Stationary solutions to the Boltzmann equation in the Hydrodynamic limit

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based on a joint work with Esposito, Kim, Marra

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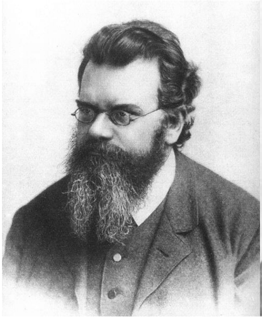
Founders of Kinetic Theory


1867, James Clerk Maxwell: the Boltzmann equation
1872, Ludwig Boltzmann: the H-Theorem (Entropy, Second law of thermodynamics)
1879, James Clerk Maxwell: Boundary Conditions
1900, David Hilbert: Hilbert's Sixth Problem

Hilbert 6 :
Mathematical treatment of the axioms of Physics

" It is therefore very desirable that the discussion of the foundations of mechanics be taken up by mathematicians also. Thus Boltzmann's work on the principles of mechanics suggests the problem of developing mathematically the limiting processes, there merely indicated, which lead from the atomistic view to the laws of motion of continua."
(David Hilbert, Mathematical Problems, 1900)

$$\boxed{\text{Boltzmann equation} \to \text{Fluid equations}}$$

The first formal descriptions were done by David Hilbert: Hilbert Expansion for the Boltzmann equation, 1916.

Plan of this Presentation

- **Scaling and the Hilbert Expansion**
- Previous Works and Main Theorem
- Mathematical Difficulties
- Key Ideas and Analysis

The Boltzmann Equation

PDE for the rarefied gas

$$\partial_t F + v \cdot \nabla_x F = Q(F, F)$$

- $F(t, x, v) \geq 0$: scalar, non-negative, density function defined on the phase space
- Spatial variable $x \in \Omega \subset \mathbf{R}^3$ with a smooth boundary $\partial\Omega$
- Velocity $v \in \mathbf{R}^3$

Q : Nonlinear Boltzmann Operator

$$Q(F_1, F_2)(v) = \int_{\mathbf{R}^3} \int_{\mathbf{S}^2} B(v-u,\omega) \Big\{ F_1(v')F_2(u') - F_1(v)F_2(u) \Big\} d\omega du$$

- Non-local operator in v (local in t, x)
- Collision kernel (hard sphere)

$$B(v-u,\omega) = \Big|(v-u)\cdot\omega\Big|.$$

- Post-collisional velocities for perfectly elastic collisions

$$v' + u' = v + u \quad \text{(conservation of momentum)}$$

$$|v'|^2 + |u'|^2 = |v|^2 + |u|^2 \quad \text{(conservation of energy)}$$

4 equations with 6 unknowns $\Longrightarrow \omega \in \mathbf{S}^2$.

Solutions of $Q(F,F)=0$

- $$\text{Equilibrium states}: Q(F,F)=0,$$

  $$\begin{aligned} &\iff F \text{ is a (local) Maxwellain} \\ &\iff F \equiv M_{\rho,u,T} = \frac{\rho}{(2\pi T)^{3/2}} \exp\left[-\frac{|v-u|^2}{2T}\right], \end{aligned}$$

  where the density ρ, the temperature T, and the mean velocity u.
- If the collision is dominant then $F \sim M_{\rho,u,T}$.

Scaling

- Macroscopic length scale: l_0, time scale t_0 and the reference temperature T_0.
- speed of sound (thermal speed) $c = \sqrt{\frac{5}{3}\frac{kT_0}{m}}$ where k the Boltzmann constant, m the molecular mass
- Mean free path λ is the scale of the average time that particles in the equilibrium spend traveling freely between two collisions.
- Rescaling: $\tilde{t} = t/t_0, \tilde{x} = x/l_0, \tilde{v} = v/c$ plus define the dimensionless collision kernel

$$\tilde{B}(\tilde{v} - \tilde{u}, \omega) \sim \text{average density} \times \frac{\lambda}{c} B(v - u, \omega).$$

- Dimensionless for of the Boltzmann equation

$$\mathrm{St}\partial_t F + v \cdot \nabla_x F = \frac{1}{\mathrm{Kn}} Q(F, F).$$

- Multiscalings: Knudsen number $\mathrm{Kn} = \frac{\lambda}{l_0}$ and Strouhal number $\mathrm{St} = \frac{l_0}{ct_0}$.

Dimensionless Form and Scalings

$$\mathrm{St}\,\partial_t F^\varepsilon + v\cdot\nabla_x F^\varepsilon = \frac{1}{\mathrm{Kn}} Q(F^\varepsilon, F^\varepsilon); \quad \mathrm{St} = \varepsilon^s, \quad \mathrm{Kn} = \varepsilon^q.$$

$$F^\varepsilon = \mu + \delta f^\varepsilon; \quad \mathrm{Ma} := \frac{u_0}{c} = \delta(\varepsilon^m), \quad \mathrm{Re} \sim \frac{\mathrm{Ma}}{\mathrm{Kn}},$$

where u_0 is the bulk velocity of the fluid.

- $q = 1,\ m = 0,\ s = 0:$ Compressible Euler ($F^\varepsilon \to F$)
- $q = 1,\ m = 1,\ s = 1:$ Incompressible Navier-Stokes-Fourier

$$\boxed{\varepsilon\partial_t F^\varepsilon + v\cdot\nabla_x F^\varepsilon = \frac{1}{\varepsilon} Q(F^\varepsilon, F^\varepsilon), \qquad F^\varepsilon = \mu + \varepsilon f^\varepsilon}$$

- $q = 1,\ m > 1,\ s = 1:$ Stokes-Fourier
- $q > 1,\ m = 1,\ s = 1:$ Incompressible Euler-Fourier

Hilbert Expansion

Plugging this Ansatz

$$F = \mu + \varepsilon\sqrt{\mu}\{f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + \cdots + \varepsilon^n f_n\},$$

in the dimensionless Boltzmann equation, and balancing the resulting coefficients of the successive powers of ε, one gets, as compatibility conditions to solve the hierarchy.

