Remark: Recalling the definition of the Taylor polynomial, and using Lagrange's formula for the remainder, we have

$$f(x)=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(x_0)(x-x_0)^k+\frac{1}{(n+1)!}f^{(n+1)}(\overline{x})(x-x_0)^{n+1}.$$

So we see that the function f is nearly equal to the Taylor polynomial, and that the remainder term has a form that is very similar to the $n+1$ terms of the polynomial. The only (important) difference, is that the derivative term within the remainder is evaluated at a different point $\overline{x}$ which lies somewhere between x_0 and x (and we do not know where exactly).

Proposition 9.4: Any Maclaurin polynomial of an even function contains only even powers. Any Maclaurin polynomial of an odd function contains only odd powers.

Proof. This is a simple proof that is left as an exercise. □

9.2 Expanding the elementary functions

We now derive the Taylor expansions of some of the most commonly used functions. Recall that a function f, the Taylor polynomial of order n at x_0 is given by

$$(Tf)_{n,x_0}(x)=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(x_0)(x-x_0)^k.$$

Hence, to construct this polynomial for a given function one needs to know the first n derivatives at a given point x_0.

The exponential function

For the function $f(x)=e^x$ we know that $f^{(k)}(x)=e^x$ for all $k\in\mathbb{N}$ and for all $x\in\mathbb{R}$. Plugging this into the expression for the Taylor polynomial, we find that

$$(Tf)_{n,x_0}(x)=\sum_{k=0}^{n}\frac{1}{k!}e^{x_0}(x-x_0)^k.$$

For simplicity, we choose $x_0=0$, so that $e^{x_0}=1$ and we're left with the *Maclaurin expansion*

$$(Tf)_{n,0}(x)=\sum_{k=0}^{n}\frac{x^k}{k!}$$

Hence, with **Peano**'s approximation of the remainder we have that

$$e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + o(x^n), \qquad x \to 0.$$

Lagrange's formula for the remainder gives us

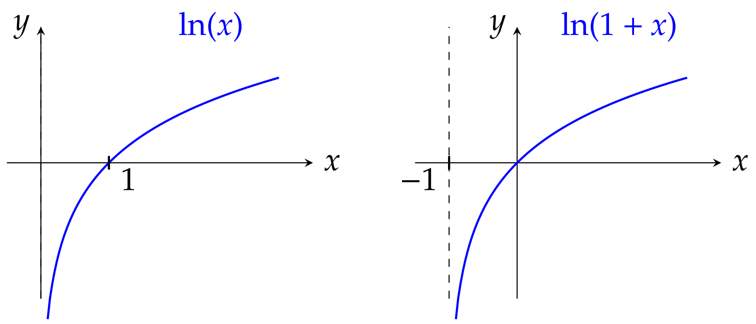
$$e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + \frac{e^{\bar{x}}}{(n+1)!} x^{n+1}$$

where $\bar{x}$ is some point between 0 and x.

Observe that by taking $x = 1$ we obtain an approximation of the number e. In fact, this approximation of e converges extremely fast. We can bound the error by bounding the error term in Lagrange's formulation.

The logarithm

We now want to write an expansion of the natural logarithm $\ln x$. It is only defined for $x > 0$, so the point x_0 where we perform the expansion must be greater than 0. As we shall see, it will be convenient to choose $x_0 = 1$. However, it will be even more convenient to translate the function and consider $\ln(1 + x)$, so that the vertical asymptote will be at -1; then we will choose to expand around 0.



Let's start with $f(x) = \ln x$. We can write the first few derivatives:

$$f'(x) = \frac{1}{x} \qquad f''(x) = -\frac{1}{x^2} \qquad f'''(x) = \frac{2}{x^3} \qquad f^{(4)}(x) = -\frac{3 \cdot 2}{x^4} \qquad \cdots$$

and the general formula is:

$$f^{(k)}(x) = (-1)^{k+1} \frac{(k-1)!}{x^k}, \qquad \forall k \in \mathbb{N}_+.$$

We therefore find that

$$
\begin{aligned}
(Tf)_{n,x_0}(x) &= \ln x_0 + \sum_{k=1}^{n} \frac{1}{k!}(-1)^{k-1}\frac{(k-1)!}{x_0^k}(x-x_0)^k \\
&= \ln x_0 + \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}\frac{(x-x_0)^k}{x_0^k}
\end{aligned}
$$

