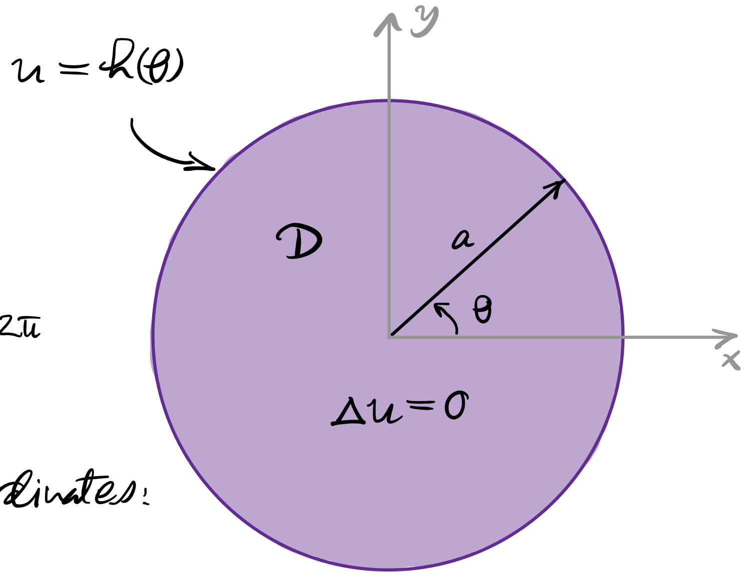
6.3 Poisson's Formula

We consider the problem

$$\begin{cases} \Delta u = 0 & r<a \\ u = h(\theta) & r=a,\ 0<\theta<2\pi \end{cases}$$



We separate variables in polar coordinates:

$$u(r,\theta) = R(r)\,\Theta(\theta).$$

The formula for Δ in polar coordinates is (we have seen this)

$$\Delta = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 .$$

Hence we get:

$$\begin{aligned} 0 = \Delta u &= u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} \\ &= R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta''. \end{aligned}$$

Multiply by r^2
Divide by $R\Theta$ $\Longrightarrow$ $r^2\dfrac{R''}{R} + r\dfrac{R'}{R} + \dfrac{\Theta''}{\Theta} = 0$

$$\Longrightarrow \quad r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$

So we find the two equations:

$$\Theta''(\theta) + \lambda\,\Theta(\theta) = 0$$
$$r^2R''(r) + rR'(r) - \lambda R(r) = 0$$

The $\Theta(\theta)$ equation: It is natural to impose periodic boundary conditions, so we have:

$$\begin{cases} \Theta''(\theta) + \lambda\,\Theta(\theta) = 0 \\ \Theta(\theta) = \Theta(\theta + 2\pi) \end{cases}$$

From our abstract theorems we know that all eigenvalues are real and non-negative. Verify that:

0 eigenvalue comes with $\Theta_0(\theta) = \text{const}$

$\lambda_n = n^2$ $\Theta_n(\theta) = A\cos(n\theta) + B\sin(n\theta)$.

The $R(r)$ equation: try $R(r) = r^\alpha$ to get:

$$\alpha(\alpha-1)r^\alpha + \alpha r^\alpha - n^2 r^\alpha = 0$$

$$\longrightarrow \quad (\alpha^2 - n^2)\, r^\alpha = 0$$

$$\longrightarrow \quad \alpha_n = \pm n$$

$$\longrightarrow \quad R_n(r) = C r^n + D r^{-n}$$

and for $\lambda = 0$ $\quad R_0(r) = C + D\ln r \quad$ (Check this)

Boundary condition at $r=0$: we can't allow functions that are unbounded (r^{-n}, $\ln r$) so we set their coefficients to 0.

$$\longrightarrow \quad \boxed{u(r,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^n \left(A_n\cos(n\theta) + B_n\sin(n\theta)\right)}$$

Inhomogeneous boundary condition:

$$h(\theta) = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty} a^n \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right)$$

$$\longrightarrow \quad \boxed{\begin{aligned} A_n &= \frac{1}{\pi a^n}\int_0^{2\pi} h(\phi)\cos(n\phi)\,d\phi \\ B_n &= \frac{1}{\pi a^n}\int_0^{2\pi} h(\phi)\sin(n\phi)\,d\phi \end{aligned}}$$

