# MATHEMATICAL ANALYSIS 1
# HOMEWORK 9

(1) Prove the following theorem: (it is Corollary 8.16 in the lecture notes)

**Theorem.** *Let $f : \mathbb{R} \to \mathbb{R}$ and suppose that $f$ is differentiable on an interval $I$. Let $x_0 \in I$ be in the interior of $I$ (not on the boundary). Then:*
- *If $f'(x) \geq 0$ to the left of $x_0$ and $f'(x) \leq 0$ to the right of $x_0$, then $x_0$ is a local maximum.*
- *If $f'(x) \leq 0$ to the left of $x_0$ and $f'(x) \geq 0$ to the right of $x_0$, then $x_0$ is a local minimum.*

(2) For each of the following functions, determine: (i) The domain. (ii) The critical points. (iii) The local maximum and minimum values. (iv) The inflection points. (v) The points where the function is not differentiable. (vi) All asymptotes. (vii) Sketch the function (or a relevant part of the function).

(a) $f(x) = x^3 - 6x^2 + 9x + 1$
(b) $f(x) = |x^2 - 4|$
(c) $f(x) = e^{\sqrt{x}}$
(d) $f(x) = \dfrac{1}{(x^2+4)^2}$
(e) $f(x) = \ln(\sin(x))$, $x \in (0, \pi)$
(f) $f(x) = x^{1/3}(x-2)^{1/3}$
(g) $f(x) = \cos(x^2)$
(h) $f(x) = \sqrt[3]{x^2 - 3x}$
(i) $f(x) = \ln(x + \sqrt{x^2+1})$
(j) $f(x) = \arctan\left(\dfrac{x}{x-1}\right)$

(3) Calculate the following limits. Identify the type of indeterminate form, and use De L'Hôpital's Theorem where appropriate.

(a) $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$
(b) $\lim_{x\to 0^+} x \ln x$
(c) $\lim_{x\to\infty} \frac{\ln(x)}{x^2}$
(d) $\lim_{x\to 0} \frac{\sin(x) - x}{x^3}$
(e) $\lim_{x\to\infty} \left(1 + \frac{3}{x}\right)^{2x}$
(f) $\lim_{x\to 0^+} x^x$
(g) $\lim_{x\to \frac{\pi}{2}^-} (\sec(x) - \tan(x))$
(h) $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{e^x - 1}\right)$

(4) Consider the function
$$f(x) = e^x(x^2 - 8|x-3| - 8)$$
Determine
(a) monotonicity intervals,
(b) local extrema and the range $\mathrm{im}(f)$,
(c) points where $f$ is not continuous; points where $f$ is not differentiable,
(d) and sketch a rough graph of $f$, highlighting the previous points.
(e) Does there exist a constant $\alpha \in \mathbb{R}$ such that the function
$$g(x) = f(x) - \alpha|x-3|$$
belongs to the functional space $\mathcal{C}^1(\mathbb{R})$?

(1) **Proof of the First Derivative Test for Local Extrema**

*Proof.* Let $f$ be differentiable on an interval $I$ and let $x_0$ be an interior point of $I$.
**Case 1: Local Maximum.** Assume $f'(x) \geq 0$ for $x < x_0$ (near $x_0$) and $f'(x) \leq 0$ for $x > x_0$ (near $x_0$).

By the Mean Value Theorem (MVT), for any $x < x_0$ in this interval, there exists a $c \in (x, x_0)$ such that:

$$f(x_0) - f(x) = f'(c)(x_0 - x).$$

Since $f'(c) \geq 0$ and $(x_0 - x) > 0$, it follows that $f(x_0) - f(x) \geq 0$, or $f(x) \leq f(x_0)$.

Similarly, for any $x > x_0$, there exists a $c \in (x_0, x)$ such that:

$$f(x) - f(x_0) = f'(c)(x - x_0).$$

Since $f'(c) \leq 0$ and $(x - x_0) > 0$, it follows that $f(x) - f(x_0) \leq 0$, or $f(x) \leq f(x_0)$.

Therefore, $f(x) \leq f(x_0)$ for all $x$ in a neighborhood of $x_0$, which means $x_0$ is a local maximum.
**Case 2: Local Minimum.** The proof is analogous. Assume $f'(x) \leq 0$ for $x < x_0$ and $f'(x) \geq 0$ for $x > x_0$. The same MVT argument shows $f(x) \geq f(x_0)$ for all $x$ near $x_0$, so $x_0$ is a local minimum. □

(2) **Curve Analysis**

(a) For $f(x) = x^3 - 6x^2 + 9x + 1$:

(i) **Domain:** $\mathbb{R}$.

(ii) **Critical Points:** $f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-1)(x-3)$. Setting $f'(x) = 0$ gives $x = 1$ and $x = 3$.

