Sculpting Vlasov Phase Space

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Overview: Vlasov-Poisson phenomena

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P. Veltri, Y. Cheng, I. Gamba, ...

Two Parts

I. Conventional Cauchy Problem

II. Dynamically Accessible Initial Conditions

Part I:

Conventional Cauchy Problem

Conventional Cauchy Problem

1D Vlasov Poisson:

$$f_t(x, v, t) = -v f_x + E f_v\,, \qquad E_x = 1 - \int_{\mathbb{R}} dv\, f$$

As usual $f : \mathbb{T} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$

Initial Condition:

$$f(x, v, 0) = f_e(v) + \delta f(x, v, 0)$$

Cases:

- Linear or nonlinear
- Stable or unstable equilibrium, e.g., $f_e(v)$
- Single mode: $\delta f(x, v, 0) = f_k(v, 0) e^{ikx} + cc$

Linear Landau Damping

Linearize, Laplace transform, $t \to \infty$, etc. or

$$f(x,v,0) = f_M(v) + f_k(v)e^{ikx - i\omega t}\,, \qquad\qquad k \in \mathbb{R}$$

Plasma dispersion function:

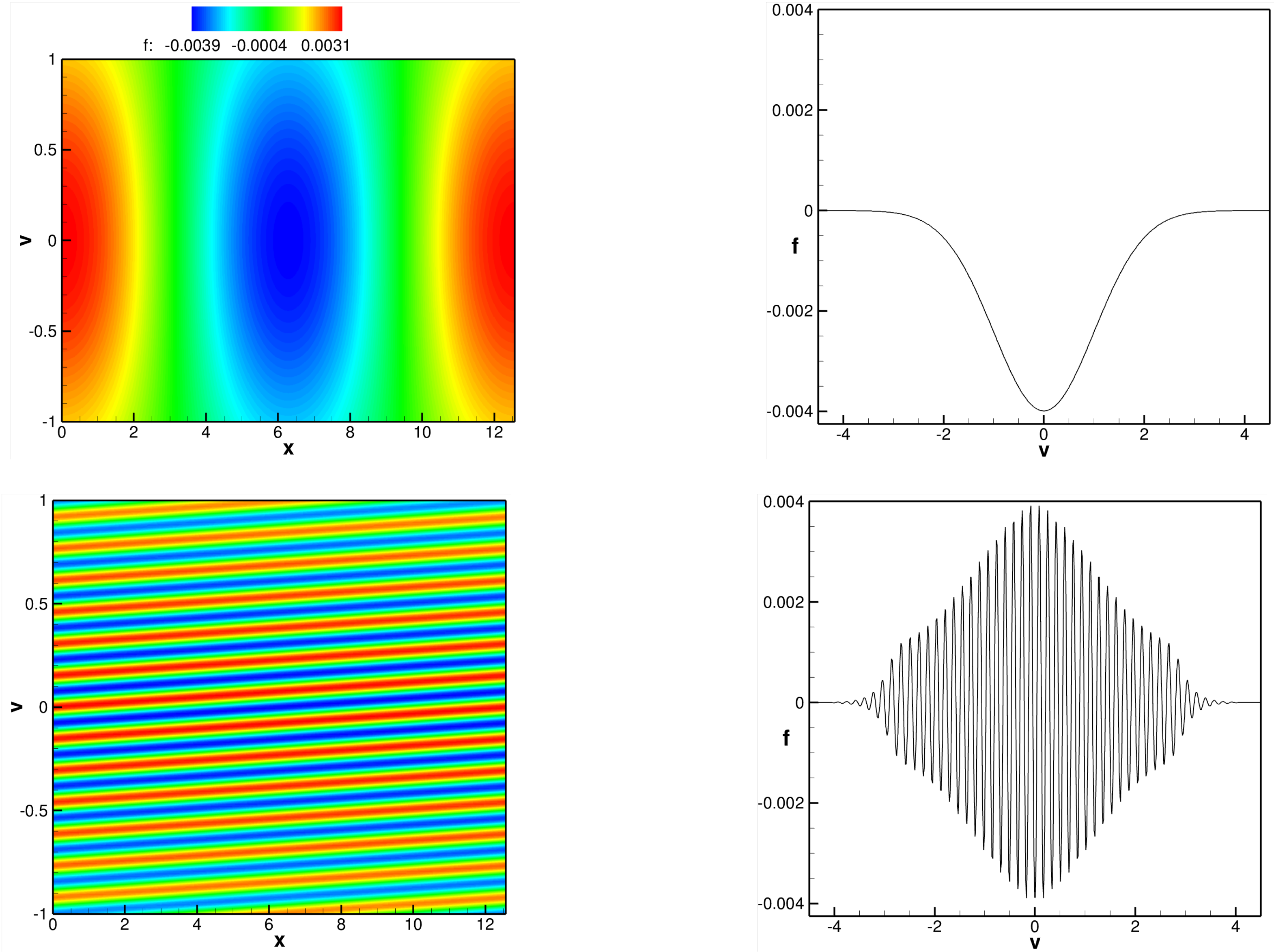
$$D(k,\omega) = 1 - \frac{1}{k^2}\int_{-\!\cup\!-} dv\, \frac{f_M'}{v - v_\phi} \qquad \text{where} \qquad v_\phi = \omega/k$$

Weak Growth/Damping:

$$D_r(\omega_r, k) = 0\,, \qquad\qquad \gamma = -\frac{D_i(\omega_r, k)}{\partial D_r(\omega_r, k)/\partial\omega_r}\,,$$

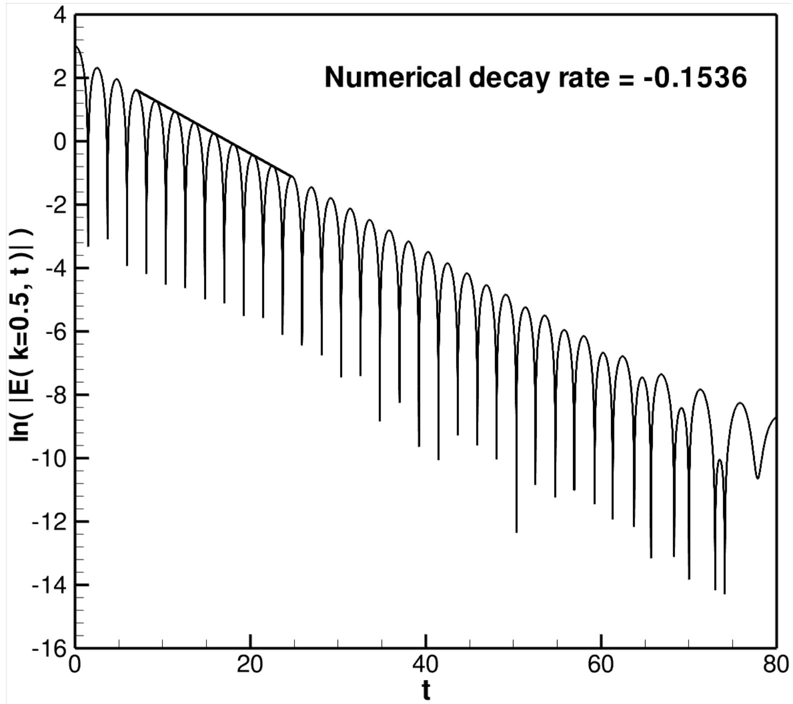
where $\omega = \omega_r + i\gamma$

$$D_r(k,\omega) = 1 - \frac{1}{k^2}\fint dv \frac{f_M'}{v - v_\phi}\,, \qquad\qquad D_i(k,\omega) = -\frac{\pi}{k^2}\, f_M'(v_\phi)$$



Contour plots (*left*) and cross-sectional plots (*right*), $x = 2\pi$, for δf at $t = 0$ (*top*) and $t = 75$ (*bottom*).

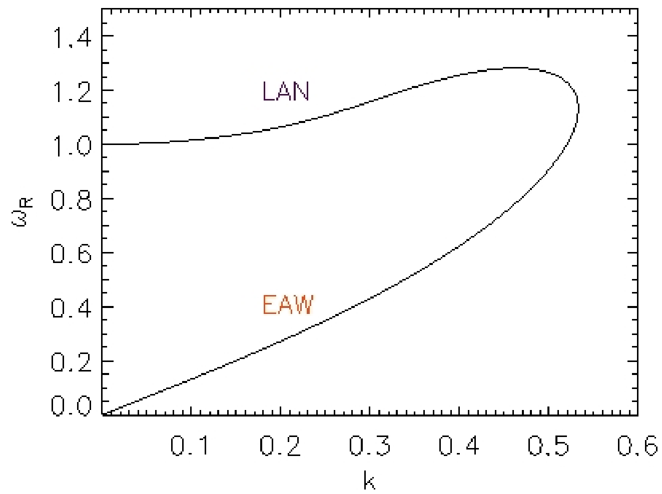
Landau Damping Maxwellian Decay Rate



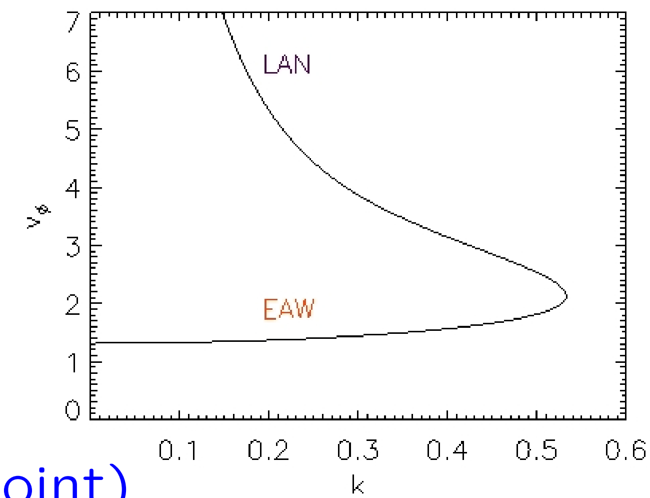
Electric field damping with fine mesh $(N_{h_x}, N_{h_v}) = (2000, 1600)$. Theoretical rate is -0.153 to three decimal-digit accuracy. After Heath (2007) and Heath et al. (2012).

