

Convergence tests

Now we give some criteria for the convergence/divergence of improper integrals of type I.

Comparison Test

Fix $a \in \mathbb{R}$. Let f, g be integrable on $[a, b]$ for any $b > a$. Assume that $0 \leq f(x) \leq g(x)$ for all $x \in [a, +\infty)$. Then

$$0 \leq \int_a^{+\infty} f(x) dx \leq \int_a^{+\infty} g(x) dx$$

Therefore

$$\begin{aligned} \int_a^{+\infty} g(x) dx < +\infty & \text{ (converges)} & \Rightarrow & \int_a^{+\infty} f(x) dx < +\infty & \text{ (converges)} \\ \int_a^{+\infty} f(x) dx = +\infty & \text{ (diverges)} & \Rightarrow & \int_a^{+\infty} g(x) dx = +\infty & \text{ (diverges)} \end{aligned}$$

Example 11.2: We determine whether the integrals

$$\int_1^{+\infty} \frac{\arctan x}{x^2} dx \quad \text{and} \quad \int_1^{+\infty} \frac{\arctan x}{x} dx$$

converge or diverge. Recall that $\arctan 1 = \frac{\pi}{4}$ and $\arctan x$ is a strictly increasing function, with $\lim_{x \rightarrow +\infty} \arctan x = \frac{\pi}{2}$. Hence

$$\frac{\pi}{4} \leq \arctan x \leq \frac{\pi}{2}, \quad \forall x \in [1, +\infty).$$

It follows that for all $x \in [1, +\infty)$,

$$\frac{\arctan x}{x^2} \leq \frac{\pi}{2} \frac{1}{x^2} \quad \text{and} \quad \frac{\pi}{4} \frac{1}{x} \leq \frac{\arctan x}{x}$$

We therefore have that

$$\int_1^{+\infty} \frac{\arctan x}{x^2} dx \leq \frac{\pi}{2} \int_1^{+\infty} \frac{1}{x^2} dx < +\infty \quad (\text{converges})$$

and

$$\int_1^{+\infty} \frac{\arctan x}{x} dx \geq \frac{\pi}{4} \int_1^{+\infty} \frac{1}{x} dx = +\infty \quad (\text{diverges})$$

If $f = O(g)$

If $f = O(g)$ are non-negative as $x \rightarrow +\infty$, then if the improper integral (type I) of g converges, then so does the integral of f .

Example 11.3: Since $e^{-x^2} = o(x^{-2})$ as $x \rightarrow +\infty$, we deduce that $\int_0^{+\infty} e^{-x^2} dx < +\infty$ (*converges*).

Absolute Convergence Test

Fix $a \in \mathbb{R}$. Suppose that both f and $|f|$ are integrable on any interval $[a, b]$ for any $b > a$. Then if the improper integral (type I) of $|f|$ converges, so does the integral of f , and

$$\left| \int_a^{+\infty} f(x) dx \right| \leq \int_a^{+\infty} |f(x)| dx.$$

Example 11.4: Consider the function $f(x) = \frac{\cos x}{x^2}$. Then $|f(x)| \leq \frac{1}{x^2}$. So $|f(x)|$ is integrable on $[1, +\infty)$ by the Comparison Test. It follows that $f(x)$ is integrable by the Absolute Convergence Test:

$$\left| \int_1^{+\infty} \frac{\cos x}{x^2} dx \right| \leq \int_1^{+\infty} \left| \frac{\cos x}{x^2} \right| dx \leq \int_1^{+\infty} \frac{1}{x^2} dx = 1.$$

Remark: The converse is not necessarily true:

$$\int_a^{+\infty} f(x) dx < +\infty \quad \text{does not imply that} \quad \int_a^{+\infty} |f(x)| dx < +\infty.$$

Asymptotic Comparison Test

Suppose that $f \sim \frac{1}{x^\alpha}$ as $x \rightarrow +\infty$. Then:

$$\begin{aligned} \alpha > 1 &\Rightarrow \int_a^{+\infty} f(x) dx < +\infty \text{ (*converges*)} \\ \alpha \leq 1 &\Rightarrow \int_a^{+\infty} f(x) dx = +\infty \text{ (*diverges*)} \end{aligned}$$

Example 11.5: Investigate

$$\int_1^{+\infty} (\pi - 2 \arctan x) dx = ?$$

We know that $\lim_{x \rightarrow +\infty} \arctan x = \frac{\pi}{2}$, so that the integrand $f(x) = \pi - 2 \arctan x$ tends to 0. Let's determine its order at $+\infty$ using De l'Hôpital:

$$\lim_{x \rightarrow +\infty} \frac{\pi - 2 \arctan x}{\frac{1}{x}} = \lim_{x \rightarrow +\infty} \frac{2x^2}{1+x^2} = 2.$$

Therefore $f(x) \sim \frac{2}{x}$ as $x \rightarrow +\infty$ and the integral diverges using the Asymptotic Comparison Test.