A vector field method for kinetic transport equations with applications

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Quick summary

1. We adapt the vector field method of Klainerman leading to a new way of proving decay estimates for velocity averages of solutions to kinetic transport equations.
2. We apply our method to the study of small data solutions of the Vlasov-Poisson and Vlasov-Norström systems.

Some motivations

- The Vlasov-Poisson system: coupling between Newtonian gravity with collisionless matter. Poisson equation: $\Delta\phi = \int_v f dv, \quad \phi := \phi(t,x), \quad f := f(t,x,v), \quad (x,v) \in \mathbb{R}^n_x \times \mathbb{R}^n_v.$
- Trivial observation: $f = 0$ gives no dynamics.
- The fully relativistic generalization of the Vlasov-Poisson system, replacing Newtonian gravity by general relativity is the so-called *Einstein-Vlasov system*.
- Poisson equation replaced by an equation of type

  $$\Box u + F(u, \partial u, \partial^2 u) = \int_v f d\mu(u,v),$$

  where $u := u(t,x)$ matrix valued and $\Box = -\partial_t^2 + \Delta$, F is some non-linearity.
- $f = 0$: non-trivial dynamics *the Einstein vacuum equations*.

1. *What are the techniques which are traditionally used in the study of the vacuum Einstein equations that can still be applied in the Einstein-Vlasov case?*

2. *Does there exist analogues of the techniques used in the study of non-linear waves for systems arising from kinetic theory ?*

3. Remark: Coupling Einstein equations to other field equations: Maxwell, scalars field, ... these system are all coupling Einstein to other wave equations. Kinetic matter is different.

Extremely brief survey of (some) literature

1. ”Small data global existence” for the Einstein vacuum equations: stability of Minkowski space (Christodoulou-Klainerman 93, Lindblad-Rodnianski 2004, ...)

2. Einstein-Vlasov system under spherically symmetry: Rendall-Rein (massive particles), Dafermos (massless particles)

3. Einstein-Vlasov system no symmetry : Martin Taylor (massless particles, see next talk).

4. Many other works on the Einstein-Vlasov system: Local existence (Choquet-Bruhat, Ringström), symmetric solutions with space variable in compact manifold (Rendall, Rein, Andréasson, Weaver, Dafermos, Fajman, J.S., ...), black hole formation/regularity (Andréasson-Rein, Dafermos-Rendall), expanding solutions (Ringström), homogeneous solutions...

Klainerman's vector field method for the wave equation

- Vector field method: robust method to prove decay estimate for solutions to $\Box\phi = 0, \Box := -\partial_t^2 + \Delta$ wave operator.

- 3 ingredients

  1. **A coercive conservation law**: for the wave equation, this the energy conservation

$$E(\phi)(t) := \int_{x\in\mathbb{R}^n} \left((\partial_t \phi)^2(t) + |\nabla\phi|^2(t) \right) dx = E(\phi)(0).$$

2. **Commutation vector fields**

Let $Z =$

- $\partial_t, \partial_{x^i}$, (translations)
- $\Omega_{ij} = x^i \partial_{x^j} - x^j \partial_{x^i}$, (rotations)
- $\Omega_{0i} = t\partial_{x^i} + x^i \partial_t$ (hyperbolic rotations)
- $S = t\partial_t + \sum_i x^i \partial_{x^i}$ (scaling)

Then, $[\Box, Z] = 0$ unless $Z = S$ in which case $[\Box, S] = 2\Box$.

The commutation vector fields Z are associated with the Lorentz invariance of the wave equations.

3. **Weighted idendities**

- The translation vector fields $\partial = \partial_t, \partial_i$ can be decomposed onto Ω_{ij}, Ω_{0i} and S:

$$(t - |x|)\partial = \sum_Z a_Z Z,$$

where Z are homogeneous coefficients of order 0 and $Z = \Omega_{ij}, \Omega_{0i}, S$.

- Away from light-cone, we gain t decay by taking usual derivatives.

