First, f needs to be continuous at $x_0 = 0$. So we need:

$$\lim_{x\to 0^-} a\sin(2x) - 4 = -4 \qquad \text{to be equal to} \qquad -b+1$$

Hence $b = 5$. Now we need the left- and right-derivatives to agree.

$$\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} (b + e^x) = 6,$$
$$\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-} 2a\cos(2x) = 2a.$$

For these to be equal, we impose $a = 3$.

8.4 Extrema and critical points

We can now dig deeper into our previous definitions of the supremum, infimum, maximum and minimum of sets.

Local maximum

A point x_0 is a **local maximum point** for f if there exists a neighborhood $I_r(x_0)$ such that

$$f(x) \le f(x_0), \qquad \forall x \in \mathrm{dom}(f) \cap I_r(x_0).$$

Then $f(x_0)$ is a **local maximum** of f.

Global maximum

A point x_0 is a **global maximum point** for f if

$$f(x) \le f(x_0), \qquad \forall x \in \mathrm{dom}(f).$$

Then $f(x_0)$ is the **global maximum** of f. The maximum is **strict** if $f(x) < f(x_0)$ for all $x \neq x_0$.

Local minimum

A point x_0 is a **local minimum point** for f if there exists a neighborhood $I_r(x_0)$ such that

$$f(x) \ge f(x_0), \qquad \forall x \in \mathrm{dom}(f) \cap I_r(x_0).$$

Then $f(x_0)$ is a **local minimum** of f.

Global minimum

A point x_0 is a **global minimum point** for f if

$$f(x) \geq f(x_0), \qquad \forall x \in \mathrm{dom}(f).$$

Then $f(x_0)$ is the **global minimum** of f. The minimum is **strict** if $f(x) > f(x_0)$ for all $x \neq x_0$.

Any one of the points described above will be called an ***extremum point*** of the function.

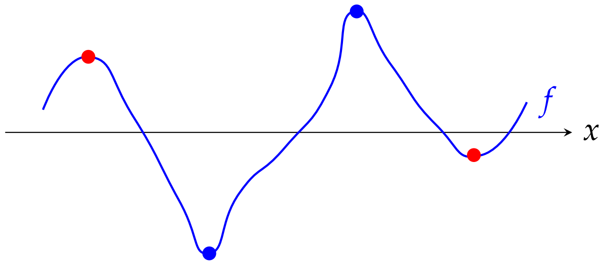

Figure 8.1: Examples of global min/max in blue, and local min/max in red

For a differentiable function, minima and maxima are points where the derivative vanishes (i.e. the tangent is parallel to the x-axis). Points where the derivative vanishes are called *critical points*:

Critical point

A **critical point** of a function $f : \mathbb{R} \to \mathbb{R}$ is a point $x_0 \in \mathbb{R}$ at which f is differentiable and $f'(x_0) = 0$.

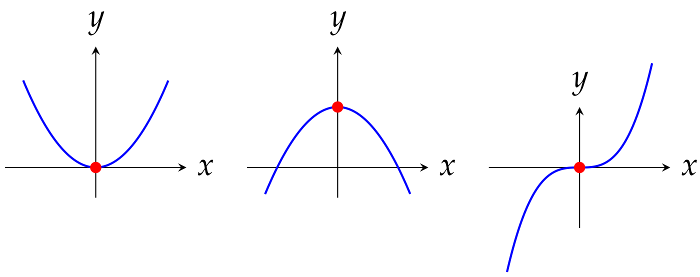

Figure 8.2: Three types of critical points: local minimum (left), local maximum (center), and inflection point (right).

Theorem 8.9 (Fermat's Theorem)**:** If f is differentiable at an extremum point x_0 then $f'(x_0) = 0$.

Proof. Suppose that x_0 is a local maximum. Let $I_r(x_0)$ be a neighborhood on which $f(x) \leq f(x_0)$ for all $x \in I_r(x_0)$. Then within this neighborhood, $\Delta f = f(x) - f(x_0) \leq 0$. We therefore have:

$$\text{The fraction} \quad \frac{f(x) - f(x_0)}{x - x_0} \quad \text{is} \quad \begin{cases} \leq 0 & \text{if } x > x_0 \\ \geq 0 & \text{if } x < x_0 \end{cases}$$

By Proposition 8.7 the left- and right-derivatives at x_0 must equal $f'(x_0)$. The only way this is possible is if they are 0. □

So we see that at an extremum the derivative (if exists) is 0, i.e. it is a critical point. An extremum might also be found at a point where f is not differentiable (think about $|x|$) or at boundary points of the domain (think about $\arcsin x$). So we summarize:

Finding extrema

To find extreme points of a function we must look at the following points:

- Critical points,
- Points where f is not differentiable,
- Points on the boundary of the domain.