Electron Acoustic Waves (EAW)

Holloway and Dorning (1991)
($\nexists$ linear limit of EAW mode)



$$
\begin{aligned}
D_r(k,\omega) &= 1 - \frac{1}{k^2}\int dv \frac{f_M'}{v - v_\phi} \\
D_i(k,\omega) &= -\frac{\pi}{k^2} f_M'(v_\phi) \\
v_\phi &= \omega/k
\end{aligned}
$$



Shadwick and pjm (1994)
($\exists$ linear EAW - stationary inflection point)

Linear Landau Damping

Linearize, Laplace transform, $t \to \infty$, etc. or

$$f(x, v, 0) = f_M(v) + f_k(v) e^{ikx - i\omega t}$$

Plasma dispersion function:

$$D(k, \omega) = 1 - \frac{1}{k^2} \int_{-\cup-} dv \, \frac{f_M'}{v - v_\phi} \qquad \text{where} \qquad v_\phi = \omega / k$$

Weak Growth Damping:

$$D_r(\omega_r, k) = 0 \,, \qquad\qquad \gamma = -\frac{D_i(\omega_r, k)}{\partial D_r(\omega_r, k) / \partial \omega_r} \,,$$

where

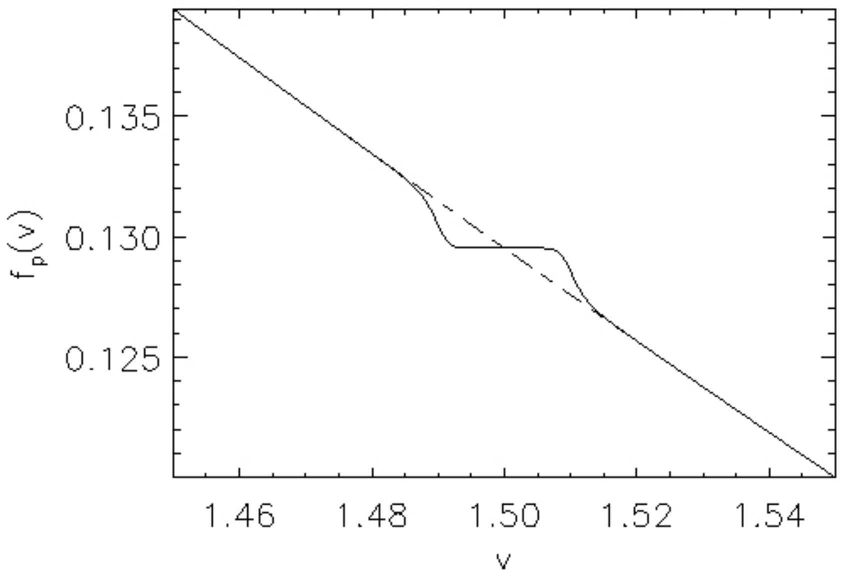
$$D_r(k, \omega) = 1 - \frac{1}{k^2} -\!\!\!\!\!\!\int dv \frac{f_M'}{v - v_\phi} \,, \qquad\qquad D_i(k, \omega) = -\frac{\pi}{k^2} \, f_M'(v_\phi)$$

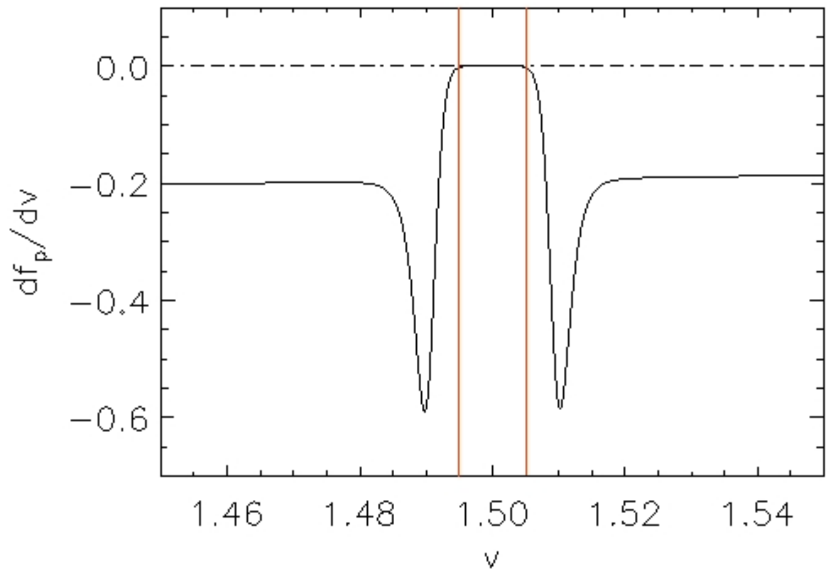
Note $\rightarrow$ $\quad \gamma \sim D_i \sim f_M'$

Corner Modes

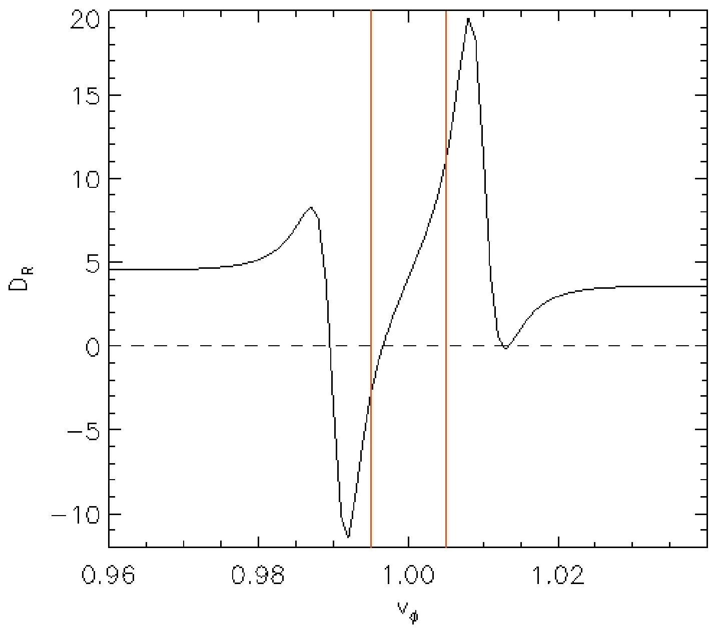
Plateau Distribution:

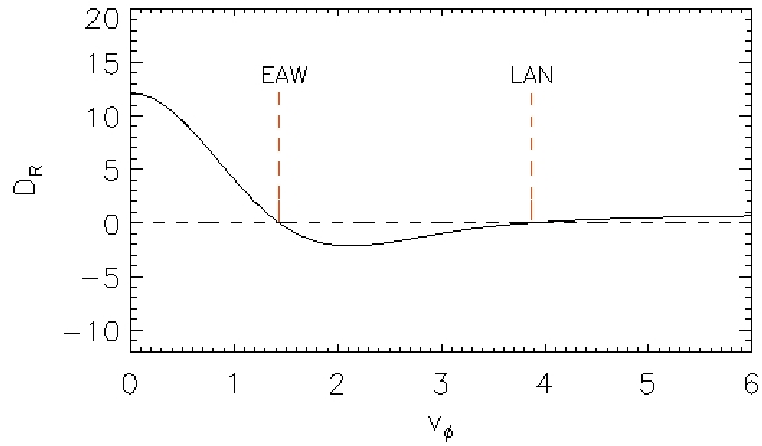
$$f_p(v) = N\left(f_M(v) - \frac{f_M(v) - f_M(V_0)}{1 + [(v - V_0)/\Delta V_p]^{n_p}}\right)$$

