Chapter 12

Ordinary differential equations

We are well acquainted with equations of a single variable, such as

$$x^2 - 1 = 0$$

whose solutions are $x = \pm 1$, as well as equations involving two variables, such as

$$x^2 + y^2 - 1 = 0$$

which describes the unit circle. In the first case, the solution was two points (± 1) in real line $\mathbb{R}$. In the second case we had a circle in the plane $\mathbb{R}^2$. As a rule of thumb, the solution to an equation of n variables is described by a set of dimension $n-1$ (but not always).

In contrast to these examples, a **differential equation** is an equation involving a *function* of the variable x and its derivatives. The **order** of an equation refers to the order of the highest derivative (so if the highest derivative appearing in the equation is the third derivative, we say that it is a third order equation).

1. The simplest example is the **first order equation**

   $$y'(x) = f(x)$$

   where f is some given (known) function. The unknown is the function $y(x)$. In this case we already know the solution: it is the antiderivative of f,

   $$y(x) = \int f(x)\,\mathrm{d}x.$$

2. A slightly more complicated first order example is the following equation, that includes both y and its own derivative:

   $$y'(x) = ky(x)$$

   where $k \neq 0$ is some fixed (known) constant. By examination we can see that a possible solution is

   $$y(x) = Ae^{kx},$$

   modeling exponential growth ($k > 0$) or decay ($k < 0$), where $A \in \mathbb{R}$ is some constant. Examples here include virus spread (exponential growth) and the decay of a radioactive material (exponential decay). We can verify the solution by plugging into the equation:

   $$y'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(Ae^{kx}\right) = kAe^{kx} = ky(x).$$

3. Another common example (which we shall discuss below) is the **second order equation**
$$y''(x) = -k^2 y(x)$$
which has the possible solutions (also, we find this by examination)
$$y_1(x) = A\cos(kx) \qquad \text{or} \qquad y_2(x) = B\sin(kx)$$
where $A, B \in \mathbb{R}$ are some constants. We can write a more general solution as
$$y(x) = A\cos(kx) + B\sin(kx).$$
Let us verify that it solves the equation:
$$\begin{aligned} y''(x) &= \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(A\cos(kx) + B\sin(kx)\right) \\ &= A\frac{\mathrm{d}^2}{\mathrm{d}x^2}\cos(kx) + B\frac{\mathrm{d}^2}{\mathrm{d}x^2}\sin(kx) \\ &= Ak^2(-\cos(kx)) + Bk^2(-\sin(kx)) \\ &= -k^2\left(A\cos(kx) + B\sin(kx)\right) \\ &= -k^2 y(x). \end{aligned}$$

Formal definitions

Ordinary differential equation

An **ordinary differential equation (ODE)** is an equation that involves the variable x, the unknown function $y = y(x)$ (this is the function we are looking for), as well as derivatives of $y(x)$ up to any finite order $n \in \mathbb{N}$. This equation can be expressed as
$$\mathcal{F}(x, y, y', \dots, y^{(n)}) = 0.$$
The **order** of the equation is the order of the highest derivative, n in this case. The ODE is called **autonomous** if $\mathcal{F}$ does not explicitly depend on x.

Note that $\mathcal{F}$ is a real-valued function taking values in the Cartesian product of $n+2$ copies of $\mathbb{R}$:
$$\mathcal{F} : \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n+2 \text{ times}} \to \mathbb{R}$$

Solution

A function $y = y(x)$ is a **solution** of the ODE $\mathcal{F}(x, y, y', \dots, y^{(n)}) = 0$ over the interval $I \subseteq \mathbb{R}$ if it is n-times differentiable on I, and if
$$\mathcal{F}(x, y(x), y'(x), \dots, y^{(n)}(x)) = 0, \qquad \forall x \in I.$$

As we have seen in the examples above, solutions need not be unique. As we shall see below, solutions also *might not exist*. These two observations make the study of ODEs delicate. In this course we are only getting a glimpse of this.

Normal form

If in the ODE $\mathcal{F}(x, y, y', \dots, y^{(n)}) = 0$ we can isolate the highest derivative of y, and rewrite it as

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)})$$

we say that it is written in **normal form**. Here f is a real-valued function of $n+1$ real variables.

Linearity and homogeneity

The ODE $\mathcal{F}(x, y, y', \dots, y^{(n)}) = 0$ is called **linear** if it can be written in the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y - g(x) = 0$$

where the coefficients $a_i(x)$ and the *forcing function* $g(x)$ are functions of the independent variable x only. If the ODE is not linear, it is called **non-linear**. If $g(x) = 0$, the equation is said to be **homogeneous**; otherwise, it is **non-homogeneous** or **inhomogeneous**.

Example 12.1: For the ODE $\mathcal{F}(x, y, y', \dots, y^{(n)}) = 0$, the following table characterizes the properties of various expressions for $\mathcal{F}$:

Expression for $\mathcal{F}$	Order	Linearity	Homogeneity	Autonomy
$y' - x^2$	1st	Linear	Non-homogeneous	Non-autonomous
$y' + y^2 - x^3$	1st	Non-linear	Non-homogeneous	Non-autonomous
$y'' + k^2 y$	2nd	Linear	Homogeneous	Autonomous
$y'' + k^2 \sin y$	2nd	Non-linear	Homogeneous	Autonomous
$xy'' + k^2 \sin y$	2nd	Non-linear	Homogeneous	Non-autonomous

