The functional spaces $C^k(I)$ and $C^\infty(I)$

Given an interval $I \subseteq \mathbb{R}$, we can define the set $C^0(I)$ of all functions that are continuous on I:

$$C^0(I) = \{f : \mathbb{R} \to \mathbb{R} \ \mid \ f \text{ is continuous on } I\}$$

and the set $C^k(I)$ of all functions that are k times differentiable ($k \in \mathbb{N}$) on I, with the k^{th} derivative being continuous in I:

$$C^k(I) = \left\{f : \mathbb{R} \to \mathbb{R} \ \mid \ f^{(j)}(x) \text{ exist } \forall j = 1, \ldots, k, \ \forall x \in I, \text{ and } f^{(k)} \in C^0(I)\right\}.$$

A function belonging to $C^k(I)$ is said to be a C^k **function on** I. We can further define

$$C^\infty(I) = \left\{f : \mathbb{R} \to \mathbb{R} \ \mid \ f^{(j)}(x) \text{ exist } \forall j \in \mathbb{N}, \ \forall x \in I\right\}.$$

the set of all infinitely-differentiable functions. A function $f \in C^\infty(I)$ is said to be a C^∞ **function on** I.

The sets $C^k(I)$ and $C^\infty(I)$ are in fact *functional spaces*, which means that they possess certain useful properties. This is not within the scope of this course, though.

8.9 Convexity and inflection points

Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable at a point $x_0 \in \mathbb{R}$. Then

$$t_{x_0}(x) = f(x_0) + f'(x_0)(x - x_0)$$

is the equation of the line tangent to the graph of f at x_0.

Convexity

The function f is called **convex at** x_0 if there exists a neighobrhood $I_r(x_0)$ of x_0 on which it lies above the tangent at x_0:

$$f(x) \geq t_{x_0}(x), \qquad \forall x \in I_r(x_0).$$

The function f is called **strictly convex at** x_0 if the inequality is strict:

$$f(x) > t_{x_0}(x), \qquad \forall x \in I_r(x_0) \setminus \{x_0\}.$$

Concavity

The function f is called **concave at** x_0 if there exists a neighobrhood $I_r(x_0)$ of x_0 on which it lies below the tangent at x_0:

$$f(x) \le t_{x_0}(x), \qquad \forall x \in I_r(x_0).$$

The function f is called **strictly concave at** x_0 if the inequality is strict:

$$f(x) < t_{x_0}(x), \qquad \forall x \in I_r(x_0) \setminus \{x_0\}.$$

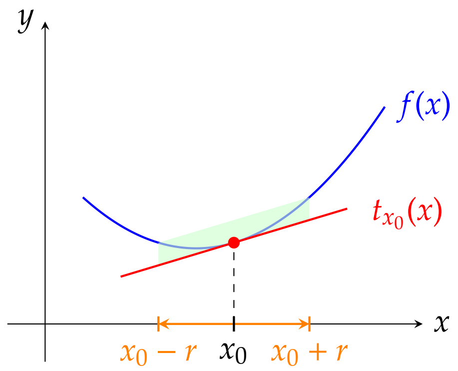

Figure 8.4: Illustration of a convex function at x_0: $f(x) \ge t_{x_0}(x)$ for all x in some neighborhood $I_r(x_0)$ of x_0 ($r > 0$ is small). The green shaded region shows where the function lies above its tangent.

Example 8.24: Let us show that $f(x) = x^2$ is strictly convex at $x_0 = 1$. The tangent line to the graph is

$$t_1(x) = 1 + 2(x-1) = 2x - 1.$$

Comparing to f we find

$$f(x) - t_1(x) = x^2 - (2x-1) = x^2 - 2x + 1 = (x-1)^2 \ge 0$$

with an equality if and only if $x = 1$. Hence f is *strictly* convex at $x_0 = 1$.

Convexity and Concavity on an Interval

A differentiable function f is called **convex on** I if it is convex at all points in I. Similarly, f is called **concave on** I if it is concave at all points in I.

Theorem 8.17: Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable on an interval $I \subseteq \mathbb{R}$. Then f is convex on I if and only if for every $x_0 \in I$,

$$f(x) \ge t_{x_0}(x) = f(x_0) + f'(x_0)(x - x_0) \qquad \forall x \in I.$$

Similarly, f is concave on I if and only if for every $x_0 \in I$,

$$f(x) \le t_{x_0}(x) = f(x_0) + f'(x_0)(x - x_0) \qquad \forall x \in I.$$

Remark: Observe that the definition of convexity/concavity is a definition at a single point x_0 (see Figure 8.4). Then we said that f is convex/concave on I it is it convex/-concave at all $x_0 \in I$. However the property remained a property of the point x_0. The novelty of this theorem is that the property is now a property of the entire interval I, not just a single point x_0. To understand this visually, let's focus on the convex case. A function is convex at x_0 if *near* x_0 its graph lies above the tangent t_{x_0} (see Figure 8.5).

