and try to identify the correct $\alpha$. We see that we get:

$$\lim_{x\to+\infty} \frac{f(x)}{\varphi^\alpha(x)} = \lim_{x\to+\infty} \frac{x^\alpha}{x} \cdot \frac{4}{\sqrt{1+\frac{3}{x^2}} + \sqrt{1-\frac{1}{x^2}}}$$

so that with $\alpha = 1$ we have

$$\lim_{x\to+\infty} \frac{f(x)}{\varphi^\alpha(x)} = \lim_{x\to+\infty} \frac{4}{\sqrt{1+\frac{3}{x^2}} + \sqrt{1-\frac{1}{x^2}}} = 2.$$

Hence $f$ is infinitesimal of order 1 at $+\infty$ with respect to $\varphi(x) = x^{-1}$. The principal part is $p(x) = 2x^{-1}$ and we can write

$$\sqrt{x^2+3} - \sqrt{x^2-1} = 2x^{-1} + o(x^{-1}), \qquad x \to +\infty.$$

**Example 6.5:** Consider the function

$$f(x) = \sqrt{9x^5 + 7x^3 - 1}$$

which is infinite as $x \to +\infty$. To determine the order we compare it with $\varphi(x) = x$:

$$\begin{aligned}\lim_{x\to+\infty} \frac{f(x)}{x^\alpha} &= \lim_{x\to+\infty} \frac{\sqrt{9x^5 + 7x^3 - 1}}{x^\alpha} \\ &= \lim_{x\to+\infty} \frac{x^{\frac{5}{2}}\sqrt{9 + 7x^{-2} - x^{-5}}}{x^\alpha}.\end{aligned}$$

This suggests choosing $\alpha = \frac{5}{2}$. Then we have:

$$\lim_{x\to+\infty} \frac{f(x)}{x^{\frac{5}{2}}} = \lim_{x\to+\infty} \sqrt{9 + 7x^{-2} - x^{-5}} = 3.$$

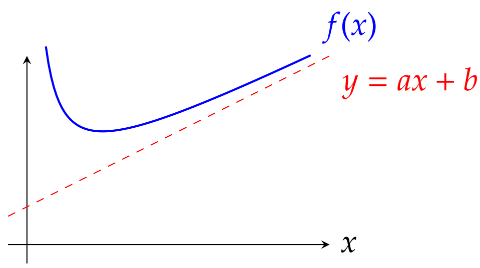
Hence $f$ is infinite of order $\frac{5}{2}$ at $+\infty$ with respect to $\varphi(x) = x$. The principal part is $p(x) = 3x^{\frac{5}{2}}$ and we can write

$$\sqrt{9x^5 + 7x^3 - 1} = 3x^{\frac{5}{2}} + o(x^{\frac{5}{2}}), \qquad x \to +\infty.$$

## 6.4 Asymptotes

We have already seen *horizontal* and *vertical* asymptotes. However it is possible to have slanted asymptotes. We say that a function $f(x)$ **behaves asymptotically as** $x \to +\infty$ like the affine function $y = ax + b$ if

$$\lim_{x\to+\infty} (f(x) - (ax + b)) = 0.$$



A similar definition can be made for $x \to -\infty$.

In principle we can compare a function $f : \mathbb{R} \to \mathbb{R}$ to any function $g : \mathbb{R} \to \mathbb{R}$ as $x \to \pm\infty$. In the case of $+\infty$, we say that $f$ behaves asymptotically as $x \to +\infty$ like $g$ if

$$\lim_{x \to +\infty} (f(x) - g(x)) = 0.$$

## Chapter 7

# Global properties of continuous functions

As opposed to the previous chapters, where we studied *local* properties of functions (that is, properties of functions as $x \to x_0$, where $x_0$ can be a finite point or $\pm\infty$), in the current chapter we study *global* properties of functions. That is, we consider functions defined on intervals $[a, b] \subset \mathbb{R}$ (where $a < b$) and try to draw some conclusions on their behavior on this entire interval.

## 7.1 Theorem of Existence of Zeroes

The first main global result is one for finding zeroes of a function $f$ – i.e. points $x_0$ where $f(x_0) = 0$:

**Theorem 7.1:** Let $f : \text{dom}(f) \subseteq \mathbb{R} \to \mathbb{R}$ and suppose it is continuous on an interval $[a, b] \subseteq \text{dom}(f)$ where $a < b$. If $f$ changes sign between $a$ and $b$ then there exists $x_0 \in (a, b)$ such that $f(x_0) = 0$. Moreover, if $f$ is *strictly* monotone on $[a, b]$ then $x_0$ is the *unique* point in $[a, b]$ where $f$ equals zero.