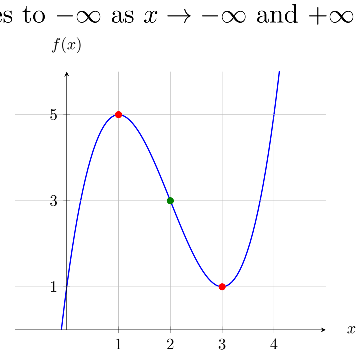
(iii) **Local Extrema:** Using the First Derivative Test:
- For $x < 1$ (e.g., $x = 0$), $f'(0) = 9 > 0$ (increasing).
- For $1 < x < 3$ (e.g., $x = 2$), $f'(2) = 3(1)(-1) = -3 < 0$ (decreasing).
- So, $x = 1$ is a **local maximum**. $f(1) = 1 - 6 + 9 + 1 = 5$.
- For $x > 3$ (e.g., $x = 4$), $f'(4) = 3(3)(1) = 9 > 0$ (increasing).
- So, $x = 3$ is a **local minimum**. $f(3) = 27 - 54 + 27 + 1 = 1$.

(iv) **Inflection Points:** $f''(x) = 6x - 12 = 6(x-2)$. Setting $f''(x) = 0$ gives $x = 2$. Since $f''(x)$ changes sign from negative ($x < 2$) to positive ($x > 2$), $x = 2$ is an inflection point. $f(2) = 8 - 24 + 18 + 1 = 3$.

(v) **Non-differentiable Points:** $f$ is a polynomial, so it is differentiable everywhere on $\mathbb{R}$.

(vi) **Asymptotes:** No vertical or horizontal asymptotes (it's a cubic polynomial).

(vii) **Sketch:** The graph passes through the points: local max at $(1,5)$, inflection at $(2,3)$, local min at $(3,1)$. It goes to $-\infty$ as $x \to -\infty$ and $+\infty$ as $x \to +\infty$.




(b) For $f(x) = |x^2 - 4|$:

(i) **Domain:** $\mathbb{R}$.

(ii), (v) **Critical Points and Non-differentiable Points:** The function is non-differentiable where $x^2 - 4 = 0$, i.e., at $x = \pm 2$. Rewrite as a piecewise function:

$$f(x) = \begin{cases} x^2 - 4, & \text{if } x \leq -2 \text{ or } x \geq 2, \\ 4 - x^2, & \text{if } -2 < x < 2. \end{cases}$$

Differentiate piecewise:

$$f'(x) = \begin{cases} 2x, & \text{if } x < -2 \text{ or } x > 2, \\ -2x, & \text{if } -2 < x < 2. \end{cases}$$

The derivative is undefined at $x = \pm 2$. Setting $f'(x) = 0$ gives $x = 0$ (in the second piece). **Critical points:** $x = -2, 0, 2$.

(iii) **Local Extrema:** Use First Derivative Test.
  - At $x = -2$: $f' < 0$ for $x < -2$, $f' > 0$ for $-2 < x < 0$. **Local minimum**. $f(-2) = 0$.
  - At $x = 0$: $f' > 0$ for $-2 < x < 0$, $f' < 0$ for $0 < x < 2$. **Local maximum**. $f(0) = 4$.
  - At $x = 2$: $f' < 0$ for $0 < x < 2$, $f' > 0$ for $x > 2$. **Local minimum**. $f(2) = 0$.

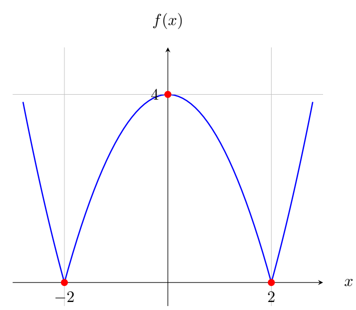
(iv) **Inflection Points:**

$$f''(x) = \begin{cases} 2, & \text{if } x < -2 \text{ or } x > 2, \\ -2, & \text{if } -2 < x < 2. \end{cases}$$

$f''(x)$ is constant on each piece and undefined at $x = \pm 2$. No sign change $\Rightarrow$ **no inflection points**.

(vi) **Asymptotes:** No vertical asymptotes. $\lim_{x \to \pm\infty} f(x) = \infty \Rightarrow$ no horizontal asymptotes.

(vii) **Sketch:** "W"-shaped curve with minima at $(-2, 0)$, $(2, 0)$ and maximum at $(0, 4)$. Corners at $x = \pm 2$.




(c) For $f(x) = e^{\sqrt{x}}$:

(i) **Domain:** $[0, \infty)$ (due to $\sqrt{x}$).

(ii) **Critical Points:** $f'(x) = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$. For $x > 0$, $f'(x) > 0$. No points with $f'(x) = 0$. At $x = 0$, derivative undefined.

