Lemma 7.6: Let $I \subseteq \mathbb{R}$ be a subset of $\mathbb{R}$ that satisfies the following condition: for any $x_1, x_2 \in I$ with $x_1 < x_2$, the entire closed interval $[x_1, x_2]$ is a subset of I. Then I is an interval, i.e. I could be an open, closed, or half-open-half-closed interval, and one or both of its endpoints can be infinite.

Proof. This is a simple proof (by contradiction) which we skip here. $\square$

Example 7.4: Let $f(x) = \tan x$. Then $f((-\frac{\pi}{2}, \frac{\pi}{2})) = (-\infty, +\infty)$.

Example 7.5: Let $f(x) = \cos x$. Then $f((-\infty, +\infty)) = [-1, 1]$.

Example 7.6: Let $f(x) = \arctan x$. Then $f((-\infty, +\infty)) = (-\frac{\pi}{2}, \frac{\pi}{2})$.

Example 7.7: Let $f(x) = e^x$. Then $f([0, +\infty)) = [1, +\infty)$.

The following theorem tells us that the image of a closed interval under a continuous function is always a closed interval:

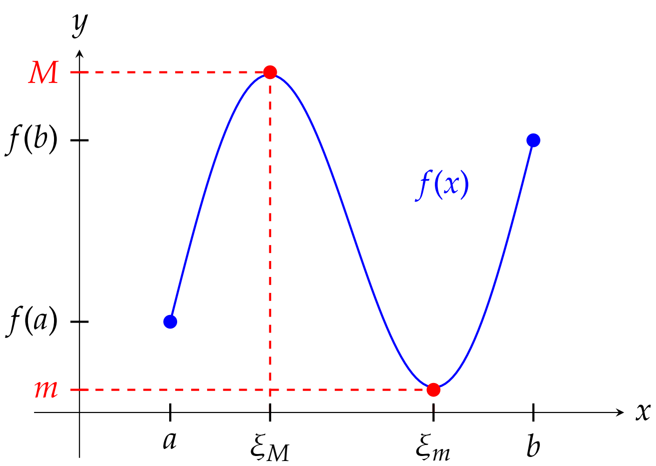
Weierstrass' Theorem

Let f be continuous on the interval $[a, b]$, where $a < b$ are real numbers. Then f is bounded on $[a, b]$ and it attains its minimum and maximum on $[a, b]$:

$$m = \min_{x \in [a,b]} f(x) \qquad \text{and} \qquad M = \max_{x \in [a,b]} f(x).$$

It follows that

$$f([a, b]) = [m, M].$$



Proof. **Step A.** The supremum. Define

$$M = \sup_{x \in [a,b]} f(x)$$

which can be a finite number or $+\infty$.

Case 1: M is finite. In this case, we know by definition of the supremum that for any $\varepsilon > 0$ there exists $x_\varepsilon \in [a, b]$ satisfying $M - \varepsilon < f(x_\varepsilon) \leq M$. Our goal is to construct a sequence $\{a_n\}_{n\in N_+}$, as follows: instead of ε take, for each $n \geq 1$, $\frac{1}{n}$, so that there exists $x_n \in [a, b]$ such that

$$M - \frac{1}{n} < f(x_n) \leq M.$$

By the Squeeze Theorem for sequences (Theorem 5.15(5)) if follows that

$$\lim_{n\to\infty} f(x_n) = M.$$

Case 2: M **is infinite.** In this case, again by the definition of the supremum, for each $n \in \mathbb{N}_+$ there exists $x_n \in [a, b]$ such that

$$f(x_n) > n.$$

By the Squeeze to $\pm\infty$ Theorem for sequences (Theorem 5.15(4)) we must have

$$\lim_{n\to\infty} f(x_n) = +\infty = M.$$

Step B. The sequence $\{x_n\}_{n\in\mathbb{N}_+}$. In both cases, we obtained a sequence $\{x_n\}_{n\in\mathbb{N}_+} \subset [a, b]$. This is a bounded sequence (it is contained in a bounded interval). By the Bolzano-Weierstrass Theorem (see below) it has a convergent subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$. Call its limit $\xi_M \in [a, b]$. Then we have:

$$\lim_{k\to\infty} x_{n_k} = \xi_M.$$

Moreover, since $f(x_n)$ converges, so does its subsequence $f(x_{n_k})$ and they share the same limit:

$$M = \lim_{n\to\infty} f(x_n) = \lim_{k\to\infty} f(x_{n_k}).$$

Using the continuity of f at the point ξ_M, we have

$$M = \lim_{k\to\infty} f(x_{n_k}) = f(\lim_{k\to\infty} x_{n_k}) = f(\xi_M).$$

But if $M = f(\xi_M)$ it must be a real number, so M cannot be $+\infty$. Furthermore, since M is attained at ξ_M, it belongs to the range f on $[a, b]$, hence

$$M = \max_{x\in[a,b]} f(x).$$

The proof for m follows the same ideas.

Finally, the fact that $f([a, b]) = [m, M]$ is an immediate consequence of Corollary 7.5. □

The Bolzano-Weierstrass Theorem

Let $\{x_n\}_{n\in\mathbb{N}}$ be a bounded sequence (i.e., there exist real numbers $a < b$ such that $a < x_n < b$ for all $n \in \mathbb{N}$). Then $\{x_n\}_{n\in\mathbb{N}}$ has a convergent subsequence: there exists a sequence of indices $\{n_k\}_{k\in\mathbb{N}} \subseteq \mathbb{N}$ such that the subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ converges.

