ON THE SOLVABILITY COMPLEXITY INDEX HIERARCHY AND TOWERS OF ALGORITHMS

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1. INTRODUCTION

This paper resolves the longstanding computational spectral problem. That is to determine the existence of algorithms that can compute spectra $\mathrm{Sp}(A)$ of classes of bounded operators $A = \{a_{ij}\}_{i,j\in\mathbb{N}} \in \mathcal{B}(l^2(\mathbb{N}))$, given the matrix elements $\{a_{ij}\}_{i,j\in\mathbb{N}}$, that are sharp in the sense that they achieve the boundary of what a digital computer can achieve. Similarly, for a Schrödinger operator $H = -\triangle + V$, determine the existence of algorithms that can compute the spectrum $\mathrm{Sp}(H)$ given point samples of the potential function V. In order to solve the problems we establish the Solvability Complexity Index (SCI) hierarchy. This is a classification hierarchy for all types of problems in computational mathematics that allows for classifications determining the boundaries of what computers can achieve in scientific computing. In addition, the SCI hierarchy provides classifications of computational problems that can be used in computer assisted proofs, see §1.3.

The SCI hierarchy captures many key computational issues in the history of mathematics including the insolvability of the quintic, Smale's problem on the existence of iterative generally convergent algorithm for polynomial root finding, the computational spectral problem, inverse problems, optimisation etc, and even mathematical logic (although this is not a paper on logic and computer science).

Given the many applications in mathematical physics, quantum chemistry, statistical mechanics, quantum mechanics, quasicrystals, optics and many other fields, the problem of computing spectra of infinite-dimensional operators has fascinated and frustrated mathematicians for several decades since the beginning of the 1950s. W. Arveson [5] pointed out in the early 1990s that: "*Unfortunately, there is a dearth of literature on this basic problem, and so far as we have been able to tell, there are no proven techniques*". Arveson considered the problem of computing spectra from matrix elements $\{a_{ij}\}_{i,j\in\mathbb{N}} \in \mathcal{B}(l^2(\mathbb{N})$ of the operators, however, the situation is not better for the Schrödinger case. In particular, despite more than 90 years of quantum mechanics, it is still unknown how to compute spectra of Schrödinger operators $-\triangle + V$ on $L^2(\mathbb{R}^d)$ given point samples from the potential function V, even when V is bounded and smooth.

We solve these problems by providing a collection of algorithms that allow for problems that before were out of reach and provide lower bounds yielding sharp classifications. The results may be surprising and link to many areas of mathematics.

Classifications and new algorithms: The SCI hierarchy induces a total ordering $\leq_{\mathrm{SCI}}$ on the family of computational spectral problems describing their difficulty. This yields, for example, given infinite matrices of the form $A = \{a_{ij}\}_{i,j\in\mathbb{N}} \in \mathcal{B}(l^2(\mathbb{N}))$, the following ordering.

$$
\begin{aligned}
\text{Computing } \mathrm{Sp}(A),\ A \text{ is diagonal} &=_{\mathrm{SCI}} \text{computing } \mathrm{Sp}(-\triangle + V) \text{ with bounded } V \\
&<_{\mathrm{SCI}} \text{computing } \mathrm{Sp}(A),\ A \text{ is compact} \\
&=_{\mathrm{SCI}} \text{computing } \mathrm{Sp}(-\triangle + V) \text{ with } V \text{ blowing up at } \infty \\
&<_{\mathrm{SCI}} \text{computing } \mathrm{Sp}(A),\ A \text{ is self-adjoint.}
\end{aligned} \tag{1.1}
$$

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Indeed, (1.1) demonstrates that computing spectra of Schrödinger operators (the first equality holds even for many non-Hermitian cases) on $L^2(\mathbb{R}^d)$ from point samples of a bounded potential function V is not harder than computing the spectrum of a diagonal infinite matrix, the easiest of infinite-dimensional spectral problems. Paradoxically, the problem of computing spectra of compact operators, for which the method has been known for decades, is strictly harder than the problem of computing spectra of such Schrödinger operators, which has been open for more than half a century. Our results finally solves this problem.

Higher part of the SCI hierarchy - why algorithms were not found: In order to compute spectra or essential spectra of arbitrary infinite matrices one needs three limits in the computation, and it is impossible with two limits - these problems are very high up in the SCI hierarchy. In particular, there does exist a family of algorithms $\{\Gamma_{n_3,n_2,n_1}\}$ such that for all $A = \{a_{ij}\}_{i,j\in\mathbb{N}} \in \mathcal{B}(l^2(\mathbb{N})$,

$$\lim_{n_k\to\infty} \lim_{n_3\to\infty} \lim_{n_1\to\infty} \Gamma_{n_3,n_2,n_1}(A) = \mathrm{Sp}(A).$$

Yet, for any family of algorithms $\{\Gamma_{n_2,n_1}\}$ based on two limits there is an A such that

$$\lim_{n_2\to\infty} \lim_{n_1\to\infty} \Gamma_{n_2,n_1}(A) \neq \mathrm{Sp}(A).$$

In the self-adjoint case, however, one needs two limits. This explains Arveson's comment, why there have been no known techniques for the general cases in the standard literature, and why it has taken substantial time to resolve the computational spectral problem. Indeed, the classical techniques in spectral computation is traditionally concerned with algorithms based on one limit. By the results above, algorithms based on one limit can never capture the general problem even in the self-adjoint case. However, results in the classical literature typically yield invaluable classification results in the lower part of the SCI hierarchy.

Computer assisted proofs: As we point out in §1.3, the recent proof of Kepler's conjecture (Hilbert's 18th problem) [55, 56], led by T. Hales, is a striking example of a computer assisted proof relying on computing undecidable problems. This may seem paradoxical, however, as the SCI hierarchy reveals and explains, there are many computational problems that are undecidable, or non-computable, that still can be used in computer assisted proofs. Another example of non-computable problems used in computer assisted proofs is the Dirac-Schwinger conjecture in mathematical physics established by C. Fefferman and L. Seco [39–47]. The SCI hierarchy provides classes of computational problems that can be utilised for computer assisted proofs, and explains why, for example, Kepler's conjecture can be resolved despite the above mentioned paradox. Moreover, our classification results and algorithms for the computational spectral problem open up for new use of computer assisted proofs in mathematical physics.

Smale's problem on the existence of iterative generally convergent algorithm: An example of how the SCI hierarchy encompasses important foundational results is the question of computing zeros of polynomials with a rational map applied iteratively (such as Newton's method). The problem with Newton's method is that it may not converge. This problem prompted S. Smale [94] to ask whether there exists an alternative to Newton's method, namely, a purely iterative generally convergent algorithm (see Section 11 for definition). Smale asked: "*Is there any purely iterative generally convergent algorithm for polynomial zero finding ?*" His conjecture was that the answer is 'no'. This problem was settled by C. McMullen in [78] as follows: yes, if the degree is three; no, if the degree is higher (see also [79, 96]). However, in [36] P. Doyle and C. McMullen demonstrated a striking phenomenon: this problem can be solved in the case of the quartic and the quintic using several limits. Indeed, Smale's question and Doyle and McMullen's results are classification problems in the SCI hierarchy.

The role of the SCI hierarchy in mathematics: As briefly mentioned above, the SCI hierarchy encompasses many areas in the mathematical sciences. An incomplete list (see §3) includes polynomial rootfinding, spectral computation, computer assisted proofs, inverse problems, optimisation, compressed sensing, statistical estimation, machine learning, foundations of computational mathematics, as well as logic.

CONTENTS

1. Introduction 1
1.1. The SCI hierarchy - an informal introduction 3
1.2. Smale's problem on iterative generally convergent algorithms and the SCI 6
1.3. Computing the non-computable - The SCI hierarchy and computer assisted proofs 6
2. The main results 8
2.1. Computing spectra of bounded operators 8
2.2. Computing spectra of Schrödinger operators on $L^2(\mathbb{R}^d)$ 10
2.3. Computational inverse problems 11
3. The role of the SCI hierarchy in mathematics 12
3.1. Connection to previous work 13
4. The Solvability Complexity Index Hierarchy and towers of algorithms 14
5. Main theorems on the general computational spectral problem 19
6. Main theorems on computational quantum mechanics 21
7. Main theorems on solving linear systems 23
8. Proof of Theorem 5.4 23
8.1. Spectra and pseudospectra of compact operators 24
8.2. Pseudospectrum 25
8.3. Spectrum 29
8.4. Essential Spectrum 34
8.5. Determining if a point z lies in $\mathrm{sp}(A)$ 39
8.6. Techniques for proving lower bounds 41
9. Proofs of Theorem 6.2 and Theorem 6.3 44
9.1. The case of bounded potential V 45
9.2. The case of unbounded potential V 56
10. Proofs of Theorem 7.2 and Theorem 7.3 66
11. Smale's problem on roots of polynomials and Doyle-McMullen towers 68
11.1. Doyle-McMullen towers 69
11.2. A height 3 tower for the quartic 69
11.3. A height 3 tower for the quintic 71
11.4. Particular initial guesses and height one towers 72
12. Proof of Proposition 4.12 73
12.1. Generalisation to Arbitrary Decision Problems 77
12.2. Intersections and Sharpness Results 79
13. Numerical examples 82
13.1. Toeplitz operators 82
13.2. Graphene 83
13.3. Schrödinger operator with quasi-periodic potential 83
13.4. The operator $f(Q)$ 85
Acknowledgements 86
References 87

1.1. **The SCI hierarchy - an informal introduction.** We give an informal description of the SCI hierarchy in order to present the main results. The detailed definitions can be found in §4. The SCI hierarchy is based on the concept of a computational problem. This is described by a function $\Xi : \Omega \to \mathcal{M}$ that we want to compute, where Ω is some domain and $\mathcal{M}$ is a metric space. For example, $\Xi(T) = \mathrm{Sp}(T)$ (the spectrum) for some operator $T \in \Omega$ and $\mathcal{M}$ is the collection of closed subsets of $\mathbb{C}$ equipped with the Hausdorff metric. The SCI was first introduced in the paper *"On the Solvability Complexity Index, the n-pseudospectrum and approximations of spectra of operators"* [59] for spectral problems in order to introduce the concept of several limits for spectral computation. The SCI of a spectral problem is the smallest number of limits needed in order to compute the solution. However, in the paper above, the main issue was left open: is it necessary to use several limits? In other words, could the SCI collapse to one for all spectral problems, or in fact for all problems in scientific computing? Moreover, as is easily seen, a hierarchy based on only the number of limits needed would not be refined enough to capture the boundaries of what is possible in spectral computation.

In this paper we introduce the general SCI hierarchy for all types of computational problems, and the mainstay of the hierarchy are the Δ_k^α classes. The α is related to the model of computation as explained below. Informally, we have the following description:

(i) Δ_0^α is the set of problems that can be computed in finite time, the SCI $= 0$.
(ii) Δ_1^α is the set of problems that can be computed using one limit, the SCI $= 1$, however one has error control and one knows an error bound that tends to zero as the algorithm progresses.
(iii) Δ_2^α is the set of problems that can be computed using one limit, the SCI $= 1$, but error control may not be possible.
(iv) Δ_{m+1}^α, for $m \in \mathbb{N}$, is the set of problems that can be computed by using m limits, the SCI $\leq m$.

In general, this hierarchy cannot be refined unless there is some extra structure on the metric space $\mathcal{M}$. The hierarchy typically does not collapse, and we have:

$$\Delta_0^\alpha \subsetneq \Delta_1^\alpha \subsetneq \Delta_2^\alpha \subsetneq \ldots \subsetneq \Delta_m^\alpha \subsetneq \ldots. \tag{1.2}$$

However, the hierarchy (1.2) may terminate for a finite m, or it may continue for arbitrary large m.

The SCI hierarchy can be refined if the metric space $\mathcal{M}$ allows for convergence from "above" and "below", for example when considering the Hausdorff metric, which is natural for spectral problems. The motivation behind the refinement is to characterise the intricate classifications of different problems. For example, consider Ω to be the class of all diagonal operators $T \in \mathcal{B}(l^2(\mathbb{N}))$ of the form

$$T = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & a_3 & \\ & & & \ddots \end{pmatrix}, \qquad a_j \in \mathbb{C}. \tag{1.3}$$

The problem of computing the spectrum $\mathrm{Sp}(T)$ of such Ts is trivially not in Δ_1^α. However, one can simply choose an algorithm Γ_n to collect $\{a_j\}_{j=1}^n$ and then one trivially has that $\Gamma_n(T) \to \mathrm{Sp}(T)$ as $n \to \infty$. Thus, the problem of computing spectra of operators in Ω is in Δ_2^α. However, we clearly have an extra feature that is not captured by the hierarchy (1.2). Indeed, we trivially have that

$$\Gamma_n(T) \subset \mathrm{Sp}(T), \quad n \in \mathbb{N}.$$

In particular, we have convergence from below, and this is much stronger than just convergence since $\Gamma_n(T)$ always produces a correct output. Such type of convergence becomes incredibly important as it provides an error control from below. Moreover, clearly, the hierarchy (1.2) does not capture this important feature. This gives the motivation behind the Σ_1^α class, which captures the concept of convergence from below. Similarly, the Π_1^α class captures a convergence from above.

Informally, for spectral problems we have the following additions to (1.2):

(1) $\Delta_0^\alpha = \Pi_0^\alpha = \Sigma_0^\alpha$ is the set of problems that can be solved in finite time, the SCI $= 0$.

(2) Σ_1^α: We have $\Delta_1^\alpha \subset \Sigma_1^\alpha \subset \Delta_2^\alpha$ and Σ_1^α is the set of problems that can be computed by passing to one limit. Error control may not be possible, however, there exists an algorithm which output is included in the true solution (up to an arbitrarily small accuracy parameter ϵ).

(3) Π_1^α: We have $\Delta_1^\alpha \subset \Pi_1^\alpha \subset \Delta_2^\alpha$ and Π_1^α is the set of problems that can be computed by passing to one limit. Error control may not be possible, however, there exists an algorithm which output includes the true solution (up to an arbitrarily small accuracy parameter ϵ).

(4) Σ_m^α is the set of problems that can be computed by passing to m limits, and computing the m-th limit is a Σ_1^α problem.

(5) Π_m^α is the set of problems that can be computed by passing to m limits, and computing the m-th limit is a Π_1^α problem.

The SCI hierarchy extends immediately to any metric space where there is a total ordering. This is for example the case for $\mathcal{M} = \mathbb{R}$ and for decision problems where $\mathcal{M} = \{0, 1\}$. Schematically, the SCI hierarchy can be viewed in the following way.

$$
\begin{array}{ccccccccccccc}
\Pi_0^\alpha & & & & \Pi_1^\alpha & & & & \Pi_2^\alpha & & & & \\
\| & & & \rotatebox{45}{$\subsetneq$} & & \rotatebox{-45}{$\supsetneq$} & & \rotatebox{45}{$\subsetneq$} & & \rotatebox{-45}{$\supsetneq$} & & \rotatebox{45}{$\subsetneq$} & \\
\Delta_0^\alpha & \subsetneq & \Delta_1^\alpha & \subsetneq & \Sigma_1^\alpha \cup \Pi_1^\alpha & \subsetneq & \Delta_2^\alpha & \subsetneq & \Sigma_2^\alpha \cup \Pi_2^\alpha & \subsetneq & \Delta_3^\alpha & \subsetneq & \cdots \\
\| & & & \rotatebox{-45}{$\subsetneq$} & & \rotatebox{45}{$\supsetneq$} & & \rotatebox{-45}{$\subsetneq$} & & \rotatebox{45}{$\supsetneq$} & & \rotatebox{-45}{$\subsetneq$} & \\
\Sigma_0^\alpha & & & & \Sigma_1^\alpha & & & & \Sigma_2^\alpha & & & &
\end{array}
\tag{1.4}
$$

Note that the Σ_1^α and Π_1^α classes become crucial in computer assisted proofs, see §1.3.

Remark 1.1 (The meaning of the α, the model of computation)**.** The α in the superscript indicates the model of computation, which is described in §4. For $\alpha = G$, the underlying algorithm is general and can use any tools at its disposal. The reader may think of a Blum-Shub-Smale (BSS) machine or a Turing machine with access to any oracle, although a general algorithm is even more powerful. However, for $\alpha = A$ this means that only arithmetic operations and comparisons are allowed. In particular, if rational inputs are considered, the algorithm is a Turing machine, and in the case of real inputs, a BSS machine. Hence, a result of the form $\notin \Delta_k^G$ is stronger than $\notin \Delta_k^A$. Indeed, a $\notin \Delta_k^G$ result is universal and holds for any model of computation. Moreover, $\in \Delta_k^A$ is stronger than $\in \Delta_k^G$, and similarly for the Π_k and Σ_k classes.

Remark 1.2 (Warning!)**.** The reader may recognise the Δ_k^α, Σ_k^α, Π_k^α notation from, for example, the arithmetical hierarchy. Indeed, classical hierarchies become special cases of the SCI hierarchy (see Proposition 4.12), and hence the similar notation is deliberate. However, there is a very big difference! In classical hierarchies the Δ_k class is defined by $\Delta_k = \Sigma_k \cap \Pi_k$. This is not the case in the SCI hierarchy. In fact, the Δ_k classes form the core of the hierarchy, and only when there is extra structure on the metric space it makes sense to define the Σ_k and the Π_k. Moreover, in the general SCI hierarchy we may have that

$$\Delta_k \neq \Sigma_k \cap \Pi_k.$$

Of course, in the special cases of the SCI hierarchy such as the Arithmetical hierarchy, then $\Delta_k = \Sigma_k \cap \Pi_k$. Also, we show that $\Delta_k^\alpha = \Sigma_k^\alpha \cap \Pi_k^\alpha$ for $k = 1, 2, 3$ and $\alpha = G, A$ in the computational spectral problem case, however, there is no reason that this should hold in general.

Remark 1.3 (The SCI ordering)**.** Note that the SCI hierarchy immediately implies a total ordering on the set of problems in the hierarchy. This is obvious when we only consider the Δ_k^α classes, but can also be extended to the general case by considering $\Sigma_k^\alpha \cup \Pi_k^\alpha$ as one class in between the Δ_k^αs. This is the ordering $\leq_{\mathrm{SCI}}$ referred to in §1.

Due to the example of the diagonal matrix in (1.3), most computational spectral problems of interest are not in Δ_1^G. Thus, most of the classical literature on spectral computation is devoted to establishing algorithms that in view of the SCI hierarchy would provide Δ_2^A classification for specific subclasses of operators. Note that according to Turing's definition of computability, problems that are not in Δ_1^A are non-computable. Hence, the field of computational spectral theory is mostly concerned with non-computable problems.

1.2. **Smale's problem on iterative generally convergent algorithms and the SCI.** S. Smale initiated a comprehensive program on the foundations of computational mathematics in the 1980s [12, 94], focusing on problems in scientific computing rather than classical computer science. One of the key problems and algorithms Smale considered was polynomial root finding as well as Newton's method. As Newton's method may not converge, even for a cubic polynomial, a natural question would be if there exists an alternative approach. This question was formulated in terms of existence of iterative generally convergent algorithms [94]. C. McMullen [78,79,96] solved the problem in the negative and together with P. Doyle [36] realised that the problem of existence could be resolved by allowing more limits resulting in several iterative convergent algorithms used consequetively. They introduced a *tower of algorithm* in order to make the mathematical statement precise, and also realised that for polynomials of degree 6 and higher, one could not handle the problem regardless of the height of the tower (number of limits used). We have adopted the name towers of algorithms, however made the concept general. The original towers of algorithms are now referred to as *Doyle-McMullen towers*, see §11. In §11 we show how Smale' problem on the existence of iterative generally convergent algorithms and the theory of McMullen and Doyle become classification problems regarding the SCI.

1.3. **Computing the non-computable - The SCI hierarchy and computer assisted proofs.** Computer assisted proofs have become essential in mathematics, and the recent computer assisted proof of Kepler's conjecture (Hilbert's 18th problem) is a striking example. However, a key question will always be; given a problem that needs to be computed in order to secure a computer assisted proof, can the computation be done with verification that is 100% reliable? In order to achieve this the instinct would normally be that the computational problem must be in Δ_1^A, or computable in the words of Turing. This is not the case. In fact, the computer assisted proof of Kepler's conjecture is done by computing non-computable problems i.e. $\notin \Delta_1^G$ as explained in the next example. Moreover, this is not unusual, in fact, there are several cases of important conjectures that have been solved by computer assisted proof, where the computational problem is higher up in the SCI hierarchy than Δ_1^G. Below follow examples of successful computer assisted proofs with the corresponding SCI hierarchy classification of the main computational problem.

Kepler's Conjecture (Hilbert's 18th problem) - SCI classification: $\in \Sigma_1^A$, $\notin \Delta_1^G$: Kepler conjectured that no packing of congruent balls in Euclidean three space has density greater than that of the face-centered cubic packing. The Flyspeck program, led by T. Hales [55, 56], provides a fully computer assisted verification, where the numerical part of the computer assisted proof of is based on deciding about 50000 linear programs with irrational inputs. More specifically, the computational problem is to decide whether there is an $x \in \mathbb{R}^N$ such that

$$\langle x, c\rangle_K \leq M \text{ subject to } Ax = y, \quad x \geq 0, \tag{1.5}$$

where

$$\langle x, c\rangle_K = \lfloor 10^K \langle x, c\rangle \rfloor 10^{-K}, \quad K \in \mathbb{N}, \quad M \in \mathbb{Q}.$$

Informally, we could think of $\langle x, c\rangle_K$ as $\langle x, c\rangle$ computed with K digits accuracy. The fact that there are irrational input numbers means that A and y are only known approximately, however, to any precision one wants (think of either a Turing machine or a BSS machine that can access $A \in \mathbb{R}^{m\times N}$ in form of an oracle $\mathcal{O}_A$ such that $|\mathcal{O}_A(i,j,k) - A_{i,j}| \leq 2^{-k}$). There are several facts about the problem (1.5) and its classification in the SCI hierarchy that may be surprising given that Kepler's conjecture is successfully proven. In a companion paper [7] to our results, as a part of the extended Smale's 9th problem, the following is proven.

(i) For any integer $\tilde{K} > 1$ there exists a class of inputs Ω such that the problem (1.5) with $K = \tilde{K}$ is $\notin \Sigma_1^G$. However, with the same input class Ω, we have that the problem (1.5), with $K = \tilde{K} - 1$ is $\in \Delta_1^A$.

(ii) The reader may ask how the computer assisted proof of Kepler's conjecture was at all possible, given that one needs to decide (1.5) for $K = 6$. Indeed, the $\notin \Sigma_1^G$ fact would suggest that no positive verification would be possible. The key is that if the inequality $\langle x, c\rangle_K \leq M$ in (1.5) is replaced by a strict inequality $\langle x, c\rangle_K < M$, then the problem is in Σ_1^A [7]. Thus, it is the latter problem that is actually verified (since it is a Σ_1^A problem), and the verification is possible because all the 50000 linear programs checked yield a strict inequality. If there had been cases where there had been actual equality, the Flyspeck program may never have resolved Kepler's conjecture.

Dirac-Schwinger conjecture - SCI classification: $\in \Sigma_1^A$, $\notin \Delta_1^G$: The Dirac-Schwinger conjecture was proven in a series of papers by C. Fefferman and L. Seco [39–47] and can be described as follows. Consider the following Hamiltonian

$$H_{dZ} = \sum_{k=1}^{d}(-\triangle_{x_k} - Z|x_k|^{-1}) + \sum_{1\leq j\leq k\leq N} |x_j - x_k|^{-1}$$

acting on antisymmetric functions in $L^2(\mathbb{R}^d)$. The ground state energy $E(d, Z)$ for d electrons and a nucleus of charge Z is then defined by

$$E(d, Z) := \inf\{\lambda \in \mathrm{Sp}(H_{dZ})\}.$$

The ground state energy of an atom is then defined as $E(Z) := \min_{d\geq 1} E(d, Z)$. The key result of C. Fefferman and L. Seco was to show asymptotic behaviour of $E(Z)$ for large Z. In particular,

$$E(Z) = -c_0 Z^{7/3} + \frac{1}{8}Z^2 - c_1 Z^{5/3} + \mathcal{O}(Z^{5/3-1/2835}),$$

for some explicitly defined constants c_0 and c_1. To prove this result, there is a crucial decision problem, namely, the verification that $F''(\omega) \leq c < 0$ for some specific function F, for some c and for all $\omega \in (0, \omega_c)$ where ω_c is specifically defined. Note that decision problems involving inequalities are in general $\notin \Delta_1^G$. Moreover, the intricate computer assisted proof hinges on several problems that are $\notin \Delta_1^G$ but $\in \Sigma_1^A$ (see for example Algorithm 3.7 and Algorithm 3.8 in [46]).

Boolean Pythagorean triples problem - SCI classification: $\in \Pi_1^A$, $\notin \Delta_1^G$: The Boolean Pythagorean triples problem asks if it is possible to colour each of the positive integers either red or blue, so that no Pythagorean triple of integers a, b, c, satisfying $a^2 + b^2 = c^2$ are all the same colour. For example, in the Pythagorean triple 3, 4 and 5 ($3^2+4^2 = 5^2$), if 3 and 4 are coloured red, then 5 must be coloured blue. This is true up to $n = 7824$. The computer assisted proof, performed by M. Heule, O. Kullmann, and V. Marek (2016) [64], is based on showing that this is not true for $n = 7825$. While it is a combinatorial task checking the problem for any finite set of integer (and hence $\in \Delta_0^A$), it is clearly not $\in \Delta_0^G$ for infinite sets of integers. Yet, the problem is clearly $\in \Pi_1^A$, which is why it was possible to verify the counterexample.

Group theory: $\mathrm{Aut}(\mathbb{F}_5)$ ***has property*** (T) ***- SCI classification :*** $\in \Sigma_1^A$, $\notin \Delta_1^G$: The fact that the automorphism group of the free group on 5 generators has Kazhdan's property (T), was shown by M. Kaluba, P. Nowak and N. Ozawa [68]. The proof relies on a decision problem involving a minimiser of a semi-definite program (actually a root of a positive definite matrix that is a minimiser). The minimiser is computed using floating point arithmetic. Hence, it is, at best (if one could do a backward error analysis), equivalent to solving the semi-definite program with inexact input. Computing minimisers to semi-definite programs with inexact, yet arbitrary small precision is $\notin \Delta_1^G$ [7]. Showing that computing a minimiser to a semi-definite program, given inexact input, is $\in \Delta_2^A$ requires an argument, which we will not discuss here. Note that there is no concept of Σ_1^A for minimisers of semi-definite programs, as the metric would simply be a norm, and in this case there is no concept of convergence from below nor above. However, given the assumption regarding Δ_2^A, the reasoning in the paper [68] regarding the verification implies that the final decision problem is $\in \Sigma_1^A$.

Remark 1.4 (Proving Σ_1^A or Π_1^A results)**.** Note that a key part in all of the examples above is that one must prove either the Σ_1^A or Π_1^A in order to demonstrate that the verification is possible. Sometimes this is trivial as in the Boolean Pythagorean triples problem, however, sometimes this may be very intricate and technical as in the proof of the Dirac-Schwinger conjecture.

2. The main results

The introduction of the SCI hierarchy implies an infinite classification theory even for the computational spectral problem, and we provide the first foundations here. The precise theorems can be found in Theorem 5.4, Theorem 6.2, Theorem 6.3, Theorem 7.2 and Theorem 7.3, however, we provide an informal and easy to read summary in this section. The fundamental question is as follows:

Given a computational problem with a domain Ω and a problem function $\Xi : \Omega \to \mathcal{M}$, where in the SCI hierarchy is the problem when Ξ represents the spectrum, essential spectrum, pseudospectrum or even a solution to an inverse problem?

Our results describing where a computational problem is in the SCI hierarchy are mainly of the form: *computational problem* $\in \mathcal{S}$ and *computational problem* $\notin \mathcal{R}$, where $\mathcal{R}, \mathcal{S}$ are of the form $\Sigma_k^{\alpha}, \Pi_k^{\alpha}, \Delta_k^{\alpha}$. This is typically written as

$$\mathcal{R} \not\ni \text{ computational problem} \in \mathcal{S},$$

where

$$\mathcal{R} = \Sigma_k^G, \Pi_k^G, \Delta_k^G, \qquad \mathcal{S} = \Sigma_j^A, \Pi_j^A, \Delta_j^A, \qquad k \leq j.$$

Note that all the upper bounds are constructive, yielding implementable algorithms. The main results are as follows:

Theorem 5.4: (Computational spectral problem, bounded operators). An informal summary follows in §2.1, and the precise formulation is in §5. Note that the $\in \Sigma_1^A$ results open up for potential use in computer assisted proofs.

Theorem 6.2 & Theorem 6.3: (Computational spectral problem, Schrödinger operators). §2.2 provides an introductory summary, however, the precise statements are in §6. The $\in \Sigma_1^A$ results open up for potential use in computer assisted proofs.

Theorem 7.2 & Theorem 7.3: (Inverse problems in the SCI hiearchy). A synopsis follows in §2.3 whereas the exact formulations can be found in §7.

2.1. Computing spectra of bounded operators.

We are given operators $T \in \mathcal{B}(l^2(\mathbb{N}))$ and the task is to compute spectral properties from the matrix elements of T. We consider the following five paramount topics in the computational spectral problem addressing Arveson's issue regarding "no known techniques" mentioned in §1.

Problem 1: Compute spectra/essential spectra/pseudospectra of general operators.

Problem 2: Compute spectra/essential spectra/pseudospectra of self-adjoint/normal/known growth of resolvent (see Def. 5.2) operators.

Problem 3: Compute spectra/essential spectra/pseudospectra of operators with off-diagonal decay (see Definition 5.1).

Problem 4: Compute spectra/pseudospectra of compact operators.

Problem 5: Determine if a given point $z \in \mathbb{C}$ lies in the spectrum.

To avoid trivialities, when considering self-adjoint classes of operators we will restrict to $z \in \mathbb{R}$ and when considering compact operators we will restrict to $z \neq 0$. Moreover, by the essential spectrum we mean the spectrum that is invariant under compact perturbation, and the pseudospectrum is defined in 5.3. This gives the following classifications.

$$\Delta_3^G \not\ni \text{Prob 1 (sp.)} \in \Pi_3^A \qquad \Delta_3^G \not\ni \text{Prob 1 (ess-sp.)} \in \Pi_3^A \qquad \Delta_2^G \not\ni \text{Prob 1 (pseudosp.)} \in \Sigma_2^A, \tag{2.1}$$

$$\Delta_2^G \not\ni \text{Prob 2 (sp.)} \in \Sigma_2^A \qquad \Delta_3^G \not\ni \text{Prob 2 (ess-sp.)} \in \Pi_3^A \qquad \Delta_2^G \not\ni \text{Prob 2 (pseudosp.)} \in \Sigma_2^A. \tag{2.2}$$

$$\Delta_2^G \not\ni \text{Prob 3 (sp.)} \in \Pi_2^A \qquad \Delta_2^G \not\ni \text{Prob 3 (ess-sp.)} \in \Pi_2^A \qquad \Delta_1^G \not\ni \text{Prob 3 (pseudosp.)} \in \Sigma_1^A. \tag{2.3}$$

Note that (2.2) means that the classification is the same for self-adjoint operators, normal operators and operators with known growth of the resolvent. These classes of operators are obviously increasingly included in each other. Continuing, we have for Problem 4

$$\Sigma_1^G \cup \Pi_1^G \not\ni \text{Prob 4 (sp.)} \in \Delta_2^A, \qquad \Sigma_1^G \cup \Pi_1^G \not\ni \text{Prob 4 (pseudosp.)} \in \Delta_2^A. \tag{2.4}$$

As for Problem 5 we have the following.

$$\Delta_2^G \not\ni \text{Prob 5 (diagonal/compact/off-diagonal decay)} \in \Pi_2^A, \tag{2.5}$$

$$\Delta_3^G \not\ni \text{Prob 5 (general/self-adjoint)} \in \Pi_3^A. \tag{2.6}$$

Finally, combining Problem 2 and Problem 3 we have

$$\Delta_1^G \not\ni \text{Prob 2} \cap \text{Prob 3 (sp.)} \in \Sigma_1^A. \tag{2.7}$$

The detailed statements can be found in Theorem 5.4.

Remark 2.1 (Solutions to the computational spectral problem for bounded operators)**.** The introduction of the SCI hierarchy means that the computational spectral problem becomes an infinite classification theory, however, we consider the results above a "solution" to this problem as they provide the sharp classifications for some of the key problems. As a response to Arveson's statement, we can now conclude that there are known techniques and algorithms, and they provide sharp classifications.

It is worth noting the subtle differences between the problems of computing spectra, essential spectra, pseudospectra or determining wether a $z \in \mathbb{C}$ is in the spectrum. Indeed, by examining (2.1), (2.2) and (2.3) we see that the classifications change for the spectrum when adding more assumptions to the operator. Fascinatingly, this is not necessarily the case for the essential spectrum and the pseudospectrum. Observe also this subtlety when comparing (2.5), (2.6) and (2.7). The infinite classification problem becomes how to characterise which extra assumptions on the classes yield which classifications.

Remark 2.2 (New algorithms and computer assisted proofs)**.** All the proofs of the upper bounds on the classifications in the SCI hierarchy are constructive yielding new algorithms that are sharp according to the classifications in the hierarchy. In order to view examples of numerical simulations using the new algorithms on problems that before were intractable, the reader is invited to consult §**??**. Note that the Σ_1^A classification in (2.7) of spectra of operators with off-diagonal decay and controlled resolvent growth means that we can compute spectra of Jacobi operators

$$J := \begin{pmatrix} b_1 & c_1 & & & \\ a_1 & b_2 & c_2 & & \\ & a_2 & b_3 & c_3 & \\ & & a_3 & b_4 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}, \tag{2.8}$$

with controlled growth of the resolvent, and guarantee that the output will be in the spectrum up to any small accuracy parameter ϵ.

Given the Σ_1^A classification mentioned means that spectra of Jacobi operators of the form (2.8) with known growth of the resolvent can be used in potential computer assisted proofs. However, note the rather subtle result that

$$\Delta_2^G \not\ni \text{ Computing essential spectra of diagonal self-adjoint operators} \in \Pi_2^A.$$

Thus, this suggests that one must have very specific assumptions on the class of operators in order to be able to use computer assisted proofs regarding essential spectra. In particular, the essential spectrum is much harder to compute than the spectrum. Note also that (2.4) reveals that general compact operators are not suited for computer assisted proofs in spectral theory. The question is which extra assumptions in addition to compactness are needed to get lower in the SCI hierarchy.

2.2. **Computing spectra of Schrödinger operators on $L^2(\mathbb{R}^d)$.** The problem of computing the spectrum of a Schrödinger operator

$$H = -\Delta + V, \qquad V : \mathbb{R}^d \to \mathbb{C}, \tag{2.9}$$

is a classical problem in computational quantum mechanics, and the case where V is a bounded (even smooth) potential has been open since the dawn of quantum mechanics. We consider computing spectra/pseudospectra of closed Schrödinger operators from point samples of the potential $V(x)$, in particular, the following problems:

Problem I: Compute spectrum/pseudospectrum of H when $\|V\|_\infty \leq M < \infty$ and $V \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d)$ (locally bounded total variation).

Problem II: Compute spectrum/pseudospectrum of H when $\|V\|_\infty \leq M < \infty$, $V \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d)$ and there is known growth of the resolvent.

Problem III: Compute spectrum/pseudospectrum of H when V is continuous, takes values in a sector of the complex plane (not containing the negative real line) and blows up at infinity.

Note that the assumption that $V \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d)$, the set of functions with locally bounded variation, is very mild as this class includes discontinuous functions and functions with arbitrary wild oscillations at infinity. One can have arbitrary oscillations elsewhere, but for each fixed V the local total variation has to be bounded. Note also that only requiring $V \in L^\infty(\mathbb{R}^d)$ and $\|V\|_\infty \leq M$ is impossible as the concept of point samples of V would not be well defined. We then have the following classifications.

$$\Delta_1^G \not\ni \text{Problem I (spectrum)} \in \Pi_2^A \qquad \Delta_1^G \not\ni \text{Problem I (pseudosp.)} \in \Sigma_1^A, \tag{2.10}$$

$$\Delta_1^G \not\ni \text{Problem II (spectrum)} \in \Sigma_1^A \qquad \Delta_1^G \not\ni \text{Problem II (pseudosp.)} \in \Sigma_1^A, \tag{2.11}$$

$$\Sigma_1^G \cup \Pi_1^G \not\ni \text{Problem III (spectrum)} \in \Delta_2^A \qquad \Sigma_1^G \cup \Pi_1^G \not\ni \text{Problem III (pseudosp.)} \in \Delta_2^A. \tag{2.12}$$

The detailed statements can be found in Theorem 6.2 and Theorem 6.3.

Remark 2.3 (Non-Hermitian Hamiltonians)**.** We emphasise that the results above are valid also for non-Hermitian quantum systems. This level of generality is important as we want the theory to include non-Hermitian quantum mechanics [10, 11, 60, 61] and the theory of resonances [93, 102].

Remark 2.4 (The solutions to the computational spectral problem for Schrödinger operators)**.** The results in (2.10) and (2.11) provide solutions to the longstanding problem of computing spectra of Schrödinger operators on $L^2(\mathbb{R}^d)$ with bounded potential. In view of (2.4) the results in (2.11) may be surprising. Indeed, the problem on computing spectra and pseudospectra of even non-Hermitian Schrödinger operators in (2.11) is actually strictly easier than computing spectra of compact operators on $l^2(\mathbb{N})$, a computational problem for which successful algorithms have been known for decades.

The upper bound for Problem III in (2.12) has been known for the self-adjoint case. Indeed, in [35] T. Digernes, V. S. Varadarajan and S. R. S. Varadhan proved convergence of spectra of J. Schwinger's finite-dimensional discretisation for this problem, which implies Δ_2^A classification in the SCI hierarchy, see §3.1.

However, the problem has been open in the general case both for spectra and pseudospectra. The classification in (2.12) provides a sharp solution to the general problem.

Remark 2.5 (New algorithms and computer assisted proofs)**.** The algorithms that come with the constructive proofs open up for new simulations of quantum systems that before have not been possible, also in the non-Hermitian case (see §13). Moreover, since we achieve the Σ_1^A classification in several cases, computer assisted proofs may be a possibility.

2.3. **Computational inverse problems.** Just as finding spectra of operators and roots of polynomials, the problem of solving linear systems of equations is at the heart of computational mathematics. For the finite-dimensional case it is easy to find an algorithm that can perform the task, but what about the infinite-dimensional case? We consider the inverse problem

$$Ax = y \qquad A \in \mathcal{B}(l^2(\mathbb{N})), \quad x, y \in l^2(\mathbb{N}),$$

where we want to compute various quantities such as x from the matrix values of A and vector components of y when A is known to be invertible. In summary, we consider the following problems.

Problem a: Compute x when A and y are arbitrary.
Problem b: Compute x when A is self-adjoint and y is arbitrary.
Problem c: Compute x when A has known off-diagonal decay and y is arbitrary.
Problem d: Compute x when A has known off-diagonal decay and y has known decay.
Problem e: Compute the norm of the inverse $\|A^{-1}\|^{-1}$.
Problem f: Determine if A is invertible.

When computing solutions to general inverse problems, as there is no concept of convergence from above and below, we only have the initial Δ_k^α classes. However, when it comes to computing the norm of the inverse and the decision problem of determining whether A is invertible or not, we do have the Σ_k^α and Π_k^α classes. In particular, we have the following classifications.

$$\Delta_2^G \not\ni \text{Problem a} \in \Delta_3^A \qquad \Delta_2^G \not\ni \text{Problem b} \in \Delta_3^A, \tag{2.13}$$

$$\Delta_1^G \not\ni \text{Problem c} \in \Delta_2^A \qquad \Delta_0^G \not\ni \text{Problem d} \in \Delta_1^A. \tag{2.14}$$

Moreover, these are the classifications for Problem e.

$$\Delta_2^G \not\ni \text{Problem e (general/self-adjoint)} \in \Pi_2^A, \tag{2.15}$$

$$\Delta_1^G \not\ni \text{Problem e (off-diagonal decay)} \in \Pi_1^A. \tag{2.16}$$

Note that Problem f is a special case of Problem 5 in §2.1. Thus, we have that

$$\Delta_2^G \not\ni \text{Problem f (diagonal/compact/off-diagonal decay)} \in \Pi_2^A,$$
$$\Delta_3^G \not\ni \text{Problem f (general/self-adjoint)} \in \Pi_3^A.$$

The detailed statements can be found in Theorem 7.2 and Theorem 7.3.

Remark 2.6 (Finite section in inverse problems)**.** Note that the results in (2.13) and (2.13) provide a simple explanation why the finite section method or any of its variants could never solve the general inverse problem. Indeed, such methods would imply at least a Δ_2^A results which is impossible. However, note that we immediately get that the class of problems for which the finite section method works are in Δ_2^A. This demonstrates the importance of the vast literature on the finite section method for classifications in the SCI hierarchy.

Finally, we note that the results in (2.15) and (2.16) shed light on the possible use of Problem e in computer assisted proofs.

3. THE ROLE OF THE SCI HIERARCHY IN MATHEMATICS

The SCI hierarchy encompasses many key computational problems in the history of mathematics and has applications in many computational areas of the mathematical sciences. This is summarised as follows.

(i) *Computer assisted proofs:* §1.3 provided examples on how non-computable problems, i.e. problems that are higher than Δ_1^A in the SCI hierarchy, can be used in computer assisted proofs such as the verification of Kepler's conjecture. Moreover, the Σ_1^A and Π_1^A problems allow for verifications, and thus the SCI hierarchy becomes a tool for understanding which problems are suitable for computer assisted proofs. In fact, one will typically prove implicitly an SCI hierarchy classification in order to demonstrate the correctness of the computer assisted proof (as done in the proof of the Dirac-Schwinger conjecture by Fefferman and Seco).

(ii) *Insolvability of the quintic:* The insolvability of the quintic becomes a classification problem in the SCI hierarchy. In particular, showing that the SCI of the problem of computing the zeros of a polynomial, when one can use arithmetic operations and radicals, is greater than 0 for polynomials of degree 5 is equivalent to the insolvability of the quintic.

(iii) *Smale's problem on the existence of generally convergent algorithms and McMullen's solutions:* §1.2 summarises how the results by McMullen and Doyle & McMullen are classification results in the SCI hierarchy.

(iv) *Optimisation (compressed sensing and the extended Smale's 9th problem, statistical estimation, machine learning):* As discussed in §1.3 and proved in a companion paper [7], deciding feasibility of linear programs given irrational inputs is not only undecidable ($\notin \Delta_1^G$) but $\notin \Sigma_1^G$. As shown in [7] , using the framework of the SCI hierarchy, similar phenomena extend to many key problems in optimisation such as finding minimisers of Basis pursuit and Lasso. These form the basis of compressed sensing, statistical estimation, areas of machine learning etc. Moreover, there is a link to the extended Smale's 9th problem [7].

(v) *Computing the exit flag (validating output of an algorithm):* Often computational routines come with a certification, a so called exit flag, that determines if the computed solution is trustworthy or not. An example is MATLABs popular routine `linprog` for solving linear programs. Paradoxically, as shown in [7], this exit flag is not trustworthy, and, paradoxically, the problem of computing the exit flag is higher up in the SCI hierarchy than computing the original problem itself.

(vi) *Spectral problems:* Arveson's comment (recall §1) regarding the lack of algorithms that could handle general spectral problems can be explained by the SCI hierarchy. As many computational spectral problems are high up in the hierarchy, none of the existing methods could handle them. Moreover, the standard methods were based on one limit approaches, and would therefore never capture the depth of the computational spectral problem.

(vii) *Inverse problems:* As established in §2.3, inverse problems have a rich classification theory in the SCI hierarchy.

(viii) *Foundations of computational mathematics:* The SCI hierarchy can be viewed as a direct continuation of Smale's program on the foundations of scientific computing, however, it allows for any computational model and any computational problem.

(ix) *Hierarchies in logic:* Classical hierarchies in logic such as the arithmetical hierarchy become special cases of the SCI hierarchy. This is not a paper in logic and computer science, however, a short discussion on connection to logic follows below.

The Baire hierarchy. The Baire hierarchy, which is closely related to the Borel hierarchy in descriptive set theory, has similarities to the SCI hierarchy, however, is fundamentally different. However, it is worth mentioning, as the Baire hierarchy does include classes of functions that are obtained as limits of functions from lower levels in the hierarchy, hence the two hierarchies share some similarities.

Recall that given metrisable spaces X, Y and a continuous function $f : X \to Y$ we say that f is of Baire class 0. We define a function $g : X \to Y$ to be in Baire class 1 if there is a sequence of functions $\{g_n\}$, all of Baire class 0, such that $g(x) = \lim_{n\to\infty} g_n(x)$ for all $x \in X$. In general, for $1 < \rho < \omega_1$ we define a function $f : X \to Y$ to be of Baire class ρ if it is the pointwise limit of a sequence of functions $f_n : X \to Y$, where f_n is of Baire class $\rho_n < \rho$. In order to understand the similarities and differences between the two hierarchies we provide a short discussion below.

Similarities between the SCI and Baire hierarchies. The main similarity between the hierarchies is the concept of pointwise limits. Indeed, for the integer values of the Baire classes, this number indeed resembles the SCI.

Differences between the SCI and Baire hierarchies. The differences between the hierarchy are due to the fact that they describe very different problems. This can be summed up as follows.

(i) (*Generality*). The SCI hierarchy is designed to be able to handle all types of computational problems such as Smale's problem on iterative algorithms for polynomial rootfinding, Doyle-McMullen towers, the insolvability of the quintic etc. This is obviously not within the scope of the Baire hierarchy, however, this was never the intension for this hierarchy.

(ii) (*Refinements*). An important difference between the hierarchies is that the SCI hierarchy, when extra structure on $\mathcal{M}$ is available, allows for the refinements in terms of the Σ_k^α and Π_k^α classes. This type of refinement is not captured by the Baire hierarchy, however, that has never been the motivation.

(iii) (*Topology vs information*). A striking difference is that the Baire hierarchy is based on metrisable topologies, whereas the SCI hierarchy is based on the information Λ available to the algorithm. The computational spectral problem is a good example to illustrate the issue. Let $\Xi : \Omega \ni A \mapsto \mathrm{Sp}(A) \in \mathcal{M}$ where Ω is the set of self-adjoint operators in $\mathcal{B}(l^2(\mathbb{N}))$ and $\mathcal{M}$ is the collection of compact subsets of $\mathbb{C}$ with the Hausdorff metric. If we equip Ω with the operator norm topology, then Ξ is Baire class 0. Yet, the SCI $= 2$ for Ξ. If one changes the metric on Ω, the Baire class will change, yet the SCI remains unchanged. Also, as a side note, the algorithms used in this paper to show that the SCI $= 2$ are not continuous in any metrisable topology. Thus, there is no metric on Ω such that these become Baire class 0.

