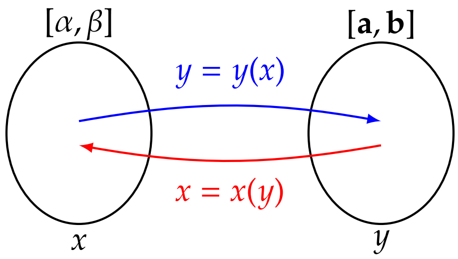
Integration by substitution (definite integrals)

Let $f(y)$ be continuous on an interval $[a,b]$ and let $y(x) : [\alpha,\beta] \to [a,b]$ belong to $C^1([\alpha,\beta])$. Then

$$\int_\alpha^\beta f(y(x))y'(x)\,\mathrm{d}x = \int_{y(\alpha)}^{y(\beta)} f(y)\,\mathrm{d}y.$$

If $y(x)$ is 1-1 then we also have

$$\int_a^b f(y)\,\mathrm{d}y = \int_{y^{-1}(a)}^{y^{-1}(b)} f(y(x))y'(x)\,\mathrm{d}x.$$



Remembering how to integrate by substitution

When asked to evaluate the definite integral $\int_\alpha^\beta g(x)\,\mathrm{d}x$ there are two options:

1. We are able identify that there exists $y = y(x)$ such that $g(x)$ has the form

   $$g(x) = f(y(x))y'(x).$$

   In this case, since $\frac{\mathrm{d}y}{\mathrm{d}x} = y'(x)$ we write $\mathrm{d}y = y'(x)\,\mathrm{d}x$ to get

   $$\int_\alpha^\beta g(x)\,\mathrm{d}x = \int_\alpha^\beta f(y(x))y'(x)\,\mathrm{d}x = \int_a^b f(y)\,\mathrm{d}y.$$

   This approach might work if g is a complicated function.

2. If we cannot identify $y = y(x)$ as above, we try to go about it the other way around: identify x as a function of y: $x = x(y)$, compute $\frac{\mathrm{d}x}{\mathrm{d}y} = x'(y)$ and write $\mathrm{d}x = x'(y)\,\mathrm{d}y$ to get:

   $$\int_\alpha^\beta g(x)\,\mathrm{d}x = \int_a^b g(x(y))x'(y)\,\mathrm{d}y.$$

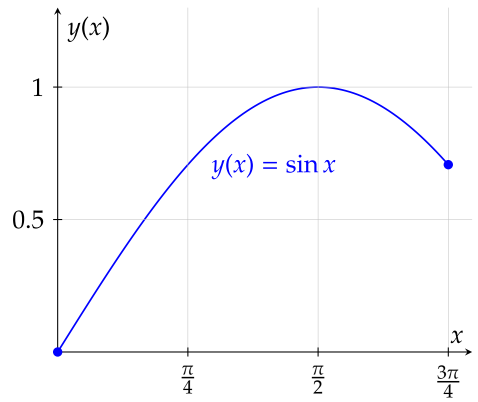
   This approach might work if g is a simple function. **In this second case, we need to be sure to check that $x = x(y)$ is invertible. This means that $x(y)$ must be strictly monotone on $[a,b]$.**

Example 10.13: Compute

$$\int_0^{\frac{3\pi}{4}} \sin^3 x \cos x \, \mathrm{d}x.$$

Define

$$y(x) = \sin x \qquad \text{so that} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \cos x \qquad \text{and} \qquad [0, \frac{3\pi}{4}] \to [0, \frac{\sqrt{2}}{2}]$$



and we have

$$\int_0^{\frac{3\pi}{4}} \sin^3 x \cos x \, \mathrm{d}x = \int_0^{\frac{\sqrt{2}}{2}} y^3 \, \mathrm{d}y = \left[\frac{1}{4}y^4\right]_0^{\frac{\sqrt{2}}{2}} = \frac{1}{16}.$$

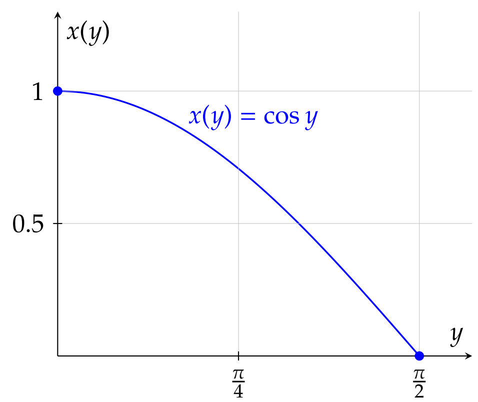
Here we observe that we used the substitution $y = y(x)$, but y is not 1-1, so it is not invertible, and we *wouldn't* have been able to use the second method.

