Strichartz estimates and moment bounds for the Vlasov-Maxwell system

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Vlasov-Maxwell system

The relativistic Vlasov-Maxwell system can then be written as

$$\begin{aligned}
&\partial_t f + \hat{p} \cdot \nabla_x f + (E + \hat{p} \times B) \cdot \nabla_p f = 0, \\
&\partial_t E = \nabla_x \times B - j, \quad \partial_t B = -\nabla_x \times E, \\
&\nabla_x \cdot E = \rho, \quad \nabla_x \cdot B = 0,
\end{aligned}$$

where

$$f : I \times \mathbb{R}^3 \times \mathbb{R}^3 \to R_{\geq 0}, \quad E, B : I \times \mathbb{R}^3 \to \mathbb{R}^3,$$

the charge and current are given by

$$\rho(t,x) \stackrel{\text{def}}{=} 4\pi \int_{\mathbb{R}^3} f(t,x,p)\,dp, \quad j_i(t,x) \stackrel{\text{def}}{=} 4\pi \int_{\mathbb{R}^3} \hat{p}_i f(t,x,p)\,dp.$$

Here,

$$\hat{p} = \frac{p}{p_0}, \quad p_0 = \sqrt{1 + |p|^2}.$$

Some previous results

Global regularity in three dimensions is an open problem. Global regularity is only known in the following cases:

- in symmetry reduced problems: Glassey-Schaeffer (spherically symmetric, 2 dimensions, $2\frac{1}{2}$ dimensions), ...
- in perturbative regime: Glassey-Strauss, Glassey-Schaeffer, Schaeffer, Rein, ...

Singularity only occurs at high velocities

Theorem 1 (Glassey-Strauss, 1984)

Let (f_0, E_0, B_0) *be regular initial data on* $\mathbb{R}^3$ *satisfying the constraints with compact momentum support. Let* (f, E, B) *be the unique classical solution in* $[0, T_*)$*. Let*

$$P(t) := \sup\{|p| : f(s, x, p) \neq 0 \text{ for some } x \in \mathbb{R}^3, s \leq t\}.$$

Assume that

$$\limsup_{t \to T_*} P(t) < \infty.$$

Then, there exists $\epsilon > 0$ *such that the solution extends uniquely and classically beyond* T_* *to an interval* $[0, T_* + \epsilon]$*.*

- See also Bouchut-Golse-Pallard, Klainerman-Staffilani.
- Most of the remainder of the talk will be discussion of quantitative refinements of this result.

Brief summary of proof of the Glassey-Strauss theorem

- It is shown that if the assumptions hold, then the solution remains C^1.
- Given a backwards lightcone, its tangent vectors together with $\partial_t + \hat{p} \cdot \nabla_x$ span the tangent space of $\mathbb{R}^{3+1}$.
- The Maxwell equations imply the following wave equations

$$\Box E = \nabla_x \rho + \partial_t j, \quad \Box B = -\nabla_x \times j,$$

which upon integration by parts and using the Vlasov equation yields, for $|K|^2 = |E|^2 + |B|^2$,

$$|K|(t,x) \lesssim \text{Data} + \int_{C_{t,x}} \frac{|K|(\int_{\mathbb{R}^3} p_0 f dp)(s,y)}{t-s}\, d\sigma_{s,y} + \int_{C_{t,x}} \int_{\mathbb{R}^3} \frac{f(s,y)}{p_0^2 (1+\hat{p}\cdot\omega)^{\frac{3}{2}} (t-s)^2} dp\, d\sigma_{s,y},$$

where the integration is over the backwards light cone of (t,x) and for $(s,y) \in C_{t,x}$, $\omega = \frac{y-x}{t-s}$.

Brief summary of proof of the Glassey-Strauss theorem

- This immediately shows that under the assumptions of the theorem, E and B are bounded.
- An additional argument, using a representation formula for the derivatives of E and B instead, shows that the derivatives of E, B and f are bounded.