For simplicity, we take $x_0 = 1$, so that $\ln x_0 = 0$, and obtain

$$
(Tf)_{n,1}(x) = \sum_{k=1}(-1)^{k-1}\frac{(x-1)^k}{k}
$$

$$
\begin{aligned}
\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots + (-1)^{n-1}\frac{(x-1)^n}{n} + o((x-1)^n) \\
= \sum_{k=1}^{n}(-1)^{k-1}\frac{(x-1)^k}{k} + o((x-1)^n), \qquad x \to 0.
\end{aligned}
$$

Now let's try $g(x) = \ln(1+x)$. Then

$$
g'(x) = \frac{1}{1+x} \qquad g''(x) = -\frac{1}{(1+x)^2} \qquad g'''(x) = \frac{2}{(1+x)^3} \qquad \cdots
$$

and the general formula is:

$$
g^{(k)}(x) = (-1)^{k+1}\frac{(k-1)!}{(1+x)^k} \qquad \forall k \in \mathbb{N}_+.
$$

We therefore find that

$$
\begin{aligned}
(Tg)_{n,x_0}(x) &= \ln(1+x_0) + \sum_{k=1}^{n} \frac{1}{k!}(-1)^{k-1}\frac{(k-1)!}{(1+x_0)^k}(x-x_0)^k \\
&= \ln(1+x_0) + \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}\frac{(x-x_0)^k}{(1+x_0)^k}
\end{aligned}
$$

Taking $x_0 = 0$, so that $g(x_0) = 0$, we obtain

$$
(Tg)_{n,0}(x) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}x^k.
$$

Hence we find that the nth degree Taylor polynomial for $\ln(1+x)$ around the point $x_0 = 0$ (hence it is also a Maclaurin polynomial) is:

$$
x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - + \cdots
$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots + (-1)^{n-1}\frac{x^n}{n} + o(x^n)$$
$$= \sum_{k=1}^{n}(-1)^{k-1}\frac{x^k}{k} + o(x^n), \qquad x \to 0.$$

The trigonometric functions $\sin x$ and $\cos x$

We know that $\sin x$ is an odd function, so by Proposition 9.4 we expect its Maclaurin polynomial to only have odd powers. Taking the first few derivatives, we have

$$\sin' x = \cos x \quad \sin'' x = -\sin x \quad \sin''' x = -\cos x \quad \sin^{(4)} x = \sin x \quad \cdots$$

and at $x_0 = 0$ we get

$$\sin' 0 = 1 \quad \sin'' 0 = 0 \quad \sin''' 0 = -1 \quad \sin^{(4)} 0 = 0 \quad \cdots$$

With **Peano**'s approximation of the remainder we have that with $n = 2m+2$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} - \cdots + (-1)^m\frac{x^{2m+1}}{(2m+1)!} + o(x^{2m+2})$$
$$= \sum_{k=1}^{m}(-1)^k\frac{x^{2k+1}}{(2k+1)!} + o(x^{2m+2}), \qquad x \to 0.$$