Plug these into the eq for u to get:

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} h(\phi)\,d\phi + \sum_{n=1}^{\infty}\frac{r^n}{\pi a^n}\int_0^{2\pi} h(\phi)\left[\cos(n\theta)\cos(n\phi) + \sin(n\theta)\sin(n\phi)\right]d\phi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} h(\phi)\Big[\underbrace{1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \underbrace{\cos(\theta-\phi)}_{\frac{e^{in(\theta-\phi)} - e^{-in(\theta-\phi)}}{2}}}\Big]$$

$$1 + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{in(\theta-\phi)} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{-in(\theta-\phi)}$$

$$= 1 + \frac{re^{i(\theta-\phi)}}{a-re^{i(\theta-\phi)}} + \frac{re^{-i(\theta-\phi)}}{a-re^{i(\theta-\phi)}}$$

geometric series

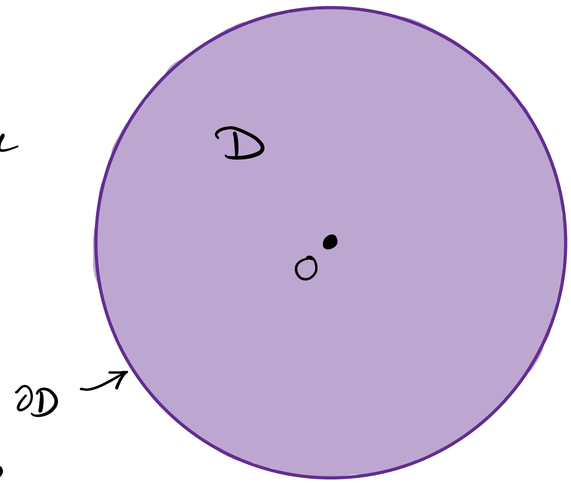
$$= \frac{a^2-r^2}{a^2-2ar\cos(\theta-\phi)+r^2}$$

$$\Longrightarrow \quad \boxed{u(r,\theta) = \frac{a^2-r^2}{2\pi}\int_0^{2\pi}\frac{h(\phi)}{a^2-2ar\cos(\theta-\phi)+r^2}\,d\phi}$$

This is called Poisson's formula. Remarkably, it gives a complete characterization of the surface $u(x,y)$ using only the knowledge of the values of h, i.e. the values of u along the boundary of the disk.

Theorem: (Mean Value Property)

Let u be a harmonic function in a disk D and continuous on $\overline{D} = D \cup \partial D$. Then the value of u at the center of D equals the average of u on its circumference ∂D.



Proof: Without loss of generality, assume that the center of D is at $(x, y) = (0,0)$. From Poisson's formula we know that

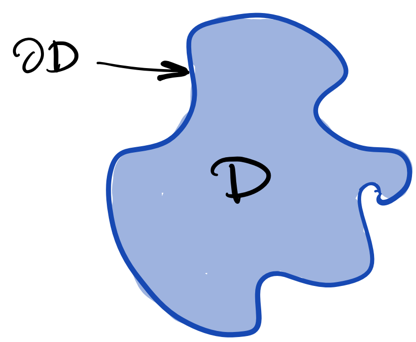
$$u(r=0) = \frac{a^2}{2\pi} \int_0^{2\pi} \frac{u(\phi)}{a^2}\, d\phi = \frac{1}{2\pi} \int_0^{2\pi} u(\phi)\, d\phi$$

which is, by definition, the average of u on ∂D. □

Theorem: (Strong Maximum Principle)

Let D be a connected and bounded open set in $\mathbb{R}^2$. Let $u(x,y)$ be harmonic in D and continuous in $\overline{D} = D \cup \partial D$. Then the max and min of u are attained on ∂D and nowhere inside D (unless u is a constant function).

