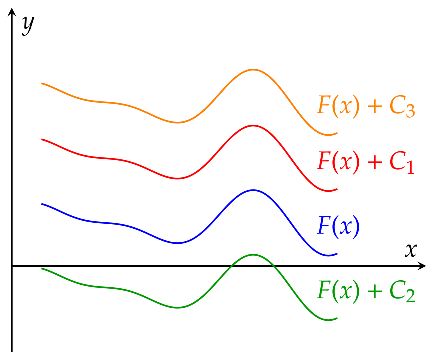

We can pinpoint which of these antiderivatives we are looking for by requiring that at some given x_0 the antiderivative has a particular value y_0: $F(x_0) = y_0$.

Example 10.2: 1. Identify the antiderivative $F(x)$ of $f(x) = \cos x$ that satisfies $F(2\pi) = 5$.

The antiderivative is $\int f(x)\,\mathrm{d}x = \sin x + C = F(x)$. If we require that $F(2\pi) = 5$ then we find that

$$\sin 2\pi + C = 5 \qquad \Rightarrow \qquad C = 5.$$

This has forced a choice of C.

2. Find the value at $x_1 = 3$ of the antiderivative of $f(x) = 6x^2 + 5x$ that vanishes at the point $x_0 = 1$.

First we identify the antiderivative:

$$\int f(x)\,\mathrm{d}x = \int \left(6x^2 + 5x\right)\,\mathrm{d}x = \frac{6x^3}{3} + \frac{5x^2}{2} + C = 2x^3 + \frac{5}{2}x^2 + C = F(x).$$

Plugging in the given condition we have:

$$0 = F(1) = 2 + \frac{5}{2} + C \qquad \Rightarrow \qquad C = -\frac{9}{2}.$$

Hence our antiderivative is:

$$F(x) = 2x^3 + \frac{5}{2}x^2 - \frac{9}{2}$$

and

$$F(3) = 2 \cdot 3^3 + \frac{5}{2} \cdot 3^2 - \frac{9}{2} = 2 \cdot 27 + \frac{5}{2} \cdot 9 - \frac{9}{2} = 72.$$

3. Find the antiderivative of $f(x) = \sin 3x$. We see that $\frac{\mathrm{d}}{\mathrm{d}x}(\cos 3x) = -3\sin 3x$, so that

$$F(x) = \int \sin 3x\,\mathrm{d}x = -\frac{1}{3}\cos 3x + C.$$

10.2 Rules of indefinite integration

Theorem 10.1 (Linearity): If $\int f(x)\,\mathrm{d}x$ and $\int g(x)\,\mathrm{d}x$ are defined, then for any $\alpha, \beta \in \mathbb{R}$, the antiderivative of $\alpha f(x) + \beta g(x)$ is also defined and satisfies:

$$\int (\alpha f(x) + \beta g(x))\,\mathrm{d}x = \alpha \int f(x)\,\mathrm{d}x + \beta \int g(x)\,\mathrm{d}x.$$

Proof. Suppose that $F(x)$ is an antiderivative of $f(x)$ and that $G(x)$ is an antiderivative of $g(x)$. Then, using the linearity of the derivative, we have:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\alpha F(x) + \beta G(x)) = \alpha F'(x) + \beta G'(x) = \alpha f(x) + \beta g(x)$$

so that $\alpha F(x) + \beta G(x)$ is an antiderivative of $\alpha f(x) + \beta g(x)$ and the claimed equality is satisfied. □

Example 10.3: Let us compute $\int \cos^2 x \,\mathrm{d}x$. We use the relation $\cos^2 x = \frac{1}{2}(1 + \cos(2x))$ to write

$$\begin{aligned}
\int \cos^2 x \,\mathrm{d}x &= \int \frac{1}{2}(1 + \cos(2x)) \,\mathrm{d}x \\
&= \frac{1}{2} \int 1 \,\mathrm{d}x + \frac{1}{2} \int \cos(2x) \,\mathrm{d}x \\
&= \frac{x}{2} + \frac{1}{2} \frac{\sin(2x)}{2} + C \\
&= \frac{x}{2} + \frac{\sin(2x)}{4} + C.
\end{aligned}$$