Finally, if we consider self-adjoint Schrödinger operators on $L^2(\mathbb{R}^d)$ with bounded potential V such that $V \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d)$ then the SCI of the spectral map is 1 if we can access point samples of V. Also, if we equip this set of operators with the natural graph metric (equivalent to norm convergence in the bounded case) the spectral map is Baire class 0. However, if one changes Λ, such that we are given matrix elements of the operator with respect to some orthonormal basis of the domain, we may get that the SCI $= \infty$, as the matrix representation may not uniquely determine the spectrum. Thus, the SCI changes with Λ whereas the Baire class changes with the metric.

3.1. **Connection to previous work.** We split the comments into four categories: foundations of computational mathematics, spectral computation, computer assisted proofs and inverse problems.

Foundations: S. Smale's seminal work [94,95] and his program on the foundations of computational mathematics and scientific computing initiated the pioneering work by C. McMullen [78,79,96] and P. Doyle & C. McMullen [36] on polynomial rootfinding. These are classification results in the SCI hierarchy and our contribution is motivated by this program and the work by L. Blum, F. Cucker, M. Shub & S. Smale [13]. Other results in this program on hierarchies include the work of F. Cucker [24] and P. Bürgisser & F. Cucker [23].

Spectral computations: The literature on computing spectra is enormous, thus, we will only emphasise the work that has been most influential on this paper. The ideas of using computational and algorithmic approaches to obtain spectral information dates back to leading physicists such as E. Schrödinger [86], P. W. Anderson [1] and J. Schwinger [87]. Schwinger introduced finite-dimensional approximations to quantum systems in infinite-dimensional spaces that allow for spectral computations. An interesting observation is that Schwinger's ideas were already present in the work by H. Weyl [101]. In [35] T. Digernes, V. S. Varadarajan and S. R. S. Varadhan proved convergence of spectra of Schwinger's finite-dimensional discretisation matrices for a specific class of Schrödinger operators with certain types of potential which yields Δ_2^A classification in the SCI hierarchy.

The finite-section method, that has been intensely studied for spectral computation and has often been viewed in connection with Toeplitz theory, is very similar to Schwinger's idea of approximating in a finite-dimensional subspace. The reader may want to consult the pioneering work by A. Böttcher [15, 16] and A. Böttcher & B. Silberman [18, 19], see also A. Böttcher, H. Brunner, A. Iserles & S. Nørsett [17], M. Marletta [75] and M. Marletta & R. Scheichl [76]. The latter papers also discuss the failure of the finite section approach for certain classes of operators, see also [57, 58]. E. B. Davies considered enclosure techniques [28] and second order spectra methods [27]. E. Shargorodsky [90] demonstrated how second order spectra methods [27] will never recover the whole spectrum.

W. Arveson [2–5] pioneered the combination of spectral computation and the C^*-algebra literature, an approach that was continued by Brown [20, 21], see also [22] where variants of finite section analysis is implicitly used. Arveson considered also spectral computation in terms of densities, which is related to Szegö's work [98] on finite section approximations. Similar results are also obtained by A. Laptev and Y. Safarov [73]. Typically, when applied to appropriate subclasses of operators, finite section approaches yield Δ_2^A classification results. There are also other approaches based on the infinite QR algorithm in connection with Toda flows with infinitely many variables pioneered by Deift, Li and Tomei [31].

The seminal work of Fefferman and Seco [39–47] on proving the Dirac-Schwinger conjecture is a striking example of computations used in order to obtain complete information about the asymptotical behaviour of the ground state of a family of Schrödinger operators. The computer assisted proof implicitly proves Σ_1^A classifications in the SCI hierarchy. Moreover, the paper [38] by Fefferman is based on similar approaches. We also want to highlight the work by L. Demanet and W. Schlag [32] as well as P. Hertel, E. Lieb and W. Thirring [63]. Finally, we would like to mention recent crucial work by M. Zworski [102, 103] on computing resonances that can be viewed in terms of the SCI hierarchy. In particular, the computational approach [103] is based on expressing the resonances as limits of non-self-adjoint spectral problems, and hence the SCI hierarchy is inevitable, see also [93].

Computer assisted proofs: The number of examples of computer assisted proofs in the literature is substantial, and thus we can only mention a few cases here. What most of them have in common is that in order to prove that the computational proof is 100% accurate one implicitly has to prove a classification in the SCI hierarchy. The work by Fefferman and Seco [39–47] can both be viewed from a computational spectral theory point of view as well as a computer assisted proofs angle, and the Σ_1^A classification is crucial. Similarly, the computer assisted proof of Kepler's conjecture, via Hale's Flyspeck program, is also relying on Σ_1^A classification. Note that these are examples of computer assisted proofs done by non-computable problems, however, there are many examples of computer assisted proofs based on Δ_A^1 classifications as well. A great example is the work of D. Gabai, R. Meyerhoff, and P. Milley [49] on hyperbolic three-manifolds.

Inverse Problems: There is a vast literature on computing solutions to certain infinite-dimensional inverse problems in one limit, typically by using the finite section method. The connection to Toeplitz theory is important an the reader may consult the foundational results in the books by A. Böttcher & B. Silberman [18,19] as well as the monograph by Lindner [74] and the references therein. Note that two-limit algorithms have been suggested by K. Gröchenig, Z. Rzeszotnik, and T. Strohmer in [53], see also [52].

4. The Solvability Complexity Index Hierarchy and towers of algorithms

Throughout this paper we assume the following:

(4.1a) $$\Omega \text{ is some set, called the } \textit{domain},$$

(4.1b) Λ is a set of complex valued functions on Ω, called the *evaluation* set,

(4.1c) $\mathcal{M}$ is a metric space,

(4.1d) Ξ is a mapping $\Omega \to \mathcal{M}$, called the *problem* function.

The set Ω is essentially the set of objects that give rise to our computational problems. It can be a family of matrices (infinite or finite), a collection of polynomials, a family of Schrödinger (or Dirac) operators with a certain potential etc. The problem function $\Xi : \Omega \to \mathcal{M}$ is what we are interested in computing. It could be the set of eigenvalues of an $n \times n$ matrix, the spectrum of a Hilbert (or Banach) space operator, root(s) of a polynomial etc. Finally, the set Λ is the collection of functions that provide us with the information we are allowed to read, say matrix elements, polynomial coefficients or pointwise values of a potential function of a Schrödinger operator, for example.

In most cases it is convenient to consider a metric space $\mathcal{M}$, however, in the case of polynomials it may be more useful to use a pseudo metric space (see Example 4.1 (III)). To explain this rather abstract setup in (4.1) we commence with the following examples:

Example 4.1. (I) **(Spectral problems)** Let $\Omega = \mathcal{B}(\mathcal{H})$, the set of all bounded linear operators on a separable Hilbert space $\mathcal{H}$, and the problem function Ξ be the mapping $A \mapsto \mathrm{sp}(A)$ (the spectrum of A). Here $(\mathcal{M}, d)$ is the set of all compact subsets of $\mathbb{C}$ provided with the Hausdorff metric $d = d_{\mathrm{H}}$ (defined precisely in (4.3)). The evaluation functions in Λ could for example consist of the family of all functions $f_{i,j} : A \mapsto \langle Ae_j, e_i\rangle$, $i, j \in \mathbb{N}$, which provide the entries of the matrix representation of A w.r.t. an orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$. Of course, Ω could be a strict subset of $\mathcal{B}(\mathcal{H})$, for example the set of self-adjoint or normal operators, and Ξ could have represented the pseudo spectrum, the essential spectrum or any other interesting information about the operator.

(II) **(Inverse problems)** Let $\Omega = \mathcal{B}_{\mathrm{inv}}(\mathcal{H}) \times \mathcal{H}$, where $\mathcal{B}_{\mathrm{inv}}(\mathcal{H})$ denotes the set of all bounded invertible operators on $\mathcal{H}$, and let the problem function Ξ be the mapping $(A, b) \mapsto A^{-1}b$, which assigns to a linear problem $Ax = b$ its solution x. The metric space $\mathcal{M}$ would simply be $\mathcal{H}$ and Λ the collection of mappings $\{f_{i,j}\}_{i\in\mathbb{N},j\in\mathbb{Z}_+}$ where $f_{i,j} : (A, b) \mapsto \langle Ae_j, e_i\rangle$ for $j \in \mathbb{N}$ and $f_{i,0} : (A, b) \mapsto \langle b, e_i\rangle$. Also here Ω could consist of operators with specific properties (off diagonal decay, self-adjointness, isometric properties).

(III) **(Polynomial root finding)** Let $\Omega = \mathbb{P}_s$, the set of polynomials of degree $\leq s$ over $\mathbb{C}$ and let the problem function Ξ be the mapping $p \mapsto \{\alpha \in \mathbb{C} \mid p(\alpha) = 0\}$ (the roots of p). Let $(\mathcal{M}, d)$ denote the collection of finite sets of points in $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ equipped with the pseudo metric $d : \mathcal{M} \times \mathcal{M} \to [0, \infty]$, defined by $d(x, y) = \min_{1\leq i\leq n, 1\leq j\leq m} |x_j - y_i|$, where $x = \{x_1, \ldots, x_n\}$, $y = \{y_1, \ldots, y_m\} \in \mathcal{M}$. The reason for the pseudo metric is that the techniques of Doyle and McMullen that we will consider are based on computing a single root of a polynomial (as for example Newton's method does). In this case Λ is the finite set of functions $\{f_j\}_{j=1}^{s}$ where $f_j : p \mapsto \alpha_j$ for $p(t) = \sum_{k=1}^{s} \alpha_k t^k$.

(IV) **(Computational quantum mechanics)** Let $\Omega = \mathrm{L}^\infty(\mathbb{R}^d) \cap \mathrm{C}(\mathbb{R}^d)$ and let $\Xi : V \mapsto \mathrm{sp}(-\Delta + V)$, where the domain $\mathcal{D}(-\Delta + V) = \mathrm{W}^{2,2}(\mathbb{R}^d)$ (the standard Sobolev space) and $-\Delta + V$ is the usual Schrödinger operator. Given that the spectra are unbounded, we cannot use the Hausdorff metric anymore, but will let $(\mathcal{M}, d_{\mathrm{AW}})$ denote the set of closed subsets of $\mathbb{C}$ equipped with the *Attouch-Wets* metric (see (4.4)). In this case a natural choice of Λ would be the set of all evaluations $f_x : V \mapsto V(x)$, $x \in \mathbb{R}^d$.

(V) **(Decision making)** Let Ω denote the set of infinite matrices with values in $\{0, 1\}$ and $\Xi : \Omega \to \mathcal{M} = \{\textit{Yes}, \textit{No}\}$ where $\mathcal{M}$ is equipped with the discrete metric d_{disc}. The evaluation functions would naturally be $f_{i,j} : A \mapsto A_{i,j}$, $i, j \in \mathbb{N}$, the (i, j)th matrix coordinate of A. A typical example of Ξ could be: $\Xi(\{A_{i,j}\})$: Does $\{A_{i,j}\}$ have a column containing infinitely many non-zero entries? Naturally, Ω can be replaced with the natural numbers including zero $\mathbb{Z}_+$ and Ξ could be a question about membership in a certain set, as in classical recursion theory. In this case the evaluation set would be $\Lambda = \{\lambda\}$ consisting of the function $\lambda : \mathbb{Z}_+ \to \mathbb{C}$, $x \mapsto x$.

Given this setup and motivation, we can now define what we mean by a computational problem.

Definition 4.2 (Computational problem)**.** Given a primary set Ω, an evaluation set Λ, a (pseudo) metric space $\mathcal{M}$ and a problem function $\Xi : \Omega \to \mathcal{M}$ we call the collection $\{\Xi, \Omega, \mathcal{M}, \Lambda\}$ a computational problem.

Our aim is to find and to study families of functions (that we will sometimes refer to as algorithms) which permit us to approximate the function Ξ. The main pillar of our framework is the concept of a tower of algorithms. However, before that we will define a general algorithm.

Definition 4.3 (General Algorithm)**.** Given a computational problem $\{\Xi, \Omega, \mathcal{M}, \Lambda\}$, a *general algorithm* is a mapping $\Gamma : \Omega \to \mathcal{M}$ such that for each $A \in \Omega$:

(i) there exists a finite subset of evaluations $\Lambda_\Gamma(A) \subset \Lambda$,
(ii) the action of Γ on A only depends on $\{A_f\}_{f\in\Lambda_\Gamma(A)}$ where $A_f := f(A)$,
(iii) for every $B \in \Omega$ such that $B_f = A_f$ for every $f \in \Lambda_\Gamma(A)$, it holds that $\Lambda_\Gamma(B) = \Lambda_\Gamma(A)$.

We will sometimes write $\Gamma(\{A_f\}_{f\in\Lambda_\Gamma(A)})$, in order to emphasize that $\Gamma(A)$ only depends on the results $\{A_f\}_{f\in\Lambda_\Gamma(A)}$ of finitely many evaluations.

Note that for a general algorithm there are no restrictions on the operations allowed. The only restriction is that it can only take a finite amount of information, though it *is* allowed to *adaptively* choose the finite amount of information it reads depending on the input (which may very well be infinite, say an infinite matrix, or a function). The condition (iii) just ensures that the algorithm is well defined and consistent since, put in simple words, changing the input A shall not affect the algorithm's action as long as the change does not affect the output of the relevant evaluations in $\Lambda_\Gamma(A)$.

Definition 4.4 (Tower of algorithms)**.** Given a computational problem $\{\Xi, \Omega, \mathcal{M}, \Lambda\}$, a *tower of algorithms of height k for* $\{\Xi, \Omega, \mathcal{M}, \Lambda\}$ is a collection of sequences of functions

$$\Gamma_{n_k} : \Omega \to \mathcal{M}, \quad \Gamma_{n_k,n_{k-1}} : \Omega \to \mathcal{M}, \ \dots\ , \Gamma_{n_k,\dots,n_1} : \Omega \to \mathcal{M},$$

where $n_k, \dots, n_1 \in \mathbb{N}$ and the functions $\Gamma_{n_k,\dots,n_1}$ at the lowest level in the tower are general algorithms in the sense of Definiton 4.3. Moreover, for every $A \in \Omega$,

$$\begin{aligned} \Xi(A) &= \lim_{n_k\to\infty} \Gamma_{n_k}(A), \\ \Gamma_{n_k}(A) &= \lim_{n_{k-1}\to\infty} \Gamma_{n_k,n_{k-1}}(A), \\ &\vdots \\ \Gamma_{n_k,\dots,n_2}(A) &= \lim_{n_1\to\infty} \Gamma_{n_k,\dots,n_1}(A), \end{aligned} \tag{4.2}$$

where $S = \lim_{n\to\infty} S_n$ means convergence $S_n \to S$ in the (pseudo) metric space $\mathcal{M}$.

In this paper we will discuss several types of towers: *Doyle-McMullen towers*, *Kleene-Shoenfield towers*, *Arithmetic towers*, *Radical towers* and *General towers*. A General tower will refer to the very general definition in Definition 4.4 specifying that there are no further restrictions as will be the case for the other towers. When we specify the type of tower, we specify requirements on the functions $\Gamma_{n_k,\dots,n_1}, \dots, \Gamma_{n_1}$ in the hierarchy, in particular, what kind of operations may be allowed. Thus, a tower of algorithms for a computational problem is essentially the toolbox allowed. The Doyle-McMullen tower appeared first in the paper of Doyle and McMullen [36] (but then only referred to as a tower of algorithms). The Kleene-Shoenfield towers describe the arithmetical hierarchy known from classical recursion theory as we will see in §**??** and can be extended (see §12.1). A radical tower, as defined below, first appeared in [59] where it was referred to as a "set of estimating functions" for computing spectra. The definition here is substantially more general and allows for the use of these types of towers for a wide range of problems.

We can now define an *arithmetic tower of algorithms* and a *radical tower of algorithms*.

Definition 4.5 (Arithmetic and radical towers)**.** Given a computational problem $\{\Xi, \Omega, \mathcal{M}, \Lambda\}$ we define the following:

(i) An *Arithmetic tower of algorithms* of height k for $\{\Xi, \Omega, \mathcal{M}, \Lambda\}$ is a tower of algorithms where the lowest functions $\Gamma = \Gamma_{n_k,\ldots,n_1} : \Omega \to \mathcal{M}$ satisfy the following: For each $A \in \Omega$ the action of Γ on A consists of only performing finitely many arithmetic operations and comparisons on $\{A_f\}_{f\in\Lambda_\Gamma(A)}$ where we remind that $A_f = f(A)$.

(ii) A *Radical tower of algorithms* of height k for $\{\Xi, \Omega, \mathcal{M}, \Lambda\}$ is a tower of algorithms where the lowest functions $\Gamma = \Gamma_{n_k,\ldots,n_1} : \Omega \to \mathcal{M}$ satisfy the following: For each $A \in \Omega$ the action of Γ on A consists of only performing finitely many arithmetic operations, comparisons and extracting radicals of $\{A_f\}_{f\in\Lambda_\Gamma(A)}$.

For arithmetic towers we let $\alpha = A$ and for radical towers we let $\alpha = R$.

Given the definition of a tower of algorithms we can now define the main concept of this paper: the Solvability Complexity Index (SCI). The SCI was first discussed in [59] for a specific spectral problem, however, this definition extends to include general problems in computations.

Definition 4.6 (Solvability Complexity Index)**.** Given a computational problem $\{\Xi, \Omega, \mathcal{M}, \Lambda\}$, it is said to have *Solvability Complexity Index* $\mathrm{SCI}(\Xi, \Omega, \mathcal{M}, \Lambda)_\alpha = k$ with respect to a tower of algorithms of type α if k is the smallest integer for which there exists a tower of algorithms of type α of height k. If no such tower exists then $\mathrm{SCI}(\Xi, \Omega, \mathcal{M}, \Lambda)_\alpha = \infty$. If there exists a tower $\{\Gamma_n\}_{n\in\mathbb{N}}$ of type α and height one such that $\Xi = \Gamma_{n_1}$ for some $n_1 < \infty$, then we define $\mathrm{SCI}(\Xi, \Omega, \mathcal{M}, \Lambda)_\alpha = 0$.

With the definition of the SCI we can define the SCI hierarchy, for which any computational problem can be classified. Without any extra structure on the metric space $\mathcal{M}$ the Δ_k^α classes are the finest refinement we can obtain in terms of the SCI. However, as described below, when more structure is allowed the hierarchy becomes much richer.

Definition 4.7 (The Solvability Complexity Index hierarchy)**.** Consider a collection $\mathcal{C}$ of computational problems and let $\mathcal{T}$ be the collection of all towers of algorithms of type α for the computational problems in $\mathcal{C}$. Define

$$\begin{aligned} \Delta_0^\alpha &:= \{\{\Xi, \Omega\} \in \mathcal{C} \mid \mathrm{SCI}(\Xi, \Omega)_\alpha = 0\} \\ \Delta_{m+1}^\alpha &:= \{\{\Xi, \Omega\} \in \mathcal{C} \mid \mathrm{SCI}(\Xi, \Omega)_\alpha \le m\}, \qquad m \in \mathbb{N}, \end{aligned}$$

as well as

$$\Delta_1^\alpha := \{\{\Xi, \Omega\} \in \mathcal{C} \mid \exists \, \{\Gamma_n\}_{n\in\mathbb{N}} \in \mathcal{T} \text{ s.t. } \forall A \; d(\Gamma_n(A), \Xi(A)) \le 2^{-n}\}.$$

When there is extra structure on the metric space $\mathcal{M}$, say $\mathcal{M} = \mathbb{R}$ or $\mathcal{M} = \{0, 1\}$ with the standard metric, one may be able to define convergence of functions from above or below. This is an extra form of structure that allows for a type of error control. As we argue below, this is for example important in computer assisted proofs, and of course, crucial in scientific computing.

Definition 4.8 (The SCI Hierarchy (totally ordered set))**.** Given the setup in Definition 4.7 and suppose in addition that $\mathcal{M}$ is a totally ordered set. Define

$$\begin{aligned} &\Sigma_0^\alpha = \Pi_0^\alpha = \Delta_0^\alpha, \\ &\Sigma_1^\alpha = \{\{\Xi, \Omega\} \in \Delta_2 \mid \exists \, \Gamma_n \in \mathcal{T} \text{ s.t. } \Gamma_n(A) \nearrow \Xi(A) \;\; \forall A \in \Omega\}, \\ &\Pi_1^\alpha = \{\{\Xi, \Omega\} \in \Delta_2 \mid \exists \, \Gamma_n \in \mathcal{T} \text{ s.t. } \Gamma_n(A) \searrow \Xi(A) \;\; \forall A \in \Omega\}, \end{aligned}$$

where $\nearrow$ and $\searrow$ denotes convergence from below and above respectively, as well as, for $m \in \mathbb{N}$,

$$\Sigma_{m+1}^{\alpha} = \{\{\Xi, \Omega\} \in \Delta_{m+2} \mid \exists\, \Gamma_{n_{m+1},\ldots,n_1} \in \mathcal{T} \text{ s.t. } \Gamma_{n_{m+1}}(A) \nearrow \Xi(A)\ \forall A \in \Omega\},$$
$$\Pi_{m+1}^{\alpha} = \{\{\Xi, \Omega\} \in \Delta_{m+2} \mid \exists\, \Gamma_{n_{m+1},\ldots,n_1} \in \mathcal{T} \text{ s.t. } \Gamma_{n_{m+1}}(A) \searrow \Xi(A)\ \forall A \in \Omega\}.$$

In the case where $\mathcal{M}$ is the collection of closed subsets of another metric space $\mathcal{M}'$ it is custom to equip $\mathcal{M}$ with the Hausdorff metric (bounded case)

$$d_{\mathrm{H}}(X, Y) = \max\{\sup_{x\in X}\inf_{y\in Y} d(x, y), \sup_{y\in Y}\inf_{x\in X} d(x, y)\}, \tag{4.3}$$

or the Attouch-Wetts metric (unbounded case)

$$d_{\mathrm{AW}}(A, B) = \sum_{i=1}^{\infty} 2^{-i} \min\{1, \sup_{|x|<i} |d(x, A) - d(x, B)|\}, \tag{4.4}$$

where A and B are closed subsets of $\mathbb{C}$, and where $d(x, A)$ is the usual Euclidean distance between the point $x \in \mathbb{C}$ and A, which is well-defined even when A is unbounded.

Definition 4.9 (The SCI Hierarchy (Attouch-Wetts/Hausdorff metric))**.** Given the setup in Definition 4.7 and suppose in addition that $\mathcal{M}$ is a metric space with the Attouch-Wetts or the Hausdorff metric induced by another metric space $\mathcal{M}'$. Define for $m \in \mathbb{N}$

$$\Sigma_0^{\alpha} = \Pi_0^{\alpha} = \Delta_0^{\alpha},$$
$$\Sigma_1^{\alpha} = \{\{\Xi, \Omega\} \in \Delta_2 \mid \exists\, \Gamma_n \in \mathcal{T} \text{ s.t. } \Gamma_n(A) \underset{\mathcal{M}'}{\subset} B_{2^{-n}}^{\mathcal{M}}(\Xi(A))\ \forall A \in \Omega\},$$
$$\Pi_1^{\alpha} = \{\{\Xi, \Omega\} \in \Delta_2 \mid \exists\, \Gamma_n \in \mathcal{T} \text{ s.t. } B_{2^{-n}}^{\mathcal{M}}(\Gamma_n(A)) \underset{\mathcal{M}'}{\supset} \Xi(A)\ \forall A \in \Omega\}.$$

where $\subset_{\mathcal{M}'}$ means inclusion in the metric space $\mathcal{M}'$, and $B_{\delta}^{\mathcal{M}}(\cdot)$ denotes the closed δ-ball in the metric space $\mathcal{M}$. There is a slight abuse of notation as we interpret $B_{2^{-n}}^{\mathcal{M}}(x)$ as a subset of $\mathcal{M}'$ by identifying it with $\bigcup\{S \subset \mathcal{M}' \mid S \in \mathcal{M}, d_{\mathcal{M}}(S, x) \leq 2^{-n}\}$. Moreover,

$$\Sigma_{m+1}^{\alpha} = \{\{\Xi, \Omega\} \in \Delta_{m+2} \mid \exists\, \Gamma_{n_{m+1},\ldots,n_1} \in \mathcal{T} \text{ s.t. } \Gamma_{n_{m+1}}(A) \underset{\mathcal{M}'}{\subset} B_{2^{-n}}^{\mathcal{M}}(\Xi(A))\ \forall A \in \Omega\},$$
$$\Pi_{m+1}^{\alpha} = \{\{\Xi, \Omega\} \in \Delta_{m+2} \mid \exists\, \Gamma_{n_{m+1},\ldots,n_1} \in \mathcal{T} \text{ s.t. } B_{2^{-n}}^{\mathcal{M}}(\Gamma_{n_{m+1}}(A)) \underset{\mathcal{M}'}{\subset} \Xi(A)\ \forall A \in \Omega\}.$$

Note that to build a Σ_1 algorithm, it is enough by taking subsequences of n to construct $\Gamma_n(A)$ such that $\Gamma_n(A) \subset \mathcal{N}_{E_n(A)}(\Xi(A))$ with some computable $E_n(A)$ that converges to zero. The same extension can be applied to the real line with the usual metric, or $\{0, 1\}$ with the discrete metric (where we interpret 1 as "Yes").

If the metric space $\mathcal{M} = \{0, 1\}$, it is clearly a totally ordered set and hence, from Definition 4.8, we get the SCI hierarchy for arbitrary decision problems. In this case we can also define the SCI hierarchy in terms of quantifiers similarly to the definition of the arithmetical hierarchy. In particular, we have the following.

Definition 4.10 (SCI hierarchy, $\mathcal{M} = \{0, 1\}$ (alternative definition))**.** Given the general setup above we define the following:

(i) We say that $\Xi : \Omega \to \mathcal{M}$ permits a representation by an alternating quantifier form of length m if

$$\Xi = (Q_m n_m) \cdots (Q_1 n_1) \Gamma_{n_m,\ldots,n_1},$$

where (Q_i) is a list of alternating quantifiers $(\forall)$ and $(\exists)$, and all $\Gamma_{n_m,\ldots,n_1} : \Omega \to \mathcal{M}$ are general algorithms in the sense of Definition 4.3.

(ii) We say that $\{\Xi, \Omega\}$ is Σ_m if an alternating quantifier form of length m exists with Q_m being $(\exists)$, and that $\{\Xi, \Omega\}$ is Π_m if an alternating quantifier form of length m exists with Q_m being $(\forall)$.

(iii) We say that $\{\Xi, \Omega\}$ is Δ_m if $\{\Xi, \Omega\}$ is Σ_m and Π_m.

It is not clear from the wordings of Definition 4.8 and Definition 4.10 that they are equivalent. However, the next propositions provides the link. We also need the following definition (which holds for standard spaces such as $\{0, 1\}$ or $\mathbb{R}$ with the usual metric).

Definition 4.11. Given a totally ordered metric space $(\mathcal{M}, d)$, we say that the metric is order respecting if for any $a, b, c \in \mathcal{M}$ with $a \leq b \leq c$ we have $d(a, b) \leq d(a, c)$.

please read the section and figure out what we should say here

Proposition 4.12 (Properties of the SCI hierarchy)**.** *Given the setup above we have the following.*

(i) The SCI hierarchy encompasses the arithmetical hierarchy.

(ii) When $\mathcal{M} = \{0, 1\}$ then definition 4.8 and Definition 4.10 are equivalent and hence the SCI encompasses generalisations of the arithmetical hierarchy. This also holds for arithmetic towers (which extends the arithmetical hierarchy to arbitrary domains).

(iii) When $\mathcal{M} = \{0, 1\}$ then $\Delta_k^\alpha = \Sigma_k^\alpha \cap \Pi_k^\alpha$ for all k and $\alpha = G, A$.

(iv) When considering the Hausdorff or Attouch-Wetts metrics or a totally ordered metric space with order respecting metric, then $\Delta_k^G = \Sigma_k^G \cap \Pi_k^G$ for $k = 1, 2, 3$. In particular, if for a problem

$$\Xi : \Omega \to \mathcal{M}$$

we have $\Delta_k^G \not\ni \{\Xi, \Omega\} \in X_k^\alpha$, where $X = \Sigma(\Pi)$ and α denotes any type of tower, then $\{\Xi, \Omega\} \notin Y_k^\alpha$, where $Y = \Pi(\Sigma)$.

Remark 4.13. Part (iv) shows that the classifications obtained in this paper are sharp in the SCI hierarchy.

5. Main theorems on the general computational spectral problem

For $A \in \Omega$, where Ω is an appropriate domain of operators, we define the problem functions

$$\Xi_{\mathrm{sp}}(A) := \mathrm{sp}(A) \quad \text{(spectrum)}, \qquad \Xi_{\text{e-sp}}(A) := \mathrm{sp}_{\mathrm{ess}}(A) \quad \text{(essential spectrum)} \tag{5.1}$$

$$\Xi_{\mathrm{sp},\epsilon}^N(A) := \mathrm{sp}_{N,\epsilon}(A) \quad \text{(pseudo-spectrum)} \qquad \Xi_{\mathrm{sp}}^z(A) := \textit{Yes} \text{ if } z \in \mathrm{sp}(A), \textit{No} \text{ otherwise.} \tag{5.2}$$

Here $\mathrm{sp}(A)$ denotes the spectrum, $\mathrm{sp}_{\mathrm{ess}}(A)$ the essential spectrum (invariant under compact perturbations) and $\mathrm{sp}_{N,\epsilon}(A)$ denotes the (N, ϵ)-pseudospectrum [18, 57, 100]

$$\mathrm{sp}_{N,\epsilon}(A) := \mathrm{cl}(\{z \in \mathbb{C} : \|(A - zI)^{-2^N}\|^{2^{-N}} > 1/\epsilon\}), \qquad N \in \mathbb{Z}_{\geq 0}, \epsilon > 0, \tag{5.3}$$

where we use the convention that $\|(A - zI)^{-2^N}\| = \infty$ when $z \in \mathrm{sp}(A)$. This set has been popular in spectral theory, analysis of pseudo differential operators and non-Hermitian quantum mechanics. For computing the spectrum/essential spectrum/(N, ϵ)-pseudospectrum, we consider computational problems $\{\Xi, \Omega, \mathcal{M}, \Lambda\}$ a la the ones in Example 4.1 in §4 (i.e. with respect to the Hausdorff metric). For the final problem of determining if $z \in \mathrm{sp}(A)$, the metric space becomes the discrete metric on $\{\textit{Yes}, \textit{No}\}$. To avoid trivialities, when considering self-adjoint classes of operators we will restrict to $z \in \mathbb{R}$ and when considering compact operators we will restrict to $z \neq 0$. The key question then becomes:

> *Given a problem function Ξ of the form* (5.1) *or* (5.2) *with a domain Ω and evaluation set Λ, where in the SCI hierarchy is the computational problem $\{\Xi, \Omega, \mathcal{M}, \Lambda\}$?*

This obviously becomes an infinite classification theory, however, we will establish some of the foundations. In order to do that we consider certain key domains such as the set of bounded self-adjoint or normal operators on $l^2(\mathbb{N})$, compact operators, operators on $l^2(\mathbb{N})$ with off-diagonal decay (bounded dispersion), operators with controlled growth of the resolvent etc. To define such domains we need a couple of definitions.

Definition 5.1 (Dispersion)**.** We say that the dispersion of $A \in \mathcal{B}(l^2(\mathbb{N}))$ is bounded by the function $f : \mathbb{N} \to \mathbb{N}$ if

$$D_{f,m}(A) := \max\{\|(I - P_{f(m)})AP_m\|, \|P_m A(I - P_{f(m)})\|\} \to 0 \quad \text{as } m \to \infty.$$

Note that for every operator A there is always a function f which is a bound for its dispersion since AP_m, $P_m A$ are compact and $\{P_n\}$ converges strongly to the identity. But there is no function f which acts as a uniform bound for all operators. Nevertheless, there are important (sub)classes of operators having well known uniform bounds, which should be mentioned:

(i) band operators with bandwidth less than d: $f(k) = k + d$.
(ii) band-dominated and weakly band-dominated operators: $f(k) = 2k$. For definitions and properties of band and band-dominated operators see [74, 83, 88]. Weakly band-dominated operators can be found in [77].
(iii) Laurent/Toeplitz operators with piecewise continuous generating function: $f(k) = k^2$ (cf. [19] and [67, Proposition 5.4]).
(iv) Let $\mathcal{F}$ be a family of bounded operators with a common bound f. Then $\tilde{f}$, given by $\tilde{f}(k) = f(k)+k$, is a common bound for all operators in the Banach algebra which is generated by $\mathcal{F}$.

Without loss of generality, we assume that f is strictly increasing and $f(n) > n$. We are also interested in operators where the control of the growth of the resolvent is bounded.

Definition 5.2 (Controlled growth of the resolvent)**.** Let $g : [0, \infty) \to [0, \infty)$ be a continuous function, vanishing only at $x = 0$ and tending to infinity as $x \to \infty$ with $g(x) \leq x$. We say that a closed operator A with non-empty spectrum on the Hilbert space $\mathcal{H}$ has controlled growth of the resolvent by g if

$$\|(A - zI)^{-1}\|^{-1} \geq g(\mathrm{dist}(z, \mathrm{sp}(A))) \quad \forall z \in \mathbb{C}, \tag{5.4}$$

where we use the convention $\|B^{-1}\|^{-1} := 0$ if B has no bounded inverse.

Notice that for every operator A there always exists such a g (define $g(\alpha) := \min\{\|(A-zI)^{-1}\|^{-1} : z \in \mathbb{C} \text{ with } \mathrm{dist}(z, \mathrm{sp}(A)) = \alpha\}$, taking continuity and compactness into account) although there is no g which works for all A.

Remark 5.3 (Assumptions on Λ)**.** In order to make the "additional knowledge" g available for the algorithms we assume that Λ contains, also the constant functions $g_{i,j} : A \mapsto g(i/j)$ $(i, j \in \mathbb{N})$, which provide the values of g in all positive rational numbers. In the case when the dispersion of the operator is known, the values $f(m)$ $(m \in \mathbb{N})$ shall be available to the algorithms as constant evaluation functions. When computing problems with SCI $= 1$ for Ω_f (and Ω_{fg}), our algorithms also require the knowledge of a null sequence c_m such that $D_{f,m}(A) \leq c_m$.

We consider the following domains defined below. In the cases of bounded dispersion or controlled growth of the resolvent we assume that we are given either f or g as above. Indeed, considering bounded operators on $l^2(\mathbb{N})$ we define the following sets.

Ω_{B} := bounded operators	Ω_{N} := bounded normal operators,
Ω_{SA} := bounded self-adjoint operators	Ω_{C} := compact operators,
Ω_f := bounded oper. w/ dispersion bounded by f	Ω_g := bounded oper. w/ contr. res. growth by g.
$\Omega_{fg} := \Omega_f \cap \Omega_g$	Ω_{D} := bounded, diagonal, self-adjoint operators.

Note that to avoid trivialities, in the case of $\{\Xi_{\mathrm{sp}}^z, \Omega_{\mathrm{D}}\}$ we take z to be real. Given the different domains, we can now state the main theorem for bounded operators.

Theorem 5.4 (The bounded computational spectral problem)**.** *Given the setup above we have the following classification results in the SCI hierarchy.*

(i) Spectrum:

$\Delta_3^G \not\ni \{\Xi_{\mathrm{sp}}, \Omega_{\mathrm{B}}\} \in \Pi_3^A$ (*all oper.*), $\Delta_2^G \not\ni \{\Xi_{\mathrm{sp}}, \Omega_{\mathrm{N}}\} \in \Sigma_2^A$ (*normal*),
$\Delta_2^G \not\ni \{\Xi_{\mathrm{sp}}, \Omega_{\mathrm{SA}}\} \in \Sigma_2^A$ (*self-adj.*), $\Sigma_1^G \cup \Pi_1^G \not\ni \{\Xi_{\mathrm{sp}}, \Omega_{\mathrm{C}}\} \in \Delta_2^A$ (*compact*),
$\Delta_2^G \not\ni \{\Xi_{\mathrm{sp}}, \Omega_f\} \in \Pi_2^A$ (*disp. bound. by f*), $\Delta_2^G \not\ni \{\Xi_{\mathrm{sp}}, \Omega_g\} \in \Sigma_2^A$ (*resolvent growth bound. by g*),
$\Delta_1^G \not\ni \{\Xi_{\mathrm{sp}}, \Omega_{fg}\} \in \Sigma_1^A$ (*res. growth bound. by g and disp. bound. by f*)

(ii) Essential spectrum:

$\Delta_3^G \not\ni \{\Xi_{\text{e-sp}}, \Omega_{\mathrm{B}}\} \in \Pi_3^A$ (*all oper.*), $\Delta_3^G \not\ni \{\Xi_{\text{e-sp}}, \Omega_{\mathrm{N}}\} \in \Pi_3^A$ (*normal*),
$\Delta_3^G \not\ni \{\Xi_{\text{e-sp}}, \Omega_{\mathrm{SA}}\} \in \Pi_3^A$ (*self-adj.*), $\Delta_2^G \not\ni \{\Xi_{\text{e-sp}}, \Omega_{\mathrm{D}}\} \in \Pi_2^A$ (*self-adj. diag.*),
$\Delta_2^G \not\ni \{\Xi_{\text{e-sp}}, \Omega_f\} \in \Pi_2^A$ (*disp. bound. by f*), $\Delta_3^G \not\ni \{\Xi_{\text{e-sp}}, \Omega_g\} \in \Pi_3^A$ (*resolvent growth bound. by g*),
$\Delta_2^G \not\ni \{\Xi_{\mathrm{sp}}, \Omega_{fg}\} \in \Pi_2^A$ (*res. growth bound. by g and disp. bound. by f*)

(iii) Pseudospectrum:

$\Delta_2^G \not\ni \{\Xi_{\mathrm{sp},\epsilon}^N, \Omega_{\mathrm{B}}\} \in \Sigma_2^A$ (*all oper.*), $\Delta_2^G \not\ni \{\Xi_{\mathrm{sp},\epsilon}^N, \Omega_{\mathrm{N}}\} \in \Sigma_2^A$ (*normal*),
$\Delta_2^G \not\ni \{\Xi_{\mathrm{sp},\epsilon}^N, \Omega_{\mathrm{SA}}\} \in \Sigma_2^A$ (*self-adj.*), $\Sigma_1^G \cup \Pi_1^G \not\ni \{\Xi_{\mathrm{sp},\epsilon}^N, \Omega_{\mathrm{C}}\} \in \Delta_2^A$ (*compact*),
$\Delta_1^G \not\ni \{\Xi_{\mathrm{sp},\epsilon}^N, \Omega_f\} \in \Sigma_1^A$ (*disp. bound. by f*), $\Delta_2^G \not\ni \{\Xi_{\mathrm{sp},\epsilon}^N, \Omega_g\} \in \Sigma_2^A$ (*resolvent growth bound. by g*),
$\Delta_1^G \not\ni \{\Xi_{\mathrm{sp}}, \Omega_{fg}\} \in \Sigma_1^A$ (*res. growth bound. by g and disp. bound. by f*)

(iv) Is z in the spectrum?:

$\Delta_3^G \not\ni \{\Xi_{\mathrm{sp}}^z, \Omega_{\mathrm{B}}\} \in \Pi_3^A$ (*all oper.*), $\Delta_3^G \not\ni \{\Xi_{\mathrm{sp}}^z, \Omega_{\mathrm{N}}\} \in \Pi_3^A$ (*normal*),
$\Delta_3^G \not\ni \{\Xi_{\mathrm{sp}}^z, \Omega_{\mathrm{SA}}\} \in \Pi_3^A$ (*self-adj.*), $\Delta_2^G \not\ni \{\Xi_{\mathrm{sp}}^z, \Omega_{\mathrm{C}}\} \in \Pi_2^A$ (*compact*),
$\Delta_2^G \not\ni \{\Xi_{\mathrm{sp}}^z, \Omega_f\} \in \Pi_2^A$ (*disp. bound. by f*), $\Delta_3^G \not\ni \{\Xi_{\mathrm{sp}}^z, \Omega_g\} \in \Pi_3^A$ (*resolvent growth bound. by g*),
$\Delta_2^G \not\ni \{\Xi_{\mathrm{sp}}^z, \Omega_{\mathrm{D}}\} \in \Pi_2^A$ (*self-adj. diag.*), $\Delta_2^G \not\ni \{\Xi_{\mathrm{sp}}, \Omega_{fg}\} \in \Pi_2^A$ (*res. growth bound. by g and disp. bound. by f*).

Remark 5.5. In order to gain the Σ_1^A algorithms for $\Xi_{\mathrm{sp},\epsilon}^N$ we need an upper bound for $\|A\|$ when $N > 0$ (without which we gain a Δ_2^A classification). No such knowledge is needed for the other towers of algorithms.

6. MAIN THEOREMS ON COMPUTATIONAL QUANTUM MECHANICS

Here we formally state the results summarised in §2.2. We consider the spectral and pseudospectral mappings Ξ_{sp}, $\Xi_{\mathrm{sp},\epsilon}$ from (5.1) for Schrödinger operators:

$$H = -\triangle + V, \qquad V : \mathbb{R}^d \to \mathbb{C}. \tag{6.1}$$

We assume that the information the algorithm can read are point samples $V(x)$ for $x \in \mathbb{R}^d$. In particular, Λ is as in 4.1 in §4. Moreover, $\mathcal{M}$ is the collection of closed subsets of $\mathbb{C}$ with the standard Attouch-Wets metric (4.4). If we fix the domain of H such that it is appropriate for a class of potentials V, the spectrum of H is uniquely determined by V. The basic question is therefore:

Given a class of Schrödinger operators $-\triangle + V \in \Omega$, let Ξ be either Ξ_{sp} or $\Xi_{\mathrm{sp},\epsilon}$, Λ and $\mathcal{M}$ as above, where in the SCI hierarchy is the computational problem $\{\Xi, \Omega, \mathcal{M}, \Lambda\}$?

Though we have stuck to the Hilbert space $\mathrm{L}^2(\mathbb{R}^d)$ for simplicity, the algorithms we construct can also be adapted for other spaces commonly found in applications such as $\mathrm{L}^2(\mathbb{R}_{>0})$.

Bounded Potentials. We first consider cases with bounded potential. In particular, let $\phi : [0,\infty) \to [0,\infty)$ be some increasing function and $M > 0$, define

$$\begin{aligned}\Omega_\phi &:= \{H : \mathcal{D}(H) = \mathrm{W}^{2,2}(\mathbb{R}^d), V \in \mathrm{BV}_\phi(\mathbb{R}^d), \|V\|_\infty \leq M\},\\ \Omega_{\phi,g} &:= \{H \in \Omega_\phi : \|(-\Delta + V - zI)^{-1}\|^{-1} \geq g(\mathrm{dist}(z, \mathrm{sp}(H)))\},\end{aligned}$$

where

$$\mathrm{BV}_\phi(\mathbb{R}^d) = \{f : \mathrm{TV}(f_{[-a,a]^d}) \leq \phi(a)\}, \tag{6.2}$$

($f_{[-a,a]^d}$ means f restricted to the box $[-a,a]^d$) with TV being the total variation of a function in the sense of Hardy and Krause (see [80]). Here as in §5, $g : [0,\infty) \to [0,\infty)$ is a continuous strictly increasing function with $g(x) \leq x$, vanishing only at $x = 0$ and tending to infinity as $x \to \infty$.

Note that the set Ω_ϕ requires a little bit more than V just being locally of bounded variation. There is a universal upper bound across the class on the growth of the total variation of the potential function as we restrict the function to a larger set. The class $\Omega_{\phi,g}$ obviously includes self-adjoint and normal operators in Ω_ϕ, however, it is much bigger.

Remark 6.1 (Assumptions on Λ)**.** As done in the case of bounded Hilbert space operators, and as discussed in Remark 5.3, the additional knowledge of g is available for the algorithms by assuming that Λ also contains the constant functions $g_{i,j} : V \mapsto g(i/j)$ $(i,j \in \mathbb{N})$, which provide the values of g in all positive rational numbers. Moreover, we need access to ϕ in a similar way, that is Λ contains the constant functions ϕ_n : $V \mapsto \phi(n)$ for $n \in \mathbb{N}$.

Theorem 6.2 (Bounded potential)**.** *Given the above set-up, we have the following classification results.*

$$\begin{aligned}&\Delta_1^G \not\ni \{\Xi_{\mathrm{sp}}, \Omega_\phi\} \in \Pi_2^A, &&\Delta_1^G \not\ni \{\Xi_{\mathrm{sp},\epsilon}, \Omega_\phi\} \in \Sigma_1^A,\\ &\Delta_1^G \not\ni \{\Xi_{\mathrm{sp}}, \Omega_{\phi,g}\} \in \Sigma_1^A, &&\Delta_1^G \not\ni \{\Xi_{\mathrm{sp},\epsilon}, \Omega_{\phi,g}\} \in \Sigma_1^A.\end{aligned}$$

As will be evident from the proof techniques, one can build towers of algorithms for operators with more general classes of potentials (for example $\mathrm{L}^\infty(\mathbb{R}^d) \cap \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d)$) or $\mathrm{L}^2(\mathbb{R}^d) \cap \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d)$), however, the height of these towers will be higher than the ones considered in this paper. The main future task is to obtain exact values of the SCI of the spectrum given the different potential classes.

Unbounded Potentials We get a rather intriguing phenomenon for sectorial operators. Namely, the SCI of both the spectrum and the pseudospectrum is one, but no type of error control is possible. In particular, suppose that we have non-negative θ_1, θ_2 such that $\theta_1 + \theta_2 < \pi$. Define

$$\Omega_\infty = \{V \in \mathrm{C}(\mathbb{R}^d) : \forall x \arg(V(x)) \in [-\theta_2, \theta_1], |V(x)| \to \infty \text{ as } x \to \infty\}. \tag{6.3}$$

We define the operator H via the minimal operator h as: $H = h^{**}$, $h = -\Delta + V$, $\mathcal{D}(h) = \mathrm{C}_c^\infty(\mathbb{R}^d)$. When $V \in \Omega_\infty$ it follows that H has compact resolvent, a result that we also establish as a part of the proof of the following theorem.

Interestingly, no constant function are needed in Λ in order to obtain the results in the following theorem, as opposed to the case where we have a bounded potential.

Theorem 6.3 (Unbounded potential)**.** *Given the above set-up, we have the following classification results*

$$\Sigma_1^G \cup \Pi_1^G \not\ni \{\Xi_{\mathrm{sp}}, \Omega_\infty\} \in \Delta_2^A, \quad \Sigma_1^G \cup \Pi_1^G \not\ni \{\Xi_{\mathrm{sp},\epsilon}, \Omega_\infty\} \in \Delta_2^A.$$

Note that the key to this result is the compact resolvent of H. It is therefore natural that these problems have the same SCI classification as for compact operators Ω_{C} (see Theorem 5.4 in §5). The continuity assumption on V in Theorem 6.3 is to make sure that the discretization used actually converges. However, by tweaking with the approximation this assumption can may be weakened to include potentials that have certain discontinuities.