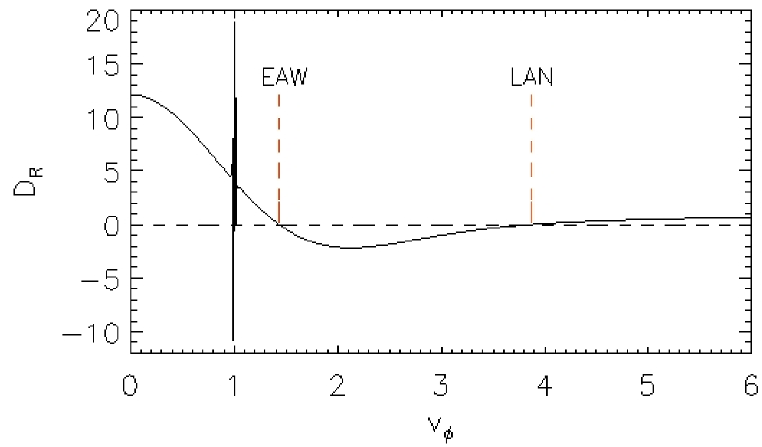




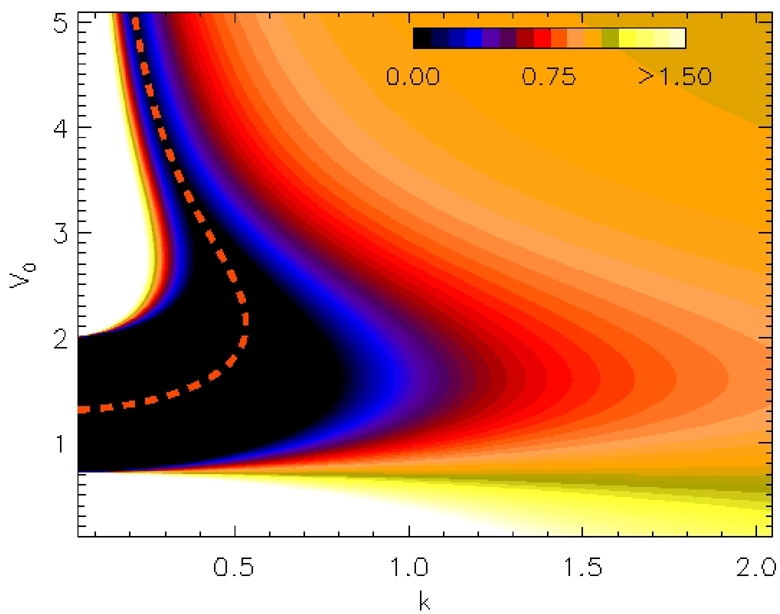
Corner Mode Sensitivity to v_ϕ

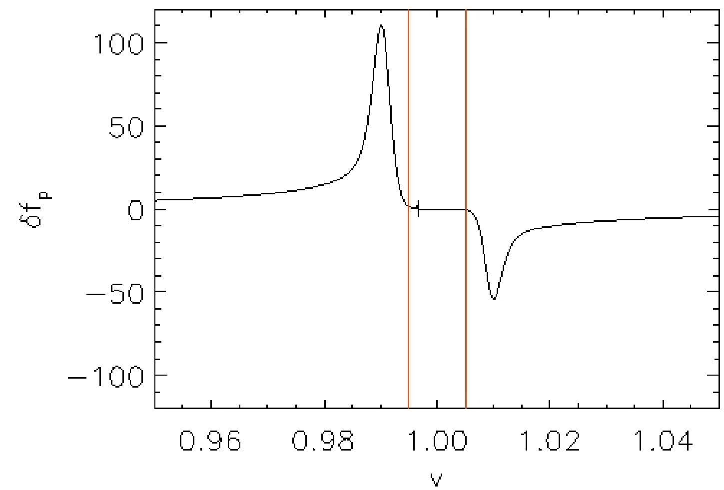


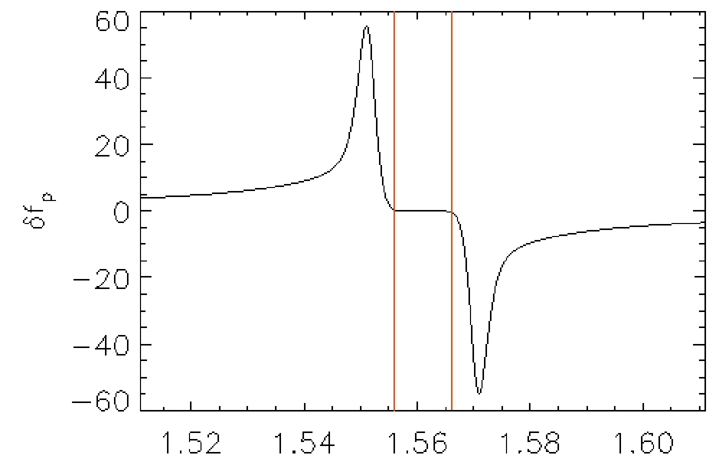




Corner Modes Again

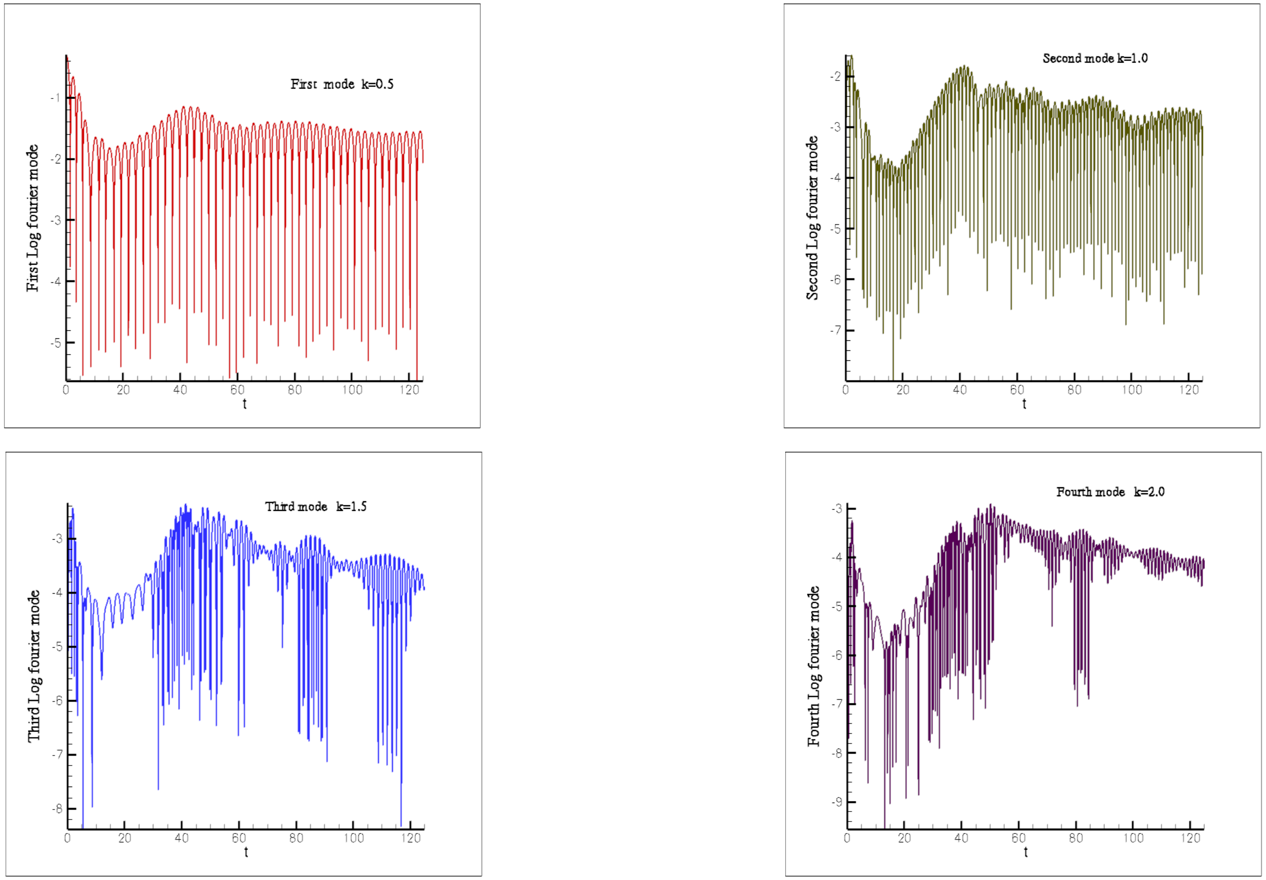






Corner modes are in the point spectrum embedded in CS.
Observed frequency shifts in non-neutral plasma experiments!

Nonlinear Landau Damping



Maxwellian, amplitude $A = .5$: k=.5 *(top left)*, k=1 *(top right)*, k=1.5 *(bottom left)* and k=2 *(bottom right)*. Bounce time ≈ 40.

Saturation by particle trapping!