- These weights can be combined with weights in the energy norms to yield weighted Sobolev inequalities.

Klainerman-Sobolev inequality

Theorem 1 (Klainerman, 85). *Let* $u \in C^\infty([0,+\infty)_t \times \mathbb{R}^n)$ *decaying sufficiently fast for large* x. *Then,*

$$(1+t+|x|)^{(n-1)}(1+|t-|x||)|u(t,x)|^2 \lesssim \sum_{|\alpha|\leq(n+2)/2} ||Z^\alpha u||^2_{L^2(\mathbb{R}^n)},$$

where Z^α *are combinations of* $|\alpha|$ *vector fields.*

Application: Let ϕ be a solution to the wave equation, arising from C^∞ data decaying sufficiently fast, then all the norms on the right-hand side are finite with $\partial\phi$ replacing u. Thus, we get decay rates for $\partial\phi$.

Sketch of proof of Klainerman-Sobolev inequality when $|x| \leq \frac{t}{2}$.
Fix (t, x) with $|x| \leq \frac{t}{2}$. Apply a local Sobolev inequality, to the function $f : y \in \mathbb{R}^n \to \phi(t, x + ty)$:

$$
\begin{aligned}
|f(0)| = |\phi(t,x)|^2 &\lesssim \sum_{|\alpha| \leq (n+2)/2} \int_{|y| \leq 1/8} |\partial_\alpha f(y)|^2 dy, \\
&\lesssim \sum_{|\alpha| \leq (n+2)/2} \int_{|y| \leq 1/8} t^{|\alpha|} |\partial_\alpha \phi(t, x + ty)|^2 dy, \\
&\lesssim \sum_{|\alpha| \leq (n+2)/2} \int_{|y| \leq 1/8} |\Gamma^\alpha \phi(t, x + ty)|^2 dy, \\
&\lesssim \frac{1}{t^n} \sum_{|\alpha| \leq (n+2)/2} \int_{|z| \leq t/8} |\Gamma^\alpha \phi(t, x + z)|^2 dz.
\end{aligned}
$$

The free transport equations

- The distribution function $f(t,x,v)$ with $t \in \mathbb{R}$, $x = (x^i) \in \mathbb{R}^n$ and $v = (v^i) \in \mathbb{R}^n$ gives the density of particles at point (t,x) and with velocity v. Typically, $f \geq 0$.
- We will consider three transport equations
  1. The classical transport equation $\partial_t f + v.\nabla_x f = 0$.
  2. The relativistic transport equation for massive particles $(m = 1)$
  $$\sqrt{1+|v|^2}\partial_t f + v.\nabla_x f = 0.$$
  3. The relativistic transport equation for massless particles
  $$|v|\partial_t f + v.\nabla_x f = 0. \quad (v \in \mathbb{R}^n \setminus \{0\})$$

Decay estimates for velocity averages

- Typically, decay estimates are obtained using the method of characteristics, following Bardos-Degond.
- Ex: Let f solves $\sqrt{1+|v|^2}\partial_t f + v.\nabla_x f = 0$ with smooth, compact data, then

$$\int_v |f|(t,x,v)\frac{dv}{\sqrt{1+|v|^2}} \lesssim \frac{C(V)}{t^n}\,||f(t=0)||_{L^1_x L^\infty_v}$$

where V is an upper bound on the size of the support in v of the data and $C(V) > 0$ is a constant depending on V such that $C(V) \to +\infty$ as $V \to +\infty$.

The massless case

We consider the equation

$$T_0(f) := |v|\partial_t f + v.\nabla_x f,$$

for $f = f(t,x,v)$ with $t \in \mathbb{R}$, $x = (x^i) \in \mathbb{R}^n$ and $v = (v^i) \in \mathbb{R}^n \setminus \{0\}$.

First ingredient: coercive conversation law:

$$\int_x \int_v |f|(t,x,v)dx\frac{dv}{|v|} = \int_x \int_v |f|(0,x,v)dx\frac{dv}{|v|}.$$

Second ingredient: good commutation vector fields.

