MATHEMATICAL ANALYSIS 1
HOMEWORK 10

(1) Prove the following proposition: (it is Proposition 9.4 in the lecture notes)

Proposition. *Any Maclaurin polynomial of an even function contains only even powers. Any Maclaurin polynomial of an odd function contains only odd powers.*

(2) Let $p_n(x) = a_0 + a_1 x + \cdots a_n x^n$ and $q_m(x) = b_0 + b_1 x + \cdots b_m x^m$ be two polynomials of orders n and m respectively. Write the formula for $p_n(x) \cdot q_m(x)$ *(pay attention: n and m might be different)*. Explain your answer.

(3) Suppose that two functions f, g can be written as

$$f(x) = p_n(x) + o(x^n) \qquad g(x) = q_m(x) + o(x^m)$$

as $x \to 0$, where p_n, q_m are as in the previous question. Express the product $f(x) \cdot g(x)$. What is the order of the error? Explain your answer.

(4) Write $(Tf)_{n,x_0}(x)$ for the following f, n, x_0:

(a) $f(x) = e^x$, $n = 4$, $x_0 = 2$
(b) $f(x) = \ln x$, $n = 3$, $x_0 = 3$
(c) $f(x) = 7 + x - 3x^2 + 5x^3$, $n = 2$, $x_0 = 1$
(d) $f(x) = \sin x$, $n = 6$, $x_0 = \frac{\pi}{2}$
(e) $f(x) = \sqrt{2x+1}$, $n = 3$, $x_0 = 4$

(5) Write the Maclaurin expansions up to the indicated order with Peano's remainder:

(a) $f(x) = x \cos 3x - 3 \sin x$, $n = 2$
(b) $f(x) = e^{x^2} \sin 2x$, $n = 5$
(c) $f(x) = \ln \frac{1+x}{1+3x}$, $n = 4$
(d) $f(x) = \frac{x}{\sqrt[6]{1+x^2}} - \sin x$, $n = 5$

(6) Prove that there exists a neighborhood of 0 on which the following inequality holds:

$$2\cos(x + x^2) \leq 2 - x^2 - 2x^3.$$

(7) (a) Write the Maclaurin polynomial of e^x of order n with Lagrange's remainder.
(b) What is the minimal n we should take if we want to approximate the number e to within $\frac{1}{1000000}$ (one millionth)? Justify your answer.

(1) Prove the following proposition: (it is Proposition 9.4 in the lecture notes)

Proposition. *Any Maclaurin polynomial of an even function contains only even powers. Any Maclaurin polynomial of an odd function contains only odd powers.*

Proof. Let $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$ be the n-th Maclaurin polynomial of $f(x)$. The proof relies on showing which derivatives $f^{(k)}(0)$ must be zero.

Case 1: f is an Even Function. If f is an even function, $f(-x) = f(x)$.
We differentiate this identity repeatedly using the chain rule:

$$\begin{aligned}
f'(-x)(-1) &= f'(x) && \Rightarrow f'(-x) = -f'(x) \quad (f' \text{ is odd}) \\
f''(-x)(-1) &= -f'(-x)(-1) = f'(-x) && \Rightarrow f''(-x) = f''(x) \quad (f'' \text{ is even}) \\
f'''(-x)(-1) &= f'''(x) && \Rightarrow f'''(-x) = -f'''(x) \quad (f''' \text{ is odd})
\end{aligned}$$

By induction, all odd derivatives $f^{(k)}$ are odd functions, and all even derivatives $f^{(k)}$ are even functions.

Since an odd function must satisfy $g(0) = -g(0)$, we must have $g(0) = 0$. Therefore, for all odd k, $f^{(k)}(0) = 0$. The Maclaurin coefficients $\frac{f^{(k)}(0)}{k!}$ are zero for all odd k, leaving only terms corresponding to $k = 0, 2, 4, \dots$ (even powers).

