MATHEMATICAL ANALYSIS 1
HOMEWORK 8

(1) Recall the translation, rescaling and reflection functions defined on $\mathbb{R}$ (with $c > 0$):

$$t_c(x) = x + c \qquad s_c(x) = cx \qquad r(x) = -x.$$

For a function $f : \mathbb{R} \to \mathbb{R}$ that is differentiable for all $x \in \mathbb{R}$, differentiate with respect to x the following functions:

(a) $(f \circ r)(x)$ (c) $(s_c \circ f)(x)$ (e) $(f \circ t_c)(x)$ (g) $(f \circ (s_c \circ r))(x)$
(b) $(r \circ f)(x)$ (d) $(f \circ s_c)(x)$ (f) $(t_c \circ f)(x)$ (h) $(f \circ (r \circ t_c))(x)$

(2) Differentiate with respect to x the following functions (wherever they are differentiable):

(a) $f(x) = 3x\sqrt[3]{1 + x^2}$ (c) $f(x) = \cos(e^{x^2+1})$ (e) $f(x) = \arcsin\sqrt{x}$
(b) $f(x) = x \ln x$ (d) $f(x) = -\frac{1}{(4x-1)^3}$ (f) $f(x) = x\tan(x^3)$

(3) Write the equation of the tangent line at x_0 to the graph of the following functions:

(a) $f(x) = \ln(3x - 2), \quad x_0 = 2$ (c) $f(x) = e^{\sqrt{2x+1}}, \quad x_0 = 0$
(b) $f(x) = \frac{x}{1+x^2}, \quad x_0 = 1$ (d) $f(x) = \sin\frac{1}{x}, \quad x_0 = \frac{1}{\pi}$

(4) Verify that $f(x) = 2x^5 + x^3 + 5x$ is invertible on $\mathbb{R}$ and that f^{-1} is differentiable on $\mathbb{R}$. Compute $(f^{-1})'(y_0)$ at $y_0 = 0$ and $y_0 = 8$.

(5) Find the maximum and minimum of the following functions on the given interval:

(a) $f(x) = \sin x + \cos x, \quad [0, 2\pi]$
(b) $f(x) = x^2 - |x + 1| - 2, \quad [-2, 1]$

(6) Verify that $f(x) = \ln(2 + x) + 2\frac{x+1}{x+2}$ vanishes only at $x_0 = -1$.

(7) Determine the number of zeroes and critical points of

$$f(x) = \frac{x \ln x - 1}{x^2}.$$

(1) (a) $(f \circ r)'(x) = -f'(-x)$
(b) $(r \circ f)'(x) = -f'(x)$
(c) $(s_c \circ f)'(x) = cf'(x)$
(d) $(f \circ s_c)'(x) = cf'(cx)$
(e) $(f \circ t_c)'(x) = f'(x+c)$
(f) $(t_c \circ f)'(x) = f'(x)$
(g) $(f \circ (s_c \circ r))'(x) = -cf'(-cx)$
(h) $(f \circ (r \circ t_c))'(x) = -f'(-(x+c))$

(2) (a) $f'(x) = 3\sqrt[3]{1+x^2} + \frac{2x^2}{\sqrt[3]{(1+x^2)^2}}$
(b) $f'(x) = \ln x + 1$
(c) $f'(x) = -2xe^{x^2+1}\sin(e^{x^2+1})$
(d) $f'(x) = \frac{12}{(4x-1)^4}$
(e) $f'(x) = \frac{1}{2\sqrt{x(1-x)}}$
(f) $f'(x) = \tan(x^3) + 3x^3(1+\tan^2(x^3))$

(3) (a) $y = \frac{3}{4}x - \frac{3}{2} + \ln 4$
(b) $y = \frac{1}{2}$
(c) $y = ex + e$
(d) $y = \pi^2 x - \pi$

(4) $f'(x) = 10x^4 + 3x^2 + 5 > 0$ for all x, so f is strictly increasing and invertible. Since $f(0) = 0$ and $f(1) = 8$, the inverse satisfies $f^{-1}(0) = 0$, $f^{-1}(8) = 1$. By the inverse function theorem:

$$(f^{-1})'(0) = \frac{1}{f'(0)} = \frac{1}{5}, \qquad (f^{-1})'(8) = \frac{1}{f'(1)} = \frac{1}{18}.$$

(5) (a) The function satisfies

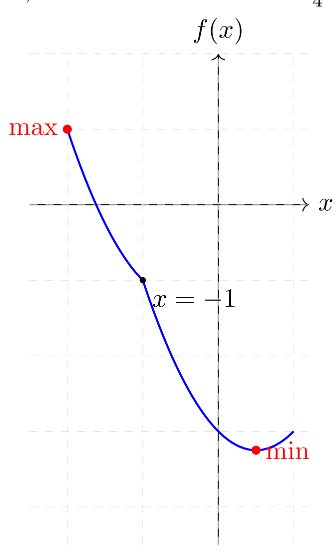
$$f'(x) = \cos x - \sin x = \sqrt{2}\cos\left(x + \frac{\pi}{4}\right),$$

so the maximum equals $\sqrt{2}$ at $x = \frac{\pi}{4}$ and the minimum equals $-\sqrt{2}$ at $x = \frac{5\pi}{4}$.

(b) For $x < -1$, $f(x) = x^2 + x - 1$; for $x \geq -1$, $f(x) = x^2 - x - 3$. The only critical point occurs at $x = \frac{1}{2}$. Evaluating the function:

$$f(-2) = 1, \quad f(-1) = -1, \quad f\left(\tfrac{1}{2}\right) = -\tfrac{13}{4}, \quad f(1) = -3.$$

Thus the maximum is 1 at $x = -2$, and the minimum is $-\frac{13}{4}$ at $x = \frac{1}{2}$.



(6) Domain: $x > -2$.

$$f'(x) = \frac{1}{x+2} + \frac{2}{(x+2)^2} = \frac{x+4}{(x+2)^2} > 0 \quad \text{for all } x > -2.$$

Hence f is strictly increasing on $(-2, \infty)$. Since $f(-1) = 0$, the function has a unique zero at $x_0 = -1$.

(7) Consider

$$f(x) = \frac{x \ln x - 1}{x^2}, \quad x > 0.$$

(a) **Zeroes:** Solve $f(x) = 0$:

$$\frac{x \ln x - 1}{x^2} = 0 \quad \Longrightarrow \quad x \ln x = 1.$$

The function $x \ln x$ is strictly increasing for $x > 1$, and $1 \ln 1 = 0 < 1$, $e \ln e = e > 1$. Hence there is exactly one solution $x_0 \in (1, e)$.

(b) **Critical points:** Compute the derivative

$$f'(x) = \frac{-x \ln x + x + 2}{x^3}.$$

It vanishes when the numerator

$$h(x) = -x \ln x + x + 2$$

is zero.

Notice that $h'(x) = -\ln x$:

- For $x \in (0, 1)$, $\ln x < 0 \implies h'(x) > 0$ (increasing),
- For $x \in (1, \infty)$, $\ln x > 0 \implies h'(x) < 0$ (decreasing).

Evaluate h at two points to locate the zero:

$$h(e) = -e \cdot 1 + e + 2 = 2 > 0, \qquad h(e^2) = -e^2 \cdot 2 + e^2 + 2 = -e^2 + 2 < 0.$$

Since $h(x)$ is continuous and strictly decreasing on $(1, \infty)$, there is exactly one $x_c \in (e, e^2)$ where $h(x_c) = 0$. Hence $f'(x_c) = 0$, giving a unique critical point in (e, e^2).