \ . \end{aligned}$$

This completes the proof. □

Theorem A.4. *Let $f, g : \mathbb{R}^d \to [0,\infty)$ be non-negative functions vanishing at infinity, and let f°, g° be their respective symmetric decreasing rearrangements. Then*

$$\int_{\mathbb{R}^d} f(x) g(x)\, \mathrm{d}x \le \int_{\mathbb{R}^d} f^\circ(x) g^\circ(x)\, \mathrm{d}x \ .$$

Proof. Firstly, observe that if $A, B \subset \mathbb{R}^d$ are Borel subsets of finite Lebesgue measure, then

$$\mathcal{L}^d(A \cap B) \le \mathcal{L}^d(A^\circ \cap B^\circ) \ .$$

Indeed, without loss of generality we may assume $\mathcal{L}^d(A) \le \mathcal{L}^d(B)$; then $A^\circ \subseteq B^\circ$, and hence we have $\mathcal{L}^d(A \cap B) \le \mathcal{L}^d(A) = \mathcal{L}^d(A^\circ) = \mathcal{L}^d(A^\circ \cap B^\circ)$.

In particular, for any $s, t > 0$, we have

$$\mathcal{L}^d\left(\{f > s\} \cap \{g > t\}\right) \le \mathcal{L}^d\left(\{f > s\}^\circ \cap \{g > t\}^\circ\right) = \mathcal{L}^d\left(\{f^\circ > s\} \cap \{g^\circ > t\}\right) \ ,$$

that is

$$\int_{\mathbb{R}^d} \mathbb{1}_{\{f>s\}}(x)\mathbb{1}_{\{g>t\}}(x)\, \mathrm{d}x \le \int_{\mathbb{R}^d} \mathbb{1}_{\{f^\circ>s\}}(x)\mathbb{1}_{\{g^\circ>t\}}(x)\, \mathrm{d}x\ .$$

Therefore, using the layer-cake representation and Tonelli's theorem, we find that

$$\begin{aligned}\int_{\mathbb{R}^d} f(x)g(x)\, \mathrm{d}x &= \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \mathbb{1}_{\{f>s\}}(x)\mathbb{1}_{\{g>t\}}(x)\, \mathrm{d}x\, \mathrm{d}t\, \mathrm{d}s \\ &\le \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \mathbb{1}_{\{f^\circ>s\}}(x)\mathbb{1}_{\{g^\circ>t\}}(x)\, \mathrm{d}x\, \mathrm{d}t\, \mathrm{d}s = \int_{\mathbb{R}^d} f^\circ(x)g^\circ(x)\, \mathrm{d}x\ .\end{aligned}$$

This completes the proof. □

Corollary A.5. *Let $\varphi : \mathbb{R}^d \to [0,\infty)$ be a radially symmetric non-decreasing function. Let $0 \le f \in L^1(\mathbb{R}^d)$, and let f° be the symmetric decreasing rearrangement of f. Then*

$$\int_{\mathbb{R}^d} \varphi(x)f(x)\, \mathrm{d}x \ge \int_{\mathbb{R}^d} \varphi(x)f^\circ(x)\, \mathrm{d}x\ .$$

Proof. Using the monotone convergence theorem, we may assume without loss of generality that φ is bounded. The claim then follows from applying Theorem A.4 to the functions $\|\varphi\|_{L^\infty} - \varphi$ and f, which are both vanishing at infinity. □

Symmetric decreasing rearrangements also satisfy Riesz's rearrangement inequality, which we have used repeatedly in the essay. Its proof, which we omit, is very long and technical, and can be found in [11], Theorem 3.7.

Theorem A.6 (Riesz's Rearrangement Inequality)**.** *Let $f, g, h : \mathbb{R}^d \to [0,\infty)$ be non-negative functions vanishing at infinity, and let $f^\circ, g^\circ, h^\circ$ be the symmetric decreasing rearrangements of f, g, h respectively. Then*

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)g(x-y)h(y)\, \mathrm{d}x\, \mathrm{d}y \le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f^\circ(x)g^\circ(x-y)h^\circ(y)\, \mathrm{d}x\, \mathrm{d}y\ , \qquad \square$$

References

[1] J. Batt, W. Faltenbacher, E. Horst: *Stationary Spherically Symmetric Models in Stellar Dynamics.* Arch. Rational Mech. Anal. **93** (1986), 159–183.