Among these we will find the extrema (but we have to check case-by-case).

8.5 The Theorems of Rolle, Lagrange and Cauchy

Theorem 8.10 (Rolle's Theorem)**:** Let $a, b \in \mathbb{R}$, $a < b$, and let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$ then there exists $x_0 \in (a, b)$ with $f'(x_0) = 0$. That is, f has at least one critical point in (a, b).

Proof. From Weierstrass' Theorem, we know that $f([a, b]) = [m, M]$ where

$$m = \min_{x \in [a,b]} f(x) = f(x_m) \qquad \text{and} \qquad M = \max_{x \in [a,b]} f(x) = f(x_M)$$

where x_m is a global minimum for f on $[a, b]$ and x_M is a global maximum for f on $[a, b]$.

If $m = M$ then f is constant on $[a, b]$ so that $f'(x) = 0$ for every $x \in [a, b]$ and the proof is done. Otherwise, $m < M$. Hence we have

$$m \leq f(a) = f(b) \leq M.$$

Since $m < M$, at least one of the $\leq$ above must be a strict inequality.

If $f(a) = f(b) < M$, then x_M cannot be a or b, so $x_M \in (a, b)$. By Fermat's Theorem (Theorem 8.9), since x_M is a differentiable extremum point, $f'(x_M) = 0$ and the proof is done. The case $m < f(a) = f(b)$ follows in a similar way. □

Remark: We have just proven that there is a critical point between a and b. It is important to note that there could be more than one critical point. The proof only shows that there exists *at least* one.

Theorem 8.11 (Mean Value Theorem, a.k.a. Lagrange's Theorem)**:** Let $a, b \in \mathbb{R}$, $a < b$, and let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists a point $x_0 \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(x_0).$$

Every such point x_0 is called a **Lagrange point** for f in (a, b).

Proof. The idea of the proof is to 'tilt' the function so that the values at the endpoints become equal, and we can apply Rolle's Theorem. To this end, define:

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a), \qquad \forall x \in [a, b].$$

Then g is continuous on $[a, b]$, differentiable on (a, b), with

$$g(a) = f(a) \qquad \text{and} \qquad g(b) = f(b) - (f(b) - f(a)) = f(a).$$

Hence Rolle's Theorem can be applied to g. Noting that

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a},$$

we see that $g'(x_0) = 0$ if and only if

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}$$

and the proof is complete. □

Theorem 8.12 (Cauchy's Theorem)**:** Let $a, b \in \mathbb{R}$, $a < b$, and let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Suppose that $g'(x) \neq 0$ for all $x \in (a, b)$. Then there exists a point $x_0 \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x_0)}{g'(x_0)}.$$

Remark: Note that this theorem generalizes the Mean Value Theorem. Indeed, by taking $g(x) = x$ we recover the Mean Value Theorem.

Proof. First we claim the $g(a) \neq g(b)$. Indeed, by contradiction, if those values were equal, then Rolle's Theorem would imply that there exists some $x_0 \in (a, b)$ such that $g'(x_0) = 0$, contrary to our assumption. Define

$$h(x) = f(x) - \frac{f(b) - f(a)}{g(b) - g(a)}(g(x) - g(a)), \qquad \forall x \in [a, b].$$

Then h is continuous on $[a, b]$, differentiable on (a, b), with

$$h(a) = f(a) \qquad \text{and} \qquad h(b) = f(b) - (f(b) - f(a)) = f(a).$$

Hence Rolle's Theorem can be applied to h. Noting that

$$h'(x) = f'(x) - \frac{f(b) - f(a)}{g(b) - g(a)}g'(x),$$

we see that $h'(x_0) = 0$ if and only if

$$f'(x_0) = \frac{f(b) - f(a)}{g(b) - g(a)}g'(x_0)$$

which completes the proof. □