Chapter 8

Differential calculus

In a bid to study the *rate of change* of a function, in this chapter we introduce the notion of the *derivative*. For instance, if $x : \mathbb{R} \to \mathbb{R}$ given by $x(t)$ describes the location of a car along a straight road at time t, then the rate of change of x gives the speed of the car. The rate of change of the speed is the acceleration.

8.1 The derivative

To define the derivative, we start by defining a small increment in x and in y. For a function $f : \mathrm{dom}(f) \subseteq \mathbb{R} \to \mathbb{R}$, we fix $x_0 \in \mathrm{dom}(f)$ and assume that f is defined in a neighborhood $I_r(x_0)$ of x_0. Then for $x \in I_r(x_0)$ we denote

$$\Delta x = x - x_0$$

which is called the **increment in** x. Correspondingly we denote

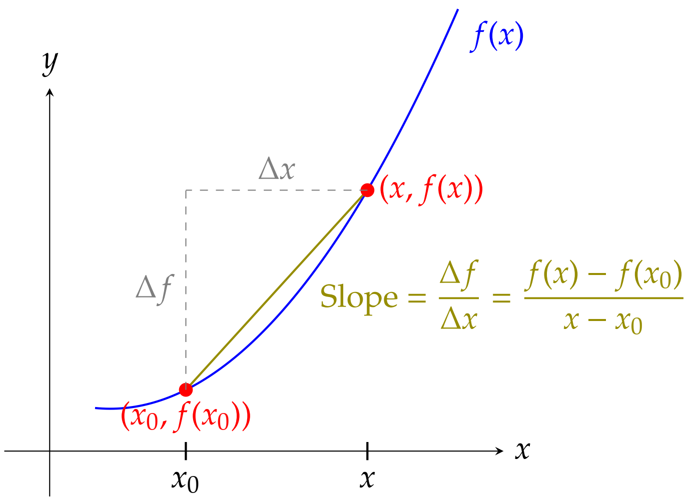
$$\Delta f = f(x) - f(x_0)$$

which is called the **increment in** f. We therefore have

$$x = x_0 + \Delta x \qquad \text{and} \qquad f(x) = f(x_0) + \Delta f.$$

The first step towards defining the derivative, is defining the **difference quotient of** f **between** x_0 **and** x:

$$\frac{\Delta f}{\Delta x} = \frac{f(x) - f(x_0)}{x - x_0} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$



The line connecting $(x_0, f(x_0))$ and another fixed point $(x_1, f(x_1))$ is called a **secant** and its equation is given by

$$s(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

By taking the limit of the secant as $x_1 \to x_0$ we obtain the *tangent line* to f at x_0, whose slope is the derivative of f at x_0:

Differentiability and the derivative

A function $f : \mathrm{dom}(f) \subseteq \mathbb{R} \to \mathbb{R}$ defined in a neighborhood of $x_0 \in \mathrm{dom}(f)$ is called **differentiable at** x_0 if $\lim_{x \to x_0} \frac{\Delta f}{\Delta x}$ exists and is finite. This number is called the **derivative of** f **at** x_0 and is denoted by $f'(x_0)$:

$$f'(x_0) = \lim_{x \to x_0} \frac{\Delta f}{\Delta x} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{\Delta x \to x_0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

Other symbols used to denote the derivative at x_0 include

$$y'(x_0) \qquad \frac{\mathrm{d}f}{\mathrm{d}x}(x_0) \qquad \frac{\mathrm{d}}{\mathrm{d}x}f(x_0) \qquad \frac{\mathrm{d}y}{\mathrm{d}x}(x_0) \qquad \frac{\mathrm{d}}{\mathrm{d}x}y(x_0) \qquad \mathrm{d}f(x_0)$$

Given $I \subseteq \mathrm{dom}(f)$, we say that f **is differentiable on** I if f is differentiable at each $x_0 \in I$.

We observe that this defines a new function:

The derivative function

The definition of the derivative at a point induces the definition of the **derivative function** f' whose value at x_0 is $f'(x_0)$. Its domain is

$$\mathrm{dom}(f') = \{x \in \mathrm{dom}(f) \ \mid \ f \text{ is differentiable at } x\}.$$

Proposition 8.1: If f is differentiable at x_0 it is also continuous at x_0.

