Energy-Conserving Discontinuous Galerkin Methods for Vlasov-Type Systems

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Outline

The one-species Vlasov-Maxwell (VM) system

The single-species VM system under the scaling of the characteristic time by the inverse of the plasma frequency ω_p^{-1} and length scaled by the Debye length λ_D, and characteristic electric and magnetic field as $\bar{E} = \bar{B} = -mc\omega_p/e$ is

$$\begin{aligned} &\partial_t f + \mathbf{v} \cdot \nabla_\mathbf{x} f + (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_\mathbf{v} f = 0 \ , \\ &\frac{\partial \mathbf{E}}{\partial t} = \nabla_\mathbf{x} \times \mathbf{B} - \mathbf{J}, \qquad \frac{\partial \mathbf{B}}{\partial t} = -\nabla_\mathbf{x} \times \mathbf{E} \ , \\ &\nabla_\mathbf{x} \cdot \mathbf{E} = \rho - \rho_i, \qquad \nabla_\mathbf{x} \cdot \mathbf{B} = 0 \ , \end{aligned}$$

where the density and current density are defined as

$$\rho(\mathbf{x}, t) = \int_{\mathbb{R}^n} f(\mathbf{x}, \mathbf{v}, t) d\mathbf{v}, \qquad \mathbf{J}(\mathbf{x}, t) = \int_{\mathbb{R}^n} f(\mathbf{x}, \mathbf{v}, t) \mathbf{v} d\mathbf{v}.$$

and ρ_i is the ion density.

The Vlasov-Ampère (VA) and Vlasov-Poisson (VP) system

In the zero-magnetic limit, the VM system becomes

$$\begin{aligned} \partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \mathbf{E} \cdot \nabla_{\mathbf{v}} f = 0 \ , \\ \frac{\partial \mathbf{E}}{\partial t} = -\mathbf{J}, \qquad \nabla_{\mathbf{x}} \cdot \mathbf{E} = \rho - \rho_i, \end{aligned} \tag{1}$$

This leads to either the Vlasov-Ampère (VA) system

$$\begin{aligned} \partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \mathbf{E} \cdot \nabla_{\mathbf{v}} f = 0 \ , \\ \frac{\partial \mathbf{E}}{\partial t} = -\mathbf{J}, \end{aligned} \tag{2}$$

or the Vlasov-Poisson (VP) system

$$\begin{aligned} \partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \mathbf{E} \cdot \nabla_{\mathbf{v}} f = 0 \ , \\ \nabla_{\mathbf{x}} \cdot \mathbf{E} = \rho - \rho_i, \end{aligned} \tag{3}$$

They are equivalent when the charge continuity equation

$$\rho_t + \nabla_{\mathbf{x}} \cdot \mathbf{J} = 0$$

is satisfied. (No external field)

The multi-species system

Multi-species VA system is formulated as

$$\partial_t f_\alpha + \mathbf{v} \cdot \nabla_\mathbf{x} f_\alpha + \frac{q_\alpha}{m_\alpha} \mathbf{E} \cdot \nabla_\mathbf{v} f_\alpha = 0 \ , \tag{4a}$$

$$\partial_t \mathbf{E} = -\frac{1}{\epsilon_0} \mathbf{J}, \tag{4b}$$

with

$$\mathbf{J} = \sum_\alpha q_\alpha \int_{\mathbb{R}^n} f_\alpha(\mathbf{x}, \mathbf{v}, t) \mathbf{v} d\mathbf{v}.$$

Two-species (electron & ion) system with electron scaling becomes

$$\partial_t f_\alpha + \mathbf{v} \cdot \nabla_\mathbf{x} f_\alpha + \mu_\alpha \mathbf{E} \cdot \nabla_\mathbf{v} f_\alpha = 0 \ , \tag{5a}$$

$$\partial_t \mathbf{E} = -\mathbf{J}, \tag{5b}$$

where $\mu_\alpha = \frac{q_\alpha m_e}{e m_\alpha}$, $\mathbf{J} = \mathbf{J}_i - \mathbf{J}_e$, with $\mathbf{J}_\alpha = \int_{\mathbb{R}^n} f_\alpha(\mathbf{x}, \mathbf{v}, t) \mathbf{v} d\mathbf{v}$.

Vlaosv solvers

- **Particle-In-Cell (PIC) methods** (Birdsall, Langdon; Hockney, Eastwood, et al.)
  Particles are solved in a Lagrangian framework, and the field equations are solved on a mesh.
- **Semi-Lagrangian methods** (Cheng and Knorr, 1976 for VP).
  Many subsequent work, Sonnendrücker, Filbet, Qiu, Christlieb, Shu, Rossmanith, Seal, Califano *et al.*
- **Eulerian solvers** Spectral methods, WENO FD, FVM, FEM, DGM have been proposed to solve VP system.
  The main computational challenges includes: high-dimensionality, multiple temporal and spatial scales, the conservation of the physical quantities due to the Hamiltonian structure of the systems.

The total energy

The total energy for VA system is defined as

$$TE = \frac{1}{2}\int_{\Omega_x}\int_{\mathbb{R}^n} f|\mathbf{v}|^2 d\mathbf{v}d\mathbf{x} + \frac{1}{2}\int_{\Omega_x} |\mathbf{E}|^2 d\mathbf{x},$$

The total energy for VM system is defined as

$$TE = \frac{1}{2}\int_{\Omega_x}\int_{\mathbb{R}^n} f|\mathbf{v}|^2 d\mathbf{v}d\mathbf{x} + \frac{1}{2}\int_{\Omega_x} |\mathbf{E}|^2 + |\mathbf{B}|^2 d\mathbf{x},$$

The total energy for two-species VA system is defined as

$$TE = \frac{1}{2}\int_{\Omega_x}\int_{\mathbb{R}^n} f_e|\mathbf{v}|^2 d\mathbf{v}d\mathbf{x} + \frac{1}{2\mu_i}\int_{\Omega_x}\int_{\mathbb{R}^n} f_i|\mathbf{v}|^2 d\mathbf{v}d\mathbf{x} + \frac{1}{2}\int_{\Omega_x} |\mathbf{E}|^2 d\mathbf{x}.$$

- Non-conservative schemes cause *plasma self heating or cooling*.
- especially for under-resolved mesh or large time steps.

Energy-conserving methods for Vlasov equations

- Energy-conserving PIC for VA equations (Chen, Chacón, Barnes, 2011). *fully implicit and charge conserving.*
- Energy-conserving PIC for VM equations (Markidis, Lapenta, 2011). *based on Yee's lattice and implicit midpoint method.*
- Charge, energy-conserving FVM for VA (Taitano, Chacón, 2015)
- Energy-conserving finite difference for VP systems (Filbet, Sonnendrücker, 2003) *based on Arakawa's method.*
- Energy-conserving DG for VP systems (Ayuso *et al.* 2011, Ayuso, Hajian, 2012, Madaule *et al.*, 2014) *semi-discrete conservation.*
- Energy-conserving DG for VM systems (Cheng, Gamba, Li, Morrison, 2013) *semi-discrete conservation.*
- Also in the literature, the symplectic methods are widely used to compute the Hamiltonian systems to achieve near conservation of the total energy.

