MATHEMATICAL ANALYSIS 1
HOMEWORK 11

(1) Using Maclaurin polynomials, determine $\alpha \in \mathbb{R}$ so that
$$f(x) = (\arctan 2x)^2 - \alpha x \sin x$$
is infinitesimal of order 4 with respect to $\varphi(x) = x$ as $x \to 0$.

(2) Compute $f^{(6)}(0)$ where f is the function
$$f(x) = \sinh(x^2 + 2\sin^4 x).$$

(3) Determine the order and the principal part as $x \to +\infty$ with respect to $\varphi(x) = \frac{1}{x}$ of the infinitesimal function
$$f(x) = \sqrt[3]{1 + 3x^2 + x^3} - \sqrt[5]{2 + 5x^4 + x^5}.$$

(4) Determine the order and the principal part as $x \to 0$ with respect to $\varphi(x) = x$ of the infinitesimal function
$$f(x) = \sqrt[3]{1 - x^2} - \sqrt{1 - \frac{2}{3}x^2} + \sin\frac{x^4}{18}.$$

(5) For different values of $\alpha \in \mathbb{R}$, as $x \to 0$, determine the order of the following infinitesimal function with respect to $\varphi(x) = x$:
$$f(x) = \ln\cos x + \ln\cosh(\alpha x).$$

(6) Compute the following indefinite integrals using integration by parts:

(a) $\int \cos^4 x \text{ dx}$
(b) $\int \ln(\sqrt[3]{1 + x^2}) \text{ dx}$
(c) $\int \ln^2 x \text{ dx}$
(d) $\int x \arctan x \text{ dx}$

(7) Compute $\int \frac{1}{(1+x^2)^2} \text{ dx}$. *[Hint: compute $\int \frac{1}{1+x^2}$ dx using integration by parts with $f(x) = \frac{1}{1+x^2}$ and $g'(x) = 1$.]*

(8) Compute the following indefinite integrals using substitution:

(a) $\int \frac{e^{2x}}{e^x+1} \text{dx}$
(b) $\int \frac{1+\cos x}{1-\cos x} \text{dx}$
(c) $\int \frac{1}{\sinh x} \text{dx}$
(d) $\int \frac{1}{e^{4x}+1} \text{dx}$

(9) (a) Recall what the *derivatives* of $\arctan x$ and of $\arctan \frac{1}{x}$ are.
(b) Use the previous part to prove that
$$\arctan\frac{1}{x} = \frac{\pi}{2} - \arctan x, \qquad \forall x > 0.$$

(1) **Determine α so that $f(x)$ is infinitesimal of order 4**

We want $f(x) = (\arctan 2x)^2 - \alpha x \sin x = O(x^4)$ as $x \to 0$, but with the coefficients of x, x^2, x^3 all zero. Compute Maclaurin expansions up to order 4.

First, $\arctan u = u - \frac{u^3}{3} + o(u^3)$. With $u = 2x$:

$$\arctan 2x = 2x - \frac{(2x)^3}{3} + o(x^3) = 2x - \frac{8x^3}{3} + o(x^3).$$

Square this:

$$\begin{aligned}(\arctan 2x)^2 &= \left(2x - \frac{8}{3}x^3 + o(x^3)\right)^2 \\ &= (2x)^2 + 2(2x)\left(-\frac{8}{3}x^3\right) + o(x^4) \\ &= 4x^2 - \frac{32}{3}x^4 + o(x^4).\end{aligned}$$

Note: the cross term gives x^4, and there is no x^3 term because the expansion of $\arctan 2x$ has no x^2 term.

Next, $x \sin x = x\left(x - \frac{x^3}{6} + o(x^3)\right) = x^2 - \frac{x^4}{6} + o(x^4)$.