Proof. We have

$$\begin{aligned}
\lim_{x \to x_0} (f(x) - f(x_0)) &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}(x - x_0) \\
&= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \lim_{x \to x_0} (x - x_0) \\
&= f'(x_0) \cdot 0 = 0.
\end{aligned}$$

□

Remark: The other implication is not true. For instance, the function $f(x) = |x|$ is continuous on $\mathbb{R}$, and in particular at $x_0 = 0$, however it is *not differentiable* at $x_0 = 0$. This can easily be seen by taking limits from the left and from the right and seeing that they are different:

$$\lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to 0^+} \frac{x}{x} = 1,$$
$$\lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to 0^-} \frac{-x}{x} = -1.$$

Example 8.1: Consider the affine function $f(x) = ax + b$. Then:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{ax + b - ax_0 - b}{x - x_0} = a \lim_{x \to x_0} \frac{x - x_0}{x - x_0} = a.$$

Since this is true for all $x_0 \in \mathbb{R}$, we find that the derivative function of $f(x)$ is $f'(x) = a$.

Example 8.2: Consider the quadratic function $f(x) = ax^2$:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{ax^2 - ax_0^2}{x - x_0} = a \lim_{x \to x_0} \frac{(x - x_0)(x + x_0)}{x - x_0} = 2ax_0.$$

Since this is true for all $x_0 \in \mathbb{R}$, we find that the derivative function of $f(x)$ is $f'(x) = 2ax$.

Example 8.3: Consider a monomial $f(x) = ax^n$. Using the formula

$$a^n - b^n = (a - b)\left(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1}\right) = (a - b)\sum_{k=0}^{n-1} a^k b^{n-1-k}$$

we have

$$\begin{aligned} f'(x_0) &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \\ &= \lim_{x \to x_0} \frac{ax^n - ax_0^n}{x - x_0} \\ &= a \lim_{x \to x_0} \frac{(x - x_0)\sum_{k=0}^{n-1} x^k x_0^{n-1-k}}{x - x_0} \\ &= anx_0^{n-1}. \end{aligned}$$

Since this is true for all $x_0 \in \mathbb{R}$, we find that the derivative function of $f(x)$ is $f'(x) = anx^{n-1}$.

Example 8.4: We can also consider the more general power function $f(x) = x^\alpha$, $\alpha \in \mathbb{R}$. Here we will use the identity (which was given in Section 5.7)

$$\lim_{y \to 0} \frac{(1 + y)^\alpha - 1}{y} = \alpha, \qquad (\alpha \in \mathbb{R})$$

which we did not prove (but we *nearly* proved, only a few minor manipulations were missing). Suppose the $x_0 \in \mathrm{dom}(f)$, then

$$
\begin{aligned}
f'(x_0) &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \\
&= \lim_{\Delta x \to 0} \frac{(x_0 + \Delta x)^\alpha - x_0^\alpha}{\Delta x} \\
&= x_0^{\alpha-1} \lim_{\Delta x \to 0} \frac{(1 + \frac{\Delta x}{x_0})^\alpha - 1}{\frac{\Delta x}{x_0}} \\
\left(\textit{substitute } y = \frac{\Delta x}{x_0}\right) \quad &= x_0^{\alpha-1} \lim_{y \to 0} \frac{(1+y)^\alpha - 1}{y} \\
&= \alpha x_0^{\alpha-1}.
\end{aligned}
$$

This is true for all x_0 for which the expression $x_0^{\alpha-1}$ is well-defined. This means that the derivative function is $f'(x) = \alpha x^{\alpha-1}$ and its domain consists of all $x \in \mathrm{dom}(f)$ for which $x^{\alpha-1}$ is well-defined.

Example 8.5: The function

$$
f(x) = \sqrt{x} = x^{\frac{1}{2}} \quad \text{has domain } \mathrm{dom}(f) = [0, +\infty)
$$

while its derivative

$$
f'(x) = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} \quad \text{has domain } \mathrm{dom}(f') = (0, +\infty).
$$

Example 8.6: For the function $f(x) = \sin x$ we use the formula

$$
\sin(\alpha) - \sin(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)
$$

as follows:

$$
\begin{aligned}
f'(x_0) &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \\
&= \lim_{x \to x_0} \frac{\sin x - \sin x_0}{x - x_0} \\
&= \lim_{x \to x_0} \frac{2\cos\left(\frac{x+x_0}{2}\right)\sin\left(\frac{x-x_0}{2}\right)}{x - x_0} \\
&= \underbrace{\lim_{x \to x_0} \cos\left(\frac{x+x_0}{2}\right)}_{=\cos x_0} \underbrace{\lim_{x \to x_0} \frac{\sin\left(\frac{x-x_0}{2}\right)}{\frac{x-x_0}{2}}}_{=1} \\
&= \cos x_0.
\end{aligned}
$$

Since this is true for all $x_0 \in \mathbb{R}$, we find that the derivative function of $f(x)$ is $f'(x) = \cos x$.

Example 8.7: Similarly it can be shown that for $f(x) = \cos x$, $f'(x) = -\sin x$.