- Plugging the Hilbert expansion into the Boltzmann equation,

$$\begin{aligned} &[\varepsilon\partial_t + v\cdot\nabla_x]\{\varepsilon f_1\sqrt{\mu} + \cdots + \varepsilon^n f_n\sqrt{\mu}\} \\ &= \frac{1}{\varepsilon}\frac{1}{\sqrt{\mu}}Q(\mu + \sqrt{\mu}\{\varepsilon f_1 + \cdots + \varepsilon^n f_n\}, \mu + \sqrt{\mu}\{\varepsilon f_1 + \cdots + \varepsilon^n f_n\}). \end{aligned}$$

- Expanding $Q(\cdots,\cdots)$, we have a linearized collision operator

$$Lf := -\frac{1}{\sqrt{\mu}}\{Q(\mu,\sqrt{\mu}f) + Q(\sqrt{\mu}f,\mu)\},$$

and the nonlinear collision operator

$$\Gamma(g,f) := \frac{1}{\sqrt{\mu}}Q(\sqrt{\mu}g,\sqrt{\mu}f).$$

- From the collision invariants

$$\text{Null } L = \left\{\sqrt{\mu},\ v\sqrt{\mu},\ \frac{|v|^2-3}{2}\sqrt{\mu}\right\}.$$

- Define the projection onto Null L:

$$\mathbf{P}f = a(f)\sqrt{\mu} + b(f)\cdot v\sqrt{\mu} + c(f)\frac{|v|^2-3}{2}\sqrt{\mu}.$$

- $\mathbf{P}L \equiv 0$, $\mathbf{P}\Gamma \equiv 0$.

- Comparing the coefficients of $\varepsilon^0, \varepsilon^1, \cdots$,

$$
\begin{aligned}
0 &= Lf_1, \\
-v \cdot \nabla_x f_1 + \Gamma(f_1, f_1) &= Lf_2, \\
-\partial_t f_1 - v \cdot \nabla_x f_2 + \Gamma(f_1, f_2) + \Gamma(f_2, f_1) &= Lf_3, \cdots
\end{aligned}
$$

- $Lf_1 = 0 \implies f_1 = [\rho_1 + u_1 \cdot v + \vartheta_1 \frac{|v|^2-3}{2}]\sqrt{\mu} \in \text{Null } L$
- Compatibility condition (Fredholm alternative: $L \sim Id -$ compact operator K) implies

$$
\mathbf{P}(-v \cdot \nabla_x f_1) = 0 \implies \nabla_x[\rho_1 + \vartheta_1] = 0, \quad \nabla_x \cdot u_1 = 0,
$$

where we can obtain the Boussinesq relation and the incompressible condition.

- $f_2 = L^{-1}(-v \cdot \nabla_x f_1 + \Gamma(f_1, f_1)) + [\rho_2 + u_2 \cdot v + \vartheta_2 \frac{|v|^2-3}{2}]\sqrt{\mu}$ with $\mathbf{P}L^{-1} = 0$.
- From $\mathbf{P}(-\partial_t f_1 - v \cdot \nabla_x f_2) = 0$,

$$\begin{aligned} \partial_t u_1 + u_1 \cdot \nabla_x u_1 + \nabla_x p_1 - \mathfrak{v}\Delta_x u &= 0, \\ \partial_t \vartheta_1 + u_1 \cdot \nabla_x \vartheta_1 - \kappa \Delta_x \vartheta_1 &= 0, \end{aligned}$$

where $\nabla_x p_1 := \nabla_x[\rho_1\vartheta_1 + \rho_2 + \vartheta_2]$ and we can compute the viscosity $\mathfrak{v}$ and the heat conductivity κ.

Physical Problems with Boundaries

Consider the rarefied gas contained in some bounded domain.

- Boundary conditions are determined by the physics of the interaction between gas particles and boundaries
- For kinetic equations, boundary conditions are ONLY imposed for incoming particles:

$$F^{\varepsilon}(t,x,v) = \text{Boundary Condition}(t,x,v), \quad x \in \partial\Omega,\ n(x)\cdot v < 0.$$

where $n(x)$ is the outward normal direction

Diffuse Reflection BC

For $x \in \partial\Omega,\ n(x) \cdot v < 0$ (incoming particles)

$$F^\varepsilon(t,x,v) = \mathrm{BC}(t,x,v).$$

- Diffusive BC

$$BC = \mathcal{M}^w(x,v) \int_{n(x)\cdot u>0} F^\varepsilon(t,x,u)\{n(x)\cdot u\}du$$

where the wall Maxwellian is given by

$$\mathcal{M}^w(x,v) = \frac{1}{2\pi[T^w(x)]^2} \exp\left[-\frac{|v|^2}{2T^w(x)}\right],$$

with a wall temperature $T^w(x)$ at $x \in \partial\Omega$.

- Since we are looking for the solution $F = \mu + \mathrm{Ma}\ f^\varepsilon$ with $\mathrm{Ma} = \varepsilon$ it is natural to set

$$T^w(x) = 1 + \varepsilon\vartheta^w(x) \quad \text{on } x \in \partial\Omega.$$

- Notation: $\mathcal{M}^\varepsilon := \frac{1}{2\pi[1+\varepsilon\vartheta^w(x)]^2} \exp\left[-\frac{|v|^2}{2[1+\varepsilon\vartheta^w(x)]}\right].$

Diffuse Reflection BC

- *The Diffusive Boundary Condition* captures the thermodynamic interaction(diffusion!) between boundary(with temperature) and gas particles.
- Null flux at the boundary :

$$\int_{\mathbf{R}^3} F^\varepsilon(x,v)\{n(x)\cdot v\}dv = 0, \quad x \in \partial\Omega.$$

- Only Total Mass is Conserved !

$$\iint_{\Omega\times\mathbf{R}^3} F^\varepsilon(t,x,v)dxdv = \iint_{\Omega\times\mathbf{R}^3} F^\varepsilon(0,x,v)dxdv.$$

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Two Modern Frameworks

$$\varepsilon \partial_t F^\varepsilon + v \cdot \nabla_x F^\varepsilon = \frac{1}{\varepsilon} Q(F^\varepsilon, F^\varepsilon)$$

$$\downarrow$$

Incompressible Navier-Stokes-Fourier system

- Diperna-Lions' renormalized solution to the Leray's weak solution: Initiated by Bardos-Golse-Levermore and further studied by Lions, Masmoudi, and others. Full result was obtained by Golse−Saint-Raymond, 2004
- Construct the truncated Hilbert expansion to the smooth solution of the fluid equations. (Caflisch, Bardos-Ukai, Guo, Esposito $\cdots$)