(iii) **Local Extrema:** $f$ strictly increasing on $(0, \infty)$. Global minimum at $x = 0$, $f(0) = 1$. No local maximum.

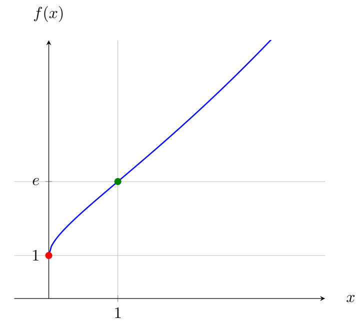
(iv) **Inflection Points:** $f''(x) = \frac{e^{\sqrt{x}}(\sqrt{x}-1)}{4x^{3/2}}$. Set $f''(x) = 0 \Rightarrow \sqrt{x} = 1 \Rightarrow x = 1$.
  - For $0 < x < 1$: $f''(x) < 0$ (concave)
  - For $x > 1$: $f''(x) > 0$ (convex)

  **Inflection point** at $(1, e)$.

(v) **Non-differentiable Points:** Not differentiable at $x = 0$ (vertical tangent).

(vi) **Asymptotes:** No vertical asymptotes. $\lim_{x \to \infty} f(x) = \infty \Rightarrow$ no horizontal asymptotes.

(vii) **Sketch:** Starts at $(0, 1)$ with vertical tangent. Increases, concave until $(1, e)$, then convex. Grows without bound.




(d) For $f(x) = \dfrac{1}{(x^2+4)^2}$:

(i) **Domain:** $\mathbb{R}$ (denominator is always positive).

(ii) **Critical Points:** $f'(x) = \frac{d}{dx}[(x^2+4)^{-2}] = -2(x^2+4)^{-3} \cdot 2x = -\frac{4x}{(x^2+4)^3}$. Setting $f'(x) = 0$ gives $x = 0$.

(iii) **Local Extrema:**

- For $x < 0$, $f'(x) > 0$ (increasing).
- For $x > 0$, $f'(x) < 0$ (decreasing).
- So, $x = 0$ is a **local maximum**. $f(0) = \frac{1}{16}$.

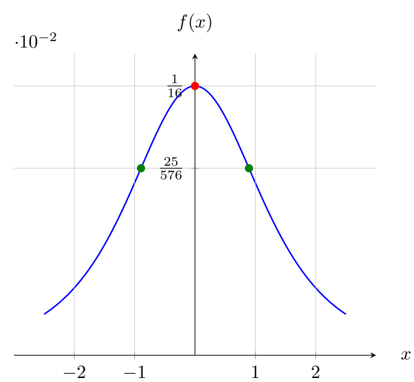
(iv) **Inflection Points:** $f''(x) = \frac{d}{dx}\left[-\frac{4x}{(x^2+4)^3}\right] = \frac{20x^2-16}{(x^2+4)^4}$. Setting $f''(x) = 0$ gives $20x^2 - 16 = 0 \Rightarrow x = \pm\frac{2}{\sqrt{5}}$. Check sign changes: convex for $|x| > \frac{2}{\sqrt{5}}$, concave for $|x| < \frac{2}{\sqrt{5}}$. **Inflection points** at $x = \pm\frac{2}{\sqrt{5}}$, $f\left(\pm\frac{2}{\sqrt{5}}\right) = \mathbf{\frac{25}{576}}$.

(v) **Non-differentiable Points:** Differentiable everywhere on $\mathbb{R}$.

(vi) **Asymptotes:**

- No vertical asymptotes.
- Horizontal asymptote: $\lim_{x\to\pm\infty} f(x) = 0$, so $y = 0$.

(vii) **Sketch:** Bell-shaped curve symmetric about $y$-axis, maximum at $(0, \frac{1}{16})$, inflection points at $(\pm\frac{2}{\sqrt{5}}, \frac{25}{576})$, approaches 0 as $x \to \pm\infty$.




(e) For $f(x) = \ln(\sin(x))$, $x \in (0, \pi)$:

(i) **Domain:** $(0, \pi)$ (since $\sin(x) > 0$ on this interval).

(ii) **Critical Points:** $f'(x) = \frac{\cos(x)}{\sin(x)} = \cot(x)$. Setting $f'(x) = 0$ gives $\cos(x) = 0 \Rightarrow x = \frac{\pi}{2}$.

(iii) **Local Extrema:**

- For $0 < x < \frac{\pi}{2}$, $f'(x) > 0$ (increasing).
- For $\frac{\pi}{2} < x < \pi$, $f'(x) < 0$ (decreasing).
- So, $x = \frac{\pi}{2}$ is a **local maximum**. $f\left(\frac{\pi}{2}\right) = 0$.