Outline

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Second order fully explicit method $S_1(\triangle t)$

An explicit second order scheme can be designed as follows

$$\frac{f^{n+1/2}-f^n}{\triangle t/2}+\mathbf{v}\cdot\nabla_{\mathbf{x}}f^n+\mathbf{E}^n\cdot\nabla_{\mathbf{v}}f^n=0\ , \tag{6a}$$

$$\frac{\mathbf{E}^{n+1}-\mathbf{E}^n}{\triangle t}=-\mathbf{J}^{n+1/2}, \quad \text{where } \mathbf{J}^{n+1/2}=\int_{\Omega_v} f^{n+1/2}\mathbf{v}d\mathbf{v} \tag{6b}$$

$$\frac{f^{n+1}-f^n}{\triangle t}+\mathbf{v}\cdot\nabla_{\mathbf{x}}f^{n+1/2}+\frac{1}{2}(\mathbf{E}^n+\mathbf{E}^{n+1})\cdot\nabla_{\mathbf{v}}f^{n+1/2}=0\ . \tag{6c}$$

Second order fully implicit method $S_2(\triangle t)$

Based on implicit midpoint method

$$\frac{f^{n+1}-f^{n}}{\triangle t}+\mathbf{v}\cdot\nabla_{\mathbf{x}}\frac{f^{n}+f^{n+1}}{2}+\frac{1}{2}(\mathbf{E}^{n}+\mathbf{E}^{n+1})\cdot\nabla_{\mathbf{v}}\frac{f^{n}+f^{n+1}}{2}=0\,, \tag{7a}$$

$$\frac{\mathbf{E}^{n+1}-\mathbf{E}^{n}}{\triangle t}=-\frac{1}{2}(\mathbf{J}^{n}+\mathbf{J}^{n+1}). \tag{7b}$$

The resulting system is nonlinearly coupled.

Second order implicit method $S_3(\triangle t)$

By Strang splitting

$$\frac{\mathbf{E}^{n+1/2}-\mathbf{E}^{n}}{\triangle t/2} = -\mathbf{J}^{n}, \tag{8a}$$

$$\frac{f^{n+1}-f^{n}}{\triangle t} + \mathbf{v}\cdot\nabla_{\mathbf{x}}\frac{f^{n}+f^{n+1}}{2} + \mathbf{E}^{n+1/2}\cdot\nabla_{\mathbf{v}}\frac{f^{n}+f^{n+1}}{2} = 0\ , \tag{8b}$$

$$\frac{\mathbf{E}^{n+1}-\mathbf{E}^{n+1/2}}{\triangle t/2} = -\mathbf{J}^{n+1}. \tag{8c}$$

The resulting system is linear.

Total energy conservation

Theorem (Total energy conservation)

With appropriate boundary conditions, the schemes above preserve the discrete total energy $TE_n = TE_{n+1}$, where

$$2\,(TE_n) = \int_{\Omega_x}\int_{\Omega_v} f^n|\mathbf{v}|^2 d\mathbf{v}d\mathbf{x} + \int_{\Omega_x} |\mathbf{E}^n|^2 d\mathbf{x}$$

in schemes $S_1(\triangle t)$, $S_2(\triangle t)$, and

$$\begin{aligned} 2\,(TE_n) &= \int_{\Omega_x}\int_{\Omega_v} f^n|\mathbf{v}|^2 d\mathbf{v}d\mathbf{x} + \int_{\Omega_x} \mathbf{E}^{n+1/2}\mathbf{E}^{n-1/2} d\mathbf{x} \\ &= \int_{\Omega_x}\int_{\Omega_v} f^n|\mathbf{v}|^2 d\mathbf{v}d\mathbf{x} + \int_{\Omega_x} |\mathbf{E}^n|^2 d\mathbf{x} - \frac{\triangle t^2}{4}\int_{\Omega_x} |\mathbf{J}^n|^2 d\mathbf{x} \end{aligned}$$

in scheme $S_3(\triangle t)$.

Energy-conserving operator splitting of VA system

To save computational cost for implicit high-dimensional applications, we can perform the operator splitting as follows:

$$\text{(a)}\begin{cases} \partial_t f + \mathbf{v}\cdot\nabla_{\mathbf{x}} f = 0 \ , \\ \partial_t \mathbf{E} = 0, \end{cases} \quad \text{(b)}\begin{cases} \partial_t f + \mathbf{E}\cdot\nabla_{\mathbf{v}} f = 0 \ , \\ \partial_t \mathbf{E} = -\mathbf{J}, \end{cases}$$

One of the main feature of this splitting is that each of the two equations is energy-conserving,

$$\frac{d}{dt}\left(\int_{\Omega_x}\int_{\Omega_v} f|\mathbf{v}|^2 d\mathbf{v}d\mathbf{x} + \int_{\Omega_x} |\mathbf{E}|^2 d\mathbf{x}\right) = 0.$$

In particular,

$$\text{(a)}\begin{cases} \frac{d}{dt}\int_{\Omega_x}\int_{\Omega_v} f|\mathbf{v}|^2 d\mathbf{v}d\mathbf{x} = 0 \ , \\ \frac{d}{dt}\int_{\Omega_x} |\mathbf{E}|^2 d\mathbf{x} = 0, \end{cases}$$

$$\text{(b)}\frac{d}{dt}\left(\int_{\Omega_x}\int_{\Omega_v} f|\mathbf{v}|^2 d\mathbf{v}d\mathbf{x} + \int_{\Omega_x} |\mathbf{E}|^2 d\mathbf{x}\right) = 0 \ .$$

Split equation

As for equation (a), we can use any implicit or explicit Runge-Kutta methods to solve it, and they all conserve the kinetic energy. To see this, consider the forward Euler

$$\frac{f^{n+1} - f^n}{\triangle t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f^n = 0,$$

or backward Euler method

$$\frac{f^{n+1} - f^n}{\triangle t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f^{n+1} = 0,$$

A simple check yields $\int_{\Omega_x} \int_{\Omega_v} f^n |\mathbf{v}|^2 d\mathbf{v} d\mathbf{x} = \int_{\Omega_x} \int_{\Omega_v} f^{n+1} |\mathbf{v}|^2 d\mathbf{v} d\mathbf{x}$. We could take the implicit midpoint method to solve this equation to achieve second order.

Split equation

Equation (b) contains the main coupling effect of the Vlasov and Ampère equations, and can be computed by the corresponding schemes from the unsplit equation, e.g.

$$\frac{f^{n+1} - f^n}{\triangle t} + \frac{1}{2}(\mathbf{E}^n + \mathbf{E}^{n+1}) \cdot \nabla_{\mathbf{v}} \frac{f^n + f^{n+1}}{2} = 0 \ , \tag{9a}$$

$$\frac{\mathbf{E}^{n+1} - \mathbf{E}^n}{\triangle t} = -\frac{1}{2}(\mathbf{J}^n + \mathbf{J}^{n+1}) \tag{9b}$$

By Strang splitting, let $S_4(\triangle t) = S^{(a)}(\triangle t/2)S^{(b)}(\triangle t)S^{(a)}(\triangle t/2)$, the method is second order for the original VA system.

Split methods

Theorem (Total energy conservation for the split methods)

With appropriate boundary conditions, the scheme $S_4(\triangle t) = S^{(a)}(\triangle t/2)S^{(b)}(\triangle t)S^{(a)}(\triangle t/2)$ *preserves the discrete total energy* $TE_n = TE_{n+1}$, *where*

$$2\,(TE_n) = \int_{\Omega_x}\int_{\Omega_v} f^n|\mathbf{v}|^2 d\mathbf{v}d\mathbf{x} + \int_{\Omega_x} |\mathbf{E}^n|^2 d\mathbf{x}.$$

Generalizations to higher order

Firstly, we directly generalize the second scheme using the method of Yoshida, 1990, Forest Ruth, 1990, De Frutos, Sanz-Serna, 1992. Let $\beta_1, \beta_2, \beta_3$ satisfy

$$\beta_1 + \beta_2 + \beta_3 = 1, \qquad \beta_1^3 + \beta_2^3 + \beta_3^3 = 1, \qquad \beta_1 = \beta_3$$

We get $\beta_1 = \beta_3 = (2 + 2^{1/3} + 2^{-1/3})/3 \approx 1.3512$, $\beta_2 = 1 - 2\beta_1 \approx -1.7024$.

