8.6 First and second finite increment formulas

For a function $f : \mathbb{R} \to \mathbb{R}$ that is differentiable at $x_0 \in \mathbb{R}$, from the definition of the derivative

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$

we can subtract $f'(x_0)$, multiply and divide it by $x - x_0$, and insert it into the limit, to obtain

$$\begin{aligned} 0 &= \lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{x - x_0} \\ &= \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0) - f'(x_0)\Delta x}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\Delta f - f'(x_0)\Delta x}{\Delta x}. \end{aligned}$$

By definition of the little o symbol, this means that the numerators in the above three expressions are little o's of their denominators, as $x \to x_0$. This leads us to

First increment formula

The first increment formula for a differentiable function f at x_0 states that

$$f(x) - f(x_0) = f'(x_0)(x - x_0) + o(x - x_0), \quad x \to x_0,$$

or, equivalently,

$$\Delta f = f'(x_0)\Delta x + o(\Delta x), \quad \Delta x \to 0.$$

For f that is differentiable at x_0, this formula gives us an approximation of $f(x)$ at nearby points x. Figure 8.3 demonstrates this formula graphically.

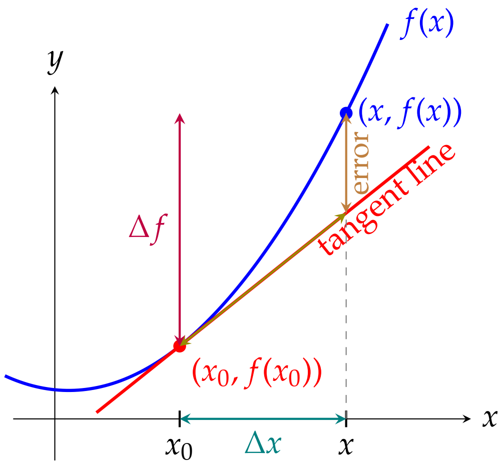


Figure 8.3: The 'error' is of order $o(\Delta x)$ as $\Delta x \to 0$

Another approximation method uses the Mean Value Theorem (Theorem 8.11). Suppose now that f is differentiable on an entire interval $I \subseteq \mathbb{R}$. Let x_1, x_2 be two points in I. Then by the Mean Value Theorem, there exists $\bar{x}$ between x_1 and x_2 such that

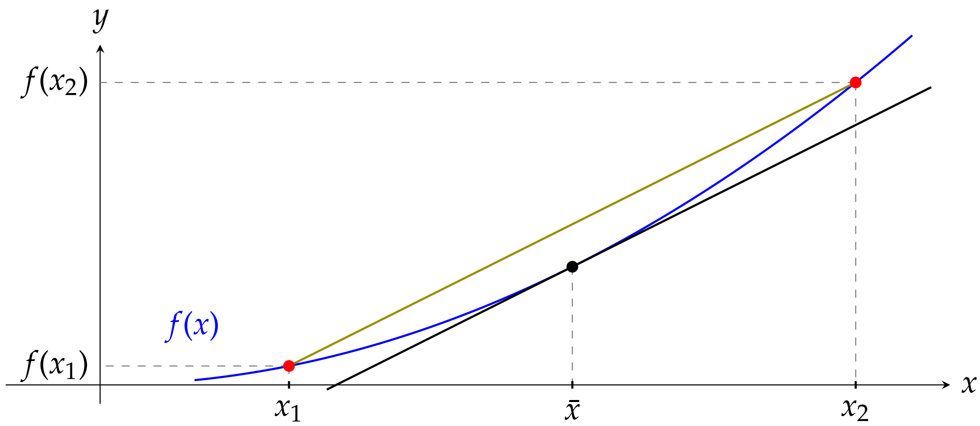
$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(\bar{x}).$$

Second increment formula

The second increment formula for a differentiable function f on an interval I with x_1, x_2 in I, states that

$$f(x_2) - f(x_1) = f'(\bar{x})(x_2 - x_1)$$

for some $\bar{x}$ between x_1 and x_2.



The following proposition follows as a consequence:

Proposition 8.13: Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable on an interval $I \subseteq \mathbb{R}$. Then

$$f \text{ is constant on } I \qquad \Leftrightarrow \qquad f'(x) = 0, \quad \forall x \in I.$$

Proof. **Direction $\Rightarrow$.** If f is constant on I, then for any $x_0 \in I$, we have $\frac{f(x)-f(x_0)}{x-x_0}$ for any other $x \in I$, so the $f'(x_0) = 0$ by the definition of the derivative.

Direction $\Leftarrow$. Take any two $x_1, x_2 \in I$. By the second increment formula

$$f(x_2) - f(x_1) = f'(\bar{x})(x_2 - x_1)$$

for some $\bar{x}$ between x_1 and x_2. But $f'(x) = 0$ for any $x \in I$ by assumption, so that $f(x_2) = f(x_1)$ and the proof is complete. □

Proposition 8.14: Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable on an interval $I \subseteq \mathbb{R}$ with bounded derivative on I. Define

$$L = \sup_{x \in I} |f'(x)|$$

which cannot be $+\infty$ since the derivative is bounded on I. Then f is Lipschitz on I with Lipschitz constant L.

