MATHEMATICAL ANALYSIS 1
HOMEWORK 6

(1) In this problem we prove the **Squeeze Theorem** for a finite point x_0, and three functions f, g, h satisfying $f \le g \le h$ near x_0, with f and h having the limit ℓ as $x \to x_0$ (you are guided in the steps below):
  (a) State the theorem (and state that you shall prove it only in the case $x_0 \in \mathbb{R}$).
  (b) Fix $\varepsilon > 0$.
  (c) With this ε write the definition of what it means that $\lim_{x\to x_0} f(x) = \ell$.
  (d) Write the definition of what it means that $\lim_{x\to x_0} h(x) = \ell$.
  (e) Using the previous two steps, find a neighborhood of x_0 (depending on ε) for which you can write a condition for convergence to ℓ for the function g.
  (f) Conclude that, since $\varepsilon > 0$ was arbitrary, the theorem follows by the definition of the limit (applied to g).

(2) Prove that
$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$$
Hint: multiply both the numerator and denominator by $1 + \cos x$*.*

(3) Compute the following limits:

  (a) $\lim_{x\to+\infty} \frac{\cos x}{\sqrt{x}}$
  (b) $\lim_{x\to+\infty} \frac{\lfloor x \rfloor}{x}$
  (c) $\lim_{x\to 0} \frac{x-\tan x}{x^2}$ *(hint: Squeeze Theorem)*
  (d) $\lim_{x\to e} \frac{\ln x - 1}{x-e}$ *(hint: take* $y = x - e$*)*
  (e) $\lim_{x\to+\infty} \frac{x+3}{x^3-2x+5}$
  (f) $\lim_{x\to 0} \frac{\sin^2 x}{x}$
  (g) $\lim_{x\to 1} \frac{\cos(\frac{\pi}{2}x)}{1-x}$ *(hint: take* $y = 1 - x$*)*
  (h) $\lim_{x\to 0} \frac{\sqrt{1+\tan x}-\sqrt{1-\tan x}}{\sin x}$
  (i) $\lim_{x\to 0+} \frac{2^{2x}-1}{2x}$
  (j) $\lim_{x\to 1} \frac{\ln x}{e^x - e}$ *(hint: take* $y = x - 1$*)*
  (k) $\lim_{x\to 0} \frac{e^x - e^{-x}}{\sin x}$
  (l) $\lim_{x\to -\infty} x e^{\sin x}$

(4) Determine how the following sequences $\{a_n\}_{n\in\mathbb{N}}$ behave for large n:

  (a) $a_n = n - \sqrt{n}$
  (b) $a_n = \frac{(2n)!}{n!}$
  (c) $a_n = \frac{(2n)!}{(n!)^2}$
  (d) $a_n = \binom{n}{3}\frac{6}{n^3}$
  (e) $a_n = 2^n \sin(2^{-n}\pi)$
  (f) $a_n = n \cos\left(\frac{n+1}{n} \cdot \frac{\pi}{2}\right)$

(5) Use the fact that $\lim_{x\to\pm\infty}(1 + \frac{1}{x})^x = e$ to prove that for $a \neq 0$
$$\lim_{x\to\pm\infty} \left(1 + \frac{a}{x}\right)^x = e^a.$$

(6) Let $f, g : \mathbb{R} \to \mathbb{R}$. Show that $f \sim g$ as $x \to x_0$ if and only if $f = g + o(g)$ as $x \to x_0$.

(7) Let $f : \mathbb{R} \to \mathbb{R}$ be infinite or infinitesimal at x_0.
  (a) State the definition of the *order* α of f at x_0 with respect to a function $\varphi : \mathbb{R} \to \mathbb{R}$.
  (b) Prove that the order α is unique.

(8) Determine the order and the principal part with respect to $\varphi(x) = \frac{1}{x}$ as $x \to +\infty$ of the function $f(x) = \sin(\sqrt{x^2-1} - x)$.

(9) As $x \to +\infty$, the function $f(x) = \ln(9 + \sin \frac{2}{x}) - 2\ln 3$ can be written as $f(x) = \frac{b}{x^\alpha} + o(x^{-\alpha})$. Find b and α.

(10) Determine the order and the principal part with respect to $\varphi(x) = x$ as $x \to 0$ of the function $f(x) = \frac{e^x}{1+x^2} - 1$.