Thus

$$f(x) = \left(4x^2 - \frac{32}{3}x^4 + o(x^4)\right) - \alpha\left(x^2 - \frac{x^4}{6} + o(x^4)\right) = (4-\alpha)x^2 + \left(-\frac{32}{3} + \frac{\alpha}{6}\right)x^4 + o(x^4).$$

For $f(x)$ to be infinitesimal of order 4, the coefficients of x^2 and x^3 (if any) must vanish. There is no x^3 term. So we require:

$$4 - \alpha = 0 \quad \Rightarrow \quad \alpha = 4.$$

With $\alpha = 4$, the x^4 coefficient becomes:

$$-\frac{32}{3} + \frac{4}{6} = -\frac{32}{3} + \frac{2}{3} = -\frac{30}{3} = -10 \neq 0.$$

Thus $f(x) = -10x^4 + o(x^4)$, which is indeed infinitesimal of order 4.

Answer: $\alpha = 4$.

(2) **Compute $f^{(6)}(0)$ for $f(x) = \sinh(x^2 + 2\sin^4 x)$**

Recall that for an analytic function, the coefficient of x^6 in its Maclaurin expansion is $\frac{f^{(6)}(0)}{6!}$. So we need the expansion of $f(x)$ up to x^6.

First, expand $\sin^4 x$. We know $\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)$. Compute $\sin^4 x = (\sin x)^4$ up to x^6. It's easier to compute step by step:

$$\begin{aligned}\sin x &= x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5). \\ \sin^2 x &= \left(x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)\right)^2 \\ &= x^2 - 2x\cdot\frac{x^3}{6} + \left(\frac{x^6}{36} + 2x\cdot\frac{x^5}{120}\right) + \dots \quad \text{(keeping up to } x^6\text{)} \\ &= x^2 - \frac{1}{3}x^4 + \left(\frac{1}{36} + \frac{1}{60}\right)x^6 + o(x^6) \\ &= x^2 - \frac{1}{3}x^4 + \left(\frac{5}{180} + \frac{3}{180}\right)x^6 + o(x^6) \\ &= x^2 - \frac{1}{3}x^4 + \frac{8}{180}x^6 + o(x^6) \\ &= x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 + o(x^6).\end{aligned}$$

Then

$$
\begin{aligned}
\sin^4 x = (\sin^2 x)^2 &= \left(x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 + o(x^6)\right)^2 \\
&= (x^2)^2 + 2x^2\left(-\frac{1}{3}x^4\right) + \left(2x^2 \cdot \frac{2}{45}x^6 + \left(-\frac{1}{3}x^4\right)^2\right) + o(x^6) \\
&= x^4 - \frac{2}{3}x^6 + \left(\frac{4}{45}x^8 + \frac{1}{9}x^8\right) + o(x^6).
\end{aligned}
$$

Terms with x^8 are $o(x^6)$, so we only keep up to x^6:

$$\sin^4 x = x^4 - \frac{2}{3}x^6 + o(x^6).$$

Now compute $u(x) = x^2 + 2\sin^4 x$:

$$u(x) = x^2 + 2\left(x^4 - \frac{2}{3}x^6 + o(x^6)\right) = x^2 + 2x^4 - \frac{4}{3}x^6 + o(x^6).$$

Next, expand $\sinh u = u + \frac{u^3}{6} + \frac{u^5}{120} + o(u^5)$. We need up to u^6 in x. Since $u = O(x^2)$, terms up to x^6 come from u, $\frac{u^3}{6}$ (since $u^3 = O(x^6)$), and possibly $\frac{u^5}{120}$ which is $O(x^{10})$, negligible.

So:

$$f(x) = u + \frac{u^3}{6} + o(x^6).$$

Compute u^3 up to x^6:

$$
\begin{aligned}
u &= x^2 + 2x^4 - \frac{4}{3}x^6 + o(x^6), \\
u^3 &= (x^2)^3 + 3(x^2)^2(2x^4) + o(x^6) \quad \text{(other terms are } O(x^8) \text{ or higher)} \\
&= x^6 + 6x^8 + o(x^6) = x^6 + o(x^6).
\end{aligned}
$$

Thus $\frac{u^3}{6} = \frac{x^6}{6} + o(x^6)$.

Therefore:

$$f(x) = \left(x^2 + 2x^4 - \frac{4}{3}x^6\right) + \frac{1}{6}x^6 + o(x^6) = x^2 + 2x^4 + \left(-\frac{4}{3} + \frac{1}{6}\right)x^6 + o(x^6).$$

Compute $-\frac{4}{3} + \frac{1}{6} = -\frac{8}{6} + \frac{1}{6} = -\frac{7}{6}$.

