The three solutions y_1, y_2 and y_3 are all legitimate solutions, and we have no method to label any one of them as 'correct' or as 'incorrect'. See Figure 12.3 for a visualization.

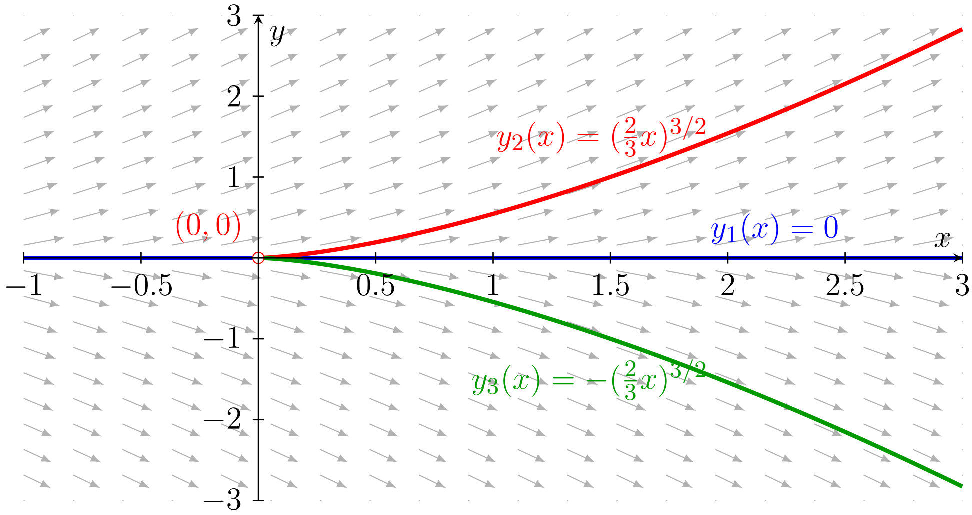

Figure 12.3: Direction field for $y' = y^{1/3}$ showing non-uniqueness at $(0, 0)$.

12.2 Second-order ODEs

12.2.1 The pendulum

In this section we demonstrate how Newton's Second Law $\mathbf{F} = m\mathbf{a}$ translates into a second-order differential equation in the case of a *physical pendulum* as in Figure 12.4.

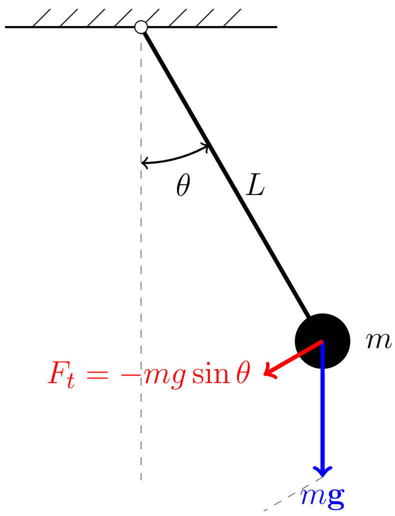

Figure 12.4: The physical pendulum. The restoring force is $mg\sin\theta$.

To derive the equation of motion for a physical pendulum of mass m and length L, we apply **Newton's Second Law** for rotation about the pivot point:

$$\tau = I\alpha$$

where τ is the *torque* (this is the force), I is the *moment of inertia* (this plays the role of the mass), and α is the *angular acceleration* (this is the acceleration).

1. Rotational Parameters

For a point mass suspended at a distance L, the moment of inertia is:

$$I = mL^2.$$

The angular acceleration α is the second derivative of the angular displacement θ with respect to time t:

$$\alpha = \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}.$$

We just need an expression for the torque.

2. The Restoring Torque

From the free-body diagram, the restoring force acting tangential to the path of motion is $F_t = -mg\sin\theta$. The resulting torque is the product of this force and the lever arm L:

$$\tau = L \cdot F_t = L \cdot (-mg\sin\theta) = -mgL\sin\theta.$$

The negative sign indicates that the torque acts in the direction opposite to the displacement, attempting to restore the pendulum to its equilibrium position ($\theta = 0$).

3. The Equation of Motion

Substituting these expressions into Newton's Second Law $I\alpha = \tau$, we obtain:

$$(mL^2)\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -mgL\sin\theta.$$

Dividing both sides by mL^2 yields the governing second-order non-linear ODE:

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{L}\sin\theta = 0.$$

Denoting the natural **frequency** of the problem

$$\omega = \sqrt{\frac{g}{L}}$$

the equation become

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \omega^2\sin\theta = 0.$$

Observe that the frequency is independent of the angle θ and of the mass m.

4. Small-Angle Approximation

For engineering applications where θ is small (typically $\theta < 10°$), we use the approximation $\sin\theta \sim \theta$ (which we've seen many times before, see Example 6.1 for instance) and the equation becomes

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \omega^2\theta = 0.$$

This procedure is called a **linearization**, as we've taken a nonlinear equation and replaced the nonlinear part ($\sin\theta$) by a linear approximation (θ).

