MATHEMATICAL ANALYSIS 1
HOMEWORK 5

(1) For the function $f(x) = \sin \frac{1}{x}$, find all $x_n \in (0, 1)$ for which $f(x_n) = 1$ and all $y_n \in (0, 1)$ for which $f(y_n) = -1$. Conclude that $\lim_{x\to 0^+} f(x)$ does not exist.
(2) Prove that the ceiling function $f(x) = \lceil x \rceil$ is left-continuous at every $x_0 \in \mathbb{R}$.
(3) Prove the following proposition:

Proposition: Let $f : \mathbb{R} \to \mathbb{R}$ be defined in a neighborhood of x_0 (possibly not at x_0 itself). Then

$$\lim_{x\to x_0} f(x) = L \qquad \Leftrightarrow \qquad \lim_{x\to x_0^+} f(x) = L \quad \text{and} \quad \lim_{x\to x_0^-} f(x) = L$$

where L can be any number or $\pm\infty$. Moreover, the function is continuous at x_0 if and only if it is both right- and left-continuous at x_0.

(4) Let $f(x) = 5 + 2x \sin \frac{1}{x}$. Does the limit $\lim_{x\to+\infty} f(x)$ exist? Prove your answer.
(5) Let $f : \mathbb{R} \to \mathbb{R}$. Prove that if $\lim_{x\to+\infty} f(x) > 0$ then there exists $M > 0$ s.t. $f > 0$ on the set $\{x \in \mathbb{R} : x > M\}$. *Hint: we proved a very similar result in class.*
(6) Consider the sequence $a_n = \arctan\left(\frac{5n+6}{n+1}\right)$, $n \in \mathbb{N}$.
   (a) Is this a monotone sequence?
   (b) Find its infimum and supremum.
   (c) Do the minimum and maximum exist? If so, what are they?
   (d) Does the limit $\lim_{n\to\infty} a_n$ exist? If so, what is it?
(7) Using the definition of the limit prove the following:
   (a) $\lim_{x\to 1}(2x^2 + 3) = 5$
   (b) $\lim_{n\to\infty} \frac{n^2}{1-2n} = -\infty$
(8) Determine the values of $\alpha \in \mathbb{R}$ for which the following functions are continuous on their domains. Explain your answers.
   (a) $f(x) = \begin{cases} \alpha \sin(x + \frac{\pi}{2}) & \text{for } x > 0 \\ 2x^2 + 3 & \text{for } x \le 0 \end{cases}$
   (b) $f(x) = \begin{cases} 3e^{\alpha x - 1} & \text{for } x \ge 1 \\ x + 2 & \text{for } x < 1 \end{cases}$
(9) Compute the following limits:
   (a) $\lim_{x\to 0} \frac{x^4-2x^3+5x}{x^5-x}$
   (b) $\lim_{x\to -1} \frac{x+1}{\sqrt{6x^2+3}+3x}$
   (c) $\lim_{x\to+\infty}(\sqrt{x+1} - \sqrt{x})$
   (d) $\lim_{x\to-\infty} \frac{3^x-3^{-x}}{3^x+3^{-x}}$
(10) Determine the domain and the behvior at the end-points of the domain of the following functions:
   (a) $f(x) = \frac{x^3-x^2+3}{x^2+3x+2}$
   (b) $f(x) = \sqrt[3]{x}e^{-x^2}$ *(hint: you may use the fact that* $\lim_{x\to+\infty}(\frac{1}{3}\ln x - x^2) = -\infty$*)*

(1) We have $f(x_n) = \sin \frac{1}{x_n} = 1$ when $\frac{1}{x_n} = \frac{\pi}{2} + 2n\pi$, so:

$$x_n = \frac{1}{\frac{\pi}{2} + 2n\pi} = \frac{2}{\pi + 4n\pi}$$

We have $f(y_n) = \sin \frac{1}{y_n} = -1$ when $\frac{1}{y_n} = \frac{3\pi}{2} + 2n\pi$, so:

$$y_n = \frac{1}{\frac{3\pi}{2} + 2n\pi} = \frac{2}{3\pi + 4n\pi}$$

As $n \to \infty$, both sequences $x_n \to 0^+$ and $y_n \to 0^+$, but $f(x_n) = 1$ and $f(y_n) = -1$. Since the function takes values 1 and -1 arbitrarily close to 0, the limit $\lim_{x\to 0^+} f(x)$ does not exist.

