The lower and upper integrals aren't necessarily equal

Consider the bounded function (called the *Dirichlet function*)

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Regardless of the partition $\{x_k\}_{k=0}^n$ of I, any subinterval I_k will include both rational and irrational points, and therefore f will attain the values 0 and 1 on any such subinterval. Hence $\mathcal{S}_f^+$ will contain step functions whose values are at least 1 and $\mathcal{S}_f^-$ will contain step functions whose values are at most 0. It is not hard to conclude that

$$\underline{\int_I} f = 0 \qquad \text{and} \qquad \overline{\int_I} f = 1.$$

This example demonstrates that it is not evident that the lower and upper integral should be equal. In fact, it motivates the following definition:

Riemann integrable functions

A bounded function $f : I \to \mathbb{R}$ is said to be (Riemann) **integrable** on I if

$$\underline{\int_I} f = \overline{\int_I} f.$$

This value, called the definite integral, is denoted $\int_a^b f(x)\,\mathrm{d}x$ or $\int_I f(x)\,\mathrm{d}x$.

Theorem 10.11: The following functions are (Riemann) integrable on I:

1. Continuous functions on I.
2. Piecewise-continuous functions on I.
3. Functions that are continuous on (a, b) and bounded on $[a, b]$.
4. Monotone functions on $[a, b]$.

Proof. We skip this proof. □

Example 10.8: The function $f(x) = x$ is Riemann integrable (it was also Cauchy integrable). We saw that the result of the Cauchy integral was $\int_0^1 x\,\mathrm{d}x = \frac{1}{2}$. The Riemann integral will give the same result, and to see that one could take the step functions

$$h_n(x) = \begin{cases} 0 & x = 0 \\ \frac{k+1}{n} & \frac{k}{n} < x \le \frac{k+1}{n},\ k = 0, \ldots, n-1 \end{cases}$$

$$g_n(x) = \begin{cases} 0 & x = 0 \\ \frac{k}{n} & \frac{k}{n} < x \leq \frac{k+1}{n},\ k = 0, \dots, n-1 \end{cases}$$

which satisfy $g_n(x) \leq f(x) \leq h_n(x)$ for any $x \in [0,1]$, and hence $h_n \in \mathcal{S}_f^+$ and $g_n \in \mathcal{S}_f^-$. It can easily be shown (can you show it?) that

$$\int_I h_n = \frac{1}{2} + \frac{1}{2n}$$

and

$$\int_I g_n = \frac{1}{2} - \frac{1}{2n}$$

so that

$$\overline{\int_I} f \leq \inf_n \int_I h_n = \frac{1}{2} \qquad \text{and} \qquad \underline{\int_I} f \geq \sup_n \int_I g_n = \frac{1}{2}$$

hence

$$\overline{\int_I} f \leq \frac{1}{2} \leq \underline{\int_I} f.$$

But we know that

$$\underline{\int_I} f \leq \overline{\int_I} f.$$

It must therefore hold that

$$\overline{\int_I} f = \frac{1}{2} = \underline{\int_I} f$$

and f is Riemann integrable on $I = [0,1]$ (and $\int_I f(x)\,\mathrm{d}x = \frac{1}{2}$).

Example 10.9: Here are two examples of functions that are Riemann integrable but not Cauchy integrable:

1. The function

$$f(x) = \begin{cases} \sin\frac{1}{x} & 0 < x \leq 1, \\ 0 & x = 0, \end{cases}$$

   is Riemann integrable (despite having infinitely many oscillations as $x \to 0$) as it falls under the third category of functions within the theorem.

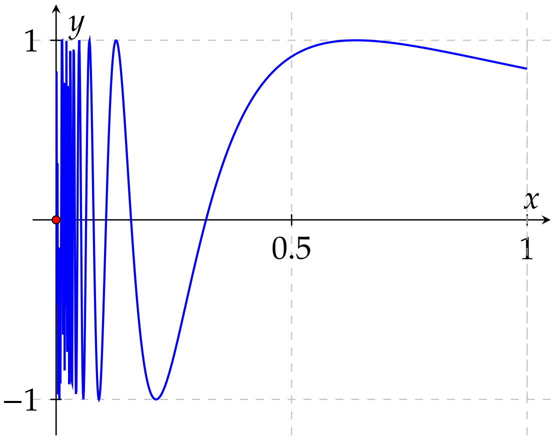
2. The function

$$f(x) = \begin{cases} \frac{1}{n} & \frac{1}{n+1} < x \leq \frac{1}{n},\ n \in \mathbb{N}_+ \\ 0 & x = 0 \end{cases}$$

