Landau damping in Gevrey regularity for Vlasov-Poisson and connections with hydrodynamic stability

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joint work with Nader Masmoudi and Clément Mouhot (fluid mechanics work with Pierre Germain and Vlad Vicol)

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September 10, 2015

Collisionless Vlasov equations

- The collisionless Vlasov equations for a distribution function $f(t,x,v)$ with $(x,v) \in \mathbb{T}^d \times \mathbb{R}^d$ (the torus of side-length 2π):

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + F(t,x) \cdot \nabla_v f = 0 \\ F(t,x) = \nabla_x W *_x (\rho - <\rho>), \\ \rho(t,x) = \int f(t,x,v) dv, \end{cases} \tag{1}$$

  here $<\rho> = (2\pi)^{-d} \int \rho(t,x) dx$ denotes the spatial average.
- An important model in plasma physics and galactic dynamics.
- In plasmas: $f(t,x,v)$ denotes the density of electrons of an electrically neutral plasma when ion acceleration and magnetic effects can be neglected.
- $F(t,x) = \frac{q}{m_e} \nabla_x \Delta_x^{-1} (\rho - <\rho>)$: Coulomb electrostatic repulsion. In this case the equations are called *Vlasov-Poisson*.

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$$\begin{cases} \partial_t h + v \cdot \nabla_x h + F(t,x) \cdot \nabla_v f^0 = 0 \\ F(t,x) = \nabla_x W *_x \rho, \\ \rho(t,x) = \int h(t,x,v) dv. \end{cases} \tag{2}$$

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- In 1946 Landau showed that (2) with $f^0(v) = (4\pi k_B T)^{-d/2} e^{-|v|^2/(k_B T)}$ and h analytic predicts $|\rho(t)| \lesssim e^{-\lambda t}$ (in the plasma case) - *Landau damping*.
- Irreversible *looking* behavior without any entropy production!
- Confirmed by experiments by Malmberg/Wharton '64 and now an integral part of plasma physics.

Landau damping as phase mixing I: decay by mixing

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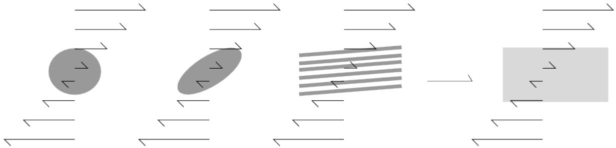
- In this case the solution is $h(t,x,v) = h^{in}(x - tv, v)$. Here is a visualization in $\mathbb{T}_x \times \mathbb{R}_v$:



- In Fourier space: $\hat{h}(t,k,\eta) = \hat{h}^{in}(k, \eta + kt)$.
- $\hat{\rho}(t,k) = \hat{h}(t,k,0) = \hat{h}^{in}(k, kt)$.

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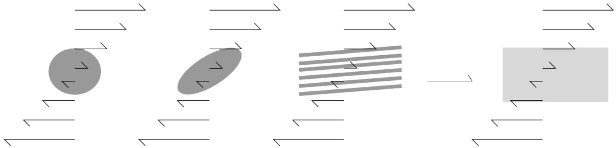

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- $\hat{\rho}(t,k) = \hat{h}(t,k,0) = \hat{h}^{in}(k, kt)$.
- Decay of Fourier transform is equivalent to *decay in time* of ρ: *regularity becomes decay*.