(2) *Proof.* Let $f(x) = \lceil x \rceil$. We must prove that $f(x)$ is left-continuous at every $x_0 \in \mathbb{R}$. For f to be left-continuous at x_0, we need $\lim_{x\to x_0^-} f(x) = f(x_0)$.

Case 1: x_0 is an integer ($x_0 = n \in \mathbb{Z}$) The function value is $f(n) = \lceil n \rceil = n$. For x in the interval $(n-1, n]$, the ceiling function is constant: $\lceil x \rceil = n$. Therefore, the left limit is:

$$\lim_{x\to n^-} \lceil x \rceil = \lim_{x\to n^-} n = n$$

Since $\lim_{x\to n^-} f(x) = f(n)$, the function is left-continuous at every integer point.

Case 2: x_0 is not an integer ($x_0 \notin \mathbb{Z}$) Let $n = \lceil x_0 \rceil$. Since x_0 is not an integer, we have $n-1 < x_0 < n$. For x sufficiently close to x_0 from the left, x is in the open interval $(n-1, n)$, so $\lceil x \rceil = n$. The function value is $f(x_0) = \lceil x_0 \rceil = n$. The left limit is:

$$\lim_{x\to x_0^-} \lceil x \rceil = \lim_{x\to x_0^-} n = n$$

Since $\lim_{x\to x_0^-} f(x) = f(x_0)$, the function is left-continuous at every non-integer point.

Since $f(x)$ is left-continuous at all integer and non-integer points, it is left-continuous at every $x_0 \in \mathbb{R}$. $\square$

(3) *Proof.* ($\Rightarrow$) If $\lim_{x\to x_0} f(x) = L$, then for any $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < |x - x_0| < \delta$ implies $|f(x) - L| < \varepsilon$. This holds for both $x > x_0$ and $x < x_0$, so both one-sided limits exist and equal L. ($\Leftarrow$) If $\lim_{x\to x_0^+} f(x) = L$ and $\lim_{x\to x_0^-} f(x) = L$, then for any $\varepsilon > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that:

$0 < x - x_0 < \delta_1$ implies $|f(x) - L| < \varepsilon$

$0 < x_0 - x < \delta_2$ implies $|f(x) - L| < \varepsilon$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then $0 < |x - x_0| < \delta$ implies $|f(x) - L| < \varepsilon$, so $\lim_{x\to x_0} f(x) = L$.

For continuity: f is continuous at x_0 if $\lim_{x\to x_0} f(x) = f(x_0)$. By the above, this is equivalent to both one-sided limits existing and equaling $f(x_0)$, i.e., both right- and left-continuity at x_0. $\square$

(4) The limit $\lim_{x\to +\infty} f(x)$ does exist and equals 7.

Proof. We have $f(x) = 5 + 2x \sin \frac{1}{x}$. Consider the limit of the second term:

$$\lim_{x\to +\infty} 2x \sin \frac{1}{x}$$

Let $u = \frac{1}{x}$. As $x \to +\infty$, $u \to 0^+$. Substituting $x = \frac{1}{u}$ into the expression yields:

$$\lim_{u\to 0^+} 2\left(\frac{1}{u}\right) \sin u = 2 \lim_{u\to 0^+} \frac{\sin u}{u}$$

Using the fundamental trigonometric limit, $\lim_{u\to 0} \frac{\sin u}{u} = 1$:

$$2 \lim_{u\to 0^+} \frac{\sin u}{u} = 2(1) = 2$$

Therefore, the limit of the entire function is:

$$\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} \left(5 + 2x \sin \frac{1}{x}\right) = 5 + 2 = 7$$

$\square$

(5) *Proof.* We consider the case of a finite limit (one should also consider the case of an infinite limit). Let $L = \lim_{x\to+\infty} f(x) > 0$. By definition, for $\varepsilon = \frac{L}{2} > 0$, there exists $M > 0$ such that for all $x > M$, we have $|f(x) - L| < \frac{L}{2}$.