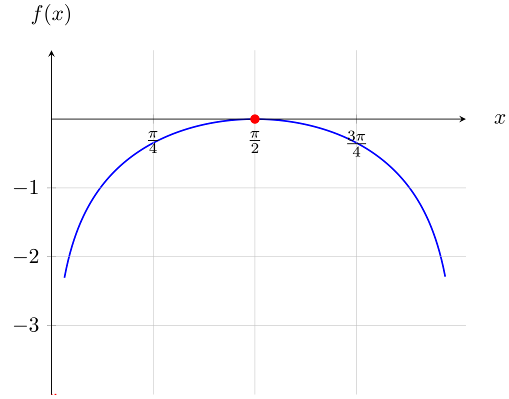
(iv) **Inflection Points:** $f''(x) = -\csc^2(x)$. This is always negative, so no inflection points.

(v) **Non-differentiable Points:** Differentiable on $(0, \pi)$.

(vi) **Asymptotes:**

- Vertical asymptotes: $\lim_{x\to 0^+} f(x) = -\infty$, $\lim_{x\to\pi^-} f(x) = -\infty$.
- No horizontal asymptotes.

(vii) **Sketch:** concave curve with maximum at $(\frac{\pi}{2}, 0)$, vertical asymptotes at $x = 0$ and $x = \pi$.




(f) For $f(x) = x^{1/3}(x-2)^{1/3}$:

(i) **Domain:** $\mathbb{R}$ (cube roots defined for all reals).

(ii) **Critical Points:** $f'(x) = \frac{1}{3}x^{-2/3}(x-2)^{1/3} + \frac{1}{3}x^{1/3}(x-2)^{-2/3} = \frac{2x-2}{3x^{2/3}(x-2)^{2/3}}$. Setting $f'(x) = 0$ gives $x = 1$. Also undefined at $x = 0, 2$.

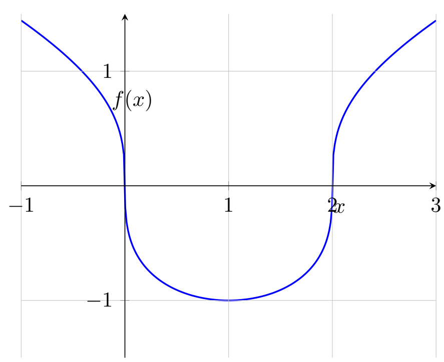
(iii) **Local Extrema:**
- For $x < 0$, $f'(x) > 0$ (increasing).
- For $0 < x < 1$, $f'(x) < 0$ (decreasing).
- For $1 < x < 2$, $f'(x) > 0$ (increasing).
- For $x > 2$, $f'(x) > 0$ (increasing).
- So, $x = 1$ is a **local minimum**. $f(1) = -1$.

(iv) **Inflection Points:** Second derivative complicated, but analysis shows inflection at $x = 0, 2$.

(v) **Non-differentiable Points:** Not differentiable at $x = 0, 2$ (vertical tangents).

(vi) **Asymptotes:** No vertical or horizontal asymptotes.

(vii) **Sketch:** S-shaped curve through $(0,0)$, minimum at $(1,-1)$, $(2,0)$, vertical tangents at $x = 0, 2$.




(g) For $f(x) = \cos(x^2)$:

(i) **Domain:** $\mathbb{R}$.

(ii) **Critical Points:** $f'(x) = -2x\sin(x^2)$. Setting $f'(x) = 0$ gives $x = 0$ or $\sin(x^2) = 0 \Rightarrow x = \pm\sqrt{n\pi}$, $n \in \mathbb{N}$.

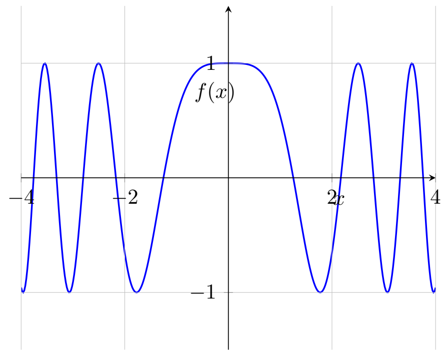
(iii) **Local Extrema:** Alternating maxima and minima at critical points.

(iv) **Inflection Points:** Many inflection points where $f''$ changes sign.

(v) **Non-differentiable Points:** Differentiable everywhere.

(vi) **Asymptotes:** No asymptotes (bounded oscillatory function).

(vii) **Sketch:** Oscillatory function with frequency increasing as $|x|$ increases, range $[-1, 1]$.




(h) For $f(x) = \sqrt[3]{x^2 - 3x}$:

(i) **Domain:** $\mathbb{R}$ (cube root defined for all reals).

(ii) **Critical Points:** $f'(x) = \frac{2x-3}{3(x^2-3x)^{2/3}}$. Setting $f'(x) = 0$ gives $x = \frac{3}{2}$. Undefined at $x = 0, 3$.