[2] J. Binney, S. Tremaine: *Galactic Dynamics* (2nd edition). Princeton University Press, Princeton, NJ (2008).

[3] J. M. Borwein, A. S. Lewis: *Convex Analysis and Nonlinear Optimization – Theory and Examples* (2nd edition). Springer Science+Business Media, New York, NY (2006).

[4] G. L. Camm: *Self-Gravitating Star Systems. II.* Mon. Not. R. Astr. Soc. **112** (1952), 155–176.

[5] Y. Guo, G. Rein: *Existence and stability of Camm type steady states in galactic dynamics.* Indiana Univ. Math. J. **48** (1999), 1237–1255.

[6] D. D. Holm, J. E. Marsden, T. Ratiu, A. Weinstein: *Nonlinear Stability of Fluid and Plasma Equilibria.* Phys. Reports **123** (1985), 1–116.

[7] J. H. Jeans: *On the Theory of Star-Streaming and the Structure of the Universe.* Mon. Not. R. Astr. Soc. **76** (1915), 70–84.

[8] M. Lemou, F. Méhats, P. Raphaël: *A new variational approach to the stability of gravitational systems.* C. R. Acad. Sci. Paris, Ser. I **347** (2009), 979–984.

[9] M. Lemou, F. Méhats, P. Raphaël: *A New Variational Approach to the Stability of Gravitational Systems*. Comm. Math. Phys. **302** (2011), 161–224.

[10] M. Lemou, F. Méhats, P. Raphaël: *Orbital stability of spherical galactic models*. Invent. Math. **187** (2012), 145–194.

[11] E. H. Lieb, M. Loss: *Analysis* (2nd edition). American Mathematical Society, Providence, RI (2001).

[12] P.-L. Lions, B. Perthame: *Propagation of moments and regularity for the 3-dimensional Vlasov-Poisson system*. Invent. Math. **105** (1991), 415–430.

[13] G. Loeper: *Uniqueness of the solution to the Vlasov-Poisson system with bounded density*. J. Math. Pures. Appl. **86** (2006), 68–79.

[14] K. Pfaffelmoser: *Global classical solutions of the Vlasov-Poisson system in three dimensions for general initial data*. J. Diff. Eqns **95** (1992), 281–303.

[15] G. Rein: *Non-Linear Stability for the Vlasov-Poisson System – The Energy-Casimir Method*. Math. Meth. Appl. Sci. **17** (1994), 1129–1140.

[16] G. Rein: *Static shells for the Vlasov-Poisson and Vlasov-Einstein systems*. Indiana Univ. Math. J. **48** (1999), 335–346.

[17] G. Rein: *Stationary and static stellar dynamic models with axial symmetry*. Nonlinear Analysis **41** (2000), 313–344.

[18] G. Rein: *Stability of Spherically Symmetric Steady States in Galactic Dynamics against General Perturbations*. Arch. Rational Mech. Anal. **161** (2002), 27–42.

[19] G. Rein: *Reduction and a Concentration-Compactness Principle for Energy-Casimir Functionals*. SIAM J. Math. Anal. **33** (2002), 896–912.

[20] G. Rein: *Collisionless Kinetic Equations from Astrophysics – The Vlasov-Poisson System*, in *Handbook of Differential Equations, Evolutionary Equations. Vol. 3.* (C. Dafermos, E. Feireisl eds.) Elsevier / North-Holland, Amsterdam (2007), 383–476.

[21] G. Rein: *Galactic dynamics in MOND – Existence of quilibria with finite mass and compact support*. arXiv:1312.3765 (2013)

[22] R. T. Rockafellar: *Convex Analysis*. Princeton University Press, Princeton, NJ (1970).

[23] J. Schaeffer: *Global existence of smooth solutions to the Vlasov-Poisson system in three dimensions*. Comm. Partial Diff. Eqns. **16** (1991), 1313–1335.

[24] J. Schaeffer: *A Class of Counterexamples to Jeans' Theorem for the Vlasov-Einstein System*. Comm. Math. Phys. **204** (1999), 313–327.

[25] A. Schulze: *Existence of Axially Symmetric Solutions to the Vlasov-Poisson System Depending on Jacobi's Integral*. Comm. Math. Sci. **6** (1999), 711–727.