## 5.4 Completeness: Convergence of Fourier Series

We continue with the operators $\mathcal{L}_D$, $\mathcal{L}_N$ and $\mathcal{L}_P$ on the interval $(a,b)$. We have seen that:

(1) There are no complex eigenvalues and all eigenfunctions can be taken to be real-valued.

(2) Any two eigenfunctions corresponding to different eigenvalues are orthogonal.

(3) There are no negative eigenvalues.

(4) There are infinitely many eigenvalues tending to $+\infty$; they can be ordered as $0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \longrightarrow +\infty$.

Let $f(x)$ be a function on $(a,b)$. Let $\mathcal{L}$ be any of $\mathcal{L}_D, \mathcal{L}_N$ or $\mathcal{L}_P$. Let $\{(\lambda_n, X_n)\}_{n=1}^{\infty}$ be eigenvalue-eigenfunction pairs (where $X_n$ are not necessarily chosen to be real, perhaps out of convenience: we've seen that complex eigenfunctions can be easier to work with).

Definition: The Fourier coefficients of $f(x)$ are

$$A_n = \frac{(f, X_n)}{(X_n, X_n)} = \frac{\int_a^b f(x)\overline{X_n(x)}\,dx}{\int_a^b |X_n(x)|^2\,dx}$$

The Fourier Series of $f(x)$ is: $f(x) = \sum_{n=1}^{\infty} A_n X_n(x)$

**Notions of Convergence:** what does the equality $f(x) = \sum_{n=1}^{\infty} A_n X_n(x)$ mean? In other words, if we consider the partial sums $S_N(x) = \sum_{n=1}^{N} A_n X_n(x)$ converge to $f(x)$ as $N \to +\infty$?

<u>Definition</u>:

(1) We say that $S_N(x)$ converges to $f(x)$ pointwise if for each $x \in (a,b)$

$$|f(x) - S_N(x)| \to 0 \quad \text{as } N \to +\infty.$$

(2) We say that $S_N(x)$ converges to $f(x)$ uniformly in $[a,b]$ if

$$\max_{a \le x \le b} |f(x) - S_N(x)| \to 0 \quad \text{as } N \to +\infty.$$

(3) We say that $S_N(x)$ converges to $f(x)$ in the $L^2$ sense if

$$\int_a^b |f(x) - S_N(x)|^2 \, dx \to 0 \quad \text{as } N \to +\infty.$$

Under various conditions on $f$ there are theorems that guarantee each of these notions of convergence. We skip that for now.

Instead we focus more on the $L^2$ theory.

# $L^2$ Theory: Bessel's Inequality and Parseval's Equality

We have seen the definition of the inner product

$$(f,g) = \int_a^b f(x)\overline{g(x)}\,dx .$$

Let us go further and define a norm:

$$\|f\| = \sqrt{(f,f)} = \sqrt{\int_a^b |f(x)|^2 dx}$$

which leads to the notion of a distance (metric):

$$\|f-g\| = \sqrt{\int_a^b |f(x)-g(x)|^2 dx}$$

Recall that our $f$ is given by $f(x) = \sum_{n=1}^{\infty} A_n X_n(x)$.

To understand the convergence we split

$$f(x) = \sum_{n=1}^{\infty} A_n X_n(x) = \underbrace{\sum_{n=1}^{N} A_n X_n(x)}_{S_N} + \sum_{n=N+1}^{\infty} A_n X_n(x)$$

$$\Longrightarrow \sum_{n=N+1}^{\infty} A_n X_n(x) = f(x) - S_N \Longrightarrow \underbrace{\Big\|\sum_{n=N+1}^{\infty} A_n X_n(x)\Big\|^2}_{\text{call this } E_N\text{, the "error"}} = \|f(x) - S_N\|^2$$

$$\begin{aligned}
\Longrightarrow E_N = \|f(x)-S_N\|^2 &= \int_a^b \Big|f(x) - \sum_{n=1}^{N} A_n X_n(x)\Big|^2 dx \\
&= \int_a^b |f(x)|^2 dx - 2\sum_{n=1}^{N}\int_a^b f(x) A_n X_n(x)\,dx + \sum_{n=1}^{N}\sum_{m=1}^{N}\int_a^b A_n A_m X_n X_m\,dx \\
&= \|f\|^2 - 2\sum_{n=1}^{N} A_n (f, X_n) + \sum_{n=1}^{N}\sum_{m=1}^{N} A_n A_m (X_n, X_m) \\
&= \|f\|^2 - 2\sum_{n=1}^{N} A_n^2 \|X_n\|^2 + \sum_{n=1}^{N} A_n^2 \|X_n\|^2 \\
&= \|f\|^2 - \sum_{n=1}^{N} A_n^2 \|X_n\|^2
\end{aligned}$$

Since $E_N$ is a norm, it is $\geq 0$, so: $\|f\|^2 - \sum_{n=1}^{N} A_n^2 \|X_n\|^2 \geq 0$

$$\Longrightarrow \sum_{n=1}^{N} A_n^2 \|X_n\|^2 \leq \|f\|^2$$

This is true for any $N$, hence all partial sums $\sum_{n=1}^{N} A_n^2 \|X_n\|^2$ are uniformly bounded, so we may take the limit $N \to +\infty$ to get:

$$\sum_{n=1}^{\infty} A_n^2 \|X_n\|^2 \leq \|f\|^2$$

This is called **Bessel's inequality**.