Hence

$$f(x) = x^2 + 2x^4 - \frac{7}{6}x^6 + o(x^6).$$

The coefficient of x^6 is $-\frac{7}{6}$. But the Maclaurin coefficient is $\frac{f^{(6)}(0)}{6!}$, so:

$$\frac{f^{(6)}(0)}{6!} = -\frac{7}{6} \quad \Rightarrow \quad f^{(6)}(0) = -\frac{7}{6} \cdot 720 = -840.$$

Answer: $f^{(6)}(0) = -840$.

(3) **Order and principal part as $x \to +\infty$ with respect to $\varphi(x) = 1/x$**

We consider $f(x)$ as $x \to +\infty$. We substitute $t = 1/x$, so $t \to 0^+$.

$$
\begin{aligned}
\sqrt[3]{1 + 3x^2 + x^3} &= x\sqrt[3]{\frac{1}{x^3} + \frac{3}{x} + 1} = x\sqrt[3]{1 + 3t + t^3}, \\
\sqrt[5]{2 + 5x^4 + x^5} &= x\sqrt[5]{\frac{2}{x^5} + \frac{5}{x} + 1} = x\sqrt[5]{1 + 5t + 2t^5}.
\end{aligned}
$$

Thus

$$f(x) = x\left(\sqrt[3]{1 + 3t + t^3} - \sqrt[5]{1 + 5t + 2t^5}\right), \quad t = 1/x.$$

We use the binomial expansion $(1+u)^a = 1 + au + \frac{a(a-1)}{2}u^2 + o(u^2)$. We must expand up to the t^2 term since the t terms are expected to cancel.

First Term Expansion (up to $o(t^2)$). For $\sqrt[3]{1+3t+t^3}$, let $u = 3t + t^3$ and $a = 1/3$. The t^2 coefficient is governed by the u^2 term, $u^2 = (3t)^2 + O(t^4) = 9t^2 + O(t^4)$.

$$\begin{aligned}(1+u)^{1/3} &= 1 + \frac{1}{3}u + \frac{\frac{1}{3}(-\frac{2}{3})}{2}u^2 + o(u^2) \\ &= 1 + \frac{1}{3}(3t+t^3) - \frac{1}{9}(9t^2) + o(t^2) \\ &= 1 + t - t^2 + o(t^2).\end{aligned}$$

Second Term Expansion (up to $o(t^2)$). For $\sqrt[5]{1+5t+2t^5}$, let $v = 5t + 2t^5$ and $a = 1/5$. The t^2 coefficient is governed by the v^2 term, $v^2 = (5t)^2 + O(t^6) = 25t^2 + O(t^4)$.

$$\begin{aligned}(1+v)^{1/5} &= 1 + \frac{1}{5}v + \frac{\frac{1}{5}(-\frac{4}{5})}{2}v^2 + o(v^2) \\ &= 1 + \frac{1}{5}(5t+2t^5) - \frac{2}{25}(25t^2) + o(t^2) \\ &= 1 + t - 2t^2 + o(t^2).\end{aligned}$$

Calculating the Difference.

$$\begin{aligned}\sqrt[3]{1+3t+t^3} - \sqrt[5]{1+5t+2t^5} &= \left(1 + t - t^2 + o(t^2)\right) - \left(1 + t - 2t^2 + o(t^2)\right) \\ &= (1-1) + (t-t) + (-t^2 - (-2t^2)) + o(t^2) \\ &= 0 + 0 + (-t^2 + 2t^2) + o(t^2) \\ &= t^2 + o(t^2).\end{aligned}$$

Therefore,

$$f(x) = x \cdot \left(t^2 + o(t^2)\right) = x \cdot \left(\frac{1}{x}\right)^2 + o\left(x \cdot \frac{1}{x^2}\right) = \frac{1}{x} + o\left(\frac{1}{x}\right).$$

Hence, as $x \to +\infty$, $f(x) \sim \frac{1}{x}$.

Order: 1 (since $f(x) = O(1/x)$).

Principal part: $\frac{1}{x}$.