$S_i^{\{4\}}(\triangle t) = S_i(\beta_1 \triangle t) S_i(\beta_2 \triangle t) S_i(\beta_3 \triangle t)$, for $i = 2, 3, 4$. We can verify that the method will be fourth order, if $S_i(\triangle t)$ is symmetric in time. For the negative time steps, some implementation details have to be considered (e.g. the flux needs to be reversed).

Generalizations to higher order

Theorem (Total energy conservation for the fourth order methods)

With appropriate boundary conditions, the schemes $S_2^{\{4\}}(\triangle t)$ *and* $S_4^{\{4\}}(\triangle t)$ *preserves the discrete total energy* $TE_n = TE_{n+1}$, *where*

$$2\,(TE_n) = \int_{\Omega_x}\int_{\Omega_v} f^n |\mathbf{v}|^2 d\mathbf{v}d\mathbf{x} + \int_{\Omega_x} |\mathbf{E}^n|^2 d\mathbf{x}.$$

Outline

Second order explicit method

We could generalize the idea to the VM system, e.g.

$$\frac{f^{n+1/2}-f^n}{\triangle t/2}+\mathbf{v}\cdot\nabla_\mathbf{x} f^n+(\mathbf{E}^n+\mathbf{v}\times\mathbf{B}^n)\cdot\nabla_\mathbf{v} f^n=0\ ,$$

$$\frac{\mathbf{B}^{n+1/2}-\mathbf{B}^n}{\triangle t/2}=-\nabla_\mathbf{x}\times\mathbf{E}^n,$$

$$\frac{\mathbf{E}^{n+1}-\mathbf{E}^n}{\triangle t}=\nabla_\mathbf{x}\times\mathbf{B}^{n+1/2}-\mathbf{J}^{n+1/2},\quad \text{where } \mathbf{J}^{n+1/2}=-\int f^{n+1/2}\mathbf{v}d\mathbf{v}$$

$$\frac{\mathbf{B}^{n+1}-\mathbf{B}^{n+1/2}}{\triangle t/2}=-\nabla_\mathbf{x}\times\mathbf{E}^{n+1},$$

$$\frac{f^{n+1}-f^n}{\triangle t}+\mathbf{v}\cdot\nabla_\mathbf{x} f^{n+1/2}+(\frac{1}{2}(\mathbf{E}^n+\mathbf{E}^{n+1})+\mathbf{v}\times\mathbf{B}^{n+1/2})\cdot\nabla_\mathbf{v} f^{n+1/2}=0\ ,$$

Second order, implicit in Maxwell, explicit in Vlasov

$$\frac{f^{n+1/2} - f^n}{\triangle t/2} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f^n + (\mathbf{E}^n + \mathbf{v} \times \mathbf{B}^n) \cdot \nabla_{\mathbf{v}} f^n = 0 \ ,$$

$$\frac{\mathbf{B}^{n+1} - \mathbf{B}^n}{\triangle t} = -\nabla_{\mathbf{x}} \times \left(\frac{\mathbf{E}^n + \mathbf{E}^{n+1}}{2} \right),$$

$$\frac{\mathbf{E}^{n+1} - \mathbf{E}^n}{\triangle t} = \nabla_{\mathbf{x}} \times \left(\frac{\mathbf{B}^n + \mathbf{B}^{n+1}}{2} \right) - \mathbf{J}^{n+1/2}, \quad \text{where } \mathbf{J}^{n+1/2} = -\int f^{n+1/2} \mathbf{v} d\mathbf{v}$$

$$\frac{f^{n+1} - f^n}{\triangle t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f^{n+1/2} + (\frac{1}{2}(\mathbf{E}^n + \mathbf{E}^{n+1}) + \frac{1}{2}\mathbf{v} \times (\mathbf{B}^n + \mathbf{B}^{n+1})) \cdot \nabla_{\mathbf{v}} f^{n+1/2} = 0 \ ,$$

Second order, explicit in Maxwell, implicit in Vlasov

$$\frac{\mathbf{E}^{n+1/2}-\mathbf{E}^n}{\triangle t/2} = \nabla_\mathbf{x} \times \mathbf{B}^n - \mathbf{J}^n, \quad \text{where } \mathbf{J}^n = -\int f^n \mathbf{v} d\mathbf{v}$$

$$\frac{\mathbf{B}^{n+1}-\mathbf{B}^n}{\triangle t} = -\nabla_\mathbf{x} \times \mathbf{E}^{n+1/2},$$

$$\frac{f^{n+1}-f^n}{\triangle t} + \mathbf{v}\cdot\nabla_\mathbf{x}\frac{f^n+f^{n+1}}{2} + (\mathbf{E}^{n+1/2} + \mathbf{v} \times \frac{\mathbf{B}^n+\mathbf{B}^{n+1}}{2})\cdot\nabla_\mathbf{v}\frac{f^n+f^{n+1}}{2} = 0\ ,$$

$$\frac{\mathbf{E}^{n+1}-\mathbf{E}^{n+1/2}}{\triangle t/2} = \nabla_\mathbf{x} \times \mathbf{B}^{n+1} - \mathbf{J}^{n+1}, \quad \text{where } \mathbf{J}^{n+1} = -\int f^{n+1} \mathbf{v} d\mathbf{v}$$

Second order fully implicit

$$
\frac{\mathbf{B}^{n+1} - \mathbf{B}^{n}}{\triangle t} = -\nabla_{\mathbf{x}} \times \left(\frac{\mathbf{E}^{n} + \mathbf{E}^{n+1}}{2} \right),
$$

$$
\frac{\mathbf{E}^{n+1} - \mathbf{E}^{n}}{\triangle t} = \nabla_{\mathbf{x}} \times \left(\frac{\mathbf{B}^{n} + \mathbf{B}^{n+1}}{2} \right) - \frac{\mathbf{J}^{n} + \mathbf{J}^{n+1}}{2},
$$

$$
\frac{f^{n+1} - f^{n}}{\triangle t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} \frac{f^{n} + f^{n+1}}{2}
$$

$$
+ (\frac{1}{2}(\mathbf{E}^{n} + \mathbf{E}^{n+1}) + \frac{1}{2}\mathbf{v} \times (\mathbf{B}^{n} + \mathbf{B}^{n+1})) \cdot \nabla_{\mathbf{v}} \frac{f^{n} + f^{n+1}}{2} = 0 \ ,
$$

Operator Splitting

$$
\text{(a)}\begin{cases}\partial_t f + \mathbf{v}\cdot\nabla_{\mathbf{x}} f = 0\ ,\\ \partial_t \mathbf{E} = 0,\\ \partial_t \mathbf{B} = 0,\end{cases}\qquad \text{(b)}\begin{cases}\partial_t f + \mathbf{E}\cdot\nabla_{\mathbf{v}} f = 0\ ,\\ \partial_t \mathbf{E} = -\mathbf{J},\\ \partial_t \mathbf{B} = 0,\end{cases}
$$

$$
\text{(c)}\begin{cases}\partial_t f + (\mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{v}} f = 0\ ,\\ \partial_t \mathbf{E} = \nabla_{\mathbf{x}}\times\mathbf{B},\\ \partial_t \mathbf{B} = -\nabla_{\mathbf{x}}\times\mathbf{E},\end{cases}
$$