Example 10.14: Compute

$$\int_0^1 \arcsin \sqrt{1 - x^2} \, \mathrm{d}x.$$

Define

$$x(y) = \cos y \qquad \text{so that} \qquad \frac{\mathrm{d}x}{\mathrm{d}y} = -\sin y \qquad \text{and} \qquad [0, 1] \to [\frac{\pi}{2}, 0]$$

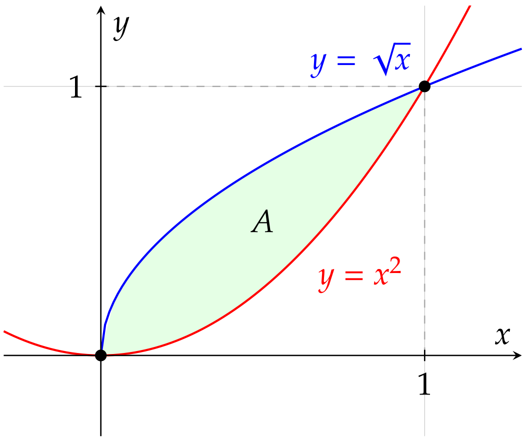


which is a *bijective* function and therefore *permitted*. So we have

$$\begin{aligned}
\int_0^1 \arcsin\sqrt{1-x^2}\,\mathrm{d}x &= \int_{\frac{\pi}{2}}^0 \arcsin\sqrt{1-\cos^2 y}\,(-\sin y)\,\mathrm{d}y \\
&= \int_0^{\frac{\pi}{2}} y\sin y\,\mathrm{d}y \\
\textit{(integration by parts)} \qquad &= [-y\cos y]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos y\,\mathrm{d}y \\
&= 0 + [\sin y]_0^{\frac{\pi}{2}} \\
&= 1.
\end{aligned}$$

Computation of areas

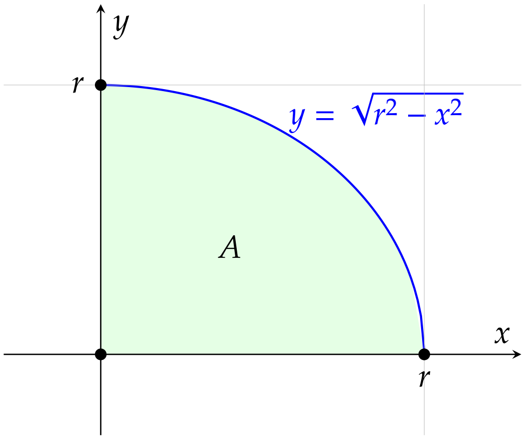
Example 10.15: Compute the area enclosed A between the graphs of $y = \sqrt{x}$ and $y = x^2$.



We identify that the intersection points are at $x = 0, 1$ and that the graph of $y = \sqrt{x}$ lies above the graph of $y = x^2$. We therefore have

$$\begin{aligned}
A &= \int_0^1 \sqrt{x}\,\mathrm{d}x - \int_0^1 x^2\,\mathrm{d}x \\
&= \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^1 - \left[\frac{1}{3}x^3\right]_0^1 \\
&= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.
\end{aligned}$$

Example 10.16: Prove that the area of a disc of radius r is $A(r) = \pi r^2$.



The area is four times the area of a quarter disc:

$$A(r) = 4\int_0^r \sqrt{r^2 - x^2}\,\mathrm{d}x$$

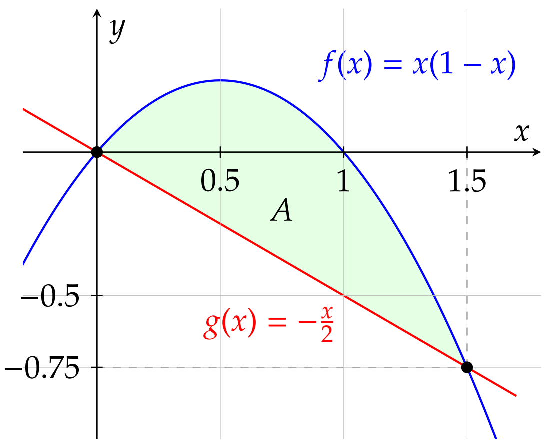
Make the (very simple, linear) change of variables:

$$x(z) = rz \qquad \text{so that} \qquad \frac{\mathrm{d}x}{\mathrm{d}z} = r \qquad \text{and} \qquad [0, r] \to [0, 1]$$