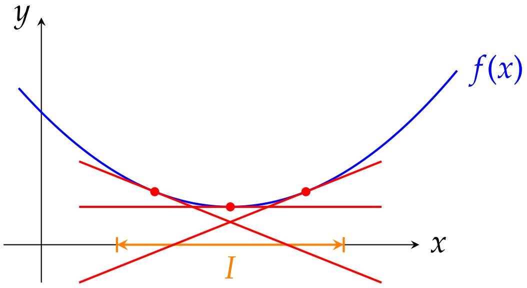

Figure 8.5: A convex function on the interval $I = [a, b]$: $f(x)$ lies above all its tangent lines within the interval, not just at a single point.

Proof of Theorem 8.17. We prove for the convex case. The concave case follows a similar argument.

(Direction $\Leftarrow$) This direction is immediate by definition of convexity.

(Direction $\Rightarrow$) We assume that f is convex on I, and want to prove that for every $x_0 \in I$, $f(x) \geq t_{x_0}(x)$ for all $x \in I$. Equivalently, define for each x_0 the function

$$g_{x_0}(x) = f(x) - t_{x_0}(x).$$

Then we want to show that $g_{x_0}(x) \geq 0$ for all $x \in I$. Observe that

$$g_{x_0}(x_0) = f(x_0) - t_{x_0}(x_0) = 0$$

and

$$g'_{x_0}(x_0) = f'(x_0) - t'_{x_0}(x_0) = 0.$$

For simplicity, assume that x_0 is not a boundary point of I.

Since f is convex at x_0, we know that there exists $r > 0$ such that $g_{x_0}(x) \geq 0$ for all $x \in I_r(x_0)$, where $I_r \subseteq I$. Let us show that $g_{x_0}(x) \geq 0$ to the *right* of x_0. The proof for the *left* of x_0 is similar.

To the right of x_0, we know that for all $x \in [x_0, x_0 + r)$, $g_{x_0}(x) \geq 0$. If this inequality can be extended to any $x \in I \cap \{x > x_0\}$ then we're done. Otherwise, define

$$P = \{x > x_0 \quad | \quad g_{x_0}(y) \geq 0, \ \forall y \in [x_0, x]\}$$

to be the set of points x to the right of x_0 such that g_{x_0} is non-negative on $[x_0, x]$. The maximal such interval $[x_0, x_1]$ has a right endpoint given by

$$x_1 = \sup P.$$

By contradiction, assume that x_1 is an internal point of I.

By definition of the supremum, immediately to the right of x_1 there are points where g_{x_0} is negative. By the continuity of g_{x_0}, it must hold that

$$g_{x_0}(x_1) = 0.$$

Claim: $g_{x_0}(x) = 0$ for all $x \in [x_0, x_1]$.

Proof of claim. **By contradiction**. If the claim is not true, then since g_{x_0} is non-negative, it must hold that $M = \max_{x\in[x_0,x_1]} g_{x_0}(x)$ is strictly positive: $M > 0$. By Weierstrass' Theorem, the maximum of a continuous function on a closed interval is attained, so that there exists $\bar{x} \in (x_0, x_1)$ such that

$$g_{x_0}(\bar{x}) = M.$$

By Fermat's Theorem (Theorem 8.9),

$$g'_{x_0}(\bar{x}) = 0$$

since $\bar{x}$ is an extremum.

Now recall that g_{x_0} is a convex function. Since $g_{x_0}(\bar{x}) = M$ and $g'_{x_0}(\bar{x}) = 0$, convexity implies that $g_{x_0}(x) \geq M$ on a neighborhood of $\bar{x}$. Since M is the maximum of g_{x_0}, it must hold that $g_{x_0}(x) = M$ on that neighborhood. How far to the right can this neighborhood extend? Define

$$Q = \{x > \bar{x} \mid g_{x_0}(y) = M, \ \forall y \in [\bar{x}, x]\}$$

and

$$x_2 = \max Q$$

(the maximum is attained since g_{x_0} is a continuous function). Observe that $x_2 < x_1$ since $g_{x_0}(x_1) = 0$. So we have found a point $x_2 \in (x_0, x_1)$ where the function g_{x_0} attains the value M, and, moreover, g_{x_0} attains the value M in a left-neighborhood of x_2. Hence the left-derivative of g_{x_0} at x_2 must be 0. Since g_{x_0} is differentiable at x_2, it must hold that $g'_{x_0}(x_2) = 0$. But then due to convexity at x_2 it must hold that $g_{x_0} \geq 0$ in a neighborhood of x_2, and in particular in a right-neighborhood. But this contradicts the definition of x_2 as $\max Q$. Therefore the claim is proved. □

We can now conclude the proof of the theorem. The claim implies that $g_{x_0}(x) = 0$ for all $x \in [x_0, x_1]$. But then, as in the preceding argument, the left-derivative of g_{x_0} at x_1 is 0. Consequently, (by convexity at x_1) it must hold that $g_{x_0} \geq 0$ to the right of x_1, in contradiction to the definition of x_1 as $\sup P$ and the assumption that x_1 is an internal point of I. Therefore it must hold that $g_{x_0} \geq 0$ on I. □

Inflection point

An inflection point is a point x_0 where the graph of the function lies above (or at) the tangent line t_{x_0} on one side, and below (or at) on the other side.