

MATHEMATICAL ANALYSIS 1
HOMEWORK 13

- (1) Compute the area under the graph of $f(x) = |\ln x|$ on the interval $[e^{-1}, e]$.
(2) Determine the following function:

$$F(x) = \int_{-1}^x (|y - 1| + 2) dy.$$

- (3) Compute the area between f and g :

- (a) $f(x) = |x|$ and $g(x) = \sqrt{1 - x^2}$.
(b) $f(x) = x^2 - 2x$ and $g(x) = -x^2 + x$.

- (4) **Improper integrals I.** Compute the following improper integrals:

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| (a) $\int_0^{+\infty} \frac{1}{x^2+3x+2} dx.$ | (c) $\int_0^{+\infty} \frac{x}{(x+1)^3} dx.$ |
| (b) $\int_2^{+\infty} \frac{1}{x\sqrt{x-2}} dx.$ | (d) $\int_{-1}^1 \frac{1}{\sqrt{x}(x-4)} dx.$ |

- (5) **Improper integrals II.** Discuss the convergence of the following improper integrals:

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| (a) $\int_0^{+\infty} \frac{\sin x}{x\sqrt{x}} dx.$ | (c) $\int_0^{+\infty} xe^{-x} dx.$ |
| (b) $\int_0^{+\infty} \frac{1}{\ln^2(2+e^x)} dx.$ | (d) $\int_0^\pi \frac{x-\frac{\pi}{2}}{\cos x \sqrt{\sin x}} dx.$ |

- (6) For which $n \in \mathbb{N}$ does the following integral converge

$$\int_2^{+\infty} \frac{x}{\sqrt{(x^2 + 3)^n}} dx \quad ?$$

Compute the integral for the smallest n for which it converges.

- (7) For which $\alpha \in \mathbb{R}$ does the integral

$$\int_2^3 \frac{x(\sin(x-2))^\alpha}{\sqrt{x^2-4}} dx$$

converge? What is its value when $\alpha = 0$?

- (8) **Geometric series.**

- (a) Let $r \in \mathbb{R}$ and let $\{r^k\}_{k=0}^{\infty} = (1, r, r^2, \dots)$ be a geometric sequence. Show that the partial sum $s_n = \sum_{k=0}^n r^k$ satisfies

$$s_n = \frac{1 - r^{n+1}}{1 - r} \quad \text{for } r \neq 1.$$

- (b) Using a geometric series, write the number $2.\overline{317} = 2.3171717\dots$ as a fraction.

- (9) **Positive-term series.** Study the convergence of the following series:

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| (a) $\sum_{k=1}^{\infty} \frac{\ln k}{k^\alpha}$ for $\alpha > 0$. | (c) $\sum_{k=1}^{\infty} \sin \frac{1}{k}$ |
| (b) $\sum_{k=1}^{\infty} \frac{1}{2^k - 1}$ | (d) $\sum_{k=1}^{\infty} \frac{k+3}{\sqrt[3]{k^9+k^2}}$ |

- (10) **Alternating sign series.** Study the convergence of the following series:

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| (a) $\sum_{k=1}^{\infty} \sin(k\pi + \frac{1}{k}).$ | (c) $\sum_{k=1}^{\infty} (-1)^k \ln \left(\frac{1}{k} + 1 \right).$ |
| (b) $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k^2}{k^3+1}.$ | (d) $\sum_{k=1}^{\infty} (-1)^k \sqrt{\frac{k^3+3}{2k^3-5}}.$ |

- (11) **Absolute convergence.** Do the following series converge absolutely?

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| (a) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1} 5^{k-1}}{(k+1)^2 4^{k+2}}.$ | (b) $\sum_{k=1}^{\infty} \frac{\cos 3k}{k^3}.$ |
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