Some remarks

- The characteristics satisfy

$$\frac{dX}{ds}(s;t,x,p) = \hat{V}(s;t,x,p),$$

$$\frac{dV}{ds}(s;t) = E(s,X(s;t)) + \hat{V}(s;t) \times B(s,X(s;t)),$$

together with the conditions

$$X(t;t,x,p) = x, \quad V(t;t,x,p) = p,$$

where $\hat{V} \stackrel{\text{def}}{=} \frac{V}{\sqrt{1+|V|^2}}$.

- Equivalently, in order to guarantee that the solution can be continued, it suffices to assume

$$\left\| \int_0^{T_*} |K(s,X(s;0,x,p))|\, ds \right\|_{L^\infty_x L^\infty_p} < \infty,$$

where X are the characteristics, and $|K|^2 := |E|^2 + |B|^2$.

Some remarks

- The condition on the integral of the electromagnetic fields makes sense even if the initial momentum support is unbounded.
- In fact it is not necessary to actually work with solutions with compact initial momentum support, but instead one can assume only sufficiently fast polynomial decay of f in p_0. The condition in the above theorem can be replaced by the condition on the integral of $|K|$ along characteristics.

Improved continuation criterion

Theorem 2 (Glassey-Strauss, 1987, 1989)

Let (f_0, E_0, B_0) be regular initial data on $\mathbb{R}^3$ satisfying the constraints. Let (f, E, B) be the unique classical solution to the Vlasov-Maxwell system in $[0, T_*)$. Assume that

$$\limsup_{t \to T_*} \| \int_{\mathbb{R}^3} p_0 f dp \|_{L^\infty_x}(t) < \infty.$$

Then, there exists $\epsilon > 0$ such that the solution extends uniquely and classically beyond T_* to an interval $[0, T_* + \epsilon]$.

- Note that $\int_{\mathbb{R}^3} p_0 f \, dp$ is the kinetic energy density.
- By the conservation law of the Vlasov-Maxwel system, we have the a priori estimate $\int_{\mathbb{R}^3} p_0 f \, dp \in L^1_x$.

Proof of Glassey-Strauss theorem

Recall

$$|K|(t,x) \lesssim \text{Data} + \int_{C_{t,x}} \frac{|K|(\int_{\mathbb{R}^3} p_0 f dp)(s,y)}{t-s}\, d\sigma_{s,y}$$
$$+ \int_{C_{t,x}} \int_{\mathbb{R}^3} \frac{f(s,y)}{p_0^2(1+\hat{p}\cdot\omega)^{\frac{3}{2}}(t-s)^2} dp\, d\sigma_{s,y}.$$

It suffices to control the L^∞_x norm of $|K|$ under the assumptions of the theorem. Observe that $\frac{1}{(1+\hat{p}\cdot\omega)^{\frac{3}{2}}} \lesssim p_0^3$, and thus

$$|K|(t,x) \leq C + C\int_0^t \sup_y |K(s,y)|\, dy.$$

Gronwall's inequality gives the result.

Pallard's improvement

- The previous proof uses the (local) $L^\infty_{t,x} - L^\infty_{t,x}$ estimates for wave equations in $3+1$-dimensions (and its obvious generalization).
- In order to improve over the previous continuation criterion, one observes the following fact for solutions to the linear wave equation: If $\Box u = F$, then the integral of u over a timelike curve is more regular than u.
- For instance, if $F \in L^1_t L^2_x$, u is no better than $u \in L^\infty_t H^1_x$ and therefore u is in general not bounded. However, it can be shown that

$$\sup_x \int_0^T |u(t,x)|^2 dt \leq C.$$

Pallard's theorem

Theorem 3 (Pallard, 2005)

Let (f_0, E_0, B_0) *be regular initial data on* $\mathbb{R}^3$ *satisfying the constraints with compact momentum support. Let* (f, E, B) *be the unique classical solution to the Vlasov-Maxwell system in* $[0, T_*)$*. Assume that*

$$\limsup_{t \to T_*} \| \int_{\mathbb{R}^3} p_0^\theta f dp \|_{L^q_x}(t) < \infty$$

holds for some $\theta > \frac{4}{q}$, $6 \leq q \leq +\infty$*. Then, there exists* $\epsilon > 0$ *such that the solution extends uniquely and classically beyond* T_* *to an interval* $[0, T_* + \epsilon]$*.*

- The case $\theta = 0$, $q = +\infty$ was later proved by Sospedra–Alfonso-Illner.