Nonlinear Two-Stream Instability

Equilibrium:

$$f_{TS}(v) \;=\; \frac{1}{\sqrt{2\pi}}\, v^2\, e^{-v^2/2}$$

Manipulations:

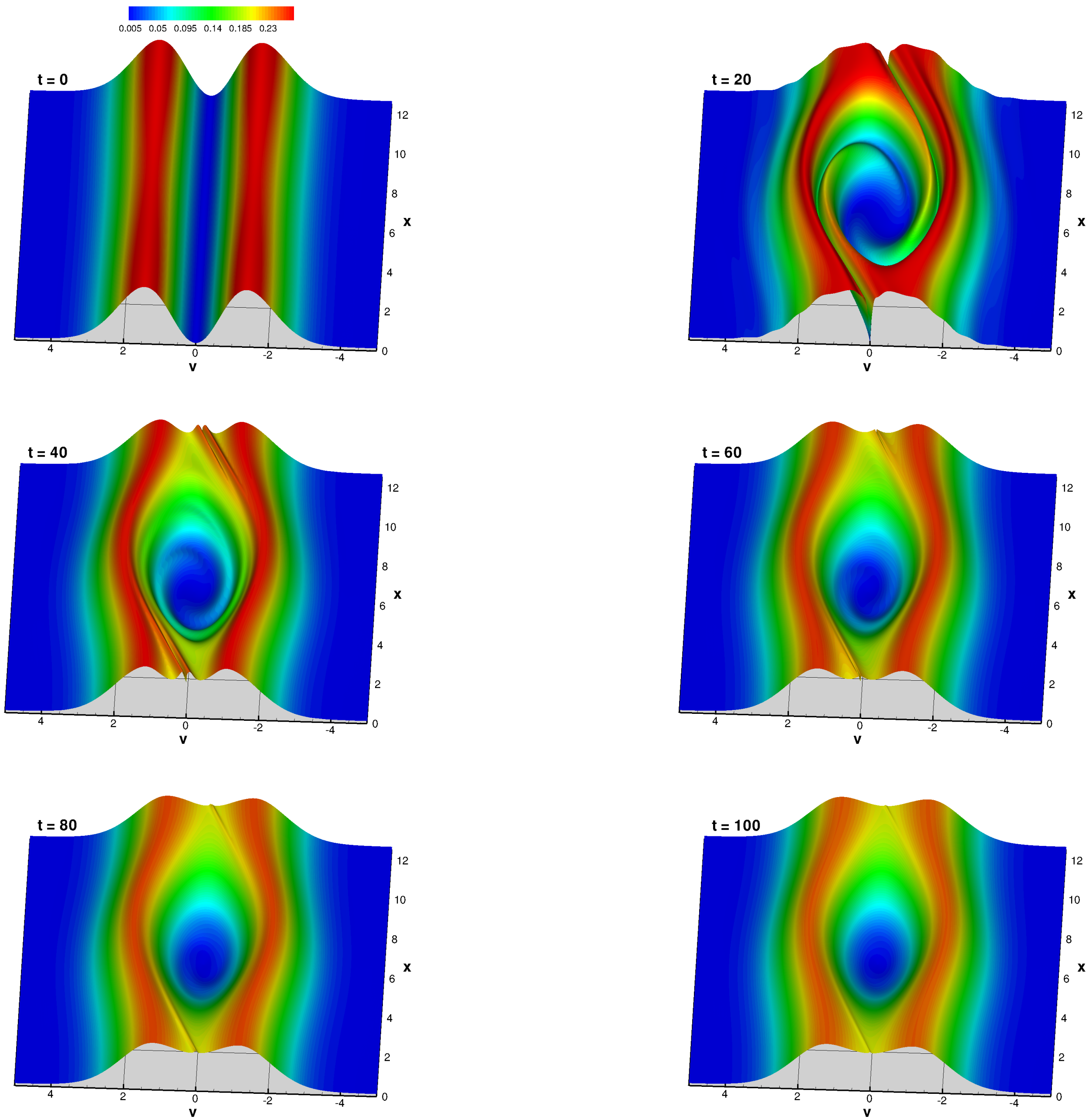
$$\varepsilon = 1 - \frac{2}{k^2}\left[1 - 2z^2 + 2zZ(z)\left(1 - z^2\right)\right] \,.$$

where $z = \omega/k$.

Plasma Z-function:

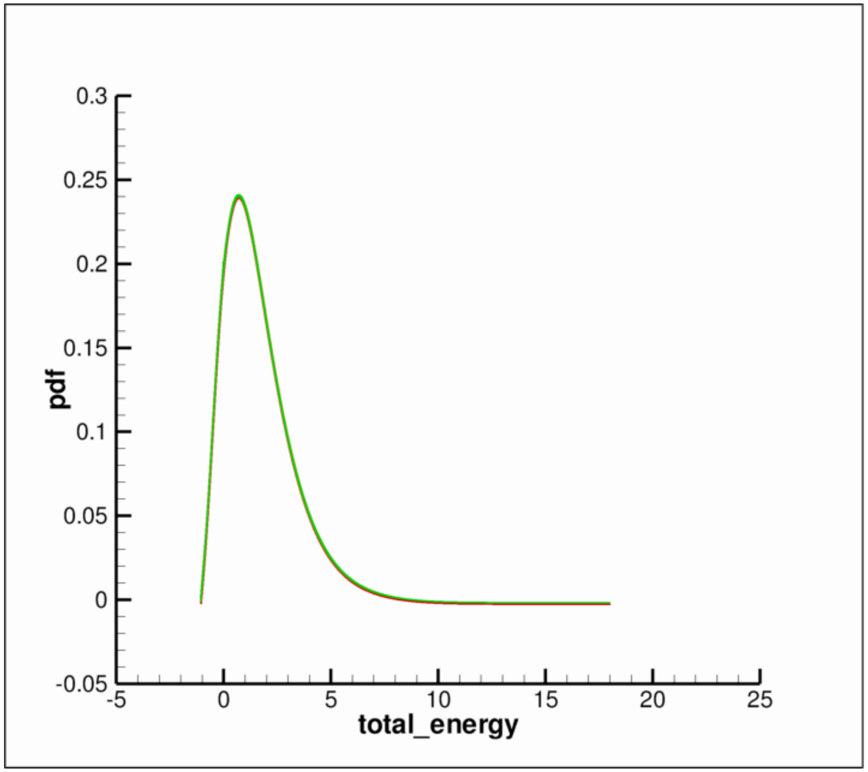
$$Z(z) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-w^2}\frac{dw}{w-z} = 2ie^{-z^2}\int_{-\infty}^{iz} e^{-t^2}dt$$

first expression $\Im(z) > 0$ and the value of Z for $\Im(z) < 0$ is obtained by analytic continuation; second expression valid for all complex z good for numerics. $\varepsilon = 0$ implies instability! γ agrees!


0.005 0.05 0.095 0.14 0.185 0.23
t = 0
t = 20
t = 40
t = 60
t = 80
t = 100
x
v

Scatter Plot f versus $\mathcal{E}(x,v)$

At $t = 100$ for every x, v, know $\Phi \Rightarrow \mathcal{E}(x,v) = v^2/2 - \Phi(x,100)$.
Make scatter plot of 9 million pairs (x,v) of f_{100} versus $\mathcal{E}(x,v)$:



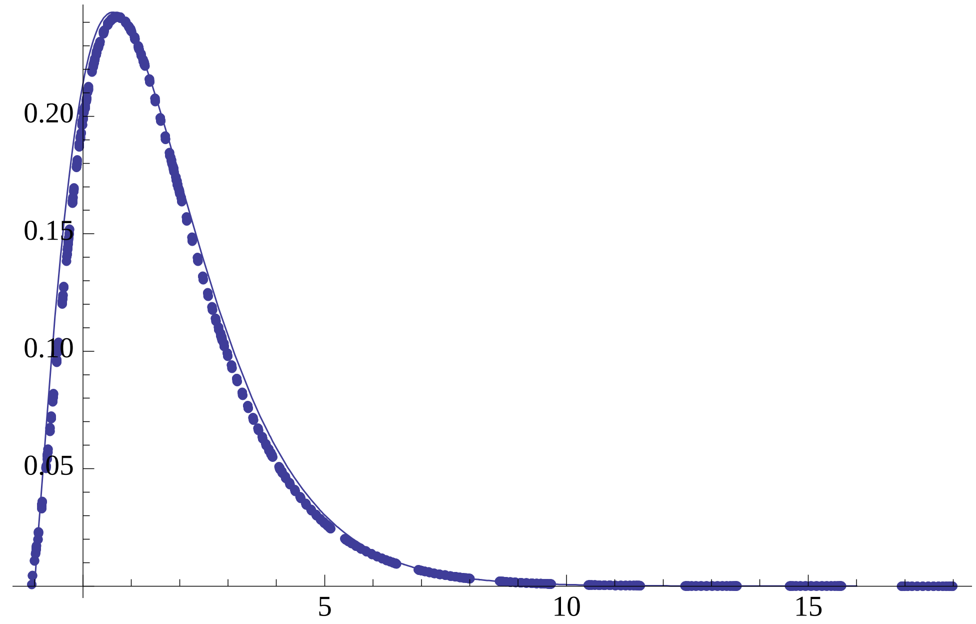
f_{100} a graph over $\mathcal{E}(x,v)$ to within line thickness. Green positive velocities; red negative velocities.

BGK Modeling

Model Distribution:

$$f_{\text{fit}} = A(\mathcal{E} + \Phi_M)(\mathcal{E} + \mathcal{E}^*)e^{-\beta\mathcal{E}}\,.$$

Here $\Phi_M = \max(\Phi)$. Since $\mathcal{E} = v^2/2 - \Phi$, $\min(\mathcal{E}) = -\Phi_M$. $f > 0 \Rightarrow \mathcal{E}^* > \Phi_M$.