Case 2: f is an Odd Function. If f is an odd function, $f(-x) = -f(x)$.
Differentiating this identity repeatedly:

$$\begin{aligned}
f'(-x)(-1) &= -f'(x) && \Rightarrow f'(-x) = f'(x) \quad (f' \text{ is even}) \\
f''(-x)(-1) &= f''(-x) && \Rightarrow f''(-x) = -f''(x) \quad (f'' \text{ is odd})
\end{aligned}$$

By induction, all even derivatives $f^{(k)}$ are odd functions, and all odd derivatives $f^{(k)}$ are even functions.

Since an odd function must be zero at 0, for all even k, $f^{(k)}(0) = 0$. (This includes $f^{(0)}(0) = f(0)$). The Maclaurin coefficients $\frac{f^{(k)}(0)}{k!}$ are zero for all even k, leaving only terms corresponding to $k = 1, 3, 5, \dots$ (odd powers). □

(2) **Product of two polynomials**
The product $p_n(x) \cdot q_m(x)$ is obtained by distributing every term of p_n with every term of q_m:

$$p_n(x) \cdot q_m(x) = \left(\sum_{i=0}^{n} a_i x^i\right)\left(\sum_{j=0}^{m} b_j x^j\right) = \sum_{i=0}^{n}\sum_{j=0}^{m} a_i b_j x^{i+j}.$$

Collecting terms with the same power $k = i + j$, we get:

$$p_n(x) \cdot q_m(x) = \sum_{k=0}^{n+m} \left(\sum_{\substack{i+j=k \\ 0\le i\le n \\ 0\le j\le m}} a_i b_j \right) x^k = \sum_{k=0}^{n+m} \left(\sum_{j=0}^{k} a_{k-j} b_j\right) x^k$$

(where the last expression is valid with our understanding that a coefficient with a non-existent index is 0). Thus the product is a polynomial of degree at most $n+m$ (exactly $n+m$ if $a_n \neq 0$ and $b_m \neq 0$). The coefficient of x^k is the sum of all products $a_i b_j$ where $i + j = k$, with i and j ranging over the appropriate indices.

(3) **Product of asymptotic expansions**
We have $f(x) = p_n(x) + o(x^n)$ and $g(x) = q_m(x) + o(x^m)$ as $x \to 0$. Their product is:

$$f(x)g(x) = \big(p_n(x) + o(x^n)\big)\big(q_m(x) + o(x^m)\big).$$

Expanding:

$$f(x)g(x) = p_n(x)q_m(x) + p_n(x)o(x^m) + q_m(x)o(x^n) + o(x^n)o(x^m).$$

We analyze each error term:

- $p_n(x) = O(1)$ as $x \to 0$ (since it's a polynomial with constant term a_0). Hence $p_n(x)o(x^m) = o(x^m)$.
- Similarly, $q_m(x) = O(1)$, so $q_m(x)o(x^n) = o(x^n)$.
- $o(x^n)o(x^m) = o(x^{n+m})$ (product of functions tending to 0 faster than x^n and x^m tends to 0 faster than x^{n+m}).

Thus the error term is $o(x^m)+o(x^n)+o(x^{n+m})$. The dominant (slowest vanishing) error is $o(x^{\min\{n,m\}})$ because $o(x^n)$ and $o(x^m)$ vanish at least as fast as $x^{\min\{n,m\}}$ (possibly faster), while $o(x^{n+m})$ vanishes even faster. More precisely, $o(x^n) + o(x^m) = o(x^{\min\{n,m\}})$. Hence:

$$\begin{aligned} f(x)g(x) &= p_n(x)q_m(x) + o(x^{\min\{n,m\}}) \\ &= \sum_{k=0}^{\min\{n,m\}} \left(\sum_{j=0}^{k} a_{k-j} b_j \right) x^k + o(x^{\min\{n,m\}}) \quad \text{as } x \to 0. \end{aligned}$$

*[** Notice the upper bound on the k sum !! **]* **Order of the error:** The error is $o(x^{\min\{n,m\}})$, i.e., it tends to zero faster than $x^{\min\{n,m\}}$.

(4) **Taylor polynomials at given points**

Recall $(Tf)_{n,x_0}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$.