7. Main theorems on solving linear systems

Suppose that $b \in l^2(\mathbb{N})$, $A \in \mathcal{B}_{\mathrm{inv}}(l^2(\mathbb{N}))$ (the set of bounded invertible operators) and $\Omega \subset \mathcal{B}_{\mathrm{inv}}(l^2(\mathbb{N})) \times l^2(\mathbb{N})$ and we define the mappings $\Xi_{\mathrm{inv}} : \Omega \ni (A, b) \mapsto A^{-1}b$, and $\Xi_{\mathrm{norm}} : A \mapsto \|A^{-1}\|^{-1}$. Depending on the problem function $\mathcal{M}$ is either $l^2(\mathbb{N})$ or $\mathbb{R}$ with the canonical metrics. We ask the following basic question:

> *Where in the SCI hierarchy are the computational problems $\{\Xi, \Omega, \mathcal{M}, \Lambda\}$ for different domains Ω when Ξ is either Ξ_{inv} or Ξ_{norm}, with the appropriate choices of $\mathcal{M}$?*

Remark 7.1 (Assumptions on Λ)**.** Here, as in Example 4.1, we again suppose that the set Λ of evaluations consists of the functions which read the matrix elements $\{\langle Ae_j, e_i\rangle\}_{i,j\in\mathbb{N}}$ and the sequence entries $\{\langle b, e_k\rangle\}_{k\in\mathbb{N}}$ of $(A, b) \in \Omega$. Also, in the case when the dispersion of the operator is known, the values $f(m)$ $(m \in \mathbb{N})$ shall be available to the algorithms as constant evaluation functions. However, if the dispersion is not known then Λ will not contain any constant functions in the theorems below.

Theorem 7.2 (Solving linear systems)**.** *Let $\mathcal{B}_{\mathrm{inv},f}(l^2(\mathbb{N}))$ denote the class of bounded invertible operators with dispersion bounded by $f : \mathbb{N} \to \mathbb{N}$, $\mathcal{B}^M_{\mathrm{inv},f}(l^2(\mathbb{N}))$ denote the class of operators $A \in \mathcal{B}_{\mathrm{inv},f}(l^2(\mathbb{N}))$ with the $\|A^{-1}\| \le M$, $\mathcal{B}_{\mathrm{inv},sa}(l^2(\mathbb{N}))$ denote the class of bounded invertible self-adjoint operators, and define the domains $\Omega_1 = \mathcal{B}_{\mathrm{inv}}(l^2(\mathbb{N})) \times l^2(\mathbb{N})$, $\Omega_2 = \mathcal{B}_{\mathrm{inv},sa}(l^2(\mathbb{N})) \times l^2(\mathbb{N})$ and $\Omega_3 = \mathcal{B}_{\mathrm{inv},f}(l^2(\mathbb{N})) \times l^2(\mathbb{N})$.*

$$\Delta^G_2 \not\ni \{\Xi_{\mathrm{inv}}, \Omega_1\} \in \Delta^A_3,$$
$$\Delta^G_2 \not\ni \{\Xi_{\mathrm{inv}}, \Omega_2\} \in \Delta^A_3,$$
$$\Delta^G_1 \not\ni \{\Xi_{\mathrm{inv}}, \Omega_3\} \in \Delta^A_2.$$

Furthermore, if we define $\Omega_4 = \mathcal{B}^M_{\mathrm{inv},f}(l^2(\mathbb{N})) \times l^2(\mathbb{N})$, and in this particular case we assume knowledge of a null sequence $\{c_m\}_{m\in\mathbb{N}}$ such that $D_{f,m}(A) \le c_m$ and $\|b - P_m b\| \le c_m$ then we have the error control

$$\Delta^G_0 \not\ni \{\Xi_{\mathrm{inv}}, \Omega_4\} \in \Delta^A_1. \tag{7.1}$$

Another problem of interest when dealing with solutions of linear systems of equations is the computation of the norm of the inverse. This is obviously related to the stability of the problem. The task of computing the norm of the inverse of an operator can also be analysed in terms of the SCI, and that is the topic of the next theorem. Note that since our metric space is $\mathbb{R}$ with the usual metric, we have a notion of Σ or Π convergence.

Theorem 7.3 (Computing norm of the inverse)**.** *Let $\Omega_1 = \mathcal{B}(l^2(\mathbb{N}))$, Ω_2 the subset of self-adjoint operators, Ω_3 the subset of operators with dispersion bounded by an $f : \mathbb{N} \to \mathbb{N}$, and let $\Xi_{\mathrm{norm}} : A \mapsto \|A^{-1}\|^{-1}$.* [1] *Then*

$$\Delta^G_2 \not\ni \{\Xi_{\mathrm{norm}}, \Omega_1\} \in \Pi^A_2, \quad \Delta^G_2 \not\ni \{\Xi_{\mathrm{norm}}, \Omega_2\} \in \Pi^A_2, \quad \Delta^G_1 \not\ni \{\Xi_{\mathrm{norm}}, \Omega_3\} \in \Pi^A_1. \tag{7.2}$$

Remark 7.4. As in the spectral case, we require the knowledge of a null sequence c_m such that $D_{f,m}(A) \le c_m$ in order to gain $\{\Xi_{\mathrm{norm}}, \Omega_3\} \in \Pi^A_1$. Without this knowledge the constructed algorithm gives a Δ^A_2 classification.

8. Proof of Theorem 5.4

We start the sections on the proofs of our main results with a simple but fundamental observation on the smallest singular values $\sigma_1(B)$ of finite matrices $B \in \mathbb{C}^{m\times n}$, which constitutes one of the corner stones for most of the general algorithms we will construct in the subsequent proofs. Note that when dealing with

[1] As usual, $\|A^{-1}\|^{-1} := 0$ if A is not invertible. We could have equally chosen to compute $\|A^{-1}\|$ with the point at infinity added to a suitable metrisation of $\mathbb{R}$. In this case we would get Σ rather than Π classification.

infinite dimensional operators, we will also use the notation σ_1 to denote the injection modulus defined, for $A \in \mathcal{B}(\mathcal{H})$ on some Hilbert space $\mathcal{H}$, as

$$\sigma_1(A) := \inf_{\|x\|=1} \|Ax\|.$$

Proposition 8.1. *Given a matrix $B \in \mathbb{C}^{m\times n}$ and a number $\epsilon > 0$ one can test with finitely many arithmetic operations of the entries of B whether the smallest singular value $\sigma_1(B)$ of B is greater than ϵ.*

Proof. The matrix B^*B is self-adjoint and positive semidefinite, hence has its eigenvalues in $[0, \infty)$. The singular values of B are the square roots of these eigenvalues of B^*B. The smallest singular value is greater than ϵ iff the smallest eigenvalue of B^*B is greater than ϵ^2, which is the case iff $C := B^*B - \epsilon^2 I$ is positive definite. It is well known that C is positive definite if and only if the pivots left after Gaussian elimination (without row exchange) are all positive. Thus, if C is positive definite, Gaussian elimination leads to pivots that are all positive, and this requires finitely many arithmetic operations. If C is not positive definite, then at some point a pivot is zero or negative, at this point the algorithm aborts. An alternative is the Cholesky decomposition. Although forming the lower triangular $L \in \mathbb{C}^{n\times n}$ (if it exists) such that $C = LL^*$ requires the use of radicals, the existence of L can be determined using finitely many arithmetic operations. This follows from the standard Cholesky algorithm, and we omit the details. $\square$

We will split the proof of Theorem 5.4 into several parts and a brief roadmap for the proof is as follows. We first deal with computing the spectra and pseudospectra of compact operators since the constructive parts of the proof uses a different (most likely more familiar) method, the finite section method, than the proof for the other classes of operators. Step I of this part also contains one of the arguments used to prove lowers bounds throughout this paper and is written out in detail for the reader's convenience. We then move onto pseudospectra where variants on the method of uneven sections are used to approximate the relevant resolvent norms. In some cases these towers are used directly to provide (with an additional limit) towers of algorithms for the spectra. The proof that $\{\Xi_{\mathrm{sp}}, \Omega_g\} \in \Sigma_2^A$ uses a very different method to those usually found in the literature, a local estimation of resolvent norm (using similar ideas to §8.2) together with the function g gives rise to upper bounds on the distance of a point to the spectrum. This is then used in a local search routine to compute the spectrum. The proof that $\{\Xi_{\mathrm{sp}}, \Omega_{\mathrm{B}}\} \notin \Delta_3^G$ relies on reducing a decision problem, known to require three limits, to $\{\Xi_{\mathrm{sp}}, \Omega_{\mathrm{B}}\}$. Proof that the decision problem requires three limits is provided in §8.6 via a Baire category argument. The constructive proofs for essential spectra build on the towers of algorithms for computing spectra but are more involved. We end with the problem Ξ_{sp}^z where the proof of lower bounds uses similar arguments for the other problem functions and the construction of towers of algorithms uses the towers constructed in §8.3 for the spectrum.

8.1. Spectra and pseudospectra of compact operators.

Proof of Theorem 5.4 for compact operators. **Step I**: $\{\Xi_{\mathrm{sp}}, \Omega_{\mathrm{C}}\} \notin \Sigma_1^G$. We argue by contradiction and suppose that there is a sequence $\{\Gamma_n\}$ of general algorithms such that $\Gamma_n(A) \to \mathrm{sp}(A)$ with $\Gamma_n(A) \subset \mathcal{N}_{2^{-n}}(\mathrm{sp}(A))$, and in particular each $\Lambda_{\Gamma_n}(A)$ is finite. Thus, for every $A \in \Omega_{\mathrm{C}}$ and every n there exists a finite number $N(A, n) \in \mathbb{N}$ such that the evaluations from $\Lambda_{\Gamma_n}(A)$ only take the matrix entries $A_{ij} = \langle Ae_j, e_i\rangle$ with $i, j \leq N(A, n)$ into account. We consider an operator of the type

$$A := A_k \oplus \mathrm{diag}\{0, 0, ...\} \quad \text{with } A_n := \begin{pmatrix} 1 & & & & 1 \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ 1 & & & & 1 \end{pmatrix} \in \mathbb{C}^{n\times n}.$$

Let $C = \mathrm{diag}\{1, 0, 0, ...\}$ then $\mathrm{sp}(C) = \{0, 1\}$ and clearly A is compact with $\mathrm{sp}(A) = \{0, 2\}$. We choose k to gain a contradiction as follows. There exists n such that $\Gamma_n(C) \cap B_{1/4}(1) \neq \emptyset$. Let $k > N(C, n)$. By this construction, it follows that

$$\Gamma_n(C) = \Gamma_n(A). \tag{8.1}$$

Indeed, since any evaluation function $f_{i,j} \in \Lambda$ just provides the (i, j)-th matrix element, it follows by the choice of k that for any evaluation functions $f_{i,j} \in \Lambda_{\Gamma_n}(A)$ we have that that $f_{i,j}(A) = f_{i,j}(C)$. Thus, by assumption (iii) in the definition of a general algorithm (Definition 4.3), we get that $\Lambda_{\Gamma_n}(A) = \Lambda_{\Gamma_n}(C)$ which, by assumption (ii) in Definition 4.3, yields (8.1). But then $\Gamma_n(A) \cap B_{1/4}(1) \neq \emptyset$, which is impossible since $\Gamma_n(A) \subset \{0, 2\} + B_{1/2^n}(0)$, a contradiction.

Step II: $\{\Xi_{\mathrm{sp}}, \Omega_{\mathrm{C}}\} \notin \Pi_1^G$. This is essentially the same argument. Assume that there exists Γ_n such that $\mathrm{sp}(A) \subset \mathcal{N}_{2^{-n}}(\Gamma_n(A))$. Let A and C be as before. But now we know that there exists n such that $\Gamma_n(C) \cap B_{3/4}(2) = \emptyset$. We argue as before, choosing $k > N(C, n)$, to get $\Gamma_n(C) = \Gamma_n(A)$. But we must have $2 \in \mathcal{N}_{2^{-n}}(\Gamma_n(A))$, a contradiction.

Step III: $\{\Xi^N_{\mathrm{sp},\epsilon}, \Omega_{\mathrm{C}}\} \notin \Pi_1^G \cup \Sigma_1^G$. For sufficiently small ϵ we have the required separation such that the above argument works for $\Xi^N_{\mathrm{sp},\epsilon}$. For larger ϵ we simply rescale the operators in the argument appropriately.

Step IV: $\{\Xi_{\mathrm{sp}}, \Omega_{\mathrm{C}}\} \in \Delta_2^A$. For $n \in \mathbb{N}$, let $G_n = \frac{1}{n}(\mathbb{Z} + i\mathbb{Z}) \cap B_n(0)$. For $A \in \Omega_{\mathrm{C}}$ let

$$\Gamma_n(A) = \{z \in G_n : \sigma_1(P_n(A - zI)P_n) \leq 1/n\},$$

where P_n denotes the orthogonal projection onto the linear span of the first n basis vectors. By Proposition 8.1, it is clear that this can be computed in finitely many arithmetical operations and comparisons.

Hence we are done if we can prove convergence, the proof of which will make clear that we can make $\Gamma_n(A)$ non-empty by replacing $\Gamma_n(A)$ with $\Gamma_{m(n)}(A)$ such that $m(n) \geq n$ is minimal with $\Gamma_{m(n)}(A) \neq \emptyset$. Let $\epsilon > 0$, then choose $N > 2/\epsilon$. If $n \geq N$ and $z \in \Gamma_n(A)$ then we must have $\sigma_1(P_n(A - zI)P_n) \leq \epsilon/2$. Hence there exists $x_n \in l^2(\mathbb{N})$ of norm 1 and with $x_n = P_n x_n$ such that $\|(P_n A - zI)x_n\| \leq \epsilon/2$. A is compact and hence we can choose N large if necessary to ensure that $\|(I - P_n)A\| \leq \epsilon/2$. It follows that $\|(A - zI)x_n\| \leq \epsilon$ and hence z is in $\mathrm{sp}_\epsilon(A)$. Note that N does not depend on the point z so for large n we have

$$\Gamma_n(A) \subset \mathrm{sp}_\epsilon(A). \tag{8.2}$$

Conversely, let $z \in \mathrm{sp}(A)$. The method of finite section converges for compact operators and hence there exists $z_n \in \mathrm{sp}(P_n A P_n)$ with $z_n \to z$. Let $w_n \in G_n$ be of minimal distance to z_n then for large n we must have $|w_n - z_n| \leq 1/(\sqrt{2}n)$ and hence $\sigma_1(P_n(A - w_n I)P_n) \leq 1/(\sqrt{2}n) < 1/n$. It follows that $w_n \in \Gamma_n(A)$. Let $\epsilon > 0$, then we can choose a finite set $S_\epsilon \subset \mathrm{sp}(A)$ with $d_H(S_\epsilon, \mathrm{sp}(A)) < \epsilon/2$. Applying the above argument to all points in S_ϵ implies that for large n we must have

$$\mathrm{sp}(A) \subset \Gamma_n(A) + B_\epsilon(0). \tag{8.3}$$

Since $\epsilon > 0$ was arbitrary, (8.2) and (8.3) imply the required convergence.

Step V: $\{\Xi^N_{\mathrm{sp},\epsilon}, \Omega_{\mathrm{C}}\} \in \Delta_2^A$. This will follow from the classification of $\{\Xi^N_{\mathrm{sp},\epsilon}, \Omega_f\}$ since we can use a dispersion bounding function $f(n) = n + 1$. Note that we do not necessarily know the dispersion bound (in the form of the null sequence $\{c_n\}$) and hence (see Remark 5.5) this provides a Δ_2^A tower (without Σ_1 error control). □

8.2. **Pseudospectrum.** Since $\Omega_{\mathrm{SA}} \subset \Omega_{\mathrm{N}} \subset \Omega_g \subset \Omega_{\mathrm{B}}, \Omega_{fg} \subset \Omega_f$ and it is clear that $\{\Xi^N_{\mathrm{sp},\epsilon}, \Omega_{fg}\} \notin \Delta_1^G$, we only need to show that $\{\Xi^N_{\mathrm{sp},\epsilon}, \Omega_{\mathrm{B}}\} \in \Sigma_2^A$, $\{\Xi^N_{\mathrm{sp},\epsilon}, \Omega_f\} \in \Sigma_1^A$, $\{\Xi^N_{\mathrm{sp},\epsilon}, \Omega_{\mathrm{SA}}\} \notin \Delta_2^G$ and $\{\Xi^N_{\mathrm{sp},\epsilon}, \Omega_f\} \notin \Delta_1^G$.

Proof of Theorem 5.4 for the pseudospectrum. **Step I:** $\{\Xi_{\mathrm{sp},\epsilon}^N, \Omega_{\mathrm{B}}\} \in \Sigma_2^A$. Let $A \in \Omega_{\mathrm{B}}$, and $\epsilon > 0$. We introduce the following continuous functions $\gamma^N : \mathbb{C} \to \mathbb{R}_+$, $\gamma_m^N : \mathbb{C} \to \mathbb{R}_+$ and $\gamma_{m,n}^N : \mathbb{C} \to \mathbb{R}_+$,

$$\gamma^N(z) := \left(\min\left\{\sigma_1\left((A-zI)^{2^N}\right), \sigma_1\left((A^*-\bar{z}I)^{2^N}\right)\right\}\right)^{2^{-N}} = \left\|(A-zI)^{-2^N}\right\|^{-2^{-N}}$$

$$\gamma_m^N(z) := \left(\min\left\{\sigma_1\left((A-zI)^{2^N}P_m\right), \sigma_1\left((A^*-\bar{z}I)^{2^N}P_m\right)\right\}\right)^{2^{-N}}$$

$$\gamma_{m,n}^N(z) := \left(\min\left\{\sigma_1\left((P_n(A-zI)P_n)^{2^N}P_m\right), \sigma_1\left((P_n(A^*-\bar{z}I)P_n)^{2^N}P_m\right)\right\}\right)^{2^{-N}},$$

where $\sigma_1(B)$ denotes the injection modulus of B, and in the terms such as $\sigma_1(P_nBP_m)$ the operator P_nBP_m is regarded as element of $\mathcal{B}(\mathrm{Ran}(P_m), \mathrm{Ran}(P_n))$. For the proof that $\gamma^N(z) = \left\|(A-zI)^{-2^N}\right\|^{-2^{-N}}$ see [59]. We define initial approximations $\hat{\Gamma}_{m,n}(A)$ for $\mathrm{sp}_{N,\epsilon}(A)$ by

$$\hat{\Gamma}_{m,n}(A) := \{z \in G_n : \gamma_{m,n}^N(z) \le \epsilon\},$$

where $G_j := (j^{-1}(\mathbb{Z}+\mathrm{i}\mathbb{Z})) \cap B_j(0)$. Writing $\gamma_m^N(z) \le \epsilon$ as $(\gamma_m^N(z))^{2^N} \le \epsilon^{2^N}$ and due to Proposition 8.1 it is clear that the computation of $\hat{\Gamma}_{m,n}(A)$ requires only finitely many arithmetic operations on finitely many evaluations $\{\langle Ae_j, e_i\rangle : i,j = 1,\dots,n\}$ of A. The problem with this tower is that it might produce the empty set.

To get round this and construct our Σ_2^A arithmetical tower there are several facts we will state that can be found in [59]. Firstly, $\gamma_{m,n}^N$ converges uniformly to γ_m^N on compact subsets of $\mathbb{C}$ as $n \to \infty$. Secondly, γ_m^N is non-increasing in m and converges uniformly to γ^N on compact subsets of $\mathbb{C}$ as $m \to \infty$. Finally, we have

$$\mathrm{cl}\{z \in \mathbb{C} : \gamma_m^N(z) < \epsilon\} = \{z \in \mathbb{C} : \gamma_m^N(A) \le \epsilon\} \tag{8.4}$$

for all $\epsilon > 0$. Now it is straightforward to show via a Neumann series argument (see the proof that $\{\Xi_{\mathrm{sp}}, \Omega_g\} \in \Sigma_2^A$ below) that there exists a compact ball K such that if $z \notin K$ then $\gamma_{m,n}^N(z) > 2\epsilon$ for all m, n. In particular, by considering the minimum of $\gamma_m^N(\cdot)$, this together with the above closure property, shows that the minimum is zero and $\{z \in \mathbb{C} : \gamma_m^N(A) \le \epsilon\} \neq \emptyset$.

Now let $z_0 \in \{z \in \mathbb{C} : \gamma_m^N(z) < \epsilon\}$. On the compact set K, and for any m, the functions $\gamma_{m,n}^N$ and γ_m^N are Lipschitz continuous with a uniform Lipschitz constant. Using this and (8.4), it follows that for large enough n, there exists $z_n \in \hat{\Gamma}_{m,n}(A)$ with $z_n \to z_0$. Furthermore, if $z_n \in \hat{\Gamma}_{m,n}(A)$ and we select a subsequence such that $z_{n_j} \to z$ as $n_j \to \infty$ it follows that $\gamma_m^N(z) \le \epsilon$. This observations together imply that

$$\lim_{n\to\infty} \hat{\Gamma}_{m,n}(A) = \{z \in \mathbb{C} : \gamma_m^N(A) \le \epsilon\} \subset \mathrm{sp}_{N,\epsilon}(A).$$

Since γ_m^N converges to γ^N uniformly on compact sets and are uniformly Lipschitz, it is easy to show that $\lim_{m\to\infty}\{z \in \mathbb{C} : \gamma_m^N(A) \le \epsilon\} = \mathrm{sp}_{N,\epsilon}(A)$. Hence in order to construct our Σ_2^A arithmetical tower we define $\Gamma_{m,n}(A) = \hat{\Gamma}_{m,j(m,n)}(A)$, where $j(m,n) \ge n$ is minimal such that $\hat{\Gamma}_{m,j(m,n)}(A) \neq \emptyset$. Such a $j(m,n)$ is guaranteed to exist and can be found by successively computing finitely many of the $\hat{\Gamma}_{m,k}(A)$'s.

Step II: $\{\Xi_{\mathrm{sp},\epsilon}^N, \Omega_f\} \in \Sigma_1^A$. Let A be such that f is a bound for its dispersion, and $\epsilon > 0$. Recall that $f(n) \ge n+1$ for every n. Define the composition $F^N := f \circ \cdots \circ f$ of 2^N copies of f. Besides the already defined functions γ^N, γ_m^N and $\gamma_{m,n}^N$ we additionally introduce $\psi_m^N := \gamma_{m,F^N(m)}^N$, i.e.

$$\psi_m^N(z) := \left(\min\left\{\sigma_1\left((P_{F^N(m)}(A-zI)P_{F^N(m)})^{2^N}P_m\right), \sigma_1\left((P_{F^N(m)}(A^*-\bar{z}I)P_{F^N(m)})^{2^N}P_m\right)\right\}\right)^{2^{-N}},$$

and we define the desired approximations $\hat{\Gamma}_m(A)$ for $\mathrm{sp}_{N,\epsilon}(A)$ by

$$\hat{\Gamma}_m(A) := \{z \in G_m : \psi_m^N(z) \le \epsilon\}.$$

Writing $\psi_m^N(z) \le \epsilon$ as $(\psi_m^N(z))^{2^N} \le \epsilon^{2^N}$ and using Proposition 8.1, we see that again the computation of $\hat{\Gamma}_m(A)$ requires only finitely many arithmetic operations on finitely many evaluations $\{\langle Ae_j, e_i\rangle : i,j = 1,\dots,F^N(m)\}$ of A.

Obviously, there exists a compact ball $K \subset \mathbb{C}$ such that $\gamma_m^N(z) > 2\epsilon$ and $\psi_m^N(z) > 2\epsilon$ for all $z \in \mathbb{C} \setminus K$ and all m. Further note that ψ_m^N converges to γ_m^N uniformly on K. Indeed, since all $z \mapsto (P_{F^N(m)}(A - zI)P_{F^N(m)})^{2^N} P_m$ and $z \mapsto (A - zI)^{2^N} P_m$ are operator-valued polynomials of the same degree whose coefficients converge in the norm due to the choice of the function F^N, we can take into account that $|\sigma_1(B + C) - \sigma_1(B)| \le \|C\|$ holds for arbitrary bounded operators B, C, and we arrive at the conclusion that $|\gamma_m^N(z) - \psi_m^N(z)| \to 0$ as $m \to \infty$ uniformly with respect to $z \in K$. To construct a Σ_1^A tower we bound this difference using the sequence $\{c_n\}$ and the constant $\|A\|$ (for the case $N > 0$ as follows).

If $N = 0$ then clearly we have $\|P_{f(m)}(A - zI)P_m - (A - zI)P_m\| \le c_m$ by definition of the $\{c_n\}$. Suppose that we have a bound

$$\|(P_{F^N(m)}(A - zI)P_{F^N(m)})^{2^N} P_m - (A - zI)^{2^N} P_m\| \le \alpha(N, m, z), \tag{8.5}$$

for some function $\alpha(N, m, z)$. We can write

$$\begin{aligned}
&(P_{F^{N+1}(m)}(A - zI)P_{F^{N+1}(m)})^{2^{N+1}} P_m - (A - zI)^{2^{N+1}} P_m \\
&\quad = \big((P_{F^{N+1}(m)}(A - zI)P_{F^{N+1}(m)})^{2^N} - (A - zI)^{2^N}\big)(P_{F^{N+1}(m)}(A - zI)P_{F^{N+1}(m)})^{2^N} P_m \\
&\quad - (A - zI)^{2^N}\big((A - zI)^{2^N} - (P_{F^{N+1}(m)}(A - zI)P_{F^{N+1}(m)})^{2^N}\big)P_m.
\end{aligned}$$

Using the fact that $F^{N+1}(m) = F^N(F^N(m))$ and $P_{F^N(m)}P_{F^{N+1}(m)} = P_{F^{N+1}(m)}$, we can bound the first of the above terms in norm by $\alpha(N, F^N(m), z)(\|A\| + |z|)^{2^N}$. Arguing similarly, we can bound the second term in norm by the same quantity. It follows that we have

$$\alpha(N, m, z) = 2\alpha(N - 1, F^{N-1}(m), z)(\|A\| + |z|)^{2^{N-1}}$$

and iterating this N times we can take

$$\alpha(N, m, z) = 2^N c_{F^{\frac{N(N-1)}{2}}(m)}(\|A\| + |z|)^{2^N - 1},$$

such that (8.5) holds. Note that this estimate can be computed with finitely many arithmetic operations and comparisons from the given data.

In order to simplify the notation we choose a sequence (δ_m) which converges monotonically to zero such that

$$\gamma_m^N(z) + \delta_m \ge \psi_m^N(z) \ge \gamma_m^N(z) - \delta_m \text{ for every } m \text{ and every } z \in K.$$

Moreover, we point out that each of the functions $z \mapsto \psi_m^N(z)$ is continuous on the compact set K, hence even uniformly continuous, and we can assume without loss of generality that, for every m,

$$|\psi_m^N(z) - \psi_m^N(y)| < \delta_m \text{ for arbitrary } z, y \in K,\ |z - y| < 1/m. \tag{8.6}$$

Now let $\zeta_\epsilon(A) := \{z \in \mathbb{C} : \gamma^N(z) \le \epsilon\}$ as well as

$$\zeta_{\epsilon,m}(A) := \{z \in \mathbb{C} : \gamma_m^N(z) \le \epsilon\}, \qquad \Psi_{\epsilon,m}(A) := \{z \in \mathbb{C} : \psi_m^N(z) \le \epsilon\}.$$

By the discussion above, we conclude for all $m \ge k$ that

$$\zeta_{\epsilon+\delta_k,m}(A) \supset \zeta_{\epsilon+\delta_m,m}(A) \supset \Psi_{\epsilon,m}(A) \supset \zeta_{\epsilon-\delta_m,m}(A) \supset \zeta_{\epsilon-\delta_k,m}(A). \tag{8.7}$$

Since, $P_m \le P_{m+1}$ and $P_m \to I$ strongly, $\gamma_m^N \to \gamma^N$ monotonically from above pointwise (and hence locally uniformly by Dini's Theorem). Thus, by [59], $\zeta_{\epsilon+\delta_k,m}(A) \to \zeta_{\epsilon+\delta_k}(A) = \mathrm{sp}_{N,\epsilon+\delta_k}(A)$ and $\zeta_{\epsilon-\delta_k,m}(A) \to \zeta_{\epsilon-\delta_k}(A) = \mathrm{sp}_{N,\epsilon-\delta_k}(A)$ as $m \to \infty$. Hence, since $\mathrm{sp}_{N,\epsilon\pm\delta_k}(A) \to \mathrm{sp}_{N,\epsilon}(A)$ as $k \to \infty$, (8.7) yields $\lim_{m\to\infty} \Psi_{\epsilon,m}(A) = \mathrm{sp}_{N,\epsilon}(A)$. To finish the convergence proof we observe that it is clear that on the one hand $\Psi_{\epsilon,m}(A) \supset \hat{\Gamma}_m(A)$. On the other hand, for sufficiently large m it holds true that for every point $x \in \Psi_{\epsilon-\delta_m,m}(A)$ there is a point $y_x \in G_m$ with $|x - y_x| < 1/m$ and, by (8.6) we get $|\psi_m^N(y_x) - \psi_m^N(x)| < \delta_m$ that is y_x even belongs to $\hat{\Gamma}_m(A)$. Thus, $\hat{\Gamma}_m(A) + B_{1/m}(0) \supset \Psi_{\epsilon-\delta_m,m}(A)$ for sufficiently large m. Combining this, we arrive at

$$\Psi_{\epsilon,m}(A) + B_{1/k}(0) \supset \hat{\Gamma}_m(A) + B_{1/m}(0) \supset \Psi_{\epsilon-\delta_m,m}(A) \supset \Psi_{\epsilon-\delta_k,m}(A),$$

for $m \geq k$ large. By the above, the sets on the left tend to $\mathrm{sp}_{N,\epsilon}(A) + B_{1/k}(0)$ as $m \to \infty$, and the sets on the right converge to $\mathrm{sp}_{N,\epsilon-\delta_k}(A)$ for every k. Since both of these sets converge to $\mathrm{sp}_{N,\epsilon}(A)$ as $k \to \infty$ this provides $\lim_{m\to\infty} \hat{\Gamma}_m(A) = \mathrm{sp}_{N,\epsilon}(A)$. This shows that (upon altering as in Step I to avoid the empty set), we can gain convergence in one limit without the knowledge of $\{c_n\}$ and $\|A\|$.

Now we have that $|(\psi_m^N(z))^{2^N} - (\gamma_m^N(z))^{2^N}| \leq \alpha(N, m, z)$. Hence we define

$$\tilde{\Gamma}_m(A) := \{z \in G_m : (\psi_m^N(z))^{2^N} \leq \epsilon^{2^N} - \alpha(N, m, z) \text{ and } \epsilon^{2^N} - \alpha(N, m, z) > 0\},$$

which can be computed in finitely many arithmetic operations and comparisons. Of course this may be empty but it has the property that

$$\tilde{\Gamma}_m(A) \subset \mathrm{sp}_{N,\epsilon}(A). \tag{8.8}$$

Suppose for a contradiction that we don't have convergence to $\mathrm{sp}_{N,\epsilon}(A)$. Without loss of generality, by taking a subsequence if necessary, there exists $z_m \in \mathrm{sp}_{N,\epsilon}(A)$, $z \in \mathrm{sp}_{N,\epsilon}(A)$ and $\delta > 0$ such that $\gamma^N(z) < \epsilon$, $z_m \to z$ but $\mathrm{dist}(z_m, \tilde{\Gamma}_m(A)) \geq \delta$. Let $\hat{z}_m \in G_m$ with $\hat{z}_m \to z$. Then for large m we must have $\gamma^N(\hat{z}_m) < \epsilon$. But $\alpha(N, m, \hat{z}_m) \to 0$ and hence $\hat{z}_m \in \tilde{\Gamma}_m(A)$ for large m, the required contradiction. To finish we simply define $\Gamma_m(A) = \hat{\Gamma}_{j(m)}(A)$, where $j(m) \geq m$ is minimal such that $\hat{\Gamma}_{j(m)}(A) \neq \emptyset$. Such a $j(m)$ must exist and we hence avoid the empty set. (8.8) ensures we have Σ_1^A convergence.

Step III: $\{\Xi_{\mathrm{sp},\epsilon}^N, \Omega_{\mathrm{SA}}\} \notin \Delta_2^G$. Assume for a contradiction that there is a sequence $\{\Gamma_k\}$ of general algorithms such that $\Gamma_k(A) \to \mathrm{sp}_{N,\epsilon}(A)$ for all $A \in \Omega_{\mathrm{SA}}$, and consider operators of the type

$$A := \bigoplus_{r=1}^{\infty} A_{l_r} \qquad \text{with } \{l_r\} \subset \mathbb{N} \text{ and } A_n := \begin{pmatrix} 1 & & & & 1 \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ 1 & & & & 1 \end{pmatrix} \in \mathbb{C}^{n\times n}. \tag{8.9}$$

Then $\mathrm{sp}(A_n) = \{0, 2\}$, hence A is bounded, self-adjoint, and $\mathrm{sp}(A) = \{0, 2\}$ as well. For sufficiently small ϵ the (N, ϵ)-pseudospectrum is a certain neighbourhood of $\{0, 2\}$ disjoint to $B_{\frac{1}{2}}(1)$, independently of the choice of $\{l_r\}$. In order to find a counterexample we simply construct an appropriate sequence $\{l_r\} \subset \mathbb{N}$ by induction: For $C := \mathrm{diag}\{1, 0, 0, 0, \ldots\}$ one obviously has $\mathrm{sp}(C) = \{0, 1\}$. Choose $k_0 := 1$ and $l_1 > N(C, k_0)$. Now, suppose that $l_1, \ldots, l_n$ are already chosen. Then we obviously have that $\mathrm{sp}\,(A_{l_1} \oplus \cdots \oplus A_{l_n} \oplus C) = \{0, 1, 2\}$, hence there exists a k_n such that

$$\Gamma_k\,(A_{l_1} \oplus \cdots \oplus A_{l_n} \oplus C) \cap B_{\frac{1}{n}}(1) \neq \emptyset$$

for every $k \geq k_n$, where $B_{\frac{1}{n}}(1)$ denotes the closed ball of radius $1/n$ and centre 1. Now, choose

$$l_{n+1} > N(A_{l_1} \oplus \cdots \oplus A_{l_n} \oplus C, k_n) - l_1 - l_2 - \ldots - l_n. \tag{8.10}$$

By this construction, it follows that

$$\Gamma_{k_n}(\oplus_{r=1}^{\infty} A_{l_r}) \cap B_{\frac{1}{n}}(1) = \Gamma_{k_n}(A_{l_1} \oplus \ldots \oplus A_{l_n} \oplus C) \cap B_{\frac{1}{n}}(1) \neq \emptyset \quad \forall\, n \in \mathbb{N}. \tag{8.11}$$

Indeed, since any evaluation function $f_{i,j} \in \Lambda$ just provides the (i, j)-th matrix element, it follows by (8.10) that for any evaluation functions $f_{i,j} \in \Lambda_{\Gamma_{k_n}}(A_{l_1} \oplus \cdots \oplus A_{l_n} \oplus C)$ we have that that

$$f_{i,j}(A_{l_1} \oplus \cdots \oplus A_{l_n} \oplus C) = f_{i,j}(\oplus_{r=1}^{\infty} A_{l_r}).$$

Thus, by assumption (iii) in the definition of a General algorithm (Definition 4.3), we get that $\Lambda_{\Gamma_{k_n}}(A_{l_1} \oplus \cdots \oplus A_{l_n} \oplus C) = \Lambda_{\Gamma_{k_n}}(\oplus_{r=1}^{\infty} A_{l_r})$ which, by assumption (ii) in Definition 4.3, yields (8.11). So, from (8.11), we see that the point 1 is contained in the partial limiting set of the sequence $\{\Gamma_k(\oplus_{r=1}^{\infty} A_{l_r})\}_{k=1}^{\infty}$ which approximates $\mathrm{sp}_{N,\epsilon}(A)$, a contradiction. For general N and ϵ, we apply the above argument after appropriate re-scaling.

Step IV: $\{\Xi_{\mathrm{sp},\epsilon}^{N}, \Omega_f\} \notin \Delta_1^G$. This is clear by considering diagonal operators. The point is that given any general Δ_1^G tower, Γ_n, and any n, $\Gamma_n(A)$ uses only finitely many matrix evaluations $\{f_{i,j}(A) : i,j \leq N_0(n,A)\}$. We can choose m large such that $m > N_0(1,0)$ and set $f_{m,m}(A) > 2\epsilon + 2$. Then $\Gamma_1(A) = \Gamma_1(0) \subset B_{1/2+\epsilon}(0)$, a contradiction since $2\epsilon + 2 \in \mathrm{sp}_{N,\epsilon}(A)$. $\square$

8.3. **Spectrum.** Again, using the inclusions $\Omega_{\mathrm{SA}} \subset \Omega_{\mathrm{N}} \subset \Omega_{\mathrm{g}}$, we only need to show that $\{\Xi_{\mathrm{sp}}, \Omega_{fg}\} \in \Sigma_1^A$, $\{\Xi_{\mathrm{sp}}, \Omega_f\} \in \Pi_2^A$, $\{\Xi_{\mathrm{sp}}, \Omega_g\} \in \Sigma_2^A$, $\{\Xi_{\mathrm{sp}}, \Omega_{\mathrm{B}}\} \in \Pi_3^A$, $\{\Xi_{\mathrm{sp}}, \Omega_{\mathrm{SA}}\} \notin \Delta_2^G$, $\{\Xi_{\mathrm{sp}}, \Omega_f\} \notin \Delta_2^G$ and $\{\Xi_{\mathrm{sp}}, \Omega_{\mathrm{B}}\} \notin \Delta_3^G$. The proof that $\{\Xi_{\mathrm{sp}}, \Omega_{\mathrm{B}}\} \notin \Delta_3^G$ relies on some results from decision making problems which we shall prove in Section 8.6.

Proof of Theorem 5.4 for the spectrum. **Step I:** We begin with the easy cases that $\{\Xi_{\mathrm{sp}}, \Omega_f\} \in \Pi_2^A$ and $\{\Xi_{\mathrm{sp}}, \Omega_{\mathrm{B}}\} \in \Pi_3^A$. To prove that $\{\Xi_{\mathrm{sp}}, \Omega_f\} \in \Pi_2^A$, let $\epsilon > 0$ and let Γ_n^ϵ denote the height one arithmetic tower to compute the (classical) pseudospectrum of operators in Ω_f. Using the fact that $\mathrm{sp}_{\mathrm{N},\epsilon}(A)$ are continuous with respect to the parameter $\epsilon > 0$, and converge to $\mathrm{sp}(A)$ as $\epsilon \to 0$ for every A, we simply set

$$\Gamma_{m,n}(A) = \Gamma_n^{1/m}(A).$$

This is a Π_2^A tower since $\mathrm{sp}_{0,1/m}(A)$ contains $\mathrm{sp}(A)$. $\{\Xi_{\mathrm{sp}}, \Omega_{\mathrm{B}}\} \in \Pi_3^A$ is similar and just requires the additional first limit.

Step II: $\{\Xi_{\mathrm{sp}}, \Omega_g\} \in \Sigma_2^A$. Let $g : [0,\infty) \to [0,\infty)$ be as in Definition 5.4, in particular, continuous, vanishing only at $x = 0$ and tending to ∞ as $x \to \infty$. Note that $g(x) \leq x$ for all x and without loss of generality we can also assume that g is strictly increasing. Then the inverse function $h(y) := g^{-1}(y) : [0,\infty) \to [0,\infty)$ is well defined, continuous, strictly increasing, $h(y) \geq y$ for every y, and $\lim_{y\to 0} h(y) = 0$.

Let $K \subset \mathbb{C}$ be a compact set and $\delta > 0$. We introduce a δ-grid for K by $G^\delta(K) := (K + B_\delta(0)) \cap (\delta(\mathbb{Z} + \mathrm{i}\mathbb{Z}))$, where $B_\delta(0)$ denotes the closed ball of radius δ centered at 0. Without loss of generality we may assume that δ^{-1} is an integer, and obviously, $G^\delta(K)$ is finite. Moreover, introduce $h_\delta(y) := \min\{k\delta : k \in \mathbb{N}, g(k\delta) > y\}$ and observe that for each y, evaluating $h_\delta(y)$ requires only finitely many evaluations of g. Also, notice that $h(y) \leq h_\delta(y) \leq h(y) + \delta$. For a given function $\zeta : \mathbb{C} \to [0,\infty)$ we define sets $\Upsilon_K^\delta(\zeta)$ as follows: For each $z \in G^\delta(K)$ let $I_z := B_{h_\delta(\zeta(z))}(z) \cap (\delta(\mathbb{Z} + \mathrm{i}\mathbb{Z}))$. Further

- If $\zeta(z) \leq 1$ then introduce the set M_z of all $w \in I_z$ for which $\zeta(w) \leq \zeta(v)$ holds for all $v \in I_z$.
- Otherwise, if $\zeta(z) > 1$, just set $M_z := \emptyset$.

Now define

$$\Upsilon_K^\delta(\zeta) := \bigcup_{z \in G^\delta(K)} M_z. \tag{8.12}$$

Notice that for the computation of $\Upsilon_K^\delta(\zeta)$ only finitely many evaluations of ζ and g are required.

Claim: Let K be a compact set containing the spectrum of A and $0 < \delta < \epsilon < 1/2$. Further assume that ζ is a function with $\|\zeta - \gamma\|_{\infty,\hat{K}} := \|(\zeta - \gamma)\chi_{\hat{K}}\|_\infty < \epsilon$ on $\hat{K} := (K + B_{h(\mathrm{diam}(K)+2\epsilon)+\epsilon}(0))$, where $\chi_{\hat{K}}$ denotes the characteristic function of $\hat{K}$. Finally, let

$$u(\xi) := \max\{h(3\xi + h(t+\xi) - h(t)) + \xi : t \in [0,1]\}. \tag{8.13}$$

Then we have that

$$d_{\mathrm{H}}(\Upsilon_K^\delta(\zeta), \mathrm{sp}(A)) \leq u(\epsilon) \text{ and } \lim_{\xi\to 0} u(\xi) = 0.$$

Proof of claim: To prove the claim, let $z \in G^\delta(K)$ and notice that $I_z \subset \hat{K}$ since, for every $v \in I_z$,

$$\begin{aligned} |z - v| &\leq h_\delta(\zeta(z)) \leq h_\delta(\gamma(z) + \epsilon) \leq h(\mathrm{dist}(z, \mathrm{sp}(A)) + \epsilon) + \delta \\ &\leq h(\mathrm{diam}(K) + \delta + \epsilon) + \delta. \end{aligned} \tag{8.14}$$

Suppose that $M_z \neq \emptyset$. Note that by (5.4), the monotonicity of h, and the compactness of $\mathrm{sp}(A)$ there is a $y \in \mathrm{sp}(A)$ of minimal distance to z with $|z - y| \leq h(\gamma(z))$. Since $\|\zeta - \gamma\|_{\infty,\hat{K}} < \epsilon$ we get $|z - y| \leq$

$h(\zeta(z)+\epsilon)$. Hence, at least one of the $v\in I_z$, let's say v_0, satisfies $|v_0-y|<h(\zeta(z)+\epsilon)-h(\zeta(z))+\delta$. Noting again that $\gamma(v_0)\le \mathrm{dist}(v_0,\mathrm{sp}(A))$, we get $\zeta(v_0)<\gamma(v_0)+\epsilon<h(\zeta(z)+\epsilon)-h(\zeta(z))+2\epsilon$. By the definition of M_z this estimate now holds for all points $w\in M_z$ and we conclude that, for all $w\in M_z$,

$$\begin{aligned}\mathrm{dist}(w,\mathrm{sp}(A)) &= h(g(\mathrm{dist}(w,\mathrm{sp}(A))))\le h(\gamma(w))\\ &\le h(\zeta(w)+\epsilon)\le h(h(\zeta(z)+\epsilon)-h(\zeta(z))+3\epsilon).\end{aligned}\tag{8.15}$$

This observation holds for every $z\in G^\delta(K)$ and all $w\in M_z$, hence all points in $\Upsilon_K^\delta(\zeta)$ are closer to $\mathrm{sp}(A)$ than $u(\epsilon)$.

Conversely, take any $y\in\mathrm{sp}(A)\subset K$. Then there is a point $z\in G^\delta(K)$ with $|z-y|<\delta<\epsilon$, hence $\zeta(z)<\gamma(z)+\epsilon\le\mathrm{dist}(z,\mathrm{sp}(A))+\epsilon<2\epsilon<1$. Thus, M_z is not empty and contains a point which is closer to y than $h(\zeta(z))+\epsilon\le h(2\epsilon)+\epsilon\le u(\epsilon)$. Finally notice that the mapping $(t,\xi)\mapsto h(h(t+\xi)-h(t)+3\xi)+\xi$ is continuous on the compact set $[0,1]\times[0,1]$, hence uniformly continuous. Moreover, for every fixed t it tends to 0 as $\xi\to 0$, thus we can conclude $u(\xi)\to 0$, and we have proved the claim. $\square$

Define the function

$$\gamma_{m,n}(z,A):=\min\{\sigma_1(P_n(A-zI)P_m),\sigma_1(P_n(A^*-\bar{z}I)P_m)\},$$

and note that we can compute this from *above* to within an accuracy $1/m$ in finitely many arithmetic operations and comparisons using Proposition 8.1 and a simple search routine. Call this approximation function $\zeta_{m,n}(z,A)$ and we can assume that it takes values in $\frac{1}{2m}\mathbb{N}$. As $n\to\infty$, $\gamma_{m,n}(\cdot,A)$ converges to $\gamma_m(z,A):=\min\{\sigma_1((A-zI)P_m),\sigma_1((A^*-\bar{z}I)P_m)\}$ monotonically from below. By taking successive maxima over n and then minima over m if necessary: $\min_{1\le j\le m}\max_{1\le k\le n}\zeta_{j,k}(z,A)$, we can assume that $\zeta_{m,n}(\cdot,A)$ is non-decreasing in n and non-increasing in m. Since $\gamma_{m,n}$ obeys these monotonicity relations, this preserves the error bound of $1/m$. It follows that $\zeta_{m,n}(\cdot,A)$ converges to $\zeta_m(\cdot,A)$ which takes values in the set $\frac{1}{2m}\mathbb{N}$ and such that $\gamma_m(z,A)\le\zeta_m(z,A)\le\gamma_m(z,A)+1/m$.

Now let

$$\hat{\Gamma}_{m,n}(A)=\Upsilon^{1/2^m}_{B_m(0)}(\zeta_{m,n}).$$

To show that this provides an arithmetic tower of algorithms, note that the computation of $\Upsilon^{1/2^m}_{B_m(0)}(\zeta_{m,n})$ requires only finitely many evaluations of $\zeta_{m,n}$, and the finite number of constants $g(k/m)\le 1$, $k=1,2,\ldots$. Since $G^{1/2^m}(B_m(0))$ is finite and we restricted values of $\zeta_{m,n}$ to $\frac{1}{2m}\mathbb{N}$, we must have that for large n, $\hat{\Gamma}_{m,n}(A)$ is constant and equal to $\Upsilon^{1/2^m}_{B_m(0)}(\zeta_m)$. Denote this eventually constant set by $\hat{\Gamma}_m(A)$. We must now adapt $\hat{\Gamma}_{m,n}$ such that the output is non-empty and such that we gain Σ_2^A convergence in the Hausdorff topology. For any $\hat{\Gamma}_{m,n}(A)$ let $S(m,n,A):=\max_{z\in\hat{\Gamma}_{m,n}(A)}\zeta_{m,n}(z,A)$, where we take the maximum over the empty set to be $+\infty$. Note that $\hat{\Gamma}_{m,n}(A)$ is empty if and only if $\zeta_{m,n}(z,A)>1$ for all $z\in G^{1/2^m}(B_m(0))$ and note also that $S(m,n,A)$ can be computed using finitely many arithmetic operations and comparisons from the given data.