Stationary Solutions with Boundary

- The Boltzmann solution cannot be Maxwellian if the boundary temperature is not constant (Non-equilibrium steady state)
- Even for fixed ε, the analog of Diperna-Lions' solution is not available. (entropy dissipation is too weak)
- Golse pointed out exactly this hydrodynamic limit problem as a challenging open problem (Hydrodynamic limits, European Congress of Mathematics, 699–717, Eur. Math. Soc., Zurich, (2005))
- For general domain with C^3 boundary, the existence, uniqueness, positivity, and dynamical asymptotic stability for fixed small ε were established in Esposito-G-Kim-Marra (2013)

Precise Setting for the Steady Problems

Construct and obtain a uniform-in-ε estimate to

$$F_s^\varepsilon = \mu + \varepsilon f_{1,s}(x)\sqrt{\mu} + \varepsilon^2 f_{2,s}(x)\sqrt{\mu} + \varepsilon^{3/2} R_s^\varepsilon \sqrt{\mu}$$

solving

$$\begin{gathered} v \cdot \nabla_x F_s^\varepsilon + \varepsilon^2 \Phi \cdot \nabla_v F_s^\varepsilon = \tfrac{1}{\varepsilon} Q(F_s^\varepsilon, F_s^\varepsilon), \\ F_s^\varepsilon(x,v)|_{\gamma_-} = \mathcal{M}^\varepsilon \int_{n(x)\cdot u>0} F_s^\varepsilon(x,u)\{n(x)\cdot u\} du \end{gathered}$$

where $f_1 = [\rho_s + u_s \cdot v + \vartheta_s \frac{|v|^2-3}{2}]\sqrt{\mu}$ solves the INSF

$$\begin{aligned} u_s \cdot \nabla_x u_s + \nabla_x p_s = \mathfrak{v} \Delta_x u_s + \Phi, \quad \nabla_x \cdot u_s = 0 \quad & \text{in} \quad \Omega, \\ u_s \cdot \nabla_x \vartheta_s = \kappa \Delta_x \vartheta_s \quad & \text{in} \quad \Omega, \\ u_s(x) = 0, \quad \vartheta_s(x) = \vartheta^w(x) \quad & \text{on} \quad \partial\Omega. \end{aligned}$$

Main Theorem: Validity of the Steady Problems

Assume $\Omega \subset \mathbb{R}^3$ is an open bounded region of $\mathbb{R}^3$ with C^3 boundary. Assume $\Phi = \Phi(x)$ and ϑ^w in some high Sobolev space. If

$$||\vartheta^w||_{H^{1+}(\partial\Omega)} + ||\Phi||_{L^{1.5+}(\Omega)} \ll 1,$$

then, for all $0 < \varepsilon \ll 1$, there exists a unique positive solution

$$F_s^\varepsilon = \mu + \varepsilon f_{1,s}(x)\sqrt{\mu} + \varepsilon^2 f_{2,s}(x)\sqrt{\mu} + \varepsilon^{3/2} R_s^\varepsilon \sqrt{\mu} \geq 0,$$

to the steady Boltzmann equation with the diffusive BC and $f_{1,s} = [\rho_s + u_s \cdot v + \vartheta_s \frac{|v|^2-3}{2}]\sqrt{\mu}$ solves the steady INSF. Moreover,

$$\begin{aligned}
||R_s^\varepsilon||_{L^2(\Omega\times\mathbb{R}^3)} + \varepsilon^{-1}||(\mathbf{I}-\mathbf{P})R_s^\varepsilon||_2 &\ll 1, \\
\varepsilon||e^{\beta|v|^2} R_s^\varepsilon||_{L^\infty(\bar{\Omega}\times\mathbb{R}^3)} &\ll 1, \\
||f_{s,1}||_{L_x^6 L_v^2 \cap L_{x,v}^\infty} + ||f_{2,s}||_{L_x^6 L_v^2 \cap L_{x,v}^\infty} &\lesssim 1,
\end{aligned}$$

for $0 < \beta \ll 1$.

Precise Setting for the Unsteady Problems

We are looking for the perturbation

$$
\begin{aligned}
F^{\varepsilon} &= F_s^{\varepsilon} + \varepsilon \tilde{f}_1 \sqrt{\mu} + \varepsilon^2 \tilde{f}_2 \sqrt{\mu} + \varepsilon^{3/2} \tilde{R}^{\varepsilon} \sqrt{\mu}, \\
\tilde{f}_1 &= [\tilde{\rho}_1 + \tilde{u}_1 \cdot v + \tilde{\vartheta}_1 \frac{|v|^2 - 3}{2}] \sqrt{\mu},
\end{aligned}
$$

where $f_{s,1} + \tilde{f}_1 = [\rho_1 + u_1 \cdot v + \vartheta_1 \frac{|v|^2-3}{2}] \sqrt{\mu}$ and $(\rho_1, u_1, \vartheta_1)$ solves the INSF. Without loss of generality we may assume $\tilde{\rho} = -\tilde{\vartheta} + \fint \tilde{\vartheta}$. We define the energy and the the dissipation as

$$
\begin{aligned}
\mathcal{E}_{\lambda}(t) &:= \sup_{0 \le s \le t} ||e^{\lambda s} \tilde{R}^{\varepsilon}(s)||^2_{L^2(\Omega \times \mathbb{R}^3)} + \sup_{0 \le s \le t} ||e^{\lambda s} \partial_t \tilde{R}^{\varepsilon}(s)||^2_{L^2(\Omega \times \mathbb{R}^3)}. \\
\mathcal{D}_{\lambda}(t) &= \frac{1}{\varepsilon^2} \int_0^t ||e^{\lambda s} (\mathbf{I} - \mathbf{P}) \tilde{R}^{\varepsilon}||_2^2 + \frac{1}{\varepsilon^2} \int_0^t ||e^{\lambda s} (\mathbf{I} - \mathbf{P}) \partial_t \tilde{R}^{\varepsilon}||_2^2 \\
&+ \int_0^t ||e^{\lambda s} \mathbf{P} \tilde{R}^{\varepsilon}||_2^2 + \int_0^t ||e^{\lambda s} \mathbf{P} \partial_t \tilde{R}^{\varepsilon}||_2^2 + \text{boundary terms}
\end{aligned}
$$

