MATHEMATICAL ANALYSIS 1
HOMEWORK 12

(1) Recall the *Fundamental Theorem of Integral Calculus*:
$F_{x_0}(x) = \int_{x_0}^{x} f(y)dy \qquad \Rightarrow \qquad F'_{x_0}(x) = f(x).$
(a) Prove the following:
Corollary. Let $G(x)$ be any antiderivative of $f(x)$. Then $F_{x_0}(x) = G(x) - G(x_0)$.
(b) Prove the following:
Corollary. Let f be continuous on $[a, b]$ and let G be any antiderivative of f. Then

$$\int_a^b f(x)dx = G(b) - G(a). \tag{$\star$}$$

For the following problems we shall use the formula ($\star$):

(2) **Calculation of Definite Integrals (I).** Compute the following integrals.

(a) $\int_0^2 (3x^2 - 4x + 1)\, dx$.
(b) $\int_1^4 \left(\sqrt{x} + \frac{1}{x^2}\right)\, dx$.
(c) $\int_0^{\pi/4} \sec^2(x)\, dx$.
(d) $\int_0^{\ln 3} 2e^x\, dx$.
(e) $\int_1^e \frac{1}{x}\, dx$.
(f) $\int_0^1 \frac{1}{1+x^2}\, dx$.
(g) $\int_0^\pi \sin\left(\frac{x}{2}\right)\, dx$.
(h) $\int_0^1 (4x - e^{2x})\, dx$.

(3) **Calculation of Definite Integrals (II).** Compute the following integrals.

(a) $\int_0^1 \frac{x^3+x^2+1}{x+1}\, dx$.
(b) $\int_0^{\pi/4} \frac{1+\sin^2 x}{\cos^2 x}\, dx$.
(c) $\int_{1/e}^e \frac{x^2+x^3+x}{x^4}\, dx$.
(d) $\int_2^4 \frac{1}{\sqrt{x^2-1}}\, dx$.
(e) $\int_0^1 \frac{1}{\sqrt{4-x^2}}\, dx$.
(f) $\int_{-1}^1 (e^{2x} - e^{-2x})\, dx$.
(g) $\int_{-\pi/4}^{\pi/4} (3\sin x + 2\tan x)\, dx$.
(h) $\int_0^1 \frac{1}{x^2+4}\, dx$.

(4) **Mean value of a function.**
(i) Compute the average $m(f; a, b)$ of $f(x)$ over the given interval $[a, b]$.
(ii) Write the equation for a point $z \in [a, b]$ such that $f(z) = m(f; a, b)$ (if there is such a z). If possible, write z explicitly.

(a) $f(x) = 2x + 1$ on $[0, 4]$.
(b) $f(x) = 3x^2 - 4x$ on $[1, 3]$.
(c) $f(x) = (x - 2)^2$ on $[0, 5]$.
(d) $f(x) = e^{2x}$ on $[0, \ln 3]$.
(e) $f(x) = \frac{1}{x}$ on $[1, e^2]$.
(f) $f(x) = \sin(x)$ on $[0, \pi]$.
(g) $f(x) = \sec^2(x)$ on $[0, \frac{\pi}{4}]$.
(h) $f(x) = \sqrt{x+1}$ on $[3, 8]$.
(i) $f(x) = \frac{x^2-1}{x^4}$ on $[1, 2]$.
(j) $f(x) = x \cdot e^{-x^2}$ on $[0, 1]$.

(5) Compute the following integral:

$$\int_{\frac{2}{41\pi}}^{\frac{2}{\pi}} \left(-\frac{1}{x^2}\cos\left(\frac{1}{x}\right)\right) dx.$$

(6) Compute the following integral:

$$\int_{-\pi}^{\pi} \frac{\sin x + x^3}{2 + \cos x + e^{x^2}} dx.$$

(7) (a) Compute the integral $\int_0^1 \frac{1}{(x+1)^2} dx$
(b) Use the previous result to prove that

$$\lim_{n\to\infty} n \left(\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(2n)^2}\right) = \frac{1}{2}$$