The commutation fields

- ∂_t, ∂_{x^i} commutes with T_0.
- On the other hand the $\Omega_{ij} = x^i \partial_{x^j} - x^j \partial_{x^j}$ do not, same for hyperbolic rotation.
- Replacement for rotation

$$^x\Omega_{ij} +^v \Omega_{ij} := x^i \partial_{x^j} - x^j \partial_{x^j} + v^i \partial_{v^j} - v^j \partial_{v^i}.$$

- Replacement for hyperbolic rotation

$$^x\Omega_{0i} +^v \Omega_{0i} := t\partial_{x^i} + x^i \partial_t + |v| \partial_{v^i},$$

- 2 scalings vector fields: $^xS = t\partial_t + x^i \partial_{x^i}$ and $^vS = v^i \partial_{v^i}$
- There is of course a “recipe” to transform commuting vector fields for $\Box$ to commuting vector fields for transport: taking the *complete lift* (classical operation in differential geometry that lifts vector fields of the tangent bundle to vector fields of the tangent bundle to the tangent bundle)

Commutation properties

Lemma 1. *Let Z be one of the above vector fields. Then $[T_0, Z] = \epsilon T_0$, where $\epsilon = 0, -1, +1$*

Lemma 2. *Let Z be one of the original vector fields (for instance $Z = \Omega^x_{ij}$) and $\tilde{Z}$ be its complete lift. Then, for any distribution function f,*

$$Z \int_v f \frac{dv}{|v|} = \int_v \tilde{Z}(f) \frac{dv}{|v|}$$

Klainerman-Sobolev inequality in L^1

Lemma 3. *Let $u \in C^\infty([0,+\infty)_t \times \mathbb{R}^n)$ decaying sufficiently fast for large x. Then,*

$$(1+t+|x|)^{(n-1)}(1+|t-|x||)|u(t,x)| \lesssim \sum_{|\alpha|\leq n} ||Z^\alpha u||_{L^1(\mathbb{R}^n)},$$

where Z^α are combinations of $|\alpha|$ vector fields .

Putting things together

Step 1: Apply Lemma 3 to $\int_v f \frac{dv}{|v|}$:

$$\int_v f(t,x,v)\frac{dv}{|v|} \lesssim \frac{1}{(1+t+|x|)^{(n-1)}(1+|t-|x||)} \sum_{|\alpha|\leq n} \left\| Z^\alpha \int_v f(t,x,v)\frac{dv}{|v|} \right\|_{L^1(\mathbb{R}^n)},$$

Step 2: Apply Lemma 2:

$$Z^\alpha \int_v f(t,x,v)\frac{dv}{|v|} = \int_v \tilde{Z}^\alpha f(t,x,v)\frac{dv}{|v|}$$

Final Step:

$$\begin{aligned}
\left\|\int_v \tilde{Z}^\alpha f(t,x,v)\frac{dv}{|v|}\right\|_{L^1(\mathbb{R}^n)} &= \int_x \left|\int_v \tilde{Z}^\alpha f(t,x,v)\frac{dv}{|v|}\right| dx \\
&= \int_x \int_v \left|\tilde{Z}^\alpha f(t,x,v)\right| \frac{dv}{|v|} dx \\
&= \int_x \int_v \left|\tilde{Z}^\alpha f(0,x,v)\right| \frac{dv}{|v|} dx,
\end{aligned}$$

using Lemma 1 (conservation law).

We have proven

Theorem 2 (Joudioux, Fajman, J.S)**.** *Let f be solution to the massless transport equation arising from smooth data decaying sufficiently fast in x, v. Then, for all $t \geq 0$,*

$$(1+t+|x|)^{n-1}(1+|t-|x||)\int_v f(t,x,v)\frac{dv}{|v|} \lesssim \sum_{|\alpha|\leq n} ||\tilde{Z}^\alpha(f)(t=0)|v|^{-1}||_{L^1_{x,v}},$$

where $\tilde{Z}^\alpha$ is a combinaison of $|\alpha|$ commutation vector fields as described above.