Main Theorem: Validity of the Dynamic Problems

Assume the same hypothesis of the previous theorem. Assume $F_0 = F_s^\varepsilon + \varepsilon\sqrt{\mu}[\tilde{f}_1(0) + \varepsilon\tilde{f}_2(0) + \varepsilon^{1/2}\tilde{R}^\varepsilon(0)] \geq 0$, and $\tilde{u}(0)$ and $\tilde{\vartheta}(0)$ in some high Sobolev space, e.g. $H^8(\Omega)$ and

$$||\tilde{u}_1(0)||_{H^4(\Omega)} + ||\tilde{\vartheta}_1(0)||_{H^4(\Omega)} \ll 1$$

If

$$\begin{aligned} \mathcal{E}(0) + \frac{1}{\varepsilon}||(\mathbf{I}-\mathbf{P})\tilde{R}_0^\varepsilon||_2 + \varepsilon^{3/2}||e^{\beta|v|^2}\partial_t\tilde{R}_0^\varepsilon||_{L^\infty(\bar{\Omega}\times\mathbb{R}^3)} &\ll 1, \\ \varepsilon||e^{\beta|v|^2}\tilde{R}_0^\varepsilon||_{L^\infty(\bar{\Omega}\times\mathbb{R}^3)} &\lesssim 1. \end{aligned}$$

Then there exists a unique positive solution

$$F^\varepsilon = F_s^\varepsilon + \varepsilon\tilde{f}_1\sqrt{\mu} + \varepsilon^2\tilde{f}_2\sqrt{\mu} + \varepsilon^{3/2}\tilde{R}^\varepsilon\sqrt{\mu}$$

to the unsteady Boltzmann equation with the diffuse BC. Moreover,

$$\begin{aligned} \mathcal{E}_\lambda(\infty) + \mathcal{D}_\lambda(\infty) + \sup_{0\leq t\leq\infty} \varepsilon^{\frac{3}{2}}||e^{\beta|v|^2}\partial_t\tilde{R}^\varepsilon(t)||_\infty &\ll 1, \\ \sup_{0\leq t\leq\infty} \varepsilon||e^{\beta|v|^2}\tilde{R}^\varepsilon(t)||_{L^\infty(\bar{\Omega}\times\mathbb{R}^3)} &\lesssim 1. \end{aligned}$$

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- Scaling and the Hilbert Expansion
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Equation for R_s^ε

$$
\begin{aligned}
& v\cdot\nabla_x R_s^\varepsilon+\varepsilon^2\frac{1}{\sqrt{\mu}}\Phi\cdot\nabla_v\left[\sqrt{\mu}R_s\right]+\varepsilon^{-1}LR_s^\varepsilon \\
= \quad & 2\Gamma(f_1+\varepsilon f_2,R_s^\varepsilon)+\varepsilon^{1/2}\Gamma(R_s^\varepsilon,R_s^\varepsilon)+\varepsilon^{1/2}A_s,
\end{aligned}
$$

and on $x\in\partial\Omega$ and $n(x)\cdot v<0$

$$
R_s^\varepsilon(x,v)=P_\gamma R_s^\varepsilon+\varepsilon^{1/2}r,
$$

where

$$
P_\gamma R_s^\varepsilon(x,v)=\sqrt{2\pi}\sqrt{\mu(v)}\int_{n(x)\cdot u>0}R_s^\varepsilon(x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}du
$$

Here

$$
\begin{aligned}
|r| \quad & \sim \quad |\nabla_x u_{s,1}|+|\nabla_x\vartheta_{s,1}| \\
|A| \quad & \sim \quad |\nabla_x^2 u_{s,1}|+|\nabla_x^2\vartheta_{s,1}|
\end{aligned}
$$

Difficulty 1: Linear operator L is not fully coercive

- L is semi-positive:

$$\|(\mathbf{I}-\mathbf{P})R_s^\varepsilon\|_2^2 \lesssim \iint_{\Omega\times\mathbb{R}^3} LR_s^\varepsilon R_s^\varepsilon$$

- From the energy estimate to the equation of R_s

$$v\cdot\nabla_x R_s^\varepsilon + \varepsilon^2 \frac{1}{\sqrt{\mu}}\Phi\cdot\nabla_v\left[\sqrt{\mu}R_s^\varepsilon\right] + \varepsilon^{-1}LR_s^\varepsilon = \text{forcing},$$

$$R_s|_{\gamma_-} = P_\gamma R_s^\varepsilon + \varepsilon^{1/2} r,$$

we obtain

$$\varepsilon^{-1}\iint_{\Omega\times\mathbb{R}^3} LR_s^\varepsilon R_s^\varepsilon + \text{"boundary term"}$$
$$\lesssim \iint_{\Omega\times\mathbb{R}^3} |\text{forcing}||R_s^\varepsilon| + \int_{n(x)\cdot v>0} |r(x,v)|^2\{n(x)\cdot v\}dS_x dv.$$

We do not have a control of $\mathbf{P}R_s^\varepsilon$. With the same reason, the unsteady problem is degenerated dissipative.

Difficulty 2: Derivatives of Boltzmann solution is bad!