(a) $f(x) = e^x$, $n = 4$, $x_0 = 2$. All derivatives are e^x, so $f^{(k)}(2) = e^2$. Thus

$$(Tf)_{4,2}(x) = e^2 \sum_{k=0}^{4} \frac{(x-2)^k}{k!} = e^2 \left[1 + (x-2) + \frac{(x-2)^2}{2} + \frac{(x-2)^3}{6} + \frac{(x-2)^4}{24} \right].$$

(b) $f(x) = \ln x$, $n = 3$, $x_0 = 3$. Compute derivatives:

$$\begin{aligned} f(x) &= \ln x, & f(3) &= \ln 3, \\ f'(x) &= \frac{1}{x}, & f'(3) &= \frac{1}{3}, \\ f''(x) &= -\frac{1}{x^2}, & f''(3) &= -\frac{1}{9}, \\ f'''(x) &= \frac{2}{x^3}, & f'''(3) &= \frac{2}{27}. \end{aligned}$$

Thus

$$(Tf)_{3,3}(x) = \ln 3 + \frac{1}{3}(x-3) - \frac{1}{9\cdot 2}(x-3)^2 + \frac{2}{27\cdot 6}(x-3)^3 = \ln 3 + \frac{x-3}{3} - \frac{(x-3)^2}{18} + \frac{(x-3)^3}{81}.$$

(c) $f(x) = 7 + x - 3x^2 + 5x^3$, $n = 2$, $x_0 = 1$. Since f itself is a polynomial, its Taylor polynomial of order 2 at $x_0 = 1$ is simply the quadratic part of f expanded in powers of $(x-1)$. Compute:

$$\begin{aligned} f(1) &= 7 + 1 - 3 + 5 = 10, \\ f'(x) &= 1 - 6x + 15x^2, & f'(1) &= 1 - 6 + 15 = 10, \\ f''(x) &= -6 + 30x, & f''(1) &= -6 + 30 = 24. \end{aligned}$$

Hence

$$(Tf)_{2,1}(x) = 10 + 10(x-1) + \frac{24}{2}(x-1)^2 = 10 + 10(x-1) + 12(x-1)^2.$$

One can check that this equals the original cubic polynomial when expanded, up to terms of order $(x-1)^2$.

(d) $f(x) = \sin x$, $n = 6$, $x_0 = \frac{\pi}{2}$. Compute derivatives:

$$\begin{aligned}
f(x) &= \sin x, & f(\pi/2) &= 1,\\
f'(x) &= \cos x, & f'(\pi/2) &= 0,\\
f''(x) &= -\sin x, & f''(\pi/2) &= -1,\\
f'''(x) &= -\cos x, & f'''(\pi/2) &= 0,\\
f^{(4)}(x) &= \sin x, & f^{(4)}(\pi/2) &= 1,\\
f^{(5)}(x) &= \cos x, & f^{(5)}(\pi/2) &= 0,\\
f^{(6)}(x) &= -\sin x, & f^{(6)}(\pi/2) &= -1.
\end{aligned}$$

Thus

$$\begin{aligned}
(Tf)_{6,\frac{\pi}{2}}(x) &= 1 + 0\cdot\left(x-\frac{\pi}{2}\right) + \frac{-1}{2!}\left(x-\frac{\pi}{2}\right)^2 + \frac{0}{3!}\left(x-\frac{\pi}{2}\right)^3\\
&\quad + \frac{1}{4!}\left(x-\frac{\pi}{2}\right)^4 + \frac{0}{5!}\left(x-\frac{\pi}{2}\right)^5 + \frac{-1}{6!}\left(x-\frac{\pi}{2}\right)^6\\
&= 1 - \frac{1}{2}\left(x-\frac{\pi}{2}\right)^2 + \frac{1}{24}\left(x-\frac{\pi}{2}\right)^4 - \frac{1}{720}\left(x-\frac{\pi}{2}\right)^6.
\end{aligned}$$