(4) **Order and principal part as $x \to 0$ with respect to $\varphi(x) = x$**

We expand each term up to sufficient order.

First, $\sqrt[3]{1-x^2} = (1-x^2)^{1/3} = 1 - \frac{1}{3}x^2 + \frac{\frac{1}{3}(-\frac{2}{3})}{2}x^4 + o(x^4) = 1 - \frac{1}{3}x^2 - \frac{1}{9}x^4 + o(x^4)$.

Second, $\sqrt{1 - \frac{2}{3}x^2} = (1 - \frac{2}{3}x^2)^{1/2} = 1 - \frac{1}{2}\cdot\frac{2}{3}x^2 + \frac{\frac{1}{2}(-\frac{1}{2})}{2}\left(\frac{2}{3}x^2\right)^2 + o(x^4) = 1 - \frac{1}{3}x^2 - \frac{1}{18}x^4 + o(x^4)$.

Third, $\sin\frac{x^4}{18} = \frac{x^4}{18} + o(x^4)$.

Now compute:

$$\begin{aligned}\sqrt[3]{1-x^2} - \sqrt{1 - \frac{2}{3}x^2} &= \left(1 - \frac{1}{3}x^2 - \frac{1}{9}x^4\right) - \left(1 - \frac{1}{3}x^2 - \frac{1}{18}x^4\right) + o(x^4) \\ &= \left(-\frac{1}{9} + \frac{1}{18}\right)x^4 + o(x^4) = -\frac{1}{18}x^4 + o(x^4).\end{aligned}$$

Then add $\sin\frac{x^4}{18}$:

$$f(x) = -\frac{1}{18}x^4 + \frac{1}{18}x^4 + o(x^4) = o(x^4).$$

So the x^4 terms cancel. We need to go to higher order to find the leading term.

Compute expansions to x^6:

$\sqrt[3]{1-x^2} = 1 - \frac{1}{3}x^2 - \frac{1}{9}x^4 + \frac{\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})}{6}x^6 + o(x^6)$. Compute the x^6 coefficient: $\frac{(1/3)(-2/3)(-5/3)}{6} = \frac{(10/27)}{6} = \frac{10}{162} = \frac{5}{81}$. Thus $\sqrt[3]{1-x^2} = 1 - \frac{1}{3}x^2 - \frac{1}{9}x^4 + \frac{5}{81}x^6 + o(x^6)$.

$\sqrt{1-\frac{2}{3}x^2} = 1-\frac{1}{3}x^2-\frac{1}{18}x^4+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}\left(\frac{2}{3}x^2\right)^3+o(x^6)$. Compute the x^6 coefficient: $\frac{(1/2)(-1/2)(-3/2)}{6}\cdot\left(\frac{2}{3}\right)^3 = \frac{(3/8)}{6}\cdot\frac{8}{27} = \frac{1}{16}\cdot\frac{8}{27} = \frac{1}{54}$. Thus $\sqrt{1-\frac{2}{3}x^2} = 1-\frac{1}{3}x^2-\frac{1}{18}x^4-\frac{1}{54}x^6+o(x^6)$.

$\sin\frac{x^4}{18} = \frac{x^4}{18}-\frac{1}{6}\left(\frac{x^4}{18}\right)^3+\cdots = \frac{x^4}{18}+o(x^6)$.

Now combine:

$$\begin{aligned}
\sqrt[3]{1-x^2}-\sqrt{1-\frac{2}{3}x^2} &= \left(1-\frac{1}{3}x^2-\frac{1}{9}x^4+\frac{5}{81}x^6\right)-\left(1-\frac{1}{3}x^2-\frac{1}{18}x^4-\frac{1}{54}x^6\right)+o(x^6)\\
&= \left(-\frac{1}{9}+\frac{1}{18}\right)x^4+\left(\frac{5}{81}+\frac{1}{54}\right)x^6+o(x^6)\\
&= -\frac{1}{18}x^4+\left(\frac{10}{162}+\frac{3}{162}\right)x^6+o(x^6)\\
&= -\frac{1}{18}x^4+\frac{13}{162}x^6+o(x^6).
\end{aligned}$$

Add $\sin\frac{x^4}{18}$:

$$f(x) = \left(-\frac{1}{18}x^4+\frac{13}{162}x^6\right)+\frac{1}{18}x^4+o(x^6) = \frac{13}{162}x^6+o(x^6).$$

Thus $f(x)\sim\frac{13}{162}x^6$ as $x\to 0$.