- The reason that we choose $\mu+\varepsilon\sqrt{\mu}[f_{s,1}+\varepsilon^2 f_{s,2}+\varepsilon^{1/2}R_s^\varepsilon]$ is to AVOID the boundary layer expansion.
- In general, for the boundary value problem, people use the boundary layer expansion:

$$F^\varepsilon=\mu+\varepsilon\sqrt{\mu}[f_1+\varepsilon f_2+\cdots]+\varepsilon\sqrt{\mu}[\varepsilon f_1^B+\varepsilon^2 f_2^B+\cdots],$$

  where $\{f_i\}$ are interior expansion (Hilbert-like expansion) and $\{f_j^B\}$ solves half space problem (Milne-line problem), whose variable is rescaled distance from the boundary.
- The equation of f_2^B contains the derivative of f_1^B which is the solution to the linearized Boltzmann equation.
- Derivatives of Boltzmann solution is expected not belong to H^1 and second order derivatives do not exists.
  [G, Kim, G-Kim-Trescases-Tonon 1,2]
- We should emphasis that the leading order term ($\varepsilon^{-1/2}$) of the boundary condition for R_s^ε due to the boundary condition $u_{s,1}=0$. (For Euler case ($u_{s,1}\cdot n|_{\partial\Omega}=0$) we should introduce the boundary layer)

Difficulty 3: The nonlinearity $\varepsilon^{1/2}|R_s^\varepsilon|^2$ is too singular!

- Drawback of small α: From $F_s^\varepsilon \sim \varepsilon^{1+\alpha} R_s^\varepsilon$ and $v \cdot \nabla_x F_s^\varepsilon \sim \varepsilon^{-1}|F_s^\varepsilon|^2$

$$v \cdot \nabla_x R_s^\varepsilon \sim \varepsilon^\alpha |R_s^\varepsilon|^2$$

- With $\alpha = \frac{1}{2}$, from the coercivity estimate

$$||\mathbf{P} R_s^\varepsilon||_2 \lesssim \varepsilon^{1/2}|R_s^\varepsilon|^2.$$

- From $L^2 - L^\infty$ bootstrap argument in Guo, Esposito-G-Kim-Marra,

$$||R_s^\varepsilon||_\infty \lesssim \frac{1}{\varepsilon^{d/2}}||R_s^\varepsilon||_2$$

- In 3D, $d = 3$,

$$||R_s^\varepsilon||_2 \lesssim \varepsilon^{1/2}||R_s^\varepsilon||_2||R_s^\varepsilon||_\infty \lesssim \frac{1}{\varepsilon}!$$

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Key 1: L^2 Coercivity Estimate

Weak formulation (Green's identity)

$$\int_\gamma \psi R_s^\varepsilon - \iint_{\Omega\times\mathbf{R}^3} v\cdot\nabla\psi R_s^\varepsilon = -\varepsilon^{-1}\iint_{\Omega\times\mathbf{R}^3}\psi L(\mathbf{I}-\mathbf{P})R_s^\varepsilon + \iint_{\Omega\times\mathbf{R}^3}\psi g$$

$$\begin{aligned} \text{bulk} \qquad R_s^\varepsilon &= \{a + v\cdot b + \frac{|v|^2-3}{2}c\}\sqrt{\mu} + (\mathbf{I}-\mathbf{P})R_s^\varepsilon \\ \text{boundary} \qquad R_s^\varepsilon|_\gamma &= P_\gamma R_s^\varepsilon + (1-P_\gamma)R_s^\varepsilon \mathbf{1}_{\gamma_+} + \varepsilon^{1/2} r \mathbf{1}_{\gamma_-} \end{aligned}$$

Idea : $v\cdot\nabla_x\psi \times (a_f, b_f, c_f)$ selects $(|a_f|^2, |b_f|^2, |c_f|^2)$ with proper

$$\psi(x,v) = \phi(v)\cdot\{1^{st}\ order\ operator(\partial_x)\}(-\Delta)^{-1}(a_f, b_f, c_f)$$

with proper boundary condition for $(-\Delta)^{-1}(a_f, b_f, c_f)$.

c_f estimate

Test functions for c_f

$$\psi_c = (|v|^2 - \beta_c)\sqrt{\mu}\{v \cdot \nabla_x\}(-\Delta_0)^{-1}c_f$$

where

$$-\Delta(-\Delta_0)^{-1}c_f = c_f$$

with Dirichlet BC
with $\int_{\mathbf{R}^3}(|v|^2 - \beta_c)v_i^2\mu(v)dv = 0$

$$\Downarrow$$

$$||c||_2^2 \lesssim |(1 - P_\gamma)f|_{2,+}^2 + \varepsilon^{-2}||(\mathbf{I} - \mathbf{P})f||_2^2 + ||g||_2^2 + ||r||_2^2$$

b_f estimate

Test functions for b_f

$$\psi_b^{i,j} = (v_i^2 - \beta_b)\sqrt{\mu}\partial_j(-\Delta_0)^{-1}(b_f)_j \quad \text{for all} \quad i,j = 1,2,\cdots d$$

with $\int_{\mathbf{R}^3}(v_i^2 - \beta_b)\mu(v)dv = 0$, for all i

$$\phi_b^{i,j} = v_i v_j |v|^2\sqrt{\mu}\partial_j(-\Delta_0)^{-1}(b_f)_i \quad \text{for all} \quad i \neq j$$

$$\Downarrow$$

$$||b||_2^2 \lesssim |(1-P_\gamma)f|_{2,+}^2 + \varepsilon^{-2}||(\mathbf{I}-\mathbf{P})f||_2^2 + ||g||_2^2 + ||r||_2^2$$

a_f estimate

Test functions for a_f

$$\psi_a = (|v|^2 - \beta_a)\{v \cdot \nabla_x\}(-\Delta_N)^{-1} a_f$$

where $(-\Delta_N)^{-1} a_f$ satisfies the Nuemann BC
$(\iint f\sqrt{\mu} = \iint a = 0)$
with $\int_{\mathbf{R}^3} (|v|^2 - \beta_a)(\frac{|v|^2}{2} - \frac{3}{2})(v_i)^2 \mu(v) dv = 0$ for all i

$$\Downarrow$$

$$||a||_2^2 \lesssim |(1 - P_\gamma) f|_{2,+}^2 + \varepsilon^{-2} ||(\mathbf{I} - \mathbf{P}) f||_2^2 + ||g||_2^2 + ||r||_2^2$$

Time-dependent case

For the time-dependent case, in the weak formulation, we have

$$\varepsilon \int_s^t \iint_{\Omega\times\mathbb{R}^3} \partial_t \psi \tilde{R}^\varepsilon$$

