**Theorem 7.8:** Let $f$ be continuous on an interval $I \subseteq \mathbb{R}$. Then

$$f \text{ is 1-1 on } I \qquad \Leftrightarrow \qquad f \text{ is strictly monotone on } I.$$

*Proof.* The direction $\Leftarrow$ was already proven in Proposition 2.1.

Hence we only need to prove the implication $\Rightarrow$. Recall that a 1-1 function is invertible, so that it is enough to prove

$$f \text{ is invertible on } I \qquad \Rightarrow \qquad f \text{ is strictly monotone on } I.$$

Let $x_1 < x_2$ be points in $I$. We'll show that if $f(x_1) < f(x_2)$ then $f$ is strictly increasing on $I$. Let $z_1, z_2$ be two points such that

$$x_1 < z_1 < z_2 < x_2.$$

We want to show that $f(z_1) < f(z_2)$.

Using Lemma 7.7 for the trio $x_1 < z_1 < x_2$, we find that

$$f(x_1) < f(z_1) < f(x_2).$$

Using Lemma 7.7 for the trio $z_1 < z_2 < x_2$, we find that

$$f(z_1) < f(z_2) < f(x_2).$$

In particular, we have found that $z_1 < z_2$ implies $f(z_1) < f(z_2)$, so $f$ is strictly increasing.

The strictly decreasing case ($f(x_1) > f(x_2)$) follows the same idea of proof, which completes the proof. □

**Theorem 7.9:** Let $f$ be continuous and invertible on an interval $I$. Then $f^{-1}$ is continuous on the interval $J = f(I)$.

*Proof.* We skip the proof. □

## 7.4 Lipschitz and uniformly continuous functions

**Lipschitz functions**

A function $f : \mathbb{R} \to \mathbb{R}$ is **Lipschitz on an interval** $I \subseteq \mathbb{R}$ if there exists a constant $L \geq 0$ such that

$$|f(x_1) - f(x_2)| \leq L|x_1 - x_2| \qquad \forall x_1, x_2 \in I.$$

For a Lipschitz function on $I$, the smallest $L$ that satisfies this inequality is called the **Lipschitz constant** of $f$ on $I$.

**Example 7.9:** The function $f(x) = x$ is Lipschitz on $\mathbb{R}$ with Lipschitz constant 1: for any $x_1, x_2 \in \mathbb{R}$,

$$|f(x_1) - f(x_2)| = |x_1 - x_2|.$$

**Example 7.10:** The function $f(x) = x^2$ is Lipschitz on the interval $[0, 2]$ with Lipschitz constant 4. Indeed, for any $x_1, x_2 \in [0, 2]$,

$$|f(x_1) - f(x_2)| = |x_1^2 - x_2^2| = |x_1 - x_2| \cdot \underbrace{|x_1 + x_2|}_{\leq 2+2=4} \leq 4|x_1 - x_2|.$$

**Example 7.11:** The function $f(x) = \sqrt{x}$ is not Lipschitz on the interval $[0, 1]$. This can be shown by taking $x_1 = 0$ and any other $x_2 \in (0, 1]$:

$$|f(0) - f(x_2)| = |\sqrt{0} - \sqrt{x_2}| = \sqrt{x_2} = \frac{1}{\sqrt{x_2}} x_2 = \frac{1}{\sqrt{x_2}} |0 - x_2|$$

As $x_2 \to 0^+$, the coefficient $\frac{1}{\sqrt{x_2}}$ tends to $+\infty$, so there's no finite $L$ satisfying the conditions of a Lipschitz function.