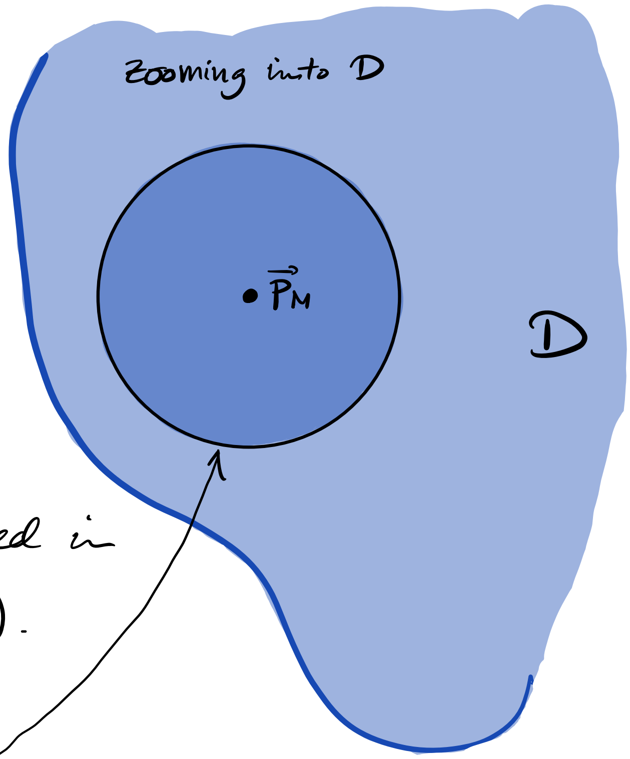
Proof: We've seen (in Section 6.1) a proof of the weak version. Now we can prove the strong version.



Suppose there's a point $\vec{p}_M$ in D where the max of u, call it M, is achieved.

That is: $\quad u(\vec{p}) \leq u(\vec{p}_M) = M$

for any $\vec{p} \in D$.



Draw a circle around $\vec{p}_M$ that is contained in D (we can do this because D is open).

Now we use the mean value property:

$$M = u(\vec{p}_M) = \text{average on circle}$$

However, the average cannot be more than the max M. So we have $\quad M = u(\vec{p}_M) = \text{average on circle} \leq M$

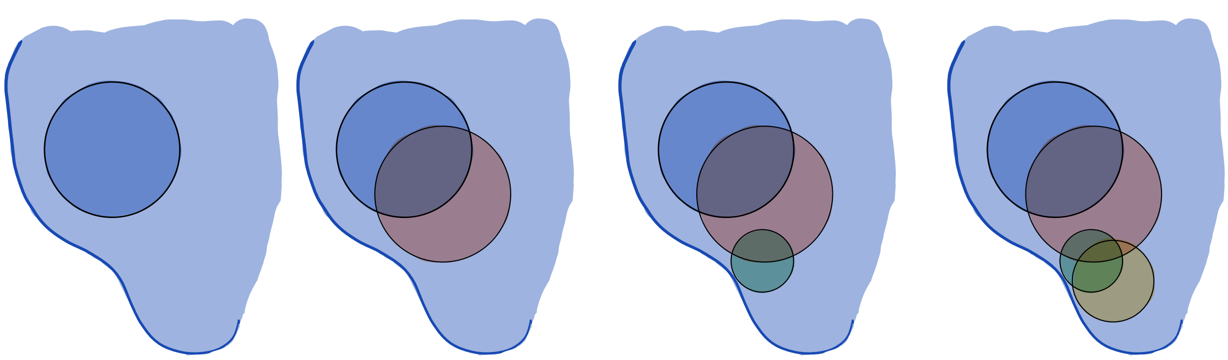
$$\longrightarrow \quad \text{average on the circle} = M$$

If there are points on the circle where $u < M$, there must be other points where $u > M$. But this would contradict M being the max.

$$\longrightarrow \quad u = M \text{ on the entire circle}$$

But we could have chosen the circle to be of any radius (so long as it is $\subset D$). So $u = M$ on the entire blue shaded disk. Now we can repeat the argument starting from any other point in the blue disk to get the red disk. Eventually, we can reach every point in D.

(here we use the fact that D is bounded and connected)



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dot!

Conclusion: $u \equiv M$ everywhere in D.

I.e.: if u attains its max inside of D, u must be constant. Otherwise, the max can only be attained on ∂D.

Similarly for the min. $\square$