is the inverse of the invertible function $z(x) = \frac{x}{r}$ (so the change of variables is allowed) and we have

$$\begin{aligned} A &= 4\int_0^r \sqrt{r^2 - x^2}\,\mathrm{d}x \\ &= 4r^2 \int_0^1 \sqrt{1 - z^2}\,\mathrm{d}z \\ \textit{(see Example 10.5(6))} \qquad &= 4r^2\left[\frac{1}{2}z\sqrt{1 - z^2} + \frac{1}{2}\arcsin z\right]_0^1 \\ &= 4r^2\left[(0 - 0) + \left(\frac{\pi}{4} - 0\right)\right] \\ &= \pi r^2. \end{aligned}$$

Example 10.17: Compute the area enclosed between the graphs of $f(x) = x(1 - x)$ and $g(x) = -\frac{x}{2}$.



Even though some of the area lies below the x axis, it is still given by

$$A = \int_0^{\frac{3}{2}} \left[x(1-x) - \left(-\frac{x}{2}\right) \right] \mathrm{d}x$$

Can you think why? In the above expression we have already computed the intersection points to be $x = 0$ and $x = \frac{3}{2}$. So we compute:

$$\begin{aligned} A &= \int_0^{\frac{3}{2}} \left(-x^2 + \frac{3}{2}x\right) \mathrm{d}x \\ &= \left[-\frac{1}{3}x^3 + \frac{3}{4}x^2\right]_0^{\frac{3}{2}} \\ &= \frac{9}{16}. \end{aligned}$$

10.10 Differentiation of Integrals with Functional Limits

The standard form of the Fundamental Theorem of Integral Calculus provides the derivative of the integral function $F(x) = \int_{x_0}^{x} f(y)\,\mathrm{d}y$, where the upper limit is x and the lower limit x_0 is a constant.

We now extend this result to cases where the limits of integration are functions of x. This extension requires the application of the **Chain Rule**. Let $f : \mathbb{R} \to \mathbb{R}$ be a function continuous on its domain, and let $u(x)$ and $v(x)$ be differentiable functions. We want to find the derivative of the function $K(x)$ defined by:

$$K(x) = \int_{v(x)}^{u(x)} f(y)\,\mathrm{d}y.$$

Case 1: The Upper Limit is a Function of x

Consider $G(x)$ defined by:

$$G(x) = \int_{x_0}^{u(x)} f(y)\,\mathrm{d}y$$

where x_0 is a constant and $u(x)$ is differentiable.

We view $G(x)$ as a composite function $G(x) = F(u(x))$, where $F(u)$ is the integral function:

$$F(u) = \int_{x_0}^{u} f(y)\,\mathrm{d}y$$

By the Chain Rule, $G'(x) = F'(u(x)) \cdot u'(x)$. So we have:

Differentiation with Upper Functional Limit

The derivative is given by:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_{x_0}^{u(x)} f(y)\,\mathrm{d}y\right] = f(u(x)) \cdot u'(x)$$

Example 10.18: Find $\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_1^{\cos x} \frac{e^y}{y+2}\,\mathrm{d}y\right]$.

Here $f(y) = \frac{e^y}{y+2}$ and $u(x) = \cos x$, so $u'(x) = -\sin x$.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_1^{\cos x} \frac{e^y}{y+2}\,\mathrm{d}y\right] &= f(\cos x)\cdot\frac{\mathrm{d}}{\mathrm{d}x}(\cos x)\\ &= \frac{e^{\cos x}}{\cos x+2}\cdot(-\sin x)\\ &= -\frac{\sin x\cdot e^{\cos x}}{\cos x+2}\end{aligned}$$

Case 2: The Lower Limit is a Function of x

Consider $H(x)$ defined by:

$$H(x) = \int_{v(x)}^{x_0} f(y)\,\mathrm{d}y$$

where x_0 is a constant and $v(x)$ is differentiable. We use the integral property $\int_a^b f(y)\,\mathrm{d}y = -\int_b^a f(y)\,\mathrm{d}y$:

$$H(x) = -\int_{x_0}^{v(x)} f(y)\,\mathrm{d}y$$

Applying the result from Case 1 (and the linearity of the derivative):

Differentiation with Lower Functional Limit

The derivative is given by:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_{v(x)}^{x_0} f(y)\,\mathrm{d}y\right] = -f(v(x))\cdot v'(x)$$