(iii) **Local Extrema:**

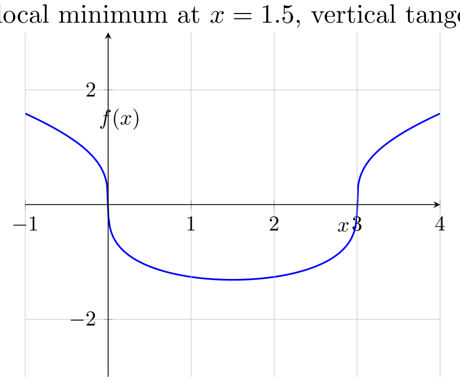
– $x = \frac{3}{2}$ is a **local minimum**. $f\left(\frac{3}{2}\right) = -\sqrt[3]{\frac{9}{4}}$.

(iv) **Inflection Points:** At $x = 0, 3$ (points of non-differentiability).

(v) **Non-differentiable Points:** Not differentiable at $x = 0, 3$ (vertical tangents).

(vi) **Asymptotes:** No asymptotes.

(vii) **Sketch:** Curve with local minimum at $x = 1.5$, vertical tangents at $x = 0, 3$.




(i) For $f(x) = \ln(x + \sqrt{x^2 + 1})$:

(i) **Domain:** $\mathbb{R}$ (argument always positive).

(ii) **Critical Points:** $f'(x) = \frac{1}{\sqrt{x^2+1}}$. Never zero, always positive.

(iii) **Local Extrema:** No local extrema (strictly increasing).

(iv) **Inflection Points:** $f''(x) = -\frac{x}{(x^2+1)^{3/2}}$, zero at $x = 0$. Inflection point at $(0, 0)$.

(v) **Non-differentiable Points:** Differentiable everywhere.

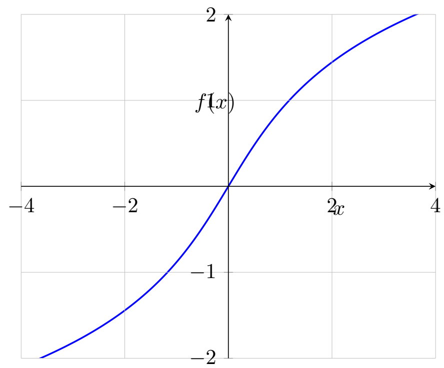
(vi) **Asymptotes:**

– As $x \to \infty$, $f(x) \sim \ln(2x) \to \infty$.

– As $x \to -\infty$, $f(x) \sim \ln(1/2|x|) \to -\infty$.

– No horizontal asymptotes.

(vii) **Sketch:** Odd function, strictly increasing, inflection at origin, asymptotic to $\ln(2x)$ for large $x$.




(j) For $f(x) = \arctan\left(\dfrac{x}{x-1}\right)$:

(i) **Domain:** $\mathbb{R} \setminus \{1\}$.

(ii) **Critical Points:** $f'(x) = \frac{-1}{(x-1)^2+x^2}$. Never zero, always negative.

(iii) **Local Extrema:** No local extrema (strictly decreasing on each interval).

(iv) **Inflection Points:** No inflection points.

(v) **Non-differentiable Points:** Not differentiable at $x = 1$ (not in domain).

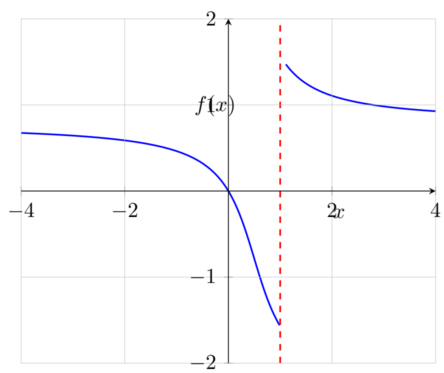
(vi) **Asymptotes:**

- Vertical asymptote at $x = 1$.
- Horizontal asymptotes: $\lim_{x\to\pm\infty} f(x) = \arctan(1) = \frac{\pi}{4}$.

(vii) **Behavior near $x = 1$:**

- As $x \to 1^-$: $\frac{x}{x-1} \to -\infty$, so $f(x) \to \arctan(-\infty) = -\frac{\pi}{2}$
- As $x \to 1^+$: $\frac{x}{x-1} \to +\infty$, so $f(x) \to \arctan(+\infty) = \frac{\pi}{2}$

(viii) **Sketch:**




(3) **Limits**

(a) $\lim_{x\to 0} \frac{e^x-1-x}{x^2}$.

- **Indeterminate Form:** $\frac{0}{0}$.
- **Solution:** Apply L'Hôpital's Theorem.

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} \overset{\text{(H)}}{=} \lim_{x\to 0} \frac{e^x - 1}{2x}.$$

This is still $\frac{0}{0}$. Apply L'Hôpital's Theorem again.

$$\lim_{x\to 0} \frac{e^x - 1}{2x} \overset{\text{(H)}}{=} \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}.$$

- **Final Answer:** $\dfrac{1}{2}$.