Remarks:

- No need for compact support in v.
- Based only on conservation laws and commutator: easy to perform in a pertubative regime.
- Get t^{n-1} decay and additional decay provided we are away from the light-cone. Relevant for non-linear applications.
- Technical difficulty: to get decay estimate for $\int_v |f|$ we cannot apply the Klainerman-Sobolev inequality because of lack of regularity of absolute values. So proof of Klainerman-Sobolev is intertwined with the above arguments.

The massive case

- The analogue in this case is the Klein-Gordon equation: $(\Box - 1)\phi = 0$.

- Scaling $t\partial_t + \sum_i x^i \partial_i$ does not commute with the Klein-Gordon operator.

- On the other hand, energy estimates gives an additional L^2 bound on ϕ itself (and not its derivatives).

- Klainerman 93 : use energy estimates on hyperboloids $\rho = \sqrt{t^2 - |x|^2} = const$, (for $t \geq |x|$).

- Key point: hyperboloids are generated by the hyperbolic rotations: there exists coordinate system (ρ, y^i) such that y^i are global coordinates on each $\rho = const$ and $t\partial_{y^i} = \Omega_{0i}$.

Theorem 3 (Decay estimates for velocity averages of massive distributions)**.** *For any regular distribution function f solution to the massive relativistic transport equation (m=1), any $t \geq \sqrt{1+|x|^2}$, we have*

$$\int_{v \in \mathbb{R}^n_v} |f|(t,x,v) \frac{dv}{\sqrt{1+|v|^2}} \lesssim \frac{1}{(1+|x|+t)^n} \sum_{|\alpha| \leq n} \left\| \tilde{Z}^\alpha(f)_{|H_1^n \times \mathbb{R}^n_v} v_\alpha \nu_1^\alpha \right\|_{L^1\left(H_1^n \times \mathbb{R}^n_v\right)},$$

where H_1^n denotes the unit hyperboloid

$$H_1^n := \left\{(t,x) \in \mathbb{R}_t \times \mathbb{R}^n_x \;/\; 1 = t^2 - x^2, t \geq 0\right\},$$

$\tilde{Z}^\alpha(f)_{|H_1^n \times \mathbb{R}^n_v}$ is the restriction to $H_1^n \times \mathbb{R}^n_v$ of $\tilde{Z}^\alpha(f)$, $v_\alpha \nu_1^\alpha$ is the contraction of the 4-velocity $(\sqrt{1+|v|^2}, v^i)$ with the unit normal ν_1 to H_1^n and where the $\hat{Z}^\alpha$ are obtained as a composition of $|\alpha|$ vector fields.

Remarks

- What happen with data given at $t = 0$ (and not on a unit hyperboloid) ? Using time translation invariance and finite speed of propagation/domain of dependence arguments, can move from $t = 0$ compact data in x to some hyperboloid with finite trace for the solution on the hyperboloid. Then, our theorem applies (and the support of the solution is entirely in the future of that hyperboloid). So the use of hyperboloids is merely technical.
- No need for compact support support in v.

Applications I: Vlasov-Poisson with small data.

Vlasov-Poisson system in $\mathbb{R}^n_x \times \mathbb{R}^n_v$:

$$\begin{aligned} \partial_t f + v.\nabla_x f \pm \nabla_x \phi.\nabla_v f &= 0, \\ \Delta \phi &= \int_v f dv := \rho(f). \end{aligned}$$

Initial data $f(t=0) = f_0$.

- Global existence in dim 3: Pfaffelmoser, Lions-Perthame, Schaeffer.
- Small data global existence : Bardos-Degond (85). Asymptotics

$$\rho(f) \sim 1/t^3.$$

- Small data sharp asymptotics for derivatives:
  Hwang-Rendall-Velasquez (2006).

$$|\partial^\alpha \rho(f)| \sim 1/t^{(3+|\alpha|)}.$$

Our method provides an alternative proof of the Hwang-Rendall-Velasquez result.