For given m,n, if $n<m$ then set $\Gamma_{m,n}(A)=\{0\}$. Otherwise, compute $S(k,n,A)$ for $m\le k\le n$. If there exists such a k with $S(k,n,A)\le g(2^{-m})$, then choose a minimal such k and set $\Gamma_{m,n}(A)=\hat{\Gamma}_{k,n}(A)$ (which must be non-empty by the definition of $S(m,n,A)$), otherwise set $\Gamma_{m,n}(A)=\{0\}$. It follows that this defines an arithmetic general algorithm mapping into the appropriate metric space (in particular it outputs a non-empty compact set). Since $\zeta_{m,n}$ increases to ζ_m and g is continuous, if $\hat{\Gamma}_l(A)\neq\emptyset$ then $S(l,n,A)$ is finite for all $n\in\mathbb{N}$. For such an l, we must have $S(l,n,A)$ non-decreasing in n, convergent to $S_l(A):=\max_{z\in\hat{\Gamma}_l(A)}g(\zeta_l(z,A))$. On the other hand if $\hat{\Gamma}_l(A)=\emptyset$ then $\zeta_l(z,A)>1$ for all $z\in G^{1/2^l}(B_l(0))$ and the fact that $\zeta_{m,n}$ increases to ζ_m shows that $S(l,n,A)=+\infty$ for large n.

Define the function

$$\gamma(z):=\min\{\sigma_1(A-zI),\sigma_1(A^*-\bar{z}I)\}=\|(A-zI)^{-1}\|^{-1}$$

To see why $\min\{\sigma_1(A-zI),\sigma_1(A^*-\overline{z}I)\} = \|(A-zI)^{-1}\|^{-1}$ see for example [59]. Now $\sigma_1(P_n(A-zI)P_m) = \inf\{\|P_n(A-zI)P_m\xi\| : \xi \in \mathrm{Ran}(P_m), \|\xi\| = 1\}$ and $\sigma_1((A-zI)P_m) = \inf\{\|(A-zI)P_m\xi\| : \xi \in \mathrm{Ran}(P_m), \|\xi\| = 1\}$. Thus, since $P_m \to I$ strongly and $P_{m+1} \geq P_m$, then $\gamma_m \to \gamma$ pointwise and monotonically from above, and by Dini's Theorem the convergence is uniform on every compact set, in particular on the ball $K := B_{m_0}(0)$, with a fixed $m_0 > 2\|A\| + 4$. Also, $\gamma_{m,n} \to \gamma_m$ pointwise monotonically from below as $n \to \infty$, hence again by Dini's Theorem it follows that the convergence is uniform on the ball $K = B_{m_0}(0)$. Outside this ball we have, for $n > m$, by a Neumann argument

$$\begin{aligned}
\gamma_{m,n}(z) &= \min\{\sigma_1(P_n(A-zI)P_nP_m), \sigma_1(P_n(A^*-\overline{z}I)P_nP_m)\} \\
&\geq \min\{\sigma_1(P_n(A-zI)P_n), \sigma_1(P_n(A^*-\overline{z}I)P_n)\} \\
&= \|(P_n(A-zI)P_n)^{-1}\|^{-1} = |z|\|(P_n - z^{-1}P_nAP_n)^{-1}\|^{-1} \geq 2.
\end{aligned}$$

For all $n > m > m_0$, the points in the finite set $G^{1/2^m}(B_m(0)) \setminus K$ lead to function values of $\zeta_{m,n}$ being larger than 1 (since $\zeta_{m,n}$ approximates $\gamma_{m,n}$ to within $1/m$), hence $\hat{\Gamma}_{m,n}(A) = \Upsilon_K^{1/2^m}(\zeta_{m,n})$. Fix $\epsilon \in (0, 1/2)$. Then there is an $m_1 > m_0$ with $m_1 > 3/\epsilon$ such that $\|\gamma - \zeta_m\|_{\infty,\hat{K}} < \epsilon/3$ on $\hat{K} := B_{h(\mathrm{diam}(K)+2\epsilon)+\epsilon}(0)$ for all $m > m_1$. Moreover, for every m there is an $n_1(m)$ such that $\|\gamma_m - \gamma_{m,n}\|_{\infty,\hat{K}} < \epsilon/3$ for all $n > n_1(m)$. This yields

$$\begin{aligned}
\|\gamma - \zeta_{m,n}\|_{\infty,\hat{K}} &\leq \|\gamma - \gamma_m\|_{\infty,\hat{K}} + \|\gamma_m - \gamma_{m,n}\|_{\infty,\hat{K}} + \|\gamma_{m,n} - \zeta_{m,n}\|_{\infty,\hat{K}} \\
&\leq \epsilon/3 + \epsilon/3 + 1/m < \epsilon
\end{aligned} \tag{8.16}$$

whenever $m > m_1$ and $n > n_1(m)$. Hence, by the above claim, we must have that

$$d_{\mathrm{H}}(\hat{\Gamma}_{m,n}(A), \mathrm{sp}(A)) \leq u(\epsilon)$$

whenever $m > m_1$ and $n > n_1(m)$. Since this bound tends to zero as $\epsilon \to 0$, it is proved that

$$\lim_{m\to\infty} \limsup_{n\to\infty} d_{\mathrm{H}}(\hat{\Gamma}_{m,n}(A), \mathrm{sp}(A)) = 0.$$

It follows that there exists $N_0 \in \mathbb{N}$ minimal such that $S_{N_0}(A) < +\infty$, equivalently such that $\hat{\Gamma}_{N_0}(A) \neq \emptyset$. Monotonicity of ζ_m in m and the fact that the grid refines itself now ensures that if $m \geq N_0$ then $S_m(A) < +\infty$. Furthermore, the above claim (as well as continuity in g) shows that $\lim_{m\to\infty} S_m(A) = 0$. Let $N_1(m) \geq m$ be minimal such that $S_{N_1(m)} \leq g(2^{-m})$. It follows that we must have

$$\lim_{n\to\infty} \Gamma_{m,n}(A) = \hat{\Gamma}_{N_1(m)}(A).$$

We must also have $\lim_{m\to\infty} \Gamma_m(A) = \mathrm{sp}(A)$. Furthermore,

$$\max_{z\in\Gamma_m(A)} g(\mathrm{dist}(z, \mathrm{sp}(A))) \leq \max_{z\in\Gamma_m(A)} \gamma(z, A) \leq S_{N_1(m)}(A) \leq g(2^{-m}). \tag{8.17}$$

But g is strictly increasing so that we must have $\Gamma_m(A) \subset \mathcal{N}_{2^{-m}}(\mathrm{sp}(A))$ and hence Σ_2^A convergence.

Step III: $\{\Xi_{\mathrm{sp}}, \Omega_{fg}\} \in \Sigma_1^A$. This is very similar to Step II, but now we use the function f to collapse the first limit. We can approximate

$$F_n(z, A) := \min\{\sigma_1(P_{f(n)}(A-zI)P_n), \sigma_1(P_{f(n)}(A^*-\bar{z}I)P_n)\} + c_n,$$

from *above* to within an accuracy $1/n$ in finitely many arithmetic operations and comparisons using Proposition 8.1 and a simple search routine. Call this approximation function $\tilde{F}_n(z, A)$. Note that by definition of $D_{f,n}$ and the fact that $D_{f,n}(A) \leq c_n$, we must have $\tilde{F}_n(z, A) \geq \gamma_n(z, A)$ and without loss of generality (take successive minima if necessary) we can assume that $\tilde{F}_n$ converges locally uniformly to γ monotonically from above.

Now let

$$\Gamma_n(A) = \Upsilon_{B_n(0)}^{1/2^n}(F_n).$$

Arguing as before, we see that this provides an arithmetic tower of algorithms, is non-empty for large n (so we can assume this holds for all n without loss of generality) and has $\lim_{n\to\infty}\Gamma_n(A)=\mathrm{sp}(A)$. Hence we only need to argue for the Σ_1^A error control. Define

$$E_n(A)=\sup_{z\in\Gamma_n(A)} h_{2^{-n}}(F_n(z,A)),$$

then since $h_{2^{-n}}\geq h$, we must have $E_n(A)\geq\sup_{z\in\Gamma_n(A)}\mathrm{dist}(z,\mathrm{sp}(A))$. Moreover, $\sup_{z\in\Gamma_n(A)}F_n(z,A)$ converges to 0 as $n\to\infty$. Since $h_{2^{-n}}\leq h+2^{-n}$, it follows that $E_n(A)\to 0$ and hence (by the usual argument of taking subsequences if necessary) we have $\{\Xi_{\mathrm{sp}},\Omega_{fg}\}\in\Sigma_1^A$.

Step IV: $\{\Xi_{\mathrm{sp}},\Omega_{\mathrm{SA}}\}\notin\Delta_2^G$. This is pretty much exactly the same argument as the pseudospectrum case. Assume that there is a sequence $\{\Gamma_k\}$ of general algorithms such that $\Gamma_k(A)\to\mathrm{sp}(A)$ for all $A\in\Omega_{\mathrm{SA}}$, and consider operators of the type (8.9). The spectrum is $\{0,2\}$ disjoint to $B_{\frac{1}{2}}(1)$, independently of the choice of $\{l_r\}$. By exactly the same procedure as before one obtains again that 1 belongs to the partial limiting set of $\Gamma_k(A)$ for a certain A, hence a contradiction.

Step V: $\{\Xi_{\mathrm{sp}},\Omega_f\}\notin\Delta_2^G$. Recall that Ω_f denotes the set of bounded operators on $l^2(\mathbb{N})$ whose dispersion is bounded by f. Thus, to show the claim, it suffices to show that for any height one general tower of algorithms $\{\Gamma_n\}_{n\in\mathbb{N}}$ for Ξ_{sp} there exists a weighted shift S, with $(Su)_1=0$ for all $u\in l^\infty(\mathbb{N})$ and $Se_n=\alpha_n e_{n+1}$ where $\alpha=\{\alpha_n\}_{n\in\mathbb{N}}\in l^\infty(\mathbb{N})$, such that $\Gamma_m(S)\nrightarrow\mathrm{sp}(S)$ when $m\to\infty$, Obviously $S\in\Omega_f$ (recall $f(n)\geq n+1$). To construct such an S we let

$$\alpha=\{0,0,\dots,0,1,0,0,\dots,0,1,1,0,0,\dots,0,1,1,1,0,\dots\},\qquad \alpha_{l_j+1},\alpha_{l_j+2},\dots,\alpha_{l_j+j}=1,$$

for some sequence $\{l_j\}_{j\in\mathbb{N}}$ where $l_{j+1}>l_j+2j$ that we will determine. Observe that regardless of the choice of $\{l_j\}_{j\in\mathbb{N}}$ we have that $\mathrm{sp}(S)=B_1(0)$ (the closed disc centred at zero with radius one). Indeed, on the one hand $\|S\|=1$, hence $\mathrm{sp}(S)\subset B_1(0)$. On the other hand, one can define the elementary shift operator $V:e_n\mapsto e_{n+1}$, $n\in\mathbb{N}$, and its left inverse $V^-:e_{n+1}\mapsto e_n$, $n\in\mathbb{N}$, $e_1\mapsto 0$. Then the shifted copies $(V^-)^{l_j}SV^{l_j}$ converge strongly to the limit operator V whose spectrum $\mathrm{sp}(V)=B_1(0)$ is necessarily contained in the essential spectrum of S (cf. [83] or [74]).

To construct S we will inductively choose $\{l_j\}_{j\in\mathbb{N}}$ with the help of another sequence $\{m_j\}_{j\in\mathbb{Z}_+}$ that will also be chosen inductively. Before we start, define, for any $A\in\Omega_f$ and $m\in\mathbb{N}$, $N(A,m)$ to be the smallest integer so that the evaluations from $\Lambda_{\Gamma_m}(A)$ only take the matrix entries $A_{ij}=\langle Ae_j,e_i\rangle$ with $i,j\leq N(A,m)$ into account. Now let $m_0=1$ and choose $l_1>N(0,m_1)$. Suppose that $l_1,\dots,l_n$ and $m_0,\dots,m_{n-1}$ are already chosen. Note that $\mathrm{sp}(P_rS)=\{0\}$, since $P_rS=P_rSP_r$ can be regarded as a $r\times r$-triangular matrix with zero-diagonal. Thus, since by assumption $\{\Gamma_m\}_{m\in\mathbb{N}}$ is a General tower of algorithms for Ξ_1, there is an m_n such that $\Gamma_m(P_{l_n+n+1}S)\subset B_{\frac{1}{2}}(0)$, for all $m\geq m_n$. Let

$$l_{n+1}>N(P_{l_n+n+1}S,m_n)\text{ such that also }l_{n+1}>l_n+2n. \tag{8.18}$$

Then, it follows that $\Gamma_{m_n}(S)=\Gamma_{m_n}(P_{l_{n+1}}S)=\Gamma_{m_n}(P_{l_n+n+1}S)$. Indeed, since any evaluation function $f_{i,j}\in\Lambda$ just provides the (i,j)-th matrix element, it follows by (8.18) that for any evaluation functions $f_{i,j}\in\Lambda_{\Gamma_{m_n}}(S)$ we have that $f_{i,j}(S)=f_{i,j}(P_{l_{n+1}}S)=f_{i,j}(P_{l_n+n+1}S)$. Thus, by assumption (iii) in the definition of a General algorithm (Definition 4.3), we get that $\Lambda_{\Gamma_{m_n}}(S)=\Lambda_{\Gamma_{m_n}}(P_{l_{n+1}}S)=\Lambda_{\Gamma_{m_n}}(P_{l_n+n+1}S)$ which, by assumption (ii) in Definition 4.3 implies the assertion. Thus, by the choice of the sequences $\{l_j\}_{j\in\mathbb{N}}$ and $\{m_j\}_{j\in\mathbb{Z}_+}$, it follows that $\Gamma_{m_n}(S)=\Gamma_{m_n}(P_{l_n+n+1}S)\subset B_{\frac{1}{2}}(0)$ for every n. Since $\mathrm{sp(S)}=B_1(0)$ we observe that $\Gamma_m(S)\nrightarrow\mathrm{sp}(S)$.

Step VI: $\{\Xi_{\mathrm{sp}}, \Omega_{\mathrm{B}}\} \notin \Delta_3^G$. To prove this we shall need one of the results from Section 8.6. Namely, if we define Ω' to be the collection of all infinite matrices $\{a_{i,j}\}_{i,j\in\mathbb{Z}}$ with entries $a_{i,j} \in \{0,1\}$ and consider

$$\Xi' : \Omega' \ni \{a_{i,j}\}_{i,j\in\mathbb{Z}} \mapsto \left(\exists D \forall j \left(\left(\forall i \sum_{k=-i}^{i} a_{k,j} < D\right) \vee \left(\forall R \exists i \sum_{k=0}^{i} a_{k,j} > R \wedge \sum_{k=-i}^{0} a_{k,j} > R\right)\right)\right)$$

("there is a bound D such that every column has either less than D 1s or is two-sided infinite")

(where we map into the discrete space $\{\mathit{Yes}, \mathit{No}\}$), then $\mathrm{SCI}(\Xi', \Omega')_{\mathrm{G}} = 3$.

We may identify $\Omega_{\mathrm{B}} = \mathcal{B}(l^2(\mathbb{N}))$ with $\Omega = \mathcal{B}(X)$, where $X = \oplus_{n=-\infty}^{\infty} X_n$ in the l^2-sense and where $X_n = l^2(\mathbb{Z})$. Consider sequences $a = \{a_i\}_{i\in\mathbb{Z}}$ over $\mathbb{Z}$ with $a_i \in \{0,1\}$, and define respective operators $B_a \in \mathcal{B}(l^2(\mathbb{Z}))$ with matrix representation $B_a = \{b_{k,i}\}$ by

$$b_{k,i} := \begin{cases} 1: & k = i \text{ and } a_k = 0 \\ 1: & k < i \text{ and } a_k = a_i = 1 \text{ and } a_j = 0 \text{ for all } k < j < i \\ 0: & \text{otherwise.} \end{cases}$$

Then B_a is again a shift on a certain subset of basis elements and the identity on the other basis elements, hence we have the following possible spectra:

- $\mathrm{sp}(B_a) \subset \{0,1\}$ if $\{a_i\}$ has finitely many 1s.
- $\mathrm{sp}(B_a) = \mathbb{T}$, the unit circle, if there are infinitely many $i > 0$ with $a_i = 1$ and infinitely many $i < 0$ with $a_i = 1$ (we say $\{a_i\}$ is two-sided infinite).
- $\mathrm{sp}(B_a) = \mathbb{D}$, the unit disc, if $\{a_i\}$ has infinitely many 1s, but only finitely many for $i < 0$ or finitely many for $i > 0$ (we say $\{a_i\}$ is one-sided infinite in that case).

Next for a matrix $\{a_{i,j}\}_{i,j\in\mathbb{Z}}$ we define the operator

$$C := \bigoplus_{k=-\infty}^{\infty} B_k \tag{8.19}$$

on X, where $B_k = B_{\{a_{i,k}\}_{i\in\mathbb{Z}}}$ corresponds to the column $\{a_{i,k}\}_{i\in\mathbb{Z}}$ in the above sense. Concerning its spectrum we have $\bigcup_{k\in\mathbb{Z}} \mathrm{sp}(B_k) \subset \mathrm{sp}(C) \subset \mathbb{D}$ since $\|C\| = 1$. Clearly, if one of the columns is one-sided infinite then $\mathrm{sp}(C) = \mathbb{D}$. The same holds true if for every $k \in \mathbb{N}$ there is a finite column with at least k 1s. Otherwise (that is if there is a number D such that for every column it holds that it either has less than D 1s or is two-sided infinite) the spectrum $\mathrm{sp}(C)$ is a subset of $\{0\} \cup \mathbb{T}$.

Suppose for a contradiction that there exists a height two tower, Γ_{n_2,n_1} solving $\{\Xi_{\mathrm{sp}}, \Omega_{\mathrm{B}}\}$. Consider the intervals $J_1 = [0, 1/8]$, and $J_2 = [3/8, \infty)$. Set $\alpha_{n_2,n_1} = \mathrm{dist}(1/2, \Gamma_{n_2,n_1}(A))$. Let $k(n_2,n_1) \leq n_1$ be maximal such that $\alpha_{n_2,k}(A) \in J_1 \cup J_2$. If no such k exists or $\alpha_{n_2,k}(A) \in J_1$ then set $\tilde{\Gamma}_{n_2,n_1}(\{a_{i,j}\}) = \mathit{No}$. Otherwise set $\tilde{\Gamma}_{n_2,n_1}(\{a_{i,j}\}) = \mathit{Yes}$. It is clear from the construction of matrix C from $\{a_{i,k}\}_{i\in\mathbb{Z}}$ that this defines a generalised algorithm. In particular, given N we can evaluate $\{f_{k,l}(C) : k, l \leq N\}$ using only finitely many evaluations of $\{a_{i,j}\}$, where we can use a bijection from $\mathbb{N}$ to $\oplus_{j=-\infty}^{\infty}\mathbb{Z}$ to view C as acting on $l^2(\mathbb{N})$. The point of the intervals J_1, J_2 is that we can show $\lim_{n_1\to\infty} \tilde{\Gamma}_{n_2,n_1}(\{a_{i,j}\}) = \tilde{\Gamma}_{n_2}(\{a_{i,j}\})$ exists (the distance to the point $1/2$ cannot oscillate infinitely often between J_1 and J_2). If $\Xi'(\{a_{i,j}\}) = \mathit{No}$ then for large n_2 we have $\lim_{n_1\to\infty} \alpha_{n_2,k}(A) < 1/8$ and hence $\lim_{n_2\to\infty} \tilde{\Gamma}_{n_2}(\{a_{i,j}\}) = \mathit{No}$. Similarly, if $\Xi'(\{a_{i,j}\}) = \mathit{Yes}$ then for large n_2 we have $\lim_{n_1\to\infty} \alpha_{n_2,k}(A) > 3/8$ and hence $\lim_{n_2\to\infty} \tilde{\Gamma}_{n_2}(\{a_{i,j}\}) = \mathit{Yes}$. Hence $\tilde{\Gamma}_{n_2,n_1}$ is a height two tower of general algorithms solving $\{\Xi', \Omega'\}$, a contradiction. $\square$

Remark 8.2. We note that in the case of self-adjoint bounded operators the spectrum $\mathrm{sp}(A)$ is real and the function g can be chosen as $x \mapsto x$. Thus, in the definition of $\Upsilon_K^\delta(\zeta)$ it suffices to consider compact $K \subset \mathbb{R}$, the real grid $G^\delta(K) := (K + [-\delta, \delta]) \cap (\delta\mathbb{Z})$, and for all $z \in G^\delta(K)$ only the two points $z_{1/2} := z \pm \zeta(z)$ in I_z. Also in the case of normal operators, where $g : x \mapsto x$ does the job again, the construction simplifies. In particular, for a given function $\zeta : \mathbb{C} \to [0, \infty)$ we may define sets $\Upsilon_K^\delta(\zeta)$ as follows: For $z \in G^\delta(K)$

consider $I_z := \{z + \zeta(z)e^{j\delta\mathrm{i}} : j = 0, 1, \ldots, \lceil 2\pi\delta^{-1}\rceil\}$ and define $\Upsilon_K^\delta(\zeta)$ again as in (8.12). The proof is then the same, up to some obvious adaptations.

8.4. **Essential Spectrum.** In this section we prove the results for the essential spectrum. Since $\Omega_{\mathrm{D}} \subset \Omega_{fg} \subset \Omega_f$ and $\Omega_{\mathrm{SA}} \subset \Omega_{\mathrm{N}} \subset \Omega_g \subset \Omega_{\mathrm{B}}$, we only need to prove that $\{\Xi_{\text{e-sp}}, \Omega_{\mathrm{D}}\} \notin \Delta_2^G$, $\{\Xi_{\text{e-sp}}, \Omega_{\mathrm{SA}}\} \notin \Delta_3^G$, $\{\Xi_{\text{e-sp}}, \Omega_{\mathrm{B}}\} \in \Pi_3^A$ and $\{\Xi_{\text{e-sp}}, \Omega_f\} \in \Pi_2^A$.

Proof of Theorem 5.4 for the essential spectrum. **Step I:** $\{\Xi_{\text{e-sp}}, \Omega_{\mathrm{D}}\} \notin \Delta_2^G$. To see this, suppose for a contradiction that a height one tower Γ_n solves the computational problem. For the contradiction we will construct $A \in \Omega_{\mathrm{D}}$ with diagonal entries in $\{0,1\}$ such that $\Gamma_n(A)$ does not converge. Let $A_n = \mathrm{diag}(0,0,...,0) \in \mathbb{C}^{n\times n}$ and $B_n = \mathrm{diag}(1,1,...,1) \in \mathbb{C}^{n\times n}$ (we let A_∞ and B_∞ be the obvious infinite analogues). We will construct

$$A = \bigoplus_{n\in\mathbb{N}} A_{a_n} \oplus B_{b_n},$$

for $a_n, b_n \in \mathbb{N}$ inductively. Suppose that $a_1, b_1, a_2, b_2, ..., a_m, b_m$ have been chosen. Then the operator

$$C_m := \Big(\bigoplus_{n=1}^m A_{a_n} \oplus B_{b_n}\Big) \oplus A_\infty$$

has essential spectrum $\{0\}$. Hence there exists $n_m \geq m$ such that $\Gamma_{n_m}(C_m) \subset B_{1/4}(0)$. But by the definition of a general tower there must exist some $N(m)$ such that $\Gamma_{n_m}(C_m)$ only uses the evaluations of matrix elements $f_{i,j}(C_m)$ with $i, j \leq N(m)$. Now choose $a_{m+1} \geq \max\{N_m - (a_1 + b_1 + ... + a_m + b_m), 1\}$ then we must have $\Gamma_{n_m}(A) = \Gamma_{n_m}(C_m)$. Similarly, if $a_1, b_1, a_2, b_2, ..., b_m, a_{m+1}$ have been chosen then we consider

$$D_m := \Big(\bigoplus_{n=1}^m A_{a_n} \oplus B_{b_n}\Big) \oplus A_{m+1} \oplus B_\infty$$

and choose b_{m+1} large so that $\Gamma_{n_m}(A) = \Gamma_{n_m}(C_m) \subset B_{1/4}(1)$. This then gives the required contradiction.

Step II: $\{\Xi_{\text{e-sp}}, \Omega_{\mathrm{SA}}\} \notin \Delta_3^G$. Suppose for a contradiction that Γ_{n_2,n_1} is a height two tower solving this problem. Let $(\mathcal{M}, d)$ be the discrete space $\{\textit{Yes}, \textit{No}\}$, let Ω' denote the collection of all infinite matrices $\{a_{i,j}\}_{i,j\in\mathbb{N}}$ with entries $a_{i,j} \in \{0,1\}$ and consider the problem function

$$\Xi'(\{a_{i,j}\}): \text{ Does } \{a_{i,j}\} \text{ have (only) finitely many columns with (only) finitely many 1s?}$$

In Section 8.6 we prove that $\mathrm{SCI}(\Xi', \Omega')_G = 3$. We will gain a contradiction by using the supposed height two tower for $\{\Xi_{\text{e-sp}}, \Omega_{\mathrm{SA}}\}$, Γ_{n_2,n_1}, to solve $\{\Xi', \Omega'\}$.

Without loss of generality, identify Ω_{SA} with self adjoint operators in $\mathcal{B}(X)$ where $X = \bigoplus_{j=1}^\infty X_j$ in the l^2-sense with $X_j = l^2(\mathbb{N})$. Now let $\{a_{i,j}\} \in \Omega'$ and for the jth column define $B_j \in \mathcal{B}(X_j)$ with the following matrix representation:

$$B_j = \bigoplus_{r=1}^{M_j} A_{l_r^j}, \quad A_m := \begin{pmatrix} 1 & & & & 1 \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ 1 & & & & 1 \end{pmatrix} \in \mathbb{C}^{m\times m},$$

where if M_j is finite then $l_{M_j}^j = \infty$ with $A_\infty = \mathrm{diag}(1,0,0,...)$. The l_r^j are defined by the relation

$$\sum_{r=1}^{\sum_{i=1}^m a_{i,j}} l_r^j = m + \sum_{i=1}^m a_{i,j}, \tag{8.20}$$

and measure the lengths $(+1)$ of successive gaps between 1's in the jth column. Define the self-adjoint operator

$$A = \bigoplus_{j=1}^{\infty} B_j.$$

We then have that

$$\mathrm{sp}_{\mathrm{ess}}(A) = \begin{cases} \{0,1,2\}, & \text{if } \Xi'(\{a_{i,j}\}) = \textit{No} \\ \{0,2\}, & \text{otherwise .} \end{cases}$$

Consider the intervals $J_1 = [0, 1/2]$, and $J_2 = [3/4, \infty)$. Set $\alpha_{n_2,n_1} = \mathrm{dist}(1, \Gamma_{n_2,n_1}(A))$. Let $k(n_2, n_1) \leq n_1$ be maximal such that $\alpha_{n_2,k}(A) \in J_1 \cup J_2$. If no such k exists or $\alpha_{n_2,k}(A) \in J_1$ then set $\tilde{\Gamma}_{n_2,n_1}(\{a_{i,j}\}) = \textit{No}$. Otherwise set $\tilde{\Gamma}_{n_2,n_1}(\{a_{i,j}\}) = \textit{Yes}$. It is clear from (8.20) and the definition of the A_m that this defines a generalised algorithm. In particular, given N we can evaluate $\{A_{k,l} : k, l \leq N\}$ using only finitely many evaluations of $\{a_{i,j}\}$, where we can use a bijection from $\mathbb{N}$ to $\oplus_{j=1}^{\infty}\mathbb{N}$ to view A as acting on $l^2(\mathbb{N})$. Again, the point of the intervals J_1, J_2 is that we can show $\lim_{n_1\to\infty} \tilde{\Gamma}_{n_2,n_1}(\{a_{i,j}\}) = \tilde{\Gamma}_{n_2}(\{a_{i,j}\})$ exists. If $\Xi'(\{a_{i,j}\}) = \textit{No}$ then for large n_2 we have $\lim_{n_1\to\infty} \alpha_{n_2,k}(A) < 1/2$ and hence $\lim_{n_2\to\infty} \tilde{\Gamma}_{n_2}(\{a_{i,j}\}) = \textit{No}$. Similarly, if $\Xi'(\{a_{i,j}\}) = \textit{Yes}$ then for large n_2 we have $\lim_{n_1\to\infty} \alpha_{n_2,k}(A) > 3/4$ and hence $\lim_{n_2\to\infty} \tilde{\Gamma}_{n_2}(\{a_{i,j}\}) = \textit{Yes}$. Hence $\tilde{\Gamma}_{n_2,n_1}$ is a height two tower of general algorithms solving $\{\Xi', \Omega'\}$, a contradiction.

Step III: $\{\Xi_{\text{e-sp}}, \Omega_{\mathrm{B}}\} \in \Pi_3^A$. We start by defining the following functions on $\mathbb{C}$, where $Q_n := I - P_n$,

$$\begin{aligned} \mu_{m,n,k} &: z \mapsto \min\{\sigma_1(P_k(A - zI)Q_m P_n), \sigma_1(P_k(A - zI)^* Q_m P_n)\} \\ \mu_{m,n} &: z \mapsto \min\{\sigma_1((A - zI)Q_m P_n), \sigma_1((A - zI)^* Q_m P_n)\} \\ \mu_{m} &: z \mapsto \min\{\sigma_1((A - zI)Q_m), \sigma_1((A - zI)^* Q_m)\}. \end{aligned}$$

Here $P_k(A - zI)Q_m P_n$ is considered as operator on $\mathrm{Ran}(Q_m P_n)$, etc. as usual. Recall from the previous proofs that, for every n, m, $\mu_{m,n,k} \to \mu_{m,n}$ pointwise and monotonically from below as $k \to \infty$ and for every m $\mu_{m,n} \to \mu_m$ pointwise and monotonically from above as $n \to \infty$. Furthermore, $\{\mu_m\}_{m\in\mathbb{N}}$ is pointwise increasing and bounded, hence converges as well. By Proposition 8.1 we can compute with finitely many arithmetic operations and comparisons, for any given z, an approximation $\tilde{\mu}_{m,n,k}(z)$ with $|\mu_{m,n,k}(z) - \tilde{\mu}_{m,n,k}(z)| \leq 1/k$. Furthermore, we can approximate from below and assume without loss of generality (by taking successive maxima) that $\tilde{\mu}_{m,n,k}(z)$ converges to $\mu_{m,n}$ pointwise and monotonically from below.

Next, we define the finite grids

$$G_n := \left\{ \frac{s + \mathrm{i}t}{2^n} : s, t \in \{-2^{2n}, \ldots, 2^{2n}\} \right\},$$

and, for $A \in \mathcal{B}(l^2(\mathbb{N}))$,

$$\hat{\Gamma}_{m,n,k}(A) := \left\{ z \in G_n : \tilde{\mu}_{m,n,k}(z) \leq \frac{1}{m} \right\}$$

$$\hat{\Gamma}_{m,n}(A) := \bigcap_{k\in\mathbb{N}} \hat{\Gamma}_{m,n,k}(A) = \lim_{k\to\infty} \hat{\Gamma}_{m,n,k}(A), \tag{8.21}$$

$$\hat{\Gamma}_{m}(A) := \overline{\bigcup_{n\in\mathbb{N}} \hat{\Gamma}_{m,n}(A)} = \lim_{n\to\infty} \hat{\Gamma}_{m,n}(A), \tag{8.22}$$

$$\hat{\Gamma}(A) := \bigcap_{m\in\mathbb{N}} \hat{\Gamma}_{m}(A) = \lim_{m\to\infty} \hat{\Gamma}_{m}(A). \tag{8.23}$$

It easily follows again that all $\hat{\Gamma}_{m,n,k}$ are general algorithms in the sense of Definition 4.3 that require only finitely many arithmetic operations. We shall show that for large enough n, the above sets are non-empty and establish the limits in (8.21), (8.22) and (8.23) and that $\hat{\Gamma}(A)$ equals $\mathrm{sp}_{\mathrm{ess}}(A)$. We will show that it is possible to choose a subsequences of n such that this holds (each output and any limits must never empty since we require convergence in the Hausdorff metric) allowing us to construct a height three arithmetic tower. The final limit will be from above and hence the Π_3^A classification.

To do that we abbreviate $\mathcal{H} := l^2(\mathbb{N})$ and first show that

$$\mu(z) := \lim_{m\to\infty} \mu_m(z) \quad \text{equals} \quad \|(A - zI + \mathcal{K}(\mathcal{H}))^{-1}\|^{-1} \quad \text{for all} \quad z \in \mathbb{C}, \tag{8.24}$$

where $A - zI + \mathcal{K}(\mathcal{H})$ denotes the element in the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ and where we use the convention $\|b^{-1}\|^{-1} := 0$ if the element b is not invertible. Clearly it suffices to consider $z = 0$. The estimate "$\leq$" is trivial in case $\mu(0) = 0$. So, Let $\mu(0) > \epsilon > 0$. Choose $m \in \mathbb{N}$ such that $\mu_m(0) \geq \mu(0) - \epsilon$. The operator $A_0 := AQ_m : \mathrm{Ran}Q_m \to \mathrm{Ran}(AQ_m)$ is invertible, hence the kernel of $A = AQ_m + AP_m$ has finite dimension. $\sigma_1(A^*Q_m) > 0$ yields that $\mathrm{Ran}A$ has finite codimension, hence both A and AQ_m are Fredholm. Let R be the orthogonal projection onto $\mathrm{Ran}AQ_m$, B_0 the inverse of A_0 and $B := B_0R$. Then

$$\begin{aligned} BA - I &= (BA - I)P_m + (BA - I)Q_m = (BA - I)P_m \quad \text{and} \\ AB - I &= (AB - I)(I - R) + (AB - I)R = (AB - I)(I - R) \end{aligned}$$

are compact, i.e. B is a regularizer for A. Now

$$\begin{aligned} \|(A + \mathcal{K}(\mathcal{H}))^{-1}\|^{-1} &\geq \|B\|^{-1} = \|B_0R\|^{-1} \\ &\geq (\|B_0\|\|R\|)^{-1} = \|B_0\|^{-1} = \sigma_1(AQ_m) \geq \mu(0) - \epsilon \end{aligned}$$

gives the estimate "$\leq$" since ϵ is arbitrary.

Conversely, there is nothing to prove if A is not Fredholm, so let $\epsilon > 0$ and $B \in (A + \mathcal{K}(\mathcal{H}))^{-1}$ be a regularizer with $\|B\| \leq \|(A + \mathcal{K}(\mathcal{H}))^{-1}\| + \epsilon$. Since the operator $K := BA - I$ is compact we get for all sufficiently large m that $\|Q_mBAQ_m - Q_m\| = \|Q_mKQ_m\|$ is so small such that $Q_m + Q_mKQ_m$ is invertible in $\mathcal{B}(\mathrm{Ran}(Q_m))$,

$$\underbrace{Q_m(Q_m + Q_mKQ_m)^{-1}Q_mB}_{=:\, B_1 \in \mathcal{B}(\mathcal{H})} AQ_m = Q_m \quad \text{and} \quad \|Q_mB - B_1\| < \epsilon.$$

We get that $\sigma_1(AQ_m) > 0$, hence the compression $AQ_m : \mathrm{Ran}(Q_m) \to \mathrm{Ran}(AQ_m)$ is invertible and the compression $B_1|_{\mathrm{Ran}(AQ_m)} : \mathrm{Ran}(AQ_m) \to \mathrm{Ran}(Q_m)$ is its (unique) inverse. Thus, we have $\|B_1\| \geq \|B_1|_{\mathrm{Ran}(AQ_m)}\| = \sigma_1(AQ_m)^{-1}$ and further $\|B\| \geq \|Q_mB\| \geq \|B_1\| - \|Q_mB - B_1\| \geq \sigma_1(AQ_m)^{-1} - \epsilon$. We conclude for sufficiently large m that $\sigma_1(AQ_m)^{-1} \leq \|B\| + \epsilon \leq \|(A+\mathcal{K}(\mathcal{H}))^{-1}\| + 2\epsilon$. Since $\epsilon > 0$ is arbitrary we arrive at $\lim_{m\to\infty} \sigma_1(AQ_m) \geq \|(A + \mathcal{K}(\mathcal{H}))^{-1}\|^{-1}$. Applying this observation to A^* we also find

$$\lim_{m\to\infty} \sigma_1(A^*Q_m) \geq \|(A^* + \mathcal{K}(\mathcal{H}^*))^{-1}\|^{-1} = \|(A + \mathcal{K}(\mathcal{H}))^{-1}\|^{-1},$$

which finishes the proof of (8.24). In particular we now can apply that all of the above functions $\mu_{m,n,k}$, $\mu_{m,n}$, μ_m, μ are continuous with respect to z, and together with the already discussed pointwise monotone convergence results, Dini's Theorem gives that the convergences are even locally uniform.

We can now establish the limits in (8.21), (8.22) and (8.23) for large enough n. Obviously, $\{\hat{\Gamma}_{m,n,k}(A)\}_k$ is decreasing. If $\hat{\Gamma}_{m,n}(A) = \emptyset$ then there must exist some finite k with $\hat{\Gamma}_{m,n,k}(A) = \emptyset$ since the sets are nested, closed and uniformly bounded. Furthermore, $\{\hat{\Gamma}_{m,n}(A)\}_n$ is increasing since, for every k, $\hat{\Gamma}_{m,n}(A) \subset \hat{\Gamma}_{m,n,k}(A) \subset \hat{\Gamma}_{m,n+1,k}(A)$. Let $z \in \mathrm{sp}_{\mathrm{ess}}(A)$. For $m \in \mathbb{N}$, $\mu_m(z) = 0$ and furthermore, there is an $n_0(m)$ and a $z_m \in G_{n_0(m)}$ with $|z - z_m| < 1/m$, $\mu_m(z_m) < 1/(2m)$ and $\mu_{m,n}(z_m) < 1/m$ for every $n \geq n_0(m)$. Then, for every k, $\hat{\mu}_{m,n,k}(z_m) < 1/m$ as well. Since the essential spectrum of a bounded linear operator is non-empty it follows that there exists a minimal $N(m)$ such that if $n \geq N(m)$ then $\hat{\Gamma}_{m,n}(A) \neq \emptyset$.

We now alter $\hat{\Gamma}_{m,n,k}$ as follows. For a given m, n and k we successively compute $\hat{\Gamma}_{m,n,k}(A), \hat{\Gamma}_{m,n+1,k}(A), \ldots$ and choose $N(m,n,k) \geq n$ minimal such that $\hat{\Gamma}_{m,N(m,n,k),k}(A) \neq \emptyset$. By the above remarks, it follows that this process must terminate. We also have that

$$\Gamma_{m,n}(A) := \lim_{k\to\infty} \Gamma_{m,n,k}(A)$$

exists (in fact $\Gamma_{m,n,k}(A)$ is eventually constant as we increase k since $\hat{\mu}_{m,n,k}$ is increasing) and also that $\Gamma_{m,n}(A) = \hat{\Gamma}_{m,\max\{n,N(m)\}}(A)$. Since $\Gamma_{m,n}(A)$ are increasing in n, it then follows that

$$\Gamma_m(A) := \lim_{n\to\infty} \Gamma_{m,n}(A) = \overline{\bigcup_{n\in\mathbb{N}} \Gamma_{m,n}(A)} = \overline{\bigcup_{n\in\mathbb{N}} \hat{\Gamma}_{m,n}(A)}.$$

Finally, $\{\Gamma_m(A)\}_m$ is decreasing. To see this, choose $z \in \Gamma_m(A)$ and a sequence (z_n) with $z_n \to z$ and $z_n \in \hat{\Gamma}_{m,n}(A)$ (for large n), respectively. The functions $\mu_{m,n}$ are non-decreasing in m and hence we have

$$\hat{\Gamma}_{m,n}(A) = \{z \in G_n : \mu_{m,n}(z) \le \frac{1}{m}\} \subset \hat{\Gamma}_{m-1,n}(A)$$

from which we conclude $z_n \in \hat{\Gamma}_{m-1,n}(A)$, hence $z \in \Gamma_{m-1}(A)$. It follows that the limit $\Gamma(A) := \lim_{m\to\infty} \Gamma_m(A)$ exists.

We are left with proving that $\Gamma(A) = \mathrm{sp}_{\mathrm{ess}}(A)$. Let $z \in \mathrm{sp}_{\mathrm{ess}}(A)$. Arguing as before, for $m \in \mathbb{N}$, $\mu_m(z) = 0$ and furthermore, there is an $n_0(m)$ and a $z_m \in G_{n_0(m)}$ with $|z-z_m| < 1/m$, $\mu_m(z_m) < 1/(2m)$ and $\mu_{m,n}(z_m) < 1/m$ for every $n \ge n_0(m)$. Then for every k $\mu_{m,n,k}(z_m) < 1/m$ as well. We conclude that $z_m \in \Gamma_m(A) \subset \Gamma_l(A)$, $l = 1, \ldots, m$. Thus the limit z of the sequence $\{z_m\}$ belongs to all $\Gamma_l(A)$ and hence $\mathrm{sp}_{\mathrm{ess}}(A) \subset \Gamma(A)$. Conversely, let $z \notin \mathrm{sp}_{\mathrm{ess}}(A)$. Then $\mu(z) > \epsilon > 0$ for a certain $\epsilon > 0$ and for all z in a certain neighborhood U of z. Moreover there is an $m_0 > 3/\epsilon$ such that $\mu_m(z) > \epsilon/2$ for all $m \ge m_0$ and $z \in U$, hence $\mu_{m,n}(z) > \epsilon/2$ for all $m \ge m_0$, all n and all $z \in U$. Further, for every $m > m_0$ and n there is a $k_0(m,n)$ such that $\mu_{m,n,k}(z) > \epsilon/3 > 1/m_0 > 1/m$ for all $k \ge k_0(m,n)$ and $z \in U$. Thus, the intersection of U and $\Gamma(A)$ is empty, in particular $z \notin \Gamma(A)$.

Step IV: $\{\Xi_{\text{e-sp}}, \Omega_f\} \in \Pi_2^A$. Knowing a bound f on the dispersion of A obviously suggests to plug it into the previously defined algorithms and define

$$\begin{aligned}
\kappa_{m,n} &: z \mapsto \min\{\sigma_1(P_{f(n)}(A - zI)Q_m P_n), \sigma_1(P_{f(n)}(A - zI)^* Q_m P_n)\} \\
\tilde{\Gamma}_{m,n}(A) &:= \left\{ z \in G_n : \hat{\kappa}_{m,n}(z) \le \frac{1}{m} \right\}.
\end{aligned}$$

Where, as usual, we will approximate $\kappa_{m,n}$ to within $1/n$ by a function $\hat{\kappa}_{m,n}$ that can be computed (using Proposition 8.1) at any point using finitely many arithmetic operations and comparisons. Unfortunately, all we know about the functions $\kappa_{m,n}, \mu_m$ is that they are Lipschitz continuous with Lipschitz constant 1 and that $\kappa_{m,n}$ converge pointwise to μ_m, but not, whether or when this convergence is monotone. Therefore we have to make a modification in order to guarantee the existence of the desired limiting sets. The following idea is similar to the use of the intervals J_1 and J_2 in Step II and avoids possible oscillations at the boundary.

Let V_m denote the square $V_m := \{z \in \mathbb{C} : |\Re(z)|, |\Im(z)| \le 2^{-(m+1)}\}$ and $V_m(z) := z + V_m$ the respective shifted copies. Moreover, set $Z_m := \{\frac{s+it}{2^m} : s, t \in \mathbb{Z}\}$ and

$$\begin{aligned}
S_{m,n}(z) &:= \{i = m+1, \ldots, n : \exists z \in V_m(z) \cap G_i : \hat{\kappa}_{m,i}(z) \le 1/m\} \\
T_{m,n}(z) &:= \{i = m+1, \ldots, n : \exists z \in V_m(z) \cap G_i : \hat{\kappa}_{m,i}(z) \le 1/(m+1)\} \\
E_{m,n}(z) &:= |S_{m,n}(z)| + |T_{m,n}(z)| - n \\
I_{m,n} &:= \{z \in Z_m : E_{m,n}(z) > 0 \text{ and } |z| \le n\} \\
\hat{\Gamma}_{m,n}(A) &:= \bigcup_{z \in I_{m,n}} V_m(z).
\end{aligned}$$

Roughly speaking, $\hat{\Gamma}_{m,n}(A)$ is the union of a family of squares $V_m(z)$ with $E_{m,n}(z)$ being positive, which is the case if "most of the $\hat{\kappa}_{m,i}$ are small on $V_m(z)$".

To make this precise, we first notice that all $\hat{\kappa}_{m,i}(z)$, $i \geq m+1$, with z outside the compact ball $K := B_{2\|A\|+2}(0)$ are larger than one, $I_{m,n}$ are finite, and all $\hat{\Gamma}_{m,n}(A)$ are contained in K, due to a simple Neumann series argument. Furthermore, $\hat{\kappa}_{m,n} \to \mu_m$ uniformly on K due to the Lipschitz continuity (uniform in n) of $\hat{\kappa}_{m,n}$ and μ_m.

We now show that for each $m \geq 5$ the sign of $E_{m,n}(z)$ are eventually constant with respect to n for every $z \in Z_m \cap K$, if n is sufficiently large. That is, for every z there is an $n(z)$ such that either $E_{m,n}(z) \leq 0$ or $E_{m,n}(z) > 0$ for all $n \geq n(z)$. For fixed z and $m \geq 5$ we have to consider three possible cases: The first one is $\mu_m(w) > 1/m$ for all $w \in V_m(z)$. Then there exists an n_0 such that $\hat{\kappa}_{m,n}(w) > 1/m$ for all $n \geq n_0$ and all $w \in V_m(z)$ (take into account that $V_m(z)$ is compact and $\hat{\kappa}_{m,n} \to \mu_m$ locally uniformly), hence $|S_{m,n}(z)|+|T_{m,n}(z)|$ is constant and $E_{m,n}(z)$ is monotonically decreasing. Secondly, assume that $\mu_m(w) < 1/m$ for all $w \in V_m(z)$. Then there exists an n_0 such that $\hat{\kappa}_{m,n}(w) < 1/m$ for all $n \geq n_0$ and all $w \in V_m(z)$, hence $|S_{m,n}(z)| = n-c$ with a certain constant c, and $E_{m,n}(z) = |T_{m,n}(z)|-c$ is monotonically increasing. Finally, assume that $1/m$ belongs to the interval $[\min\{\mu_m(w) : w \in V_m(z)\}, \max\{\mu_m(w) : w \in V_m(z)\}]$ and notice that the length of that interval is at most 2^{-m} which is less than $1/m-1/(m+1)$ for $m \geq 5$. Then there exists an n_0 such that $\hat{\kappa}_{m,n}(w) > 1/(m+1)$ for all $n \geq n_0$ and all $w \in V_m(z)$, hence $\{|T_{m,n}(z)|\}_{n \geq n_0}$ is constant, and

$$E_{m,n}(z) = (|S_{m,n}(z)| - n) + |T_{m,n}(z)|$$

is monotonically decreasing.