(b) $\lim_{x\to 0^+} x \ln x$.

- **Indeterminate Form:** $0 \cdot (-\infty)$.

- **Solution:** Rewrite as a quotient: $x \ln x = \frac{\ln x}{1/x}$.

$$\lim_{x \to 0^+} \frac{\ln x}{1/x} \text{ is of the form } \frac{-\infty}{\infty}.$$

Apply L'Hôpital's Theorem.

$$\lim_{x \to 0^+} \frac{\ln x}{1/x} \overset{\text{(H)}}{=} \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} (-x) = 0.$$

- **Final Answer:** 0.

(c) $\lim_{x \to \infty} \frac{\ln(x)}{x^2}$.
- **Indeterminate Form:** $\frac{\infty}{\infty}$.
- **Solution:** Apply L'Hôpital's Theorem.

$$\lim_{x \to \infty} \frac{\ln(x)}{x^2} \overset{\text{(H)}}{=} \lim_{x \to \infty} \frac{1/x}{2x} = \lim_{x \to \infty} \frac{1}{2x^2} = 0.$$

- **Final Answer:** 0.

(d) $\lim_{x \to 0} \frac{\sin(x) - x}{x^3}$.
- **Indeterminate Form:** $\frac{0}{0}$.
- **Solution:** Apply L'Hôpital's Theorem.

$$\lim_{x \to 0} \frac{\sin(x) - x}{x^3} \overset{\text{(H)}}{=} \lim_{x \to 0} \frac{\cos(x) - 1}{3x^2}.$$

This is still $\frac{0}{0}$. Apply L'Hôpital's Theorem again.

$$\lim_{x \to 0} \frac{\cos(x) - 1}{3x^2} \overset{\text{(H)}}{=} \lim_{x \to 0} \frac{-\sin(x)}{6x}.$$

This is still $\frac{0}{0}$. Apply L'Hôpital's Theorem again.

$$\lim_{x \to 0} \frac{-\sin(x)}{6x} \overset{\text{(H)}}{=} \lim_{x \to 0} \frac{-\cos(x)}{6} = -\frac{1}{6}.$$

- **Final Answer:** $-\dfrac{1}{6}$.

(e) $\lim_{x \to \infty} \left(1 + \frac{3}{x}\right)^{2x}$.
- **Indeterminate Form:** $1^\infty$.
- **Solution:** Let $L = \lim_{x \to \infty} \left(1 + \frac{3}{x}\right)^{2x}$. Then:

$$\ln L = \lim_{x \to \infty} 2x \ln\left(1 + \frac{3}{x}\right) = \lim_{x \to \infty} \frac{2\ln\left(1 + \frac{3}{x}\right)}{1/x}.$$

This is of the form $\frac{0}{0}$. Apply L'Hôpital's Theorem.

$$\ln L = \lim_{x \to \infty} \frac{2 \cdot \frac{1}{1 + \frac{3}{x}} \cdot \left(-\frac{3}{x^2}\right)}{-1/x^2} = \lim_{x \to \infty} \frac{2 \cdot \frac{1}{1 + \frac{3}{x}} \cdot 3}{1} = \lim_{x \to \infty} \frac{6}{1 + \frac{3}{x}} = 6.$$

Thus $L = e^6$.

- **Final Answer:** $e^6$.

(f) $\lim_{x \to 0^+} x^x$.
- **Indeterminate Form:** $0^0$.
- **Solution:** Let $L = \lim_{x \to 0^+} x^x$. Then:

$$\ln L = \lim_{x \to 0^+} x \ln x = 0 \quad \text{(from part 3(b))}.$$

Thus $L = e^0 = 1$.

- **Final Answer:** 1.

(g) $\lim_{x \to \frac{\pi}{2}^-} (\sec(x) - \tan(x))$.
- **Indeterminate Form:** $\infty - \infty$.

- **Solution:** Rewrite using trigonometric identities:
$$\sec(x) - \tan(x) = \frac{1}{\cos(x)} - \frac{\sin(x)}{\cos(x)} = \frac{1-\sin(x)}{\cos(x)}.$$
As $x \to \frac{\pi}{2}^-$, this is $\frac{0}{0}$. Apply L'Hôpital's Theorem.
$$\lim_{x\to\frac{\pi}{2}^-} \frac{1-\sin(x)}{\cos(x)} \overset{(\mathrm{H})}{=} \lim_{x\to\frac{\pi}{2}^-} \frac{-\cos(x)}{-\sin(x)} = \lim_{x\to\frac{\pi}{2}^-} \frac{\cos(x)}{\sin(x)} = 0.$$
- **Final Answer:** 0.