Such an $x_0$ is called a **zero of** $f$.

*Proof.* Assume that $f(a) < 0 < f(b)$. The proof for the case $f(a) > 0 > f(b)$ will follow the exact same ideas. We split the proof into several steps.

**Step A.** Constructing sequences $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$ in search of $x_0$:

**Step A0.** Define $a_0 = a$ and $b_0 = b$. Define the center point between $a_0$ and $b_0$ to be $c_0 = \frac{a_0+b_0}{2}$. If $f(c_0) = 0$ then $x_0 = c_0$ and the proof is done. Otherwise, if $f(c_0) < 0$ we define $a_1 = c_0$ and $b_1 = b_0$ and if $f(c_0) > 0$ we define $a_1 = a_0$ and $b_1 = c_0$.

**Step A1.** Define the center point between $a_1$ and $b_1$ to be $c_1 = \frac{a_1+b_1}{2}$. If $f(c_1) = 0$ then $x_0 = c_1$ and the proof is done. Otherwise, if $f(c_1) < 0$ we define $a_2 = c_1$ and $b_2 = b_1$ and if $f(c_1) > 0$ we define $a_2 = a_1$ and $b_2 = c_1$.

**Step A2.** Define the center point between $a_2$ and $b_2$ to be $c_2 = \frac{a_2+b_2}{2}$. If $f(c_2) = 0$ then $x_0 = c_2$ and the proof is done. Otherwise, if $f(c_2) < 0$ we define $a_3 = c_2$ and $b_3 = b_2$ and if $f(c_2) > 0$ we define $a_3 = a_2$ and $b_3 = c_2$.

...and so on...

**Step A$n$.** The points $a_n$ and $b_n$ have been defined in the previous step. Define the center point between $a_n$ and $b_n$ to be $c_n = \frac{a_n+b_n}{2}$. If $f(c_n) = 0$ then $x_0 = c_n$ and the proof is done. Otherwise, if $f(c_n) < 0$ we define $a_{n+1} = c_n$ and $b_{n+1} = b_n$ and if $f(c_n) > 0$ we define $a_{n+1} = a_n$ and $b_{n+1} = c_n$.

...and so on...

**Step B.** Properties of the sequences $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$.

If a zero $x_0$ hasn't been located in any finite step, then by induction we have constructed sequences $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$ satisfying:

$$
\begin{aligned}
&a_0 \le a_1 \le \cdots \le a_n \le \cdots \le b_n \le \cdots b_1 \le b_0 \\
&[a_0, b_0] \supset [a_1, b_1] \supset \cdots \supset [a_n, b_n] \supset \cdots \\
&f(a_n) < 0 < f(b_n) \quad \text{and} \quad b_n - a_n = \frac{a_0 - b_0}{2^n}, \quad \forall n \in \mathbb{N}.
\end{aligned}
$$

By the first property, $\{a_n\}_{n\in\mathbb{N}}$ is a bounded, monotonically increasing sequence (it is bounded by $b_0$, for instance). Hence its limit exists, denote it by $a_\infty \in \mathbb{R}$:

$$\lim_{n\to\infty} a_n = a_\infty.$$

Similarly, $\{b_n\}_{n\in\mathbb{N}}$ is bounded and monotonically decreasing, hence its limit exists. We denote it by $b_\infty \in \mathbb{R}$:

$$\lim_{n\to\infty} b_n = b_\infty.$$

**Step C.** Conclusion.

Observe that

$$b_\infty - a_\infty = \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n = \lim_{n\to\infty} (b_n - a_n) = \lim_{n\to\infty} \frac{b_0 - a_0}{2^n} = 0$$

so that $b_\infty = a_\infty$. Denote this number by $x_0$:

$$x_0 = b_\infty = a_\infty.$$

Since $f$ is continuous, we may use Theorem 5.15(7a) (the substitution theorem for a continuous function of a convergent sequence) to get:

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(b_n) = f(x_0).$$

However we know that

$$f(a_n) < 0 < f(b_n), \quad \forall n \in \mathbb{N}.$$

By Theorem 5.15(4),

$$f(x_0) = \lim_{n\to\infty} f(a_n) \le 0 \quad \text{and} \quad 0 \le \lim_{n\to\infty} f(b_n) = f(x_0)$$

Hence it necessarily follows that

$$f(x_0) = 0.$$

Finally, if $f$ is strictly monotone on $[a, b]$, then by Proposition 2.1, $f$ is injective on $[a, b]$. This means that for every $y \in f([a, b])$ there exists a unique $x \in [a, b]$ such that $f(x) = y$. Hence $x_0$ is the unique zero of $f$ in $[a, b]$. □

**Example 7.1:** The function

$$f(x) = \begin{cases} 1 & x \geq 10 \\ -1 & x < 10 \end{cases}$$

changes sign on the interval $[9, 11]$. However, there is no zero in $[9, 11]$ (i.e. there is no $x \in [9, 11]$ such that $f(x) = 0$. Why? The problem with this $f$ is that it is not continuous on $[9, 11]$ (there's a jump discontinuity at $x = 10$).

**Example 7.2:** The function $f(x) = x^2$ is a continuous function on $\mathbb{R}$ that has a zero at $x = 0$, however it is always non-negative (i.e. it does not change sign). This shows that a continuous function can have a zero without changing sign. That is, changing sign is a *sufficient* condition for a continuous function to have a zero, but not a *necessary* condition.

**Example 7.3:** The function $f(x) = e^x + \sin x$ is continuous on $\mathbb{R}$. Let's look at the interval $[-\frac{\pi}{2}, 0]$. For $x = -\frac{\pi}{2}$ the function is negative: $f(-\frac{\pi}{2}) = e^{-\pi/2} + \sin(-\frac{\pi}{2}) = e^{-\pi/2} - 1 < e^0 - 1 = 1 - 1 = 0$ and for $x = 0$ the function is positive: $f(0) = e^0 - \sin 0 = 1 - 0 = 1 > 0$. Hence there exists $x_0 \in (-\frac{\pi}{2}, 0)$ such that $f(x_0) = 0$. Moreover, $e^x$ is strictly increasing on $\mathbb{R}$, and $\sin x$ is strictly increasing on $[-\frac{\pi}{2}, 0]$, so that $x_0$ is the unique zero within this interval.

**Corollary 7.2:** Let $(\alpha, \beta) \subseteq \mathbb{R}$ be an interval ($\alpha$ may be $-\infty$ and $\beta$ may be $+\infty$), and let $f : \mathbb{R} \to \mathbb{R}$ be defined on $(a, b)$, satisfy

$$\lim_{x \to \alpha^+} f(x) = \ell_\alpha \in \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$$
$$\lim_{x \to \beta^-} f(x) = \ell_\beta \in \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$$

with $\ell_\alpha$ and $\ell_\beta$ having opposite signs. Then $f$ has a zero $x_0$ in $(\alpha, \beta)$: $f(x_0) = 0$. Moreover, if $f$ is strictly monotone in $(\alpha, \beta)$ then $x_0$ is unique.

*Proof.* This proof is an exercise. □

**Corollary 7.3:** Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous on an interval $[a, b]$. If

$$f(a) < g(a) \quad \text{and} \quad f(b) > g(b)$$

or

$$f(a) > g(a) \quad \text{and} \quad f(b) < g(b)$$

then there exists a point $x_0 \in (a, b)$ satisfying

$$f(x_0) = g(x_0).$$

Moreover, if $f$ and $g$ are strictly monotone then $x_0$ is unique.

*Proof.* This proof is very simple: we consider the function

$$h(x) = f(x) - g(x).$$

Then $h$ is continuous on $[a, b]$ (since it is the difference of two continuous functions on $[a, b]$). Furthermore, $h(a)$ and $h(b)$ have different sign (due to the assumptions on $f$ and $g$). So $h$ satisfies the conditions of Theorem 7.1, and there exists $x_0 \in (a, b)$ such that $h(x_0) = 0$. But this means (by definition of $h$) that $f(x_0) = g(x_0)$.

The strictly monotone case is also a consequence of Theorem 7.1 (can you think why if $f$ and $g$ are both strictly monotone, then $h$ is strictly monotone? it is not immediately evident. Look at Lemma 2.2). □