(e) $f(x) = \sqrt{2x+1} = (2x+1)^{1/2}$, $n = 3$, $x_0 = 4$. Compute derivatives:

$$\begin{aligned}
f(x) &= (2x+1)^{1/2}, & f(4) &= \sqrt{9} = 3,\\
f'(x) &= \frac{1}{2}(2x+1)^{-1/2}\cdot 2 = (2x+1)^{-1/2}, & f'(4) &= \frac{1}{3},\\
f''(x) &= -\frac{1}{2}(2x+1)^{-3/2}\cdot 2 = -(2x+1)^{-3/2}, & f''(4) &= -\frac{1}{27},\\
f'''(x) &= \frac{3}{2}(2x+1)^{-5/2}\cdot 2 = 3(2x+1)^{-5/2}, & f'''(4) &= \frac{3}{243} = \frac{1}{81}.
\end{aligned}$$

Thus

$$(Tf)_{3,4}(x) = 3 + \frac{1}{3}(x-4) - \frac{1}{27\cdot 2}(x-4)^2 + \frac{1}{81\cdot 6}(x-4)^3 = 3 + \frac{x-4}{3} - \frac{(x-4)^2}{54} + \frac{(x-4)^3}{486}.$$

(5) **Maclaurin expansions with Peano remainder**

Recall: $f(x) = P_n(x) + o(x^n)$ as $x \to 0$, where P_n is the nth Maclaurin polynomial.

(a) $f(x) = x\cos 3x - 3\sin x$, $n = 2$. Use known expansions:

$$\begin{aligned}
\cos 3x &= 1 - \frac{(3x)^2}{2!} + o(x^2) = 1 - \frac{9x^2}{2} + o(x^2),\\
x\cos 3x &= x\left(1 - \frac{9x^2}{2} + o(x^2)\right) = x - \frac{9}{2}x^3 + o(x^3) = x + o(x^2),\\
\sin x &= x - \frac{x^3}{6} + o(x^3) = x + o(x^2),\\
3\sin x &= 3x + o(x^2).
\end{aligned}$$

Thus

$$f(x) = (x + o(x^2)) - (3x + o(x^2)) = -2x + o(x^2).$$

So the Maclaurin expansion up to order 2 is $f(x) = -2x + o(x^2)$. (The quadratic term is zero.)

(b) $f(x) = e^{x^2}\sin 2x$, $n = 5$. We need expansions up to x^5:

$$\begin{aligned}
e^{x^2} &= 1 + x^2 + \frac{x^4}{2!} + o(x^5) = 1 + x^2 + \frac{x^4}{2} + o(x^5),\\
\sin 2x &= 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + o(x^5) = 2x - \frac{8x^3}{6} + \frac{32x^5}{120} + o(x^5) = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 + o(x^5).
\end{aligned}$$

Now multiply, keeping terms up to x^5:

$$\begin{aligned}
e^{x^2}\sin 2x &= \left(1 + x^2 + \frac{x^4}{2} + o(x^5)\right)\left(2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 + o(x^5)\right) \\
&= 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 \\
&\quad + x^2 \cdot 2x - x^2 \cdot \frac{4}{3}x^3 \quad \text{(only terms up to } x^5\text{)} \\
&\quad + \frac{x^4}{2} \cdot 2x \quad \text{(gives } x^5\text{)} \\
&\quad + o(x^5) \\
&= 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 + 2x^3 - \frac{4}{3}x^5 + x^5 + o(x^5) \\
&= 2x + \left(-\frac{4}{3} + 2\right)x^3 + \left(\frac{4}{15} - \frac{4}{3} + 1\right)x^5 + o(x^5) \\
&= 2x + \frac{2}{3}x^3 + \left(\frac{4}{15} - \frac{20}{15} + \frac{15}{15}\right)x^5 + o(x^5) \\
&= 2x + \frac{2}{3}x^3 - \frac{1}{15}x^5 + o(x^5).
\end{aligned}$$

Thus $f(x) = 2x + \frac{2}{3}x^3 - \frac{1}{15}x^5 + o(x^5)$.