Proof of Pallard's theorem I

We will only consider the $q = \infty$ case. For

$$K_1 := \int_{C_{t,x}} \frac{|K|(\int_{\mathbb{R}^3} p_0 f dp)(s,y)}{t-s}\, d\sigma_{s,y},$$

we have

$$\begin{aligned}
&\int_0^{T_*} |K_1(s, X(s))| ds \\
\lesssim &\sup_s (1 + \sqrt{|\log(1 - |X'(s)|)})|| \, ||K|(\int_{\mathbb{R}^3} p_0 f dp)||_{L^\infty_t L^2_x} \\
\lesssim &\sup_{t\in[0,T_*)} P(t)^{1-\epsilon} \log P(t) || \int_{\mathbb{R}^3} p_0^\epsilon f dp||_{L^\infty_t L^\infty_x}.
\end{aligned}$$

Proof of Pallard's theorem II

For
$$K_2 := \int_{C_{t,x}} \int_{\mathbb{R}^3} \frac{f(s,y)}{p_0^2(1+\hat{p}\cdot\omega)^{\frac{3}{2}}(t-s)^2} dp\, d\sigma_{s,y},$$

we have
$$\int_0^{T_*} |K_2(s,X(s))| ds \lesssim \| \int_{\mathbb{R}^3} p_0 f dp \|_{L^\infty_t L^4_x}$$
$$\lesssim \sup_{t\in[0,T_*)} P(t)^{1-\epsilon} \| \| \int_{\mathbb{R}^3} p_0^\epsilon f dp \|_{L^\infty_t L^4_x}.$$

Combining and using the assumption of the theorem, we get
$$\sup_{t\in[0,T_*)} P(t) \lesssim 1+ \int_0^{T_*} |K(s,X(s))| ds \lesssim 1+ \sup_{t\in[0,T_*)} P(t)^{1-\epsilon} \log P(t),$$

which gives the desired result.

Main result

Theorem 4 (L.-Strain)

Let (f_0, E_0, B_0) be regular initial data on $\mathbb{R}^3$ satisfying the constraints. Let (f, E, B) be the unique classical solution to the Vlasov-Maxwell system in $[0, T_*)$. Assume that

$$\limsup_{t \to T_*} \| \int_{\mathbb{R}^3} p_0^\theta f dp \|_{L_x^q}(t) < \infty$$

holds for some $\theta > \frac{2}{q}$, $2 \leq q \leq +\infty$. Then, there exists $\epsilon > 0$ such that the solution extends uniquely and classically beyond T_* to an interval $[0, T_* + \epsilon]$.

- Note that for $q = \infty$, we recover the $q = \infty$ case of Pallard's theorem.
- The result for $q \approx 2$ says that if the kinetic energy density remains in L_x^p for $p > 2$, then the solution can be continued.

Strichartz estimates

There is another way to see that solutions to linear wave equation have better integrability in x after integration in t:

Theorem 5 (Strichartz estimates in $\mathbb{R}^3$)

Let u be a solution to the linear inhomogeneous wave equation in $\mathbb{R}^3$ with zero initial data: Then, the following estimates hold

$$\|u\|_{L_t^{q_1} L_x^{r_1}} \lesssim \|F\|_{L_t^{q_2'} L_x^{r_2'}},$$

where

$$\frac{1}{q_1} + \frac{3}{r_1} = \frac{1}{q_2'} + \frac{3}{r_2'} - 2, \quad \frac{1}{q_1} \leq \frac{1}{2} - \frac{1}{r_1}, \quad \frac{1}{q_2'} \geq \frac{3}{2} - \frac{1}{r_2'},$$

and

$$2 \leq q_1, q_2 \leq \infty, \quad 2 \leq r_1, r_2 < \infty.$$

The end-point Strichartz estimates

Notice that the end-point case $(q_1, r_1, q_2', r_2') = (2, \infty, 1, 2)$ is false! (This is unlike the estimate for $\sup_x \int_0^T |u(t,x)|^2 dt$ previously , where integral is taken before the supremum.) It is however illuminating to see what we can prove **if it were true**.

