The Riccati and Bernoulli equations are *first-order non-linear* equations that frequently appear in control theory and fluid dynamics.

Second-order equations like those of Bessel, Legendre, and Hermite are essential for solving partial differential equations in cylindrical, spherical, or quantum mechanical domains, giving rise to "Special Functions" that describe everything from electromagnetic wave propagation to atomic orbitals.

Finally, *second order non-linear* models like the Van der Pol oscillator illustrate complex behaviors such as limit cycles, which arise in self-sustaining biological and electronic systems.

12.1 First-order differential equations in normal form

A first-order ODE which can be written in normal form is

$$y' = f(x, y).$$

The function $f(x, y)$ associates to each point $(x, y) \in \mathbb{R}^2$ in the plane a number which is the slope of the solution at that point. This can be visualized as a 'direction field', see Figure 12.1.

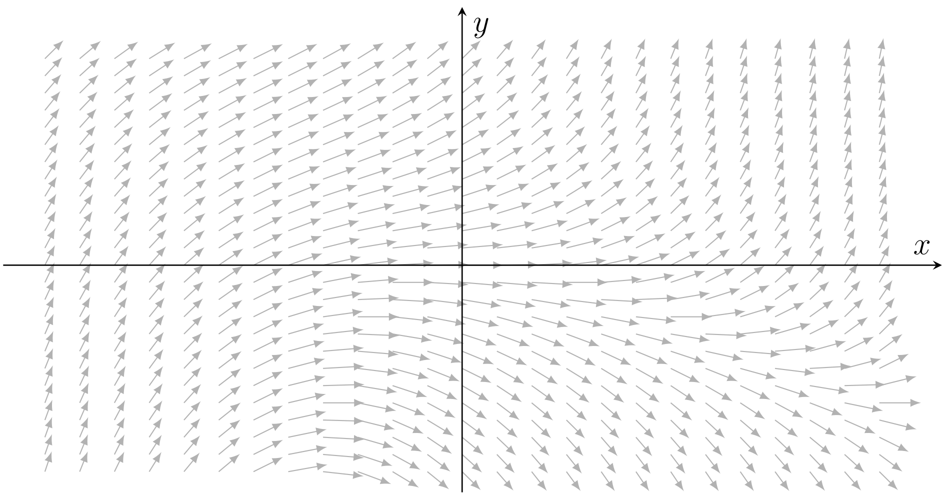

Figure 12.1: Direction field for the ODE $y' = (1 + x)y + x^2$.

By choosing a starting point $(x_0, y_0) \in \mathbb{R}^2$ in the plane and following the arrows, we construct a solution $y = y(x)$. Different starting points will lead to different solutions. This leads us to consider the *initial-value problem*.

Initial-value problem

For the normal form ODE $y' = f(x, y)$, the **initial-value problem (IVP)** (also called the **Cauchy problem**) for an interval $I \subseteq \mathbb{R}$ is

$$\begin{cases} y' = f(x, y) & \text{in } I, \\ y(x_0) = y_0, \end{cases}$$

where $x_0 \in I$ and $y_0 \in \mathbb{R}$.

For the initial-value problem presented above, the solution is obtained by following the arrows starting from the point (x_0, y_0) and drawing the resulting curve for as long as possible. But that is not always possible. Sometimes we might be led by the arrows in various directions, leading to different solutions. The following theorem tells us when the arrows will lead us in a unique direction, starting from (x_0, y_0).

