Since f is continuous, the Integral Mean Value Theorem (Theorem 10.13) implies that there exists z between x and $x + \Delta x$ such that $m(f; x, x + \Delta x) = f(z)$. So we have

$$\frac{F(x + \Delta x) - F(x)}{\Delta x} = f(z).$$

Observe that z depends on the choice of Δx, so we should write $z = z(\Delta x)$. Necessarily

$$\lim_{\Delta x \to 0} z(\Delta x) = x$$

(by the Squeeze Theorem). Since f is continuous, we have

$$\lim_{\Delta x \to 0} f(z(\Delta x)) = f(\lim_{\Delta x \to 0} z(\Delta x)) = f(x).$$

Hence we have:

$$F'(x) = \lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x} = \lim_{\Delta x \to 0} f(z(\Delta x)) = f(x).$$

This proves the theorem for the case where x lies in the interior of I. If x is on the boundary then we must take one-sided limits, but the details are very similar. □

This theorem tells us how to define a (specific) antiderivative (depending on our choice of x_0): $F(x) = \int_{x_0}^{x} f(y)\,\mathrm{d}y$. Now we can state a result that links this to any other antiderivative:

Corollary 10.14: If we define $F_{x_0}(x) = \int_{x_0}^{x} f(y)\,\mathrm{d}y$, then

$$F_{x_0}(x) = G(x) - G(x_0)$$

for any other G which is an antiderivative of f.

Proof. The proof is immediate by plugging in $x = x_0$ in the above expression, since we know that all antiderivatives differ by a constant □

Important corollary

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $[a, b]$ and let G be any antiderivative. Then

$$\int_a^b f(x)\,\mathrm{d}x = G(b) - G(a).$$

Proof. Let F_a be the antiderivative defined with the choice $x_0 = a$. Then

$$\int_a^b f(x)\,\mathrm{d}x = F_a(b).$$

By Corollary 10.14 we then further have

$$\int_a^b f(x)\,\mathrm{d}x = F_a(b) = G(b) - G(a)$$

for any antiderivative G. □

Notation

Instead of $G(b) - G(a)$, we often write

$$\Big[G(x)\Big]_a^b \qquad \text{or} \qquad G(x)\Big|_a^b$$

Example 10.11:

$$\int_0^1 x^2\,\mathrm{d}x = \left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3}\cdot 1^3 - \frac{1}{3}\cdot 0^3 = \frac{1}{3}$$

$$\int_\pi^{2\pi} \sin x\,\mathrm{d}x = [-\cos x]_\pi^{2\pi} = -\cos 2\pi - (-\cos\pi) = -1 - 1 = -2$$

$$\int_2^6 \frac{1}{x}\,\mathrm{d}x = [\ln x]_2^6 = \ln 6 - \ln 2 = \ln 3$$

$$\int_{-1}^1 e^{2x}\,\mathrm{d}x = \left[\frac{1}{2}e^{2x}\right]_{-1}^1 = \frac{1}{2}e^2 - \frac{1}{2}e^{-2} = \frac{e^2 - e^{-2}}{2} = \sinh(2)$$

$$\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x = [\arctan x]_0^1 = \arctan 1 - \arctan 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

$$\int_0^4 \sqrt{x}\,\mathrm{d}x = \left[\frac{2}{3}x^{3/2}\right]_0^4 = \frac{2}{3}(4^{3/2}) - \frac{2}{3}(0) = \frac{2}{3}(8) = \frac{16}{3}$$

$$\int_1^3 \frac{x}{x^2+1}\,\mathrm{d}x = \left[\frac{1}{2}\ln(x^2+1)\right]_1^3 = \frac{1}{2}\ln(10) - \frac{1}{2}\ln(2) = \frac{1}{2}\ln(5)$$

$$\int_0^\pi \cos x\,\mathrm{d}x = [\sin x]_0^\pi = \sin\pi - \sin 0 = 0 - 0 = 0$$

Corollary 10.15: If $f \in C^1(I)$ (that is, f is differentiable on I and f' is continuous on I), then for any $x_0 \in I$,

$$f(x) = f(x_0) + \int_{x_0}^x f'(y)\,\mathrm{d}y.$$

Proof. This follows immediately from the previous corollary, and the proof is left as an exercise. ◻

Application to Maclaurin expansions

The following lemma will help us express the Maclaurin polynomials of some functions whose derivative is easy to study.

Lemma 10.16: Integration increases by 1 the order of decay of an infinitesimal function:

$$\int_0^x o(y^\alpha)\,\mathrm{d}y = o(x^{\alpha+1}), \qquad \text{as } x \to 0.$$

Proof. We skip this proof. ◻

The function $f(x) = \arctan x$.