Rough guess:
$\Phi_M = 1$ and $\mathcal{E}^* = 2$ uniformly good fit. $f'(\mathcal{E}_M) = 0$. For $\beta = 1$, $\mathcal{E}_M = 1/\gamma$, where γ is the golden mean!

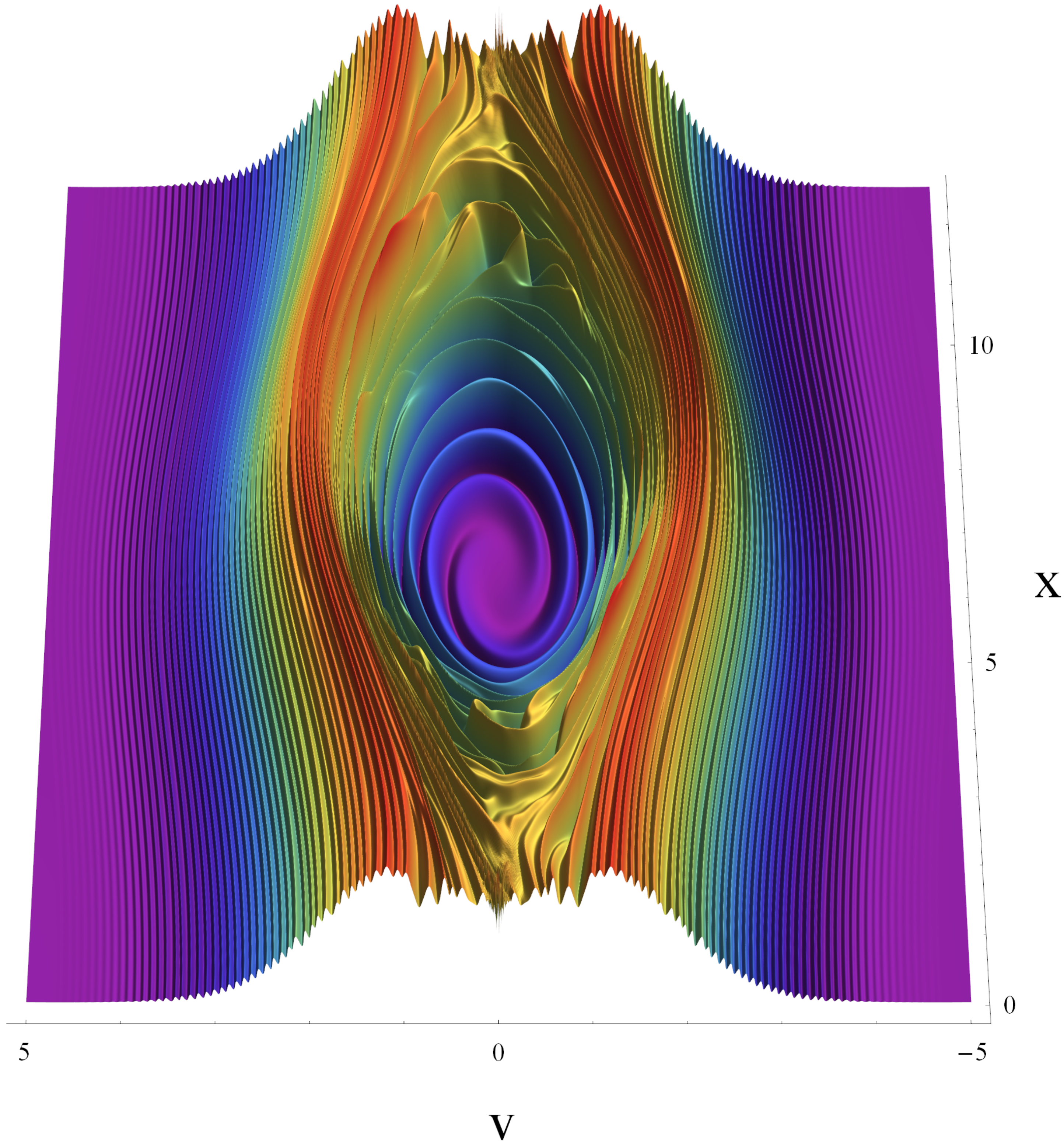
Convergence?

Asymptotic convergence in Hamiltonian system?

What really happens?

What kind of convergence?

Z. Stone $\rightarrow$ at $t = 160$!



10
5
0
X
5
0
−5
V

Part II:

Dynamically Accessible Initial Conditions

Dynamically Accessible IC

- Forces cause perturbations: δf vs. δE

- Dynamically Accessible ICs preserve constraints.
  Measure preserving rearrangement, Casimirs,
  level set topology, etc.

- Lie series generation, e.g.: $f_{DA} = e^{s[H_{DA},\,\cdot\,]} f_0$
  where $[f,g] = f_x g_v - f_v g_x$ and $H_{DA}(x,v,t)$ ‘arbitrary’.

PJM & D. Pfirsch (1989)

Dynamically Accessible IC $\leftarrow$ Driving

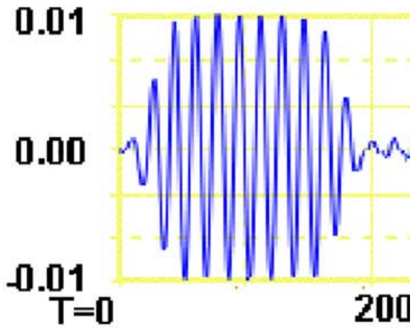
Vlasov with Drive:

$$f_t = -v f_x + \Big(E + E_d(x,t)\Big) f_v\,, \qquad E_x = 1 - \int_{\mathbb{R}} dv\, f$$

External Drive:

$$E_d(x,t) = E_{DA}\, g(t) \cos(kx - \omega t)$$

Drive Created IC:



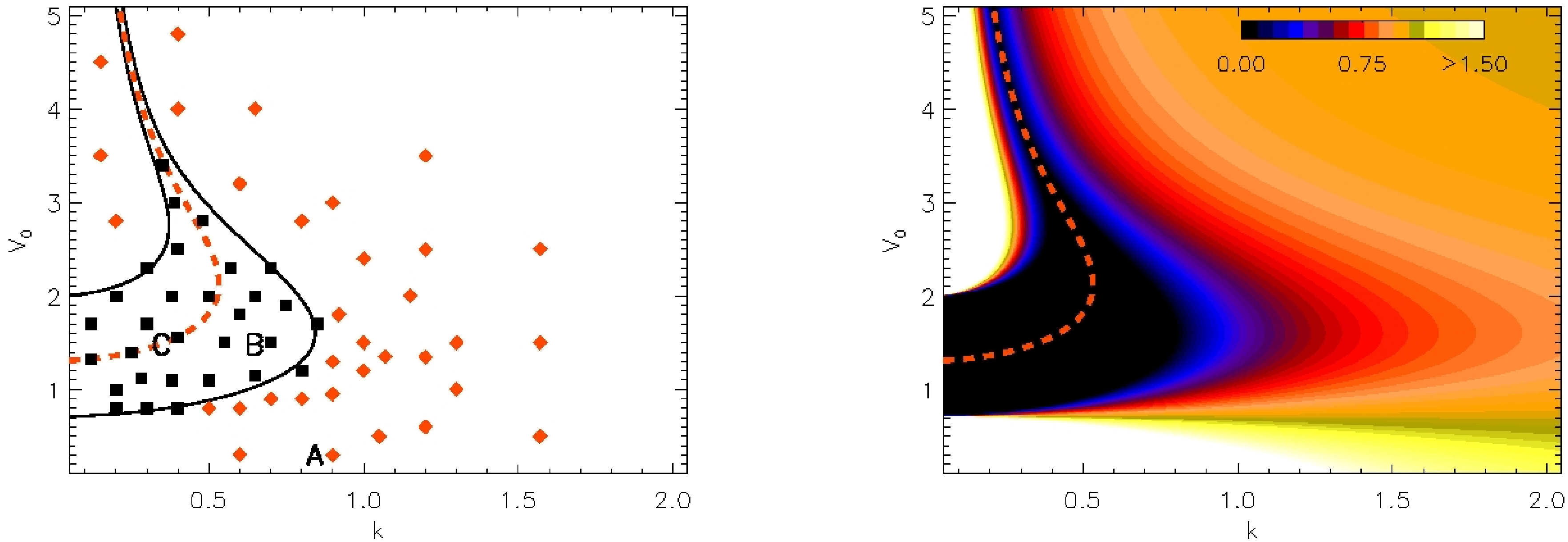
$E_{DA}(t) = .052,\ T_{d,r} = 200, 50$

pjm & Pfirsch, Johnston et al., Afeyan, ...

Dichotomies:

Weak vs. Strong - Adiabatic vs. Slap - Short vs. Long (ω, k)

Driving f_p: Weak-Adiabatic-Shortish



Nonlinear simulations $\rightarrow$ linear theory. (Valentini et al. 2012)

Toward a Taxonomy

DA ICs gives rise to many interesting structures.

Examples to follow.

How to classify and understand?

The Program

- Vlasov is Hamiltonian wrt noncanonical Poisson Bracket, e.g. Vlasov-Poisson (pjm 1980)

$$\{F, G\} = \int dxdv\, f \left[\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f} \right]$$

- Do for infinite degree-of-freedom Hamiltonian systems that which can be done for finite. KAM etc.