Taking the maximum N of the finite set $\{n(z) : z \in Z_m \cap K\}$ then yields that the $\hat{\Gamma}_{m,n}(A)$, $n \geq N$, are constant, hence converge (if this constant set is non empty) as $n \to \infty$. If $z_0 \in \mathrm{sp}_{\mathrm{ess}}(A)$ then $\mu(z_0) = 0$, hence $\mu_m(z_0) = 0$ for all m. So, for fixed m, we have $\hat{\kappa}_{m,n}(z) < 1/(m+1)$ for all sufficiently large n and all z in the neighborhood $U_{1/(2m)}(z_0)$. Choose $z \in Z_m$ such that $z_0 \in V_m(z) \subset U_{1/(2m)}(z_0)$. This is possible since $m \geq 5$. Then it is immediate from the definitions that $E_{m,n}(z) = n - c$ with a constant c for all sufficiently large n, hence $z_0 \in \Gamma_{m,n}(A)$ for n large. Now given m, n, successively compute $\hat{\Gamma}_{m+5,n}(A), \hat{\Gamma}_{m+5,n+1}(A), \ldots$ and let $N(m,n) \geq n$ be minimal such that $\hat{\Gamma}_{m+5,N(m,n)}(A) \neq \emptyset$. Define

$$\Gamma_{m,n}(A) = \hat{\Gamma}_{m+4,N(m,n)}(A).$$

The above arguments, in particular the fact that $\mathrm{sp}_{\mathrm{ess}}(A) \neq \emptyset$, demonstrate that this sequence of computations halts and $\Gamma_{m,n}$ is an arithmetical algorithm. Note also that $\Gamma_m(A) := \lim_{n\to\infty} \Gamma_{m,n}(A)$ exists and the above argument shows that it contains the essential spectrum. Note also that $\Gamma_{m,n}(A)$ is in fact equal to $\Gamma_m(A)$ for large n.

We claim that $\{\Gamma_m(A)\}_m$ is a decreasing nested sequence, hence converges as well. Indeed, let $z \in \Gamma_{m+1}(A)$, then $z \in \hat{\Gamma}_{m+5,n}(A)$ for large n, i.e. $z \in V_{m+5}(w)$ for a $w \in I_{m+5,n}$, i.e. $w \in Z_{m+5}$ and $E_{m+5,n}(w) > 0$. Clearly, (for large enough n) there exists a $w_0 \in Z_{m+4}$ with $V_{m+5}(w) \subset V_{m+4}(w_0)$, and further (since we can assume without loss of generality by computing maxima over successive m that $\hat{\kappa}_{m+4,i}(z) \leq \hat{\kappa}_{m+5,i}(z)$ holds whenever $n > m+5$)

$$\begin{aligned} S_{m+5,n}(w) &= \{i = m+6, \ldots, n : \exists z \in V_{m+5}(w) \cap G_i : \hat{\kappa}_{m+5,i}(z) \leq 1/(m+5)\} \\ &\subset \{i = m+5, \ldots, n : \exists z \in V_{m+4}(w_0) \cap G_i : \hat{\kappa}_{m+4,i}(z) \leq 1/(m+4)\} = S_{m+4,n}(w_0) \end{aligned}$$

and analogously $T_{m+5,n}(w) \subset T_{m+4,n}(w_0)$. Therefore $E_{m+5,n}(w) \leq E_{m+4,n}(w_0)$, which shows that $w_0 \in I_{m+4,n}$ and thus $z \in \Gamma_m(A)$.

It remains to prove that the final limiting set $\lim_{m\to\infty} \Gamma_m(A)$ coincides with the essential spectrum. We have already proven that it must contain the essential spectrum. Conversely, let $z_0 \notin \mathrm{sp}_{\mathrm{ess}}(A)$, i.e. $\mu(z_0) > 0$. Then, for large m_0, there exists an $\epsilon > 3/m_0$ such that $\mu_m(z_0) > \epsilon$ and $\hat{\kappa}_{m,n}(z_0) > \epsilon/2$ for $m \geq m_0$ and large n, and then also $\hat{\kappa}_{m,n}(z) > \epsilon/3 > 1/m_0$ for all z in a certain neighbourhood U of z_0. For all sufficiently large $m \geq m_0$ all $V_m(z)$ which contain z_0 are subsets of U, hence $E_{m,n}(z) = d - n$

with a constant d for large n, that is $\lim_{n\to\infty}\hat{\Gamma}_{m,n}(A)$ and $\{z_0\}$ are separated. But since the $\{\Gamma_m(A)\}_m$ are nested, it follows z_0 is not in the limiting set $\lim_{m\to\infty}\Gamma_m(A)$. This finishes the proof. □

8.5. **Determining if a point z lies in $\mathrm{sp}(A)$.** Recall that for this problem, we restrict to $z\in\mathbb{R}$ when considering Ω_{D} or Ω_{SA}. We also restrict to $z\neq 0$ when considering Ω_{C}. Since $\Omega_{\mathrm{D}}\subset\Omega_{fg}\subset\Omega_f$, $\Omega_{\mathrm{C}}\subset\Omega_f$ and $\Omega_{\mathrm{SA}}\subset\Omega_{\mathrm{N}}\subset\Omega_g\subset\Omega_B$ it is enough to prove that $\{\Xi_{\mathrm{sp}}^z,\Omega_{\mathrm{SA}}\}\notin\Delta_3^G$, $\{\Xi_{\mathrm{sp}}^z,\Omega_B\}\in\Pi_3^A$, $\{\Xi_{\mathrm{sp}}^z,\Omega_f\}\in\Pi_2^A$, $\{\Xi_{\mathrm{sp}}^z,\Omega_{\mathrm{D}}\}\notin\Delta_2^G$ and $\{\Xi_{\mathrm{sp}}^z,\Omega_{\mathrm{C}}\}\notin\Delta_2^G$.

Proof. **Step I:** $\{\Xi_{\mathrm{sp}}^z,\Omega_{\mathrm{SA}}\}\notin\Delta_3^G$. By considering the shift $A-zI$, we can without loss of generality assume that $z=0$. Suppose for a contradiction that Γ_{n_2,n_1} is a height two tower solving $\{\Xi_{\mathrm{sp}}^0,\Omega_{\mathrm{SA}}\}$. Let $(\mathcal{M},d)$ be the discrete space $\{\textit{Yes},\textit{No}\}$, let Ω' denote the collection of all infinite matrices $\{a_{i,j}\}_{i,j\in\mathbb{N}}$ with entries $a_{i,j}\in\{0,1\}$ and consider the problem function

$$\Xi'(\{a_{i,j}\}):\ \text{Does } \{a_{i,j}\} \text{ have (only) finitely many columns with (only) finitely many 1s?}$$

In Section 8.6 we prove that $\mathrm{SCI}(\Xi',\Omega')_G=3$. Our strategy will be the same as the proof that $\{\Xi_{\mathrm{sp}},\Omega_{\mathrm{B}}\}\notin\Delta_3^G$ - we will gain a contradiction by using the supposed height two tower Γ_{n_2,n_1} to solve $\{\Xi',\Omega'\}$.

First we need a certain periodic semi-infinite Jacobi matrix which gives rise to spectral pollution when applying the finite section method. Define

$$A_\infty:=\begin{pmatrix}0&3&&&&\\3&0&1&&&\\&1&0&3&&\\&&3&0&1&\\&&&1&0&\ddots\\&&&&\ddots&\ddots\end{pmatrix}$$

It is well known that $\mathrm{sp}(A_\infty)=[-4,-2]\cup[2,4]$ (see for instance [33]). However, an easy check shows that 0 is an eigenvalue of the finite truncated matrix $P_nA_\infty P_n$ whenever n is odd. With an abuse of notation we also define

$$A_n:=P_nA_\infty P_n\oplus C_\infty\in\mathcal{B}(l^2(\mathbb{N})),$$

where C_n denotes the $n\times n$ diagonal matrix with diagonal entries equal to -4.

Without loss of generality, we identify Ω_{SA} with self adjoint operators in $\mathcal{B}(X)$ where $X=\bigoplus_{j=1}^\infty X_j$ in the l^2-sense with $X_j=l^2(\mathbb{N})$. Now let $\{a_{i,j}\}\in\Omega'$ and for the jth column define $B_j\in\mathcal{B}(X_j)$ as follows. Let $I_j=\{i\in\mathbb{N}:a_{i,j}=1\}$ and $J_j=\{i\in\mathbb{N}:a_{i,j}=0\}$. We partition $\mathbb{N}$ into two sets:

$$N_1(j)=\{1\}\cup\{2k,2k+1:k\in I_j\},\quad N_2(j)=\{2k,2k+1:k\in J_j\}.$$

On $\mathrm{span}\{e_k:k\in N_1(j)\}$ we let B_j act as $A_{|N_1(j)|}$, whereas on $\mathrm{span}\{e_k:k\in N_2(j)\}$ we let B_j act as $C_{|N_2(j)|}$ (both with respect to the natural bases and ordering). It is clear that B_j is unitarily equivalent to $A_{|N_1(j)|}\oplus C_{|N_2(j)|}$. Hence $\mathrm{sp}(B_j)$ is equal to $[-4,-2]\cup[2,4]\cup K_j$, where $K=\{0\}$ if $\sum_i a_{i,j}<\infty$ and $K_j=\emptyset$ otherwise.

Next we define the operator

$$C:=\bigoplus_{j=1}^\infty\left(B_j+\frac{1}{2j}I\right)$$

on X. Concerning its spectrum, we note that any non-zero point of $\mathrm{sp}(C)$ inside the interval $[-1,1]$ is equal to $1/(2j)$ corresponding to precisely when the column $\{a_{i,j}\}_{i\in\mathbb{N}}$ has finitely many 1's. It is also clear that $0\in\mathrm{sp}(C)$ precisely when this happens infinitely many times (0 is a limit point of a descending sequence in the spectrum). Hence $\Xi_{\mathrm{sp}}^0(C)=\textit{Yes}$ if and only if $\Xi'(\{a_{i,j}\})=\textit{No}$.

We then define $\tilde{\Gamma}_{n_2,n_1}(\{a_{i,j}\})=\textit{Yes}$ if $\Gamma_{n_2,n_1}(C)=\textit{No}$ and $\tilde{\Gamma}_{n_2,n_1}(\{a_{i,j}\})=\textit{No}$ if $\Gamma_{n_2,n_1}(C)=\textit{Yes}$. Given N we can evaluate $\{f_{k,l}(C):k,l\leq N\}$ using only finitely many evaluations of $\{a_{i,j}\}$, where we can use a bijection from $\mathbb{N}$ to $\oplus_{j=1}^{\mathbb{N}}\mathbb{N}$ to view C as acting on $l^2(\mathbb{N})$. This follows since given any finite i, we can compute the sets $\{1,...,i\}\cap N_1(j)$ and $\{1,...,i\}\cap N_2(j)$. Hence $\tilde{\Gamma}_{n_2,n_1}$ defines a generalised algorithm and provides a height two tower of general algorithms solving $\{\Xi',\Omega'\}$, a contradiction.

Step II: $\{\Xi_{\mathrm{sp}}^z,\Omega_B\}\in\Pi_3^A$. By considering the shift $A-zI$, we can without loss of generality assume that $z=0$. Define the numbers

$$\begin{aligned}\gamma &:= \min\{\sigma_1(A),\sigma_1(A^*)\}\\ \gamma_m &:= \min\{\sigma_1(AP_m),\sigma_1(A^*P_m)\}\\ \gamma_{m,n} &:= \min\{\sigma_1(P_nAP_m),\sigma_1(P_nA^*P_m)\}\\ \delta_{m,n} &:= \min\{2^{-m}k : k\in\mathbb{N}, 2^{-m}k\geq\sigma_1(P_nAP_m) \text{ or } 2^{-m}k\geq\sigma_1(P_nA^*P_m)\}.\end{aligned}$$

As pointed out before, A is invertible if and only if $\gamma>0$. Furthermore, note that $\gamma_m\downarrow_m\gamma$, and that $\gamma_{m,n}\uparrow_n\gamma_m$ for every fixed m. The sequences $\{\delta_{m,n}\}_n$ are bounded and monotonically non-decreasing, and $\gamma_{m,n}\leq\delta_{m,n}\leq\gamma_{m,n}+2^{-m}\leq\gamma_m+2^{-m}$. Thus, for $\epsilon>0$ there is an m_0, and for every $m\geq m_0$ there is an $n_0=n_0(m)$ such that

$$|\gamma-\delta_{m,n}|\leq|\gamma-\gamma_m|+|\gamma_m-\gamma_{m,n}|+|\gamma_{m,n}-\delta_{m,n}|\leq\epsilon/3+\epsilon/3+2^{-m}\leq\epsilon \tag{8.25}$$

whenever $m\geq m_0$ and $n\geq n_0(m)$. So we see that the numbers $\delta_{m,n}$ converge monotonically from below for every m as $n\to\infty$, and the respective limits form a non-increasing sequence with respect to m, tending to γ. Moreover, each $\delta_{m,n}$ can be computed with finitely many arithmetic operations by Proposition 8.1. Thus, if we define $\Gamma_{k,m,n}(A):=(\delta_{m,n}<k^{-1})$, the monotonicity ensure that

$$\Gamma_k(A):=\lim_{m\to\infty}\lim_{n\to\infty}\Gamma_{k,m,n}(A)$$

exists. Moreover, if $\gamma<k^{-1}$ then $\Gamma_k(A)=$ *Yes*. If $\Gamma_k(A)=$ *No* then we must have that $\gamma\geq k^{-1}$ and hence $\Xi_{\mathrm{sp}}^0(A)=$ *No*. Finally, if $\gamma>k^{-1}$ then $\Gamma_k(A)=$ *No*. Hence $\Gamma_{k,m,n}$ provides a Π_3^A tower.

Step III: $\{\Xi_{\mathrm{sp}}^z,\Omega_f\}\in\Pi_2^A$. Again, by considering the shift $A-zI$, we can without loss of generality assume that $z=0$. If one considers operators for which a bound f on their dispersion is known, then choosing $n=f(m)$ turns (8.25) into

$$|\gamma-\delta_{m,f(m)}|\leq|\gamma-\gamma_m|+|\gamma_m-\gamma_{m,f(m)}|+|\gamma_{m,f(m)}-\delta_{m,f(m)}|\leq\epsilon/3+\epsilon/3+2^{-m}\leq\epsilon \tag{8.26}$$

for large m taking $|\sigma_1(BP_m)-\sigma_1(P_{f(m)}BP_m)|\leq\|(I-P_{f(m)})BP_m\|$ into account. Therefore, a natural first guess for our general algorithms could be $\tilde{\Gamma}_{k,m}(A):=(\delta_{m,f(m)}<k^{-1})$. Unfortunately, although $\delta_{m,f(m)}$ converges to γ as $m\to\infty$ by (8.26), this is not monotone in general. Hence, it might be the case that $\gamma=k^{-1}$, but $\delta_{m,f(m)}$ oscillates around k^{-1} such that $\{\tilde{\Gamma}_{k,m}(A)\}_m$ may not converge. To overcome this drawback, we can use the same interval trick as before. Define $J_k^1=[0,k^{-1}]$ and $J_k^2=[2k^{-1},\infty)$. For any given m, let $j(m)\leq m$ be maximal such that $\delta_{j,f(j)}\in J_k^1\cup J_k^2$. If no such j exists or $\delta_{j,f(j)}\in J_k^2$ then set $\Gamma_{k,m}(A)=$ *No*, otherwise set $\Gamma_{k,m}(A)=$ *Yes*. By our now standard argument, this converges as $m\to\infty$. If $\gamma>0$, then for large enough k (such that $2k^{-1}<\gamma$), $\Gamma_{k,m}(A)=$ *No* for large m. Conversely, if $\gamma=0$ then for any k, $\delta_{m,f(m)}\in J_k^1$ for large m and hence $\Gamma_{k,m}(A)=$ *Yes* for large m. This gives Π_2^A convergence.

Step IV: $\{\Xi_{\mathrm{sp}}^z,\Omega_{\mathrm{D}}\}\notin\Delta_2^G$. Again, by considering the shift $A-zI$, we can without loss of generality assume that $z=0$. If we assume that there is a general height-one-tower of algorithms $\{\Gamma_n\}$ over Ω_{D} then we can again construct counterexamples very easily: For a decreasing sequence $\{a_i\}$ of positive numbers we consider the diagonal operator $A:=\mathrm{diag}\{a_i\}$. Clearly, 0 belongs to the spectrum of A if and only if the a_is tend to zero. As a start, set $\{a_i^1\}:=\{1,1,\ldots\}$, choose n_1 such that $\Gamma_n(\mathrm{diag}\{a_i^1\})=$ *No* for all $n\geq n_1$, and i_1 such that $\Gamma_{n_1}(\mathrm{diag}\{a_i^1\})$ does not see the diagonal entries a_i^1 with indices $i\geq i_1$. This is possible by (iii) in Definition 4.3 and our now standard argument. Then set $\{a_i^2\}:=\{1,1,\ldots,1,1/2,1/2,\ldots\}$ with $1/2$s starting at the i_1th position. If $n_1,\ldots,n_{k-1}$ and $i_1,\ldots,i_{k-1}$ are already chosen then pick n_k such that

$\Gamma_n(\mathrm{diag}\{a_i^k\}) = \mathit{No}$ for all $n \geq n_k$, and i_k such that $\Gamma_{n_k}(\mathrm{diag}\{a_i^k\})$ doesn't see the diagonal entries a_i^k with indices $i \geq i_k$, and modify $\{a_i^k\}$ to $\{a_i^{k+1}\} := \{1, \dots, 2^{-k}, 2^{-k}, \dots\}$ with 2^{-k}s starting at the i_kth position. Now, the contradiction is as in the previous proofs and we see that $\{\Xi^0, \Omega_{\mathrm{D}}\} \notin \Delta_2^G$.

Step V: $\{\Xi_{\mathrm{sp}}^z, \Omega_{\mathrm{C}}\} \notin \Delta_2^G$. Recall in this case that $z \neq 0$. By scaling any $A \in \Omega_{\mathrm{C}}$ by the factor $3/(2z)$, we can assume without loss of generality that $z = 3/2$. Suppose for a contradiction that a general height-one-tower of algorithms $\{\Gamma_n\}$ solves $\{\Xi_{\mathrm{sp}}^{\frac{3}{2}}, \Omega_{\mathrm{C}}\}$. Consider the arrowhead matrix:

$$A_n(\epsilon) := \begin{pmatrix} 1 & \epsilon & \epsilon^2 & \cdots & \epsilon^n \\ \epsilon & 0 & & & \\ \epsilon^2 & & \ddots & & \\ \vdots & & & 0 & \\ \epsilon^n & & & & 0 \end{pmatrix},$$

where $\epsilon \in (0,1)$. A simple calculation yields that the eigenvalues of $A_n(\epsilon)$ are $\{0, 1/2 \pm \sqrt{1 + 4a_n(\epsilon)}/2\}$, where

$$a_n(\epsilon) = \frac{\epsilon^2(1 - \epsilon^{2n})}{1 - \epsilon^2}.$$

In particular, we choose $\epsilon = \sqrt{3/7}$ for which the only positive eigenvalue is

$$b_n := \frac{1 + \sqrt{1 + 3(1 - \frac{3^n}{7^n})}}{2}.$$

We now choose an increasing sequence of integers (greater than 1) $r_1, r_2, ...$ inductively, and define $A \in \Omega_{\mathrm{C}}$ such that when projected onto the span of the basis vectors $\{e_1, e_{r_1}, ..., e_{r_n}\}$ (with the natural order), with projection denoted by Q_n, $Q_n A Q_n$ has matrix $A_n(\sqrt{3/7})$. We also enforce that if $j \notin \{r_n\}_{n \in \mathbb{N}} \cup \{1\}$, then the $j$the column and row of A are zero. In other words, $A_{1,r_n} = A_{r_n,1} = (\sqrt{3/7})^n$, $A_{1,1} = 1$ and all other entries are 0. It follows that $\mathrm{sp}(A) = \{0, 1/2 \pm 1\}$ and hence $\Xi_{\mathrm{sp}}^{\frac{3}{2}}(A) = \mathit{Yes}$. However, we choose $\{r_n\}$ such that there is an increasing sequence $\{c_n\}$ with $\Gamma_{c_n}(A) = \mathit{No}$, yielding the contradiction.

Suppose that $r_1, ..., r_n$ have been chosen. Then let B_n be the infinite matrix with $Q_n B_n Q_n$ having matrix $A_n(\sqrt{3/7})$ and zeros elsewhere. Clearly the only positive eigenvalue of B_n is $b_n < 3/2$ and hence $\Xi_{\mathrm{sp}}^{\frac{3}{2}}(B_n) = \mathit{No}$. So let $c_n > r_n$ have $\Gamma_{c_n}(B_n) = \mathit{No}$. But by our now standard argument using the Definition 4.3 of a general algorithm, we can choose $r_{n+1} > r_n$ large such that $\Gamma_{c_n}(A) = \Gamma_{c_n}(B_n)$.

□

8.6. **Techniques for proving lower bounds.** Here we collect two results concerning decision making problems which are used to show lower bounds for two of our spectral problems. Within this section we exclusively deal with problems (functions)

$$\Xi : \Omega \to \mathcal{M} := \{\mathit{Yes}, \mathit{No}\},$$

where $\mathcal{M}$ is equipped with the discrete metric. This means that for such problems we search for General algorithms $\Gamma_{n_k,\dots,n_1} : \Omega \to \mathcal{M}$ which, for a given input $\omega \in \Omega$, answer *Yes* or *No*. We will refer to such problems as decision making problems. Clearly, a sequence $\{m_i\} \subset \mathcal{M}$ of such "answers" converges to $m \in \mathcal{M}$ if and only if finitely many m_i are different from m. Let Ω_1 denote the collection of all infinite matrices $\{a_{i,j}\}_{i,j \in \mathbb{N}}$ with entries $a_{i,j} \in \{0,1\}$ and let Ω_2 denote the collection of all infinite matrices $\{a_{i,j}\}_{i,j \in \mathbb{Z}}$ with entries $a_{i,j} \in \{0,1\}$. Consider the following two problems:

$\Xi_1 : \Omega_1 \ni \{a_{i,j}\}_{i,j \in \mathbb{N}} \mapsto$ Does $\{a_{i,j}\}$ have (only) finitely many columns with (only) finitely many 1s?

$$\Xi_2 : \Omega_2 \ni \{a_{i,j}\}_{i,j \in \mathbb{Z}} \mapsto \left(\exists D \forall j \left(\left(\forall i \sum_{k=-i}^{i} a_{k,j} < D \right) \vee \left(\forall R \exists i \sum_{k=0}^{i} a_{k,j} > R \wedge \sum_{k=-i}^{0} a_{k,j} > R \right) \right) \right)$$

("there is a bound D such that every column has either less than D 1s or is two-sided infinite")

Theorem 8.3 (Decision making problems)**.** *Given the setup above we have*

$$\mathrm{SCI}(\Xi_1,\Omega_1)_{\mathrm{G}} = \mathrm{SCI}(\Xi_1,\Omega_1)_{\mathrm{A}} = 3,$$
$$\mathrm{SCI}(\Xi_2,\Omega_2)_{\mathrm{G}} = \mathrm{SCI}(\Xi_2,\Omega_2)_{\mathrm{A}} = 3.$$

Remark 8.4. Note that the SCI of the decision problems above are considered with respect to general and arithmetic towers. These towers do not assume any computability model, but only a model on the mathematical tools allowed (arithmetic operations in the case of arithmetic tower) and the way the algorithm can read information (only finite amount of input). However, the SCI framework with towers of algorithms fit naturally into the classical theory of computability and the Arithmetical Hierarchy.

To prove this we need to introduce some helpful background. Equip the set of all sequences $\{x_i\}_{i\in\mathbb{N}} \subset \{0,1\}$ with the following metric:

$$d_{\mathrm{B}}(\{x_i\},\{y_i\}) := \sum_{n\in\mathbb{N}} 3^{-n}|x_n - y_n|. \tag{8.27}$$

The resulting metric space is known as the Cantor space. By the usual enumeration of the elements of $\mathbb{N}^2$ this metric translates to a metric on the set Ω_1 of all matrices $A = \{a_{i,j}\}_{i,j\in\mathbb{N}}$ with entries in $\{0,1\}$. Similarly, we do this for the set Ω_2 of all matrices $A = \{a_{i,j}\}_{i,j\in\mathbb{Z}}$ with entries in $\{0,1\}$. In each case this gives a complete metric space, hence a so called Baire space, i.e. it is of second category (in itself). To make this precise we recall the following definitions:

Definition 8.5 (Meager set)**.** A set $S \subset \Omega$ in a metric space Ω is nowhere dense if every open set $U \subset \Omega$ has an open subset $V \subset U$ such that $V \cap S = \emptyset$, i.e. if the interior of the closure of S is empty. A set $S \subset \Omega$ is meager (or of first category) if it is an at most countable union of nowhere dense sets. Otherwise S is nonmeager (or of second category).

Notice that every subset of a meager set is meager, as is every countable union of meager sets. By the Baire category theorem, every (nonempty) complete metric space is nonmeager.

Definition 8.6 (Initial segment)**.** We call a finite matrix $\sigma \in \mathbb{C}^{n\times m}$ an initial segment for an infinite matrix $A \in \Omega_2$ and say that A is an extension of σ if σ is in the upper left corner of A. In particular, $\sigma = P_nAP_m$ for some $n, m \in \mathbb{N}$, where we, with slight abuse of notation, consider $P_nAP_m \in \mathbb{C}^{n\times m}$. P_n is as usual the projection onto $\mathrm{span}\{e_j\}_{j=1}^n$, where $\{e_j\}_{j\in\mathbb{N}}$ is the canonical basis for $l^2(\mathbb{N})$.

Similarly, a finite matrix $\sigma \in \mathbb{C}^{(2n+1)\times(2m+1)}$ is an initial segment for an infinite matrix $B \in \Omega_3$ if σ is in the center of B i.e. $\sigma = \tilde{P}_nB\tilde{P}_m$ where $\tilde{P}_n$ is the projection onto $\mathrm{span}\{e_j\}_{j=-n}^n$, where $\{e_j\}_{j\in\mathbb{Z}}$ is the canonical basis for $l^2(\mathbb{Z})$. We denote that A is an extension of σ by $\sigma \subset A$, and the set of all extensions of σ by $E(\sigma)$.

The notion of extension extends in an obvious way to finite matrices.

Notice that the set $E(\sigma)$ of all extensions of σ is a nonempty open and closed neighborhood for every extension of σ.

Lemma 8.7. *Let $\{\Gamma_n\}_{n\in\mathbb{N}}$ be a sequence of General algorithms mapping $\Omega_1 \to \mathcal{M}$, $T \subset \Omega_1$ be a nonempty closed set, and $S \subset T$ be a nonmeager set (in T) such that $\xi = \lim_{n\to\infty}\Gamma_n(A)$ exists and is the same for all $A \in S$. Then there exists an initial segment σ and a number n_0 such that $E^T(\sigma) := T \cap E(\sigma)$ is not empty, and such that $\Gamma_n(A) = \xi$ for all $A \in E^T(\sigma)$ and all $n \geq n_0$. The same statement is true if we consider Ω_2 instead of Ω_1.*

Proof. We are in a complete metric space T. Since $S = \bigcup_{k\in\mathbb{N}} S_k$ with $S_k := \{A \in S : \Gamma_n(A) = \xi\ \forall n \geq k\}$ and S is nonmeager, not all of the S_k can be meager, hence there is a nonmeager S_k, and we set $n_0 := k$. Now, let A be in the closure $\overline{S_{n_0}}$, i.e. there is a sequence $\{A_j\} \subset S_{n_0}$ converging to A. Note that by assumption (i) in Definition 4.3 and the fact that Γ_n are General algorithms, we have that, for every fixed

$n \geq n_0$, $|\Lambda_{\Gamma_n}(A)| < \infty$. Thus, by (ii) in Definition 4.3, the General algorithm Γ_n only depends on a finite part of A, in particular $\{A_f\}_{f\in\Lambda_{\Gamma_n}(A)}$ where $A_f = f(A)$. Since each $f \in \Lambda_{\Gamma_n}(A)$ represents a coordinate evaluation of A and by the definition of the metric d_B in (8.27), it follows that for all sufficiently large j, $f(A) = f(A_j)$ for all $f \in \Lambda_{\Gamma_n}(A)$. By assumption (iii) in Definition 4.3, it then follows that $\Lambda_{\Gamma_n}(A_j) = \Lambda_{\Gamma_n}(A)$ for all sufficiently large j. Hence, by assumption (ii) in Definition 4.3, we have that $\Gamma_n(A) = \Gamma_n(A_j) = \xi$ for all sufficiently large j. Thus, $\Gamma_n(A) = \xi$ for all $n \geq n_0$ and all $A \in \overline{S_{n_0}}$. Since S_{n_0} is not nowhere dense, we can choose a point $\tilde{A}$ in the interior of $\overline{S_{n_0}}$ and fix a sufficiently large initial segment σ of $\tilde{A}$ such that $E^T(\sigma)$ is a subset of $\overline{S_{n_0}}$. The assertion of the lemma now follows. The extension of the proof to Ω_2 is clear. $\square$

Roughly speaking, this shows that there is a nice open and closed nonmeager subspace of T for which $\lim_{n\to\infty} \Gamma_n(A)$ exists even in a uniform manner. Note that this result particularly applies to the case $T = \Omega$.

Proof of Theorem 8.3. **Step I:** $\mathrm{SCI}(\Xi_1, \Omega_2)_{\mathrm{G}} \geq 3$. We argue by contradiction and assume that there is a height two tower $\{\Gamma_r\}$, $\{\Gamma_{r,s}\}$ for Ξ_1, where Γ_r denote, as usual, the pointwise limits $\lim_{s\to\infty} \Gamma_{r,s}$. We will inductively construct initial segments $\{\sigma_n\}$ with $\sigma_{n+1} \supset \sigma_n$ yielding an infinite matrix $A \supset \sigma_n$ for all $n \in \mathbb{N}$, such that $\lim_{r\to\infty} \Gamma_r(A)$ does not exist. We construct $\{\sigma_n\}$ with the help of two sequences of subsets $\{T_n\}$ and $\{S_n\}$ of Ω, with the properties that $T_{n+1} \subset T_n$, each T_n is closed, and either $T_n = \Omega_1$ or there is an initial segment $\sigma \in \mathbb{C}^{m\times m}$ where $m \geq n$ such that T_n is the set of all extensions of σ with all the remaining entries in the first n columns being zero.

Suppose that we have chosen T_n. Note that the subset of all matrices in T_n with one particular entry being fixed is closed in T_n, hence the set of all matrices with one particular column being fixed is closed (as an intersection of closed sets). The latter set has no interior points in T_n, hence is nowhere dense in T_n. This provides that the set of all matrices in T_n for which a particular column has only finitely many 1s is a countable union of nowhere dense sets in T_n, hence is meager in T_n. Hence the set of all matrices in $A \in T_n$ with $\Xi_1(A) =$ *No* (i.e matrices with infinitely many "finite columns") is meager in T_n as well. Let R be its complement in T_n, i.e. the nonmeager set of all matrices $A \in T_n$ with $\Xi_1(A) =$ *Yes*.

Clearly, $R = \bigcup_{r\in\mathbb{N}} R_r$ with $R_r := \{A \in R : \Gamma_k(A) = \textit{Yes}\ \forall k \geq r\}$, and there is an r_n such that $S_n := R_{r_n}$ is nonmeager in T_n. Note that $\Gamma_{r_n,s}$ are General algorithms and $\Gamma_{r_n}(A) = \lim_{s\to\infty} \Gamma_{r_n,s}(A) =$ *Yes* for all $A \in S_n$. Thus, Lemma 8.7 applies and yields an initial segment σ_n, such that

$$E^{T_n}(\sigma_n) \neq \emptyset \quad \text{and } \Gamma_{r_n}(A) = \textit{Yes} \text{ for all } A \in E^{T_n}(\sigma_n). \tag{8.28}$$

Now, let $T_{n+1} \subset T_n$ be the (closed) set of all matrices in $E^{T_n}(\sigma_n)$ with all remaining [2] entries in the first $n+1$ columns being zero. Letting $T_0 = \Omega_1$ we have completed the construction.

The nested initial segments $\sigma_{n+1} \supset \sigma_n$ obviously yield a matrix $A \in \cap_{n=0}^{\infty} T_n$ and this A has only finitely many 1s in each of its columns. However, by the construction of $\{T_n\}$, we have that $A \in E^{T_n}(\sigma_n)$ for all $n \in \mathbb{N}$. Thus, $\Xi_1(A) =$ *No*, but by (8.28), $\Gamma_k(A) =$ *Yes* for infinitely many k.

Step II: $\mathrm{SCI}(\Xi_2, \Omega_2)_{\mathrm{G}} \geq 3$. The proof is very similar to the proof of Step I. In particular, we argue by contradiction and assume that there is a height two tower $\{\Gamma_r\}$, $\{\Gamma_{r,s}\}$ for Ξ_2. As above, we inductively construct initial segments $\{\sigma_n\}$ with $\sigma_{n+1} \supset \sigma_n$ yielding an infinite matrix $A \supset \sigma_n$ for all $n \in \mathbb{N}$, such that $\lim_{r\to\infty} \Gamma_r(A)$ does not exist. We construct $\{\sigma_n\}$ with the help of two sequences of subsets $\{T_n\}$ and $\{S_n\}$ of Ω_2, with the properties that $T_{n+1} \subset T_n$, each T_n is closed, and either $T_n = \Omega_2$ or there is an initial segment $\sigma \in \mathbb{C}^{(2m+1)\times(2m+1)}$ where $m \geq n$ such that T_n is the set of all extensions of σ with all $\pm n$th semi-columns being filled by n additional 1s and infinitely many 0s, and and all the other kth columns,

[2] I.e. outside the initial segment σ_n.

$|k| \leq n-1$, are being filled with zeros. In particular, if $\{a_{i,j}\}_{i,j\in\mathbb{Z}} \in T_n$ then

$$\begin{aligned} \{a_{i,\pm n}\}_{i\in\mathbb{Z}} &= \{\dots,0,\ \underbrace{1,\dots,1}_{n\text{ times}},\sigma_{-m,\pm n},\dots,\sigma_{m,\pm n},\ \underbrace{1,\dots,1}_{n\text{ times}},0,\dots\}^T, \\ \{a_{i,k}\}_{i\in\mathbb{Z}} &= \{\dots,0,\sigma_{-m,k},\dots,\sigma_{m,k},0,\dots\}^T, \quad k\in\mathbb{Z}_+,\ |k|\leq n-1. \end{aligned} \tag{8.29}$$

Suppose that we have chosen T_n. We argue as in Step I and deduce that for $k \in \mathbb{Z}$ the set of all matrices in T_n with one of the two kth semi-columns being fixed is nowhere dense in T_n, hence the set of all matrices in T_n with (one of the two) kth semi-columns having finitely many 1s is meager in T_n. We conclude that the set of all matrices in T_n with one semi-column having finitely many 1s is meager, thus its complement in T_n, the set of all matrices with all semi-columns having infinitely many 1s, is nonmeager. Therefore the same holds for the superset $\{A \in T_n : \Xi_2(A) = \textit{Yes}\}$. Denoting this set by R we obviously have $R = \bigcup_{r\in\mathbb{N}} R_r$ with $R_r := \{A \in R : \Gamma_k(A) = \textit{Yes}\ \forall k \geq r\}$, and there is an r_n such that $S_n := R_{r_n}$ is nonmeager in T_n. Note that $\Gamma_{r_n,s}$ are General algorithms and $\Gamma_{r_n}(A) = \lim_{s\to\infty}\Gamma_{r_n,s}(A) = \textit{Yes}$ for all $A \in S_n$. Thus, Lemma 8.7 applies and yields an initial segment σ_n, such that

$$E^{T_n}(\sigma_n) \neq \emptyset \quad \text{and } \Gamma_{r_n}(A) = \textit{Yes} \text{ for all } A \in E^{T_n}(\sigma_n). \tag{8.30}$$

Now, let $T_{n+1} \subset T_n$ be the (closed) set of all matrices $\{a_{i,j}\}_{i,j\in\mathbb{N}}$ in $E^{T_n}(\sigma_n)$ with the property that (8.29) holds with $\sigma = \sigma_n$. Letting $T_0 = \Omega_2$ concludes the construction. The nested sequence $\{\sigma_n\}$again defines a matrix $A \in \cap_{n=0}^{\infty} T_n$ with the property that A has finitely many but at least k non-zero entries in the each of its kth semi-column which gives $\Xi_2(A) = \textit{No}$, but, by (8.30), $\Gamma_k(A) = \textit{Yes}$ for infinitely many k, a contradiction.

Step III: $\mathrm{SCI}(\Xi_1,\Omega_1)_{\mathrm{A}} \leq 3$ and $\mathrm{SCI}(\Xi_2,\Omega_2)_{\mathrm{A}} \leq 3$. This can again be proved by defining an appropriate tower of height 3 directly. For Ξ_1

$$\Gamma_{k,m,n}(\{a_{i,j}\}_{i,j\in\mathbb{N}}) = \textit{Yes} \quad \Leftrightarrow \quad |\{j=1,\dots,m : \sum_{i=1}^{n} a_{i,j} < m\}| < k.$$

For Ξ_2

$$\Gamma_{k,m,n}(\{a_{i,j}\}_{i,j\in\mathbb{Z}}) = \textit{Yes} \quad \Leftrightarrow \quad |\{j=-m,\dots,m : k < \sum_{i=1}^{n} a_{i,j} < m \text{ or } k < \sum_{i=-n}^{-1} a_{i,j} < m\}| = 0.$$

It is straightforward to show these provide height three arithmetical towers. ☐

The lower bounds of the SCI of the decision problems Ξ_1 and Ξ_2 allow us to obtain the lower bounds of the SCI of spectra and essential spectra of operators.

9. Proofs of Theorem 6.2 and Theorem 6.3

Remark 9.1 (Fourier Transform)**.** In this section we require the Fourier transform on $\mathrm{L}^2(\mathbb{R}^d)$, which will be denoted by $\mathcal{F} : \mathrm{L}^2(\mathbb{R}^d) \to \mathrm{L}^2(\mathbb{R}^d)$. Our definition of $\mathcal{F}$ is as follows:

$$[\mathcal{F}\psi](\xi) = \int_{\mathbb{R}^d} \psi(x) e^{-2\pi i x\cdot\xi}\, dx.$$

For brevity we may write $\hat{\psi}$ instead of $\mathcal{F}\psi$. With this definition $\mathcal{F}$ is unitary on $\mathrm{L}^2(\mathbb{R}^d)$.

Remark 9.2 (The Attouch-Wets Topology)**.** In (4.4) we introduced the Attouch-Wets metric d_{AW} on the space $\mathcal{M}$ of closed subsets of $\mathbb{C}$. Since it is not convenient to work with d_{AW} directly, we make note of the following simple characterization of convergence w.r.t. d_{AW}. Let $A \subset \mathbb{C}$ and $A_n \subset \mathbb{C}$, $n = 1,2,\dots$ be closed and non-empty. Then:

$$d_{\mathrm{AW}}(A_n, A) \to 0 \quad \text{if and only if} \quad d_{\mathcal{K}}(A_n, A) \to 0 \text{ for any compact } \mathcal{K} \subset \mathbb{C}, \tag{9.1}$$

where

$$d_{\mathcal{K}}(S,T) = \max\left\{\sup_{s\in S\cap\mathcal{K}} d(s,T), \sup_{t\in T\cap\mathcal{K}} d(t,S)\right\}, \tag{9.2}$$

where we use the convention that $\sup_{s\in S\cap\mathcal{K}} d(s,T) = 0$ if $S\cap\mathcal{K} = \emptyset$. We refer to [9, Chapter 3] for details and further discussion. Equivalently, we observe that

$$\begin{aligned} &d_{\mathrm{AW}}(A_n, A) \to 0 \\ &\qquad \text{if and only if} \\ &\forall\delta > 0,\ \mathcal{K}\subset\mathbb{C} \text{ compact}, \exists N \text{ s.t. } \forall n > N,\ A_n\cap\mathcal{K}\subset\mathcal{N}_\delta(A) \text{ and } A\cap\mathcal{K}\subset\mathcal{N}_\delta(A_n)\end{aligned} \tag{9.3}$$

where $\mathcal{N}_\delta(X)$ is the usual open δ-neighborhood of the set X. In this section we will simply use the notation $A_n \to A$ to denote this convergence, since there is no room for confusion.

9.1. **The case of bounded potential V.** We will split the proof of Theorem 6.2 into two sections:

a. $\mathrm{SCI}(\Xi_{\mathrm{sp}}, \Omega_{\phi,g})_{\mathrm{A}} = 1$*:* Whilst the proof of this is somehat long and technical, it is done via similar steps to the proof of Theorem 5.4 in §8.3, namely through approximations of the resolvent norm. However, some work is needed to convert point samples of V into approximations of the relevant matrices with respect to a Gabor basis. Lemmas 9.7 and 9.8 are technical lemmas needed to achieve this, whereas Lemma 9.9 concerns the approximations obtained via discretisations of the relevant inner products (and is need to gain the Σ_1^A classification).
b. *Error control and rest of proof:* Lemma 9.9 is used to prove $\{\Xi_{\mathrm{sp}}, \Omega_{\phi,g}\}\in\Sigma_1^A$. To prove the rest of the theorem, we argue that it is enough to prove $\{\Xi_{\mathrm{sp},\epsilon}, \Omega_\phi\}\in\Sigma_1^A$. This is done via Lemma 9.11 which uses the approximations of $\gamma(z)$ constructed in part (a).

Before we embark on the proof, the reader unfamiliar with the concept of Halton sequences may want to review this material. A great reference is [80] (see p. 29 for definition). We will also be needing the following definition and theorem in order to prove Theorem 6.2.

Definition 9.3. Let $\{t_1,\dots t_N\}$ be a sequence in $[0,1]^d$. Then we define the *star discrepancy* of $\{t_1,\dots t_N\}$ to be

$$D_N^*(\{t_1,\dots t_N\}) = \sup_{K\in\mathcal{K}}\left|\frac{1}{N}\sum_{k=1}^N \chi_K(t_k) - \nu(K)\right|,$$

where $\mathcal{K}$ denotes the family of all subsets of $[0,1]^d$ of the form $\prod_{k=1}^d [0,b_k)$, χ_K denotes the characteristic function on K, $b_k\in(0,1]$ and ν denotes the Lebesgue measure.

Theorem 9.4 ([80])**.** *If $\{t_k\}_{k\in\mathbb{N}}$ is the Halton sequence in $[0,1]^d$in the pairwise relatively prime bases $b_1,\dots,b_d$, then*

$$D_N^*(\{t_1,\dots t_N\}) < \frac{d}{N} + \frac{1}{N}\prod_{k=1}^d\left(\frac{b_k-1}{2\log(b_k)}\log(N) + \frac{b_k+1}{2}\right) \qquad N\in\mathbb{N}. \tag{9.4}$$

For a proof of this theorem see [80], p. 29. Note that as the right hand side of (9.4) is rather cumbersome to work with, it is convenient to define the following constant.

Definition 9.5. Define $C^*(b_1,\dots,b_d)$ to be the smallest integer such that for all $N\in\mathbb{N}$

$$\frac{d}{N} + \frac{1}{N}\prod_{k=1}^d\left(\frac{b_k-1}{2\log(b_k)}\log(N) + \frac{b_k+1}{2}\right) \le C^*(b_1,\dots,b_d)\frac{\log(N)^d}{N}$$

where $b_1,\dots,b_d$ are as in Theorem 9.4.

Further to these definitions, we shall require a Gabor basis which is the core in the discretisation carried out to produce the tower of algorithms. In particular, let

$$\psi_{k,l}(x) = e^{2\pi ikx}\chi_{[0,1]}(x-l), \qquad k,l \in \mathbb{Z}. \tag{9.5}$$

It is well-known that $\psi_{k,l}$ form an orthonormal basis for $\mathrm{L}^2(\mathbb{R})$. Thus, by applying the Fourier transform,

$$\{\hat{\psi}_{k_1,l_1} \otimes \hat{\psi}_{k_2,l_2} \otimes \cdots \otimes \hat{\psi}_{k_d,l_d} : k_1, l_1, \ldots, k_d, l_d \in \mathbb{Z}\} \tag{9.6}$$

forms an orthonormal basis for $L^2(\mathbb{R}^d)$ since the Fourier transform $\mathcal{F}$ is unitary. Let $\{\varphi_j\}_{j\in\mathbb{N}}$ be an enumeration of the collection of functions above, define

$$\mathcal{S} = \mathrm{span}\{\varphi_j\}_{j\in\mathbb{N}} \tag{9.7}$$

and let

$$\theta : \mathbb{N} \ni j \mapsto (k_1,l_1) \times \ldots \times (k_d,l_d) \in \mathbb{Z}^{2d} \tag{9.8}$$

be the bijection used in this enumeration. Define

$$\begin{aligned} \tilde{k}(m,d) &:= \max\{|k_p| : (k_p,l_p) = \theta(j)_p, p \in \{1,\ldots,d\}, j \in \{1,\ldots,m\}\}, \\ \tilde{l}(m,d) &:= \max\{|l_p| : (k_p,l_p) = \theta(j)_p, p \in \{1,\ldots,d\}, j \in \{1,\ldots,m\}\}, \end{aligned} \tag{9.9}$$

and let

$$C_1(m,d,a) := d^2 \left(4\frac{(\max\{\tilde{l}(m,d)^2 + \tilde{l}(m,d) + 1/3, 1\})^2}{|a - \tilde{k}(m,d)| + 1}\right)^d, \qquad m,d,a \in \mathbb{N}, \tag{9.10}$$

$$C_2(m,d) := 2^d \left(2((\tilde{l}(m,d)+1)^4 + \tilde{l}(m,d)^4)^2(2(\tilde{k}(m,d)+1)+2)\right)^d, \qquad m,d \in \mathbb{N}. \tag{9.11}$$

The quantities $C_1(m,d,a)$ and $C_2(m,d)$ may seem to come out of the blue. They stem from Lemma 9.7 and Lemma 9.8 that are technical lemmas needed in order to construct the tower of algorithms. However, $C_1(m,d,a)$ and $C_2(m,d)$ occur in the main proof and thus it is advantageous to introduce them here to prepare the reader.

