

# MATHEMATICAL ANALYSIS 1

## THINGS TO REMEMBER

### 1. TRIGONOMETRY

#### 1.1. Fundamental trigonometric identities.

$$\begin{aligned}
 \sin^2(\alpha) + \cos^2(\alpha) &= 1 \\
 \tan^2(\alpha) + 1 &= \sec^2(\alpha) \\
 1 + \cot^2(\alpha) &= \csc^2(\alpha) \\
 \sin(\alpha \pm \beta) &= \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \\
 \cos(\alpha \pm \beta) &= \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \\
 \sin(2\alpha) &= 2\sin(\alpha)\cos(\alpha) \\
 \cos(2\alpha) &= \cos^2(\alpha) - \sin^2(\alpha) = 2\cos^2(\alpha) - 1 = 1 - 2\sin^2(\alpha) \\
 \sin(-\alpha) &= -\sin(\alpha) \quad (\text{sine is odd}) \\
 \cos(-\alpha) &= \cos(\alpha) \quad (\text{cosine is even}) \\
 \sin(\alpha) - \sin(\beta) &= 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \\
 \cos(\alpha) - \cos(\beta) &= -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)
 \end{aligned}$$

$\theta$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\pi$
$\sin \theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1
$\tan \theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	-	0

1.2. Inverse trigonometric functions. The inverse trigonometric functions are defined by restricting the domains appropriately:

- $\arcsin : [-1, 1] \rightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$  is the inverse of  $\sin$  restricted to  $[-\frac{\pi}{2}, \frac{\pi}{2}]$ .
- $\arccos : [-1, 1] \rightarrow [0, \pi]$  is the inverse of  $\cos$  restricted to  $[0, \pi]$ .
- $\arctan : \mathbb{R} \rightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$  is the inverse of  $\tan$  restricted to  $(-\frac{\pi}{2}, \frac{\pi}{2})$ .

### 2. COMPLEX NUMBERS

Property	Formula
Modulus	$ z  = \sqrt{a^2 + b^2}$
Argument	$\arg(z) = \arctan(b/a)$
Conjugate	$\bar{z} = a - bi$
Real part	$\operatorname{Re}(z) = \frac{z + \bar{z}}{2}$
Imaginary part	$\operatorname{Im}(z) = \frac{z - \bar{z}}{2i}$
Polar form	$z = r(\cos \theta + i \sin \theta)$
Exponential form	$z = re^{i\theta}$
Multiplication	$z \cdot w = rs e^{i(\theta + \phi)}$
Division	$\frac{z}{w} = \frac{r}{s} e^{i(\theta - \phi)}$
De Moivre's	$z^n = r^n e^{in\theta}$
nth roots	$\sqrt[n]{z} = \sqrt[n]{r} e^{i(\theta + 2\pi k)/n}$

### 3. DERIVATIVES

$$\frac{d}{dx}x^\alpha = \alpha x^{\alpha-1} \quad (\forall \alpha \in \mathbb{R})$$

$$\frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \cos x = -\sin x$$

$$\frac{d}{dx} \tan x = 1 + \tan^2 x = \frac{1}{\cos^2 x}$$

$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$$

$$\frac{d}{dx} a^x = (\ln a) a^x \quad (\forall a > 0) \quad \text{in particular, } \frac{d}{dx} e^x = e^x$$

$$\frac{d}{dx} \log_a |x| = \frac{1}{(\ln a) x} \quad (\forall a > 0, a \neq 1) \quad \text{in particular, } \frac{d}{dx} \ln |x| = \frac{1}{x}$$

### 4. TAYLOR

$$e^x = \sum_{k=0}^n \frac{x^k}{k!} + o(x^n), \quad x \rightarrow 0.$$

$$\begin{aligned} \ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots + (-1)^{n-1} \frac{x^n}{n} + o(x^n) \\ &= \sum_{k=1}^n (-1)^{k-1} \frac{x^k}{k} + o(x^n), \quad x \rightarrow 0. \end{aligned}$$

$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} - \cdots + (-1)^m \frac{x^{2m+1}}{(2m+1)!} + o(x^{2m+2}) \\ &= \sum_{k=1}^m (-1)^k \frac{x^{2k+1}}{(2k+1)!} + o(x^{2m+2}), \quad x \rightarrow 0. \end{aligned}$$

$$\begin{aligned} \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} - \cdots + (-1)^m \frac{x^{2m}}{(2m)!} + o(x^{2m+1}) \\ &= \sum_{k=0}^m (-1)^k \frac{x^{2k}}{(2k)!} + o(x^{2m+1}), \quad x \rightarrow 0. \end{aligned}$$

$$\begin{aligned} (1+x)^\alpha &= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \cdots + \binom{\alpha}{n} x^n + o(x^n) \\ &= \sum_{k=0}^n \binom{\alpha}{k} x^k + o(x^n) \end{aligned}$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots + (-1)^n x^n + o(x^n) = \sum_{k=0}^n (-1)^k x^k + o(x^n)$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + o(x^3)$$

## 5. INTEGRALS

$$\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + C \quad (\alpha \neq -1)$$

$$\int \frac{1}{x} dx = \log|x| + C \quad (\text{for } x > 0 \text{ or } x < 0)$$

$$\int \sin x dx = -\cos x + C$$

$$\int \cos x dx = \sin x + C$$

$$\int e^x dx = e^x + C$$

$$\int \frac{1}{1+x^2} dx = \arctan x + C$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$$

$$\int \sinh x dx = \cosh x + C$$

$$\int \cosh x dx = \sinh x + C$$