Order: 6.

Principal part: $\dfrac{13}{162}x^6$.

(5) **Determine the order of the infinitesimal function $f(x)=\ln\cos x+\ln\cosh(\alpha x)$ with respect to $\varphi(x)=x$ as $x\to 0$, for different values of $\alpha\in\mathbb{R}$.**

1. Maclaurin Expansions of Individual Components. We use the known Maclaurin expansions for $\cos x$ and $\cosh u$ with Peano's remainder up to order 4:

$$\cos x = 1-\frac{x^2}{2}+\frac{x^4}{24}+o(x^5)$$
$$\cosh u = 1+\frac{u^2}{2}+\frac{u^4}{24}+o(u^5)$$

Substitute $u=\alpha x$ into the cosh expansion:

$$\cosh(\alpha x) = 1+\frac{\alpha^2x^2}{2}+\frac{\alpha^4x^4}{24}+o(x^5).$$

Now we use the expansion $\ln(1+u)=u-\frac{u^2}{2}+o(u^2)$.

2. Expansion of $\ln(\cos x)$. Let $u_1=\cos x-1=-\frac{x^2}{2}+\frac{x^4}{24}+o(x^5)$. Since u_1 starts with x^2, we need the expansion up to u_1^2 to get terms up to x^4.

$$\begin{aligned}
\ln(\cos x) = \ln(1+u_1) &= u_1-\frac{1}{2}u_1^2+o(u_1^2)\\
&= \left(-\frac{x^2}{2}+\frac{x^4}{24}\right)-\frac{1}{2}\left(-\frac{x^2}{2}+\frac{x^4}{24}\right)^2+o(x^4)\\
&= -\frac{x^2}{2}+\frac{x^4}{24}-\frac{1}{2}\left(\frac{x^4}{4}+o(x^6)\right)+o(x^4)\\
&= -\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^4}{8}+o(x^4)\\
&= -\frac{x^2}{2}+\left(\frac{1}{24}-\frac{3}{24}\right)x^4+o(x^4)\\
&= -\frac{x^2}{2}-\frac{1}{12}x^4+o(x^4).
\end{aligned}$$

3. Expansion of $\ln(\cosh(\alpha x))$. Let $u_2 = \cosh(\alpha x) - 1 = \frac{\alpha^2 x^2}{2} + \frac{\alpha^4 x^4}{24} + o(x^5)$. Again, we expand up to u_2^2 to get terms up to x^4.

$$\begin{aligned}
\ln(\cosh(\alpha x)) &= \ln(1+u_2) = u_2 - \frac{1}{2}u_2^2 + o(u_2^2) \\
&= \left(\frac{\alpha^2 x^2}{2} + \frac{\alpha^4 x^4}{24}\right) - \frac{1}{2}\left(\frac{\alpha^2 x^2}{2} + \frac{\alpha^4 x^4}{24}\right)^2 + o(x^4) \\
&= \frac{\alpha^2 x^2}{2} + \frac{\alpha^4 x^4}{24} - \frac{1}{2}\left(\frac{\alpha^4 x^4}{4} + o(x^6)\right) + o(x^4) \\
&= \frac{\alpha^2 x^2}{2} + \frac{\alpha^4 x^4}{24} - \frac{\alpha^4 x^4}{8} + o(x^4) \\
&= \frac{\alpha^2 x^2}{2} + \left(\frac{1}{24} - \frac{3}{24}\right)\alpha^4 x^4 + o(x^4) \\
&= \frac{\alpha^2 x^2}{2} - \frac{\alpha^4}{12}x^4 + o(x^4).
\end{aligned}$$