Vector fields in the classical case

- The Lorentz invariance is replaced by the Galilean invariance.
- The vector fields commuting with the classical transport operator $\partial_t + v.\nabla_x$ are simply
  1. translations ∂_t, ∂_{x^i},
  2. uniform motions $t\partial_{x^i} + \partial_{v^i}$,
  3. rotations $\Omega^x_{i,j} + \Omega^v_{i,j}$,
  4. scaling in space $\sum_{i=1}^n x^i\partial_{x^i} + v^i\partial_{v^i}$,
  5. scaling in space and time $t\partial_t + \sum_{i=1}^n x^i\partial_{x^i}$,

A norm

Let $\delta > 0$ and N sufficiently large depending only on n.

$$E_{N,\delta}[g] := \sum_{|\alpha| \le N} ||Z^\alpha(g)||_{L^1(\mathbb{R}^n_x \times \mathbb{R}^n_v)} + \sum_{|\alpha| \le N} ||(1+|v|^2)^{\frac{\delta(\delta+n)}{2(1+\delta)}} Z^\alpha(g)||_{L^{1+\delta}(\mathbb{R}^n_x \times \mathbb{R}^n_v)}.$$

Results in $4d$

Theorem 4 (J.S)**.** *Let $n \geq 4$, $0 < \delta$ and $N \geq 5n/2 + 2$. Then, there exists $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$, if $E_{N,\delta}[f_0] \leq \epsilon$, then*

1. *Global bounds*
$$\forall t \in \mathbb{R}, \quad E_{N,\delta}[f(t)] \lesssim \epsilon.$$
2. *Space and time pointwise decay of averages of f*
$$\text{for any } \alpha \text{ of order } |\alpha| \leq N - n, \quad |\rho(Z^\alpha f)(t,x)| \lesssim \frac{\epsilon}{(1 + |t| + |x|)^n},$$
as well as the improved decay estimates
$$|\rho(\partial_x^\alpha f)(t,x)| \lesssim \frac{\epsilon}{(1 + |t| + |x|)^{n+|\alpha|}}.$$
3. *+ Boundedness and decay for $\partial Z^\alpha \phi$ and $\partial^2 Z^\alpha \phi$*

Commutation

Let $Z = t\partial_x + \partial_v$ and commute the transport equation.

$$\partial_t Z(f) + v.\nabla_x Z(f) + \nabla_x \phi.\nabla_v Z(f) = -\nabla_x Z(\phi).\nabla_v f$$

∂_{v^i} does not commute with free transport but we can rewrite it as

$$\partial_{v^i} = (t\partial_{x^i} + \partial_{v^i}) - t\partial_{x^i}$$

Worse error term to derive approximate conservation laws is of the form

$$\int_t \int_x \int_v |\partial Z(\phi) t \partial_x f| dt dx dv.$$

The expected decay in $3d$ for $\partial Z(\phi)$ is $1/t^2$ and thus gives log growth. However, it closes in $4d$.

3d case: modified vector fields

Consider the vector field $Y = t\partial_{x^i} + \partial_{v^i} - \sum_{i=1} \Phi_j(t,x,v)\partial_{x^j}$ and repeat the commutation

$$T_\phi(Z(f)) = -Y(\nabla_x(\phi)).\nabla_v f + \sum_{i=1} T_\phi(\Phi_j)\partial_{x^j} + ...$$

where $T_\phi := \partial_t + v.\nabla_x + \nabla_x\phi.\nabla_v$. Using again $\partial_{v^i} = (t\partial_{x^i} + \partial_{v^i}) - t\partial_{x^i}$, recall that the bad term is of the form

$$t\partial_x(Z\phi).\partial_x f$$

So we impose $T_\phi(\Phi_j) = t\partial_{x^j} Z\phi$ to cancel the worse term.