Remark 9.6 (Assumptions on Λ)**.** As mentioned in Remark 6.1 we will now specify the assumption on the constants in Λ. In particular, Λ will include

$$\{\theta(j)_p : p \leq d, j \in \mathbb{N}\} \cup \{C^*(b_1,\ldots,b_d)\} \cup \{\log(k\phi(k))\}_{k=1}^{\infty} \cup \{\phi(k)\}_{k=1}^{\infty},$$

where ϕ is the function describing the bound on the local bounded variation in (6.2). Moreover, Λ will also include

$$\left\{\frac{\partial^s \hat{\psi}_{k,l}}{\partial \xi^s}(\xi) : \xi \in \mathbb{R},\, k,l \in \mathbb{Z},\, s = 0,2\right\}. \tag{9.12}$$

Note that it is easy to derive closed form expressions for $\hat{\psi}_{k,l}$ and $\frac{\partial^2 \hat{\psi}_{k,l}}{\partial \xi^2}$, and these expressions will be variations of products of exponential functions and functions of the form $x \mapsto 1/x^p$ for $p = 1,2,3$. Any of the general algorithms $\Gamma : \Omega \to \mathcal{M}$ (where Ω is the appropriate domain), used in the lowest level of the tower, will satisfy the assumptions in Remark 6.1. In particular, the constant functions in $\Lambda_\Gamma(A)$ are the same for different inputs $A, B \in \Omega$. We will also assume that Λ contains an upper bound on $\|V\|_\infty$.

Proof that $\mathrm{SCI}(\Xi_{\mathrm{sp}}, \Omega_{\phi,g})_{\mathrm{A}} = 1$**.**

The proof will make clear that we do not need to worry about the algorithm outputting the empty set - given m, simply compute $\Gamma_{j(m)}(V)$ with $j(m) \geq m$ minimal such that $\Gamma_{j(m)}(V) \neq \emptyset$.

Proof of $\mathrm{SCI}(\Xi_{\mathrm{sp}}, \Omega_{\phi,g})_{\mathrm{A}} = 1$. **Step I: Defining** $\Gamma_m(\{V_\rho\}_{\rho\in\Lambda_{\Gamma_m}(V)})$ **and** $\Lambda_{\Gamma_m}(V)$. To do so recall $\mathcal{S}$ from (9.7). Note that since $\mathcal{D}(H) = \mathrm{W}^{2,2}(\mathbb{R}^d)$ it is easy to show that $\mathcal{S}$ is a core for H. Let P_m, $m \in \mathbb{N}$, be the projection onto $\mathrm{span}\{\varphi_j\}_{j=1}^m$, and let $z \in \mathbb{C}$. Define

$$S_m(V,z) := (-\Delta + V - zI)P_m \quad \text{and} \quad \tilde{S}_m(V,z) := (-\Delta + \overline{V} - \overline{z}I)P_m.$$

Let

$$\sigma_1(S_m(V,z)) := \min\{(\langle S_m(V,z)f, S_m(V,z)f\rangle)^{\frac{1}{2}} : f \in \mathrm{Ran}(P_m), \|f\| = 1\}$$

and $\sigma_1(\tilde{S}_m(V,z)) := \min\{(\langle \tilde{S}_m(V,z)f, \tilde{S}_m(V,z)f\rangle)^{\frac{1}{2}} : f \in \mathrm{Ran}(P_m), \|f\| = 1\}$, and define

$$\gamma_m(z) := \min\{\sigma_1(S_m(V,z)), \sigma_1(\tilde{S}_m(V,z))\}\}. \tag{9.13}$$

Note that if we could evaluate γ_m at any point z using only finitely many arithmetic operations of elements of the form $V(x)$, $x \in \mathbb{R}^d$, we could have defined a general algorithm as desired by using $\Upsilon_{B_m(0)}^{1/m}(\gamma_m)$ where $\Upsilon_{B_m(0)}^{1/m}$ is defined in (8.12). Unfortunately, such evaluation is not possible (γ_m may depend on infinitely many samples of V), and we will now focus on finding an approximation to γ_m.

Let $S = \{t_k\}_{k\in\mathbb{N}}$, where $t_k \in [0,1]^d$ is a Halton sequence (see [80] p. 29 for definition) in the pairwise relatively prime bases $b_1, \ldots, b_d$ (note that the particular choice of the b_js is not important). Define, for $a > 0$ and $N \in \mathbb{N}$, the discrete inner product

$$\langle f, u\rangle_{a,N} = \frac{(2a)^d}{N}\sum_{k=1}^{N} f^a(t_k)\overline{u^a(t_k)}, \qquad f, u \in L^2(\mathbb{R}^d) \cap \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d)), \tag{9.14}$$

where we have defined the rescaling function on $[0,1]^d$ by

$$f^a = f(a(2\cdot -1), \ldots, a(2\cdot -1))|_{[0,1]^d}, \tag{9.15}$$

(we will throughout the proof use the superscript a on a function to indicate (9.15)), where $\mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d)) = \{f : \mathrm{TV}(f|_{[-b,b]^d}) < \infty, \forall b > 0\}$ and $\mathrm{TV}(f|_{[-b,b]^d})$ denotes the total variation, in the sense of Hardy and Krause (see [80]), of f restricted to $[-b,b]^d$. Note that since $V \in \mathrm{L}^\infty(\mathbb{R}^d) \cap \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d)$ and any $f \in \mathrm{Ran}(P_m)$ is smooth we have that $S_m(V,z)f \in L^2(\mathbb{R}^d) \cap \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d))$. Hence, we can define for $n, m \in \mathbb{N}$

$$\begin{aligned} \sigma_{1,n}(S_m(V,z)) &:= \min\{(\langle S_m(V,z)f, S_m(V,z)f\rangle_{n,N(n)})^{\frac{1}{2}} : f \in \mathrm{Ran}(P_m), \|f\| = 1\} \\ \sigma_{1,n}(\tilde{S}_m(V,z)) &:= \min\{(\langle \tilde{S}_m(V,z)f, \tilde{S}_m(V,z)f\rangle_{n,N(n)})^{\frac{1}{2}} : f \in \mathrm{Ran}(P_m), \|f\| = 1\}, \end{aligned} \tag{9.16}$$

where $N(n) := \lceil n\phi(n)^4\rceil$ and where ϕ comes from the definition of Ω_ϕ. Let

$$\zeta_m(z) := \min\{k/m : k \in \mathbb{N},\ k/m \geq \min\{\sigma_{1,n(m)}(S_m(V,z)), \sigma_{1,n(m)}(\tilde{S}_m(V,z))\}\}, \tag{9.17}$$

$$n(m) := \min\{n : \tilde{\tau}(m,n) \leq \frac{1}{m^3}\}, \tag{9.18}$$

and

$$\begin{aligned} \tilde{\tau}(m,n) := &(m+1)mC_1(m,d,n) \\ &+ d^2\left(m^2 + \sigma^2\phi^2(n) + 2(\sigma m + 1)(\phi(n)+1)\right) \\ &\times \left(1 + \sigma^2 + 2\sigma\right)C_2(m,d)C^*(b_1,\ldots,b_d)\frac{\log(N(n))^d}{N(n)}, \quad N(n) = \lceil n\phi(n)^4\rceil, \end{aligned} \tag{9.19}$$

$\sigma = 3^d - 2^{d+1} + 2$, $C_1(m,d,n)$ is defined in (9.10), $C_2(m,d)$ is defined in (9.11) and $C^*(b_1,\ldots,b_d)$ is defined in Definition 9.5. First, note that the choice of $N(n)$ in (9.19) implies that $\tilde{\tau}(m,n) \to 0$ as $n \to \infty$. Thus, $n(m)$ is well defined. Second, note that it is clear that $\tilde{\tau}$, and hence also $n(m)$, can be evaluated by using finitely many arithmetic operations and comparisons from the set

$$\tilde{\Lambda}_1 = \{\theta(j)_p : p \leq d, j \leq m\} \cup \{C^*(b_1,\ldots,b_d)\} \cup \{\log(k\phi(k))\}_{k=1}^r \cup \{\phi(k)\}_{k=1}^r, \tag{9.20}$$

where r is some finite integer depending on m. Recall from Remark 9.6 that we have that $\tilde{\Lambda}_1 \subset \Lambda$.

The function $\tilde{\tau}$ may seem to come somewhat out of the blue, however, it stems from certain bounds in (9.41) (see also (9.42)) on errors of discrete integrals related to (9.16). We can now define

$$\Gamma_m(V) = \Gamma_m(\{V_\rho\}_{\rho\in\Lambda_{\Gamma_m}(V)}) := \Upsilon^{1/m}_{B_m(0)}(\zeta_m),$$

where $\Upsilon^{1/m}_{B_m(0)}(\zeta_m)$ is defined in (8.12) and

$$\Lambda_{\Gamma_m}(V) = \{\rho_x : x \in L_m\} \cup \tilde{\Lambda}_1 \cup \tilde{\Lambda}_2. \tag{9.21}$$

Here $\{\rho_x : x \in L_m\}$ is the set of all point evaluations $\rho_x(V) := V(x)$ at the points in

$$L_m := \{(n(2t_{k,1} - 1), \dots, n(2t_{k,d} - 1)) : k = 1, \dots, N(n) = \lceil n\phi(n)^4\rceil, n = n(m)\},$$

where $t_k = \{t_{k,1}, \dots, t_{k,d}\}$, $n = n(m)$ is defined in (9.18) and $\tilde{\Lambda}_2$ is a finite set of constant functions that will be determined in (9.27) in Step II.

To show that this provides an arithmetic tower of algorithms for Ξ_{sp} note that each of the mappings $V \mapsto \Gamma_m(V)$ is an algorithm as desired for arithmetic towers of algorithms. The computation of $\Upsilon^{1/m}_{B_m(0)}(\zeta_m)$ requires only finitely many evaluations of ζ_m, hence it suffices to demonstrate the following.

Step II: For a single $z \in \mathbb{C}$, the evaluation of $\zeta_m(z)$ requires finitely many arithmetic operations of the elements $\{V_\rho\}_{\rho\in\Lambda_{\Gamma_m}(V)}$. To see this we proceed as follows. For $z \in \mathbb{C}$, form the matrices $Z_m(z), \tilde{Z}_m(z) \in \mathbb{C}^{m\times m}$ by considering the orthonormal basis $\{\varphi_j\}_{j\in\mathbb{N}}$ constructed in the beginning of Step I. More precisely,

$$\begin{aligned} Z_m(z)_{ij} &= \langle S_m(V,z)\varphi_j, S_m(V,z)\varphi_i\rangle_{n,N}, \quad i,j \le m, \\ \tilde{Z}_m(z)_{ij} &= \langle \tilde{S}_m(V,z)\varphi_j, \widetilde{S}_m(V,z)\varphi_i\rangle_{n,N}, \quad i,j \le m, \quad N = N(n) = \lceil n\phi(n)^4\rceil, \end{aligned} \tag{9.22}$$

where $n = n(m)$ is defined in (9.18). Note that forming $Z_m(z)_{ij}$ and $\tilde{Z}_m(z)_{ij}$ require only finitely many arithmetic operations and comparisons on the elements $\{V_\rho\}_{\rho\in\Lambda_{\Gamma_m}(V)}$, where we will now specify $\tilde{\Lambda}_2$ in (9.21). Indeed,

$$\begin{aligned} \langle S_m(V,z)\varphi_j, S_m(V,z)\varphi_i\rangle_{n,N} =& \langle \Delta\varphi_j, \Delta\varphi_i\rangle_{n,N} - \langle V\varphi_j, \Delta\varphi_i\rangle_{n,N} - \langle \Delta\varphi_j, V\varphi_i\rangle_{n,N} \\ &+ \langle V\varphi_j, V\varphi_i\rangle_{n,N} - 2\Re(z)(\langle \Delta\varphi_j, \varphi_i\rangle_{n,N} \\ &+ \langle V\varphi_j, \varphi_i\rangle_{n,N}) + |z|^2\langle \varphi_j, \varphi_i\rangle_{n,N}. \end{aligned} \tag{9.23}$$

for $i,j \le m$. Observe that for $s,t \in \{0,1\}$, $\tilde{s},\tilde{t} \in \{0,2\}$ and $u \in \{V, \overline{V}, |V|^2\}$ it follows that

$$\langle u\,\Delta^s\varphi_j, \Delta^t\varphi_i\rangle_{n,N} = \frac{(2n)^d}{N}\sum_{k=1}^{N}\left(u^n(t_k)\sum_{p\in\Phi(s),q\in\Phi(t)} h^n_{i,j,p,q}(t_k)\right), \qquad i,j \le m, \tag{9.24}$$

$$\begin{aligned} h_{i,j,p,q}(x) := &\left(\hat{\psi}_{\theta(j)_1}(x_1)\cdots\frac{\partial^{\tilde{s}}\hat{\psi}_{\theta(j)_p}}{\partial x_p^{\tilde{s}}}(x_p)\cdots\hat{\psi}_{\theta(j)_d}(x_d)\right) \\ &\times\left(\overline{\hat{\psi}_{\theta(i)_1}(x_1)\cdots\frac{\partial^{\tilde{t}}\hat{\psi}_{\theta(i)_q}}{\partial x_q^{\tilde{t}}}(x_q)\cdots\hat{\psi}_{\theta(i)_d}(x_d)}\right), \end{aligned} \tag{9.25}$$

$$\Phi(t) = \begin{cases} \{1,\dots,d\}, & t = 1 \\ \{1\}, & t = 0, \end{cases} \tag{9.26}$$

where $\tilde{s} = 2s$ and $\tilde{t} = 2t$. Note that because of the choice of $\psi_{k,l}$ in (9.5) we have explicit formulas for $\hat{\psi}_{\theta(j)_p}$ and $\frac{\partial^{\tilde{s}}\hat{\psi}_{\theta(j)_p}}{\partial x_p^{\tilde{s}}}$ that are variants of exponential functions. Thus, since $n(m)$ can be evaluated with finitely many arithmetic operations and comparisons of the elements in $\tilde{\Lambda}_1$, and by (9.24), (9.25) and (9.26), it follows that $\langle u\,\Delta^s\varphi_j, \Delta^t\varphi_i\rangle_{n,N}$ can be evaluated by using finitely many arithmetic operations and comparisons of elements in $\{\rho(V) : \rho \in \Lambda_{\Gamma_m}(V)\}$ where $\Lambda_{\Gamma_m}(V)$ is defined in (9.21) and

$$\tilde{\Lambda}_2 = \left\{ \rho_x\left(\frac{\partial^{\tilde{s}}\hat{\psi}_{\theta(j)_p}}{\partial x_p^{\tilde{s}}}\right) : x \in L_m,\, 1 \le j \le m,\, 1 \le p \le d,\, \tilde{s} = 0, 2 \right\}. \tag{9.27}$$

(As discussed in the assumption in Remark 9.6, we treat the numbers in $\tilde{\Lambda}_2$ as constant functions on Ω). Hence, it follows that forming $Z_m(z)_{ij}$ requires only finitely many arithmetic operations and comparisons of the elements $\{V_\rho\}_{\rho\in\Lambda_{\Gamma_m}(V)}$. The argument for $\tilde{Z}_m(z)_{ij}$ using $\langle \tilde{S}_m(V,z)\varphi_j, \widetilde{S}_m(V,z)\varphi_i\rangle_{a,N}$ is identical.

When $Z_m(z)$ and $\tilde{Z}_m(z)$ are formed, we proceed as follows in order to compute $\zeta_m(z)$. For $k \in \mathbb{N}$, we start with $k = 1$, then:

- Check whether $\min\{\sigma_1(Z_m(z)), \sigma_1(\tilde{Z}_m(z))\} \le k/m$.
- If not let $k = k + 1$ and repeat, otherwise $\zeta_m(z) = k/m$.

Note that the first step requires finitely many arithmetic operations of $\{Z_m(z)_{ij}\}_{i,j\le m}$ and $\{\tilde{Z}_m(z)_{ij}\}_{i,j\le m}$ by Proposition 8.1, and the loop will clearly terminate for a finite k and thus compute $\zeta_m(z)$. Hence, we have proven the assertion that the evaluation of $\zeta_m(z)$ requires finitely many arithmetic operations and comparisons of the elements $\{V_\rho\}_{\rho\in\Lambda_{\Gamma_m}(V)}$ and we conclude that Γ_m are general algorithms as desired for arithmetic towers of algorithms.

Step III: Finally, we show that $\Gamma_m(V) \to \Xi_{\mathrm{sp}}(V)$, as $m \to \infty$. Note that, by the properties of the Attouch-Wets topology, and as discussed in Remark 9.2, it suffices to show that for any compact set $\mathcal{K} \subset \mathbb{C}$

$$d_{\mathcal{K}}(\Gamma_m(V), \Xi_{\mathrm{sp}}(V)) \longrightarrow 0, \quad m \to \infty, \tag{9.28}$$

where $d_{\mathcal{K}}$ is defined in (9.2). To show (9.28) we start by defining

$$\begin{aligned}\gamma(z) := \min\big\{ &\inf\{\|(-\Delta + V - zI)\psi\| : \psi \in \mathrm{W}^{2,2}(\mathbb{R}^d), \|\psi\| = 1\},\\ &\inf\{\|(-\Delta + \overline{V} - \overline{z}I)\psi\| : \psi \in \mathrm{W}^{2,2}(\mathbb{R}^d), \|\psi\| = 1\}\big\} = \|(-\Delta + V - zI)^{-1}\|^{-1},\end{aligned} \tag{9.29}$$

where we use the convention that $\|(-\Delta+V-zI)^{-1}\|^{-1} = 0$ when $z \in \mathrm{sp}(-\Delta+V)$ and proceed similarly to the proof of Theorem 5.4 with the following claim. Before we state the claim recall h from the definition of $\Upsilon_K^\delta(\zeta)$ in Step II of the proof of Theorem 5.4 in §8.3.

Claim: Let $\mathcal{K} \subset \mathbb{C}$ be any compact set, and let K be a compact set containing $\mathcal{K}$ such that $\mathrm{sp}(-\Delta+V)\cap K \neq \emptyset$ and $0 < \delta < \epsilon < 1/2$. Suppose that ζ is a function with $\|\zeta - \gamma\|_{\infty,\hat{K}} := \|(\zeta-\gamma)\chi_{\hat{K}}\|_\infty < \epsilon$ on $\hat{K} := (K + B_{h(\mathrm{diam}(K)+2\epsilon)+\epsilon}(0))$, where $\chi_{\hat{K}}$ denotes the characteristic function of $\hat{K}$ and h is the inverse of g. Finally, let u be defined as in (8.13). Then $\lim_{\xi\to 0} u(\xi) = 0$ and

$$d_{\mathcal{K}}(\Upsilon_K^\delta(\zeta), \mathrm{sp}(-\Delta+V)) \le u(\epsilon). \tag{9.30}$$

To prove the claim, we first show that

$$\sup_{s\in\Upsilon_K^\delta(\zeta)\cap\mathcal{K}} \mathrm{dist}(s, \mathrm{sp}(-\Delta+V)) \le u(\epsilon). \tag{9.31}$$

If $\Upsilon_K^\delta(\zeta)\cap\mathcal{K} = \emptyset$ then there is nothing to prove, thus we assume that $\Upsilon_K^\delta(\zeta)\cap\mathcal{K} \neq \emptyset$. Let $z \in G^\delta(K)$ and recall $G^\delta(K)$, h_δ and $I_z = B_{h_\delta(\zeta(z))}(z) \cap (\delta(\mathbb{Z}+\mathrm{i}\mathbb{Z}))$ from the definition of $\Upsilon_K^\delta(\zeta)$ in Step II of the proof of Theorem 5.4 in §8.3. Notice that we may argue exactly as in (8.14) and deduce that $I_z \subset \hat{K}$. Suppose that $M_z \neq \emptyset$. Note that from

$$\|(-\Delta+V-zI)^{-1}\|^{-1} \ge g(\mathrm{dist}(z, \mathrm{sp}(H))),$$

the monotonicity of h, and the compactness of $\mathrm{sp}(-\Delta+V)\cap K \neq \emptyset$ there is a $y \in \mathrm{sp}(-\Delta+V)$ of minimal distance to z with $|z-y| \le h(\gamma(z))$. Since $\|\zeta-\gamma\|_{\infty,\hat{K}} < \epsilon$, and by using the monotonicity of h, we get $|z-y| \le h(\zeta(z)+\epsilon)$. Hence, at least one of the $v \in I_z$, say v_0, satisfies $|v_0 - y| < h(\zeta(z)+\epsilon) - h(\zeta(z)) + 2\delta$. Thus, by noting that $\gamma(v_0) \le \mathrm{dist}(v_0, \mathrm{sp}(-\Delta+V))$, and by the assumption that $\delta < \epsilon$, we get $\zeta(v_0) < \gamma(v_0) + \epsilon < h(\zeta(z) + \epsilon) - h(\zeta(z)) + 3\epsilon$. By the definition of M_z, this estimate now holds for all points $w \in M_z$. Thus, we may argue exactly as in (8.15) and deduce that $\mathrm{dist}(w, \mathrm{sp}(-\Delta + V)) \leq h(h(\zeta(z) + \epsilon) - h(\zeta(z)) + 3\epsilon)$ which yields (9.31). To see that

$$\sup_{t \in \mathrm{sp}(-\Delta+V)\cap\mathcal{K}} \mathrm{dist}(\Upsilon_K^\delta(\zeta), t) \leq u(\epsilon), \tag{9.32}$$

(where we assume that $\mathrm{sp}(-\Delta + V) \cap \mathcal{K} \neq \emptyset$) take any $y \in \mathrm{sp}(-\Delta + V) \cap \mathcal{K} \subset K$. Then there is a point $z \in G^\delta(K)$ with $|z - y| < \delta < \epsilon$, hence $\zeta(z) < \gamma(z) + \epsilon \leq \mathrm{dist}(z, \mathrm{sp}(-\Delta + V)) + \epsilon < 2\epsilon < 1$. Thus, M_z is not empty and contains a point which is closer to y than $h(\zeta(z)) + \epsilon \leq h(2\epsilon) + \epsilon \leq u(\epsilon)$, and this yields (9.32). The fact that $\lim_{\xi\to 0} u(\xi) = 0$ is shown in Step II of the proof of Theorem 5.4 in §8.3, and we have proved the claim.

Armed with this claim we continue on the path to prove (9.28). We define

$$\gamma_{m,n}(z) := \min\{\sigma_{1,n}(S_m(V, z)), \sigma_{1,n}(\tilde{S}_m(V, z))\}. \tag{9.33}$$

Then $\zeta_m = \gamma_{m,n(m)}$ where $n(m)$ is defined as in (9.18). By Lemma 9.9 (below), $\zeta_m \to \gamma$ locally uniformly, when $m \to \infty$. Let m_0 be large enough so that $B_{m_0}(0) \supset \mathcal{K}$. Then, for all $m \geq m_0$, $\Gamma_m(V) \cap \mathcal{K} = \Upsilon^{1/m}_{B_{m_0}(0)}(\zeta_m) \cap \mathcal{K}$. Choose $K = B_{m_0}(0)$ and $\epsilon \in (0, 1/2)$ as in the claim. Then, by the claim, there is an $m_1 > m_0$ such that for every $m > m_1$, by (9.30), $d_\mathcal{K}(\Gamma_m(V), \Xi_{\mathrm{sp}}(V)) \leq u(\epsilon)$. Since $\lim_{\xi\to 0} u(\xi) = 0$ then 9.28) follows. ☐

To finish this step of the proof, we need to establish the convergence of the functions γ_m, ζ_m and $\gamma_{m,n}$.

Lemma 9.7. *Consider the functions $\gamma_{m,n}$ and γ_m defined in (9.33) and (9.13) respectively. Then $\gamma_{m,n} \to \gamma_m$, locally uniformly as $n \to \infty$.*

Proof. Note that we will be using the notation $\mathrm{TV}_{[-a,a]^d}(f) = \mathrm{TV}(f|_{[-a,a]^d})$. Let, for $s, t \in \{0, 1\}$, $i, j \leq m$ and $u \in \{V, \overline{V}, |V|^2\}$

$$I(u, \Delta^s \varphi_j, \Delta^t \varphi_i) = \int_{\mathbb{R}^d} u(x) \sum_{p\in\Phi(s),q\in\Phi(t)} h_{i,j,p,q}(x)\, dx,$$

where $h_{i,j,p,q}$ is defined in (9.25) and Φ is defined in (9.26) (recall that $\{\varphi_j\}_{j\in\mathbb{N}}$ is an enumeration of $\{\hat{\psi}_{k_1,l_1} \otimes \hat{\psi}_{k_2,l_2} \otimes \cdots \otimes \hat{\psi}_{k_d,l_d} : k_1, l_1, \ldots, k_d, l_d \in \mathbb{Z}\}$ from (9.6)). Observe that by the definition of $\gamma_{m,n}$ and γ_m in (9.33) and (9.13) the lemma follows if we can show that

$$I(u, \Delta^s \varphi_j, \Delta^t \varphi_i) - \frac{(2n)^d}{N} \sum_{k=1}^{N} u^n(t_k) \sum_{p\in\Phi(s),q\in\Phi(t)} h^n_{i,j,p,q}(t_k)) \longrightarrow 0, \qquad n \to \infty, \tag{9.34}$$

where $N = N(n)$ is from (9.22), $i, j \leq m$, $s, t \in \{0, 1\}$ and u is either $V, \overline{V}, |V|^2$ (recall the notation V^a from (9.15)). Note that, by the multi-dimensional Koksma-Hlawka inequality (Theorem 2.11 in [80]) it follows that

$$\begin{aligned} &|I(u, \Delta^s \varphi_j, \Delta^t \varphi_i) - \frac{(2a)^d}{N} \sum_{k=1}^{N} u^n(t_k) \sum_{p\in\Phi(s),q\in\Phi(t)} h^n_{i,j,p,q}(t_k)| \\ &\leq \|u \sum_{p\in\Phi(s),q\in\Phi(t)} h_{i,j,p,q}\chi_{R(n)}\|_{\mathrm{L}^1} + \mathrm{TV}_{[-n,n]^d}\left(u^n \sum_{p\in\Phi(s),q\in\Phi(t)} h^n_{i,j,p,q}\right) D^*_N(t_1, \ldots, t_N), \end{aligned} \tag{9.35}$$

where $R(n) = ([-n, n]^d)^c$. To bound the first part of the right hand side of (9.35) note that

$$\left\| u \sum_{p\in\Phi(s),q\in\Phi(t)} h_{i,j,p,q}\chi_{R(n)} \right\|_{\mathrm{L}^1} \leq \|u\|_\infty K_{i,j}(n), \tag{9.36}$$

where

$$K_{i,j}(n) := \sum_{p\in\Phi(s),q\in\Phi(t)} \left\langle |\chi_{([-n,n]^d)^c}\hat{\psi}_{\theta(j)_1}\cdots\frac{\partial^{\tilde{s}}\hat{\psi}_{\theta(j)_p}}{\partial x_p^{\tilde{s}}}\cdots\hat{\psi}_{\theta(j)_d}|, |\hat{\psi}_{\theta(i)_1}\cdots\frac{\partial^{\tilde{t}}\hat{\psi}_{\theta(i)_q}}{\partial x_q^{\tilde{t}}}\cdots\hat{\psi}_{\theta(i)_d}|\right\rangle,$$

(recall θ from (9.8)) where $\chi_{([-n,n]^d)^c}$ denotes the characteristic function on $([-n,n]^d)^c$. To bound $K_{i,j}(n)$, note that it follows by the definition of $\psi_{k,l}$ with $k,l\in\mathbb{Z}$ in (9.5) and some straightforward integration that for $1\leq p\leq d$ and $(k_p,l_p)=\theta(j)_p$ we have

$$\left|\hat{\psi}_{k_p,l_p}(x_p)\right| \leq \begin{cases} 1 & \text{when } k_p-1\leq x_p\leq k_p+1, \\ \frac{1}{|x_p-k_p|+1} & \text{otherwise,}\end{cases} \tag{9.37}$$

$$\left|\frac{\partial^2\hat{\psi}_{k_p,l_p}}{\partial x_p^2}(x_p)\right| \leq \begin{cases} l_p^2+l_p+\frac{1}{3} & \text{when } k_p-1\leq x_p\leq k_p+1, \\ \frac{l_p^2+l_p+\frac{1}{3}}{|x_p-k_p|+1} & \text{otherwise.}\end{cases} \tag{9.38}$$

Hence, if

$$\begin{aligned}\tilde{k} = \tilde{k}(m,d) &:= \max\{|k_p| : (k_p,l_p)=\theta(j)_p, p\in\{1,\dots,d\}, j\in\{1,\dots,m\}\},\\ \tilde{l} = \tilde{l}(m,d) &:= \max\{|l_p| : (k_p,l_p)=\theta(j)_p, p\in\{1,\dots,d\}, j\in\{1,\dots,m\}\},\end{aligned}$$

and $n>\tilde{k}$, then it follows that

$$\begin{aligned} K_{i,j}(n) \leq d^2\max\Big\{\Big\langle &|\chi_{([-n,n]^d)^c}\hat{\psi}_{\theta(j)_1}\cdots\frac{\partial^{2s}\hat{\psi}_{\theta(j)_p}}{\partial x_p^{2s}}\cdots\hat{\psi}_{\theta(j)_d}|,\\ &|\hat{\psi}_{\theta(i)_1}\cdots\frac{\partial^{2t}\hat{\psi}_{\theta(i)_q}}{\partial x_q^{2t}}\cdots\hat{\psi}_{\theta(i)_d}|\Big\rangle : p\in\Phi(s), q\in\Phi(t), s,t\in\{0,1\}\Big\} \\ &\leq C_1(m,d,n), \qquad C_1(m,d,n) = d^2\left(4\frac{(\max\{\tilde{l}^2+\tilde{l}+1/3,1\})^2}{|n-\tilde{k}|+1}\right)^d. \end{aligned} \tag{9.39}$$

To bound the second part of the right hand side of (9.35) observe that, by Lemma 9.8 we have

$$\begin{aligned} &\mathrm{TV}_{[-n,n]^d}\left(u^n\sum_{p\in\Phi(s),q\in\Phi(t)} h^n_{i,j,p,q}\right)\\ &\leq d^2\big(\|u\|_\infty\|h_{i,j,p,q}\|_\infty + \sigma^2\mathrm{TV}_{[-n,n]^d}(u)\mathrm{TV}_{[-n,n]^d}(h_{i,j,p,q})\\ &\quad + \sigma\left(\mathrm{TV}_{[-n,n]^d}(u)\|h_{i,j,p,q}\|_\infty + \mathrm{TV}_{[-n,n]^d}(h_{i,j,p,q})\|u\|_\infty\right)\big)\\ &\leq d^2\max\left\{\|V\|_\infty, \|V^2\|_\infty, \mathrm{TV}_{[-n,n]^d}(V), \mathrm{TV}_{[-n,n]^d}(|V|^2)\right\}\left(1+\sigma^2+2\sigma\right)C_2(m,d), \end{aligned} \tag{9.40}$$

where $\sigma = 3^d-2^{d+1}+2$ and $C_2(m,d)$ is defined in (9.11). Thus, by (9.35), (9.36), (9.39), (9.40), Lemma 9.8 and Theorem 9.4 (recall that $\{t_k\}_{k\in\mathbb{N}}$ is a Halton sequence) we get

$$\begin{aligned} &|I(u,\Delta^s\varphi_j,\Delta^t\varphi_i) - \frac{(2n)^d}{N}\sum_{k=1}^N V^n(t_k)\sum_{p\in\Phi(s),q\in\Phi(t)} h^n_{i,j,p,q}(t_k)|\\ &\leq \max\{\|V\|_\infty,\|V\|_\infty^2\}C_1(m,d,n) + d^2\max\left\{\|V\|_\infty, \|V^2\|_\infty, \mathrm{TV}_{[-n,n]^d}(V), \mathrm{TV}_{[-n,n]^d}(|V|^2)\right\}\\ &\qquad\times\left(1+\sigma^2+2\sigma\right)C_2(m,d)\left(\frac{d}{N}+\frac{1}{N}\prod_{k=1}^d\left(\frac{b_k-1}{2\log(b_k)}\log(N)+\frac{b_k+1}{2}\right)\right)\\ &\leq \tau(\|V\|_\infty, m, n), \end{aligned} \tag{9.41}$$

where the last inequality uses the bound on the total variation of V from (6.2) and

$$\begin{aligned}\tau(\|V\|_\infty, m, n) &:= (\|V\|_\infty + 1)\|V\|_\infty C_1(m,d,n)\\ &\quad + d^2\left(\|V\|_\infty^2 + \sigma^2\phi^2(n) + 2(\sigma\|V\|_\infty + 1)(\phi(n)+1)\right)\\ &\quad \times \left(1+\sigma^2+2\sigma\right) C_2(m,d)C^*(b_1,\ldots,b_d)\frac{\log(N)^d}{N}, \quad N(n) = \lceil n\phi(n)^4\rceil\end{aligned} \tag{9.42}$$

(recall (9.16)) where $C^*(b_1,\ldots,b_d)$ is defined in Definition 9.5. Finally, note that, by the definition of $C_1(m,d,n)$ and the fact that we have chosen $N(n)$ according to (9.42), it follows that $\tau(\|V\|_\infty, m, n) \to 0$ as $n \to \infty$. Hence, (9.34) follows via (9.42), and the proof is finished. □

Lemma 9.8. *For all $a > 0$, $i, j \leq n_2$ and $m, n \leq d$:*

(i) $\mathrm{TV}(h^a_{i,j,m,n}) \leq C_2(m,d)$,

(ii) $\|h^a_{i,j,m,n}\|_\infty \leq C_2(m,d)$,

(iii) for $u \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d)$ and $\sigma = 3^d - 2^{d+1} + 2$ we have that

$$\begin{aligned}\mathrm{TV}(u^a h^a_{i,j,p,q}) &\leq \|u\|_\infty\|h_{i,j,p,q}\|_\infty + \sigma^2\mathrm{TV}_{[-a,a]^d}(u)\mathrm{TV}_{[-a,a]^d}(h_{i,j,p,q})\\ &\quad + \sigma\left(\mathrm{TV}_{[-a,a]^d}(u)\|h_{i,j,p,q}\|_\infty + \mathrm{TV}_{[-a,a]^d}(h_{i,j,p,q})\|u\|_\infty\right),\end{aligned}$$

(iv) $\mathrm{TV}_{[-a,a]^d}(|g|^2) \leq \|g\|_\infty^2 + \sigma^2\mathrm{TV}^2_{[-a,a]^d}(g) + 2\sigma\|g\|_\infty\mathrm{TV}_{[-a,a]^d}(g)$

where

$$C_2(m,d) := 2^d\left(2((\tilde{l}+1)^4 + \tilde{l}^4)^2(2(\tilde{k}+1)+2)\right)^d,$$

and $\tilde{k}, \tilde{l}$ are defined in (9.9).

Proof. To prove both (i) and (ii) we will use the easy facts that $\mathrm{TV}(h^a_{i,j,p,q}) = \mathrm{TV}_{[-a,a]^d}(h_{i,j,p,q})$ and $\mathrm{TV}(g^a h^a_{i,j,p,q}) = \mathrm{TV}_{[-a,a]^d}(gh_{i,j,p,q})$. To prove (i) of the claim let us first recall (see for example [80], p. 19) that when $\psi \in C^1([-a,a]^d)$ then

$$\mathrm{TV}_{[-a,a]^d}(\psi) = \sum_{k=1}^{d}\sum_{1\leq i_1<\ldots<i_k\leq d} V^{(k)}(\psi; i_1,\ldots,i_k), \tag{9.43}$$

where $V^{(k)}(\psi; i_1,\ldots,i_k) = V^{(k)}(\psi_{i_1,\ldots,i_k})$ and

$$\psi_{i_1,\ldots,i_k} : (y_1,\ldots,y_k) \mapsto \psi(\tilde{y}_1,\ldots,\tilde{y}_d), \qquad \tilde{y}_j = a,\ j \neq i_1,\ldots,i_k, \quad \tilde{y}_{i_j} = y_j,$$

$$V^{(k)}(\varphi) = \int_{-a}^{a}\cdots\int_{-a}^{a}\left|\frac{\partial^k\varphi}{\partial x_1\cdots\partial x_k}\right| dx_1\ldots dx_k, \qquad \varphi \in C^1([-a,a]^k).$$

Note that from (9.25) and (9.5) it follows that $h^a_{i,j,p,q} \in C^\infty([0,1]^d)$, so by the definition of h in (9.25) we have that, for $k \in \{1,\ldots,d\}$ and $1 \leq i_1 < \ldots < i_k \leq d$,

$$\begin{aligned}&V^{(k)}(h^a_{i,j,p,q}; i_1,\ldots,i_k)\\ &\leq \prod_{\mu=1}^{d}\max\left[\max_{s,t=0,2}\int_{-a}^{a}\left|\frac{\partial}{\partial x_\mu}\left(\frac{\partial^s\hat{\psi}_{\theta(j)_\mu}}{\partial x_\mu^s}(x_\mu)\overline{\frac{\partial^t\hat{\psi}_{\theta(i)_\mu}}{\partial x_\mu^t}(x_\mu)}\right)\right| dx_\mu,\right.\\ &\qquad\left.\max_{\substack{s,t=0,2\\ x_\mu\in[-a,a]}}\left|\frac{\partial^s\hat{\psi}_{\theta(j)_\mu}}{\partial x_\mu^s}(x_\mu)\overline{\frac{\partial^t\hat{\psi}_{\theta(i)_\mu}}{\partial x_\mu^t}(x_\mu)}\right|\right], \quad \forall k,p,q \leq d.\end{aligned} \tag{9.44}$$

We will now focus on bounding the right hand side of (9.44). Note that by using the definition of $\psi_{k,l}$ with $k, l \in \mathbb{Z}$ in (9.5) and some straightforward integration it follows that for $1 \leq p \leq d$ and $(k_p, l_p) = \theta(j)_p$ we have

$$\left|\frac{\partial\hat{\psi}_{k_p,l_p}}{\partial x_p}(x_p)\right| \leq \begin{cases} l_p + \frac{1}{2} & \text{when } k_p - 1 \leq x_p \leq k_p + 1,\\ \frac{l+\frac{1}{2}}{|x_p - k_p| + 1} & \text{otherwise,}\end{cases} \tag{9.45}$$

$$\left|\frac{\partial^3 \hat{\psi}_{k_p,l_p}}{\partial x_p^3}(x_p)\right| \leq \begin{cases} \frac{(l_p+1)^4-l_p^4}{4} & \text{when } k_p-1 \leq x_p \leq k_p+1, \\ \frac{(l_p+1)^4-l_p^4}{4(|x_p-k_p|+1)} & \text{otherwise.} \end{cases} \tag{9.46}$$

Thus, by using (9.37), (9.38), (9.45) and (9.46) it follows that

$$\begin{aligned} \max_{s,t=0,2} \int_{-a}^{a} \left| \frac{\partial}{\partial x_\mu} \left(\frac{\partial^s \hat{\psi}_{\theta(j)_\mu}}{\partial x_\mu^s}(x_\mu) \overline{\frac{\partial^t \hat{\psi}_{\theta(i)_\mu}}{\partial x_\mu^t}(x_\mu)} \right) \right| dx_\mu \\ &\leq 2 \max_{s,t=0,1,2,3} \int_{-\infty}^{\infty} \left| \frac{\partial^s \hat{\psi}_{\theta(j)_\mu}}{\partial x_\mu^s}(x_\mu) \overline{\frac{\partial^t \hat{\psi}_{\theta(i)_\mu}}{\partial x_\mu^t}(x_\mu)} \right| dx_\mu \\ &\leq 2((\tilde{l}+1)^4 + \tilde{l}^4)^2 \left(2(\tilde{k}+1) + \int_{[-\infty,-1]\cup[1,\infty]} \frac{1}{y^2}\, dy \right) \\ &= 2((\tilde{l}+1)^4 + \tilde{l}^4)^2 \left(2(\tilde{k}+1) + 2 \right), \end{aligned} \tag{9.47}$$

where $\tilde{k} := \max\{|k_p| : (k_p, l_p) = \theta(j)_p, p \in \{1,\dots,d\}, j \in \{1,\dots,n\}\}$, $\tilde{l} := \max\{|l_p| : (k_p, l_p) = \theta(j)_p, p \in \{1,\dots,d\}, j \in \{1,\dots,n\}\}$. Moreover, by (9.37) and (9.38)

$$\max_{\substack{s,t=0,2 \\ x_\mu \in [-a,a]}} \left| \frac{\partial^s \hat{\psi}_{\theta(j)_\mu}}{\partial x_\mu^s}(x_\mu) \overline{\frac{\partial^t \hat{\psi}_{\theta(i)_\mu}}{\partial x_\mu^t}(x_\mu)} \right| \leq \max\{\tilde{l}^2 + \tilde{l} + 1/3, 1\}, \qquad i,j \leq m, \quad 1 \leq \mu \leq d. \tag{9.48}$$

Hence, from (9.44), (9.47) and (9.48) it follows that for $k \in \{1,\dots,d\}$ and $1 \leq i_1 < \dots < i_k \leq d$,

$$V^{(k)}(h^a_{i,j,p,q}; i_1, \dots, i_k) \leq \Big(2((\tilde{l}+1)^4 + \tilde{l}^4)^2 (2(\tilde{k}+1)+2) \Big)^d$$

and thus, by (9.43) we get that

$$\begin{aligned} \mathrm{TV}_{[-a,a]^d}(h_{i,j,p,q}) &\leq \Big(2((\tilde{l}+1)^4 + \tilde{l}^4)^2 (2(\tilde{k}+1)+2) \Big)^d \sum_{k=1}^{d} \binom{d}{k} \\ &\leq 2^d \Big(2((\tilde{l}+1)^4 + \tilde{l}^4)^2 (2(\tilde{k}+1)+2) \Big)^d, \end{aligned}$$

and thus we have proved (i) in the claim.

To prove (ii) in the claim, we observe that by (9.5), (9.25) and (9.48) it follows that

$$\|h^a_{i,j,p,q}\|_\infty \leq \prod_{\mu=1}^{d} \max_{\substack{s,t=0,2 \\ x_\mu \in [-\infty,\infty]}} \left| \frac{\partial^s \hat{\psi}_{\theta(j)_\mu}}{\partial x_\mu^s}(x_\mu) \overline{\frac{\partial^t \hat{\psi}_{\theta(i)_\mu}}{\partial x_\mu^t}(x_\mu)} \right| \leq \Big(\max\{\tilde{l}^2 + \tilde{l} + 1/3, 1\} \Big)^d,$$

for $i,j \leq m$ and $p,q \leq d$. Obviously, the last part of the above inequality is bounded by $C(m,d)$, which yields the assertion.

To prove (iii) and (iv) we will use the fact (see [14]) that

$$\mathcal{A} = \{f \in \mathcal{M}([-a,a]^d) : \|f\|_\infty + \mathrm{TV}_{[-a,a]^d}(f) < \infty\},$$

where $\mathcal{M}([-a,a]^d)$ denotes the set of measurable functions on $[-a,a]^d$, is a Banach algebra when $\mathcal{A}$ is equipped with the norm $\|f\|_{\mathcal{A}} = \|f\|_\infty + \sigma \mathrm{TV}_{[-a,a]^d}(f)$, where $\sigma > 3^d - 2^{d+1} + 1$. We will let $\sigma = 3^d - 2^{d+1} + 2$. Hence, we get, by the Banach algebra property of the norm and (i) and (ii) that we already have proved, that

$$\begin{aligned} \mathrm{TV}_{[-a,a]^d}(u h_{i,j,p,q}) &\leq \|u\|_\infty \|h_{i,j,p,q}\|_\infty + \sigma^2 \mathrm{TV}_{[-a,a]^d}(u) \mathrm{TV}_{[-a,a]^d}(h_{i,j,p,q}) \\ &\quad + \sigma \left(\mathrm{TV}_{[-a,a]^d}(u) \|h_{i,j,p,q}\|_\infty + \mathrm{TV}_{[-a,a]^d}(h_{i,j,p,q}) \|u\|_\infty \right), \quad u \in \mathcal{A}, \end{aligned}$$

finally proving (iii). The proof of (iv) is almost identical. □

Lemma 9.9. *Let ζ_m be defined as in (9.17). Then, $\zeta_m \to \gamma$ locally uniformly, where γ is defined in (9.29). Furthermore, if $m \geq \|V\|_\infty$ then we have*

$$\zeta_m(z) \geq \gamma_m(z) - \frac{3+|z|}{m}.$$

Proof. Let γ_m be as defined in (9.13). Also, observe that $\gamma_m \to \gamma$ locally uniformly as $m \to \infty$. Indeed, let $\mathcal{T} = \{\|(-\Delta + V + zI)\psi\| : \psi \in W^{2,2}(\mathbb{R}^d), \|\psi\| = 1\}$. Then, since $\mathcal{S}$ is a core for H (recall $\mathcal{S}$ from Step I of the proof of $\mathrm{SCI}(\Xi_{\mathrm{sp}}, \Omega_{\phi,g})_{\mathrm{A}} = 1$) then every element in $\mathcal{T}$ can be approximated arbitrarily well by $\|(-\Delta + V + zI)\tilde{\varphi}\|$ for some $\tilde{\varphi} \in \mathcal{S}$, thus it follows from (9.29) that we have convergence. Note that the convergence must be monotonically from above by the definition of P_m, and thus Dini's Theorem assures the locally uniform convergence. Thus, it suffices to show that $|\zeta_m - \gamma_m| \to 0$ locally uniformly as $m \to \infty$.

Note that if we define, for $z \in \mathbb{C}$, the operator matrices

$$\begin{aligned} Z_m(z)_{ij} &= \langle S_m(V,z)\varphi_j, S_m(V,z)\varphi_i\rangle_{n,N}, \quad i,j \leq m, \\ \tilde{Z}_m(z)_{ij} &= \langle \tilde{S}_m(V,z)\varphi_j, \widetilde{S}_m(V,z)\varphi_i\rangle_{n,N}, \quad i,j \leq m, \quad N = \lceil n\phi(n)^4 \rceil, \end{aligned}$$

where $n = n(m)$ is defined in (9.18) and

$$\begin{aligned} W_m(z)_{ij} &= \langle S_m(V,z)\varphi_j, S_m(V,z)\varphi_i\rangle, \quad i,j \leq m, \\ \tilde{W}_m(z)_{ij} &= \langle \tilde{S}_m(V,z)\varphi_j, \widetilde{S}_m(V,z)\varphi_i\rangle, \quad i,j \leq m, \end{aligned}$$

the desired convergence follows if we can show that $\|Z_m(z) - W_m(z)\|$ and $\|\tilde{Z}_m(z) - \tilde{W}_m(z)\|$ tend to zero as m tends to infinity for all z in some compact set. However, this follows by the choice of $n(m) = \min\{n : \tilde{\tau}(m,n) \leq \frac{1}{m^3}\}$ in (9.18). In particular, $\tilde{\tau}(m,n) = \tau(m,m,n)$ and clearly $\tau(\|V\|_\infty, m, n) \leq \tau(m,m,n)$ for $\|V\|_\infty \leq m$ (recall τ from (9.42)). Thus it follows immediately by (9.41) and (9.23) that

$$\begin{aligned} \max\left\{ |Z_m(z)_{ij} - W_m(z)_{ij}|, |\tilde{Z}_m(z)_{ij} - \tilde{W}_m(z)_{ij}|\right\} &\leq (4(|z|+1) + |z|^2)\tau(\|V\|_\infty, m, n) \\ &\leq \frac{4(|z|+1)+|z|^2}{m^3}. \end{aligned}$$

Using the fact that the operator norm of a matrix is bounded by its Frobenius norm $\|\cdot\|_F$, it follows that for $z \in K \subset \mathbb{C}$, where K is compact, $\|Z_m(z) - W_m(z)\|_F = \mathcal{O}(\frac{1}{m^2})$ and $\|\tilde{Z}_m(z) - \tilde{W}_m(z)\|_F = \mathcal{O}(\frac{1}{m^2})$ for sufficiently large m. To see the explicit bound, note that the above shows (the $1/m$ comes from the discretisation in the search routine in the definition of ζ_m)

$$\gamma_m(z)^2 \leq \frac{4(|z|+1)+|z|^2}{m^2} + \left(\zeta_m(z) + \frac{1}{m}\right)^2 \leq \left\{\zeta_m(z) + \frac{1}{m} + \frac{\sqrt{4(|z|+1)+|z|^2}}{m}\right\}^2$$

Taking square roots and re-arranging gives the result. ☐

Error control and rest of proof of Theorem 6.2.