Existence and uniqueness of solutions: Picard-Lindlöf Theorem

Consider the initial value problem:

$$\begin{cases} y' = f(x,y) & \text{in } I, \\ y(x_0) = y_0, \end{cases}$$

where $x_0 \in I$ and $y_0 \in \mathbb{R}$. Suppose $f(x,y)$ is a continuous function in a rectangular region R defined by $|x - x_0| \leq a$ and $|y - y_0| \leq b$ where $a, b > 0$ and $[x_0 - a, x_0 + a] \subseteq I$. If f is Lipschitz with respect to y in R, i.e. if there exists a constant $L > 0$ such that:

$$|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2| \qquad \forall (x, y_1), (x, y_2) \in R$$

then there exists $h > 0$ such that there **exists** a **unique** solution $y(x)$ to the IVP on $[x_0 - h, x_0 + h]$. Moreover, the solution is *stable*, in the sense that a small perturbation of the initial value (x_0, y_0) will result in a small perturbation of the solution curve $y(x)$ (there's no 'butterfly effect').

We do not pursue the proof here, as it is beyond the scope of this course. This theorem generalizes to higher order ODEs that are not in normal form, but we omit this here.

Remark: The Picard-Lindelöf Theorem is crucial for modeling. By requiring the function $f(x,y)$ to be Lipschitz continuous, we ensure that the system's evolution is uniquely determined by its initial state. For linear equations, this condition is naturally satisfied if the coefficients are continuous. However, in non-linear analysis, checking the Lipschitz constant L is vital to ensure that numerical simulations – like those using Runge-Kutta methods – converge to a single, physically meaningful result rather than branching into multiple mathematical possibilities. See Example 12.5 for an example of an IVP with three possible solutions.

Equations with separable variables: $f(x,y) = h(x)g(y)$

Here is a general strategy for solving the IVP

$$\begin{cases} y' = f(x,y) & \text{in } I, \\ y(x_0) = y_0, & x_0 \in I, \end{cases}$$

in the case where $f(x,y) = h(x)g(y)$:

1. In the expression for f separate the x variable and the y variable, writing $f(x,y) = h(x)g(y)$ (note that this can only be done in special cases), so that the ODE becomes
$$\frac{\mathrm{d}y}{\mathrm{d}x} = h(x)g(y).$$
[Note: any $\tilde{y} \in \mathbb{R}$ for which $g(\tilde{y}) = 0$ will provide the (uninteresting) solution of the ODE: $y(x) = \tilde{y}$ for all x.]

2. For all y for which $g(y) \neq 0$, we can move $g(y)$ to the left hand side and the $\mathrm{d}x$ to the right hand side (this is a formal step, lacking rigor, but practical) to obtain
$$\frac{\mathrm{d}y}{g(y)} = h(x)\,\mathrm{d}x.$$

3. Compute the indefinite integrals of both sides (one involving only x and the other involving only y):
$$\int \frac{\mathrm{d}y}{g(y)} = \int h(x)\,\mathrm{d}x$$
to obtain a relation between y and x. In this step a constant C appears as well. We call it a *degree of freedom* of the equation.

4. Impose the condition $y(x_0) = y_0$ in order to determine the value of C.

Example 12.2: Let us solve the IVP
$$\begin{cases} y' = -5y \\ y(10) = 3 \end{cases}$$
in $\mathbb{R}$. Following the strategy we write
$$\frac{\mathrm{d}y}{y} = -5\,\mathrm{d}x$$
and take the antiderivatives of both sides to get
$$\ln y = -5x + C.$$
Taking the exponential of both sides we have
$$y(x) = e^{-5x+C} = e^{C}e^{-5x}.$$
Plugging the initial condition:
$$3 = e^{C}e^{-50} \qquad \Rightarrow \qquad e^{C} = 3e^{50}.$$
So the solution is:
$$y(x) = 3e^{50}e^{-5x}$$
which is valid for all $x \in \mathbb{R}$.

Example 12.3: Solve the IVP

$$\begin{cases} y' = \frac{e^x+1}{e^y+1} \\ y(2) = 3. \end{cases}$$

Following the strategy we have

$$(e^y + 1)\,\mathrm{d}y = (e^x + 1)\,\mathrm{d}x$$

and take the antiderivatives of both sides to get

$$e^y + y = e^x + x + C.$$

Plugging the initial condition:

$$e^3 + 3 = e^2 + 2 + C \qquad \Rightarrow \qquad C = e^3 - e^2 + 1.$$

So the solution is given implicitly by:

$$e^y + y = e^x + x + e^3 - e^2 + 1$$

which is valid for all $x \in \mathbb{R}$. Note that because the equation $e^y + y = f(x)$ is transcendental, it is not possible to express $y(x)$ explicitly in terms of elementary functions.