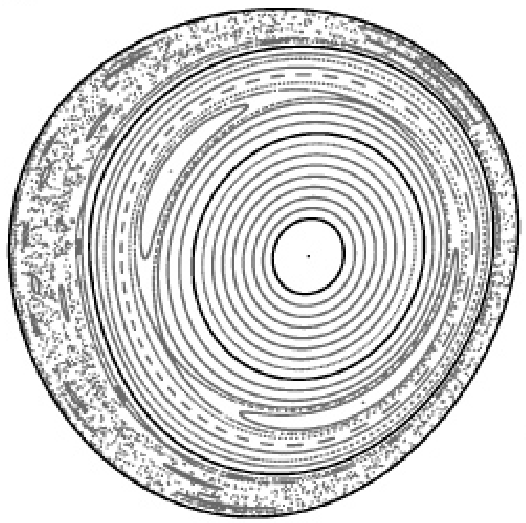
- Example: Krein-Moser theorem. Discrete spectrum pretty easy. Continuous spectrum? Not so easy. Analysis necessary. Signature pjm Pfirsch (1992); Krein's theorem *G. Hagstrom and pjm (2011, 2013).*

- **Here**: Lyapunov, Weinstein, Moser, ... Theorem about periodic orbits

LWM Theorem

• Finite-Dimensional Hamiltonian Systems:

▷ $\exists$ other periodic orbits near stable periodic orbit (equilibrium)

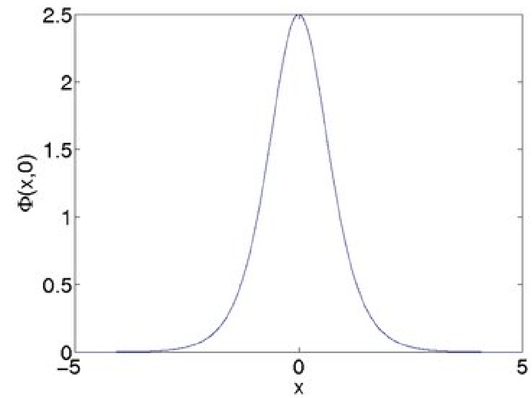


• Infinite-Dimensional Hamiltonian Systems:

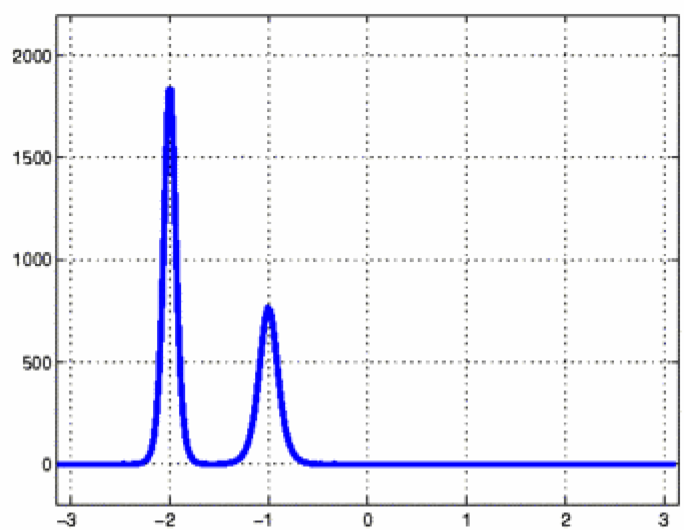
▷ precedent $\rightarrow$ for KdV soliton solution $\exists$ N-soliton solution, i.e., motion on an N-torus