Operator Splitting

$$
\text{(a)} \begin{cases} \frac{d}{dt}\int\int f|\mathbf{v}|^2 d\mathbf{v}d\mathbf{x} = 0 \ , \\ \frac{d}{dt}\int |\mathbf{E}|^2 d\mathbf{x} = 0, \\ \frac{d}{dt}\int |\mathbf{B}|^2 d\mathbf{x} = 0, \end{cases} \quad \text{(b)} \begin{cases} \frac{d}{dt}(\int\int f|\mathbf{v}|^2 d\mathbf{v}d\mathbf{x} + \int |\mathbf{E}|^2 d\mathbf{x}) = 0 \ , \\ \frac{d}{dt}\int |\mathbf{B}|^2 d\mathbf{x} = 0, \end{cases}
$$

$$
\text{(c)} \begin{cases} \frac{d}{dt}\int\int f|\mathbf{v}|^2 d\mathbf{v}d\mathbf{x} = 0 \ , \\ \frac{d}{dt}\int(|\mathbf{E}|^2 + |\mathbf{B}|^2) d\mathbf{x} = 0, \end{cases}
$$

Outline

Second order explicit method

Such methodologies can be easily generalized to multi-species

$$\frac{f_\alpha^{n+1/2} - f_\alpha^n}{\Delta t/2} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_\alpha^n + \mu_\alpha \mathbf{E}^n \cdot \nabla_{\mathbf{v}} f_\alpha^n = 0, \quad \alpha = e, i \tag{14a}$$

$$\frac{\mathbf{E}^{n+1} - \mathbf{E}^n}{\Delta t} = -\mathbf{J}^{n+1/2}, \quad \text{where } \mathbf{J}^{n+1/2} = \int_{\mathbb{R}^n} (f_i^{n+1/2} - f_e^{n+1/2}) \mathbf{v} d\mathbf{v} \tag{14b}$$

$$\frac{f_\alpha^{n+1} - f_\alpha^n}{\Delta t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_\alpha^{n+1/2} + \frac{1}{2} \mu_\alpha (\mathbf{E}^n + \mathbf{E}^{n+1}) \cdot \nabla_{\mathbf{v}} f_\alpha^{n+1/2} = 0. \tag{14c}$$

Second order fully implicit method

Define

$$\frac{f_\alpha^{n+1} - f_\alpha^n}{\Delta t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} \frac{f_\alpha^n + f_\alpha^{n+1}}{2} = 0, \tag{15a}$$

$$\frac{\mathbf{E}^{n+1} - \mathbf{E}^n}{\Delta t} = 0, \tag{15b}$$

as **Scheme-a**(Δt) , and

$$\frac{f_\alpha^{n+1} - f_\alpha^n}{\Delta t} + \frac{1}{2}\mu_\alpha(\mathbf{E}^n + \mathbf{E}^{n+1}) \cdot \nabla_{\mathbf{v}} \frac{f_\alpha^n + f_\alpha^{n+1}}{2} = 0 \ , \quad \alpha = e, i \tag{16a}$$

$$\frac{\mathbf{E}^{n+1} - \mathbf{E}^n}{\Delta t} = -\frac{1}{2}(\mathbf{J}^n + \mathbf{J}^{n+1}), \tag{16b}$$

as **Scheme-b**(Δt). Then the fully implicit method is given by Strang Splitting

Scheme-a$(\Delta t/2)$**Scheme-b**(Δt)**Scheme-a**$(\Delta t/2)$.

Outline

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DG methods with explicit scheme

The scheme is formulated as follows:

$$\int_K \frac{f_h^{n+1/2} - f_h^n}{\triangle t/2} g d\mathbf{x} d\mathbf{v} - \int_K f_h^n \mathbf{v} \cdot \nabla_{\mathbf{x}} g d\mathbf{x} d\mathbf{v} - \int_K f_h^n \mathbf{E}_h^n \cdot \nabla_{\mathbf{v}} g d\mathbf{x} d\mathbf{v}$$
$$+ \int_{K_v} \int_{\partial K_x} \widehat{f_h^n \mathbf{v} \cdot \mathbf{n}}_x g ds_x d\mathbf{v} + \int_{K_x} \int_{\partial K_v} (\widehat{f_h^n \mathbf{E}_h^n \cdot \mathbf{n}}_{\mathbf{v}}) g ds_v dx = 0 \ , \tag{17a}$$

$$\frac{\mathbf{E}_h^{n+1} - \mathbf{E}_h^n}{\triangle t} = -\mathbf{J}_h^{n+1/2} \quad \text{where } \mathbf{J}_h^{n+1/2} = \int_{\Omega_v} f_h^{n+1/2} \mathbf{v} d\mathbf{v} \ , \tag{17b}$$

$$\int_K \frac{f_h^{n+1} - f_h^n}{\triangle t} g d\mathbf{x} d\mathbf{v} - \int_K f_h^{n+1/2} \mathbf{v} \cdot \nabla_{\mathbf{x}} g d\mathbf{x} d\mathbf{v} - \frac{1}{2} \int_K f_h^{n+1/2} (\mathbf{E}_h^n + \mathbf{E}_h^{n+1}) \cdot \nabla_{\mathbf{v}} g d\mathbf{x} d\mathbf{v}$$
$$+ \int_{K_v} \int_{\partial K_x} \widehat{f_h^{n+1/2} \mathbf{v}} \cdot \mathbf{n}_x g ds_x d\mathbf{v} + \frac{1}{2} \int_{K_x} \int_{\partial K_v} (f_h^{n+1/2} (\widehat{\mathbf{E}_h^n + \mathbf{E}_h^{n+1}}) \cdot \mathbf{n}_{\mathbf{v}}) g ds_v dx = 0 \ , \tag{17c}$$

Schemes with other time discretizations can be formulated similarly.

Properties

Theorem (Total particle number conservation)

The scheme (17) *preserves the total particle number of the system, i.e.*

$$\int_{\Omega_x}\int_{\Omega_v} f_h^{n+1} d\mathbf{v} d\mathbf{x} = \int_{\Omega_x}\int_{\Omega_v} f_h^{n} d\mathbf{v} d\mathbf{x}.$$

This also holds for DG methods with time integrators $S_2(\triangle t)$, $S_3(\triangle t)$, $S_2^{\{4\}}(\triangle t)$, $S_3^{\{4\}}(\triangle t)$.

Theorem (Total energy conservation)

If $k \geq 2$, the scheme (17) *preserves the discrete total energy $TE_n = TE_{n+1}$, where*

$$2\,(TE_n) = \int_{\Omega_x}\int_{\Omega_v} f_h^n |\mathbf{v}|^2 d\mathbf{v} d\mathbf{x} + \int_{\Omega_x} |\mathbf{E}_h^n|^2 d\mathbf{x}.$$

Properties

Theorem (L^2 stability)

The DG methods with fully implicit schemes satisfy

$$\int_{\Omega_x}\int_{\Omega_v} |f_h^{n+1}|^2 d\mathbf{v} d\mathbf{x} = \int_{\Omega_x}\int_{\Omega_v} |f_h^{n}|^2 d\mathbf{v} d\mathbf{x}$$

for central flux, and

$$\int_{\Omega_x}\int_{\Omega_v} |f_h^{n+1}|^2 d\mathbf{v} d\mathbf{x} \le \int_{\Omega_x}\int_{\Omega_v} |f_h^{n}|^2 d\mathbf{v} d\mathbf{x}$$

for upwind flux.