However, $f(x) = \sqrt{x}$ *is* Lipschitz on any interval whose lower bound is greater than 0. For instance, consider $I = (\frac{1}{2}, 3]$, then for any $x_1, x_2 \in I$:

$$|f(x_1) - f(x_2)| = |\sqrt{x_1} - \sqrt{x_2}| = \frac{1}{\sqrt{x_1} + \sqrt{x_2}} |x_1 - x_2| \leq \frac{1}{\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{2}}} |x_1 - x_2| = \frac{\sqrt{2}}{2} |x_1 - x_2|.$$

**Example 7.12:** The function $f(x) = \sin x$ is Lipschitz on $\mathbb{R}$ with constant $L = 1$ (which means that the Lipschitz constant is *at most* 1). Indeed, we have

$$\begin{aligned} |\sin x_1 - \sin x_2| &= 2 \left| \sin \frac{x_1 - x_2}{2} \right| \left| \cos \frac{x_1 + x_2}{2} \right| \\ &\leq 2 \left| \sin \frac{x_1 - x_2}{2} \right| \\ &\leq |x_1 - x_2|. \end{aligned}$$

**Proposition 7.10:** If $f$ is Lipschitz on $I$, then it is also continuous on $I$.

*Proof.* Fix some point $x_0 \in I$. Since $f$ is Lipschitz, there exists some $L \geq 0$ such that

$$|f(x_0) - f(x)| \leq L|x_0 - x|, \qquad \forall x \in I.$$

Let $\varepsilon > 0$ and define $\delta = \frac{\varepsilon}{L}$. Then whenever $|x_0 - x| < \delta$ we have

$$|f(x_0) - f(x)| \leq L|x_0 - x| < L\delta = \varepsilon$$

which proves that $f$ is continuous at $x_0$. Since $x_0$ was arbitrary, $f$ is continuous on $I$. □

**Uniform continuity**

A function $f : \mathbb{R} \to \mathbb{R}$ is **uniformly continuous on an interval** $I \subseteq \mathbb{R}$ if for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\forall x_1, x_2 \in I, \qquad |x_1 - x_2| < \delta \quad \Rightarrow \quad |f(x_1) - f(x_2)| < \varepsilon.$$

The crucial aspect in the above definition is that the same $\delta(\varepsilon)$ works for the *entire interval* $I$, not just at a single point. Recall that the definition of continuity held only at a given point $x_0$!

**Example 7.13:** The function $f(x) = \frac{1}{x}$ is *not* uniformly continuous on $(0, 1)$: as $x \to 0^+$, compare, for instance $x$ and $\sqrt{x}$:

$$\lim_{x \to 0^+} |x - \sqrt{x}| = 0$$

while

$$\lim_{x \to 0^+} |f(x) - f(\sqrt{x})| = \lim_{x \to 0^+} \left| \frac{1}{x} - \frac{1}{\sqrt{x}} \right| = \lim_{x \to 0^+} \left| \frac{1 - \sqrt{x}}{x} \right| = +\infty.$$

So, points that become arbitrarily close on the $x$-axis, have their images become arbitrarily distant on the $y$-axis.

However, $f(x) = \frac{1}{x}$ is uniformly continuous on any interval further away from 0. For instance, consider $I = [a, +\infty)$ where $a > 0$. Then

$$|f(x_1) - f(x_2)| = \left| \frac{1}{x_1} - \frac{1}{x_2} \right| = \frac{|x_1 - x_2|}{x_1 x_2} \leq \frac{1}{a^2} |x_1 - x_2|.$$

Hence for any $\varepsilon > 0$ we can take $\delta = a^2 \varepsilon$ to get that if $|x_1 - x_2| < \delta$ then $|f(x_1) - f(x_2)| \leq \frac{1}{a^2}|x_1 - x_2| < \frac{1}{a^2}\delta = \frac{1}{a^2}a^2\varepsilon = \varepsilon$.

**Example 7.14:** The function $f(x) = \sin(\frac{1}{x})$ is also *not* uniformly continuous on $(0, 1)$. Indeed, as $x \to 0^+$ there are points that are arbitrarily close on the $x$-axis, while their distance on the $y$-axis is 2 (we're thinking here about points where the sine function achieves the values $\pm 1$).

**Theorem 7.11 (Heine-Cantor Theorem):** If $f$ is continuous on a closed interval $I = [a, b]$ then is it uniformly continuous there.

*Proof.* We prove *by contradiction*. If the claim is not true, then $f$ is not uniformly continuous on $I$. That is, $\exists \varepsilon > 0$ such that $\forall \delta > 0$, $\exists x', x'' \in I$, such that $|x' - x''| < \delta$ and $|f(x') - f(x'') \geq \varepsilon$.

