Section 2.3 Q4: Consider the diffusion eq. $u_t = u_{xx}$ in $(x,t) \in (0,1) \times (0,\infty)$ with $u(0,t) = u(1,t) = 0$ and $u(x,0) = 4x(1-x)$.

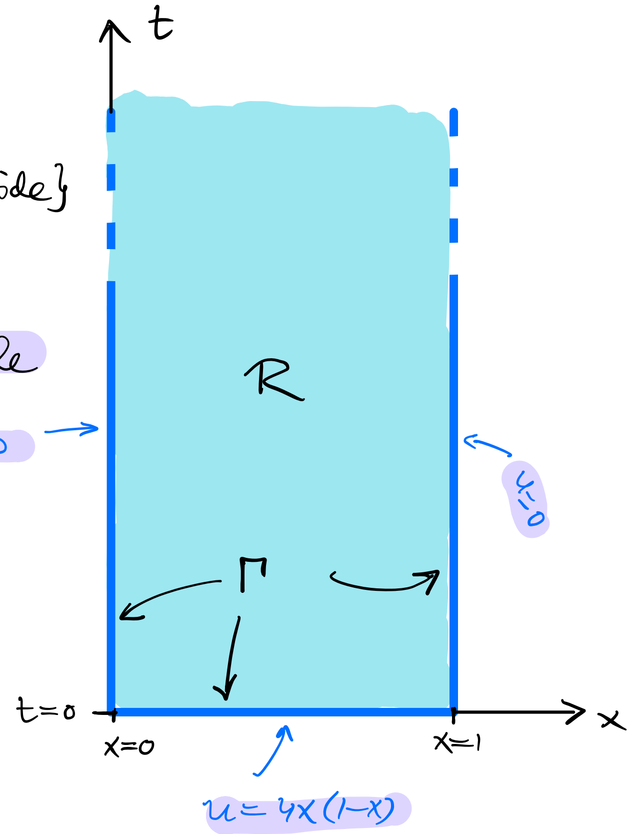
a) Show that $0 < u(x,t) < 1$ $\forall t > 0$, $0 < x < 1$.

b) Show that $u(x,t) = u(1-x,t)$ $\forall t \geq 0$, $0 \leq x \leq 1$.

c) Use the energy method to show that $\int_0^1 u(x,t)^2\,dx$ is a strictly decreasing function of t.

Denote $R = [0,1] \times [0,\infty)$,

$\Gamma = \{\text{bottom}\} \cup \{\text{left side}\} \cup \{\text{right side}\}$



a) By the strong maximum principle the max of $u(x,t)$ in R is achieved on the boundary Γ.

On the sides $u=0$. On the bottom $u(x,0) = 4x(1-x)$,

$u(\frac{1}{2}, 0) = 4 \cdot \frac{1}{2} \cdot \frac{1}{2} = \underline{1}$.

It is easy to see that this is the max.

So $u(x,t) < 1$ (strict $<$) inside R, i.e. for $x \in (0,1)$, $t > 0$.

Similarly, by the strong minimum principle, u achieves its minimum on Γ. Since $u \geq 0$ on Γ, it must hold that $u(x,t) > 0$ (strict $>$) inside R. So inside R

$$0 < u(x,t) < 1.$$

b) Let $v(x,t) = u(1-x,t)$. Then:

$$v_t = u_t, \quad v_x = -u_x, \quad v_{xx} = -(-u_x)_x = u_{xx}.$$

Hence $v_t - v_{xx} = u_t - u_{xx} = 0.$

Moreover: $v(0,t) = u(1,t) = 0$

$v(1,t) = u(0,t) = 0$

$v(x,0) = u(1-x,0) = 4(1-x)x$

So v solves the same problem like u. We know that solutions are unique ("Uniqueness of Solutions" theorem) so that u and v must be the same: $u(x,t) = v(x,t) = u(1-x,t)$ for all $t \geq 0$ and $0 \leq x \leq 1$.

c) The energy method is the method where we multiply the eq. by u and integrate:

The eq. is: $u_t - u_{xx} = 0$

Multiply: $u(u_t - u_{xx}) = 0$

Integrate: $\int_0^1 u(x,t)[u_t(x,t) - u_{xx}(x,t)]\,dx = 0.$

$$\Rightarrow \quad 0 = \int_0^1 u u_t \,dx - \int_0^1 u u_{xx}\,dx \quad \text{int. by parts}$$

$$= \frac{1}{2}\frac{d}{dt}\int_0^1 u^2 dx + \int_0^1 u_x^2 dx - \underbrace{[u u_x]_{x=0}^1}_{=0 \text{ since } u(0,t)=u(1,t)=0}$$

$$\Rightarrow \quad \frac{d}{dt}\int_0^1 u^2 dx = -2\underbrace{\int_0^1 u_x^2 dx}_{>0} < 0$$