Example 12.4 (IVP with finite-time 'blow up')**:** Solve the IVP

$$\begin{cases} y' = y^2 \\ y(1) = 5. \end{cases}$$

Following the strategy we have

$$\frac{\mathrm{d}y}{y^2} = \mathrm{d}x.$$

Taking antiderivatives:

$$-\frac{1}{y} = x + C.$$

Rearranging we get

$$y(x) = -\frac{1}{x+C}$$

which is not defined for $x = -C$ (vertical asymptote). Plugging in the initial condition we have:

$$5 = -\frac{1}{1+C} \qquad \Rightarrow \qquad C = -\frac{6}{5}$$

so that our solution is

$$y(x) = \frac{1}{\frac{6}{5} - x}$$

and is valid for all $x < \frac{6}{5}$ (which is the part of the domain that includes $x_0 = 1$). This solution (as well as the direction field) is visualized in Figure 12.2. In this case the solution does not exist for all $x \in \mathbb{R}$: it *blows up* as $x \to \frac{6}{5}$ from the left.

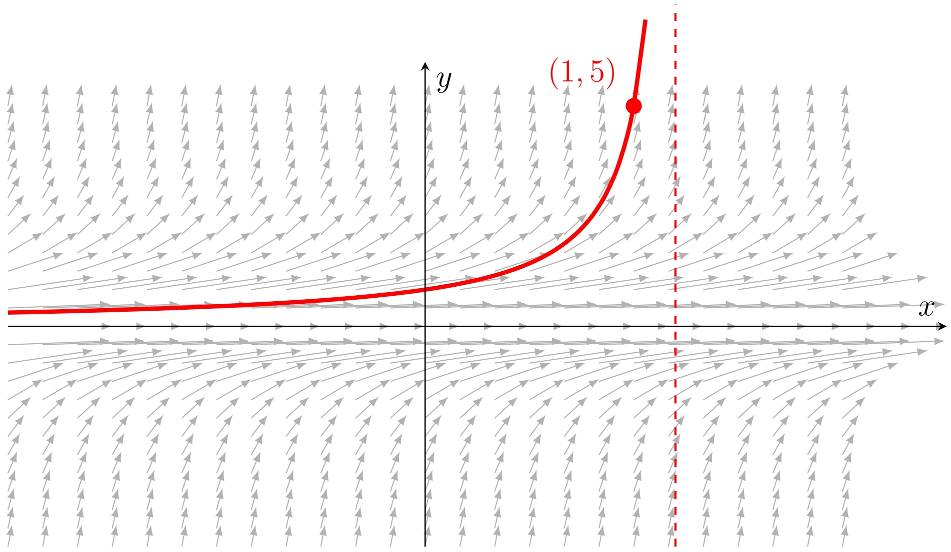

Figure 12.2: Direction field for the ODE $y' = y^2$ with particular solution $y(x) = \frac{1}{\frac{6}{5}-x}$ passing through $(1, 5)$.

Example 12.5 (IVP with more than one solution)**:** The IVP

$$\begin{cases} y' = y^{1/3} \\ y(0) = 0 \end{cases}$$

doesn't satisfy the Lipschitz requirement appearing in the Picard-Lindlöf theorem (the function $y^{1/3}$ is not Lipschitz at $y = 0$). As a result it does not have a unique solution. Let us show this:

- The trivial solution
$$y_1(x) = 0, \qquad \forall x \in \mathbb{R},$$
satisfies the IVP: $y'(x) = 0 = y^{1/3}(x)$ for all $x \in \mathbb{R}$.