- Recall that $\psi \sim \nabla_x(-\Delta_x)^{-1}\mathbf{P}\tilde{R}^\varepsilon$ and therefore
$$\partial_t\psi \sim \nabla_x(-\Delta_x)^{-1}\partial_t\mathbf{P}\tilde{R}^\varepsilon$$
- From
$$\begin{aligned}
&\mathbf{P}\{\varepsilon\partial_t\tilde{R}^\varepsilon + v\cdot\nabla_x\tilde{R}^\varepsilon + \varepsilon^2\Phi\cdot\nabla_v\tilde{R} + \cdots\} \\
= \ &\int_{\mathbb{R}^3} dv[1, v, \frac{|v|^2-3}{2}]\sqrt{\mu}\{\varepsilon\partial_t\tilde{R}^\varepsilon + v\cdot\nabla_x\tilde{R}^\varepsilon + \varepsilon^2\Phi\cdot\nabla_v\tilde{R} + \cdots \\
= \ &\varepsilon\mathbf{P}\partial_t\tilde{R}^\varepsilon + \nabla_x f + \cdots
\end{aligned}$$
- $||\varepsilon\mathbf{P}\partial_t\tilde{R}^\varepsilon||_{H^{-1}(\Omega)} \lesssim ||\nabla_x f||_{H^{-1}(\Omega)} \lesssim ||f||_{L^2}$
-
$$\begin{aligned}
\varepsilon||\partial_t\psi||_{L^2} &\lesssim \varepsilon||\nabla_x(-\Delta_x)^{-1}\partial_t\mathbf{P}\tilde{R}^\varepsilon||_{L^2} \\
&\lesssim ||\varepsilon\mathbf{P}\partial_t\tilde{R}^\varepsilon||_{H^{-1}(\Omega)} \lesssim ||\tilde{R}^\varepsilon||_{L^2}.
\end{aligned}$$

Key 2: Qualitative $L^p - L^\infty$ Estimate

- Qualitative $L^p - L^\infty$ estimate in Guo and Esposito-G-Kim-Marra

$$||R_s^\varepsilon||_{L^\infty(\mathbb{R}^d)} \lesssim \frac{1}{\varepsilon^{d/p}}||R_s^\varepsilon||_{L^p(\mathbb{R}^d)} \tag{1}$$

- In 3D with $L^3(\mathbb{R}^3)$,

$$||R_s^\varepsilon||_{L^\infty(\mathbb{R}^3)} \lesssim \frac{1}{\varepsilon}||R_s^\varepsilon||_{L^3(\mathbb{R}^3)}$$

- On the other hand

$$\begin{aligned} \varepsilon^{1/2}||\Gamma(R_s^\varepsilon, R_s^\varepsilon)||_{L^2} &\lesssim \varepsilon^{1/2}||\Gamma((\mathbf{I}-\mathbf{P})R_s^\varepsilon, R_s^\varepsilon)||_{L^2} \\ &+ \varepsilon^{1/2}||\Gamma(\mathbf{P}R_s^\varepsilon, \mathbf{P}R_s^\varepsilon)||_{L^2}. \end{aligned}$$

- IF $\mathbf{P}R_s^\varepsilon \in L^3$

$$\begin{aligned} \varepsilon^{1/2}|||\mathbf{P}R_s^\varepsilon|^2||_{L^2} &\lesssim \varepsilon^{1/2}||\mathbf{P}R_s^\varepsilon||_{L^3}||\mathbf{P}R_s^\varepsilon||_{L^6} \\ &\lesssim ||\mathbf{P}R_s^\varepsilon||_{L^3}||\mathbf{P}R_s^\varepsilon||_{L^3}^{1/2}\left[\varepsilon||R_s^\varepsilon||_{L^\infty}\right]^{1/2} \end{aligned}$$

Key 3: Gain of Integrability: $\mathbf{P}R_s^\varepsilon \in L^3$

Let

$$v \cdot \nabla_x f + \varepsilon^2 \Phi \cdot \nabla_v f = g.$$

We define

$$f_\delta(x, v) = \text{Non-Grazing part of } f$$

Then

$$\Big|\Big| \int_{\mathbb{R}^3} |f_\delta| \langle v \rangle^4 \mu^{1/2} dv \Big|\Big|_{L^3(\Omega)} \lesssim ||f||_{L^2} + ||g||_{L^2} + \text{boundary term}.$$

c.f. This estimate is well-known for $\int_{\mathbb{R}^3} f \phi dv$ with $\phi(v) \in C_c^\infty$ by Golse-Lions-Perthame-Sentis and Diperna-Lions when $\Phi = 0$ and $\Omega = \mathbb{R}^3$ and by the sharp Sobolev embedding $H^{1/2}(\mathbb{R}^3) \subset L^3(\mathbb{R}^3)$. On the other hand with field, $\int_{\mathbb{R}^3} f \phi dv \in H_x^{1/4} \notin L^3$.

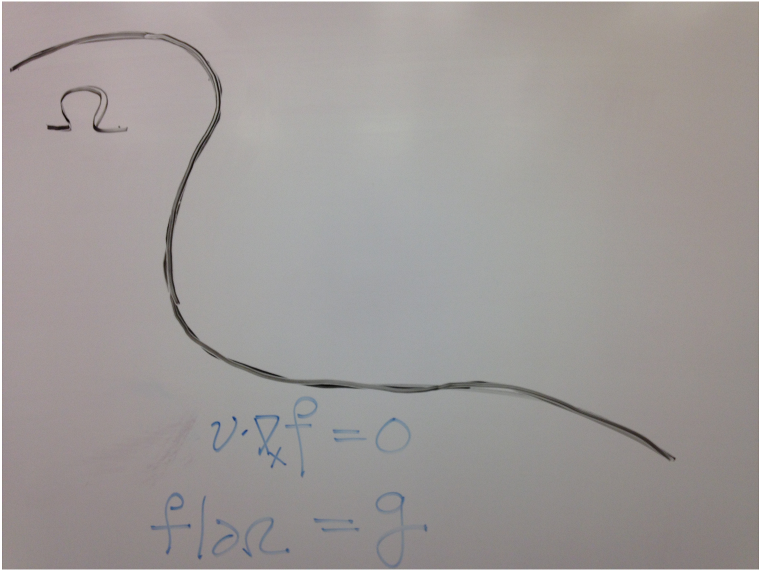
- Step 1: Extension $\bar{f}$: A priori bound of $(\mathbf{I} - \mathbf{P})f$ is crucial.
- Step 2: We use the method of Jabin-Vega: utilize characteristics to obtain sharp L^3 directly