- Once we control $|K|$ in $L^2_t L^\infty_x$, then we are done (since in particular, after using Cauchy-Schwarz, this controls the integral of $|K|$ over any characteristics).
- Notice that bounding $|K|$ in $L^2_t L^\infty_x$ is stronger than controlling the integral over all characteristics. Proving a stronger bound means that we can potentially use this stronger bound in the proof (say via a Gronwall argument).

We will only consider the $q = 2$ case. We will assume that the data have compact momentum support and try to show that under the assumptions of the theorem, the momentum support remains bounded in $[0, T_*)$.
Consider the term

$$K_1 := \int_{C_{t,x}} \frac{|K|(\int_{\mathbb{R}^3} p_0 f dp)(s, y)}{t - s} \, d\sigma_{s,y}$$

Notice that this is (up to a constant factor) a convolution with the wave kernel. **If the $(2, \infty)$-endpoint Strichartz estimates were true,**

$$\|K_1\|_{L_t^2 L_x^\infty} \lesssim \|K \int_{\mathbb{R}^3} p_0 f dp\|_{L_t^1 L_x^2} \lesssim \|K\|_{L_t^1 L_x^\infty} \| \int_{\mathbb{R}^3} p_0 f dp\|_{L_t^\infty L_x^2}.$$

Estimates for the electromagnetic field II

In the other term, we do not have the wave kernel

$$\int_{C_{t,x}} \int_{\mathbb{R}^3} \frac{f(s,y)}{p_0^2(1+\hat{p}\cdot\omega)^{\frac{3}{2}}(t-s)^2}\, dp\, d\sigma_{s,y},$$

but for every $\epsilon > 0$, we can control this term up to a constant $C = C(\epsilon)$ by

$$\Big(\int_{C_{t,x}} \frac{(\int_{\mathbb{R}^3} p_0^{3+\epsilon} f dp)^{1+\frac{\epsilon}{2}}}{(t-s)}\, d\sigma\Big)^{\frac{1}{2+\epsilon}}.$$

Again, **if the end-point Strichartz estimates were true**, then

$$\|K_2\|_{L^2_t L^\infty_x} \lesssim \|(\int_{\mathbb{R}^3} p_0^{3+\epsilon} f dp)^{1+\frac{\epsilon}{2}}\|^{\frac{1}{2+\epsilon}}_{L^1_t L^2_x} \lesssim \|P\|_{L^{1+}_t} \|\int_{\mathbb{R}^3} p_0^{1+} f dp\|^{\frac{1}{2}}_{L^\infty_t L^{2+}_x},$$

where $P(t)$ is the supremum of the momentum support at time t.

Estimates for the electromagnetic field III

Combining, we get the **if the endpoint Strichartz estimates were true**,

$$\|K\|_{L^2_t L^\infty_x} \lesssim 1 + \|K\|_{L^1_t L^\infty_x} \| \int_{\mathbb{R}^3} p_0 f dp \|_{L^\infty_t L^2_x} + \|P\|_{L^{1+}_t} \| \int_{\mathbb{R}^3} p_0^{1+} f dp \|^{\frac{1}{2}-}_{L^\infty_t L^{2+}_x}.$$

Under the assumption that $\| \int_{\mathbb{R}^3} p_0^{1+} f dp \|_{L^\infty_t L^2_x}$ is bounded, we get

$$\|K\|_{L^2_t L^\infty_x} \lesssim 1 + \|K\|_{L^1_t L^\infty_x} + \|P\|_{L^{1+}_t}.$$

A Gronwall-type argument gives.

$$P(t) \lesssim \|K\|_{L^1_t L^\infty_x} \lesssim \|K\|_{L^2_t L^\infty_x} \lesssim 1 + \|P\|_{L^{1+}_t}$$

and another Gronwall-type argument concludes the "proof".

Moment bounds

- Of course, we have to deal with the issue that the $(2, \infty)$-endpoint Strichartz estimate is false. We can replace it with (q_1, r_1), where $q_1 > 2$ and $r_1 < \infty$ and r_1 can be chosen arbitrarily large.
- On the other hand, controlling the $L^{r_1}_x$ norm is not sufficient to bound the momentum support. One needs to work with a weaker norm.