In order to gain Σ_1^A error control for $\{\Xi_{\mathrm{sp}}, \Omega_{\phi,g}\}$, consider $\hat{\Gamma}_m(A) = \Gamma_{m+\lceil \|V\|_\infty \rceil}(V)$ where we now use the fact that an upper bound on $\|V\|_\infty$ is included in the evaluation functions. From Lemma 9.9, if $z \in \hat{\Gamma}_m(A)$ then

$$\mathrm{dist}(z, \mathrm{sp}(-\Delta + V)) \leq g^{-1}\left(\zeta_{m+\lceil \|V\|_\infty \rceil}(z) + \frac{3+|z|}{m}\right).$$

This can be approximated from above to within an error that converges to zero as $m \to \infty$ using finitely many evaluations of the function g at rational points. Taking the maximum over all $z \in \hat{\Gamma}_m(A)$ gives us an error bound which converges to 0 uniformly on compact subsets of $\mathbb{C}$ as $m \to \infty$. The following shows this is enough for the Σ_1^A error control.

Lemma 9.10. *Let Ξ be a problem function mapping to the metric space $(\mathcal{C}(\mathbb{C}), d_{AW})$ and suppose that there is a function E_m provided by an arithmetic algorithm (and hence computable in finitely many arithmetic operations and comparisons), converging uniformly to zero on compact subsets, such that*

$$\mathrm{dist}(z, \Xi(A)) \leq E_m(z), \forall z \in \Gamma_m(A).$$

If $\Gamma_m(A)$ is finite for each m, A then, given $\Gamma_m(A)$, we can compute in finitely many arithmetic operations and comparisons a sequence of non-negative numbers $b_m \to 0$ such that

$$\Gamma_m(A) \subset A_m$$

for some $A_m \in \mathcal{C}(\mathbb{C})$ with $d_{AW}(A_m, \Xi(A)) \leq b_m$. Hence, by taking subsequences if necessary, we can build an arithmetic Σ_1^A tower.

Proof. Let $a_m^n = \sup\{E_m(z) : z \in \Gamma_m(A) \cap B_n(0)\}$. Define

$$A_m^n = \big((\Xi(A) + B_{a_m^n}(0)) \cap B_n(0)\big) \cup (\Gamma_m(A) \cap \{z : |z| \geq n\}).$$

It is clear that $\Gamma_m(A) \subset A_m^n$ and given $\{\Gamma_m(A), E_m(A)\}$ (we assume $\Gamma_m(A) \neq \emptyset$), we can easily compute a lower bound n_1 such that $\Xi(A) \cap B_{n_1}(0) \neq \emptyset$. Compute this from $\Gamma_1(A)$ and then fix it. Suppose that $n \geq 4n_1$, and suppose that $|z| < \lfloor n/4 \rfloor$. Then the points in A_m^n and $\Xi(A)$ nearest to z must lie in $B_n(0)$ and hence

$$\mathrm{dist}(z, A_m^n) \leq \mathrm{dist}(z, \Xi(A)), \quad \mathrm{dist}(z, \Xi(A)) \leq \mathrm{dist}(z, A_m^n) + a_m^n.$$

It follows that

$$d_{AW}(A_m^n, \Xi(A)) \leq a_m^n + 2^{-\lfloor n/4 \rfloor}.$$

We now choose a sequence $n(m)$ such that setting $A_m = A_m^{n(m)}$ and $b_m = a_n^{n(m)} + 2^{-\lfloor n(m)/4 \rfloor}$ proves the result. Clearly it is enough to ensure that b_m is null. If $m < 4n_1$ then set $n(m) = 4n_1$, otherwise consider $4n_1 \leq k \leq m$. If such a k exists with $a_m^k \leq 2^{-k}$ then let $n(m)$ be the maximal such k and finally if no such k exists then set $n(m) = 4n_1$. For a fixed n, $a_m^n \to 0$ as $m \to \infty$. It follows that for large m, we must have $a_m^{n(m)} \leq 2^{-n(m)}$ and that $n(m) \to \infty$. $\square$

Note that its is clear that none of the problems lie in Δ_1^G. Hence to finish the proof of Theorem 6.2, we must show that $\{\Xi_{\mathrm{sp},\epsilon}, \Omega_\phi\} \in \Sigma_1^A$ since by taking $\epsilon \downarrow 0$ this will show $\{\Xi_{\mathrm{sp}}, \Omega_\phi\} \in \Pi_2^A$ and we have $\Omega_{\phi,g} \subset \Omega_\phi$. Note that through the use of ζ_m and Lemma 9.9 we can compute, using finitely many arithmetic operations and comparisons for any z, a function $\hat{\gamma}_m(z)$ that converges uniformly to $\gamma(z)$ on any compact subset of $\mathbb{C}$ with $\hat{\gamma}_m(z) \geq \gamma(z)$. The next Lemma then says that this is enough.

Lemma 9.11. *Suppose that $\hat{\gamma}_m(z) \geq \gamma(z)$ converge uniformly to $\left\|(-\Delta + V - zI)^{-1}\right\|^{-1}$ on compact subsets of $\mathbb{C}$. Set*

$$\Gamma_m(V) = (B_m(0) \cap \frac{1}{m}(\mathbb{Z} + i\mathbb{Z})) \cap \{z : \hat{\gamma}_m(z) < \epsilon\}.$$

For large m, $\Gamma_m(V) \neq \emptyset$ so we can assume this without loss of generality. $d_{AW}(\Gamma_m(V), \mathrm{sp}_\epsilon(-\Delta+V)) \to 0$ as $m \to \infty$ and clearly $\Gamma_m(V) \subset \mathrm{sp}_\epsilon(-\Delta + V)$.

Proof. Since the pseudospectrum is non-empty, for large m, $\Gamma_m(V) \neq \emptyset$ so we may assume that this holds for all m without loss of generality. We use the characterisation of the Attouch-Wets topology where it is enough to consider closed balls. Suppose that n is large such that $B_n(0) \cap \mathrm{sp}_\epsilon(-\Delta + V) \neq \emptyset$. Since $\Gamma_m(V) \subset \mathrm{sp}_\epsilon(-\Delta + V)$, we must show that given $\delta > 0$, there exists N_1 such that if $m > N_1$ then $\mathrm{sp}_\epsilon(-\Delta+V) \cap B_n(0) \subset \Gamma_m(V) + B_\delta(0)$. Suppose for a contradiction that this were false. Then there exists $z_j \in \mathrm{sp}_\epsilon(-\Delta+V) \cap B_n(0)$, $\delta > 0$ and $m_j \to \infty$ such that $\mathrm{dist}(z_j, \Gamma_{m_j}(V)) \geq \delta$. Without loss of generality, we can assume that $z_j \to z \in \mathrm{sp}_\epsilon(-\Delta + V)$. There exists some w with $\left\|(-\Delta + V - wI)^{-1}\right\|^{-1} < \epsilon$ and $|z - w| \leq \delta/2$. Assuming $m_j > n+\delta$, there exists $y_{m_j} \in (B_{m_j}(0) \cap \frac{1}{m_j}(\mathbb{Z}+i\mathbb{Z}))$ with $\left|y_{m_j} - w\right| \leq 1/m_j$. It follows that

$$\hat{\gamma}_{m_j}(y_{m_j}) \leq \left|\hat{\gamma}_{m_j}(y_{m_j}) - \gamma(y_{m_j})\right| + \left|\gamma(w) - \gamma(y_{m_j})\right| + \left\|(-\Delta + V - wI)^{-1}\right\|^{-1}.$$

But γ is continuous and $\hat{\gamma}_{m_j}$ converges uniformly to γ on compact subsets. Hence for large m_j, $\hat{\gamma}_{m_j}(y_{m_j}) < \epsilon$ so that $y_{m_j} \in \Gamma_{m_j}(V)$. But $\left|y_{m_j} - z\right| \leq |z - w| + \left|y_{m_j} - w\right| \leq \delta/2 + 1/m_j$ which is smaller than δ for large m_j. This gives the required contradiction. $\square$

9.2. **The case of unbounded potential V.** In this section we prove Theorem 6.3 on the SCI of spectra and pseudospectra of Schrödinger operators with unbounded potentials. First of all we will build the Δ_2^A algorithms. Let us outline the steps of the proof first:

a. *Compactness of the resolvent:* The assumptions on the potential imply that the operator H has a compact resolvent $R(H,z)$ (see Proposition 9.22). Therefore the spectrum is countable consisting of eigenvalues with finite dimensional invariant subspaces.
b. *Finite-dimensional approximations:* The main part of the proof centers around showing that it is possible to construct, with finite amount of evaluations of V, square matrices $\widetilde{H}_n$ whose resolvents (when suitably embedded into the large space) converge to $R(H,z_0)$ in norm at a suitable point z_0 (see Theorem 9.24). Note that this technique is very different from the techniques used so far in the paper and is only possible due to compactness.
c. *Convergence of the spectrum and pseudospectrum:* We use the convergence at z_0 to show convergence at other points z in the resolvent set.

Once this is done, we proved proofs that neither problem lies in $\Sigma_1^G \cup \Pi_1^G$.

As the argument is otherwise independent of the particular set-up, we start with a general discussion. In the end we demonstrate the construction of the matrices $\widetilde{H}_n$ and the convergence of the resolvents. We assume the following:

(i) Assumptions on the operator A: Given a closed densely defined operator A in a Hilbert space $\mathcal{H}$ such that at $z_0 \in \mathbb{C}$ the resolvent operator $R(z_0) = (A - z_0)^{-1}$ is compact $R(z_0) \in \mathcal{K}(\mathcal{H})$. Thus $\mathrm{sp}(A) = \{\lambda_j\}$, the spectrum of A, is at most countable with no finite accumulation points.

(ii) Assumptions on the approximations A_n: Suppose A_n is a finite rank approximation to A such that if E_n is the orthogonal projection onto the range of A_n, then $A_n = A_n E_n$. We put further $\mathcal{H}_n = E_n\mathcal{H}$ and denote by $\widetilde{A}_n$ the matrix representing A_n when restricted to the invariant subspace $\mathcal{H}_n$ w.r.t. some orthonormal basis. Now, take the resolvent $(A_nE_n - zE_n)^{-1}$ of this restriction, extend it to $\mathcal{H}_n^\perp$ by zero, and denote this extension by $R_n(z)$. Then $R_n(z) = R_n(z)E_n$, and $R_n(z) = (A_n - z)^{-1} + (I - E_n)z^{-1}$ for all $z \neq 0$ for which the inverse exists. Finally we assume that $R_n(z_0)$ exist and

$$\|R_n(z_0) - R(z_0)\| \longrightarrow 0, \quad n \to \infty. \tag{9.49}$$

Convergence of the spectrum and pseudospectrum. The first step is to conclude that if the finite rank approximations to the resolvent converge in operator norm at one point z_0, then they also converge locally uniformly away from the spectrum of A. To that end denote by $U_r(\mu)$ the open disc at center μ and radius r.

Proposition 9.12. *Suppose $R(z)$ and $R_n(z)$ are as above and satisfy* (9.49). *Let $\mathcal{K} \subset \mathbb{C}$ be compact, $r > 0$ and define $\mathcal{K}_r = \mathcal{K} \backslash \bigcup_j U_r(\lambda_j)$. Then for large enough n, $R_n(z)$ exists for all $z \in \mathcal{K}_r$ and $\sup_{z\in\mathcal{K}_r} \|R_n(z) - R(z)\| \to 0$ as $n \to \infty$.*

Proof. Clearly $R(z) = R(z_0)(I - (z - z_0)R(z_0))^{-1}$ and $R_n(z) = R_n(z_0)(I - (z - z_0)R_n(z_0))^{-1}$ for all z in which $R(z)$, resp. $R_n(z)$, exist. By (9.49) it suffices to prove the existence of $R_n(z)$ and

$$\sup_{z\in\mathcal{K}_r} \|(I - (z - z_0)R_n(z_0))^{-1} - (I - (z - z_0)R(z_0))^{-1}\| \to 0.$$

However, we know that $(I - (z - z_0)R(z_0))^{-1}$ is meromorphic in the whole plane and hence analytic in the compact set $\mathcal{K}_r$ and in particular uniformly bounded. But this means that it is sufficient to show that the inverses converge, which in turn is immediate from (9.49) since

$$\sup_{z\in\mathcal{K}_r} \|(I-(z-z_0)R_n(z_0))-(I-(z-z_0)R(z_0))\| \leq \|R_n(z_0)-R(z_0)\| + \sup_{z\in\mathcal{K}_r} |z-z_0|\ \|R_n(z_0)-R(z_0)\|.$$

To see that this suffices, write $T_n(z) = (I - (z - z_0)R_n(z_0))$, $T(z) = (I - (z - z_0)R(z_0))$ and

$$T_n(z) = T(z)[I + T(z)^{-1}(T_n(z) - T(z))].$$

Then for large enough n and $z \in \mathcal{K}_r$ by a Neumann series argument

$$\|T_n(z)^{-1} - T(z)^{-1}\| \leq \|T(z)^{-1}\| \, [(1 - \|T(z)^{-1}\|\|T_n(z) - T(z)\|)^{-1} - 1].$$

□

Proposition 9.13. *Let $\mathcal{K} \subset \mathbb{C}$ be compact and $\delta > 0$. Then, for all large enough n,*

$$\mathrm{sp}(A) \cap \mathcal{K} \subset \mathcal{N}_\delta(\mathrm{sp}(A_n)), \qquad \mathrm{sp}(A_n) \cap \mathcal{K} \subset \mathcal{N}_\delta(\mathrm{sp}(A)).$$

Proof. Since the eigenvalues are exactly the poles of the resolvents, the claim follows immediately from the previous proposition. □

The last proposition gives the convergence of the spectra. The discussion on pseudospectra is somewhat more involved. We need to know that the norm of the resolvent is not constant in any open set. The following is a theorem due to J.Globevnik, E.B.Davies and E.Shargorodsky which we formulate here as a lemma:

Lemma 9.14 ([50] and [29]). *Suppose A is a closed densely defined operator in $\mathcal{H}$ such that the resolvent $R(z) = (A - z)^{-1}$ is compact. Let $\Omega \subset \mathbb{C}$ be open and connected, and assume that, for all $z \in \Omega$, $\|R(z)\| \leq M$. Then, for all $z \in \Omega$, $\|R(z)\| < M$. This is particularly true if $\mathcal{H}$ is finite dimensional.*

The theorem in [29] is formulated for Banach spaces X with the extra assumption that X or its dual is complex strictly convex, a condition which holds for Hilbert spaces. The case $\mathcal{H}$ being of finite dimension is already settled by [50]. We put $\gamma(z) = 1/\|R(z)\|$ and $\gamma_n(z) = 1/\|R_n(z)\|$ and summarize the properties of γ and γ_n as follows:

Lemma 9.15. *If (i) and (ii) hold, then $\gamma_n(z) \to \gamma(z)$ uniformly on compact sets. Neither γ, nor γ_n is constant in any open set and they have local minima only where they vanish. Additionally,*

$$\gamma(z) \leq \mathrm{dist}(z, \mathrm{sp}(A)). \tag{9.50}$$

Consequently,

$$\mathrm{sp}_\epsilon(A) = \{z \,:\, \gamma(z) \leq \epsilon\} = \mathrm{cl}\{z \,:\, \gamma(z) < \epsilon\}, \quad \mathrm{sp}_\epsilon(A_n) = \{z \,:\, \gamma_n(z) \leq \epsilon\} = \mathrm{cl}\{z \,:\, \gamma_n(z) < \epsilon\}.$$

Proof. Observe first that (9.50) is just a reformulation of a general property of resolvents. Next, notice that $\|R_n(z)\| = \|R(A_n, z)\|$ and that the norms of resolvents are subharmonic away from spectrum and therefore γ and γ_n cannot have proper local minima, except when they vanish. Furthermore, they cannot be constant in an open set by Lemma 9.14.

To conclude the local uniform convergence, let M be such that along the curve $\{|z| = M\}$ there are no eigenvalues of A and choose $\mathcal{K}$ as the set $\{|z| \leq M\}$. Choose any ϵ, small enough so that the discs $\{|z - \lambda_j|\} \leq \epsilon/3$ separate the eigenvalues inside $\mathcal{K}$. By Proposition 9.12 we may assume that n is large enough so that for $z \in \mathcal{K}_{\epsilon/3}$ (recall $\mathcal{K}_r$ from Proposition 9.12) we have $|\gamma_n(z) - \gamma(z)| \leq \epsilon/3$. On the other hand, if $|z - \lambda_j| \leq \epsilon/3$ then $\gamma(z) \leq \epsilon/3$ and, since γ_n has to vanish also somewhere in that disc (again for large enough n), we have $\gamma_n(z) \leq 2\epsilon/3$ in that disc, hence $|\gamma_n(z) - \gamma(z)| \leq \gamma_n(z) + \gamma(z) \leq \epsilon$. Thus we have $|\gamma_n(z) - \gamma(z)| \leq \epsilon$ for all $z \in \mathcal{K}$.

Finally, to justify the equivalence of the characterizations of pseudospectra just notice that the level sets $\{z \,:\, \gamma(z) = \epsilon\}$ and $\{z \,:\, \gamma_n(z) = \epsilon\}$ cannot contain open subsets or isolated points. □

Lemma 9.16. *Assume φ_n and φ are continuous nonnegative functions in $\mathbb{C}$ which have local minima only when they vanish, are not constant in any open set and φ_n converges to φ uniformly in compact sets. Set*

$\mathcal{S} := \{z : \varphi(z) \leq 1\}$ and $\mathcal{S}_n := \{z : \varphi_n(z) \leq 1\}$. Let $\mathcal{K}$ be compact and $\delta > 0$. Then the following hold for all large enough n

$$\mathcal{S} \cap \mathcal{K} \subset \mathcal{N}_\delta(\mathcal{S}_n), \qquad \mathcal{S}_n \cap \mathcal{K} \subset \mathcal{N}_\delta(\mathcal{S}). \tag{9.51}$$

Proof. Consider the first part of (9.51) and assume that the left hand side is not empty. Due to compactness of $\mathcal{S} \cap \mathcal{K}$ there are points $z_i \in \mathcal{S} \cap \mathcal{K}$ for $i = 1, \ldots, m$ such that $\mathcal{S} \cap \mathcal{K} \subset \bigcup_{i=1}^m U_{\delta/2}(z_i)$. Notice that $\varphi(z_i) \leq 1$. If $\varphi(z_i) < 1$, set $y_i = z_i$. Otherwise, $\varphi(z_i) = 1$, in which case z_i cannot be a local minimum, but since φ is not constant in any open set, there exists a point $y_i \in U_{\delta/2}(z_i)$ such that $\varphi(y_i) < 1$. But since φ_n converges uniformly in compact sets to φ we conclude that for all large enough n and all i we have $\varphi_n(y_i) < 1$. Hence $z_i \in \mathcal{N}_{\delta/2}(\mathcal{S}_n)$ and so $\mathcal{S} \cap \mathcal{K} \subset \bigcup_{i=1}^m U_{\delta/2}(z_i) \subset \mathcal{N}_\delta(\mathcal{S}_n)$.

Consider now the second part of (9.51). If it would not hold, there would exist a sequence $\{n_j\}$ and points $z_{n_j} \in \mathcal{S}_{n_j} \cap \mathcal{K}$ such that $z_{n_j} \notin \mathcal{N}_\delta(\mathcal{S})$. Suppose $z_{n_{j_k}} \to \hat{z}$. Then $\mathrm{dist}(\hat{z}, \mathcal{S}) \geq \delta$ as well. However, writing $\varphi(\hat{z}) \leq |\varphi(\hat{z}) - \varphi(z_{n_{j_k}})| + |\varphi(z_{n_{j_k}}) - \varphi_{n_{j_k}}(z_{n_{j_k}})| + \varphi_{n_{j_k}}(z_{n_{j_k}})$ we obtain $\varphi(\hat{z}) \leq 1$ as the first term on the right tends to zero because φ is continuous, the second term converge to zero as φ_n approximate φ uniformly in compact sets, and $\varphi_{n_{j_k}}(z_{n_{j_k}}) \leq 1$. Hence $\hat{z} \in \mathcal{S} \cap \mathcal{K}$ which is a contradiction. ☐

Combining these we can state the following result.

Proposition 9.17. *Let $\mathcal{K} \subset \mathbb{C}$ be compact and $\delta > 0$. Then, for all large enough n,*

$$\mathrm{sp}_\epsilon(A) \cap \mathcal{K} \subset \mathcal{N}_\delta(\mathrm{sp}_\epsilon(A_n)), \qquad \mathrm{sp}_\epsilon(A_n) \cap \mathcal{K} \subset \mathcal{N}_\delta(\mathrm{sp}_\epsilon(A)).$$

The general algorithms. Here A, A_n are operators in $\mathcal{H}$ as in (i), (ii) above, while $\widetilde{A}_n$ is the matrix representing A_n when restricted to the finite dimensional invariant subspace $\mathcal{H}_n = E_n\mathcal{H}$. In particular $\|R_n(z)\| = \|(\widetilde{A}_n - z)^{-1}\|$. Denoting by σ_1 the smallest singular value of a square matrix we have $\gamma_n(z) = 1/\|R_n(z)\| = \sigma_1(\widetilde{A}_n - zI)$. Let $r > 0$ and define $G_r := B_r(0) \cap (\frac{1}{2r}(\mathbb{Z} + \mathbf{i}\mathbb{Z}))$. Define Γ_n^1 and Γ_n^2 by

$$\Gamma_n^1(A) = \left\{z \in G_n : \sigma_1(\widetilde{A}_n - zI) \leq \frac{1}{n}\right\}, \qquad \Gamma_n^2(A) = \left\{z \in G_n : \sigma_1(\widetilde{A}_n - zI) \leq \epsilon\right\}, \tag{9.52}$$

which we shall prove to be the towers of algorithms for Ξ_{sp} and $\Xi_{\mathrm{sp},\epsilon}$ (as defined in Theorem 6.3), respectively. Observe that $\Gamma_n^1(A)$ and $\Gamma_n^2(A)$ can be executed with finite amount of arithmetic operations, if the matrices $\widetilde{A}_n$ are available. Also note that our proof will show that $\Gamma_n^i(A) \neq \emptyset$ for large n. Hence by our usual trick of searching for minimal $n(m) \geq m$ such that this is so, we can assume without loss of generality this holds for all n.

Proposition 9.18. *The algorithms satisfy the following:*

$$\Gamma_n^1(A) \longrightarrow \mathrm{sp}(A), \qquad \Gamma_n^2(A) \longrightarrow \mathrm{sp}_\epsilon(A), \quad n \to \infty. \tag{9.53}$$

Proof. We begin with the second part of (9.53). It suffices to show that given δ and a compact ball $\mathcal{K}$, for large n:

$$\text{(i) } \mathrm{sp}_\epsilon(\widetilde{A}_n) \cap G_n \cap \mathcal{K} \subset \mathcal{N}_\delta(\mathrm{sp}_\epsilon(A)), \qquad \text{(ii) } \mathrm{sp}_\epsilon(A) \cap \mathcal{K} \subset \mathcal{N}_\delta(\mathrm{sp}_\epsilon(\widetilde{A}_n) \cap G_n).$$

The first inclusion follows immediately from Proposition 9.17 . To see (ii) we argue by contradiction and suppose not. Then by possibly passing to an increasing subsequence $\{k_n\}_{n\in\mathbb{N}} \subset \mathbb{N}$ there is a sequence $z_n \in (\mathrm{sp}_\epsilon(A) \cap \mathcal{K}) \setminus \mathcal{N}_\delta(\mathrm{sp}_\epsilon(\widetilde{A}_{k_n}) \cap G_{k_n})$ for all n. Since $\mathrm{sp}_\epsilon(A) \cap \mathcal{K}$ is a compact set, by possibly extracting a subsequence, we have that $z_n \to z_0 \in \mathrm{sp}_\epsilon(A) \cap \mathcal{K}$. Consider the open ball $U_{\delta/3}(z_0)$ which must contain all z_n for n sufficiently large. Since $\gamma(z)$ is continuous, positive, not constant in any open set and without nontrivial local minima, it follows that $\mathrm{sp}_\epsilon(A)$ equals the closure of its interior points. In particular $\mathrm{int}(\mathrm{sp}_\epsilon(A)) \cap U_{\delta/3}(z_0) \neq \emptyset$. Suppose then $r > 0$ and y_0 are such that the closure of the open ball $U_r(y_0)$ is inside this open set: $B_r(y_0) \subset \mathrm{int}(\mathrm{sp}_\epsilon(A)) \cap U_{\delta/3}(z_0)$. We claim that $\mathrm{sp}_\epsilon(\widetilde{A}_n) \cap U_r(y_0) = U_r(y_0)$ for all large enough n. Indeed, since $U_r(y_0)$ bounded away from the boundary of the pseudospectrum of A, we have $\gamma(z) \leq \epsilon - s$ for some $s > 0$ and for all $z \in U_r(y_0)$ Now the claim follows from the locally uniform convergence of γ_n.

By the definition of G_n we have that $U_r(y_0) \subset \mathcal{N}_{\delta/3}(U_r(y_0) \cap G_n)$ for large n, so, by the claim, $U_r(y_0) \subset \mathcal{N}_{\delta/3}(\mathrm{sp}_\epsilon(\widetilde{A}_n) \cap G_n)$. Hence, since $U_r(y_0) \subset U_{\delta/3}(z_0)$, it follows that

$$z_n \in U_{\delta/3}(z_0) \subset \mathcal{N}_{2\delta/3}(U_r(y_0)) \subset \mathcal{N}_\delta(\mathrm{sp}_\epsilon(\widetilde{A}_n) \cap G_n),$$

for large n, contradicting $z_n \notin \mathcal{N}_\delta(\mathrm{sp}_\epsilon(\widetilde{A}_n) \cap G_n)$.

To prove the first part of (9.53) we argue as follows. Given $\delta > 0$ and compact $\mathcal{K}$, we need to show that for large n:

$$\text{(iii) } \mathrm{sp}(A) \cap \mathcal{K} \subset \mathcal{N}_\delta(\mathrm{sp}_{1/n}(\widetilde{A}_n) \cap G_n) \qquad \text{(iv) } \mathrm{sp}_{1/n}(\widetilde{A}_n) \cap G_n \cap \mathcal{K} \subset \mathcal{N}_\delta(\mathrm{sp}(A)).$$

To show (iii), we start by defining $\widetilde{G}_n := \frac{1}{2n}(\mathbb{Z} + \mathbf{i}\mathbb{Z})$ and note that for $\lambda_j \in \mathrm{sp}(\widetilde{A}_n)$ we have that $\mathcal{N}_{1/n}(\{\lambda_j\}) \cap \widetilde{G}_n \neq \emptyset$ for every n. Hence, $\mathrm{sp}(\widetilde{A}_n) \subset \mathcal{N}_{1/n}\left(\mathcal{N}_{1/n}\left(\mathrm{sp}(\widetilde{A}_n)\right) \cap \widetilde{G}_n\right)$. Thus, since $\mathcal{N}_{1/n}(\mathrm{sp}(\widetilde{A}_n)) \subset \mathrm{sp}_{1/n}(\widetilde{A}_n)$, compare (9.50), it follows that $\mathrm{sp}(\widetilde{A}_n) \subset \mathcal{N}_{1/n}\left(\mathrm{sp}_{1/n}(\widetilde{A}_n) \cap \widetilde{G}_n\right)$. Now by the first part of Proposition 9.13 we have that $\mathrm{sp}(A) \cap \mathcal{K} \subset \mathcal{N}_{\delta/2}(\mathrm{sp}(\widetilde{A}_n))$ for large n. Thus, combining the previous observations, we have that

$$\mathrm{sp}(A) \cap \mathcal{K} \subset \mathcal{N}_{\delta/2+1/n}\left(\mathrm{sp}_{1/n}(\widetilde{A}_n) \cap \widetilde{G}_n\right) \subset \mathcal{N}_\delta\left(\mathrm{sp}_{1/n}(\widetilde{A}_n) \cap \widetilde{G}_n\right),$$

for large n. However, since $\mathcal{K}$ is bounded we have that there exists an $r > 0$ such that if $\lambda \in \widetilde{G}_n \cap U_r(0)^c$ then $\mathcal{N}_\delta(\{\lambda\}) \cap \mathrm{sp}(A) \cap \mathcal{K} = \emptyset$ for all n. Hence, $\mathrm{sp}(A) \cap \mathcal{K} \subset \mathcal{N}_\delta\left(\mathrm{sp}_{1/n}(\widetilde{A}_n) \cap G_n\right)$ as desired.

To see (iv), let $r > 0$ be so large that $\mathcal{N}_\delta(U_r(0)^c) \cap \mathcal{K} = \emptyset$. Note that $\mathrm{sp}_\epsilon(A) \to \mathrm{sp}(A)$ as $\epsilon \to 0$. Thus, $\mathrm{sp}_{\epsilon_1}(A) \cap B_r(0) \subset \mathcal{N}_{\delta/2}(\mathrm{sp}(A))$ for a sufficiently small ϵ_1. Also, by the second part of Proposition 9.17 it follows that $\mathrm{sp}_{\epsilon_1}(\widetilde{A}_n) \cap \mathcal{K} \subset \mathcal{N}_{\delta/2}(\mathrm{sp}_{\epsilon_1}(A))$ for large n. However, by the choice of r we have that $\mathrm{sp}_{\epsilon_1}(\widetilde{A}_n) \cap \mathcal{K} \subset \mathcal{N}_{\delta/2}(\mathrm{sp}_{\epsilon_1}(A) \cap B_r(0))$. Clearly, $\mathrm{sp}_{1/n}(\widetilde{A}_n) \cap \mathcal{K} \subset \mathrm{sp}_{\epsilon_1}(\widetilde{A}_n) \cap \mathcal{K}$ for large n. Thus, by patching the above inclusions together we get that

$$\mathrm{sp}_{1/n}(\widetilde{A}_n) \cap \mathcal{K} \subset \mathrm{sp}_{\epsilon_1}(\widetilde{A}_n) \cap \mathcal{K} \subset \mathcal{N}_{\delta/2}(\mathrm{sp}_{\epsilon_1}(A) \cap B_r(0)) \subset \mathcal{N}_\delta(\mathrm{sp}(A)),$$

for large n, as desired. This finishes the proof of Proposition 9.18. □

Next, we pass from these general considerations to the Schrödinger case.

Compactness of the resolvent. We first show that the resolvent of the Schrödinger operator $H \in \Omega_\infty$ is compact. To prove this we recall some well known lemmas and definitions from [69].

Definition 9.19. An operator A on the Hilbert space $\mathcal{H}$ is *accretive* if the $\mathrm{Re}\langle Ax, x\rangle \geq 0$ for $x \in \mathcal{D}(A)$. It is called m-accretive if there exists no proper accretive extension. If A (possibly after shifting with a scalar) is m-accretive and additionally there exists $\beta < \pi/2$ such that $|\arg\langle Ax, x\rangle| \leq \beta$ for all $x \in \mathcal{D}(A)$, then A is m-sectorial.

Lemma 9.20 ([69, VI-Theorem 3.3]). *Let A be m-sectorial with $B = \mathrm{Re}\, A$. A has compact resolvent if and only if B has.*

Lemma 9.21 ([69, V-Theorem 3.2]). *If T is closed and the complement of $\mathrm{Num}(T)$ is connected, then for every ζ in the complement of the closure of $\mathrm{Num}(T)$ the following hold: the kernel of $T - \zeta$ is trivial and the range of $T - \zeta$ is closed with constant codimension.*

Proposition 9.22. *Suppose V is continuous $\mathbb{R}^d \to \mathbb{C}$ satisfying the following: $V(x) = |V(x)|e^{i\varphi(x)}$ such that $|V(x)| \to \infty$ as $x \to \infty$, and there exist nonnegative θ_1, θ_2 such that $\theta_1 + \theta_2 < \pi$ and $-\theta_2 \leq \varphi(x) \leq$*

θ_1. Denote by h the operator $h = -\Delta + V$ with domain $\mathcal{D}(h) = C_c^\infty(\mathbb{R}^d)$ and put in $L^2(\mathbb{R}^d)$ $H = h^{**}$. Then $H = -\Delta + V$ is a densely defined operator with compact resolvent.

Proof. The proof goes as follows: Notice first that the numerical range of H lies in a sector with opening $2\beta < \pi$. Then we turn the sector into the symmetric position around the positive real axis to get the operator $a(\alpha)$. It is clearly enough to show that $A(\alpha) = a(\alpha)^{**}$ is an m-sectorial operator with half-angle $\beta = (\theta_1 + \theta_2)/2$ which has a compact resolvent. Next, since the numerical range of $a(\alpha)$ is not the whole plane, the operator is closable. Then we conclude that every point away from the numerical range belongs to the resolvent set. This is done based on the fact that the adjoint shares the same key properties as $A(\alpha)$. Then the compactness of the resolvent follows by considering the resolvent of the real part of $A(\alpha)$.

Here is the notation. Put $\alpha = (\theta_1 - \theta_2)/2$ so that $|\alpha| < \pi/2$. Then with

$$\vartheta(x) = \varphi(x) - \alpha \tag{9.54}$$

we have $a(\alpha) := e^{-i\alpha}h = -e^{-i\alpha}\Delta + |V(x)|e^{i\vartheta(x)}$ and after extending $A(\alpha) = a(\alpha)^{**}$, in particular $H(\alpha) := \mathrm{Re}A(\alpha) = -\cos\alpha\,\Delta + \cos\vartheta(x)|V(x)|$.

We claim that the operator $A(\alpha) := e^{-i\alpha}H$ is m-sectorial with half-angle $\beta = (\theta_1 + \theta_2)/2$. Indeed, it is immediate that the numerical range satisfies the following $\mathrm{Num}(a(\alpha)) \subset \{z = re^{i\theta} : \ |\theta| \le \beta,\ r \ge 0\}$, which is not the whole complex plane, and we can therefore (by [69, V-Theorem 3.4 on p. 268]) consider the extended closed operator $A(\alpha)$ instead. The next thing is to conclude that points away from this closed sector are in the resolvent set of $A(\alpha)$. Take any point $\zeta = re^{i\varphi}$ with $\beta < |\varphi| \le \pi$, $r > 0$. We need to conclude that $\zeta \notin \mathrm{sp}(A(\alpha))$. Since the complement of $\mathrm{Num}(A(\alpha))$ is connected, the following holds (by Lemma 9.21): the operator $A(\alpha) - \zeta$ has closed range with constant codimension. Thus, we need that the range is the whole space. Put for that purpose $T = A(\alpha) - \zeta$. Suppose there is $g \neq 0$ such that $g \in \mathrm{Ran}(T)^\perp$. Then for all $f \in \mathcal{D}(T)$ we have $\langle Tf, g\rangle = 0$ which means, as $\mathcal{D}(T)$ is dense, that $T^*g = 0$. But that is not the case as $A(\alpha)^* - \overline{\zeta}$ is also closed whose complement of the numerical range is connected and hence does not have a nontrivial kernel.

The proof of Proposition 9.22 can now be completed by invoking Lemma 9.20 since it is well known ([84], Theorem XIII.67) that (since $\alpha < \pi/2$) the self-adjoint operator $H(\alpha)$ has compact resolvent when the potential $|V(x)|$ tends to infinity with x. □

We shall next consider the discretisation of H and of $A(\alpha)$. It shall be clear that the discrete versions have their numerical ranges inside the same sectors, where the numerical range of an operator T is denoted by $\mathrm{Num}(T)$. Thus all resolvents can be estimated using the fact that if $(T-\zeta)^{-1}$ is regular outside the closure of $\mathrm{Num}(T)$, then $\|(T-\zeta)^{-1}\| \le 1/\mathrm{dist}(\zeta, \mathrm{Num}(T))$.

Discretizing the Schrödinger operator. We shall show how to assemble the matrices $\widetilde{H}_n$ mentioned above. The underlying Hilbert space is again $L^2(\mathbb{R}^d)$ and we start with approximating the Laplacian. Let $1 \le j \le d$, $t \in \mathbb{R}$ and define $U_{j,t}$ to be the one-parameter unitary group of translations

$$U_{j,t}\psi(x_1, \dots, x_d) = \psi(x_1, \dots, x_j - t, \dots, x_d)$$

and let P_j be the infinitesimal generator of $U_{j,t}$ so that $U_{j,t} = e^{\mathrm{i}tP_j}$ and $P_j = \lim_{t\to 0} \frac{1}{\mathrm{i}t}(U_{j,t} - I)$. Thus, defining $\Phi_n(x) = \frac{n}{\mathrm{i}}(e^{\mathrm{i}\frac{1}{n}x} - 1)$ with $n \in \mathbb{N}$ and $x \in \mathbb{R}$, it follows that

$$|\Phi_n|^2(P_j)\psi(x) = n^2(-\psi(x_1, \dots, x_j + 1/n, \dots x_d) - \psi(x_1, \dots, x_j - 1/n, \dots x_d) + 2\psi(x)) \tag{9.55}$$

is the discretized Laplacian in the j direction. The full discretized Laplacian is therefore $\sum_{j=1}^d |\Phi_n|^2(P_j)$. Now we replace V by an appropriate approximation. Consider the lattice $(\frac{1}{n}\mathbb{Z})^d$ as a subset of $\mathbb{R}^d$ and for $y \in (\frac{1}{n}\mathbb{Z})^d$ define the box

$$Q_n(y) = \left\{ x = (x_1, \dots, x_d) : x_j \in \left[y_j - \frac{1}{2n}, y_j + \frac{1}{2n} \right), \, 1 \le j \le d \right\}. \tag{9.56}$$

Let $S_n = [-\lfloor\sqrt{n}\rfloor, \lfloor\sqrt{n}\rfloor]^d \subset \mathbb{R}^d$ and define E_n to be the orthogonal projection onto the subspace

$$\left\{ \psi \in L^2(\mathbb{R}^d) : \psi = \sum_{y \in (\frac{1}{n}\mathbb{Z})^d \cap S_n} \alpha_y \chi_{Q_n(y)},\, \alpha_y \in \mathbb{C} \right\}, \tag{9.57}$$

where $\chi_{Q_n(y)}$ denotes the characteristic function on $Q_n(y)$. Define the approximate potential as

$$V_n(x) = \begin{cases} V(y) & x \in Q_n(y) \cap S_n \text{ for some } y \in (\frac{1}{n}\mathbb{Z})^d, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $V_n = E_n V_n E_n$, but that, in general, $V_n \neq E_n V E_n$. Finally, we define the approximate Schrödinger operator $H_n : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ defined as

$$H_n = E_n \sum_{j=1}^{d} |\Phi_n|^2(P_j) E_n + V_n. \tag{9.58}$$

Remark 9.23. Note that the restriction $H_n|_{\mathrm{Ran}(E_n)}$ of H_n to the image of E_n has a matrix representation $\widetilde{H}_n \in \mathbb{C}^{m\times m}$ (where $m = \dim(\mathrm{Ran}(E_n))$) defined as follows. First, for $y_1, y_2 \in (\frac{1}{n}\mathbb{Z})^d \cap S_n$,

$$\langle |\Phi_n|^2(P_j) E_n n^{d/2} \chi_{Q_n(y_1)}, n^{d/2} \chi_{Q_n(y_2)} \rangle = \begin{cases} 2n^2 & y_1 = y_2 \\ -n^2 & y_1 - y_2 = \pm 1/n e_j \\ 0 & \text{otherwise} \end{cases}$$

and $\langle V_n n^{d/2} \chi_{Q_n(y_1)}, n^{d/2} \chi_{Q_n(y_2)} \rangle = V(y_1)$ when $y_1 = y_2$ and zero otherwise. Thus, we can form the matrix representation of $H_n|_{\mathrm{Ran}(E_n)}$ with respect to the orthonormal basis $\{n^{d/2}\chi_{Q_n(y)}\}_{y \in (\frac{1}{n}\mathbb{Z})^d \cap S_n}$. It is important to note that calculating the matrix elements of $\widetilde{H}_n$ requires knowledge only of $\{V_f\}_{f \in \Lambda_n}$ where we have $\Lambda_n := \{f_y : y \in (n^{-1}\mathbb{Z})^d \cap S_n\}$ and $V_{f_y} = f_y(V) = V(y)$.

We have so far shown that the Assumption (i) holds, and we are left to show that the discretization we have chosen satisfies Assumption (ii). In particular, we need to demonstrate that our discretization satisfies (9.49). That is the topic of the following theorem.

Theorem 9.24. *Let $V \in C(\mathbb{R}^d)$ be sectorial as defined in* (6.3) *satisfying $|V(x)| \to \infty$ as $|x| \to \infty$, and let $h = -\Delta + V$ with $\mathcal{D}(h) = C_c^\infty(\mathbb{R}^d)$ and let $H = h^{**}$. Let H_n be as in* (9.58)*. Then there exists z_0 such that $\|(H - z_0)^{-1} - (H_n - z_0)^{-1}E_n\| \to 0$, as $n \to \infty$.*

Remark 9.25 (Proof of Theorem 6.3)**.** Note that we immediately have

$$\text{Theorem 9.24 } + \text{ Proposition 9.18 } \Rightarrow \{\Xi_{\mathrm{sp}}, \Omega_\infty\} \in \Delta_2^A, \{\Xi_{\mathrm{sp},\epsilon}, \Omega_\infty\} \in \Delta_2^A.$$

Thus, the rest of the section is devoted to prove Theorem 9.24.

We shall treat the discretizations in a similar way as the continuous case, namely by "rotating" the operator into symmetric position with respect to the real axis and then, by taking the real part, we are dealing with a sequence of self-adjoint invertible operators. Before we prove this theorem we will need a couple of lemmas. We recall the following definition.

Definition 9.26 (Collectively compact)**.** A set $\mathcal{T} \subset B(\mathcal{H})$ is called *collectively compact* if the set $\{Tx : T \in \mathcal{T}, \|x\| \leq 1\}$ has compact closure.

Lemma 9.27. *Let $\{K_n\}$ be a collectively compact operator sequence and $K_n^* \to 0$ strongly. Then $\|K_n\| \to 0$.*

Proof. It is well known that on any compact set $\mathcal{B}$ the strong convergence $K_n^* \to 0$ turns into norm convergence: $\sup\{\|K_n^* x\| : x \in \mathcal{B}\} \to_n 0$. Since $\mathcal{B} := \mathrm{cl}\{K_n x : \|x\| \leq 1, n \in \mathbb{N}\}$ is compact, we get

$$\|K_n\|^2 = \|K_n^* K_n\| = \sup\{\|K_n^* K_n x\| : \|x\| \leq 1\} \leq \sup\{\|K_n^* y\| : y \in \mathcal{B}\} \to 0 \quad \text{as} \quad n \to \infty.$$

□

We also need a modification of Lemma 9.20.

Lemma 9.28. *Let $\{A_n\}$ be m-sectorial with common semi-angle $\beta < \pi/2$ and denote $B_n = \mathrm{Re}\, A_n$. Assume that $\{E_n\}$ is a sequence of orthogonal projections, converging strongly to identity and such that $A_n E_n = E_n A_n E_n$ and $B_n E_n = E_n B_n E_n$. Assume further that $\{B_n^{-1}\}$ is uniformly bounded. If $\{B_n^{-1} E_n\}$ is collectively compact, then so is $\{A_n^{-1} E_n\}$.*

Proof. Denote by $B_n^{1/2}$ the unique self-adjoint non-negative square root of B_n. By [69, VI-Theorem 3.2 on p.337] for each A_n there exists a bounded symmetric operator C_n satisfying $\|C_n\| \leq \tan(\beta)$ and such that $A_n = B_n^{1/2}(1 + \mathrm{i}\, C_n) B_n^{1/2}$. Writing

$$A_n^{-1} = \int_0^\infty e^{-tA_n}\, dt$$

we conclude that $E_n A_n^{-1} E_n = A_n^{-1} E_n$ and likewise for B_n^{-1}. Assume now that $\{B_n^{-1} E_n\}$ is collectively compact. But then so is $\{(B_n + t)^{-1} E_n\} = \{B_n^{-1} E_n (I + t B_n^{-1})^{-1} E_n\}$ and writing, compare [69, V (3.43) on p.282],

$$B_n^{-1/2} E_n = \frac{1}{\pi} \int_0^\infty t^{-1/2} (B_n + t)^{-1} E_n\, dt$$

we see that $\{B_n^{-1/2} E_n\}$ is also collectively compact and $B_n^{-1/2} E_n = E_n B_n^{-1/2} E_n$. Finally $\{A_n^{-1} E_n\}$ is then collectively compact as well since $A_n^{-1} E_n$ is of the form $B_n^{-1/2} E_n T_n$ with T_n uniformly bounded. □

Proof of Theorem 9.24. Note that it is clear from the definition of H_n and the assumption on V that $\mathrm{Num}(H_n) \subset \{re^{i\rho} : -\theta_2 \leq \rho \leq \theta_1, r \geq 0\}$ for all n. Thus, since H_n is bounded and by Proposition 9.22 we can choose any point $z_0 \in \mathbb{C}$ such that z_0 has a positive distance d to the closed sector $\{re^{i\rho} : -\theta_2 \leq \rho \leq \theta_1, r \geq 0\}$, and both $R(H, z_0) = (H - z_0)^{-1}$ and $R(H_n, z_0) = (H_n - z_0)^{-1}$ for every n will exist. Moreover, $R(H_n, z_0)$ are uniformly bounded for all n, since for every x, $\|x\| = 1$,

$$\|(H_n - z_0)x\| \geq |\langle (H_n - z_0)x, x\rangle| \geq |\langle H_n x, x\rangle - z_0| \geq d.$$

Note that by Lemma 9.27 it suffices to show that (i) $R(H_n, z_0)^* E_n \to R(H, z_0)^*$ strongly, and (ii) $\{R(H_n, z_0)E_n - R(H, z_0)\}$ is collectively compact.