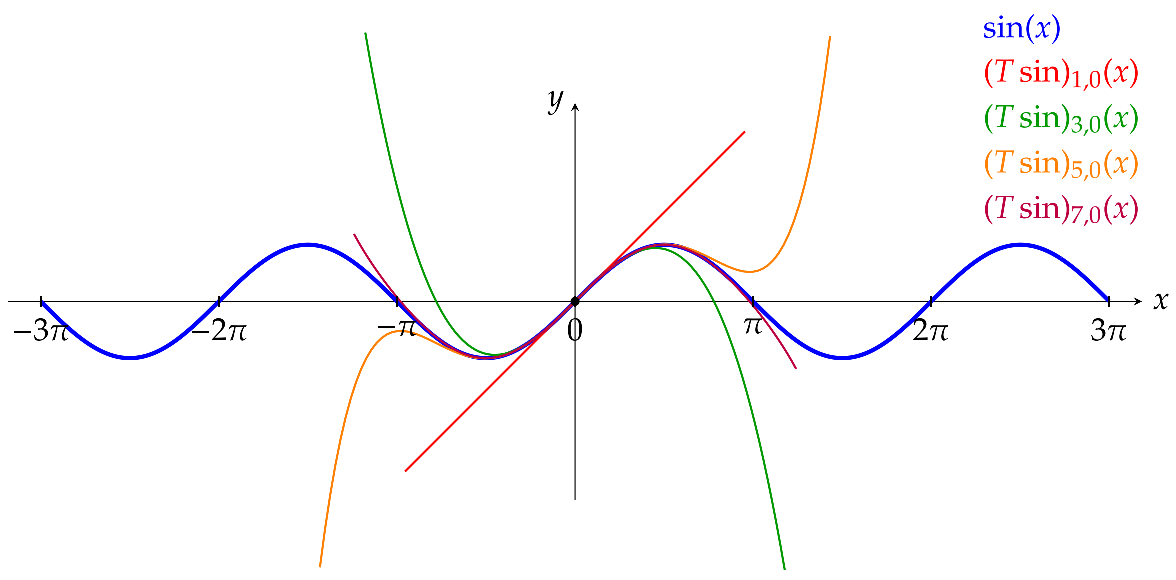


Figure 9.1: The function $\sin(x)$ (blue) and its Maclaurin polynomials $(T\sin)_{n,0}(x)$ of orders $n = 2m+1 = 1, 3, 5, 7$. Higher order polynomials provide better approximations.

We know that $\cos x$ is an even function, so by Proposition 9.4 we expect its Maclaurin polynomial to only have even powers. Taking the first few derivatives, we have

$$\cos' x = -\sin x \quad \cos'' x = -\cos x \quad \cos''' x = \sin x \quad \cos^{(4)} x = \cos x \quad \cdots$$

and at $x_0 = 0$ we get

$$\cos' 0 = 0 \quad \cos'' 0 = -1 \quad \cos''' 0 = 0 \quad \cos^{(4)} 0 = 1 \quad \cdots$$

With **Peano**'s approximation of the remainder we have that with $n = 2m + 1$

$$\begin{aligned}\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} - \cdots + (-1)^m \frac{x^{2m}}{(2m)!} + o(x^{2m+1}) \\ &= \sum_{k=0}^{m} (-1)^k \frac{x^{2k}}{(2k)!} + o(x^{2m+1}), \qquad x \to 0.\end{aligned}$$

Power functions

We would want to study functions of the form x^α for $\alpha \in \mathbb{R}$. However, since power functions of this form are defined only for $x > 0$, we translate the function by 1 and then study the Maclaurin polynomials (just like we did for the natural logarithm). We therefore define

$$f(x) = (1 + x)^\alpha$$

so that

$$f'(x) = \alpha(1+x)^{\alpha-1} \quad f''(x) = \alpha(\alpha-1)(1+x)^{\alpha-2} \quad f'''(x) = \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3} \quad \cdots$$

and we get

$$f'(0) = \alpha \quad f''(0) = \alpha(\alpha-1) \quad f'''(0) = \alpha(\alpha-1)(\alpha-2) \quad \cdots$$

Thus, the coefficients that will appear in the Maclaurin polynomial will be:

$$f(0) = 1 \qquad \text{and} \qquad \frac{f^{(k)}(0)}{k!} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!} \quad \text{for } k \geq 1.$$

To abbreviate these expressions, we use their similarity to the binomial coefficients, to define:

$$\binom{\alpha}{0} = 1 \qquad \text{and} \qquad \binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!} \quad \text{for } k \geq 1.$$

Therefore:

$$
\begin{aligned}
(1+x)^\alpha &= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots + \binom{\alpha}{n}x^n + o(x^n)\\
&= \sum_{k=0}^{n}\binom{\alpha}{k}x^k + o(x^n)
\end{aligned}
$$

Remark: If $\alpha \in \mathbb{N}$ is an integer, then $(1+x)^\alpha$ can be expanded as a polynomial using the standard binomial formula. In this case, the Maclaurin polynomials will coincide with this polynomial if their order is at least α, otherwise they will only contain the first α terms of the polynomial. For example, consider

$$f(x) = (1+x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5.$$

Then it expansion of order 3 is

$$(1+x)^5 = \underbrace{1 + 5x + 10x^2 + 10x^3}_{(Tf)_{3,0}(x)} + o(x^3)$$

but its expansion of any order ≥ 5 is identical to f itself:

$$(Tf)_{n,0}(x) = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5 = f(x) \qquad \forall n \geq 5.$$

Remark: If $\alpha \notin \mathbb{N}$, then, there will be nontrivial Maclaurin polynomials of arbitrary order (i.e. there will be arbitrarily high powers of x that won't vanish). This is in contrast to the case $\alpha \in \mathbb{N}$.

Let us highlight some special cases:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 - \cdots + (-1)^n x^n + o(x^n) = \sum_{k=0}^{n}(-1)^k x^k + o(x^n)$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + o(x^3)$$

We stop at x^3, because the next terms aren't as nice: the coefficient of x^4 is $-\frac{5}{128}$ and the coefficient of x^5 is $\frac{7}{256}$.