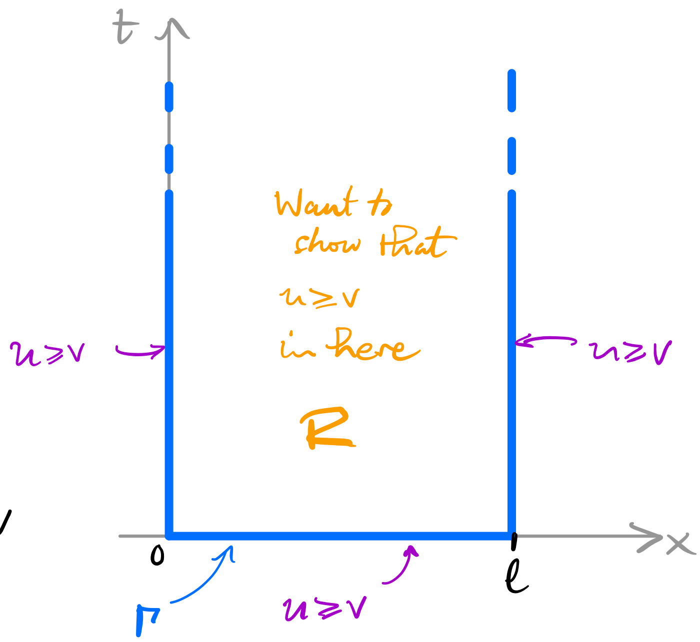
But we know that $\to$ >0 strictly (i.e. it is not 0).

How do we know this? From part (a) we know that $0 < u < 1$ inside R, yet $u = 0$ on the sides.

This means that it is impossible for u_x to always be 0 along lines of constant t.

Hence $\int_0^1 u^2\,dx$ strictly decreases in time.

Section 2.3 Q6: Prove the comparison principle for the diffusion eq: if u and v are two solutions and if $u \le v$ for $t=0$, $x=0$, $x=l$, then $u \le v$ for $t \ge 0$ and $x \in [0,l]$.



Define $w = u - v$.

Then $w \le 0$ on

$\Gamma = \{\text{bottom}\} \cup \{\text{right}\} \cup \{\text{left}\}$.

By linearity of the diffusion eq., w is also a solution.

By the maximum principle, $w \le 0$ within the infinite rectangle $R = [0,l] \times [0,\infty)$.

So $u - v = w \le 0 \longrightarrow u \le v$ in R.

Section 2.4 Q1: Solve the diffusion eq. with the initial condition $\phi(x) = \begin{cases} 1 & |x|<l \\ 0 & |x|>l \end{cases}$

We know that the formula is

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \phi(y)\, dy.$$

In our case this simplifies to

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-l}^{l} e^{-\frac{(x-y)^2}{4kt}} dy$$

To express in terms of the error function, make the change of variables $p = \frac{y-x}{\sqrt{4kt}}$ so that

$$dp = \frac{dy}{\sqrt{4kt}} \longrightarrow dy = \sqrt{4kt}\, dp$$

$$\longrightarrow u(x,t) = \frac{1}{\sqrt{\pi}} \int_{\frac{-l-x}{\sqrt{4kt}}}^{\frac{l-x}{\sqrt{4kt}}} e^{-p^2} dp$$

$$= \frac{1}{\sqrt{\pi}} \int_0^{\frac{l-x}{\sqrt{4kt}}} e^{-p^2} dp - \frac{1}{\sqrt{\pi}} \int_0^{\frac{-l-x}{\sqrt{4kt}}} e^{-p^2} dp$$

$$= \frac{1}{2} \operatorname{Erf}\left(\frac{l-x}{\sqrt{4kt}}\right) - \frac{1}{2} \operatorname{Erf}\left(\frac{-l-x}{\sqrt{4kt}}\right).$$

Section 2.4 Q6: Compute $\int_0^{\infty} e^{-x^2} dx$.

We've done this in class!

Section 2.4 Q7: Show that $\int_{-\infty}^{\infty} e^{-p^2} dp = \sqrt{\pi}$ and that $\int_{-\infty}^{\infty} S(x,t)\, dx = 1$.

We've done this in class too!

Section 2.4 Q18: Solve the heat eq with convection:

$$\begin{cases} u_t - k u_{xx} + V u_x = 0 & t>0 \quad -\infty<x<\infty \\ u(x,0) = \phi(x) & -\infty<x<\infty \end{cases}$$

where V is a constant.

Make the substitution $y = x - Vt$, $x = y + Vt$:

Define $v(y,t) = u(y+Vt, t)$.

Then $v_t = u_x \cdot V + u_t$

$v_x = u_x$

$v_{xx} = u_{xx}$

So: $0 = \underbrace{u_t + V u_x}_{v_t} - k \underbrace{u_{xx}}_{v_{xx}} = v_t - k v_{xx}$

So v satisfies the usual diffusion eq. with the initial condition $v(y,0) = u(y,0) = \phi(y)$. Hence

$$v(y,t) = \int_{-\infty}^{\infty} S(y-w, t)\, \phi(w)\, dw$$

$$\Longrightarrow \quad u(x,t) = v(x-Vt, t) = \int_{-\infty}^{\infty} S(x-Vt-w, t)\, \phi(w)\, dw.$$