(h) $\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{e^x-1}\right)$.
- **Indeterminate Form:** $\infty - \infty$.
- **Solution:** Combine into a single fraction:
$$\frac{1}{x} - \frac{1}{e^x-1} = \frac{e^x-1-x}{x(e^x-1)}.$$
As $x \to 0$, this is $\frac{0}{0}$. Apply L'Hôpital's Theorem.
$$\lim_{x\to 0} \frac{e^x-1-x}{x(e^x-1)} \overset{(\mathrm{H})}{=} \lim_{x\to 0} \frac{e^x-1}{e^x-1+xe^x}.$$
This is still $\frac{0}{0}$. Apply L'Hôpital's Theorem again.
$$\lim_{x\to 0} \frac{e^x-1}{e^x-1+xe^x} \overset{(\mathrm{H})}{=} \lim_{x\to 0} \frac{e^x}{e^x+e^x+xe^x} = \lim_{x\to 0} \frac{e^x}{2e^x+xe^x} = \frac{1}{2}.$$
- **Final Answer:** $\frac{1}{2}$.

(4) For $f(x) = e^x(x^2 - 8|x-3| - 8)$.

(a) **Monotonicity Intervals:** First, write $f$ as a piecewise function.
$$f(x) = \begin{cases} e^x(x^2-8(3-x)-8) = e^x(x^2+8x-32), & x<3, \\ e^x(x^2-8(x-3)-8) = e^x(x^2-8x+16) = e^x(x-4)^2, & x \geq 3. \end{cases}$$
Now differentiate piecewise. For $x<3$: $f'(x) = e^x(x^2+8x-32)+e^x(2x+8) = e^x(x^2+10x-24) = e^x(x+12)(x-2)$. For $x>3$: $f'(x) = e^x(x-4)^2 + e^x \cdot 2(x-4) = e^x(x-4)[(x-4)+2] = e^x(x-4)(x-2)$.
Analyze the sign of $f'(x)$:
- On $(-\infty, 3)$: $f'(x) = e^x(x+12)(x-2)$. $e^x > 0$ always.
  - $f'(x) = 0$ at $x=-12$ and $x=2$.
  - Sign chart: $f'(x) > 0$ for $x < -12$, $f'(x) < 0$ for $-12 < x < 2$, $f'(x) > 0$ for $2 < x < 3$.
- On $(3, \infty)$: $f'(x) = e^x(x-4)(x-2)$.
  - $f'(x) = 0$ at $x = 4$. (Note: $x = 2$ is not in this interval).
  - Sign chart: For $3 < x < 4$, $f'(x) < 0$. For $x > 4$, $f'(x) > 0$.

**Conclusion:** $f$ is increasing on $(-\infty, -12)$, decreasing on $(-12, 2)$, increasing on $(2, 3)$, decreasing on $(3, 4)$, and increasing on $(4, \infty)$.

(b) **Local Extrema and Range:**
- $x = -12$: $f'$ changes from $+$ to $-$, so **local maximum**. $f(-12) = e^{-12}((-12)^2+8(-12)-32) = e^{-12}(144-96-32) = 16e^{-12}$.
- $x = 2$: $f'$ changes from $-$ to $+$, so **local minimum**. $f(2) = e^2(4+16-32) = -12e^2$.
- $x = 4$: $f'$ changes from $-$ to $+$, so **local minimum**. $f(4) = e^4(4-4)^2 = 0$.
- At $x = 3$, we must check from the left and right. $\lim_{x\to 3^-} f(x) = e^3(9+24-32) = e^3$, and $f(3) = e^3(3-4)^2 = e^3$. The function is continuous at $x = 3$. Since $f$ is increasing on $(2, 3)$ and decreasing on $(3, 4)$, $x = 3$ is a **local maximum**. $f(3) = e^3$.

To find the range, consider the limits and values. As $x \to -\infty$, $f(x) \to 0^+$ (since $e^x \to 0$ and the polynomial dominates to $+\infty$? Let's check: for $x << 0$, $x^2+8x-32 > 0$, but $e^x$ decays to 0, so limit is $0^+$). As $x \to \infty$, $f(x) \to \infty$. The minimum values are $f(2) = -12e^2$ and $f(4) = 0$.

The maximum values are $f(-12) = 16e^{-12}$ and $f(3) = e^3$. Since the function is continuous on each piece and takes on all values between the min and max on the intervals between these critical points, the **range** is $[-12e^2, \infty)$.