Proof. Our goal is simple: verify that

$$|f(x_1) - f(x_2)| \le L|x_1 - x_2| \qquad \forall x_1, x_2 \in I$$

with L being the *optimal* (i.e. smallest) constant satisfying this inequality. By the second increment formula, for any $x_1, x_2 \in I$ there exists some $\overline{x}$ between x_1 and x_2 such that $f(x_2) - f(x_1) = f'(\overline{x})(x_2 - x_1)$. Taking absolute values and estimating, we have

$$|f(x_2) - f(x_1)| = |f'(\overline{x})| \cdot |x_2 - x_1| \le L|x_2 - x_1|$$

where the inequality follows from our assumption. This is enough to prove that f is Lipschitz on I, but it doesn't prove that L is the *optimal (or Lipschitz)* constant in the inequality. Denote by L_{opt} the optimal constant. Then we know that

$$L_{\text{opt}} \le L.$$

Now we'll show that $L_{\text{opt}} \ge L$, thus concluding that $L_{\text{opt}} = L$. Fix $x_0 \in I$. We know that

$$\left|\frac{f(x) - f(x_0)}{x - x_0}\right| = \frac{|f(x) - f(x_0)|}{|x - x_0|} \le L_{\text{opt}} \qquad \forall x, x_0 \in I,\ x \ne x_0.$$

Taking the limit $x \to x_0$, we have

$$|f'(x_0)| = \left|\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}\right| = \lim_{x \to x_0} \left|\frac{f(x) - f(x_0)}{x - x_0}\right| \le L_{\text{opt}}$$

[Note that we are allowed to exchange the order of the limit and the absolute value due to the Substitution Theorem (Theorem 5.12) using the fact that the limit of $\frac{f(x)-f(x_0)}{x-x_0}$ exists.] Taking the supremum over all $x_0 \in I$ of the above and using the definition of L, we arrive at

$$L \le L_{\text{opt}}$$

and the proof is complete. □

8.7 Monotonicity intervals

Theorem 8.15: Let $f : \mathbb{R} \to \mathbb{R}$ and suppose that f is differentiable on an interval I. Then (a)

$$f'(x) \text{ has the same sign (or 0) throughout } I \qquad \Leftrightarrow \qquad f \text{ is monotone on } I$$

and (b)

$$f'(x) \text{ has a strict sign throughout } I \qquad \Rightarrow \qquad f \text{ is strictly monotone on } I.$$

Proof. **We start by proving (a)(⇐).** Suppose that f is monotone increasing on I (the monotone decreasing case will be similar). Let $x_0 \in I$ and assume that it is not at the boundary of I (i.e. there are other points to its left and to its right that are in I).

So for any $x \in I$ with $x \le x_0$ we have $f(x) - f(x_0) \le 0$. Therefore $\frac{f(x)-f(x_0)}{x-x_0} \ge 0$.

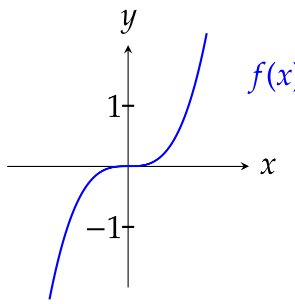
For any $x \in I$ with $x \ge x_0$ we have $f(x) - f(x_0) \ge 0$. Again, $\frac{f(x)-f(x_0)}{x-x_0} \ge 0$.

Therefore, for any $x_0, x \in I$
$$\frac{f(x) - f(x_0)}{x - x_0} \geq 0.$$
By Theorem 5.3 (local sign of limits) the limit as $x \to x_0$ is also non-negative:
$$f'(x_0) \geq 0.$$
This proves the assertion for all x_0 that are not on the boundary of I. If x_0 is on the boundary of I, the same argument can be repeated with one-sided limits.

Now we prove (a)($\Rightarrow$). Let $x_1, x_2 \in I$ with $x_1 < x_2$. By the second increment formula, there exists $\bar{x} \in (x_1, x_2)$ such that
$$f(x_2) - f(x_1) = f'(\bar{x})(x_2 - x_1).$$
By assumption, $f'(\bar{x}) \geq 0$ and $x_2 - x_1 > 0$, so that $f(x_2) \geq f(x_1)$ which completes the proof.

The proof of (b) follows immediately, since in the above argument $f'(\bar{x}) > 0$, hence $f(x_2) > f(x_1)$. □

Observe that part (b) has a one-sided implication; the other implication is not true. For example, the function $f(x) = x^3$ is strictly increasing on $\mathbb{R}$, however its derivative function is not strictly positive (it vanishes at $x = 0$).



Corollary 8.16: Let $f : \mathbb{R} \to \mathbb{R}$ and suppose that f is differentiable on an interval I. Let $x_0 \in I$ be in the interior of I (not on the boundary). Then:

- If $f'(x) \geq 0$ to the left of x_0 and $f'(x) \leq 0$ to the right of x_0, then x_0 is a local maximum.
- If $f'(x) \leq 0$ to the left of x_0 and $f'(x) \geq 0$ to the right of x_0, then x_0 is a local minimum.

Proof. This simple proof is left as an exercise. □

Finding extrema and monotonicity intervals of a function

Using Theorem 8.15 and Corollary 8.16, we see that to find extrema and monotonicity intervals of a function, all we need to do is to know the sign and zeroes of its derivative.