This inequality is equivalent to:

$$L - \frac{L}{2} < f(x) < L + \frac{L}{2}$$
$$\frac{L}{2} < f(x) < \frac{3L}{2}$$

Since $L > 0$, we have $f(x) > \frac{L}{2} > 0$ for all $x > M$. Therefore, $f(x) > 0$ on the set $\{x \in \mathbb{R} : x > M\}$. $\square$

(6) (a) Let's examine if the sequence is monotone. Consider:

$$a_n = \arctan\left(\frac{5n+6}{n+1}\right) = \arctan\left(\frac{5(n+1)+1}{n+1}\right) = \arctan\left(5 + \frac{1}{n+1}\right)$$

As n increases, $\frac{1}{n+1}$ decreases, so $5 + \frac{1}{n+1}$ decreases. Since arctan is an increasing function, a_n is a decreasing sequence. Therefore, the sequence is monotone.

(b) Since the sequence is decreasing, the supremum is the first term, and the infimum is the limit:

$$\inf a_n = \lim_{n\to\infty} a_n = \arctan 5$$
$$\sup a_n = a_0 = \arctan 6$$

(c) The maximum exists and is attained at $n = 0$: $a_0 = \arctan 6$. The minimum does not exist since the sequence approaches $\arctan 5$ but never equals it for finite n.

(d) Yes, the limit exists:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \arctan\left(5 + \frac{1}{n+1}\right) = \arctan 5$$

(7) (a) *Proof.* Let $\varepsilon > 0$ be given. We want to find $\delta > 0$ such that $0 < |x-1| < \delta$ implies $|(2x^2+3)-5| < \varepsilon$. Note that $|2x^2 + 3 - 5| = |2x^2 - 2| = 2|x^2 - 1| = 2|x - 1||x + 1|$.
If we restrict $|x - 1| < 1$, then $0 < x < 2$, so $|x + 1| < 3$.
Choose $\delta = \min\left\{1, \frac{\varepsilon}{6}\right\}$. Then if $0 < |x - 1| < \delta$:

$$|2x^2 + 3 - 5| = 2|x - 1||x + 1| < 2 \cdot \frac{\varepsilon}{6} \cdot 3 = \varepsilon$$

Thus $\lim_{x\to1}(2x^2 + 3) = 5$. $\square$

(b) *Proof.* We want to show that for any $M > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $\frac{n^2}{1-2n} < -M$. Note that for $n > 1$:

$$\frac{n^2}{1 - 2n} = \frac{n}{-2 + 1/n}$$

Since $-2 + 1/n > -2$ for $n > 1$, and the denominator is negative, we have:

$$\frac{n}{-2 + 1/n} < \frac{n}{-2} = -\frac{n}{2}$$

We want $-\frac{n}{2} < -M$, which is equivalent to $n > 2M$. Choose $N = \lceil 2M \rceil$. Then for $n > N$:

$$\frac{n^2}{1 - 2n} < -\frac{n}{2} < -\frac{N}{2} \leq -M$$

Thus $\lim_{n\to\infty} \frac{n^2}{1-2n} = -\infty$. $\square$

(8) (a) For continuity at $x = 0$, we need:

$$\lim_{x\to0^-} f(x) = \lim_{x\to0^+} f(x) = f(0)$$

Left limit: $\lim_{x\to0^-}(2x^2 + 3) = 3$ Right limit: $\lim_{x\to0^+} \alpha \sin(x + \frac{\pi}{2}) = \alpha \sin(\frac{\pi}{2}) = \alpha$ Function value: $f(0) = 2(0)^2 + 3 = 3$
For continuity, we need $\alpha = 3$.

(b) For continuity at $x = 1$, we need:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$$

Left limit: $\lim_{x\to 1^-}(x+2) = 3$ Right limit and Function value: $f(1) = \lim_{x\to 1^+} 3e^{\alpha x-1} = 3e^{\alpha-1}$
For continuity, we need $3 = 3e^{\alpha-1}$, so $e^{\alpha-1} = 1$, thus $\alpha - 1 = 0$, so $\alpha = 1$.