Outline

Split methods

For all $f_h \in \mathcal{S}_h^k$, we can pick a few nodal points to represent the degree of freedom for that element, Let the nodes in K_x and K_v are $\mathbf{x}_{K_x}^{(l)}$, $\mathbf{v}_{K_v}^{(m)}$, $l = 1, \ldots, dof(k1)$, $m = 1, \ldots, dof(k2)$, respectively, then any $g \in \mathcal{S}_h^k$ can be uniquely represented as $g = \sum_{l,m} g(\mathbf{x}_{K_x}^{(l)}, \mathbf{v}_{K_v}^{(m)}) L_x^{(l)}(\mathbf{x}) L_v^{(m)}(\mathbf{v})$ on K, where $L_x^{(l)}(\mathbf{x}), L_v^{(m)}(\mathbf{v})$ denote the l-th and m-th Lagrangian interpolating polynomials in K_x and K_v, resp.

Then, for equation (a), we could fix a nodal point in $\mathbf{v}$, say $\mathbf{v}_{K_v}^{(m)}$, then solve $\partial_t f(\mathbf{v}_{K_v}^{(m)}) + \mathbf{v}_{K_v}^{(m)} \cdot \nabla_{\mathbf{x}} f(\mathbf{v}_{K_v}^{(m)}) = 0$ by a DG methods in the $\mathbf{x}$ direction. For equation (b), we can fix a nodal point in $\mathbf{x}$, say $\mathbf{x}_{K_x}^{(l)}$, then solve

$$\begin{cases} \partial_t f(\mathbf{x}_{K_x}^{(l)}) + \mathbf{E}(\mathbf{x}_{K_x}^{(l)}) \cdot \nabla_{\mathbf{v}} f(\mathbf{x}_{K_x}^{(l)}) = 0 \ , \\ \partial_t \mathbf{E}(\mathbf{x}_{K_x}^{(l)}) = -\mathbf{J}(\mathbf{x}_{K_x}^{(l)}), \end{cases}$$

in $\mathbf{v}$ direction.

Solver for equation (a)

To solve from t^n to t^{n+1}

$$\text{(a)} \begin{cases} \partial_t f + v f_x = 0 \ , \\ \partial_t E = 0, \end{cases}$$

1. For each $j = 1, \dots N_v, m = 1, \dots, k+1$, we seek $g_j^{(m)}(x) \in \mathcal{U}_h^k$, such that

$$\int_{J_i} \frac{g_j^{(m)}(x) - f_h^n(x, v_j^{(m)})}{\triangle t} \varphi_h \, dx - \int_{J_i} v_j^{(m)} \frac{g_j^{(m)}(x) + f_h^n(x, v_j^{(m)})}{2} (\varphi_h)_x \, dx$$
$$+ v_j^{(m)} \frac{\widehat{g_j^{(m)}(x_{i+\frac{1}{2}}) + f_h^n(x_{i+\frac{1}{2}}, v_j^{(m)})}}{2} (\varphi_h)^-_{i+\frac{1}{2}}$$
$$- v_j^{(m)} \frac{\widehat{g_j^{(m)}(x_{i-\frac{1}{2}}) + f_h^n(x_{i-\frac{1}{2}}, v_j^{(m)})}}{2} (\varphi_h)^+_{i-\frac{1}{2}} = 0$$

   holds for any test function $\varphi_h(x,t) \in \mathcal{U}_h^k$.
2. Let f_h^{n+1} be the unique polynomial in $\mathcal{S}_h^k$, such that $f_h^{n+1}(x_i^{(l)}, v_j^{(m)}) = g_j^{(m)}(x_i^{(l)}), \ \forall i, j, l, m.$

Solver for equation (b)

To solve from t^n to t^{n+1}, we use Jacobian-free Newton-Krylov solver (KINSOL) to solve

$$\text{(b)} \begin{cases} \partial_t f + E f_v = 0 , \\ \partial_t E = -J, \end{cases}$$

1. For each $i = 1, \dots N_x, l = 1, \dots, k+1$, we seek $g_i^{(l)}(v) \in \mathcal{Z}_h^k$ and $E_i^{(l)}$, such that

$$\begin{aligned} &\int_{K_j} \frac{g_i^{(l)}(v) - f_h^n(x_i^{(l)}, v)}{\triangle t} \varphi_h \, dv - \int_{K_j} \frac{E_h^n(x_i^{(l)}) + E_i^{(l)}}{2} \frac{g_i^{(l)}(v) + f_h^n(x_i^{(l)}, v)}{2} (\varphi_h)_v \, dv \\ &+ \frac{E_h^n(x_i^{(l)}) + E_i^{(l)}}{2} \frac{\widehat{g_i^{(l)}(v_{j+\frac{1}{2}}) + f_h^n(x_i^{(l)}, v_{j+\frac{1}{2}})}}{2} (\varphi_h)^-_{j+\frac{1}{2}} \\ &- \frac{E_h^n(x_i^{(l)}) + E_i^{(l)}}{2} \frac{\widehat{g_i^{(l)}(v_{j-\frac{1}{2}}) + f_h^n(x_i^{(l)}, v_{j-\frac{1}{2}})}}{2} (\varphi_h)^+_{j-\frac{1}{2}} = 0 \\ &\frac{E_i^{(l)} - E_h^n(x_i^{(l)})}{\triangle t} = -\frac{1}{2}(J_h^n(x_i^{(l)}) + J_i^{(l)}), \end{aligned} \tag{18}$$

   holds for any test function $\varphi_h(x, t) \in \mathcal{Z}_h^k$.
2. Let f_h^{n+1} be the unique polynomial in $\mathcal{S}_h^k$, such that $f_h^{n+1}(x_i^{(l)}, v_j^{(m)}) = g_i^{(l)}(x_j^{(m)})$, $\forall i, j, l, m$. Let E_h^{n+1} be the unique polynomial in $\mathcal{U}_h^k$, such that $E_h^{n+1}(x_i^{(l)}) = E_i^{(l)}$, $\forall i, l$.

Properties

Theorem (Total particle number conservation)

The DG schemes with $S_4(\triangle t)$, $S_4^{\{4\}}(\triangle t)$ preserve the total particle number of the system, i.e.

$$\int_{\Omega_x}\int_{\Omega_v} f_h^{n+1} d\mathbf{v} d\mathbf{x} = \int_{\Omega_x}\int_{\Omega_v} f_h^{n} d\mathbf{v} d\mathbf{x}.$$

Proof. We only need to show that each of the operators $S^{(a)}(\triangle t)$ and $S^{(b)}(\triangle t)$ conserves mass. For example, for $S^{(a)}(\triangle t)$, let $\varphi_h = 1$ in (18), and sum over all element J_i, we get

$$\int_{\Omega_x} g_j^{(m)}(x) d\mathbf{x} = \int_{\Omega_x} f_h^n(x, v_j^{(m)}) d\mathbf{x}.$$

Therefore for any j, m,

$$\int_{\Omega_x} f_h^{n+1}(x, v_j^{(m)}) d\mathbf{x} = \int_{\Omega_x} f_h^n(x, v_j^{(m)}) d\mathbf{x},$$

and because the (k+1) Gauss quadrature formula is exact for polynomial with degree less than $2k+2$,

$$\int_{\Omega_x} \int_{\Omega_v} f_h^{n+1} d\mathbf{v} d\mathbf{x} = \sum_j \sum_m w_m \int_{\Omega_x} f_h^{n+1}(x, v_j^{(m)}) d\mathbf{x} \triangle v_j$$
$$= \sum_j \sum_m w_m \int_{\Omega_x} f_h^n(x, v_j^{(m)}) d\mathbf{x} \triangle v_j = \int_{\Omega_x} \int_{\Omega_v} f_h^n d\mathbf{v} d\mathbf{x},$$

where w_m is the corresponding Gauss quadrature weights. □

Properties

Theorem (Total energy conservation)

