MATHEMATICAL ANALYSIS 1
HOMEWORK 2

(1) Prove Newton's binomial formula. *Hint: prove by induction.* In your proof you may also use the formula:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

(2) Let $f : X \to Y$ be a function between two sets X and Y.
   (a) Prove that $A \subseteq f^{-1}(f(A))$ for any $A \subseteq X$.
   (b) Give an example of when $A \neq f^{-1}(f(A))$.

(3) Describe the following subsets of $\mathbb{R}$, specify what are their infimum and supremum, and determine whether their minimum and maximum are attained (explain your answers).
   (a) $A = \{x \in \mathbb{R} \mid (x+1)(x-1)(x-5) < 0\} \cap \{x \in \mathbb{R} \mid \frac{3x+1}{x-2} \geq 0\}$
   (b) $B = \{x \in \mathbb{R} \mid x - 4 \geq \sqrt{x^2 - 6x + 5}\} \cup \{x \in \mathbb{R} \mid x + 2 > \sqrt{x-1}\}$
   (c) $C = \{x \in \mathbb{R} \mid x = \frac{1}{n-2},\ n = 3, 4, 5, \dots\}$

(4) Describe and sketch the following subsets of $\mathbb{R}^2$:
   (a) $A = \{(x, y) \in \mathbb{R}^2 \mid xy \geq 0\}$
   (b) $B = \{(x, y) \in \mathbb{R}^2 \mid 1 + xy > 0\}$
   (c) $C = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 > 0\}$
   (d) $D = \{(x, y) \in \mathbb{R}^2 \mid x - y \neq 0\}$
   (e) $E = \{(x, y) \in \mathbb{R}^2 \mid |x - y| < 2\}$
   (f) $F = \{(x, y) \in \mathbb{R}^2 \mid |x - y| < -2\}$

(5) Describe the following sets: (explain your answers)
   (a) $\mathrm{dom}\,(f)$ where $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = \frac{1}{\sin x}$.
   (b) $\mathrm{im}\,(f)$ where $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = \frac{1}{\sin x}$.
   (c) $\mathrm{dom}\,(f)$ where $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = \frac{1}{2+\sin x}$.
   (d) $\mathrm{dom}(f)$ where $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = 10^x$.
   (e) $\mathrm{im}(f)$ where $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = 10^x$.
   (f) $\mathrm{dom}(f)$ where $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = \log_{10}(x)$.
   (g) $\mathrm{im}(f)$ where $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = \log_{10}(x)$.

(6) For the following functions f and subsets $B \subseteq \mathbb{R}$, describe $f^{-1}(B)$.
   (a) $f : \{3, 4, 5, \dots\} \to \mathbb{R}$ defined by $f(n) = \frac{1}{n-2}$, and $B = (0, 1)$.
   (b) In the previous question, what if $B = [0, 1]$?
   (c) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^4$, $B = [1, 16)$.
   (d) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 + 1$, $B = (-26, -7]$.
   (e) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \cos x$, $B = \{0\}$.

(7) Compute $\frac{100!}{98!}$. Explain your answer.

Solutions

(1) We prove by induction on n that $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$. For $n = 0$, we have $(a+b)^0 = 1 = \binom{0}{0} a^0 b^0$. Assume the formula holds for $n-1$. Then

$$\begin{aligned}
(a+b)^n = (a+b)(a+b)^{n-1} &= (a+b)\sum_{k=0}^{n-1} \binom{n-1}{k} a^k b^{n-1-k} \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} a^{k+1} b^{n-1-k} + \sum_{k=0}^{n-1} \binom{n-1}{k} a^k b^{n-k} \\
&= \sum_{k=1}^{n} \binom{n-1}{k-1} a^k b^{n-k} + \sum_{k=0}^{n-1} \binom{n-1}{k} a^k b^{n-k} \\
&= a^n + \sum_{k=1}^{n-1} \left(\binom{n-1}{k-1} + \binom{n-1}{k} \right) a^k b^{n-k} + b^n \\
&= \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k},
\end{aligned}$$

where we used the given formula $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ for $1 \le k \le n-1$.

(2) (a) Let $x \in A$. Then $f(x) \in f(A)$, so $x \in f^{-1}(f(A))$.

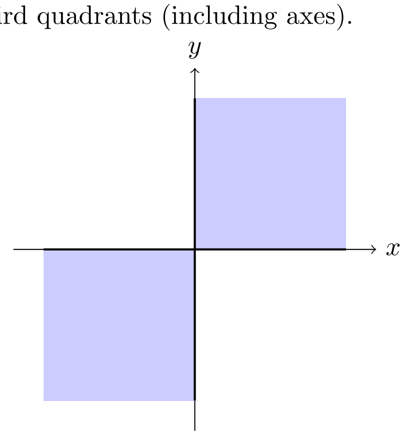
(b) Let $X = \{1, 2\}$, $Y = \{a\}$, $f(1) = f(2) = a$, and $A = \{1\}$. Then $f(A) = \{a\}$ and $f^{-1}(f(A)) = f^{-1}(\{a\}) = \{1, 2\} \ne A$.

(3) (a) $(x+1)(x-1)(x-5) < 0$ gives $x \in (-\infty, -1) \cup (1, 5)$. For $\frac{3x+1}{x-2} \ge 0$, we need $(3x+1)(x-2) \ge 0$ and $x \ne 2$, giving $x \in (-\infty, -\frac{1}{3}] \cup (2, \infty)$. Thus $A = (-\infty, -1) \cup (2, 5)$. We have $\inf A = -\infty$, $\sup A = 5 \notin A$. The minimum is not attained, the maximum is not attained.