Integration by parts

Let $f(x)$ and $g(x)$ be differentiable functions over an interval $I \subseteq \mathbb{R}$. Then

$$\int f(x)g'(x) \,\mathrm{d}x = f(x)g(x) - \int f'(x)g(x) \,\mathrm{d}x$$

if the right hand side is defined (i.e. if $f'(x)g(x)$ is integrable on I).

Proof. Since $f'(x)g(x)$ is integrable on I, we let $H(x)$ be an antiderivative, i.e. $H'(x) = f'(x)g(x)$. Then, using the linearity of the derivative and the product rule, we have:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\big(f(x)g(x) - H(x)\big) &= \frac{\mathrm{d}}{\mathrm{d}x}\big(f(x)g(x)\big) - H'(x) \\
&= f'(x)g(x) + f(x)g'(x) - f'(x)g(x) = f(x)g'(x)
\end{aligned}$$

which implies that $f(x)g(x) - H(x)$ is an antiderivative of $f(x)g'(x)$, as has been claimed. □

An easy way to remember this is as follows, starting from the product rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(f(x)g(x)\big) = f'(x)g(x) + f(x)g'(x)$$
$$\Downarrow$$
$$f(x)g'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\big(f(x)g(x)\big) - f'(x)g(x)$$
$$\Downarrow$$
$$\int f(x)g'(x)\,\mathrm{d}x = \int \frac{\mathrm{d}}{\mathrm{d}x}\big(f(x)g(x)\big)\,\mathrm{d}x - \int f'(x)g(x)\,\mathrm{d}x$$
$$\Downarrow$$
$$\int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\,\mathrm{d}x$$

A fundamental concept

Integration by parts is a fundamental concept in mathematical physics and beyond, as it allows us to transfer a derivative from one function to another:

$$\int f(x)\mathbf{g}'(\mathbf{x})\,\mathrm{d}x = f(x)\mathbf{g}(\mathbf{x}) - \int f'(x)\mathbf{g}(\mathbf{x})\,\mathrm{d}x$$

Observe that on the left hand side, g has a derivative and f does not, while on the right hand side g doesn't have a derivative and f does. In equations coming from mathematical physics (for instance, the wave equation, the heat equation or Schrödinger's equation) we often use this idea to prove certain properties of solutions. For instance, this is used to prove that the energy of an electromagnetic wave is conserved, while the energy carried by heat dissipates over time.

Example 10.4: 1. Determine $\int xe^x\,\mathrm{d}x$. We have two functions — x and e^x — one of which will need to take the role of $f(x)$ and the other will take the role of $g'(x)$. We observe that the following choice can work quite nicely:

$$\begin{aligned} f(x) &= x & f'(x) &= 1 \\ g'(x) &= e^x & g(x) &= e^x \end{aligned}$$

so that we get

$$\int xe^x\,\mathrm{d}x = xe^x - \int e^x\,\mathrm{d}x = xe^x - e^x + C = e^x(x-1) + C.$$

Observe that if we had made the other choice we would have ended up with a more complicated integral. Indeed, choose

$$
\begin{aligned}
f(x) &= e^x & f'(x) &= e^x \\
g'(x) &= x & g(x) &= \frac{1}{2}x^2
\end{aligned}
$$

so that

$$\int xe^x \,\mathrm{d}x = \frac{1}{2}x^2e^x - \frac{1}{2}\int x^2e^x \,\mathrm{d}x = ?$$