Landau damping as phase mixing II: transient growth by unmixing and the Orr mechanism

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- Imagine $\hat{h}^{in}(k,\eta)$ is concentrated in frequency near k and η with $\eta \gg k > 0$. Then:

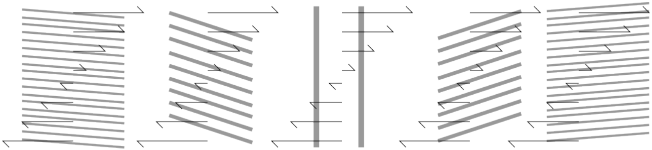

Figure : The center image occurs at $t \sim \eta/k$ – in fluid mechanics this is called the *critical time*

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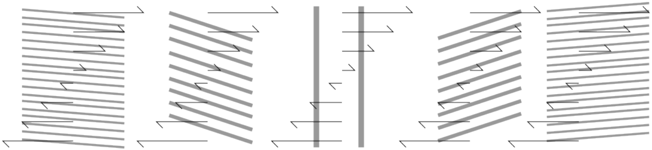

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- The density variations (and hence electric field) become large at $t \sim \eta/k$: particles can just as easily mix as they can unmix.
- Decay *costs* regularity!
- In fluid mechanics, this effect was identified by Orr in 1907 and is known as the *Orr mechanism*.

Landau damping in linearized Vlasov

- That picture is easily extended to the linearized Vlasov if $f^0(v)$ or W are 'small' but not generally the case in plasmas.
- Landau proved damping via a Laplace transform (see also: van Kampen '55, Case '59, Penrose '60, Degond '86, Morrison '00...)
- I will not discuss the linear theory, as this has been well-understood for a long time...

Nonlinear effects

- There is a class of exact nonlinear traveling wave solutions called BGK modes (discovered by Bernstein/Green/Kruskal '57) which are solitons with particles trapped in potential wells.
- There also seems to exist more complicated nonlinear coherent structures nearby in phase space (see e.g. KEEN waves: Afeyan, Casas, Crouseilles et. al. '14).

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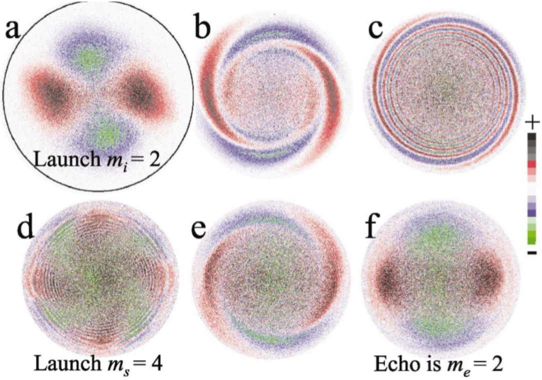
- Lin and Zeng '11 show using the BGK modes that you need at least $h^{in} \in H^{\sigma}$, $\sigma > 3/2$ to get damping for general small data.

Plasma echoes

- There are weakly nonlinear resonances called *plasma echoes* - discovered by Malmberg/Wharton/Gould/O'Neil in '68.
- This arises from the repeated nonlinear interaction of mixed modes with the electrostatic field, exciting oscillations which *unmix* in the future.
- Bad interactions between non-normal transient growth and nonlinearity is by now a well-established idea in fluid mechanics as well, see Trefethen et. al. '93.
- We will see the echoes as problematic off-diagonal terms in a non-local integral operator.

A hydrodynamic echo in 2D Euler produced by a pure electron plasma

This figure is from J.H. Yu, C.F. Driscoll and T.M. O'Neil, *Phase mixing and echoes in a pure electron plasma*, Phys. Plasmas **12**, 055701 (2005)



Mathematical Results

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- Mouhot/Villani '11 showed that damping holds for *all* sufficiently small h which are Gevrey-$\frac{1}{s}$ for $1 \approx s \leq 1$:

$$\|h\|_{\lambda;s}^2 = \sum_{k \in \mathbb{Z}^d} \int_{\mathbb{R}^d} \left|\hat{h}(k,\eta)\right|^2 e^{2\lambda \langle k,\eta \rangle^s} d\eta.$$

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- Use a Newton iteration scheme that successively linearizes Vlasov around iterate $h^{(j)}$ to construct $h^{(j+1)}$, each step lowering the analytic regularity (similar to the KAM theorem and Nash-Moser iterations).
- Faou/Rousset in '14 show that if W is a finite sum of sines and cosines then there is a reasonable proof of Landau damping for all small data in finite regularity – this is because this model only permits finitely many echoes.