Lemma 6

For $N > 0$ we have the estimate

$$\|p_0^N f\|^{\frac{1}{N+3}}_{L^\infty_t([0,T);L^1_x L^1_p)} \lesssim \|p_0^N f_0\|^{\frac{1}{N+3}}_{L^1_x L^1_p} + \|K\|_{L^1_t([0,T);L^{N+3}_x)}$$

We will use this lemma for some large N.

Estimates for the electromagnetic field IV

Strichartz estimates gives (without being precise for the Strichartz exponents):

$$\|K\|_{L_t^{2+}([0,T_*);L_x^{\infty-})} \lesssim 1 + \||K| \int_{\mathbb{R}^3} p_0 f dp\|_{L_t^1([0,T_*);L_x^{2+})} + \|(\int_{\mathbb{R}^3} p_0^{3+\epsilon} f dp)^{1+\frac{\epsilon}{2}}\|^{\frac{1}{2+\epsilon}}_{L_t^\infty([0,T_*);L_x^{2+})}.$$

Again, there are two terms to estimate.

Estimates for the electromagnetic field V

For the first term, we have

$$
\begin{aligned}
&\| |K| \int_{\mathbb{R}^3} p_0 f dp \|_{L^1_t([0,T_*);L^{2+}_x)} \\
&\quad \lesssim \|K\|_{L^1_t([0,T_*);L^{\infty--}_x)} \| \int_{\mathbb{R}^3} p_0 f dp \|_{L^1_t([0,T_*);L^{2++}_x)} \\
&\quad \lesssim \|K\|^{1-}_{L^1_t([0,T_*);L^{\infty-}_x)} \| \int_{\mathbb{R}^3} p_0 f dp \|_{L^1_t([0,T_*);L^{2++}_x)} \\
&\quad \lesssim \|K\|^{1-}_{L^1_t([0,T_*);L^{\infty-}_x)} \| \int_{\mathbb{R}^3} p_0^{1+} f dp \|_{L^1_t([0,T_*);L^{2}_x)},
\end{aligned}
$$

where we have used the L^2 conservation law for K.

Estimates for the electromagnetic field VI

First notice that

$$\begin{aligned}
&\|\int_{\mathbb{R}^3} p_0^3 f dp\|_{L^\infty_t([0,T_*);L^2_x)} \\
&\quad\lesssim \|\int_{\mathbb{R}^3} p_0 f dp\|^{\frac{N-5}{N-1}}_{L^\infty_t([0,T_*);L^2_x)} \|\int_{\mathbb{R}^3} p_0^{\frac{N+1}{2}} f dp\|^{\frac{4}{N-1}}_{L^\infty_t([0,T_*);L^2_x)} \\
&\quad\quad\lesssim \|\int_{\mathbb{R}^3} p_0 f dp\|^{\frac{N-5}{N-1}}_{L^\infty_t([0,T_*);L^2_x)} \|\int_{\mathbb{R}^3} p_0^N f dp\|^{\frac{2}{N-1}}_{L^\infty_t([0,T_*);L^1_x)}.
\end{aligned}$$

The second term in the earlier inequality can be estimated as follows:

$$\begin{aligned}
&\|(\int_{\mathbb{R}^3} p_0^{3+\epsilon} f dp)^{1+\epsilon}\|^{\frac{1}{2+\epsilon}}_{L^\infty_t([0,T_*);L^{2+}_x)} \\
&\quad\quad\lesssim \|\int_{\mathbb{R}^3} p_0^{1+} f dp\|^{\beta}_{L^\infty_t([0,T_*);L^2_x)} \|p_0^N f\|^{\frac{\alpha}{N+3}}_{L^\infty_t([0,T_*);L^1_x L^1_p)}
\end{aligned}$$

for any sufficiently large N after choosing $0<\alpha<1$ and $\beta>0$ appropriately.