(c) $f(x) = \ln\frac{1+x}{1+3x} = \ln(1+x) - \ln(1+3x)$, $n = 4$. Use $\ln(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + o(u^4)$:

$$\begin{aligned}
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + o(x^4), \\
\ln(1+3x) &= 3x - \frac{(3x)^2}{2} + \frac{(3x)^3}{3} - \frac{(3x)^4}{4} + o(x^4) \\
&= 3x - \frac{9x^2}{2} + 9x^3 - \frac{81x^4}{4} + o(x^4).
\end{aligned}$$

Subtract:

$$\begin{aligned}
f(x) &= (x - 3x) + \left(-\frac{1}{2} + \frac{9}{2}\right)x^2 + \left(\frac{1}{3} - 9\right)x^3 + \left(-\frac{1}{4} + \frac{81}{4}\right)x^4 + o(x^4) \\
&= -2x + 4x^2 + \left(\frac{1}{3} - \frac{27}{3}\right)x^3 + \left(\frac{80}{4}\right)x^4 + o(x^4) \\
&= -2x + 4x^2 - \frac{26}{3}x^3 + 20x^4 + o(x^4).
\end{aligned}$$

(d) $f(x) = \frac{x}{\sqrt[6]{1+x^2}} - \sin x$, $n = 5$. Write $\sqrt[6]{1+x^2} = (1+x^2)^{1/6}$. Use binomial expansion $(1+u)^\alpha = 1 + \alpha u + \frac{\alpha(\alpha-1)}{2}u^2 + o(u^2)$ with $u = x^2$, $\alpha = 1/6$:

$$\begin{aligned}
(1+x^2)^{1/6} &= 1 + \frac{1}{6}x^2 + \frac{\frac{1}{6}\left(-\frac{5}{6}\right)}{2}x^4 + o(x^4) \\
&= 1 + \frac{1}{6}x^2 - \frac{5}{72}x^4 + o(x^4).
\end{aligned}$$

Then

$$\frac{1}{(1+x^2)^{1/6}} = 1 - \frac{1}{6}x^2 + \left(\frac{5}{72} + \frac{1}{36}\right)x^4 + o(x^4) = 1 - \frac{1}{6}x^2 + \frac{7}{72}x^4 + o(x^4),$$

using $(1+u)^{-1} = 1 - u + u^2 + o(u^2)$ applied to $u = \frac{1}{6}x^2 - \frac{5}{72}x^4 + \cdots$. More systematically, we can compute the reciprocal of the series. Up to x^4, we have $(1+x^2)^{1/6} = 1 + ax^2 + bx^4 + o(x^4)$ with $a = 1/6$, $b = -5/72$. Its reciprocal is $1 - ax^2 + (a^2 - b)x^4 + o(x^4)$. Indeed:

$$\frac{1}{1 + ax^2 + bx^4} = 1 - ax^2 + (a^2 - b)x^4 + o(x^4).$$

Plugging $a = 1/6$, $b = -5/72$ gives $a^2 - b = \frac{1}{36} + \frac{5}{72} = \frac{2}{72} + \frac{5}{72} = \frac{7}{72}$.
Now,

$$\frac{x}{\sqrt[6]{1+x^2}} = x\left(1 - \frac{1}{6}x^2 + \frac{7}{72}x^4 + o(x^4)\right) = x - \frac{1}{6}x^3 + \frac{7}{72}x^5 + o(x^5).$$

Subtract $\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)$:

$$\begin{aligned} f(x) &= \left(x - \frac{1}{6}x^3 + \frac{7}{72}x^5\right) - \left(x - \frac{x^3}{6} + \frac{x^5}{120}\right) + o(x^5) \\ &= \left(-\frac{1}{6} + \frac{1}{6}\right)x^3 + \left(\frac{7}{72} - \frac{1}{120}\right)x^5 + o(x^5) \\ &= 0 \cdot x^3 + \left(\frac{35}{360} - \frac{3}{360}\right)x^5 + o(x^5) \\ &= \frac{32}{360}x^5 + o(x^5) = \frac{4}{45}x^5 + o(x^5). \end{aligned}$$

Thus $f(x) = \frac{4}{45}x^5 + o(x^5)$.

(6) **Inequality proof using Taylor expansion**

Consider the function $h(x) = 2\cos(x + x^2) - (2 - x^2 - 2x^3)$. We want to show $h(x) \le 0$ for x near 0.