Nonlinear Landau damping

[1] Penrose criterion + $\sum_{\alpha \leq M} \|v^{\alpha} f^0\|_{\bar{\lambda};1} < \infty$ for some $\bar{\lambda} > 0$.

[2] For $s = 1$, you have to require that λ_0 is less than the radius of analyticity of f^0.

Nonlinear Landau damping

- We obtain the conjectured regularity and we have a significantly simpler proof (the latter basically gives us the former for free).
- Our proof was also adapted to relativistic plasmas (Young '14).

Theorem (JB, Masmoudi, Mouhot 2013)

Let f^0 satisfy a linear stability condition[1] *and let $\frac{1}{3} < s \leq 1$, $\lambda_0 > \lambda' > 0$ be arbitrary*[2] *and $M > d/2$ be an integer. Then there exists an $\epsilon_0 = \epsilon_0(d, M, f^0, \lambda_0, \lambda', s)$ such that if h_{in} is mean zero and*

$$\sum_{\alpha \in \mathbb{N}^d : |\alpha| \leq M} \| v^\alpha h_{in} \|^2_{\lambda_0;s} < \epsilon^2 \leq \epsilon_0^2,$$

then there exists a mean-zero h_∞ satisfying

$$\| h(t, x + vt, v) - h_\infty(x, v) \|_{\lambda';s} \lesssim \epsilon e^{-\frac{1}{2}(\lambda_0 - \lambda') t^s}, \tag{3a}$$

$$\left| e^{\lambda' \langle k, kt \rangle^s} \hat{\rho}_k(t) \right| \lesssim \epsilon e^{-\frac{1}{2}(\lambda_0 - \lambda') t^s}. \tag{3b}$$

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[2] For $s = 1$, you have to require that λ_0 is less than the radius of analyticity of f^0.

A brief aside about inviscid damping in the 2D Euler equations

- The 2D incompressible Euler equations perturbed around the Couette flow for $(x, y) \in \mathbb{T} \times \mathbb{R}$:

$$\begin{cases} \omega_t + y\partial_x \omega + U \cdot \nabla \omega = 0 \\ U = \nabla^\perp \Delta^{-1} \omega \end{cases} \tag{4}$$

- The vorticity ω, and the velocity U it creates through the Biot-Savart law, are the perturbation from the background shear flow.
- Mixing in phase space now becomes regular mixing – like milk into coffee except for vorticity.

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- Linearization predicts that $(y, 0) + U(t, x, y) \to (y + u_\infty(y), 0)$ *strongly* in L^2 (Orr 1907) - is sometimes called *inviscid damping* by some physicists (and now us) due to the analogy with Landau damping.

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- Similarities: regularity vs. decay, transient growth, nonlinear echoes...

Asymptotic stability of nearly-Couette shear flows

Theorem (JB, Masmoudi 2013)

For all $1/2 < s \leq 1$, $\lambda > \lambda' > 0$, *there exists an* $\epsilon_0 = \epsilon_0(\lambda, \lambda', s) \leq 1/2$ *such that if*

$$\|U^{in}\|_{L^2} + \|\omega^{in}\|_{\lambda;s} = \epsilon < \epsilon_0,$$

then velocity field U *converges strongly in* L^2 *to a shear flow* $(y + u_\infty(y), 0)$:

$$\|U^x(t) - u_\infty\|_2 \lesssim \frac{\epsilon}{\langle t \rangle} \tag{5a}$$

$$\|U^y(t)\|_2 \lesssim \frac{\epsilon}{\langle t \rangle^2}, \tag{5b}$$

and the vorticity mixes like a passive scalar in the sense that: there exists an f_∞ *and* $u_\infty(y)$ *such that*

$$\|\omega(t, x + ty + u_\infty(y)t, y) - f_\infty(x, y)\|_{\lambda';s} \lesssim \frac{\epsilon^2}{\langle t \rangle}. \tag{5c}$$