If $k \geq 2$, the split scheme $S(\triangle t) = S^{(a)}(\triangle t/2) S^{(b)}(\triangle t) S^{(a)}(\triangle t/2)$ preserves the discrete total energy $TE_n = TE_{n+1}$, where

$$2\,(TE_n) = \int_{\Omega_x} \int_{\Omega_v} f_h^n |\mathbf{v}|^2 d\mathbf{v} d\mathbf{x} + \int_{\Omega_x} |\mathbf{E}_h^n|^2 d\mathbf{x}.$$

Proof. For $S^{(a)}(\triangle t)$, we have that for any j, m,

$$\int_{\Omega_x} f_h^{n+1}(x, v_j^{(m)}) d\mathbf{x} = \int_{\Omega_x} f_h^n(x, v_j^{(m)}) d\mathbf{x},$$

and because the (k+1) Gauss quadrature formula is exact for polynomial with degree less than $2k+2$,

$$\int_{\Omega_x} \int_{\Omega_v} f_h^{n+1} |\mathbf{v}|^2 d\mathbf{v} d\mathbf{x} = \sum_j \sum_m w_m \int_{\Omega_x} f_h^{n+1}(x, v_j^{(m)}) d\mathbf{x}\, |v_j^{(m)}|^2 \triangle v_j$$

$$= \sum_j \sum_m w_m \int_{\Omega_x} f_h^n(x, v_j^{(m)}) d\mathbf{x}\, |v_j^{(m)}|^2 \triangle v_j = \int_{\Omega_x} \int_{\Omega_v} f_h^n |\mathbf{v}|^2 d\mathbf{v} d\mathbf{x}.$$

As for $S^{(b)}(\triangle t)$, $|\mathbf{E}_h|^2$ is a polynomial of degree at most $2k$,

$$
\begin{aligned}
\int_{\Omega_x} |\mathbf{E}_h^{n+1}|^2 - |\mathbf{E}_h^n|^2 dx &= \sum_i \sum_l w_l \left((E_h^{n+1}(x_i^{(l)}))^2 - (E_h^n(x_i^{(l)}))^2 \right) \triangle x_i \\
&= \sum_i \sum_l w_l \left((E_i^{(l)}))^2 - (E_h^n(x_i^{(l)}))^2 \right) \triangle x_i \\
&= \sum_i \sum_l w_l \left(E_i^{(l)}) + E_h^n(x_i^{(l)}) \right) \left(-\frac{\triangle t}{2} (J_h^n(x_i^{(l)}) + J_i^{(l)}) \right) \triangle x_i
\end{aligned}
$$

Because $k \geq 2$, we can take $\varphi_h = v^2$ in (18), and

$$
\begin{aligned}
& \int_{\Omega_x} \int_{\Omega_v} f_h^{n+1} |\mathbf{v}|^2 d\mathbf{v} d\mathbf{x} - \int_{\Omega_x} \int_{\Omega_v} f_h^n |\mathbf{v}|^2 d\mathbf{v} d\mathbf{x} \\
= & \sum_i \sum_l w_l \left(\int_{\Omega_v} f_h^{n+1}(x_i^{(l)}, v) v^2 dv - \int_{\Omega_v} f_h^n(x_i^{(l)}, v) v^2 dv \right) \triangle x_i \\
= & \sum_i \sum_l w_l \left(\int_{\Omega_v} (g_i^{(l)} - f_h^n(x_i^{(l)}, v)) v^2 dv \right) \triangle x_i \\
= & \sum_i \sum_l w_l \left(\triangle t \int_{\Omega_v} \frac{E_h^n(x_i^{(l)}) + E_i^{(l)}}{2} (g_i^{(l)}(v) + f_h^n(x_i^{(l)}, v)) v \, dv \right) \triangle x_i \\
= & \sum_i \sum_l w_l \left(\triangle t \frac{E_h^n(x_i^{(l)}) + E_i^{(l)}}{2} (J_h^n(x_i^{(l)}) + J_i^{(l)}) \right) \triangle x_i
\end{aligned}
$$

Properties

Theorem (L^2 stability)

The DG schemes with $S_4(\triangle t)$, $S_4^{\{4\}}(\triangle t)$ satisfy

$$\int_{\Omega_x}\int_{\Omega_v} |f_h^{n+1}|^2 d\mathbf{v}d\mathbf{x} = \int_{\Omega_x}\int_{\Omega_v} |f_h^{n}|^2 d\mathbf{v}d\mathbf{x}$$

for central flux, and

$$\int_{\Omega_x}\int_{\Omega_v} |f_h^{n+1}|^2 d\mathbf{v}d\mathbf{x} \leq \int_{\Omega_x}\int_{\Omega_v} |f_h^{n}|^2 d\mathbf{v}d\mathbf{x}$$

for upwind flux.

Extensions

- For VM system, an energy-conserving DG scheme has to be used for the Maxwell's equation (e.g. those with alternating or central flux).
- Some care needs to be taken if there is external field present, which may cause coupling in the sub equations.

Outline

Selected numerical tests

1. 1D1V, one-species VA: Landau damping

$$f(x,v,0) = f_M(v)(1 + A\cos(\kappa x)), x \in [0,L],\ v \in [-V_c, V_c],$$

with $A = 0.5,\ \kappa = 0.5, L = 4\pi,\ V_c = 8$, and $f_M = \frac{1}{\sqrt{2\pi}} e^{-v^2/2}$.

2. 1D2V, one-species VM: streaming Weibel instability

$$f(x_2, v_1, v_2, 0) = \frac{1}{\pi\beta} e^{-v_2^2/\beta}[\delta e^{-(v_1 - v_{0,1})^2/\beta} + (1-\delta)e^{-(v_1+v_{0,2})^2/\beta}],$$

$$E_1(x_2,0) = E_2(x_2,0) = 0, \qquad B_3(x_2,0) = b\sin(k_0 x_2),$$

$$\beta = 0.01, b = 0.001$$

$$\delta = 0.5,\ v_{0,1} = v_{0,2} = 0.3,\ k_0 = 0.2; \quad \text{(initially symmetric beams)}$$

3. 1D1V, two-species VA: electron holes. Initial condition is constructed by Schamel distribution.

Accuracy test (One-species VA)

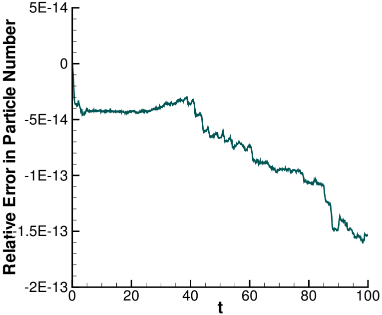
Table : Time discretization S_4, the upwind flux. L^2 error and order.

Space \ Mesh		20 × 20	40 × 40		80 × 80		160 × 160	
		Error	Error	Order	Error	Order	Error	Order
$\mathcal{S}_h^1, \mathcal{W}_h^1$	f	4.03E-03	1.08E-03	1.90	3.46E-04	1.64	1.05E-04	1.72
	E	1.16E-03	1.68E-04	2.79	3.13E-05	2.42	5.87E-06	2.41
$\mathcal{S}_h^2, \mathcal{W}_h^2$	f	8.03E-04	1.65E-04	2.28	2.82E-05	2.55	2.73E-06	3.37
	E	6.11E-05	3.49E-06	4.13	2.31E-07	3.92	1.39E-08	4.05
$\mathcal{S}_h^3, \mathcal{W}_h^3$	f	1.54E-04	1.22E-05	3.66	1.13E-06	3.43	6.64E-08	4.09
	E	3.96E-06	6.94E-08	5.83	9.26E-10	6.23	8.72E-10	0.09

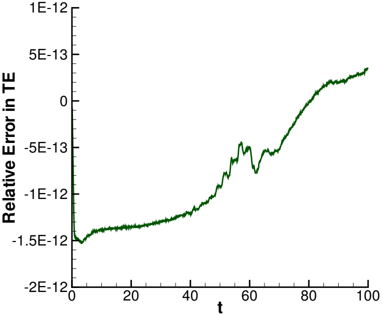
Table : Time discretization $S_4^{\{4\}}$, upwind flux. L^2 error and order.