Compute the Maclaurin expansion of $2\cos(x + x^2)$ up to order 3. Let $u = x + x^2$. Then $\cos u = 1 - \frac{u^2}{2} + \frac{u^4}{24} + \cdots$. Up to x^3:

$$u = x + x^2, \quad u^2 = x^2 + 2x^3 + x^4, \quad u^3 = O(x^3).$$

Thus

$$\begin{aligned} \cos(x + x^2) &= 1 - \frac{1}{2}(x^2 + 2x^3 + x^4) + o(x^3) \\ &= 1 - \frac{x^2}{2} - x^3 + o(x^3). \end{aligned}$$

Multiply by 2:

$$2\cos(x + x^2) = 2 - x^2 - 2x^3 + o(x^3).$$

Therefore,

$$h(x) = \big(2 - x^2 - 2x^3 + o(x^3)\big) - \big(2 - x^2 - 2x^3\big) = o(x^3).$$

This means $h(x) = \varepsilon(x)x^3$ where $\varepsilon(x) \to 0$ as $x \to 0$. For x sufficiently small, $|\varepsilon(x)| < 1$, so $|h(x)| < |x^3|$.

Now, for $x > 0$ small, $x^3 > 0$, so $h(x)$ could be positive or negative but is bounded in absolute value by x^3. However, we need to check the sign more precisely. Let's compute the next term to determine local behavior. Compute expansion to x^4 to see the leading non-zero term of $h(x)$ beyond the $o(x^3)$:

$$u^2 = x^2 + 2x^3 + x^4, \quad u^4 = x^4 + O(x^5).$$

Then

$$\cos u = 1 - \frac{u^2}{2} + \frac{u^4}{24} + o(x^4) = 1 - \frac{x^2}{2} - x^3 - \frac{x^4}{2} + \frac{x^4}{24} + o(x^4) = 1 - \frac{x^2}{2} - x^3 - \frac{11}{24}x^4 + o(x^4).$$

Thus

$$2\cos(x + x^2) = 2 - x^2 - 2x^3 - \frac{11}{12}x^4 + o(x^4).$$

Then

$$h(x) = -\frac{11}{12}x^4 + o(x^4).$$

Since $-\frac{11}{12}x^4$ is negative for $x \neq 0$, and $o(x^4)$ is negligible compared to x^4 as $x \to 0$, there exists $\delta > 0$ such that for $0 < |x| < \delta$, $h(x) < 0$. That is,

$$2\cos(x + x^2) < 2 - x^2 - 2x^3.$$

For $x = 0$, equality holds. Hence, in a neighborhood of 0, $2\cos(x + x^2) \leq 2 - x^2 - 2x^3$.

(7) (a) **Maclaurin polynomial of e^x with Lagrange remainder**

For $f(x) = e^x$, we have $f^{(k)}(x) = e^x$ for all k. The Maclaurin polynomial of order n is

$$P_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}.$$

The Lagrange form of the remainder is: for any x, there exists ξ between 0 and x such that

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1} = \frac{e^{\xi}}{(n+1)!} x^{n+1}.$$

Thus

$$e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + \frac{e^{\xi}}{(n+1)!} x^{n+1}.$$

(b) **Approximating e with error $< 10^{-6}$**

We want to approximate $e = e^1$ using the Maclaurin polynomial at $x = 1$. The remainder term is

$$R_n(1) = \frac{e^{\xi}}{(n+1)!}, \quad \text{where } 0 < \xi < 1.$$

Since $e^{\xi} < e < 3$ (we could use a tighter bound, but $e < 3$ suffices for finding n), we have

$$|R_n(1)| < \frac{3}{(n+1)!}.$$

We require $|R_n(1)| < 10^{-6}$. So it suffices to find n such that

$$\frac{3}{(n+1)!} < 10^{-6} \quad \Leftrightarrow \quad (n+1)! > 3 \times 10^6.$$

Compute factorials:

$$9! = 362880,$$
$$10! = 3628800 > 3 \times 10^6.$$

Thus $(n+1)! > 3 \times 10^6$ when $n + 1 \geq 10$, i.e., $n \geq 9$. Therefore, the minimal n is 9.
Thus the Maclaurin polynomial $\sum_{k=0}^{9} \frac{1}{k!}$ approximates e with error less than 10^{-6}.