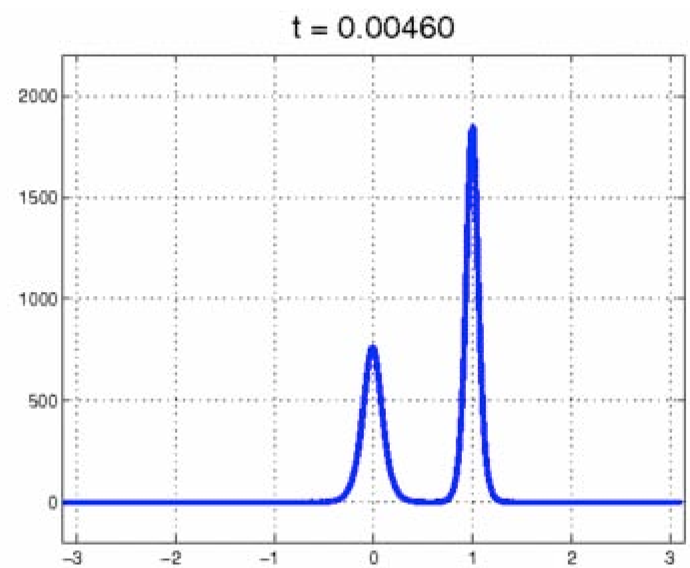
Solitons on N-Tori

1-soliton $\rightarrow$

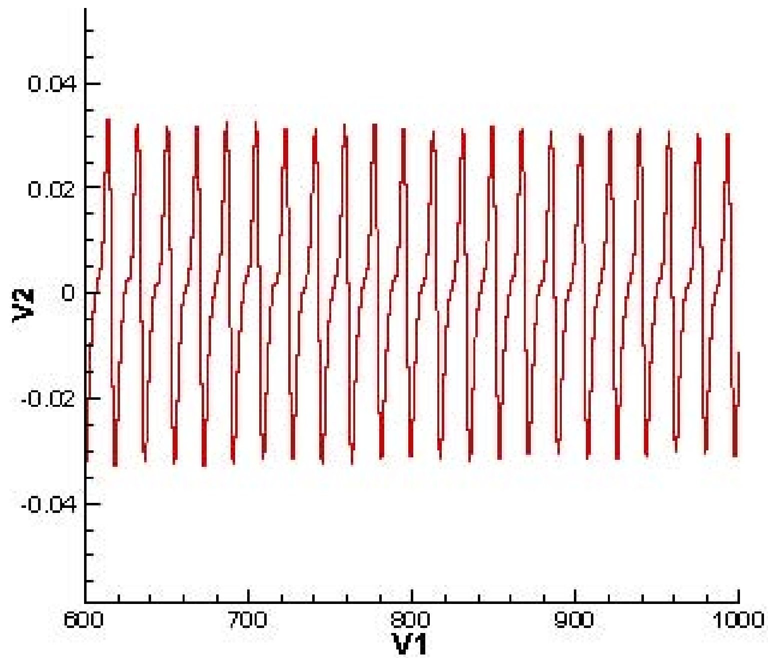


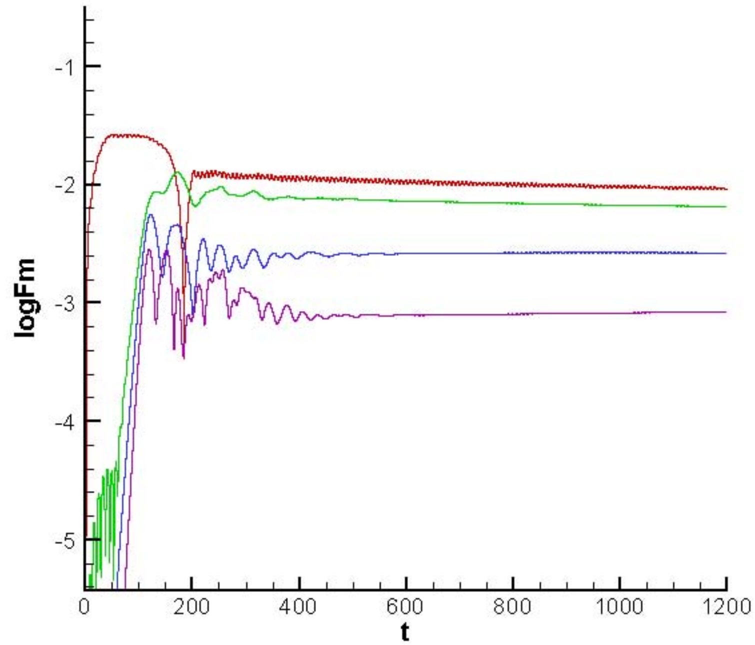
2-soliton $\downarrow$





Driving f_M: Weak - Adiabatic

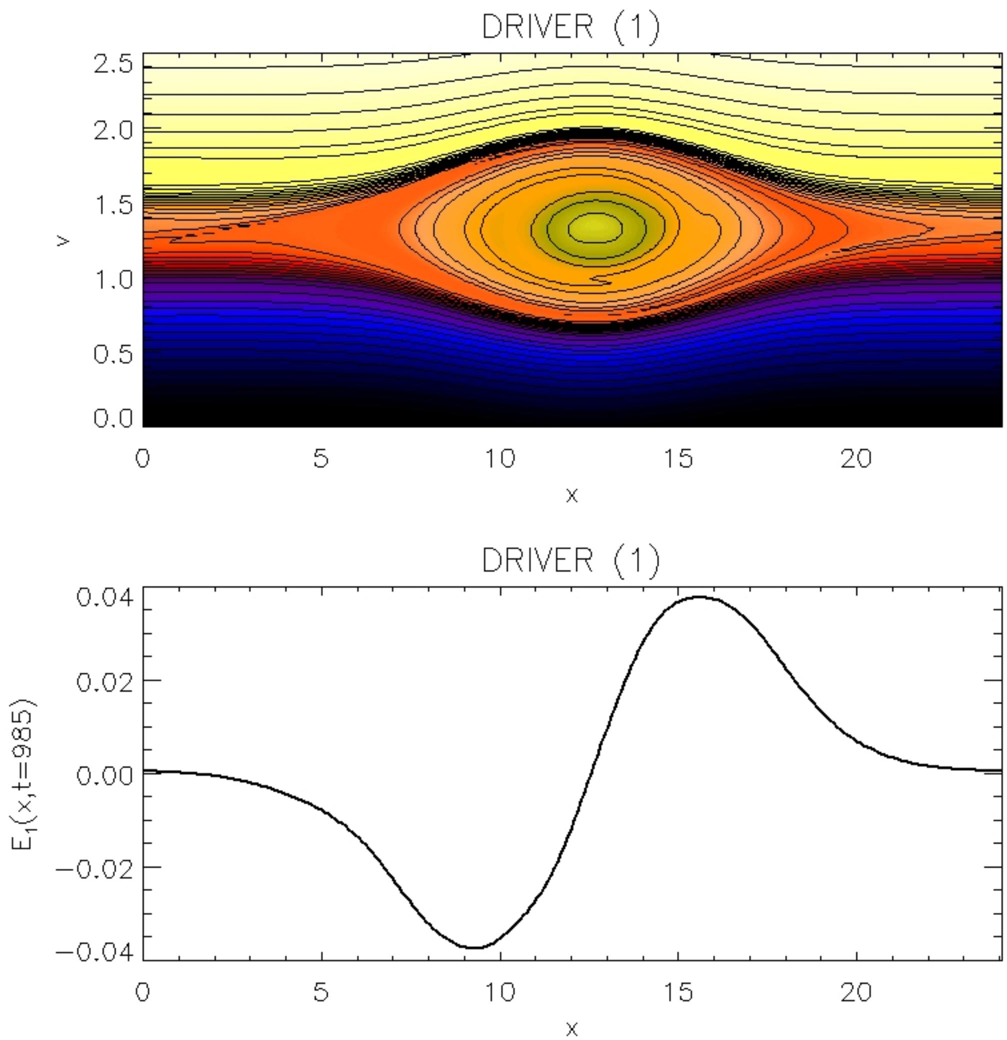




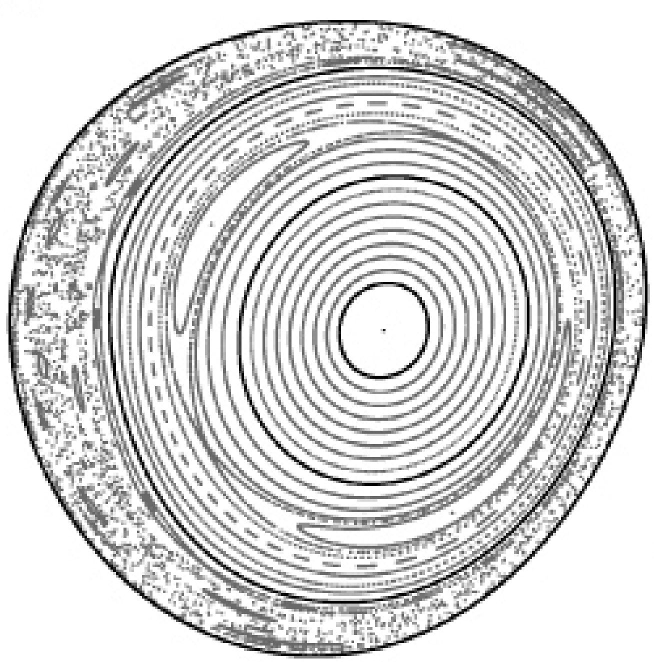
$$A_d(t) = .052 \text{ and } T_d = 200$$

Appears to settle into <u>periodic orbit</u> – travelling BGK hole.