Space \ Mesh		20 × 20	40 × 40		80 × 80		160 × 160	
		Error	Error	Order	Error	Order	Error	Order
$\mathcal{S}_h^3, \mathcal{W}_h^3$	f	1.25E-04	7.54E-06	4.05	5.12E-07	3.88	2.40E-08	4.42
	E	3.03E-06	6.65E-08	5.51	4.15E-09	4.00	3.02E-10	3.78

Conservation on a coarse mesh (Landau damping)



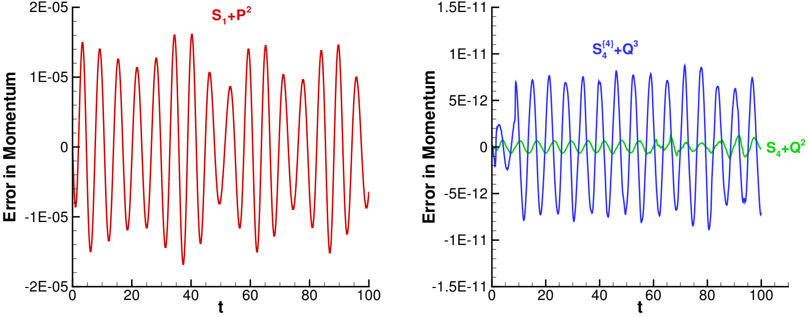
(a) Total particle number.



(b) Total energy.

Figure : Evolution of the relative error in conserved quantities by **Scheme-4** and Q^2 for Landau damping. 40×80 mesh. $CFL = 2$. Upwind flux.

Conservation: momentum

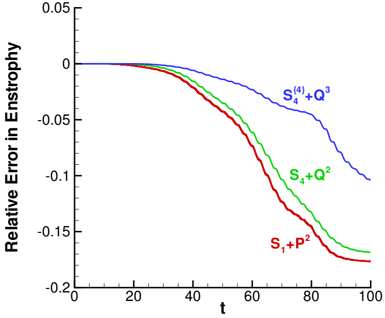


(a) Landau Damping.

(b) Landau Damping.

Figure : Evolution of error in momentum as a function of time. 100×200 mesh. Upwind Flux.

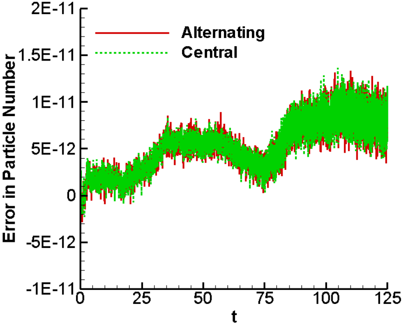
Conservation: enstrophy



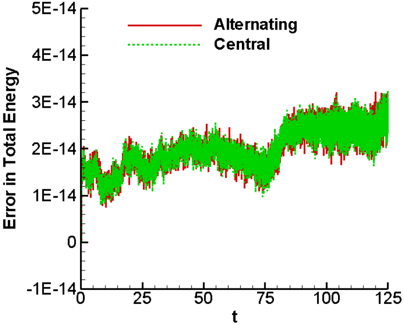
(a) Landau Damping.

Figure : Evolution of the relative error in enstrophy a function of time. 100×200 mesh. Upwind Flux.

Conservation for VM



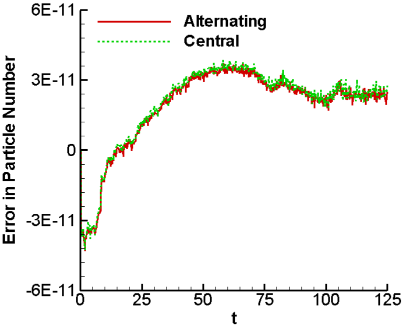
(a) Total particle number. Run1



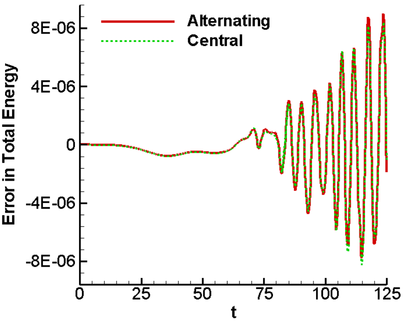
(b) Total energy. Run1

Figure : Evolution of the error in total particle number and total energy computed by semi-implicit scheme and P^2 with indicated fluxes. 100^3 mesh. $CFL = 0.15$.

Conservation for VM



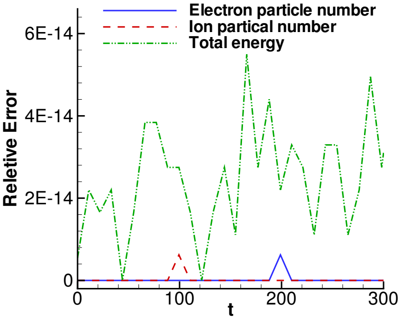
(a) Total particle number. Run1



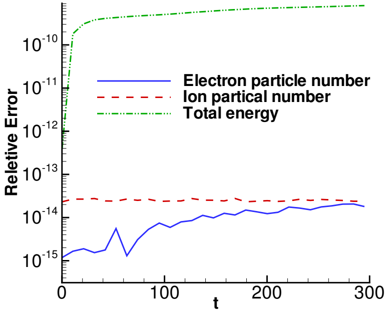
(b) Total energy. Run1

Figure : Evolution of the error in total particle number and total energy computed by fully implicit scheme and Q^2 with indicated fluxes. 80^3 mesh. $\Delta t = 0.2$. Tolerance number $\epsilon_{tol} = 10^{-8}$.

Conservation for two-species



(a) **Scheme-1**. Explicit method.



(b) **Scheme-2**. Implicit method.

Figure : Evolution of absolute value of relative errors in total particle number and total energy.

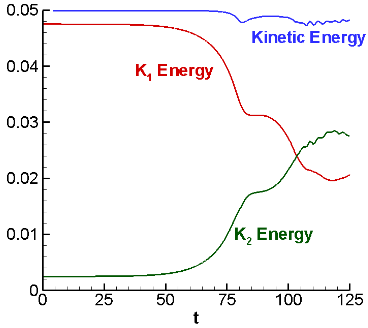
Convergence of Newton-Krylov solver (one-species)

We implement the fully implicit scheme on a 100×200 mesh and integrate up to $T = 50$ using piecewise P^1 polynomials.
nni = average of the total numbers of nonlinear iterations.
nli = average of the total number of linear (Krylov) iterations.
It failed to converge at $cfl = 300$.

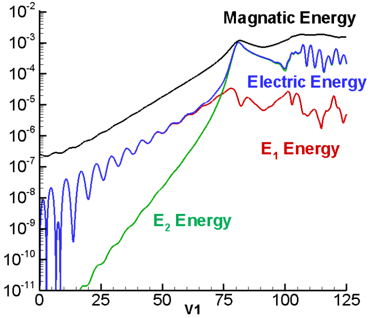
Table : Relation of the number of Newton and Krylov iterations per step with the CFL number. $S_4 + Q^1$. 100×200 mesh. $\epsilon_{tol} = 10^{-12}$.