Differences with Vlasov

- Asymptotically passive transport in a shear flow

$$\omega(t,x,y) \sim f_\infty\left(x - ty - u_\infty(y)t, y\right), \quad \text{when } t \to \infty,$$

  but the shear flow $(y + u_\infty(y), 0)$ is *determined by the solution* and *not* known a priori: the long-time behavior is *quasi-linear* whereas for Vlasov it is *not*.
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- The techniques used in 2D Euler do not apply to Vlasov or vice-versa.
- Since then, we have studied *mixing-enhanced dissipation* in 2D Navier-Stokes (JB,Masmoudi,Vicol '14), in 3D Navier-Stokes (JB,Germain,Masmoudi '15), and passive scalars (JB,Coti Zelati,Glatt-Holtz '15).
- The 3D works show that mixing-enhanced dissipation and inviscid damping play a key role in understanding the subcritical transition of 3D shear flows, a very classical problem in fluid mechanics.
- In particular, we were able to analytically determine the "transition threshold" for 3D Couette flow for Gevrey regular initial data.

Back to Landau damping...

- Start by modding out by free transport: define $z = x - tv$ and $f(t,z,v) = h(t, z+tv, v)$ solves

$$\partial_t f + F(t, z+tv) \cdot (\nabla_v - t\nabla_z)(f + f^0) = 0.$$

- Now $\hat{\rho}_k(t) = \hat{f}_k(t, kt)$ so uniform regularity control on f implies damping.

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- Fourier transform $(z, v) \mapsto (k, \eta)$:

$$\partial_t \hat{f}_k(t,\eta) = \hat{\rho}_k(t)\widehat{W}(k)k \cdot (\eta - kt)\widehat{f^0}(\eta - kt) + \sum_{\ell \in \mathbb{Z}^d_*} \hat{\rho}_\ell(t)\widehat{W}(\ell)\ell \cdot (\eta - kt)\hat{f}_{k-\ell}(t, \eta - t\ell).$$

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- Integrating in time and evaluating at $\eta = kt$ (the critical time!) implies:

$$\begin{aligned}\hat{\rho}_k(t) &= \hat{h}_{in}(k, kt) + \int_0^t \hat{\rho}_k(\tau)\widehat{W}(k)k \cdot k(t - \tau)f^0\left(k(t - \tau)\right) d\tau \\ &\quad + \int_0^t \sum_{\ell \in \mathbb{Z}^d_*} \hat{\rho}_\ell(\tau)\widehat{W}(\ell)\ell \cdot k(t - \tau)\hat{f}_{k-\ell}(\tau, kt - \tau\ell)d\tau.\end{aligned}$$

Paraproducts and the density

- Paraproducts (introduced by Bony in '81) can be thought of as a linearization of higher frequencies around lower frequencies[3]:

$$\hat{\rho}_k(t) = \hat{h}_{in}(k,kt) + \int_0^t \hat{\rho}_k(\tau)\,|k|^2\,\widehat{W}(k)(t-\tau)\hat{f}^0(k(t-\tau))d\tau$$
$$+ \int_0^t \sum_{\ell\in\mathbb{Z}^d_*} \hat{\rho}_\ell(\tau)\widehat{W}(\ell)\ell\cdot k(t-\tau)\hat{f}_{k-\ell}(\tau,kt-\ell\tau)\mathbf{1}_{|\ell,\ell\tau|<<|k-\ell,kt-\tau\ell|}d\tau$$
$$+ \int_0^t \sum_{\ell\in\mathbb{Z}^d_*} \hat{\rho}_\ell(\tau)\widehat{W}(\ell)\ell\cdot k(t-\tau)\hat{f}_{k-\ell}(\tau,kt-\ell\tau)\mathbf{1}_{|\ell,\ell\tau|>>|k-\ell,kt-\tau\ell|}d\tau$$
$$+ \int_0^t \sum_{\ell\in\mathbb{Z}^d_*} \hat{\rho}_\ell(\tau)\widehat{W}(\ell)\ell\cdot k(t-\tau)\hat{f}_{k-\ell}(\tau,kt-\ell\tau)\mathbf{1}_{|\ell,\ell\tau|\approx|k-\ell,kt-\tau\ell|}d\tau$$