Step 1: Extension $\bar{f}$

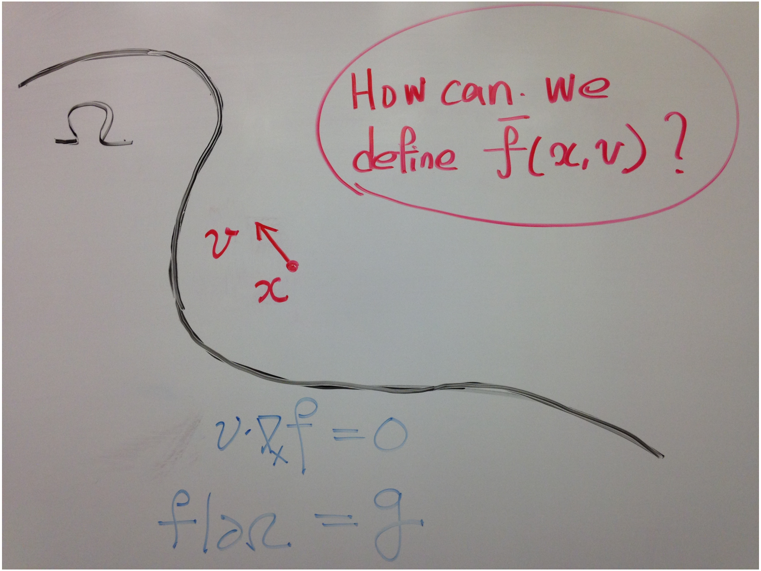
- The key is that we already have a control of $(\mathbf{I}-\mathbf{P})f$
- Since $f = \mathbf{P}f \times \text{nice test function } + (\mathbf{I}-\mathbf{P})f$, in order to have a bound of $\|\mathbf{P}f\|_{L^3(\Omega)}$ we only need to control