CFL	1	10	20	40	80	100	150	200	250
nni	4.29	4.94	5.16	5.50	6.10	6.79	7.81	8.97	9.86
nli	6.26	13.36	19.82	33.02	59.36	82.01	106.64	122.27	147.71

Energy exchange for streaming Weibel instability



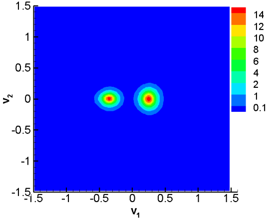
(a) Kinetic energy.



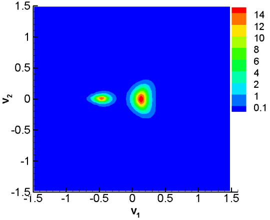
(b) Electric and magnetic energy.

Figure : Evolution of the kinetic, electric and magnetic energies for the streaming Weibel instability.

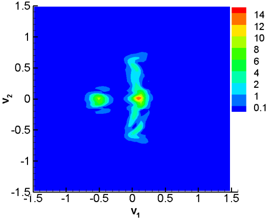
Evolution of pdf in streaming Weibel instability



(a) $t = 55$.



(b) $t = 82$.



(c) $t = 125$.

Figure : 2D contour plots of the distribution function at selected location $x_2 = 0.0625\pi$ and time t. Semi-implicit scheme and P^2. 80^3 mesh. $CFL = 0.15$. Alternating flux for Maxwell solver.

Evolution of electron hole in electron-oxygen-ion plasmas

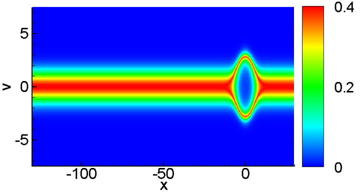
The EH is an important Bernstein-Greene-Kurskal (BGK) state in plasmas, and represents electrons that are trapped in a self-created positive electrostatic potential. Let $T_e/T_i = 1$, and $m_i/m_e = 29500$. Initial condition is constructed by Schamel distribution.

$$f_e = \begin{cases} \frac{1}{\sqrt{2\pi}} exp\left(-\frac{1}{2}[(|v-M|^2 - 2\phi)^{\frac{1}{2}} + M]^2\right), & v - M > \sqrt{2\phi} \\ \frac{1}{\sqrt{2\pi}} exp\left(-\frac{1}{2}[-(|v-M|^2 - 2\phi)^{\frac{1}{2}} + M]^2\right), & v - M < -\sqrt{2\phi} \\ \frac{1}{\sqrt{2\pi}} exp\left(-\frac{1}{2}[\beta(|v-M|^2 - 2\phi) + M^2]\right), & |v - M| \le \sqrt{2\phi} \end{cases} \tag{19}$$

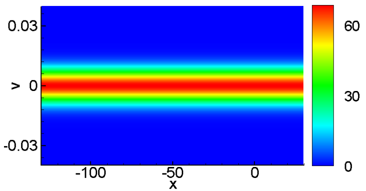
$$f_i = \frac{1}{\sqrt{2\pi\gamma}} e^{-v^2/2\gamma}, \tag{20}$$

Then we add a perturbation to f_e.

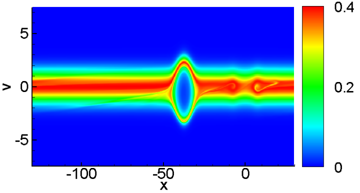
A single hole in electron-oxygen-ion plasmas



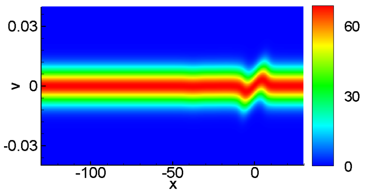
(a) $t = 0.$ Electron distribution



(b) $t = 0.$ Ion distribution

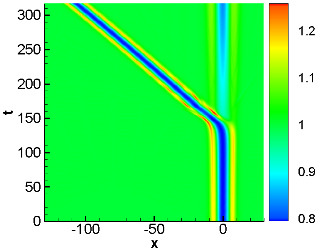


(c) $t = 200.$ Electron distribution

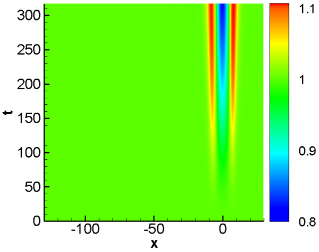


(d) $t = 200.$ Ion distribution

A single hole in electron-oxygen-ion plasmas

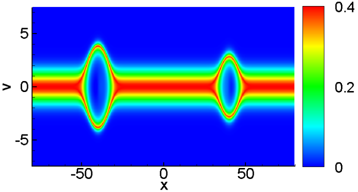


(e) Electron density

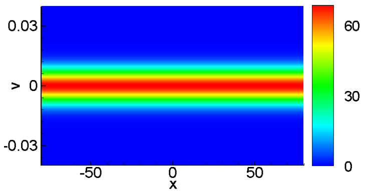


(f) Ion density

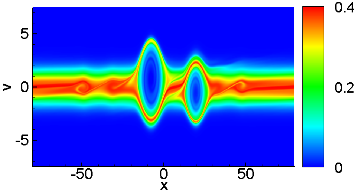
Two holes in electron-oxygen-ion plasmas



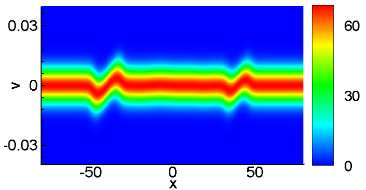
(g) $t = 0$. Electron distribution



(h) $t = 0$. Ion distribution

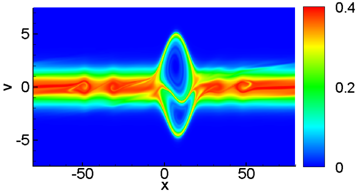


(i) $t = 155$. Electron distribution

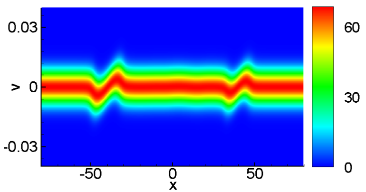


(j) $t = 155$. Ion distribution

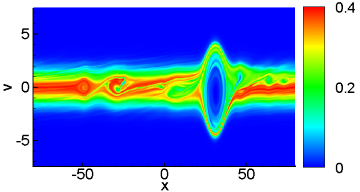
Two holes in electron-oxygen-ion plasmas



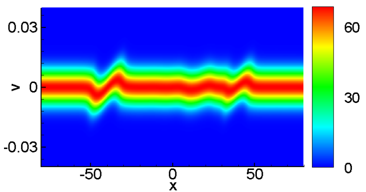
(k) $t = 175.$ Electron distribution



(l) $t = 175.$ Ion distribution



(m) $t = 251.$ Electron distribution



(n) $t = 251.$ Ion distribution

Conclusion

We design Eulerian solvers to treat the one-species, multi-species VA & VM equations with the following features:

- Conserves the total particle number and energy of the system on the fully discrete level. (For fully implicit method, it is subject to the tolerance parameter in the nonlinear solve).
- L^2 stability, and fully discrete L^2 stability for the implicit methods.
- High order, implicit or explicit in time.
- Can work on unstructured mesh in $\mathbf{x}$ and $\mathbf{v}$ space.

The END!

Thank You!