- *transport*, *reaction* and *remainder* (T_k, R_k and $\mathcal{R}_k$).
- The reaction term contains the plasma echoes: this term is our main enemy and where the requirement $s > 1/3$ will be used.

[3] For those familiar with the proof of MV '11, the idea that paradifferential calculus can be useful in replacing Nash-Moser iterations goes back at least to Hörmander '92

Bootstrap

- Our norms are built on the Fourier multiplier:

$$A(t,k,\eta) = e^{\lambda(t)\langle k,\eta\rangle^s}\langle k,\eta\rangle^\sigma$$

- We use the following notation:

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- We prove the following bootstrap: for $\epsilon \ll 1$, the LHS for $0 < t < T$ implies the RHS (on same interval):

$$\begin{aligned}
\sum_{|\alpha|\leq M} \|\langle\nabla\rangle A(v^\alpha f)(t)\|_2^2 \leq 4K_1\langle t\rangle^7\epsilon^2 & \qquad & \sum_{|\alpha|\leq M} \|\langle\nabla\rangle A(v^\alpha f)(t)\|_2^2 < 2K_1\langle t\rangle^7\epsilon^2 \\
\sum_{|\alpha|\leq M} \|\langle\nabla\rangle^{-3} A(v^\alpha f)(t)\|_2^2 \leq 4K_2\epsilon^2 & \Rightarrow & \sum_{|\alpha|\leq M} \|\langle\nabla\rangle^{-3} A(v^\alpha f)(t)\|_2^2 < 2K_2\epsilon^2 \\
\int_0^t \|A\rho(\tau)\|_2^2 d\tau \leq 4K_3\epsilon^2 & \qquad & \int_0^t \|A\rho(\tau)\|_2^2 d\tau < 2K_3\epsilon^2.
\end{aligned}$$

The $L^2(dt)$ estimate: transport term

- After the dust settles, the linear lemma tells us:

$$\int_0^T \|A\rho(t)\|_2^2 dt \lesssim \epsilon^2 + \int_0^T \sum_{k\in\mathbb{Z}_*^d} \|AT_k(t)\|^2 dt + \int_0^T \sum_{k\in\mathbb{Z}_*^d} \|AR_k(t)\|^2 dt + \int_0^T \sum_{k\in\mathbb{Z}_*^d} \|A\mathcal{R}_k(t)\|^2 dt.$$

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- To deal with the transport term: growth of the high norm on f is counter-balanced by time-decay on ρ.
- After some fiddling around:

$$\int_0^T \sum_{k\in\mathbb{Z}^d_*} \|AT_k(t)\|^2 dt \lesssim K_1\epsilon^2 \int_0^T \|A\rho(t)\|_2^2 dt.$$

- Absorbed on the LHS by picking ϵ small.
- Remainder can be treated in more or less the same way and gives the same bound.