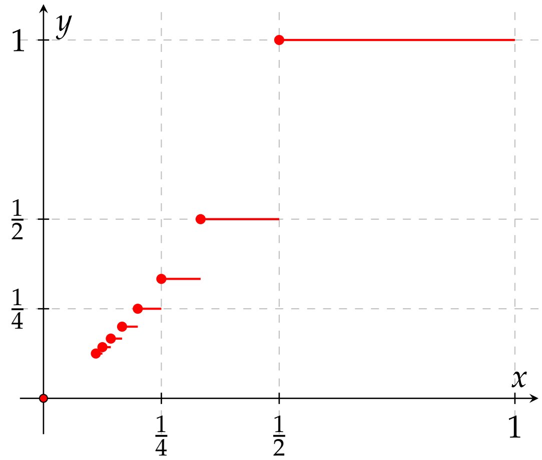
   is Riemann integrable (despite having infinitely many points with a jump discontinuity) as it falls under the fourth category of functions within the theorem.

Here are figures of both functions. Note that for the figure on the right, there are infinitely many 'steps' descending all the way to 0, but they become difficult to draw.

$$f(x) = \begin{cases} \sin\frac{1}{x} & 0 < x \le 1 \\ 0 & x = 0 \end{cases}$$



$$f(x) = \begin{cases} \frac{1}{n} & \frac{1}{n+1} < x \le \frac{1}{n}, n \in \mathbb{N}_+ \\ 0 & x = 0 \end{cases}$$



10.6 Properties of definite integrals

In this section we give some basic properties of the definite integral, without proof. Denote

$$\mathcal{R}([a,b]) = \text{the set of all integrable functions on } [a,b]$$

Definition 10.12: We define

$$\int_b^a f(x)\,\mathrm{d}x = -\int_a^b f(x)\,\mathrm{d}x$$

and if $a = b$ then

$$\int_a^a f(x)\,\mathrm{d}x = 0.$$

Properties (definite) integrable functions I

1. If $f \in \mathcal{R}([a,b])$ then it is also integrable on any subinterval of $[a,b]$.
2. If $f \in \mathcal{R}([a,b])$ then also $|f| \in \mathcal{R}([a,b])$.

Properties (definite) integrable functions II

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$$

$$\int_a^b (\alpha f(x) + \beta g(x))\,\mathrm{d}x = \alpha\int_a^b f(x)\,\mathrm{d}x + \beta\int_a^b g(x)\,\mathrm{d}x$$

Properties (definite) integrable functions III

$$f \geq 0,\ a < b \quad \Rightarrow \quad \int_a^b f(x)\,\mathrm{d}x \geq 0 \quad \text{(equality iff } f \equiv 0)$$

$$f \leq g,\ a < b \quad \Rightarrow \quad \int_a^b f(x)\,\mathrm{d}x \leq \int_a^b g(x)\,\mathrm{d}x$$

$$a < b \quad \Rightarrow \quad \left|\int_a^b f(x)\,\mathrm{d}x\right| \leq \int_a^b |f(x)|\,\mathrm{d}x$$

10.7 Integral mean value

Average (mean value)

Let $f \in \mathcal{R}([a, b])$. The **average (mean value) of** f **on** $[a, b]$ is the number

$$m(f; a, b) = \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x$$

it is often denoted

$$⨍_a^b f(x)\,\mathrm{d}x.$$

This can be rewritten as

$$(b-a)m(f; a, b) = \int_a^b f(x)\,\mathrm{d}x.$$

Theorem 10.13 (Integral Mean Value Theorem)**:** Let $f \in \mathcal{R}([a, b])$. Then

$$\inf_{x\in[a,b]} f(x) \leq m(f; a, b) \leq \sup_{x\in[a,b]} f(x).$$

If $f \in C^0([a, b])$ (i.e., if f is continuous) then there exists $z \in [a, b]$ such that

$$m(f; a, b) = f(z).$$

Proof. We skip this proof. □

Example 10.10: Let f be the following *continuous* function on $[0, 2]$:

$$f(x) = \begin{cases} 2x & 0 \leq x \leq 1, \\ 2 & 1 < x \leq 2. \end{cases}$$

Then:

$$\begin{aligned}\int_0^2 f(x)\,\mathrm{d}x &= \int_0^1 f(x)\,\mathrm{d}x + \int_1^2 f(x)\,\mathrm{d}x \\ &= \int_0^1 2x\,\mathrm{d}x + \int_1^2 2\,\mathrm{d}x = 2\int_0^1 x\,\mathrm{d}x + 2\int_1^2 1\,\mathrm{d}x = 1 + 2 = 3.\end{aligned}$$

Therefore
$$m(f;0,2) = \frac{3}{2}$$
and since f is continuous, it follows that there exists $z \in [0,2]$ such that $f(z) = \frac{3}{2}$. Indeed, this is true for $z = \frac{3}{4}$.