4. Combining Terms and Determining Order. We add the two expansions:

$$\begin{aligned}
f(x) &= \ln\cos x + \ln\cosh(\alpha x) \\
&= \left(-\frac{x^2}{2} + \left(\frac{1}{24} - \frac{1}{8}\right)x^4\right) + \left(\frac{\alpha^2 x^2}{2} + \left(\frac{1}{24} - \frac{1}{8}\right)\alpha^4 x^4\right) + o(x^4) \\
&= \left(\frac{\alpha^2}{2} - \frac{1}{2}\right)x^2 + \left[\left(\frac{1}{24} - \frac{1}{8}\right)x^4 + \left(\frac{1}{24} - \frac{1}{8}\right)\alpha^4 x^4\right] + o(x^4) \\
&= \frac{1}{2}(\alpha^2 - 1)x^2 + \left(\frac{1}{4!} - \frac{1}{8}\right)(1+\alpha^4)x^4 + o(x^4).
\end{aligned}$$

We analyze the coefficient of the leading term, $\frac{1}{2}(\alpha^2 - 1)$.

- **Case 1:** $\alpha^2 - 1 \neq 0$ **(i.e.,** $\alpha \neq 1$ **and** $\alpha \neq -1$**)** The leading term is $\frac{1}{2}(\alpha^2 - 1)x^2$.

$$f(x) \sim \frac{1}{2}(\alpha^2 - 1)x^2 \quad \text{as } x \to 0.$$

  The order of $f(x)$ with respect to $\varphi(x) = x$ is **2**.
- **Case 2:** $\alpha = 1$ **or** $\alpha = -1$ If $\alpha^2 = 1$, the x^2 term cancels. We use the fact that $\left(\frac{1}{4!} - \frac{1}{8}\right) = -\frac{1}{12}$.

$$f(x) = 0 \cdot x^2 + \left(-\frac{1}{12}\right)(1+1^4)x^4 + o(x^4) = -\frac{2}{12}x^4 + o(x^4) = -\frac{1}{6}x^4 + o(x^4).$$

$$f(x) \sim -\frac{1}{6}x^4 \quad \text{as } x \to 0.$$

  The order of $f(x)$ with respect to $\varphi(x) = x$ is **4**.

(6) **Indefinite integrals by parts**

(a) $\int \cos^4 x\,dx$.

Using integration by parts: let $f(x) = \cos^3 x$ and $g'(x) = \cos x$. Then $f'(x) = -3\cos^2 x \sin x$ and $g(x) = \sin x$. Thus:

$$\int \cos^4 x\,dx = \cos^3 x \sin x + 3\int \cos^2 x \sin^2 x\,dx.$$

Now, $\sin^2 x = 1 - \cos^2 x$, so $\cos^2 x \sin^2 x = \cos^2 x - \cos^4 x$. Hence:

$$\int \cos^4 x\,dx = \cos^3 x \sin x + 3\int (\cos^2 x - \cos^4 x)\,dx.$$

Let $I = \int \cos^4 x\,dx$. Then:

$$I = \cos^3 x \sin x + 3\int \cos^2 x\,dx - 3I \quad \Rightarrow \quad 4I = \cos^3 x \sin x + 3\int \cos^2 x\,dx.$$

We compute $\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}dx = \frac{x}{2} + \frac{\sin 2x}{4} + C$. Therefore:

$$4I = \cos^3 x \sin x + 3\left(\frac{x}{2} + \frac{\sin 2x}{4}\right) + C.$$

Thus:

$$I = \frac{1}{4}\cos^3 x \sin x + \frac{3x}{8} + \frac{3\sin 2x}{16} + C.$$

This is the result using integration by parts. (One may also express it as $\frac{3x}{8} + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$ using trigonometric identities.)