The $L^2(dt)$ estimate: reaction term

- After some different fiddling and applying Schur's test, we get:

$$\int_0^T \sum_{k\in\mathbb{Z}^d_*} |AR_k(t)|^2 \, dt \lesssim$$

$$\epsilon^2 \left(\sup_{t\geq 0} \sup_{k\in\mathbb{Z}^d_*} \int_0^t \sum_{\ell\in\mathbb{Z}^d_*} \bar{K}_{k,\ell}(t,\tau) d\tau \right) \left(\sup_{\tau\geq 0} \sup_{\ell\in\mathbb{Z}^d_*} \sum_{k\in\mathbb{Z}^d_*} \int_\tau^T \bar{K}_{k,\ell}(t,\tau) dt \right) \int_0^T \|A\rho(t)\|_2^2 dt,$$

where (using the low norm control on f) for some $\delta > 0$:

$$\bar{K}_{k,\ell}(t,\tau) \lesssim \frac{\langle \tau \rangle |k-\ell| + |kt - \ell\tau|}{|\ell|} e^{-\delta \langle k-\ell, kt-\ell\tau\rangle^s} e^{(\lambda(t)-\lambda(\tau))\langle k,kt\rangle^s}.$$

The $L^2(dt)$ estimate: reaction term

- After some different fiddling and applying Schur's test, we get:

$$\int_0^T \sum_{k \in \mathbb{Z}^d_*} |AR_k(t)|^2 \, dt \lesssim$$

$$\epsilon^2 \left(\sup_{t \geq 0} \sup_{k \in \mathbb{Z}^d_*} \int_0^t \sum_{\ell \in \mathbb{Z}^d_*} \bar{K}_{k,\ell}(t,\tau) d\tau \right) \left(\sup_{\tau \geq 0} \sup_{\ell \in \mathbb{Z}^d_*} \sum_{k \in \mathbb{Z}^d_*} \int_\tau^T \bar{K}_{k,\ell}(t,\tau) dt \right) \int_0^T \|A\rho(t)\|_2^2 dt,$$

  where (using the low norm control on f) for some $\delta > 0$:

$$\bar{K}_{k,\ell}(t,\tau) \lesssim \frac{\langle \tau \rangle |k - \ell| + |kt - \ell\tau|}{|\ell|} e^{-\delta \langle k-\ell, kt - \ell\tau \rangle^s} e^{(\lambda(t) - \lambda(\tau)) \langle k, kt \rangle^s}.$$

- The game is to now find s and $\lambda(t)$ such that:

$$\int_0^T \sum_{k \in \mathbb{Z}^d_*} |AR_k(t)|^2 \, dt \lesssim \epsilon^2 \int_0^T \|A\rho(t)\|_2^2 dt.$$

Echo estimate in Gevrey class

- Choose a such that $a = \frac{3s-1}{2} > 0$ and write
$$\lambda(t) = \lambda\langle t\rangle^{-a} + \lambda_\infty.$$
- Then, for example, (reducing to 1D case),
$$\begin{aligned}
\int_0^t \sum_\ell \bar{K}_{k,\ell}(t,\tau)d\tau &\lesssim \sum_{\ell>k} \int_{|kt-\ell\tau|<\frac{t}{2}} \frac{\langle\tau\rangle}{|\ell|} e^{-\delta\langle k-\ell, kt-\ell\tau\rangle^s} e^{(\lambda(t)-\lambda(\tau))|kt|^s} d\tau + Easy \\
&\lesssim \sum_{\ell>k} \int_{|kt-\ell\tau|<\frac{t}{2}} \frac{\langle\tau\rangle}{|\ell|} e^{-\delta\langle k-\ell, kt-\ell\tau\rangle^s} e^{-\frac{\delta'}{\ell^{1-a}}|kt|^{s-a}} d\tau + Easy \\
&\lesssim \sum_\ell \frac{kt}{|\ell|^3} e^{-\frac{\delta'}{|\ell|^{1-a}}|kt|^{s-a}} e^{-c\delta\langle k-\ell\rangle^s} + Easy \\
&\lesssim \sum_\ell e^{-c\delta\langle k-\ell\rangle^s} + Easy.
\end{aligned}$$
- We used in the last line that $3(s-a) = 1-a$, which required $s > 1/3$. This inequality is the only place in the proof where Gevrey regularity is important (of any index but also in particular 1/3).

Thanks for your attention!