Proof. We do not give the full proof here, just a sketch. The proof follows the same bisection method that we've seen in the proof of Theorem 7.1:

There exist infinitely many points of the sequence in $[a, b]$. Split $[a, b]$ into two halves, left and right. At least one of them will contain infinitely many points of the sequence. Pick n_1 such that x_{n_1} belongs to that half. Divide that half into two. At least one of those two halves contains infinitely many points of the sequence. Pick n_2 such that x_{n_2} belongs to that half. If we keep dividing the interval and picking points, we obtain a subsequence

$$x_{n_1}, x_{n_2}, x_{n_3}, x_{n_4}, \ldots, x_{n_k}, \ldots$$

where the indices $\{n_k\}_{n\in\mathbb{N}_+} \subseteq \mathbb{N}_+$ form an infinite subset of the integers. By construction, this subsequence will have a limit. □

Example 7.8: The sequence $a_n = (-1)^n$, $n \in \mathbb{N}$, has values

$$1, -1, 1, -1, 1, -1, \ldots, (-1)^n, (-1)^{n+1}, \ldots$$

This sequence is indeterminate. However, if we choose only the even indices we find that a_{n_k} (where $n_k = 2k$) is the subsequence $a_0, a_2, a_4, \cdots = 1, 1, 1, \ldots$ does have the trivial limit of 1. Similarly, the odd indices ($n_k = 2k + 1$) will give us a subsequence with the limit -1.

Remark: The above example demonstrates that a bounded sequence can have more than one convergent subsequence. The Bolzano-Weierstrass theorem tells us that there's *at least* one.

7.3 Invertibility of continuous functions

Lemma 7.7: Let f be continuous and invertible on an interval I. Let $x_1 < x_2 < x_3$ be points in I. Then exactly one of the following holds:

$$(i) \quad f(x_1) < f(x_2) < f(x_3)$$

or

$$(ii) \quad f(x_1) > f(x_2) > f(x_3).$$

Proof. Since f is invertible, it is 1-1. Therefore, since $x_1 \neq x_3$, also $f(x_1) \neq f(x_3)$.

Consider the case $f(x_1) < f(x_3)$ and assume *by contradiction* that (i) isn't satisfied. So $f(x_2) \notin (f(x_1), f(x_3))$.

Suppose that $f(x_2)$ lies to the right of the interval $(f(x_1), f(x_3))$, so that $f(x_1) < f(x_3) < f(x_2)$. Consider f on $[x_1, x_2]$. Since it is continuous there, by Theorem 7.4 (The Intermediate Value Theorem), $f|_{[x_1,x_2]}$ must assume all values in $[f(x_1), f(x_2)]$. Since $f(x_1) < f(x_3) < f(x_2)$, there will be $z \in [x_1, x_2]$ such that $f(z) = f(x_3)$. But this contradicts the fact that f is 1-1. Hence (i) is satisfied and $f(x_1) < f(x_2) < f(x_3)$.

The case that $f(x_2)$ lies to the left of the interval $(f(x_1), f(x_3))$ is handled in similar way.

The proof for the case $f(x_1) > f(x_3)$ follows the same ideas, and the lemma follows. □

Theorem 7.8: Let f be continuous on an interval $I \subseteq \mathbb{R}$. Then

$$f \text{ is 1-1 on } I \qquad \Leftrightarrow \qquad f \text{ is strictly monotone on } I.$$

Proof. The direction $\Leftarrow$ was already proven in Proposition 2.1.

Hence we only need to prove the implication $\Rightarrow$. Recall that a 1-1 function is invertible, so that it is enough to prove

$$f \text{ is invertible on } I \qquad \Rightarrow \qquad f \text{ is strictly monotone on } I.$$

Let $x_1 < x_2$ be points in I. We'll show that if $f(x_1) < f(x_2)$ then f is strictly increasing on I. Let z_1, z_2 be two points such that

$$x_1 < z_1 < z_2 < x_2.$$

We want to show that $f(z_1) < f(z_2)$.

Using Lemma 7.7 for the trio $x_1 < z_1 < x_2$, we find that

$$f(x_1) < f(z_1) < f(x_2).$$

Using Lemma 7.7 for the trio $z_1 < z_2 < x_2$, we find that

$$f(z_1) < f(z_2) < f(x_2).$$

In particular, we have found that $z_1 < z_2$ implies $f(z_1) < f(z_2)$, so f is strictly increasing.

The strictly decreasing case ($f(x_1) > f(x_2)$) follows the same idea of proof, which completes the proof. □

Theorem 7.9: Let f be continuous and invertible on an interval I. Then f^{-1} is continuous on the interval $J = f(I)$.

Proof. We skip the proof. □

7.4 Lipschitz and uniformly continuous functions

Lipschitz functions

A function $f : \mathbb{R} \to \mathbb{R}$ is **Lipschitz on an interval** $I \subseteq \mathbb{R}$ if there exists a constant $L \geq 0$ such that

$$|f(x_1) - f(x_2)| \leq L|x_1 - x_2| \qquad \forall x_1, x_2 \in I.$$

For a Lipschitz function on I, the smallest L that satisfies this inequality is called the **Lipschitz constant** of f on I.

Example 7.9: The function $f(x) = x$ is Lipschitz on $\mathbb{R}$ with Lipschitz constant 1: for any $x_1, x_2 \in \mathbb{R}$,

$$|f(x_1) - f(x_2)| = |x_1 - x_2|.$$