2. Determine $\int \ln x \,\mathrm{d}x$. Here our two functions are $\ln x$ and the constant function 1! Here is our choice for f and g:

$$
\begin{aligned}
f(x) &= \ln x & f'(x) &= \frac{1}{x} \\
g'(x) &= 1 & g(x) &= x
\end{aligned}
$$

which implies

$$\int \ln x \,\mathrm{d}x = x \ln x - \int \frac{x}{x} \,\mathrm{d}x = x \ln x - x + C.$$

3. Determine $I_1 = \int e^x \sin x \,\mathrm{d}x$. We choose

$$
\begin{aligned}
f(x) &= e^x & f'(x) &= e^x \\
g'(x) &= \sin x & g(x) &= -\cos x
\end{aligned}
$$

so that

$$I_1 = \int e^x \sin x \,\mathrm{d}x = -e^x \cos x + \underbrace{\int e^x \cos x \,\mathrm{d}x}_{I_2}.$$

We still cannot determine the integral on the right hand side, so we perform another integration by parts with

$$
\begin{aligned}
f(x) &= e^x & f'(x) &= e^x \\
g'(x) &= \cos x & g(x) &= \sin x
\end{aligned}
$$

hence

$$I_2 = \int e^x \cos x \,\mathrm{d}x = e^x \sin x - \underbrace{\int e^x \sin x \,\mathrm{d}x}_{I_1}$$

and we observe that we're back to the original integral I_1. Putting everything together, we found that

$$I_1 = -e^x \cos x + I_2 = -e^x \cos x + e^x \sin x - I_1.$$

Rearranging, we finally get

$$I_1 = \frac{1}{2}e^x (\sin x - \cos x) + C.$$

Integration by substitution

Let $f(y)$ be integrable on an interval J and let $F(y)$ be an antiderivative. Suppose that $\varphi(x) : I \to J$ is differentiable. Then $f(\varphi(x))\varphi'(x)$ is integrable on I and

$$\int f(\varphi(x))\varphi'(x)\,\mathrm{d}x = F(\varphi(x)) + C.$$

A simpler way to remember this formula is by writing $y = \varphi(x)$ to get:

$$\int f(\varphi(x))\varphi'(x)\,\mathrm{d}x = \int f(y)\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int f(y)\,\mathrm{d}y.$$

Proof. From the chain rule (Theorem 8.3) we know that

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(F(\varphi(x))\big) = F'(\varphi(x))\varphi'(x) = f(\varphi(x))\varphi'(x).$$

By definition of the antiderivative the result follows. □

Example 10.5: 1. Determine $\int \sin(k(x - x_0))\,\mathrm{d}x$, where $0 \neq k \in \mathbb{R}$ and $x_0 \in \mathbb{R}$. We let

$$y = \varphi(x) = k(x - x_0) \qquad \text{so that} \qquad \varphi'(x) = k.$$

Then we have

$$\begin{aligned}\int \sin(k(x - x_0))\,\mathrm{d}x = \int \sin(\varphi(x))\,\mathrm{d}x &= \int \sin(\varphi(x))\frac{k}{k}\,\mathrm{d}x \\ &= \frac{1}{k}\int \sin(\varphi(x))\varphi'(x)\,\mathrm{d}x \\ &= \frac{1}{k}\int \sin y\,\mathrm{d}y \\ &= -\frac{1}{k}\cos y + C \\ &= -\frac{1}{k}\cos(k(x - x_0)) + C.\end{aligned}$$

2. Determine $\int xe^{x^2}\,\mathrm{d}x$. We let

$$y = \varphi(x) = x^2 \qquad \text{so that} \qquad \varphi'(x) = 2x.$$

Then we have

$$\begin{aligned}\int xe^{x^2}\,\mathrm{d}x = \frac{1}{2}\int 2xe^{x^2}\,\mathrm{d}x &= \frac{1}{2}\int \varphi'(x)e^{\varphi(x)}\,\mathrm{d}x \\ &= \frac{1}{2}\int e^y\,\mathrm{d}y \\ &= \frac{1}{2}e^y + C \\ &= \frac{1}{2}e^{x^2} + C.\end{aligned}$$