$$f(x,v)\mathbf{1}_{v\notin \text{ small set}}$$

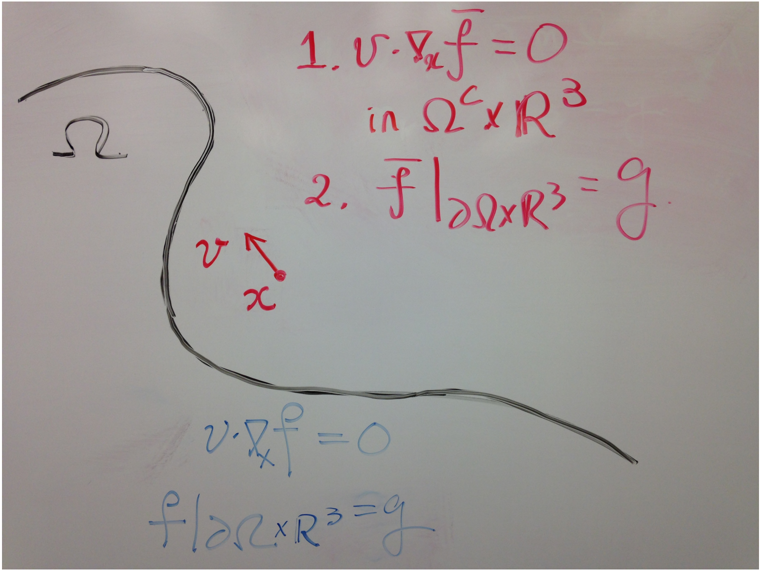
Step 1: Extension $\bar{f}$



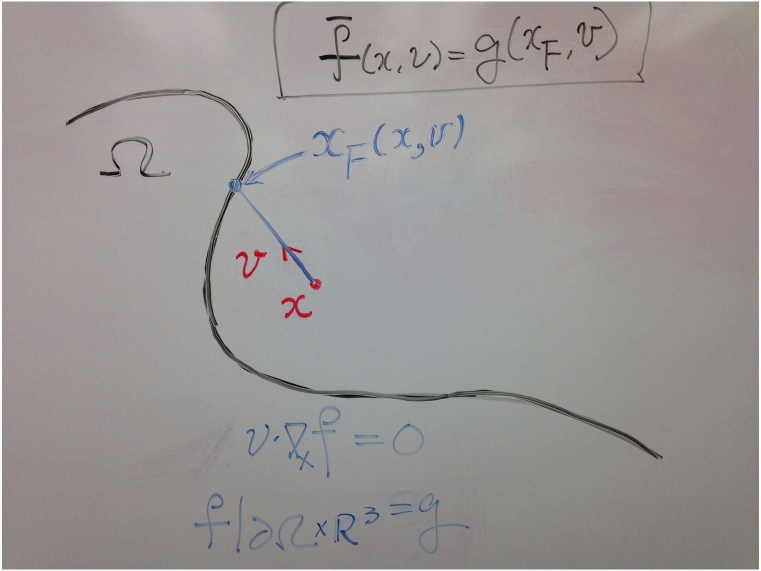
Step 1: Extension $\bar{f}$



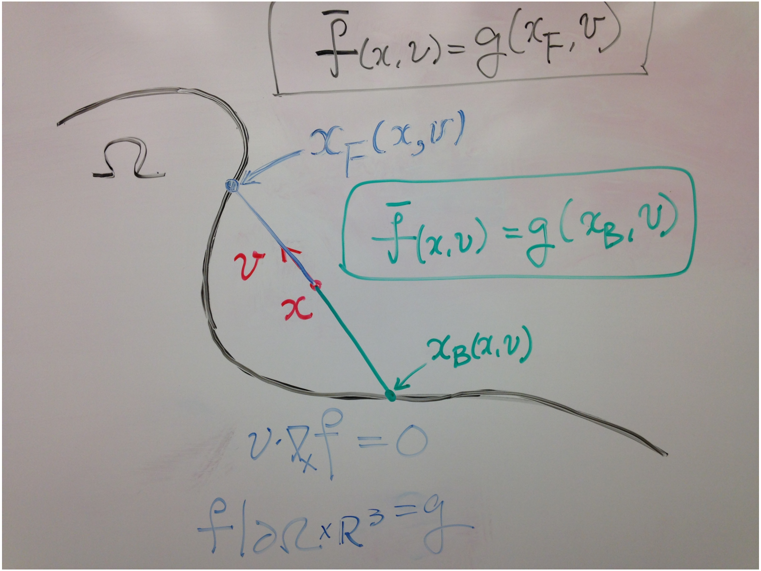
Step 1: Extension $\bar{f}$



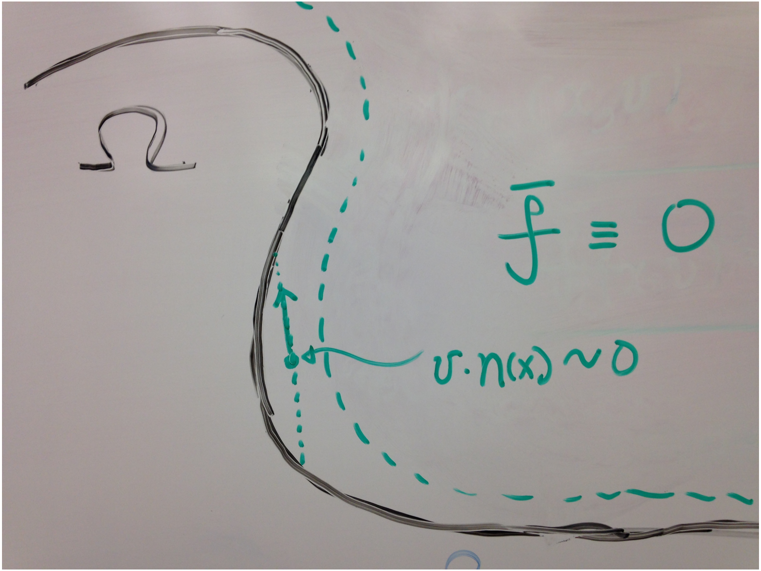
Step 1: Extension $\bar{f}$



Step 1: Extension $\bar{f}$



Step 1: Extension $\bar{f}$



Details of Step 2

- Along the characteristics $[X(s;0,x,v), V(s;0,x,v)]$

$$\bar{f}(x,v) = \int_0^T g(X(s;0,x,v), V(s;0,x,v))ds$$

- We consider, for $\phi = \phi(v) \in C_c^\infty$,

$$Sg(x) := \int_0^T \int_{\mathbb{R}^3} g(X(s;0,x,v), V(s;0,x,v))\phi(v)dvds$$

- In order to adopt S^*S trick, we define the dual

$$S^*(h)(x,v) = \int_0^T h(X(-s;0,x,v))\phi(V(-s;0,x,v))ds$$

- Indeed, by $(x,v) \leftrightarrow (X(s;0,x,v), V(s;0,x,v))$

$$(Sg,h)_{L^2_x} = (g, S^*h)_{L^2_{x,v}}.$$

- $S : L^2 \to L^3$ and $S^* : L^{3/2} \to L^2$ so that $SS^* : L^{3/2} \to L^3$.
- For $1/p + 1/p' = 1$,

$$\begin{aligned} ||Sg||_{L^p_x} &= \sup_{||h||_{L^{p'}} \le 1} (Sg, h)_{L^2_x} = \sup_{||h||_{L^{p'}} \le 1} (g, S^*h)_{L^2_x} \\ &\le ||g||_{L^2_{x,v}} \sup_{||h||_{L^{p'}_x} \le 1} ||S^*h||_{L^2_{x,v}} \end{aligned}$$

- To $||Sg||_{L^p_x} \lesssim ||g||_{L^2_{x,v}}$ It suffice to prove $||S^*h||_{L^2_{x,v}} \lesssim ||h||_{L^{p'}_x}$
- From

$$||S^*h||^2_{L^2_{x,v}} = (S^*h, S^*h)_{L^2_{x,v}} = (SS^*h, h)_{L^2_x} \le ||SS^*h||_{L^p_x} ||h||_{L^{p'}_x},$$

we only need to prove

$$||SS^*h||_{L^p_x} \lesssim ||h||_{L^{p'}_x}. \tag{2}$$

- $$SS^*g(x) = \int_0^T \int_0^T \int_{\mathbb{R}^3} g(X(-s+\tau;0,x,v)) \\ \times \phi(V(-s+\tau;0,x,v))\phi(v)dvd\tau ds$$

- For simplicity, ignore Φ. Then we apply

$$v \mapsto y := X(-s+\tau;0,x,v)$$

  with $dv \lesssim \frac{1}{|s-\tau|^2}dy$
- Moreover $|V(-s+\tau;0,x,v)| \sim \frac{|y-x|}{|s-\tau|}$ and $|v| \sim \frac{|y-x|}{|s-\tau|}$,

$$|SS^*g(x)| \lesssim \int_{\mathbb{R}^3} |g(y)| \int_0^T \int_0^T \left|\phi(\frac{|y-x|}{|s-\tau|})\right|^2 \frac{1}{|s-\tau|^3} dsd\tau dy$$

- $$M(x-y) := \int_0^T \int_0^T \left|\phi\left(\frac{|x-y|}{|s-\tau|}\right)\right|^2 \frac{1}{|s-\tau|^3} ds d\tau$$

  By $(s,\tau) \mapsto (s,t)$ with $t = |s-\tau|$

  $$\begin{aligned} M(x-y) &\lesssim \int_0^T \frac{1}{t^3}\left|\phi\left(\frac{|y-x|}{t}\right)\right|^2 dt \\ &\lesssim \int_0^\infty \frac{w}{|y-x|^2}\phi(w)dw \\ &\lesssim \frac{1}{|y-x|^2}, \end{aligned}$$

  where we have used $w = \frac{|y-x|}{t}$ with $dw = \frac{|y-x|}{t^2}dt$.

- Now $|SS^*g| = |g| * M$ where $M \in L_w^{3/2}$. By the weak Young's inequality (HLS)

  $$SS^* : L^q \to L^{q'}$$

  with $1 + \frac{1}{q'} = \frac{1}{q} + \frac{2}{3}$ and $q' = 3$.

Thank you !