MATHEMATICAL ANALYSIS 1
HOMEWORK 7

Since we are at the mid-point of the semester, this assignment is minimal. Just one exercise, to deepen your understanding of the theorems that we've seen this week:

(1) In this problem we prove the **corollary to the Existence of Zeroes theorem** regarding a zero of a continuous function on an open interval that may be infinite (you are guided in the steps below):
   (a) State the corollary (it is Corollary 7.2 in the lecture notes).
   (b) Suppose that $\ell_\alpha < 0 < \ell_\beta$ (state that the case $\ell_\alpha > 0 > \ell_\beta$ follows a similar proof, and is therefore omitted).
   (c) By Theorem 5.2 of the lecture notes there exists a neighborhood of α on which $f < 0$ (here you need to consider the two cases of α being finite or $-\infty$).
   (d) A similar argument is applied for a neighborhood of β.
   (e) In these neighborhoods pick points a and b.
   (f) Argue that Theorem 7.1 (existence of zeroes) is now applicable for f on $[a, b]$.
   (g) Conclude the proof.

Solution

To prove the corollary, we follow the prescribed steps:

(a) We shall prove the following Corollary:

Corollary. Let $(\alpha, \beta) \subseteq \mathbb{R}$ be an interval (α may be $-\infty$ and β may be $+\infty$), and let $f : \mathbb{R} \to \mathbb{R}$ be defined on (α, β), satisfy

$$\lim_{x \to \alpha^+} f(x) = \ell_\alpha \in \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$$
$$\lim_{x \to \beta^-} f(x) = \ell_\beta \in \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$$

with ℓ_α and ℓ_β having opposite signs. Then f has a zero x_0 in (α, β): $f(x_0) = 0$. Moreover, if f is strictly monotone in (α, β) then x_0 is unique.

Now we start the proof.

(b) Suppose that $\ell_\alpha < 0 < \ell_\beta$ (the case $\ell_\alpha > 0 > \ell_\beta$ follows a similar proof, and is therefore omitted).

(c) By Theorem 5.2 of the lecture notes there exists a neighborhood of α on which $f < 0$:
- If $\alpha = -\infty$, then there exists $M < 0$ s.t. $f < 0$ on $I_\alpha = \{x < M\}$.
- If $\alpha \in \mathbb{R}$, then $f < 0$ on a neighborhood I_α of α (potentially excluding α itself).

(d) By Theorem 5.2 of the lecture notes there exists a neighborhood of β on which $f > 0$:
- If $\beta = +\infty$, then there exists $M > 0$ s.t. $f > 0$ on $I_\beta = \{x > M\}$.
- If $\beta \in \mathbb{R}$, then $f > 0$ on a neighborhood I_β of β (potentially excluding β itself).

(e) Pick any $a \in I_\alpha$ and $b \in I_\beta$, where I_α and I_β are as defined above. Then (by definition of I_α and I_β)

$$f(a) < 0 \qquad \text{and} \qquad f(b) > 0.$$

(f) Since $a, b \in \mathbb{R}$, f is continuous on $[a, b]$, and $f(a) < 0 < f(b)$, Theorem 7.1 (existence of zeroes) is now applicable for f on $[a, b]$.

(g) By Theorem 7.1 there exists $x_0 \in (a, b)$ such that $f(x_0) = 0$. Finally, if f is strictly monotone on (α, β), then by Proposition 2.1, f is injective on (α, β). This means that for every $y \in f((\alpha, \beta))$ there exists a unique $x \in (\alpha, \beta)$ such that $f(x) = y$. Hence x_0 is the unique zero of f in (α, β). This concludes that proof.