- We can find another solution using our recipe from before. We rewrite the ODE as
$$\frac{\mathrm{d}y}{y^{1/3}} = \mathrm{d}x$$
and integrate, to get
$$\frac{3}{2}y^{2/3} = x + C.$$
The initial condition implies that $C = 0$ so that a second solution is
$$y_2(x) = \left(\frac{2}{3}x\right)^{3/2}, \qquad \forall x \geq 0.$$

- In fact, there's a third solution with a negative sign:
$$y_3(x) = -\left(\frac{2}{3}x\right)^{3/2}, \qquad \forall x \geq 0.$$

The three solutions y_1, y_2 and y_3 are all legitimate solutions, and we have no method to label any one of them as ‘correct’ or as ‘incorrect’. See Figure 12.3 for a visualization.

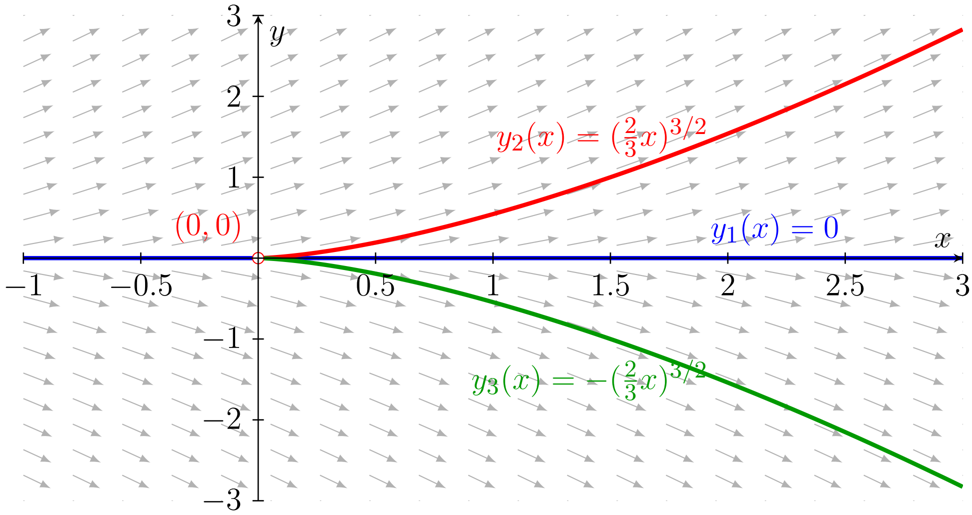

Figure 12.3: Direction field for $y' = y^{1/3}$ showing non-uniqueness at $(0, 0)$.

12.2 Second-order ODEs

12.2.1 The pendulum

In this section we demonstrate how Newton’s Second Law $\mathbf{F} = m\mathbf{a}$ translates into a second-order differential equation in the case of a *physical pendulum* as in Figure 12.4.

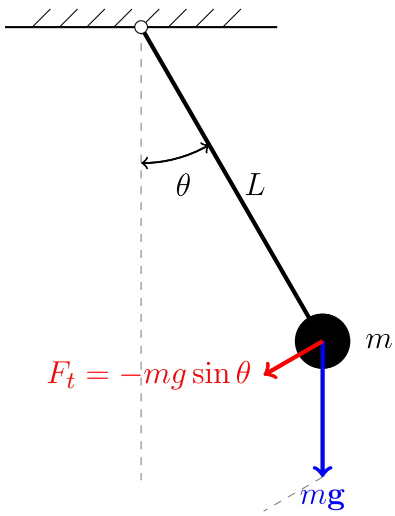

Figure 12.4: The physical pendulum. The restoring force is $mg\sin\theta$.

To derive the equation of motion for a physical pendulum of mass m and length L, we apply **Newton’s Second Law** for rotation about the pivot point:

$$\tau = I\alpha$$

where τ is the *torque* (this is the force), I is the *moment of inertia* (this plays the role of the mass), and α is the *angular acceleration* (this is the acceleration).