(9) (a)

$$\lim_{x\to 0} \frac{x^4 - 2x^3 + 5x}{x^5 - x} = \lim_{x\to 0} \frac{x(x^3 - 2x^2 + 5)}{x(x^4 - 1)} = \lim_{x\to 0} \frac{x^3 - 2x^2 + 5}{x^4 - 1} = \frac{0 - 0 + 5}{0 - 1} = -5$$

(b)

$$\lim_{x\to -1} \frac{x+1}{\sqrt{6x^2+3}+3x} = \lim_{x\to -1} \frac{x+1}{\sqrt{6x^2+3}+3x} \cdot \frac{\sqrt{6x^2+3}-3x}{\sqrt{6x^2+3}-3x}$$
$$= \lim_{x\to -1} \frac{(x+1)(\sqrt{6x^2+3}-3x)}{6x^2+3-9x^2} = \lim_{x\to -1} \frac{(x+1)(\sqrt{6x^2+3}-3x)}{3-3x^2}$$
$$= \lim_{x\to -1} \frac{(x+1)(\sqrt{6x^2+3}-3x)}{3(1-x)(1+x)} = \lim_{x\to -1} \frac{\sqrt{6x^2+3}-3x}{3(1-x)}$$
$$= \frac{\sqrt{6(-1)^2+3}-3(-1)}{3(1-(-1))} = \frac{\sqrt{9}+3}{3(2)} = \frac{3+3}{6} = 1$$

(c)

$$\lim_{x\to +\infty} (\sqrt{x+1} - \sqrt{x}) = \lim_{x\to +\infty} \frac{(\sqrt{x+1}-\sqrt{x})(\sqrt{x+1}+\sqrt{x})}{\sqrt{x+1}+\sqrt{x}} = \lim_{x\to +\infty} \frac{(x+1)-x}{\sqrt{x+1}+\sqrt{x}} = \lim_{x\to +\infty} \frac{1}{\sqrt{x+1}+\sqrt{x}} = 0$$

(d)

$$\lim_{x\to -\infty} \frac{3^x - 3^{-x}}{3^x + 3^{-x}} = \lim_{x\to -\infty} \frac{3^x/3^{-x} - 3^{-x}/3^{-x}}{3^x/3^{-x} + 3^{-x}/3^{-x}} = \lim_{x\to -\infty} \frac{3^{2x}-1}{3^{2x}+1} = \frac{0-1}{0+1} = -1$$

(10) (a) Domain: $x^2 + 3x + 2 \neq 0$, so $(x+1)(x+2) \neq 0$, thus $x \neq -1, -2$. Domain: $\mathbb{R} \setminus \{-2, -1\}$.
Behavior at endpoints:
- As $x \to -2$: Vertical Asymptote. $\lim_{x\to -2^-} f(x) = -\infty$, $\lim_{x\to -2^+} f(x) = +\infty$.
- As $x \to -1$: Vertical Asymptote. $\lim_{x\to -1^-} f(x) = -\infty$, $\lim_{x\to -1^+} f(x) = +\infty$.
- As $x \to \pm\infty$: $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{x^3}{x^2} = \lim_{x\to\pm\infty} x = \pm\infty$.

(b) Domain: $\sqrt[3]{x}$ is defined for all $x \in \mathbb{R}$, so domain is $\mathbb{R}$.
Behavior at endpoints:
- As $x \to +\infty$: $\lim_{x\to+\infty} \sqrt[3]{x}e^{-x^2} = \lim_{x\to+\infty} e^{\frac{1}{3}\ln x - x^2} = 0$ using the hint.
- As $x \to -\infty$: Observe that $\sqrt[3]{x}e^{-x^2}$ is an odd function, so if $\lim_{x\to+\infty} \sqrt[3]{x}e^{-x^2} = 0$, then also $\lim_{x\to-\infty} \sqrt[3]{x}e^{-x^2} = 0$