So $\varepsilon$ is fixed, but we can choose $\delta$. Let $\delta = \frac{1}{n}$, $n \in \mathbb{N}_+$. Then for each $n$ we have

$$|x'_n - x''_n| < \frac{1}{n} \qquad \text{and} \qquad |f(x'_n) - f(x''_n)| \geq \varepsilon.$$

We immediately observe that

$$\lim_{n \to \infty} |x'_n - x''_n| = 0,$$

while the corresponding function values will always remain at least $\varepsilon$ apart, thus cannot converge to 0. Both sequences $\{x'_n\}_{n \in \mathbb{N}_+}$ and $\{x''_n\}_{n \in \mathbb{N}_+}$ are bounded, so the Bolzano-Weierstrass theorem applies.

Let's start with the sequence $\{x'_n\}_{n \in \mathbb{N}_+}$: it is bounded, and therefore there is a convergent subsequence $\{x'_{n_k}\}_{k \in \mathbb{N}_+}$. Call its limit $\overline{x}$:

$$\lim_{k \to \infty} x'_{n_k} = \overline{x}.$$

We note that since the interval $[a, b]$ is closed, the limit point $\bar{x}$ must also belong to $[a, b]$.

Now we consider the sequence $\{x''_n\}_{n\in\mathbb{N}_+}$: it is also bounded, but we need to be cautious about working with it. We know that $|x'_n - x''_n| < \frac{1}{n}$, where it is important that both elements have the same index $n$. For the first sequence we already took a subsequence with index $\{n_k\}_{k\in\mathbb{N}_+}$. This forces us to only consider the elements of $\{x''_n\}_{n\in\mathbb{N}_+}$ that also have the indices $\{n_k\}_{k\in\mathbb{N}_+}$. Hence we should start with the subsequence $\{x''_{n_k}\}_{k\in\mathbb{N}_+}$. This is again a bounded sequence, so it has a convergent subsequence $\{x''_{n_{k_j}}\}_{j\in\mathbb{N}_+}$. Let us show that its limit must also be $\bar{x}$. First, we summarize all that we have established so far:

$$
\begin{aligned}
&1) \lim_{n\to\infty} |x'_n - x''_n| = 0,\\
&2) \lim_{k\to\infty} x'_{n_k} = \bar{x},\\
&3) \lim_{j\to\infty} x''_{n_{k_j}} \text{ exists.}
\end{aligned}
$$

So we have:

$$
\begin{aligned}
\lim_{j\to\infty}\left(x''_{n_{k_j}} - \bar{x}\right) &= \lim_{j\to\infty}\left(x''_{n_{k_j}} - x'_{n_{k_j}} + x'_{n_{k_j}} - \bar{x}\right)\\
&= \lim_{j\to\infty}\left(x''_{n_{k_j}} - x'_{n_{k_j}}\right) + \lim_{j\to\infty}\left(x'_{n_{k_j}} - \bar{x}\right) = 0
\end{aligned}
$$

So, indeed,

$$\lim_{j\to\infty} x''_{n_{k_j}} = \bar{x}.$$

The function $f$ is continuous on $I$, so in particular it is continuous at $\bar{x} \in I$. This means that:

$$\lim_{j\to\infty} f(x'_{n_{k_j}}) = f(\lim_{j\to\infty} x'_{n_{k_j}}) = f(\bar{x}) = f(\lim_{j\to\infty} x''_{n_{k_j}}) = \lim_{j\to\infty} f(x''_{n_{k_j}}).$$

So we get that

$$\lim_{j\to\infty}\left(f(x'_{n_{k_j}}) - f(x''_{n_{k_j}})\right) = 0,$$

in contradiction to the assumption that

$$|f(x'_n) - f(x''_n)| \geq \varepsilon > 0, \qquad \forall n \in \mathbb{N}_+.$$

□

**Example 7.15:** The function $f(x) = \sqrt{x}$ is uniformly continuous on $[0, 1]$ due to the Heine-Cantor theorem. However, as we saw in Example 7.11 it is not Lipschitz on $[0, 1]$.