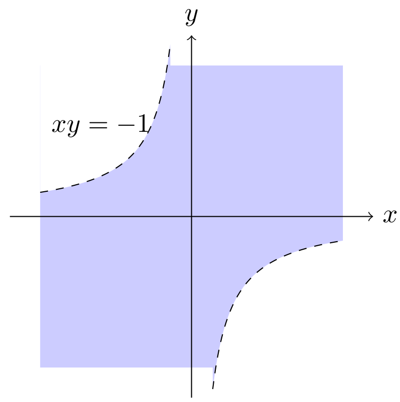
(b) For $x - 4 \ge \sqrt{x^2 - 6x + 5}$, we need $x^2 - 6x + 5 \ge 0$ (so $x \le 1$ or $x \ge 5$) and $(x-4)^2 \ge x^2 - 6x + 5$, giving $x \ge \frac{21}{8}$. So the first set is $[\frac{21}{8}, \infty)$. For $x + 2 > \sqrt{x-1}$, we need $x \ge 1$ and $(x+2)^2 > x - 1$, giving $x^2 + 3x + 5 > 0$, which holds for all $x \ge 1$. Thus $B = [1, \infty)$, with $\inf B = \min B = 1$, $\sup B = +\infty$.

(c) $C = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$. We have $\inf C = 0 \notin C$, $\sup C = \max C = 1 \in C$.

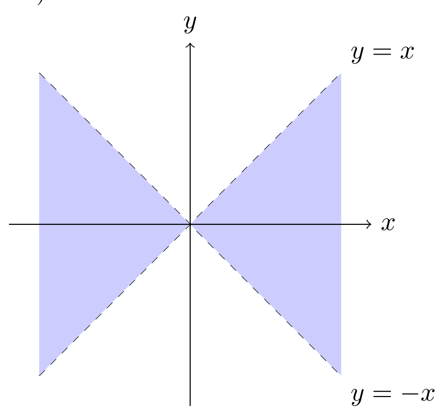
(4) (a) A consists of the first and third quadrants (including axes).



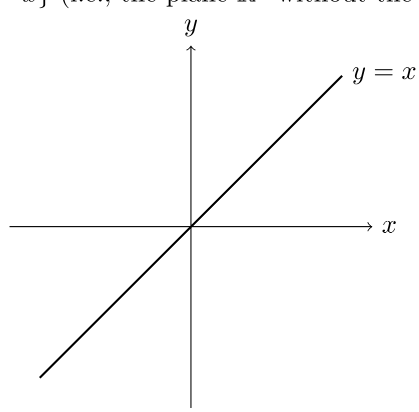
(b) B is the region between the two branches of the hyperbola $xy = -1$ (excluding the hyperbola itself)



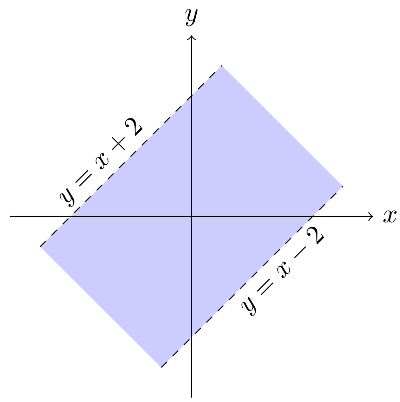
(c) C consists of regions where $|x| > |y|$, i.e., the regions to the left and right of the lines $y = \pm x$ (excluding the lines themselves).



(d) $D = \mathbb{R}^2 \setminus \{(x, y) \in \mathbb{R}^2 \quad | \quad y = x\}$ (i.e., the plane $\mathbb{R}^2$ without the line $y = x$).



(e) E is the strip between the parallel lines $y = x - 2$ and $y = x + 2$ (excluding these lines).



(f) $F = \emptyset$ since $|x - y| \geq 0$ cannot be < -2.

(5) (a) $\mathrm{dom}(f) = \mathbb{R} \setminus \{k\pi \mid k \in \mathbb{Z}\}$ (where $\sin x \neq 0$).

(b) $\mathrm{im}(f) = (-\infty, -1] \cup [1, \infty)$ (since $|\sin x| \leq 1$ with equality attained).

(c) $\mathrm{dom}(f) = \mathbb{R}$ (since $2 + \sin x \geq 1 > 0$ for all x).

(d) $\mathrm{dom}(f) = \mathbb{R}$.

(e) $\mathrm{im}(f) = (0, \infty)$.

(f) $\mathrm{dom}(f) = (0, \infty)$.

(g) $\mathrm{im}(f) = \mathbb{R}$.

(6) (a) We need $\frac{1}{n-2} \in (0, 1)$, so $\frac{1}{n-2} > 0$ and $\frac{1}{n-2} < 1$. This gives $n > 2$ and $n > 3$, so $n \geq 4$. Thus $f^{-1}(B) = \{4, 5, 6, \dots\}$.

(b) $f^{-1}([0, 1]) = \{3, 4, 5, \dots\}$ (including $n = 3$ where $f(3) = 1$).

(c) $x^4 \in [1, 16)$ gives $|x| \in [1, 2)$, so $f^{-1}(B) = (-2, -1] \cup [1, 2)$.

(d) $x^3 + 1 \in (-26, -7]$ gives $x^3 \in (-27, -8]$, so $x \in (-3, -2]$. Thus $f^{-1}(B) = (-3, -2]$.

(e) $\cos x = 0$ when $x = \frac{\pi}{2} + k\pi$ for $k \in \mathbb{Z}$. Thus $f^{-1}(\{0\}) = \{\frac{\pi}{2} + k\pi \mid k \in \mathbb{Z}\}$.

(7) $\frac{100!}{98!} = \frac{100 \cdot 99 \cdot 98!}{98!} = 100 \cdot 99 = 9900$.