(b) $\int \ln(\sqrt[3]{1+x^2})\,dx = \frac{1}{3}\int \ln(1+x^2)\,dx$.
Integrate by parts: let $f(x) = \ln(1+x^2)$ and $g'(x) = 1$, so $f'(x) = \frac{2x}{1+x^2}$ and $g(x) = x$. Then

$$\int \ln(1+x^2)\,dx = x\ln(1+x^2) - \int \frac{2x^2}{1+x^2}\,dx.$$

Now $\frac{2x^2}{1+x^2} = 2 - \frac{2}{1+x^2}$. Thus

$$\int \frac{2x^2}{1+x^2}\,dx = \int 2\,dx - 2\int \frac{1}{1+x^2}\,dx = 2x - 2\arctan x.$$

Hence

$$\int \ln(1+x^2)\,dx = x\ln(1+x^2) - 2x + 2\arctan x + C.$$

Therefore,

$$\int \ln(\sqrt[3]{1+x^2})\,dx = \frac{1}{3}\left(x\ln(1+x^2) - 2x + 2\arctan x\right) + C.$$

(c) $\int \ln^2 x\,dx$.
Integrate by parts: let $f(x) = \ln^2 x$ and $g'(x) = 1$, so $f'(x) = 2\ln x \cdot \frac{1}{x}$ and $g(x) = x$. Then

$$\int \ln^2 x\,dx = x\ln^2 x - \int 2\ln x\,dx.$$

Now $\int \ln x\,dx = x\ln x - x + C$. So

$$\int \ln^2 x\,dx = x\ln^2 x - 2(x\ln x - x) + C = x\ln^2 x - 2x\ln x + 2x + C.$$

(d) $\int x\arctan x\,dx$.
Integrate by parts: let $f(x) = \arctan x$ and $g'(x) = x$, so $f'(x) = \frac{1}{1+x^2}$ and $g(x) = \frac{x^2}{2}$. Then

$$\int x\arctan x\,dx = \frac{x^2}{2}\arctan x - \int \frac{x^2}{2(1+x^2)}\,dx.$$

Now $\frac{x^2}{1+x^2} = 1 - \frac{1}{1+x^2}$. Thus

$$\int \frac{x^2}{1+x^2}\,dx = \int 1\,dx - \int \frac{1}{1+x^2}\,dx = x - \arctan x.$$

Hence

$$\int x\arctan x\,dx = \frac{x^2}{2}\arctan x - \frac{1}{2}(x - \arctan x) + C = \frac{1}{2}(x^2+1)\arctan x - \frac{x}{2} + C.$$

(7) $\int \frac{1}{(1+x^2)^2}\,dx$.

Following the hint, first compute $\int \frac{1}{1+x^2}\,dx$ by parts with $f(x) = \frac{1}{1+x^2}$, $g'(x) = 1$. Then $f'(x) = -\frac{2x}{(1+x^2)^2}$, $g(x) = x$. So

$$\int \frac{1}{1+x^2}\,dx = \frac{x}{1+x^2} + \int \frac{2x^2}{(1+x^2)^2}\,dx.$$

But note $\frac{2x^2}{(1+x^2)^2} = \frac{2(1+x^2)-2}{(1+x^2)^2} = \frac{2}{1+x^2} - \frac{2}{(1+x^2)^2}$. Thus

$$\int \frac{1}{1+x^2}\,dx = \frac{x}{1+x^2} + 2\int \frac{1}{1+x^2}\,dx - 2\int \frac{1}{(1+x^2)^2}\,dx.$$

Rearranging:

$$\int \frac{1}{1+x^2}\,dx - 2\int \frac{1}{1+x^2}\,dx = \frac{x}{1+x^2} - 2\int \frac{1}{(1+x^2)^2}\,dx,$$

so

$$-\int \frac{1}{1+x^2}\,dx = \frac{x}{1+x^2} - 2\int \frac{1}{(1+x^2)^2}\,dx.$$

Thus

$$2\int \frac{1}{(1+x^2)^2}\,dx = \frac{x}{1+x^2} + \int \frac{1}{1+x^2}\,dx = \frac{x}{1+x^2} + \arctan x + C.$$

Therefore,

$$\int \frac{1}{(1+x^2)^2}\,dx = \frac{1}{2}\left(\frac{x}{1+x^2} + \arctan x\right) + C.$$

(8) **Indefinite integrals by substitution**

(a) $\int \frac{e^{2x}}{e^x+1}\,dx$.

Let $y = e^x$, so $dy = e^x dx$, and $e^{2x} = y^2$. Then

$$\int \frac{e^{2x}}{e^x+1}\,dx = \int \frac{y^2}{y+1}\cdot\frac{dy}{y} = \int \frac{y}{y+1}\,dy = \int \left(1 - \frac{1}{y+1}\right)dy = y - \ln|y+1| + C = e^x - \ln(e^x+1) + C.$$

(b) $\int \frac{1+\cos x}{1-\cos x}\,dx$.