We recall that

$$\frac{\mathrm{d}}{\mathrm{d}x}\arctan(x) = \frac{1}{1+x^2}$$

so that

$$\arctan x = \int_0^x \frac{1}{1+y^2}\,\mathrm{d}y.$$

The Maclaurin polynomial of $\frac{1}{1+y^2}$ is known to be (see Section 9.2):

$$\frac{1}{1+y^2} = 1 - y^2 + y^4 - y^6 - \cdots + (-1)^n y^{2n} + o(y^{2n+1}) = \sum_{k=0}^{n}(-1)^k y^{2k} + o(y^{2n+1}).$$

Combining these two facts, and using the lemma, we find that as $x \to 0$,

$$\boxed{\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + o(x^{2n+2}) = \sum_{k=0}^{n}(-1)^k \frac{x^{2k+1}}{2k+1} + o(x^{2n+2})}$$

The function $f(x) = \arcsin x$.

We recall that

$$\frac{\mathrm{d}}{\mathrm{d}x}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$$

so that

$$\arcsin x = \int_0^x \frac{1}{\sqrt{1-y^2}}\,\mathrm{d}y.$$

The Maclaurin polynomial of $\frac{1}{\sqrt{1-y^2}}$ is known to be (see Section 9.2):

$$\frac{1}{\sqrt{1-y^2}} = 1 + \frac{1}{2}y^2 + \frac{3}{8}y^4 + \frac{5}{16}y^6 + \cdots + \left|\binom{-\frac{1}{2}}{n}\right| y^{2n} + o(y^{2n+1}) = \sum_{k=0}^{n}\left|\binom{-\frac{1}{2}}{k}\right| y^{2k} + o(y^{2n+1}).$$

Combining these two facts, and using the lemma, we find that as $x \to 0$,

$$\boxed{\begin{aligned}\arcsin x &= x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \cdots + \left|\binom{-\frac{1}{2}}{n}\right| \frac{x^{2n+1}}{2n+1} + o(x^{2n+2}) \\ &= \sum_{k=0}^{n}\left|\binom{-\frac{1}{2}}{k}\right| \frac{x^{2k+1}}{2k+1} + o(x^{2n+2})\end{aligned}}$$

Application to the remainder of a Taylor expansion

We have seen Peano's remainder:

$$f(x) - (Tf)_{n,x_0}(x) = o((x-x_0)^n), \qquad x \to x_0.$$

and Lagrange's remainder:

$$f(x) - (Tf)_{n,x_0}(x) = \frac{1}{(n+1)!} f^{(n+1)}(\overline{x})(x-x_0)^{n+1},$$

where $\overline{x}$ is some point between x and x_0. These lead us to:

Taylor formula with integral remainder

Let $n \in \mathbb{N}$ and suppose that $f \in C^{n+1}(I)$ in some neighborhood I of x_0. Then

$$f(x) - (Tf)_{n,x_0}(x) = \frac{1}{n!} \int_{x_0}^{x} f^{(n+1)}(y)(x-y)^n \, \mathrm{d}y.$$

Proof. The proof, which we skip here, relies on induction. □

Remark: Observe that for $n = 0$ this result is precisely Corollary 10.15:

$$f(x) - f(x_0) = \int_{x_0}^{x} f'(y) \, \mathrm{d}y.$$

Example 10.12: Let us compare approximation of the number e using Lagrange's remainder and the integral remainder. Taking order $n = 1$, we have

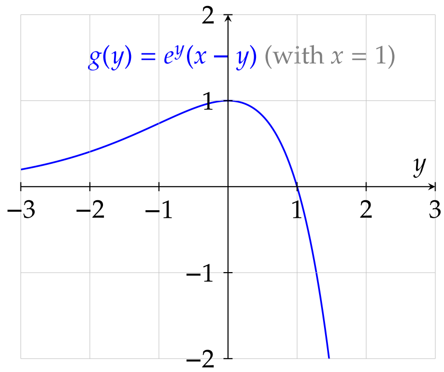
$$\text{Lagrange:} \qquad e^x = 1 + x + \frac{1}{2} e^{\overline{x}} x^2,$$

$$\text{Integral:} \qquad e^x = 1 + x + \int_0^x e^y (x-y) \, \mathrm{d}y.$$

Lagrange: Since the exponential function is strictly increasing, we can deduce that

$$0 < e^x - (1+x) = \frac{1}{2} e^{\overline{x}} x^2 < \frac{1}{2} e^x x^2$$

Integral: consider the integrand $g(y) = e^y(x-y)$. We have $g'(y) = e^y(x-y-1)$. Searching for extrema for $x \geq 1$ we impose $g'(y) = 0$ to find $y = x-1$. For $y < x-1$, $g'(y) > 0$ and for $y > x-1$, $g'(y) < 0$. Hence $y = x-1$ is a global maximum.