Remark: We can change the order of a and b in the definition of the average and the sign remains (since we get two minuses)
$$m(f;a,b) = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x = \frac{1}{a-b}\int_b^a f(x)\,\mathrm{d}x = m(f;b,a).$$
This makes sense: the average of a function shouldn't depend on whether we measure it going on the x axis to the right or to the left.

10.8 Fundamental Theorem of Integral Calculus

Integral function

Let $f:\mathbb{R}\to\mathbb{R}$ be integrable on any finite interval. Fix some $x_0 \in \mathbb{R}$. We define the **integral function of** f to be
$$F(x) = \int_{x_0}^{x} f(y)\,\mathrm{d}y.$$
Observe that the integral function depends on the choice of x_0, *and* $F(x_0) = 0$.

Fundamental Theorem of Integral Calculus

Let $f:\mathbb{R}\to\mathbb{R}$ be defined and continuous on an interval (possibly unbounded) $I \subseteq \mathbb{R}$. Then for any $x_0 \in I$, $F(x)$ is differentiable on I and
$$F'(x) = f(x), \qquad \forall x \in I.$$

Proof. Fix x in the interior of I (i.e. it is not on the boundary of I). Let Δx be small enough (positive or negative) so that $x + \Delta x$ belongs to I. Then, using the definition of F and the fact that
$$F(x+\Delta x) = \int_{x_0}^{x+\Delta x} f(y)\,\mathrm{d}y$$
we have
$$\begin{aligned}\frac{F(x+\Delta x)-F(x)}{\Delta x} &= \frac{1}{\Delta x}\left(\int_{x_0}^{x+\Delta x} f(y)\,\mathrm{d}y - \int_{x_0}^{x} f(y)\,\mathrm{d}y\right)\\ &= \frac{1}{\Delta x}\int_{x}^{x+\Delta x} f(y)\,\mathrm{d}y\\ &= m(f;x,x+\Delta x).\end{aligned}$$

Since f is continuous, the Integral Mean Value Theorem (Theorem 10.13) implies that there exists z between x and $x + \Delta x$ such that $m(f; x, x + \Delta x) = f(z)$. So we have

$$\frac{F(x + \Delta x) - F(x)}{\Delta x} = f(z).$$

Observe that z depends on the choice of Δx, so we should write $z = z(\Delta x)$. Necessarily

$$\lim_{\Delta x \to 0} z(\Delta x) = x$$

(by the Squeeze Theorem). Since f is continuous, we have

$$\lim_{\Delta x \to 0} f(z(\Delta x)) = f(\lim_{\Delta x \to 0} z(\Delta x)) = f(x).$$

Hence we have:

$$F'(x) = \lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x} = \lim_{\Delta x \to 0} f(z(\Delta x)) = f(x).$$

This proves the theorem for the case where x lies in the interior of I. If x is on the boundary then we must take one-sided limits, but the details are very similar. □

This theorem tells us how to define a (specific) antiderivative (depending on our choice of x_0): $F(x) = \int_{x_0}^{x} f(y)\, \mathrm{d}y$. Now we can state a result that links this to any other antiderivative:

Corollary 10.14: If we define $F_{x_0}(x) = \int_{x_0}^{x} f(y)\, \mathrm{d}y$, then

$$F_{x_0}(x) = G(x) - G(x_0)$$

for any other G which is an antiderivative of f.

Proof. The proof is immediate by plugging in $x = x_0$ in the above expression, since we know that all antiderivatives differ by a constant □

Important corollary

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $[a, b]$ and let G be any antiderivative. Then

$$\int_a^b f(x)\, \mathrm{d}x = G(b) - G(a).$$

Proof. Let F_a be the antiderivative defined with the choice $x_0 = a$. Then

$$\int_a^b f(x)\, \mathrm{d}x = F_a(b).$$

By Corollary 10.14 we then further have

$$\int_a^b f(x)\, \mathrm{d}x = F_a(b) = G(b) - G(a)$$

for any antiderivative G. □