5. Solving the Linearized Problem

The general solution is a linear combination of sines and cosines, as we've already seen:

$$\theta(t) = A\cos(\omega t) + B\sin(\omega t).$$

The motion is perfectly periodic with **period**

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}.$$

Indeed,

$$\begin{aligned}
\theta(t+T) &= A\cos(\omega(t+T)) + B\sin(\omega(t+T)) \\
&= A\cos(\omega t + 2\pi) + B\sin(\omega t + 2\pi) \\
&= A\cos(\omega t) + B\sin(\omega t) \\
&= \theta(t).
\end{aligned}$$

The two constants A and B are determined by the *initial conditions*. Let us specify the IVP, which now must have *two* conditions:

$$\begin{cases} \ddot{\theta} + \omega^2\theta = 0, \\ \theta(0) = \theta_0, \\ \dot{\theta}(0) = \omega_0. \end{cases}$$

Physically, this means that at time $t = 0$ the pendulum was

- at an angle θ_0, and
- had angular velocity ω_0.

Then we have:

$$\theta_0 = \theta(t=0) = A\cos 0 + B\sin 0 = A$$

and, taking a derivative of the expression for $\theta(t)$ and evaluating at $\theta = 0$,

$$\omega_0 = \frac{\mathrm{d}\theta}{\mathrm{d}t}(t=0) = -A\omega\sin 0 + B\omega\cos 0 = B\omega.$$

Hence these initial conditions dictate:

$$A = \theta_0 \qquad \text{and} \qquad B = \frac{\omega_0}{\omega}$$

and the solution is

$$\theta(t) = \theta_0\cos(\omega t) + \frac{\omega_0}{\omega}\sin(\omega t).$$

In particular, for a pendulum released from rest at an initial angle θ_0, the solution simplifies to

$$\theta(t) = \theta_0\cos(\omega t).$$

In this linearized regime, the period is *isochronous*—it remains constant regardless of the amplitude θ_0. This predictability is the foundation for classical timekeeping.

12.2.2 Second Order Linear Homogeneous ODEs with Constant Coefficients

Consider the second order equation with constant coefficients $a, b, c \in \mathbb{R}$:

$$ay'' + by' + cy = 0.$$

We can view the left hand side as a mapping that takes y and sends it to $ay'' + by' + cy$. This is called an **operator**, and we write it as

$$\mathcal{L} = a\frac{\mathrm{d}^2}{\mathrm{d}x^2} + b\frac{\mathrm{d}}{\mathrm{d}x} + c.$$

Thus our ODE becomes

$$\mathcal{L}y = 0.$$

We search for solutions of the form $y(x) = e^{rx}$, where $r \in \mathbb{C}$. Applying the operator to this function gives us

$$\begin{aligned}\mathcal{L}y &= \left(a\frac{\mathrm{d}^2}{\mathrm{d}x^2} + b\frac{\mathrm{d}}{\mathrm{d}x} + c\right) e^{rx} \\ &= \left(ar^2 + br + c\right) e^{rx}.\end{aligned}$$

Hence the ODE becomes

$$\left(ar^2 + br + c\right) e^{rx} = 0.$$

Since e^{rx} is always nonzero, we can divide by it, and our ODE has therefore been converted into a quadratic algebraic equation:

$$ar^2 + br + c = 0.$$

This is called the **characteristic equation**. Since we are working in $\mathbb{C}$, we distinguish three cases based on the roots r_1, r_2:

1. **Distinct Real Roots** ($b^2 - 4ac > 0$):

$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}.$$

2. **Repeated Real Root** ($b^2 - 4ac = 0$): Letting $r = r_1 = r_2$,

$$y(x) = (C_1 + C_2 x)e^{rx}.$$

3. **Complex Conjugate Roots** ($b^2 - 4ac < 0$): Letting $r = \alpha \pm i\beta$, we obtain the real-valued solution

$$y(x) = e^{\alpha x}(C_1 \cos(\beta x) + C_2 \sin(\beta x)).$$

Example 12.6: Solve the IVP

$$\begin{cases} y'' + 2y' + 5y = 0 \\ y(0) = 2,\ y'(0) = 0. \end{cases}$$

The characteristic equation for the operator $\mathcal{L}$ is $r^2 + 2r + 5 = 0$. Using the quadratic formula we find

$$r = \frac{-2 \pm \sqrt{4 - 20}}{2} = -1 \pm 2i.$$

So the general solution is $y(x) = e^{-x}(C_1 \cos 2x + C_2 \sin 2x)$. Plugging the first initial condition $y(0) = 2$:

$$2 = e^0(C_1 \cos 0 + C_2 \sin 0) \qquad \Rightarrow \qquad C_1 = 2.$$

To use the second condition, we compute the derivative

$$y'(x) = -e^{-x}(C_1 \cos 2x + C_2 \sin 2x) + e^{-x}(-2C_1 \sin 2x + 2C_2 \cos 2x).$$

Plugging $y'(0) = 0$:

$$0 = -C_1 + 2C_2 \qquad \Rightarrow \qquad 2C_2 = 2 \qquad \Rightarrow \qquad C_2 = 1.$$

The solution is $y(x) = e^{-x}(2 \cos 2x + \sin 2x)$, which is valid for all $x \in \mathbb{R}$.