- We need modified vector fields for all the vector fields expect the translations.
- The coefficients $\Phi_j(t, x, v)$ all grow logarithmically in time.
- Nonetheless, we prove that the conservation laws are still coercive and that modified Klainerman-Sobolev inequalities still holds.
- Some care are needed to close the high-order estimates (when the number of commutation is large).
- The final results are then similar to the $4d$ case expect that in the global norm $E_{N,\delta}$ all the Z vector fields are replaced by their modified versions.
- $E_{N,\delta}$ mesures a deviation from symmetry of the solution. Thus, in $3d$, *globally*, the solution deviates logarithmically in time from total Galilean symmetry, while in $4d$ and greater, the total asymmetry of the solutions stays at size ϵ.

Results in $3d$

Theorem 5 (J.S). *Let $0 < \delta$ and $N \geq 14$. Then, there exists $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$, if $E_{N,\delta}[f_0] \leq \epsilon$, then*

1. *Global bounds*

$$\forall t \in \mathbb{R}, \quad E_{N,\delta}[f(t)] \lesssim \epsilon.$$

2. *Space and time pointwise decay of averages of f*

$$\text{for any } \alpha \text{ of order } |\alpha| \leq N - n, \quad |\rho(Z^\alpha f)(t,x)| \lesssim \frac{\epsilon}{(1 + |t| + |x|)^n},$$

as well as the improved decay estimates

$$|\rho(\partial_x^\alpha f)(t,x)| \lesssim \frac{\epsilon}{(1 + |t| + |x|)^{n+|\alpha|}}.$$

3. *+ Boundedness and decay for $\partial Z^\alpha \phi$ and $\partial^2 Z^\alpha \phi$*

In the above theorem, the norm $E_{N,\delta}$ is similar to the $4d$ case, expect that the vector fields are replaced by the modified ones.

Applications to the Vlasov-Norström system

The system

$$
\begin{aligned}
\Box\phi &= -m^2 \int_v f \frac{dv}{v^0} \\
T_m(f) - \left(T_m(\phi)v^i + m^2\nabla^i\phi\right)\frac{\partial f}{\partial v^i} &= (n+1)T_m(\phi)f
\end{aligned}
$$

$$
T_m = \sqrt{m^2 + |v|^2}\partial_t + v.\nabla_x.
$$

- Global existence in $3d$: Calogero.
- Small data global existence and asymptotics for $\rho(f)$: Stephan Friedrich.
- Other words: Calogero-Rein, Andrasson-Calogero-Rein, Pallard,..
- In the massless case, the equations degenerate to a perturbed linear transport equation.
- We prove sharp asymptotics for solutions and their derivatives in the following case:
  1. Massless case: $4d$ even for large data, $3d$ small data on ϕ, exploiting a form of the *null condition*.
  2. Massive case: $4d$ small data. The $3d$ case in progress.

Massive case: The high order estimates difficulty

Consider the model problem

$$\begin{aligned} \sqrt{1+|v|^2}\partial_t + v.\nabla_x + \nabla_x\phi.\nabla_v f &= 0, \\ \Box\phi &= \int_v f \frac{dv}{\sqrt{1+|v|^2}} \end{aligned}$$

To prove decay of ϕ, we also commute the wave equation by vector fields. Imagine our initial energy contains N commutation vector fields, then we are allowed to commute N times the wave equation

$$\Box Z^N \phi = \int_v Z^N f \frac{dv}{\sqrt{1+|v|^2}}$$

Now, to prove just an energy estimate for this equation, we need the right-hand side to decay (in time and space), but to prove decay of the right-hand side with our method, we need to control norms for $Z^{N+n} f$. This would force us to commute the transport equation more, which would requires more control on ϕ...

Instead, we give another proof of decay for $\int_v Z^N f \frac{dv}{\sqrt{1+|v|^2}}$ based on our vector field method and a proof of decay in L^2_x for inhomogeneous transport equation of the form

$$T(f) = g \nabla \psi$$

assuming only L^2_x bounds on $\nabla \psi$ and that g is itself solution to an inhomogeneous transport equation

$$T(g) = Err$$

where the Err is regular and integrable and the data for g is regular.