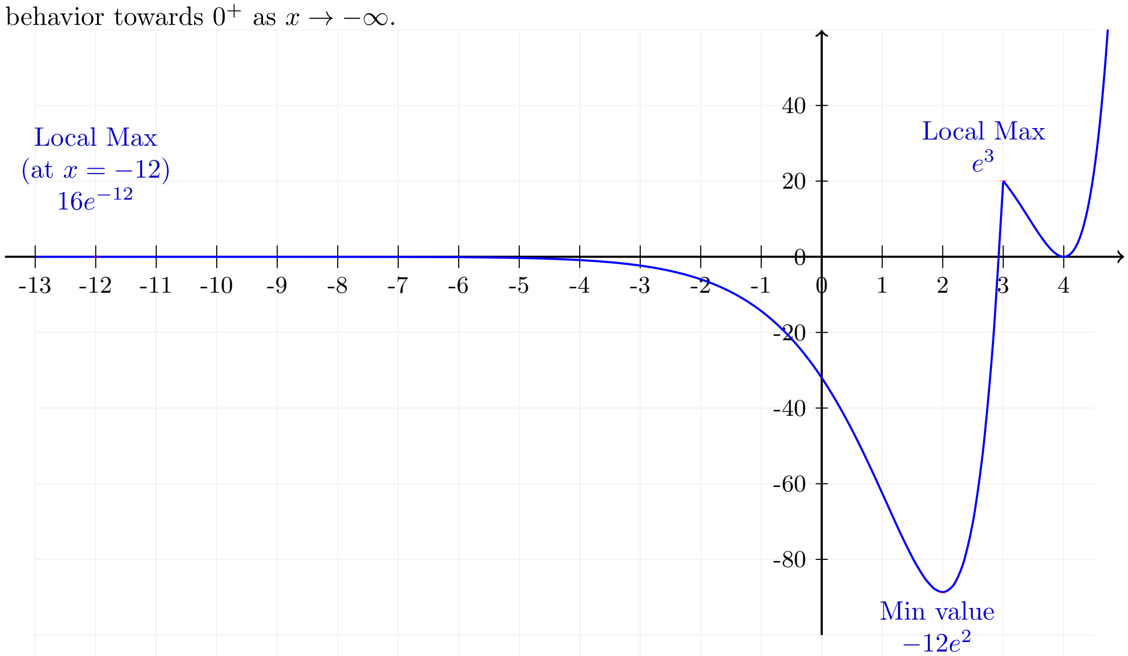
(c) **Continuity and Differentiability:** $f$ is continuous everywhere (composition of continuous functions). It is differentiable everywhere **except** possibly at $x = 3$ due to the absolute value. Let's check the derivative from left and right at $x = 3$.

$$f'_-(3) = \lim_{h\to 0^-} \frac{f(3+h) - f(3)}{h}. \quad \text{Using the left piece: } f(3+h) = e^{3+h}((3+h)^2 + 8(3+h) - 32).$$

$$f'_+(3) = \lim_{h\to 0^+} \frac{f(3+h) - f(3)}{h}. \quad \text{Using the right piece: } f(3+h) = e^{3+h}((3+h) - 4)^2 = e^{3+h}(h-1)^2.$$

It's easier to use the piecewise derivatives we computed. $f'_-(3) = e^3(3+12)(3-2) = e^3 \cdot 15 \cdot 1 = 15e^3$. $f'_+(3) = e^3(3-4)(3-2) = e^3 \cdot (-1) \cdot 1 = -e^3$. Since $f'_-(3) \neq f'_+(3)$, $f$ is **not differentiable** at $x = 3$.

(d) **Sketch:** Highlight the points: $(-12, 16e^{-12})$ (local max), $(2, -12e^2)$ (local min), $(3, e^3)$ (local max, cusp), $(4, 0)$ (local min). Show the increasing/decreasing intervals and the asymptotic behavior towards $0^+$ as $x \to -\infty$.




(e) **Find $\alpha$ for $g \in \mathcal{C}^1(\mathbb{R})$:** We have $g(x) = f(x) - \alpha|x-3|$. We know $f$ is not differentiable at $x = 3$ because of the $|x-3|$ term. For $g$ to be differentiable at $x = 3$, the coefficient in front of $|x-3|$ in the piecewise definition of $g$ must be zero when we consider the jump in derivative. More precisely, the derivative jump of $g$ at $x = 3$ is:

$$g'_+(3) - g'_-(3) = \big(f'_+(3) - \alpha \cdot (1)\big) - \big(f'_-(3) - \alpha \cdot (-1)\big) = (f'_+(3) - f'_-(3)) - 2\alpha.$$

We have $f'_+(3) - f'_-(3) = -e^3 - 15e^3 = -16e^3$. For $g$ to be differentiable at $x = 3$, we need this jump to be zero:

$$-16e^3 - 2\alpha = 0 \quad \Rightarrow \quad \alpha = -8e^3.$$

With this $\alpha$, $g'(3)$ exists and $g$ is differentiable everywhere. Since $f$ and $|x-3|$ are continuously differentiable on $\mathbb{R} \setminus \{3\}$, and we have removed the discontinuity in the derivative at $x = 3$, $g$ is in fact $\mathcal{C}^1(\mathbb{R})$.