Convention: time-dependent problems

When the independent variable is a *time* variable, we typically denote each derivative by a *dot* above the function, instead of a *prime*:

$$\dot{y}(t) \qquad \text{instead of} \qquad y'(t)$$

and

$$\ddot{y}(t) \qquad \text{instead of} \qquad y''(t).$$

Important ODEs

Basic ODEs

$$\begin{aligned}
\text{Exponential Growth/Decay:} \quad & \frac{\mathrm{d}y}{\mathrm{d}t} = ky \\
\text{Newton's Law of Cooling:} \quad & \frac{\mathrm{d}T}{\mathrm{d}t} = -k(T - T_a) \\
\text{Logistic Growth Equation:} \quad & \frac{\mathrm{d}P}{\mathrm{d}t} = rP\left(1 - \frac{P}{K}\right) \\
\text{Non-linear Pendulum Equation:} \quad & \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \omega^2 \sin\theta = 0 \\
\text{Simple Harmonic Motion:} \quad & \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \omega^2\theta = 0 \\
\text{Damped Harmonic Oscillator:} \quad & m\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + c\frac{\mathrm{d}\theta}{\mathrm{d}t} + \omega^2\theta = 0
\end{aligned}$$

First-order models, such as the Exponential Growth and Newton's Law of Cooling, utilize the relation $\frac{\mathrm{d}y}{\mathrm{d}t} = ky$ to represent physical processes like population dynamics and thermal equilibration. The Logistic equation refines these by introducing a carrying capacity K to model resource-limited growth.

Second-order equations, such as the Simple Harmonic Motion and Damped Oscillator characterize mechanical and electrical vibrations, where acceleration $\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$ is influenced by restoring forces and resistive damping.

Specialized ODEs

$$\begin{aligned}
\text{Bernoulli Equation:} \quad & \frac{\mathrm{d}y}{\mathrm{d}x} = p(x)y^\alpha + q(x)y, \qquad \alpha \neq 0, 1 \\
\text{Riccati Equation:} \quad & \frac{\mathrm{d}y}{\mathrm{d}x} = p(x)y^2 + q(x)y + r(x) \\
\text{Clairaut Equation:} \quad & y = x\frac{\mathrm{d}y}{\mathrm{d}x} + f\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) \\
\text{Bessel's Equation:} \quad & x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} + (x^2 - \nu^2)y = 0 \\
\text{Legendre's Equation:} \quad & (1 - x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + n(n+1)y = 0 \\
\text{Hermite Equation:} \quad & \frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + 2ny = 0 \\
\text{Airy Equation:} \quad & \frac{\mathrm{d}^2y}{\mathrm{d}x^2} - xy = 0 \\
\text{Van der Pol Oscillator:} \quad & \frac{\mathrm{d}^2x}{\mathrm{d}t^2} - \mu(1 - x^2)\frac{\mathrm{d}x}{\mathrm{d}t} + x = 0
\end{aligned}$$

More specialized ODEs often arise in higher-dimensional physics and non-linear systems where solutions cannot be expressed via elementary functions.

The Riccati and Bernoulli equations are *first-order non-linear* equations that frequently appear in control theory and fluid dynamics.

Second-order equations like those of Bessel, Legendre, and Hermite are essential for solving partial differential equations in cylindrical, spherical, or quantum mechanical domains, giving rise to "Special Functions" that describe everything from electromagnetic wave propagation to atomic orbitals.

Finally, *second order non-linear* models like the Van der Pol oscillator illustrate complex behaviors such as limit cycles, which arise in self-sustaining biological and electronic systems.

12.1 First-order differential equations in normal form

A first-order ODE which can be written in normal form is

$$y' = f(x, y).$$

The function $f(x, y)$ associates to each point $(x, y) \in \mathbb{R}^2$ in the plane a number which is the slope of the solution at that point. This can be visualized as a 'direction field', see Figure 12.1.

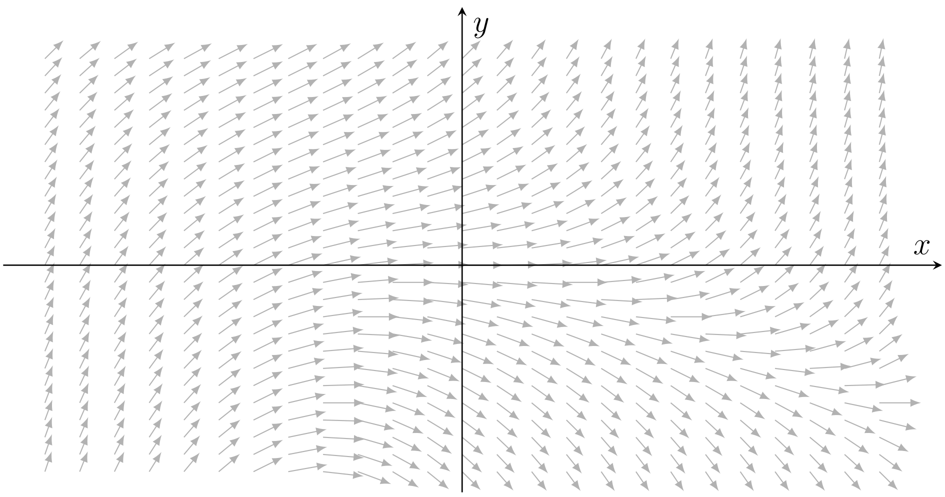

Figure 12.1: Direction field for the ODE $y' = (1 + x)y + x^2$.

By choosing a starting point $(x_0, y_0) \in \mathbb{R}^2$ in the plane and following the arrows, we construct a solution $y = y(x)$. Different starting points will lead to different solutions. This leads us to consider the *initial-value problem*.

Initial-value problem

For the normal form ODE $y' = f(x, y)$, the **initial-value problem (IVP)** (also called the **Cauchy problem**) for an interval $I \subseteq \mathbb{R}$ is

$$\begin{cases} y' = f(x, y) & \text{in } I, \\ y(x_0) = y_0, \end{cases}$$

where $x_0 \in I$ and $y_0 \in \mathbb{R}$.