**Theorem:** The Fourier series of $f$ converges to $f$ in $L^2$ if and only if there's an equality in Bessel's inequality.

Proof: By definition, $S_N(x)$ converge to $f$ in the $L^2$ sense if and only if $\underbrace{\int_a^b |f(x) - S_N(x)|^2 dx}_{\text{this is exactly } E_N} \longrightarrow 0$.

However, from our calculations above, $E_N \to 0$ as $N \to +\infty$ if and only if $\|f\|^2 - \sum_{n=1}^{N} A_n^2 \|X_n\|^2 \longrightarrow 0$ as $N \to +\infty$, which is true if and only if

$$\sum_{n=1}^{\infty} A_n^2 \|X_n\|^2 = \|f\|^2$$

This is known as **Parseval's equality**.

**Definition:** The set of orthogonal functions $\{X_i(x)\}_{i=1}^{\infty}$ is called complete if Parseval's equality is true for any $f$ with $\|f\|^2 = \int_a^b |f(x)|^2 dx < \infty$.

**Theorem:** ($L^2$ convergence, without proof)

$\{X_i(x)\}_{i=1}^{\infty}$ coming from $\mathcal{L}_D, \mathcal{L}_N$ or $\mathcal{L}_P$ are complete.

Therefore Parseval's equality holds whenever $\|f\|^2 < \infty$.

**Theorem:** (uniform convergence)

The Fourier series $\sum_{n=1}^{N} A_n X_n(x)$ converges to $f(x)$ uniformly on $[a,b]$ provided that:

(i) $f(x), f'(x)$ exist and are continuous on $[a,b]$

(ii) $f(x)$ satisfies the BCs coming from $\mathcal{L}$.

**Proof:** We prove for the case of the full Fourier series on $(-l,l)$ with periodic BCs. To simplify further, take $l=\pi$.

Write:
$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}[A_n \cos(nx) + B_n \sin(nx)]$$
$$f'(x) = \frac{1}{2}\tilde{A}_0 + \sum_{n=1}^{\infty}[\tilde{A}_n \cos(nx) + \tilde{B}_n \sin(nx)]$$

$$\Longrightarrow \quad A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \overset{\text{int. by parts}}{=} \underbrace{\cancel{\frac{1}{n\pi} f(x)\sin(nx)\Big|_{x=-\pi}^{\pi}}}_{0} \underbrace{- \frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x)\sin(nx)\,dx}_{-\frac{1}{n}\tilde{B}_n}$$

$$\Longrightarrow \quad A_n = -\frac{1}{n}\tilde{B}_n$$

similarly we can find that $B_n = \frac{1}{n}\tilde{A}_n$

here the periodicity and continuity of $f, f'$ are used!

$$\sum_{n=1}^{\infty}\left(|A_n\cos(nx)|+|B_n\sin(nx)|\right) \leqslant \sum_{n=1}^{\infty}\left(|A_n|+|B_n|\right)$$

$$= \sum_{n=1}^{\infty}\frac{1}{n}\left(|\tilde{A}_n|+|\tilde{B}_n|\right)$$

Cauchy-Schwarz

$$\leqslant \left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{1/2}\left(\sum_{n=1}^{\infty}\left(|\tilde{A}_n|+|\tilde{B}_n|\right)^2\right)^{1/2}$$

$$\leqslant \underbrace{\phantom{xx}}_{\text{Const}} \cdot \underbrace{\left(\sum_{n=1}^{\infty}2\left(|\tilde{A}_n|^2+|\tilde{B}_n|^2\right)\right)^{1/2}}_{\text{this is finite by Parseval's inequality}}$$

$$\Rightarrow \sum_{n=1}^{\infty}\left(|A_n\cos(nx)|+|B_n\sin(nx)|\right) < \infty$$

$\Rightarrow$ The Fourier series of $f$ converges absolutely.

$$\Rightarrow \max_{-\pi\leq x\leq\pi}\left|f(x)-\frac{1}{2}A_0-\sum_{n=1}^{N}\left[A_n\cos(nx)+B_n\sin(nx)\right]\right| \leqslant$$

$$= \max_{-\pi\leq x\leq\pi}\left|\sum_{n=N+1}^{\infty}\left[A_n\cos(nx)+B_n\sin(nx)\right]\right|$$

$$\leqslant \max_{-\pi\leq x\leq\pi}\sum_{n=N+1}^{\infty}\left|A_n\cos(nx)+B_n\sin(nx)\right|$$

$$\leqslant \underbrace{\sum_{n=N+1}^{\infty}\left(|A_n|+|B_n|\right)}_{\substack{\text{This is the tail of a convergent series, so} \\ \text{it tends to 0 as } N\to+\infty.}} < \infty .$$

$\Rightarrow$ The Fourier series converges to $f(x)$ both absolutely and uniformly.

$\square$