Therefore:

$$0 < \int_0^x e^y(x-y)\,\mathrm{d}y < e^{x-1}\int_0^x \mathrm{d}y = e^{x-1}x = \frac{1}{e}e^x x$$

and it follows that

$$0 < e^x - (1+x) = \int_0^x e^y(x-y)\,\mathrm{d}y < \frac{1}{e}e^x x, \quad \forall x \geq 1.$$

Comparing the two bounds we've obtained, we have

$$\begin{aligned} \text{Lagrange:} \quad & 0 < e^x - (1+x) < \frac{1}{2}e^x x^2 \\ \text{Integral:} \quad & 0 < e^x - (1+x) < \frac{1}{e}e^x x \end{aligned}$$

We see that for $x \geq 1$, the error with the integral remainder is smaller, since $\frac{1}{e} < \frac{1}{2}$ and $x \leq x^2$.

10.9 Rules of definite integration

Even and odd functions

Proposition 10.17: Let f be integrable on the interval $[-a, a]$ (where $a > 0$). Then

$$\begin{aligned} f \text{ even} \quad &\Rightarrow \quad \int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_0^a f(x)\,\mathrm{d}x, \\ f \text{ odd} \quad &\Rightarrow \quad \int_{-a}^{a} f(x)\,\mathrm{d}x = 0. \end{aligned}$$

Proof. This is left as an exercise. □

Integration by parts and by substitution

We now want to write the formulas for integration by parts and for integration by substitution for definite integrals:

Integration by parts (definite integrals)

Let $f, g \in C^1([a,b])$. Then

$$\int_a^b f(x)g'(x)\,\mathrm{d}x = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)\,\mathrm{d}x.$$

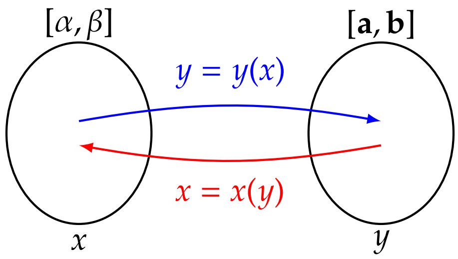
Integration by substitution (definite integrals)

Let $f(y)$ be continuous on an interval $[a,b]$ and let $y(x) : [\alpha,\beta] \to [a,b]$ belong to $C^1([\alpha,\beta])$. Then

$$\int_{\alpha}^{\beta} f(y(x))y'(x)\,\mathrm{d}x = \int_{y(\alpha)}^{y(\beta)} f(y)\,\mathrm{d}y.$$

If $y(x)$ is 1-1 then we also have

$$\int_{a}^{b} f(y)\,\mathrm{d}y = \int_{y^{-1}(a)}^{y^{-1}(b)} f(y(x))y'(x)\,\mathrm{d}x.$$



Remembering how to integrate by substitution

When asked to evaluate the definite integral $\int_{\alpha}^{\beta} g(x)\,\mathrm{d}x$ there are two options:

1. We are able identify that there exists $y = y(x)$ such that $g(x)$ has the form

   $$g(x) = f(y(x))y'(x).$$

   In this case, since $\frac{\mathrm{d}y}{\mathrm{d}x} = y'(x)$ we write $\mathrm{d}y = y'(x)\,\mathrm{d}x$ to get

   $$\int_{\alpha}^{\beta} g(x)\,\mathrm{d}x = \int_{\alpha}^{\beta} f(y(x))y'(x)\,\mathrm{d}x = \int_{a}^{b} f(y)\,\mathrm{d}y.$$

   This approach might work if g is a complicated function.

2. If we cannot identify $y = y(x)$ as above, we try to go about it the other way around: identify x as a function of y: $x = x(y)$, compute $\frac{\mathrm{d}x}{\mathrm{d}y} = x'(y)$ and write $\mathrm{d}x = x'(y)\,\mathrm{d}y$ to get:

   $$\int_{\alpha}^{\beta} g(x)\,\mathrm{d}x = \int_{a}^{b} g(x(y))x'(y)\,\mathrm{d}y.$$

   This approach might work if g is a simple function. **In this second case, we need to be sure to check that $x = x(y)$ is invertible. This means that $x(y)$ must be strictly monotone on $[a,b]$.**