Case 3: Both Limits are Functions of x

To find the derivative of $K(x) = \int_{v(x)}^{u(x)} f(y)\,\mathrm{d}y$, we split the integral at any constant c in the domain of f:

$$K(x) = \int_{v(x)}^{c} f(y)\,\mathrm{d}y + \int_{c}^{u(x)} f(y)\,\mathrm{d}y$$

Rewriting the first term using the reversal property:

$$K(x) = -\int_{c}^{v(x)} f(y)\,\mathrm{d}y + \int_{c}^{u(x)} f(y)\,\mathrm{d}y$$

Differentiating term-by-term using the results from Case 1 and Case 2:

$$\frac{\mathrm{d}}{\mathrm{d}x}K(x) = -\left(f(v(x))\cdot v'(x)\right) + \left(f(u(x))\cdot u'(x)\right)$$

General Case: The Leibniz Integral Rule

Let $u(x)$ and $v(x)$ be differentiable functions. The derivative is:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_{v(x)}^{u(x)} f(y)\,\mathrm{d}y\right] = f(u(x))u'(x) - f(v(x))v'(x)$$

Example 10.19: Find $\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_{\sqrt{x}}^{x^2} \sin(y^3)\,\mathrm{d}y\right]$ for $x > 0$.

Here, $f(y) = \sin(y^3)$, $u(x) = x^2$ (so that $u'(x) = 2x$), and $v(x) = \sqrt{x}$ (so that $v'(x) = \frac{1}{2\sqrt{x}}$).

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}K(x) &= f(u(x))u'(x) - f(v(x))v'(x) \\
&= \sin((x^2)^3)\cdot(2x) - \sin((\sqrt{x})^3)\cdot\left(\frac{1}{2\sqrt{x}}\right) \\
&= 2x\sin(x^6) - \frac{1}{2\sqrt{x}}\sin(x^{3/2})
\end{aligned}$$

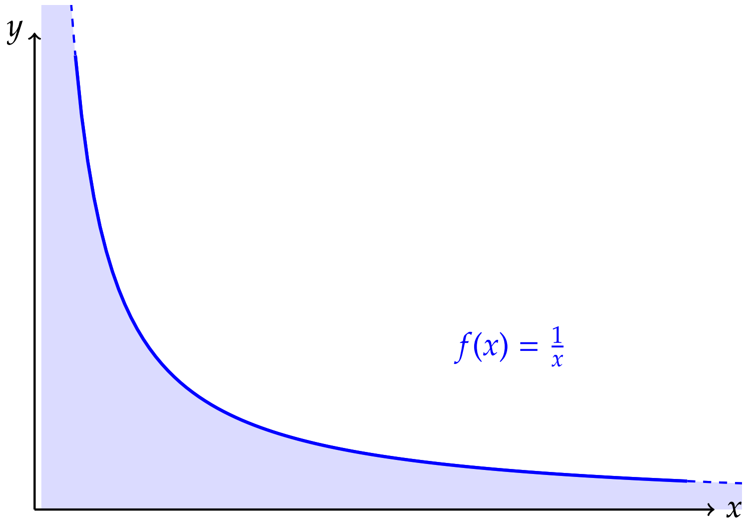
Chapter 11

Improper integrals and numerical series

In this chapter we will extend the notion of a definite integral $\int_a^b f(x)\,\mathrm{d}x$ to cases where either

- one (or both) of the endpoints is at $\pm\infty$,
- $f(x)$ has a vertical asymptote at a and/or b.

These are called **improper integrals**.



This will allow us to ask what is the area under the graph of functions such as $f(x) = \frac{1}{x}$ between 0 and $+\infty$:

$$\int_0^{+\infty} \frac{1}{x}\,\mathrm{d}x = ?$$

A topic directly related to this is that of (infinite) **numerical series**: we could ask (very much related to the previous question):

$$\sum_{n=1}^{\infty} \frac{1}{n} = ?$$

11.1 Improper integrals

11.1.1 Type I improper integrals

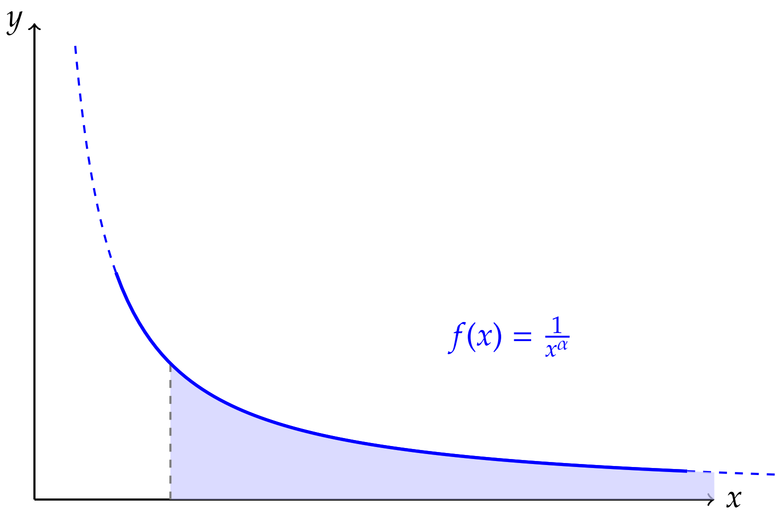
We start with improper integrals that have a limit at infinity:

Improper integral (type I)

Let $f : \mathbb{R} \to \mathbb{R}$ be a function integrable on $[a, b]$ for any $b > a$. We define the **improper integral** (type I) of f on $[a, +\infty)$ to be

$$\int_a^{+\infty} f(x)\,\mathrm{d}x = \lim_{b\to+\infty} \int_a^b f(x)\,\mathrm{d}x.$$

It will *converge*, *diverge* or be *indeterminate* depending on the limit on the right hand side.



Example 11.1: Investigate the convergence of the integral

$$\int_1^{+\infty} \frac{1}{x^\alpha}\,\mathrm{d}x,$$

where $\alpha > 0$. We use $a = 1$ to avoid issues at $x = 0$.

We must analyze two main cases: $\alpha = 1$ and $\alpha \neq 1$.

Case 1: $\alpha = 1$**.** In this case, the integrand is $f(x) = \frac{1}{x}$. We compute the definite integral first:

$$\int_1^b \frac{1}{x}\,\mathrm{d}x = [\ln|x|]_1^b = \ln|b| - \ln|1| = \ln b$$

Now we take the limit as $b \to +\infty$:

$$\int_1^{+\infty} \frac{1}{x}\,\mathrm{d}x = \lim_{b\to+\infty} (\ln b) = +\infty.$$

Therefore, the integral *diverges* for $\alpha = 1$.

Case 2: $\alpha \neq 1$**.** We have:

$$\int_1^b \frac{1}{x^\alpha}\,\mathrm{d}x = \left[\frac{1}{1-\alpha}x^{1-\alpha}\right]_1^b = \frac{1}{1-\alpha}\left(b^{1-\alpha} - 1\right)$$

Now we analyze the limit $\lim_{b\to+\infty} b^{1-\alpha}$ based on the sign of the exponent $1-\alpha$:

- If $\mathbf{1} - \alpha < \mathbf{0}$ (i.e., $\alpha > \mathbf{1}$): Since $1-\alpha$ is negative, we rewrite $b^{1-\alpha}$ by moving it to the denominator with a positive exponent:

  $$b^{1-\alpha} = \frac{1}{b^{-(1-\alpha)}} = \frac{1}{b^{\alpha-1}}$$

  Since $\alpha - 1 > 0$, as $\lim_{b\to+\infty} \frac{1}{b^{\alpha-1}} = 0$. The limit is:

  $$\int_1^{+\infty} \frac{1}{x^\alpha}\,\mathrm{d}x = \lim_{b\to+\infty} \frac{1}{1-\alpha}\left(\frac{1}{b^{\alpha-1}} - 1\right) = \frac{1}{1-\alpha}(0-1) = \frac{-1}{1-\alpha} = \frac{1}{\alpha-1}.$$

  The integral *converges* to $\frac{1}{\alpha-1}$ when $\alpha > \mathbf{1}$.

- If $\mathbf{1} - \alpha > \mathbf{0}$ (i.e., $\alpha < \mathbf{1}$): Since $1-\alpha$ is positive, as $\lim_{b\to+\infty} b^{1-\alpha} = +\infty$. The limit is:

  $$\int_1^{+\infty} \frac{1}{x^\alpha}\,\mathrm{d}x = \lim_{b\to+\infty} \frac{1}{1-\alpha}(b^{1-\alpha} - 1) = +\infty \quad (\text{since } 1-\alpha \neq 0).$$

  The integral *diverges* for $\alpha < \mathbf{1}$.

The choice of lower bound 1 was convenient but arbitrary. We could choose any $a > 0$. We therefore conclude:

$$\int_a^{+\infty} \frac{1}{x^\alpha}\,\mathrm{d}x \qquad \begin{cases} \text{converges} & \text{if } \alpha > 1, \\ \text{diverges} & \text{if } 0 < \alpha \leq 1. \end{cases}$$

Convergence tests

Now we give some criteria for the convergence/divergence of improper integrals of type I.