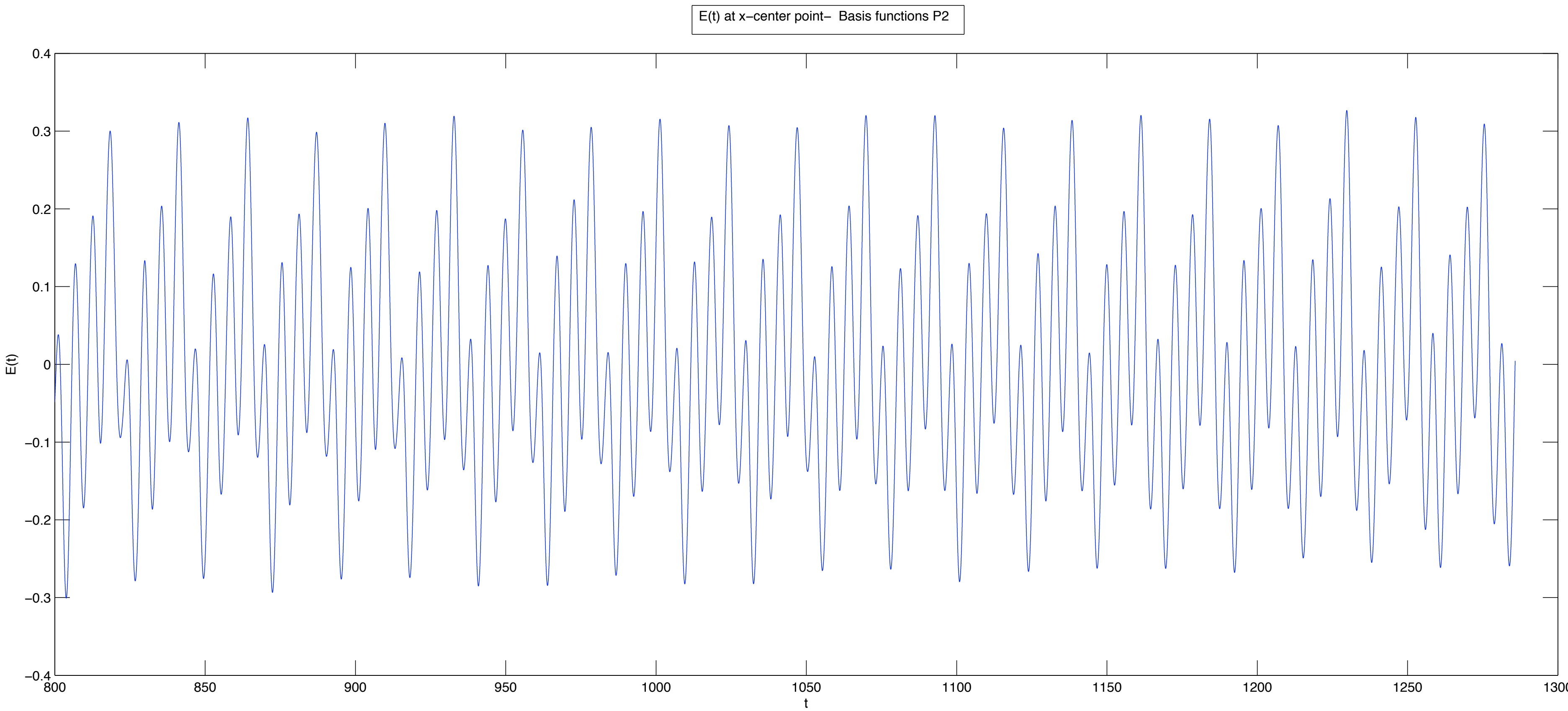
Driving f_M: Weak - Adiabatic



Central periodic orbit is BGK mode



Strong Drive

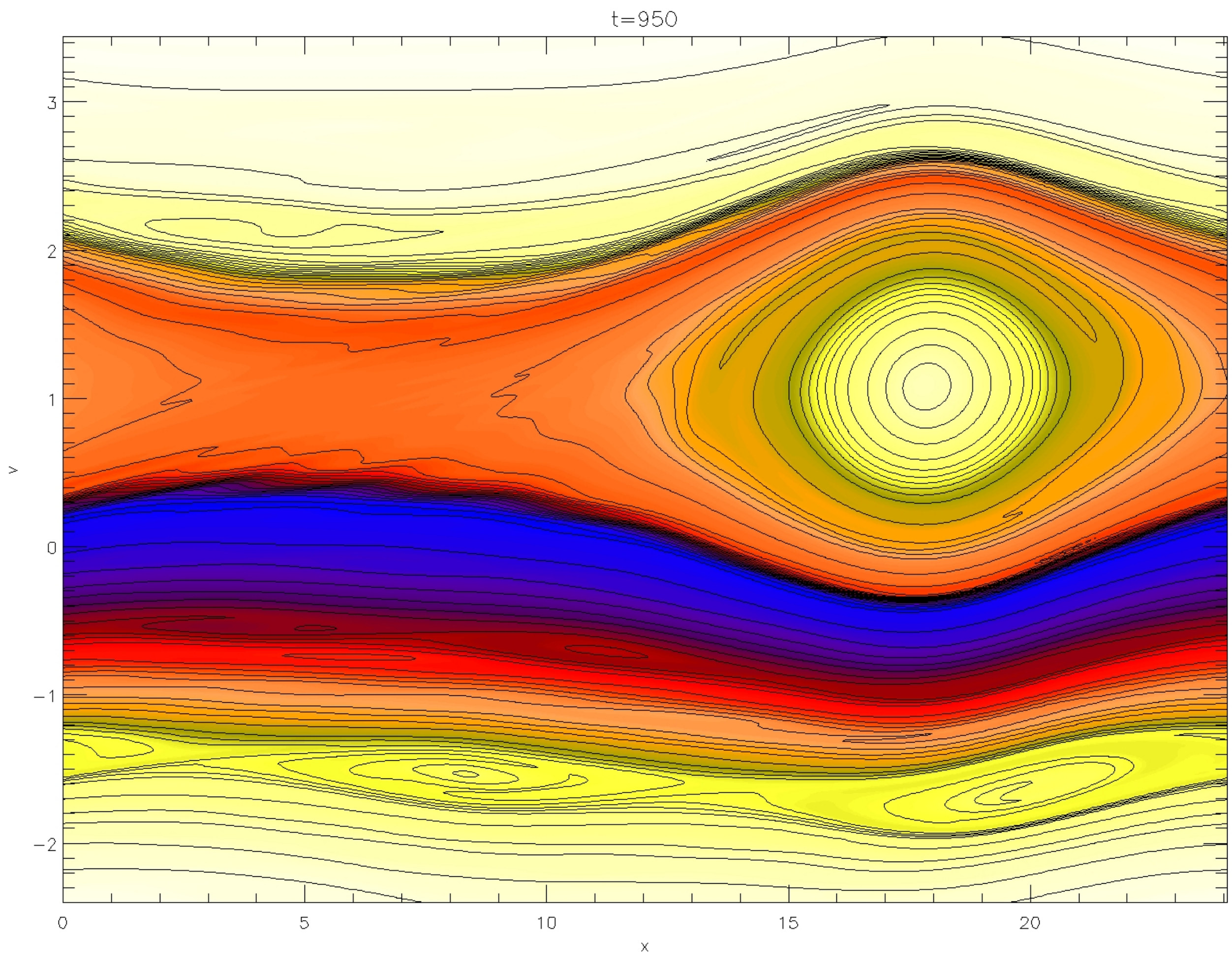


$$A_d(t) = .4 \text{ and } T_d = 200$$

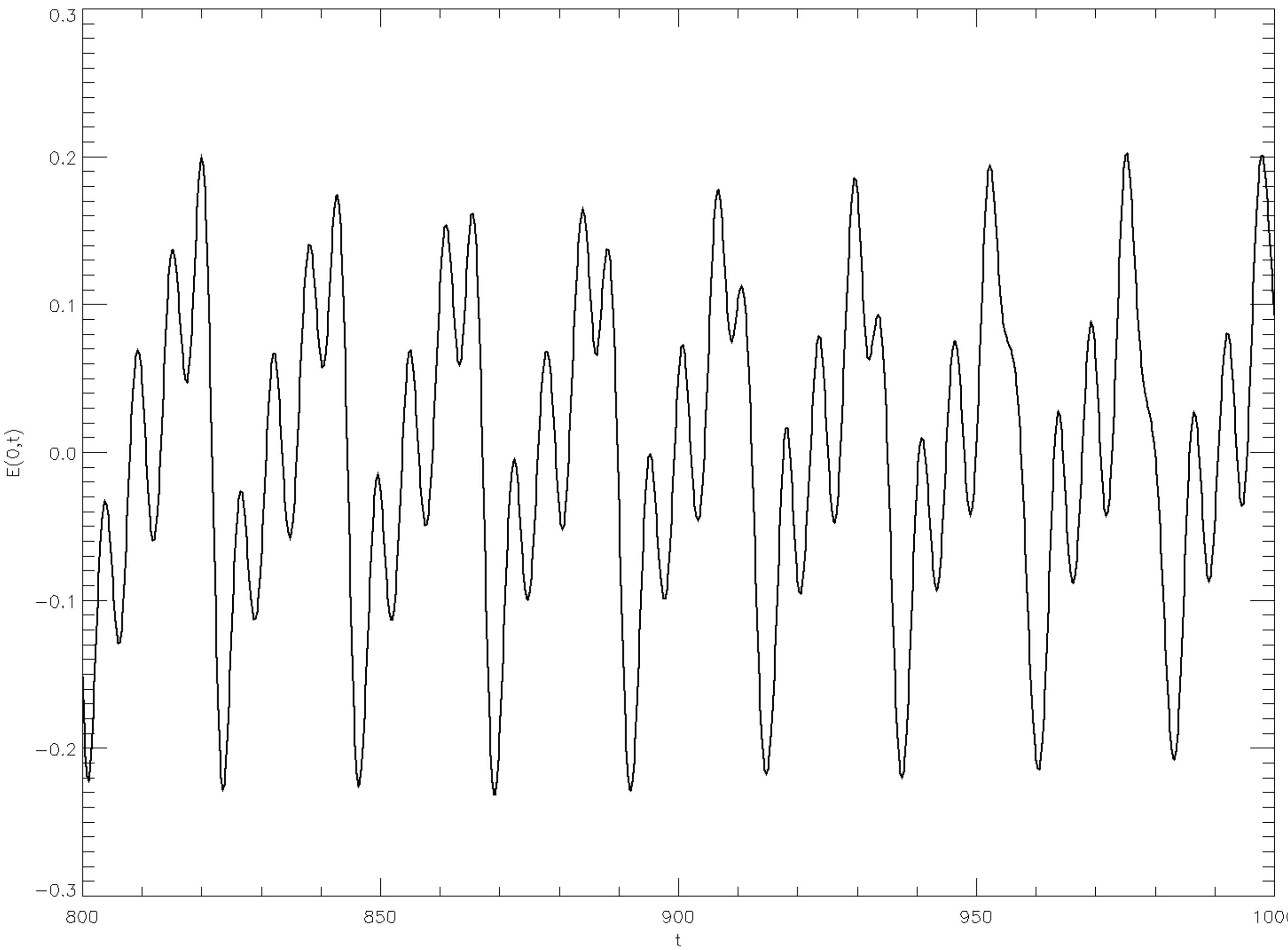
Higher Order Periodic/Quasiperiodic Orbit: $E(t) = A(t)E_0(t)$

$A(t) = A(t + T/4)$ with $E_0(t) = E_0(t + T)$

$E_0(t)$ like weak drive

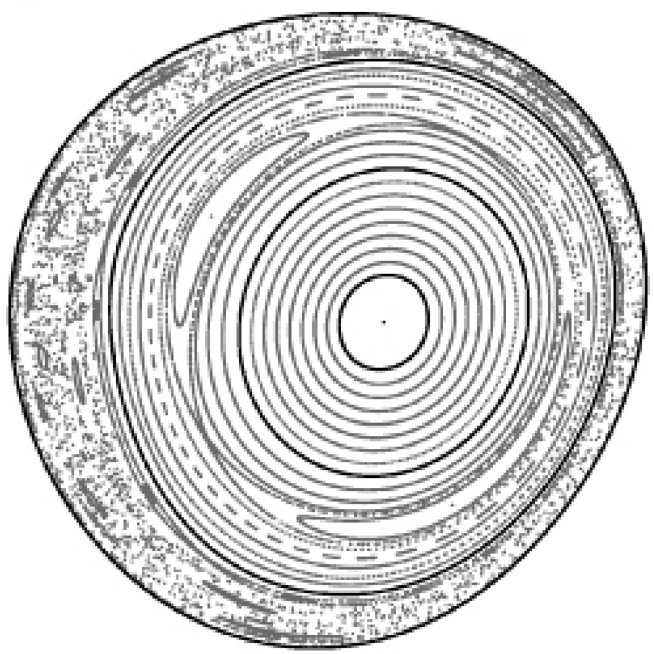


similar but different from Demeio and Zweifel (1990) mpg1(phase_space_2.mpg,shade_surf.mpg)



0.3
0.2
0.1
0.0
−0.1
−0.2
−0.3
E(0,t)
800
850
900
950
1000
t

Periodic and Quasiperiodic orbits



Multimode Drive

$$E_d(x,t) = E_{DA}^{(1)} g_1(t)\cos(k_1 x - \omega_1 t) + E_{DA}^{(2)} g_2(t)\cos(k_2 x + \omega_2 t)$$

mpg2(movie_1)

Recalcitrance

Strong Slap

$$E_d(x,t) = E_{DA}^{(1)} g_1(t) \cos(k_1 x - \omega_1 t)$$

mpg2(movie_2)

Summary

Cauchy vs. DA.

Lots of phenomenology.

Is any of it true?

<u>Some Refs</u>:

DG for Landau and BGK
Heath et al., J. Comp. Phys **231**, 1140 (2012).
Undamped electrostatic plasma waves
Valentini et al., Phys. Plasmas **19**, 092103 (2012)
Nonlinear DA structures
Yingda Cheng, et al., J. Sci. Computing **56**, 319 (2013) and Valentini et al. in progress.