Applying moment bounds

Combining, and using the assumptions of the theorem, we get

$$\|K\|_{L_t^{2+}([0,T_*);L_x^{N+3})} \lesssim 1+\|K\|^{1-}_{L_t^1([0,T_*);L_x^{N+3})}+\|p_0^N f\|^{\frac{\alpha}{N+3}}_{L_t^\infty([0,T_*);L_x^1 L_p^1)},$$

which implies

$$\|K\|_{L_t^{2+}([0,T_*);L_x^{N+3})} \lesssim 1 + \|p_0^N f\|^{\frac{\alpha}{N+3}}_{L_t^\infty([0,T_*);L_x^1 L_p^1)}.$$

The moment bounds then imply that for some $\alpha \in (0,1)$,

$$\|p_0^N f\|_{L_t^\infty([0,T_*);L_x^1 L_p^1)} \lesssim 1 + \|p_0^N f\|^{\alpha}_{L_t^\infty([0,T_*);L_x^1 L_p^1)}.$$

This implies the boundedness of the moments and the Strichartz norm for K.

Conclusion of the proof

By choosing N sufficiently large, the argument above shows that $|K|$ and $\int_{\mathbb{R}^3} p_0 f dp$ are in L^p_x for some very large p. This does not immediately imply the boundedness of $|K|$, but arguing as in Pallard's theorem, this controls the integral of $|K|$ over characteristics.

$$\begin{aligned}&\| \int_0^{T_*} |K(s, X(s;t,x,p))| ds \|_{L^\infty_t([0,T_*);L^\infty_x L^\infty_p)} \\ &\qquad\qquad \lesssim \|K f p_0\|_{L^1_t([0,T_*);L^4_x L^1_p)} + \|f p_0\|_{L^1_t([0,T_*);L^4_x L^1_p)}.\end{aligned}$$

The proof can be concluded using the Glassey-Strauss result.

Remarks

- Since we control the moments of f, it is not necessary that the initial data for f have compact momentum support.
- On the other hand, the proof requires that the initial data satisfy
$$\|f_0 p_0^N\|_{L^1_x L^1_p} \leq C_N < \infty, \quad \text{for all } N.$$
- See also recent works of Pallard, Kunze.

Epilogue: The 2- and $2\frac{1}{2}$-D case

In 2- and $2\frac{1}{2}$-dimensions, global regularity is known for regular (large) initial data with compact initial momentum support.

Theorem 7 (Glassey-Schaeffer, 1997, 1998)

Let (f_0, E_0, B_0) be regular initial data satisfying the constraints with compact momentum support for the 2D or $2\frac{1}{2}D$ Vlasov-Maxwell system. Then there exists a unique global-in-time solution.

Epilogue: The 2- and $2\frac{1}{2}$-D case

Using moment estimates instead of bounds for the momentum support, we also obtain the following global regularity in 2D and $2\frac{1}{2}$D without the assumption on compact initial momentum support:

Theorem 8 (L.-Strain)

Let (f_0, E_0, B_0) be regular initial data satisfying the constraints for the 2D or $2\frac{1}{2}$D Vlasov-Maxwell system and verifying

$$|f_0(x,p)| \leq C p_0^{-(16+\epsilon)},$$

and

$$|\nabla_{x,p} f_0(x,p)| \leq C p_0^{-6} \log^{-2}(1+p_0)$$

for some $\epsilon > 0$. Then there exists a unique global-in-time solution.

Outline of proof

- Consider the 2D case. The first step is the bound the electromagnetic field in terms of solutions to homogeneous wave equation. For every small $\epsilon > 0$,

$$|K| \lesssim \text{Data} + \Box^{-1}\left(|K| \int_{\mathbb{R}^2} \frac{f}{p_0} dp\right)$$
$$+ \epsilon^{-\frac{1}{10}} \left(\Box^{-1}\left(\int_{\mathbb{R}^2} p_0^2 f dp\right)\right)^{\frac{2}{5}} + \epsilon^{\frac{3}{10}} \left(\Box^{-1}\left(\int_{\mathbb{R}^2} p_0^4 f dp\right)\right)^{\frac{2}{5}}.$$

- This can be achieved based on the Glassey-Schaeffer work and estimating the kernel separately near and away from the light cone.
- Then apply Strichartz estimates and moment bounds in a similar way as in the 3D case.
- Need to use the improved Strichartz estimates for inhomogeneous wave equations by Foschi.

Thank you!