Let $t = \tan(x/2)$, then $\cos x = \frac{1-t^2}{1+t^2}$, $dx = \frac{2}{1+t^2}dt$. Then

$$\frac{1+\cos x}{1-\cos x} = \frac{1+\frac{1-t^2}{1+t^2}}{1-\frac{1-t^2}{1+t^2}} = \frac{\frac{2}{1+t^2}}{\frac{2t^2}{1+t^2}} = \frac{1}{t^2}.$$

Thus

$$\int \frac{1+\cos x}{1-\cos x}\,dx = \int \frac{1}{t^2}\cdot\frac{2}{1+t^2}\,dt = 2\int \frac{1}{t^2(1+t^2)}\,dt = 2\int \left(\frac{1}{t^2} - \frac{1}{1+t^2}\right)dt = 2\left(-\frac{1}{t} - \arctan t\right) + C.$$

Substitute back $t = \tan(x/2)$:

$$= -2\cot(x/2) - 2\arctan(\tan(x/2)) + C = -2\cot(x/2) - x + C.$$

(c) $\int \frac{1}{\sinh x}\,dx$.

Recall $\sinh x = \frac{e^x - e^{-x}}{2}$. Multiply numerator and denominator by e^x:

$$\int \frac{1}{\sinh x}\,dx = \int \frac{2}{e^x - e^{-x}}\,dx = \int \frac{2e^x}{e^{2x}-1}\,dx.$$

Let $y = e^x$, so $dy = e^x dx$. Then

$$\int \frac{2e^x}{e^{2x}-1}\,dx = \int \frac{2}{y^2-1}\,dy = \int \left(\frac{1}{y-1} - \frac{1}{y+1}\right)dy = \ln|y-1| - \ln|y+1| + C = \ln\left|\frac{e^x-1}{e^x+1}\right| + C.$$

(d) $\int \frac{1}{e^{4x}+1}\,dx$.

Let $y = e^{4x}$, then $dy = 4e^{4x}dx = 4y\,dx$, so $dx = \frac{dy}{4y}$. Then

$$\int \frac{1}{e^{4x}+1}\,dx = \int \frac{1}{y+1}\cdot\frac{dy}{4y} = \frac{1}{4}\int \frac{1}{y(y+1)}\,dy.$$

Using partial fractions: $\frac{1}{y(y+1)} = \frac{1}{y} - \frac{1}{y+1}$. Thus

$$\frac{1}{4}\int \left(\frac{1}{y} - \frac{1}{y+1}\right)dy = \frac{1}{4}\left(\ln|y| - \ln|y+1|\right) + C = \frac{1}{4}\ln\left|\frac{y}{y+1}\right| + C.$$

Substitute back $y = e^{4x}$:

$$= \frac{1}{4} \ln\left(\frac{e^{4x}}{e^{4x}+1}\right) + C = \frac{1}{4} \ln\left(1 - \frac{1}{e^{4x}+1}\right) + C = x - \frac{1}{4} \ln(e^{4x}+1) + C.$$

(9) (a) Derivatives:

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2}, \qquad \frac{d}{dx} \arctan \frac{1}{x} = \frac{1}{1+(1/x)^2} \cdot \left(-\frac{1}{x^2}\right) = -\frac{1}{x^2+1}.$$

(b) Proof of identity.
Let $h(x) = \arctan \frac{1}{x} + \arctan x$ for $x > 0$. Then

$$h'(x) = -\frac{1}{1+x^2} + \frac{1}{1+x^2} = 0.$$

Thus $h(x)$ is constant on $(0, \infty)$. To find the constant, evaluate at $x = 1$:

$$h(1) = \arctan 1 + \arctan 1 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}.$$

Hence for all $x > 0$,

$$\arctan \frac{1}{x} + \arctan x = \frac{\pi}{2} \quad \Rightarrow \quad \arctan \frac{1}{x} = \frac{\pi}{2} - \arctan x.$$