To see (i) observe that $C_c^\infty(\mathbb{R}^d)$ is a common core for H and for H_n. Hence by [69, VIII-Theorem 1.5 on p.429], the strong resolvent convergence $R(H_n, z_0)^* \to R(H, z_0)^*$ will follow if we show that $H_n^* \psi \to H^* \psi$ as $n \to \infty$ for any $\psi \in C_c^\infty(\mathbb{R}^d)$. Then the strong convergence $R(H_n, z_0)^* E_n \to R(H, z_0)^*$ follows as well. Note that

$$\|H_n^* \psi - H^* \psi\| \leq \left\| \sum_{j=1}^d |\Phi_n|^2(P_j) E_n \psi - \sum_{j=1}^d P_j^2 \psi \right\| + \|(\overline{V}_n - \overline{V})\psi\|. \tag{9.59}$$

Also, $|\Phi_n|^2(P_j) = n(\tau_{-1/ne_j} - I) n(\tau_{1/ne_j} - I)$, where $\tau_z \psi(x) = \psi(x - z)$ and $\{e_j\}$ is the canonical basis for $\mathbb{R}^d$. Moreover, for $\psi \in C_c^\infty(\mathbb{R}^d)$,

$$E_n \psi = \sum_{y \in (\frac{1}{n}\mathbb{Z})^d \cap S_n} (\Psi_n * \psi)(y) \chi_{Q_n(y)}, \quad \Psi_n = \rho_n \otimes \ldots \otimes \rho_n, \quad \rho_n = n\chi_{[-\frac{1}{2n}, \frac{1}{2n})},$$

where S_n was defined in (9.57). Thus, it follows from easy calculus manipulations and basic properties of convolution that $|\Phi_n|^2(P_j) E_n \psi = \sum_{y \in (\frac{1}{n}\mathbb{Z})^d} (\Psi_n * \tilde{\rho}_1 *_j \tilde{\rho}_2 *_j \psi'')(y) \chi_{Q_n(y)}$, where $\tilde{\rho}_1 = n\chi_{[-1/n, 0]}$, $\tilde{\rho}_2 = n\chi_{[0, 1/n]}$ and $*_j$ denotes the convolution operation in the jth variable. By standard properties of the convolution we have that $\Psi_n * \tilde{\rho}_1 *_j \tilde{\rho}_2 *_j \psi'' \to \psi''$ uniformly as $n \to \infty$. Thus, since $\psi \in C_c^\infty(\mathbb{R}^d)$, the first part of the right hand side of (9.59) tends to zero as $n \to \infty$. Due to the continuity of V and the bounded support of ψ it also follows easily that $\|(\overline{V}_n - \overline{V})\psi\| \to 0$ as $n \to \infty$.

To see (ii) we use the same trick as in the proof of Proposition 9.22. In particular, first set $z_0 = -e^{i\alpha}$ (which is clearly in the resolvent set of H_n for $\alpha = (\theta_1 - \theta_2)/2$) then let $A_n(\alpha) = e^{-i\alpha}(H_n - z_0)$ and further $H_n(\alpha) = \mathrm{Re}\, A_n(\alpha)$. Note that, by Lemma 9.28, we would be done if we could show that $\{H_n(\alpha)^{-1}\}$ is uniformly bounded and $\{H_n(\alpha)^{-1}E_n\}$ is collectively compact as that would yield collective compactness of $\{A_n(\alpha)^{-1}E_n\}$ and hence of $\{R(H_n, z_0)E_n\}$. To establish the uniform bound, note that

$$H_n(\alpha) = \cos\alpha\, E_n \sum_{j=1}^{d} |\Phi_n|^2(P_j)E_n + \cos\vartheta(x)|V_n(x)| + 1, \tag{9.60}$$

where ϑ is defined in (9.54). Thus $\|H_n(\alpha)^{-1}\| \leq 1$ and by applying Lemma 9.29 we are now done. □

Lemma 9.29. *Let $H_n(\alpha)$ be given by (9.60). Then the set $\{H_n(\alpha)^{-1}E_n\}$ is collectively compact.*

Proof. We shall show that if we choose an arbitrary sequence $\{\psi_n\} \subset L^2(\mathbb{R}^d)$ satisfying $\|\psi_n\| \leq 1$, then the sequence $\{\varphi_n\}$ where $\varphi_n = H_n(\alpha)^{-1}E_n\psi_n$, is relatively compact in $L^2(\mathbb{R}^d)$. The compactness argument is based on the Rellich's criterion.

Lemma 9.30 (Rellich's criterion ([84] Theorem XIII.65))**.** *Let $F(x)$ and $G(\omega)$ be two measurable nonnegative functions becoming larger than any constant for all large enough $|x|$ and $|\omega|$. Then*

$$S = \{\varphi \;:\; \int |\varphi(x)|^2 dx \leq 1,\; \int F(x)|\varphi(x)|^2 dx \leq 1,\; \int G(\omega)|\mathcal{F}\varphi(\omega)|^2 d\omega \leq 1\}$$

is a compact subset of $L^2(\mathbb{R}^d)$.

To prove Lemma 9.29 we proceed as follows. First we conclude that $\{\varphi_n\}$ is a bounded sequence itself. Then, in order to be able to define suitable functions F, G we need to approximate the sequence by another one of the form $\Psi_n * \varphi_n$. This approximation shall satisfy $\lim_{n\to\infty} \|\Psi_n * \varphi_n - \varphi_n\| = 0$ and this is very similar to the standard result on local uniform convergence of mollifications of continuous functions. Then the Rellich's criterion holds for $\Psi_n * \varphi_n$ with $F(x)$ essentially given by $|V(x)|$ and $G(\omega)$ by $|\omega|^2$. We then conclude that the sequence $\{\Psi_n * \varphi_n\}$ is relatively compact. But since $\lim_{n\to\infty} \|\Psi_n * \varphi_n - \varphi_n\| = 0$, the sequence $\{\varphi_n\}$ is relatively compact as well, completing the argument.

More precisely, since $|\vartheta(x)| \leq \alpha < \pi/2$ we have from (9.60)

$$|\langle H_n(\alpha)\varphi_n, \varphi_n\rangle| \geq \cos\alpha \left(\langle \sum_{j=1}^{d} |\Phi_n|^2(P_j)\varphi_n, \varphi_n\rangle + \langle |V_n|\varphi_n, \varphi_n\rangle \right) + \|\varphi_n\|^2. \tag{9.61}$$

But $|\langle H_n(\alpha)\varphi_n, \varphi_n\rangle|$ is bounded not only from below but also from above. Indeed, $|\langle H_n(\alpha)\varphi_n, \varphi_n\rangle| = |\langle E_n\psi_n, \varphi_n\rangle| \leq \|H_n(\alpha)^{-1}E_n\|\|\psi_n\|^2$. Thus, we conclude first from (9.61) that the sequence $\{\varphi_n\}$ is bounded. Next, in view of (9.61), there exist constants $C_1, C_2 > 0$ such that for all $n \in \mathbb{N}$

$$\langle \sum_{j=1}^{d} |\Phi_n|^2(P_j)\varphi_n, \varphi_n\rangle \leq C_1, \tag{9.62a}$$

$$\langle |V_n|\varphi_n, \varphi_n\rangle \leq C_2. \tag{9.62b}$$

First we use the bound (9.62a). Letting $\mathcal{F}$ denote the Fourier transform , we have that $(\mathcal{F}\Phi_n(P_j)\varphi_n)(\omega) = \Phi_n(\omega_j)(\mathcal{F}\varphi_n)(\omega)$, for a.e. ω and for $1 \leq j \leq d$. Letting $\Theta_n(\omega) = \frac{\sin(\omega/2n)}{\omega/2n}$, an application of the Fourier transform to (9.62a) along with Plancherel's theorem yield

$$\int_{\mathbb{R}^d} |(\mathcal{F}\varphi_n)(\omega)|^2 \sum_{1\leq j\leq d} |\omega_j \Theta_n(\omega_j)|^2\, d\omega \leq C_1.$$

Moreover, since $|\Theta_n(\omega)| \leq 1$ for all ω, we get

$$\int_{\mathbb{R}^d} |\omega|^2 |\Theta_n(\omega_1)\cdots\Theta_n(\omega_d)|^2 |(\mathcal{F}\varphi_n)(\omega)|^2 \, d\omega \leq C_1. \tag{9.63}$$

We now define the approximation $\Psi_n * \varphi_n$. Let $\Psi_1(z) = \chi_{[-1/2,1/2]^d}(z)$ and further $\Psi_n(z) = n^d \Psi_1(nz)$, where $\chi_A(z)$ is the usual characteristic function for the set A. We shall prove below that $\lim_{n\to\infty} \|\Psi_n * \varphi_n - \varphi_n\| = 0$, which in particular shows that the sequence $\{\Psi_n * \varphi_n\}$ is bounded. Observe then that $(\mathcal{F}\Psi_n)(\omega) = \Theta_n(\omega_1)\cdots\Theta_n(\omega_d)$. Therefore we obtain from (9.63)

$$\int_{\mathbb{R}^d} |\omega|^2 |\mathcal{F}(\Psi_n * \varphi_n)(\omega)|^2 d\omega \leq C_1,$$

which shows that we can choose $G(\omega)$ to be (a constant times) $|\omega|^2$.

We still need to establish the growth function $F(x)$ for $\Psi_n * \varphi_n$. Consider φ_n. It is of the form $\varphi_n = (E_n + E_n B_n E_n)^{-1} E_n \psi_n$ and hence $E_n \varphi_n = \varphi_n$. Therefore φ_n vanishes outside S_n and we can essentially replace V_n by V in the inequality (9.62b). To that end, put $F(x) = \min_{|y|\geq|x|} |V(y)|$. Then with some constant C_3

$$\int_{\mathbb{R}^d} F(x) |(\Psi_n * \varphi_n)(x)|^2 dx \leq C_3. \tag{9.64}$$

In view of the bounds (9.63), (9.64) and since the sequence $\{\Psi_n * \varphi_n\}_{n\in\mathbb{N}}$ is bounded in L^2, Rellich's criterion implies that $\{\Psi_n * \varphi_n\}_{n\in\mathbb{N}}$ is a relatively compact sequence and it therefore follows that $\{\varphi_n\}_{n\in\mathbb{N}}$ is relatively compact, thus finishing the proof. Hence, our only remaining obligation is to show that $\lim_{n\to\infty} \|\Psi_n * \varphi_n - \varphi_n\| = 0$. This result is very similar to the standard result on local uniform convergence of mollifications of continuous functions.

Let $z \in \mathbb{R}^d$ and define the shift operator τ_z on $L^2(\mathbb{R}^d)$ by $\tau_z f(x) = f(x - z)$. Now observe that by Minkowski's inequality for integrals it follows that

$$\|\Psi_n * \varphi_n - \varphi_n\| \leq \int_{\mathbb{R}^d} \|\tau_{\frac{1}{n}z}\varphi_n - \varphi_n\| |\Psi_1(z)| \, dz = \int_{[-1/2,1/2]^d} \|e^{i\frac{z_d}{n}P_d} \dots e^{i\frac{z_1}{n}P_1}\varphi_n - \varphi_n\| \, dz. \tag{9.65}$$

The claim follows from an ϵ/d argument and (9.65) combined with the dominated convergence theorem (recall that $\{\varphi_n\}$ is bounded): we need to show that for fixed $z \in [-1/2, 1/2]^d$ and for any $1 < j \leq d$,

$$\lim_{n\to\infty} \left\| e^{i\frac{z_j}{n}P_j} \dots e^{i\frac{z_1}{n}P_1}\varphi_n - e^{i\frac{z_{j-1}}{n}P_{j-1}} \dots e^{i\frac{z_1}{n}P_1}\varphi_n \right\| = 0, \quad \lim_{n\to\infty} \left\| e^{i\frac{z_1}{n}P_1}\varphi_n - \varphi_n \right\| = 0. \tag{9.66}$$

Since $e^{i\frac{z_j}{n}P_j} e^{i\frac{z_k}{n}P_k} = e^{i\frac{z_k}{n}P_k} e^{i\frac{z_j}{n}P_j}$ and $\|e^{i\frac{z_j}{n}P_j} \cdots e^{i\frac{z_1}{n}P_1}\| \leq 1$ for $1 \leq j, k \leq d$, (9.66) will follow if we can show that $\|(e^{i\frac{z_j}{n}P_j} - I)\varphi_n\| \to 0$ as $n \to \infty$. Note that, by the choice of the projections E_n, it follows that for $1 \leq j \leq d$, $|((e^{i\frac{z_j}{n}P_j} - I)\varphi_n)(x)| \leq |((e^{i\frac{1}{n}P_j} - I)\varphi_n)(x)|$, for $0 \leq z_j \leq 1/2$ and $x \in \mathbb{R}^d$. Also, $|((e^{i\frac{z_j}{n}P_j} - I)\varphi_n)(x)| \leq |((e^{-i\frac{1}{n}P_j} - I)\varphi_n)(x)|$ for $-1/2 \leq z_j < 0$. However the bound $\sum_{1\leq j\leq d} \|\Phi_n(P_j)\varphi_n\|^2 \leq C_1$ implies that $\lim_{n\to\infty} \|(e^{\pm i\frac{1}{n}P_j} - I)\varphi_n)\| = 0$, which proves the claim. □

Proof that neither problem lies in $\Sigma_1^G \cup \Pi_1^G$. Finally, we shall complete the proof of Theorem 6.3 by showing that $\{\Xi_{\mathrm{sp}}, \Omega_\infty\} \notin \Sigma_1^G \cup \Pi_1^G$ and $\{\Xi_{\mathrm{sp},\epsilon}, \Omega_\infty\} \notin \Sigma_1^G \cup \Pi_1^G$.

Proof. **Step I:** $\{\Xi_{\mathrm{sp}}, \Omega_\infty\} \notin \Sigma_1^G$. Suppose for a contradiction that there exists a Σ_1^G tower Γ_n which solves the computational problem $\{\Xi_{\mathrm{sp}}, \Omega_\infty\}$. Now let V be any (real valued) positive potential in the class Ω_∞ such that the corresponding Schrödinger operator is self-adjoint and has a unique ground state (the operator must be bounded below). Call the associated operator H_0. For instance in one dimension this could be the quantum harmonic oscillator $V(x) = x^2$ and examples in arbitrary dimension (the harmonic oscillator in $d > 1$ dimensions does not have a unique ground state) are well known in the physics literature. In this case, let ϕ_0 be the normalised ground state and E be the orthogonal complement of the span of this function intersected with the domain of H_0. Assume that $H_0\phi_0 = c\phi_0$. Denoting the standard $L^2(\mathbb{R}^d)$ inner product by $\langle\cdot,\cdot\rangle$, it follows that there exists some $\eta > 0$ such that

$$\langle H_0\phi, \phi\rangle \geq (c+\eta)\|\phi\|^2, \quad \forall \phi \in E.$$

There exists n such that there is a point $z_n \in \Gamma_n(V)$ with $|z_n - c| \leq \eta/20$ and such that $\Gamma_n(V)$ guarantees there is a point in the spectrum $\Xi_{\mathrm{sp}}(V)$ of distance at most $\eta/20$ to z_n. Hence $\Gamma_n(V)$ guarantees there is a point in the spectrum $\Xi_{\mathrm{sp}}(V)$ of of distance at most $\eta/10$ from c. There also exists a finite set $S = \{x^1, ..., x^{M(n)}\}$ such that the computation of $\Gamma_n(V)$ only depends on the potential V evaluated at points in S. Let V_m be a sequence of real-valued continuous potentials such that $0 \leq V_m(x) \leq 1$, $V_m(x^j) = 0$ $\forall x^j \in S$ and such that V_m converges pointwise almost everywhere to 1 as $m \to \infty$. By construction and the definition of a general algorithm (Definition 4.3) we must have for all $a \in \mathbb{R}_+$ that $\Gamma_n(V + aV_m) = \Gamma_n(V)$. In particular, this includes the guarantee of a point in the spectrum $\Xi_{\mathrm{sp}}(V + aV_m)$ of distance at most $\eta/10$ from c. We will show that this gives rise to a contradiction for a choice of $a \in \mathbb{R}_+$ and m.

Indeed, choose m large such that

$$\langle V_m\phi_0, \phi_0\rangle \geq \frac{10}{11},$$

and set $a = \eta/2$. It is well known that the minimum of the spectrum $\Xi_{\mathrm{sp}}(V + aV_m)$ is given by

$$\inf_{\phi\in\mathcal{D}(H_0):\|\phi\|=1} \langle (H_0 + aV_m)\phi, \phi\rangle.$$

In particular, $H_0 + aV_m$ and H_0 have the same domain as V_m is bounded. Now let $\phi \in \mathcal{D}(H_0)$ of norm 1. Without loss of generality by a change of phase, we can write

$$\phi = \delta\phi_0 + \sqrt{1-\delta^2}\phi_1,$$

with $\phi_1 \in E$ of unit norm and $\delta \in [0, 1]$. Using the fact that $H_0\phi_0 = c\phi_0$ and H_0 is self-adjoint and $\langle\phi_0, \phi_1\rangle = 0$, we have that

$$\begin{aligned}\langle (H_0 + aV_m)\phi, \phi\rangle &= \delta^2 c + (1-\delta^2)\langle H_0\phi_1, \phi_1\rangle + \delta^2 a\langle V_m\phi_0, \phi_0\rangle \\ &\quad + a(1-\delta^2)\langle V_m\phi_1, \phi_1\rangle + 2\mathrm{Re}(a\delta\sqrt{1-\delta^2}\langle V_m\phi_0, \phi_1\rangle) \\ &\geq c + (1-\delta^2)\eta + \frac{10}{11}\delta^2 a - 2a\delta\sqrt{1-\delta^2},\end{aligned}$$

where we have used that V_m is positive to throw away the $\langle V_m\phi_1, \phi_1\rangle$ term. It follows that the minimum of the spectrum of $H_0 + aV_m$ is at least

$$c + \inf_{\delta\in[0,1]} \eta(1 - (1 - 5/11)\delta^2 - \delta\sqrt{1-\delta^2}) > c + \frac{\eta}{10},$$

the required contradiction.

Step II: $\{\Xi_{\mathrm{sp}}, \Omega_\infty\} \notin \Pi_1^G$. We argue as in Step I but now the proof is less involved. Suppose for a contradiction that there exists a Π_1^G tower Γ_n which solves the computational problem $\{\Xi_{\mathrm{sp}}, \Omega_\infty\}$. We let H_0, V, ϕ_0 and E be as in Step I, where we also assume as before that $H_0\phi_0 = c\phi_0$. We also assume that $c \geq 0$ and $V(x) \geq 1$.

Arguing as before, there exists some n such that $\Gamma_n(V)$ guarantees that the spectrum is disjoint from the interval $[c - 3/2, c - 1/2]$. Again, there exists a finite set $S = \{x^1, ..., x^{M(n)}\}$ such that the computation of $\Gamma_n(V)$ only depends on the potential V evaluated at points in S. Let V_m be a sequence of real-valued continuous potentials such that $-1 \leq V_m(x) \leq 0$, $V_m(x^j) = 0$ $\forall x^j \in S$ but now such that V_m converges pointwise almost everywhere to -1 as $m \to \infty$. Note that we must have $V + V_m \in \Omega_\infty$ since we assume the pointwise inequality $V(x) \geq 1$. By construction and the definition of a general algorithm (Definition 4.3) we must have that $\Gamma_n(V + V_m) = \Gamma_n(V)$. In particular, this includes the guarantee that the spectrum of $H_0 + V_m$ is disjoint from the interval $[c - 3/2, c - 1/2]$. But we have that

$$\langle (H_0 + V_m - (c-1))\phi_0, \phi_0\rangle = \langle V_m\phi_0, \phi_0\rangle + 1 \to 0,$$

as $m \to \infty$. It follows for some large m that $\|R(c-1, H_0+V_m)\|^{-1} \leq 1/4$ and hence that the spectrum of H_0+V_m intersects the interval $[c-3/2, c-1/2]$, since the operator is self-adjoint. But this contradicts the Π_1^G guarantee.

Step III: $\{\Xi_{\mathrm{sp},\epsilon}, \Omega_\infty\} \notin \Pi_1^G \cup \Sigma_1^G$. The arguments are the same as in Steps I and II. We note that the pseudospectrum is simply the ϵ ball neighbourhood of the spectrum in these self-adjoint cases. The arguments work once we scale the operators by N/ϵ for some large N in order to gain the relevant separations. $\square$

10. Proofs of Theorem 7.2 and Theorem 7.3

Proof of Theorem 7.2. We have that $\mathrm{SCI}(\Xi_{\mathrm{inv}}, \Omega_1)_{\mathrm{A}} \geq \mathrm{SCI}(\Xi_{\mathrm{inv}}, \Omega_1)_{\mathrm{G}} \geq \mathrm{SCI}(\Xi_{\mathrm{inv}}, \Omega_2)_{\mathrm{G}}$ and $\mathrm{SCI}(\Xi_{\mathrm{inv}}, \Omega_3)_{\mathrm{G}} \geq 1$. It is also clear that $\{\Xi_{\mathrm{inv}}, \Omega_4\} \notin \Delta_0^G$. Hence it is enough to prove that $\mathrm{SCI}(\Xi_{\mathrm{inv}}, \Omega_2)_{\mathrm{G}} \geq 2$, $\mathrm{SCI}(\Xi_{\mathrm{inv}}, \Omega_1)_{\mathrm{A}} \leq 2$, $\mathrm{SCI}(\Xi_{\mathrm{inv}}, \Omega_3)_{\mathrm{A}} \leq 1$ and $\{\Xi_{\mathrm{inv}}, \Omega_4\} \in \Delta_1^A$.

Step I: We start by showing that $\mathrm{SCI}(\Xi_{\mathrm{inv}}, \Omega_2)_{\mathrm{G}} \geq 2$. For $n, m \in \mathbb{N} \setminus \{1\}$ let

$$B_{n,m} := \begin{pmatrix} 1/m & & & & 1 \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ 1 & & & & 1/m \end{pmatrix} \in \mathbb{C}^{n\times n}$$

and for a sequence $\{l_n\}_{n\in\mathbb{N}} \subset \mathbb{N} \setminus \{1\}$ set

$$A := \bigoplus_{n=1}^{\infty} B_{l_n, n+1}.$$

Clearly, A defines an invertible operator on $l^2(\mathbb{N})$. Furthermore, we define $b = \{b_j\} \in l^2(\mathbb{N})$ such that

$$b_j = \begin{cases} \dfrac{1}{n+2} & j = \sum_{i=1}^{n} l_i + 1, \quad n \in \mathbb{N}_0 \\ 0 & \text{otherwise.} \end{cases}$$

Let also $C_m := \mathrm{diag}\{1/m, 1, 1, \ldots\}$ and note that its inverse is given by $\mathrm{diag}\{m, 1, 1, \ldots\}$. We argue by contradiction and suppose that there is a General tower of algorithms Γ_n of height one such that $\Gamma_n(A, b) \to \Xi_{\mathrm{inv}}(A, b)$ as $n \to \infty$ for $(A, b) \in \Omega_2$. For such A, b and $k \in \mathbb{N}$ let $N(A, b, k)$ denote the smallest integer such that the evaluations from $\Lambda_{\Gamma_k}(A, b)$ only take the matrix entries $A_{ij} = \langle Ae_j, e_i\rangle$ with $i, j \leq N(A, b, k)$ and the entries b_i with $i \leq N(A, b, k)$ into account. To obtain a particular counterexample (A, b) we construct sequences $\{l_n\}_{n\in\mathbb{N}}$ and $\{k_n\}_{n\in\mathbb{Z}_+}$ inductively such that A and b are given by $\{l_n\}$ as above but $\Gamma_{k_n}(A, b) \nrightarrow \Xi_{\mathrm{inv}}(A, b)$. As a start, set $k_0 = l_0 := 1$. The sequence $\{x_j^{(1)}\}_{j\in\mathbb{N}} := (C_2)^{-1} P_1 b$ has a 1 at its first entry and since, by assumption, $\Gamma_k \to \Xi_{\mathrm{inv}}$, there is a k_1 such that, for all $k \geq k_1$, the first entry of $\Gamma_k(C_2, P_1 b)$ is closer to 1 than $1/2$. Then, choose $l_1 > N(C_2, P_1 b, k_1) - l_0$. Now, for $n > 1$, suppose that $l_0, \ldots, l_{n-1}$ and $k_0, \ldots, k_{n-1}$ are already chosen. Set $s_n := \sum_{i=0}^{n-1} l_i$. Then also $P_{s_n} b$ is already determined and

$$x_{s_n+1}^{(n)} = 1, \quad \text{where} \quad \{x_j^{(n)}\}_{j\in\mathbb{N}} := (B_{l_1,2} \oplus B_{l_2,3} \oplus \ldots \oplus B_{l_{n-1},n} \oplus C_{n+1})^{-1} P_{s_n} b.$$

Since, by assumption, $\Gamma_k \to \Xi_{\mathrm{inv}}$, there is a k_n such that for all $k \geq k_n$

$$|x_{s_n+1}^{(n,k)} - 1| \leq 1/2, \quad \text{where} \quad \{x_j^{(n,k)}\}_{j\in\mathbb{N}} := \Gamma_k(B_{l_1,2} \oplus B_{l_2,3} \oplus \ldots \oplus B_{l_{n-1},n} \oplus C_{n+1}, P_{s_n} b).$$

Now, choose $l_n > N(B_{l_1,2} \oplus B_{l_2,3} \oplus \ldots \oplus B_{l_{n-1},n} \oplus C_{n+1}, P_{s_n} b, k_n) - l_0 - l_1 - \ldots - l_{n-1}$. By this construction we get for the resulting A and b that for every n

$$\Gamma_{k_n}(A, b) = \Gamma_{k_n}(B_{l_1,2} \oplus B_{l_2,3} \oplus \ldots \oplus B_{l_{n-1},n} \oplus C_{n+1}, P_{s_n} b).$$

In particular $\lim_{k\to\infty} \Gamma_k(A, b)$ does not exist in $l^2(\mathbb{N})$, a contradiction.

Step II: To show that $\mathrm{SCI}(\Xi_{\mathrm{inv}}, \Omega_1)_{\mathrm{A}} \le 2$, let A be invertible and $Ax = b$ with the unknown x. Since P_m are compact projections converging strongly to the identity, we get that the ranks $\mathrm{rk}\, P_m = \mathrm{rk}(AP_m) = \mathrm{rk}(P_n A P_m)$ for every m and all $n \ge n_0$ with an n_0 depending on m and A. Then, obviously, $P_m A^* P_n A P_m$ is an invertible operator on $\mathrm{Ran}(P_m)$, and we can define

$$\Gamma_{m,n}(A,b) := \begin{cases} \{0\}_{j\in\mathbb{N}} & \text{if } \sigma_1(P_m A^* P_n A P_m) \le \frac{1}{m} \\ (P_m A^* P_n A P_m)^{-1} P_m A^* P_n b & \text{otherwise.} \end{cases}$$

Note that for every A, b, m, n in view of Proposition 8.1 and any standard algorithm for finite dimensional linear problems, these approximate solutions can be computed by finitely many arithmetic operations on finitely many entries of A and b, hence $\Gamma_{m,n}$ are general algorithms in the sense of Definition 4.3 and require only a finite number of arithmetic operations. Moreover, they converge to $y_m := (P_m A^* A P_m)^{-1} P_m A^* b$ as $n \to \infty$. It is well known that y_m is also a (least squares) solution of the optimization problem $\|AP_m y - b\| \to \min$, that is

$$\|AP_m y_m - b\| \le \|AP_m x - b\| \le \|A\|\|P_m x - A^{-1}b\| = \|A\|\|P_m x - x\| \to 0$$

as $m \to \infty$. Therefore $\|y_m - x\| = \|P_m y_m - x\|$ is not greater than

$$\|A^{-1}\|\|A(P_m y_m - x)\| = \|A^{-1}\|\|AP_m y_m - b\| \le \|A^{-1}\|\|A\|\|P_m x - x\| \to 0,$$

which yields the convergence $y_m \to x$ and finishes the proof of Step II.

Step III: Let f be a bound on the dispersion of A. The smallest singular values of the operators AP_m are uniformly bounded below by $\|A^{-1}\|^{-1}$ which, together with $\|P_{f(m)} A P_m - AP_m\| \to 0$, yields that the limit inferior of the smallest singular values of $P_{f(m)} A P_m$ is positive, hence the inverses of the operators $B_m := P_m A^* A P_m$ and $C_m := P_m A^* P_{f(m)} A P_m$ on the range of P_m exist for sufficiently large m and have uniformly bounded norm. Moreover, $\|B_m^{-1} - C_m^{-1}\| \le \|B_m^{-1}\|\|C_m - B_m\|\|C_m^{-1}\|$ tend to zero as $m \to \infty$.

This particularly implies that the norms $\|y_m - (P_m A^* P_{f(m)} A P_m)^{-1} P_m A^* b\|$ with y_m as above tend to zero as $m \to \infty$, and we easily conclude that the norms $\|y_m - \Gamma_{m,f(m)}(A,b)\|$ tend to zero as well. With the convergence $\|y_m - x\| \to 0$ from the previous proof, now also $\|x - \Gamma_{m,f(m)}(A,b)\| \to 0$ holds as $m \to \infty$, which is the assertion $\mathrm{SCI}(\Xi_{\mathrm{inv}}, \Omega_3)_{\mathrm{A}} \le 1$.

Step IV: We prove that $\{\Xi_{\mathrm{inv}}, \Omega_4\} \in \Delta_1^A$. To do this we take the algorithm constructed in Step III, and note that by increasing m if necessary, we can assume that $\sigma_1(P_m A^* P_{f(m)} A P_m) > 1/m$. Hence we only need to bound the error of the approximation. We have that

$$\begin{aligned} \|A\Gamma_{m,f(m)}(A,b) - b\| &\le \|P_{f(m)} A P_m \Gamma_{m,f(m)}(A,b) - P_m b\| \\ &\quad + \|P_m b - b\| + \|(I - P_{f(m)}) A P_m\|\|\Gamma_{m,f(m)}(A,b)\| \\ &\le \|P_{f(m)} A P_m \Gamma_{m,f(m)}(A,b) - P_m b\| + c_m(1 + \|\Gamma_{m,f(m)}(A,b)\|) \end{aligned}$$

and hence the bound

$$\|\Gamma_{m,f(m)}(A,b) - A^{-1}b\| \le M\big[\|P_{f(m)} A P_m \Gamma_{m,f(m)}(A,b) - P_m b\| + c_m(1 + \|\Gamma_{m,f(m)}(A,b)\|)\big].$$

Note that this final bound converges to zero and it is also clear that we can approximate it to arbitrary accuracy using finitely many arithmetic operations and comparisons. □

Remark 10.1. The technique used with uneven sections to obtain the bound $\mathrm{SCI}(\Xi_{\mathrm{inv}}, \Omega_1)_{\mathrm{A}} \le 2$ is also referred to as asymptotic Moore-Penrose inversion as well as modified (or non-symmetric) finite section method in the literature, although written in a different form, and is widely used (see e.g. [53,54,62,89,92]). Also the idea to exploit bounds on the off diagonal decay is considered e.g. in [52] or in the theory of band-dominated operators and operators of Wiener type (cf. [74,83,88]).

Proof of Theorem 7.3. Clearly $\{\Xi_{\mathrm{norm}}, \Omega_3\} \notin \Delta_1^G$ and $\Omega_2 \subset \Omega_1$, so it is enough to prove $\{\Xi_{\mathrm{norm}}, \Omega_1\} \in \Pi_2^A$, $\{\Xi_{\mathrm{norm}}, \Omega_3\} \in \Pi_1^A$ and $\{\Xi_{\mathrm{norm}}, \Omega_2\} \notin \Delta_2^G$. We start with the latter. Let $\{l_n\}_{n\in\mathbb{N}}$ be some sequence of integers $l_n \geq 2$. Define

$$A := \bigoplus_{n=1}^{\infty} B_{l_n} - I, \qquad B_n := \begin{pmatrix} 1 & & & & 1 \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ 1 & & & & 1 \end{pmatrix} \in \mathbb{C}^{n\times n}.$$

Clearly, such A are invertible and their inverses have norm one. Suppose that $\{\Gamma_k\}$ is a height-one General tower of algorithms which in its kth step only reads information contained in the first $N(A,k) \times N(A,k)$ entries of the input A. In order to find a counterexample we again construct an appropriate sequence $\{l_n\} \subset \mathbb{N} \setminus \{1\}$ by induction: For $C := \mathrm{diag}\{1,0,0,0,\ldots\}$ one obviously has $\|(C-I)^{-1}\|^{-1} = 0$. As a start, choose $k_0 := 1$ and $l_1 > N(C-I,k_0)$. Now, suppose that $l_1,\ldots,l_n$ are already chosen. Then the operator given by the matrix $B_{l_1} \oplus \ldots \oplus B_{l_n} \oplus C - I$ is not invertible, hence there exists a k_n such that, for every $k \geq k_n$,

$$\Gamma_k(B_{l_1} \oplus \ldots \oplus B_{l_n} \oplus C - I) < \frac{1}{2}.$$

Now finish the construction by choosing $l_{n+1} > N(B_{l_1} \oplus \ldots \oplus B_{l_n} \oplus C - I, k_n) - l_1 - l_2 - \ldots - l_n$.

So, we see that

$$\Gamma_{k_n}(A) = \Gamma_{k_n}(B_{l_1} \oplus \ldots \oplus B_{l_n} \oplus C - I) \nrightarrow \|A^{-1}\|^{-1} = 1, \quad n \to \infty,$$

a contradiction. Thus $\{\Xi_{\mathrm{norm}}, \Omega_2\} \notin \Delta_2^G$. In order to prove $\{\Xi_{\mathrm{norm}}, \Omega_1\} \in \Pi_2^A$ we introduce the numbers

$$\begin{aligned} \gamma &:= \|A^{-1}\|^{-1} = \min\{\sigma_1(A), \sigma_1(A^*)\} \\ \gamma_m &:= \min\{\sigma_1(AP_m), \sigma_1(A^*P_m)\} \\ \gamma_{m,n} &:= \min\{\sigma_1(P_nAP_m), \sigma_1(P_nA^*P_m)\} \\ \delta_{m,n} &:= \min\{k/m : k \in \mathbb{N}, k/m \geq \sigma_1(P_nAP_m) \text{ or } k/m \geq \sigma_1(P_nA^*P_m)\} \end{aligned}$$

and note that $\gamma_m \downarrow_m \gamma$, and $\gamma_{m,n} \uparrow_n \gamma_m$ for every fixed m. Moreover, $\{\delta_{m,n}\}_n$ is bounded and monotone, and $\gamma_{m,n} \leq \delta_{m,n} \leq \gamma_{m,n} + 1/m$. Thus, $\{\delta_{m,n}\}_n$ converges for every m, and for $\epsilon > 0$ there is an m_0, and for every $m \geq m_0$ there is an $n_0 = n_0(m)$ such that

$$|\gamma - \delta_{m,n}| \leq |\gamma - \gamma_m| + |\gamma_m - \gamma_{m,n}| + |\gamma_{m,n} - \delta_{m,n}| \leq \epsilon/3 + \epsilon/3 + 1/m \leq \epsilon \tag{10.1}$$

whenever $m \geq m_0$ and $n \geq n_0(m)$. Since $\delta_{m,n}$ and hence $\Gamma_{m,n}(A) := \delta_{m,n}$ can again be computed with finitely many arithmetic operations by Proposition 8.1 this provides an Arithmetic tower of algorithms of height two, with the final convergence from above and hence easily completes the proof that $\{\Xi_{\mathrm{norm}}, \Omega_1\} \in \Pi_2^A$. On Ω_3 we apply (10.1) with $n = f(m)$ and straightforwardly check that $\Gamma_m(A) := \delta_{m,f(m)}$ provides a height 1 tower. If we wish to have Π_1^A convergence (i.e. convergence from above) then we need to use the sequence $\{c_n\}$ that bounds the dispersion to bound the difference between $\delta_{m,f(m)}$ and γ_m and choose $\Gamma_m(A) := \delta_{m,f(m)} + c_m$. □

11. SMALE'S PROBLEM ON ROOTS OF POLYNOMIALS AND DOYLE-MCMULLEN TOWERS

WARNING: unsure what Smale3 ref should be - clash in bib file

In this section we recall the definition of a tower of algorithms from [36]. We will name this type of tower a Doyle-McMullen tower and demonstrate how the results in [78] and [36] can be put in a framework of the SCI. In particular, we will demonstrate how the construction of the Doyle-McMullen tower in [36] can be viewed as a tower of algorithms defined in Definition 4.4.

As mentioned in the introduction, one can compute zeros of a polynomial if one allows arithmetic operations and radicals and can pass to a limit. However, what if one cannot use radicals, but rather iterations of a rational map? A natural choice of such a rational map would be Newton's method. The only problem is that the iteration may not converge, and that motivated the question by Smale quoted in the introduction.

As we now know from [78] the answer is no, however, the results in [36] show that the quartic and the quintic can be solved with several rational maps and limits while this is not the case for higher degree polynomials. Below we first quote their results and then specify a particular tower of height three in the form that it can be viewed as a tower of algorithm in the sense of this paper.

11.1. **Doyle-McMullen towers.** A *purely iterative algorithm* [94] is a rational map [3]

$$T : \mathbb{P}_d \to \mathrm{Rat}_m,\ p \mapsto T_p$$

which sends any polynomial p of degree $\leq d$ to a rational function T_p of a certain degree m. An important example of a purely iterative algorithm is *Newton's method*. Furthermore, Doyle and McMullen call a purely iterative algorithm *generally convergent* if

$$\lim_{n\to\infty} T_p^n(z) \text{ exists for } (p,z) \text{ in an open dense subset of } \mathbb{P}_d \times \hat{\mathbb{C}}.$$

Here $T_p^n(z)$ denotes the nth iterate $T_p^n(z) = T_p(T_p^{n-1}(z))$ of T_p. For instance, Newton's method is generally convergent *only* when $d = 2$. However, given a cubic polynomial $p \in \mathbb{P}_3$ one can define an appropriate rational function $q \in \mathrm{Rat}_3$ whose roots coincide with the roots of p, and for which Newton's method *is* generally convergent (see [78], Proposition 1.2). In [36] the authors provide a definition of a tower of algorithms, which we quote verbatim:

Definition 11.1 (Doyle-McMullen tower)**.** A tower of algorithms is a finite sequence of generally convergent algorithms, linked together serially, so the output of one or more can be used to compute the input to the next. The final output of the tower is a single number, computed rationally from the original input and the outputs of the intermediate generally convergent algorithms.

Theorem 11.2 (McMullen [78]; Doyle and McMullen [36])**.** *For $\mathbb{P}_d$ there exists a generally convergent algorithm only for $d \leq 3$. Towers of algorithms exist additionally for $d = 4$ and $d = 5$ but not for $d \geq 6$.*

Note that, as shown in [91], there are generally convergent algorithms if one in addition allows the operation of complex conjugation. In the following we present how the Doyle-McMullen towers can be recast in the form of a general tower as defined in Definition 4.4.

11.2. **A height 3 tower for the quartic.** In the following $X, Y, \dots$ denote variables in the polynomials while $x, y, \dots \in \mathbb{C}$. We build the tower following the standard reduction path, see e.g. [34]. Given

$$p(X) := X^4 + a_1X^3 + a_2X^2 + a_3X + a_4$$

one first transforms the equation by change of variable $Y = X + a_1/4$ to arrive into the polynomial

$$q(Y) := Y^4 + b_2Y^2 + b_3Y + b_4,$$

which one writes, with help of a parameter z, as $q(Y) = (Y^2 + z)^2 - r(Y,z)$ where

$$r(Y,z) = (2z - b_2)Y^2 - b_3Y + z^2 - b_4.$$

Here one wants a value of z such that $r(Y,z)$ becomes a square which requires the discriminant to vanish: $4(2z-b_2)(z^2-b_4)-b_3^2 = 0$. Viewing this as polynomial in Z, making a change of variable $W = Z+(1/6)b_2$ and scaling the polynomial to monic we arrive at asking for a root of

$$s(W) := W^3 + c_2W + c_3. \tag{11.1}$$

[3]I.e. it's a rational map of the coefficients of p.

As all these are rational computations on the coefficients of p, we shall not express them explicitly.

We denote by $N(f,\xi_0)$ the function in Newton's iteration with initial value ξ_0:

$$\xi_{j+1} := N(f(\xi_j)) \text{ where } N(f(\xi)) = \xi - \frac{f(\xi)}{f'(\xi)}$$

and further by N_j the mapping from initial data to the j^{th} iterate $N_j : (f,\xi_0) \mapsto \xi_j$. We shall apply Newton's iteration to the rational function [36]

$$t(W) := \frac{s(W)}{3c_2W^2 + 9c_3W - c_2^2}.$$

Thus $w_j = N_j(t, w_0)$ denotes the j^{th} iterate w_j for a zero for $s(w) = 0$. This iteration converges in an open dense set of initial data. Denote $w_\infty := \lim_{j\to\infty} w_j$. Now we change the variable $Z = W - (1/6)b_2$ and, denoting by z_j and z_∞ the corresponding values, we obtain $r(Y, z_\infty)$ as a square:

$$r(Y, z_\infty) = (2z_\infty - b_2)\left(Y - \frac{b_3}{2(2z_\infty - b_2)}\right)^2.$$

To find a zero of $q(Y)$ we shall need to have a generally convergent iteration for $\sqrt{2z - b_2}$. Thus, we set $u_j(V) := V^2 + b_2 - 2z_j$ and apply Newton's method for this, starting with initial guess v_0 and iterating k times and set $v_{k,j} := N_k(u_j, v_0)$. From $q(Y) = (Y^2 + z_\infty)^2 - r(Y, z_\infty) = 0$ we move to solve one of the factors

$$Q(Y) = Y^2 + z_\infty - \sqrt{2z_\infty - b_2}\left(Y - \frac{b_3}{2(2z_\infty - b_2)}\right) = 0.$$

However, we can do this only based on approximative values for the parameters, so we set

$$Q_{k,j}(Y) = Y^2 + z_j - v_{k,j}\left(Y - \frac{b_3}{2(2z_j - b_2)}\right) = 0.$$

Now apply Newton's iteration to this, say n times, using starting value y_0 and denote the output by $y_{n,k,j}$:

$$y_{n,k,j} = N_n(Q_{k,j}, y_0).$$

Finally, we set $x_{n,k,j} = y_{n,k,j} - a_1/4$ in order to get an approximation to a root of p. Suppose now $j = n_1, k = n_2, n = n_3$. If $n_1 \to \infty$ then $w_{n_1} \to w_\infty$ and hence $z_{n_1} \to z_\infty$, too. It is natural to denote $u(V) := V^2 + b_2 - 2z_\infty$ and correspondingly $v_{n_2} := N_{n_2}(u, v_0)$ and

$$Q_{n_2}(Y) = Y^2 + z_\infty - v_{n_2}\left(Y - \frac{b_3}{2(2z_\infty - b_2)}\right) = 0.$$

Then in an obvious manner $x_{n_3,n_2} = N_{n_3}(Q_{n_2}, y_0) - a_1/4$. Then we have $\lim_{n_1\to\infty} x_{n_3,n_2,n_1} = x_{n_3,n_2}$. If we denote $x_{n_3} = N_{n_3}(Q, y_0) - a_1/4$, then clearly $\lim_{n_2\to\infty} x_{n_3,n_2} = x_{n_3}$. Finally $x_\infty = \lim_{n_3\to\infty} x_{n_3}$ is a root of p.

The link to the SCI. One special feature of these towers which are build on generally convergent algorithms is the following: in addition to the polynomial p, the initial values for the iterations have to be read into the process via evaluation functions. Denoting the initial values for the three different Newton's iterations by $d_0 = (w_0, v_0, y_0) \in \mathbb{C}^3$ we can now put this Doyle-McMullen tower in the form of a general tower as defined in Definition 4.4, with the slight weakening that, for each $p \in \mathbb{P}_4$, the tower might converge only at a dense subset of initial values. In particular, set

$$\Gamma_{n_3} : \mathbb{P}_4 \times \mathbb{C}^3 \to \mathbb{C}, \text{ by } (p, d_0) \mapsto x_{n_3},$$
$$\Gamma_{n_3,n_2} : \mathbb{P}_4 \times \mathbb{C}^3 \to \mathbb{C} \text{ by } (p, d_0) \mapsto x_{n_3,n_2},$$
$$\Gamma_{n_3,n_2,n_1} : \mathbb{P}_4 \times \mathbb{C}^3 \to \mathbb{C} \text{ by } (p, d_0) \mapsto x_{n_3,n_2,n_1}.$$

Thus, if we let $\Omega = \mathbb{P}_4 \times \mathbb{C}^3$ and $\Xi, \mathcal{M}$ be as in Example 4.1 (III), and complement Λ by the mappings that read w_0, v_0, y_0 from the input, then by the construction above and Theorem 11.2 we have that

$$\mathrm{SCI}(\Xi, \Omega)_{\mathrm{DM}} \in \{2, 3\}.$$

11.3. **A height 3 tower for the quintic.** Let

$$p(X) = X^5 + a_1X^4 + a_2X^3 + a_3X^2 + a_4X + a_5$$

be the given quintic. Doyle and McMullen [36] give a generally convergent algorithm for the quintic in Brioschi form. Thus, one needs first to bring the general quintic to Brioschi form, then apply the iteration and finally construct at least one root for $p(X)$. In the following we outline a path for doing this, which follows L. Kiepert [70] except that the Brioschi quintic is solved by Doyle-McMullen iteration rather than by using Jacobi sextic. This path can be found in [71].

One begins applying a Tschirnhaus transformation $Y = X^2 - uX + v$ to arrive into *principal* form

$$q(Y) = Y^5 + b_3Y^2 + b_4Y + b_5.$$

Here v is obtained from a linear equation but to solve u one needs to solve a quadratic equation $Q(U) = U^2 + \alpha U + \beta$, where the coefficients α, β are rational expressions of the coefficients of $p(X)$, (see for example p. 100, eq. (6.2-9) in [71]).

Here is the first application of Newton's method. We are given an initial value u_0 and iterate j times $u_j = N_j(Q, u_0)$. We may assume that v is known exactly but we only have an approximation u_j to make the transformation. So, suppose the Newton iteration converges to u_∞. Thus, we make the transformation using u_j and *force* the coefficients $b_{2,j} = b_{1,j} = 0$ while keep the others as they appear. The transformation being continuous yields polynomials

$$q_j(Y) = Y^5 + b_{3,j}Y^2 + b_{4,j}Y + b_{5,j},$$

whose roots shall converge to those of $q(Y)$. The next step is to transform $q_j(Y)$ into Brioschi form. Let the Brioschi form corresponding to the exact polynomial $q(Y)$ be denoted by $B(Z)$

$$B(Z) = Z^5 - 10CZ^3 + 45C^2Z - C^2 = 0, \tag{11.2}$$

while with $B_j(Z)$ we denote the exact Brioschi form corresponding to $q_j(Y)$. The transformation from $q(Y)$ to $B(Z)$ is of the form

$$Y = \frac{\lambda + \mu Z}{(Z^2/C) - 3}. \tag{11.3}$$

Here λ satisfies a quadratic equation with coefficients being polynomials of the coefficients in the principal form (p. 107, eq. (6.3-28) in [71]). Let us denote that quadratic by $R(L)$ when it comes from $q(Y)$ and by $R_j(L)$ when it comes from $q_j(Y)$ respectively. Thus here we meet our second application of Newton's method. So, we denote by

$$\lambda_{k,j} := N_k(R_j, \lambda_0)$$

the output of iterating k times for a solution of $R_j(L) = 0$. And, in a natural manner, we denote also

$$\lambda_k = N_k(R, \lambda_0) \quad \text{and} \quad \lambda = \lim_{k\to\infty} N_k(R, \lambda_0).$$

The corresponding values of $\mu_{k,j}, \mu_k$ and μ are then obtained by simple substitution (p. 107, eq. (6.3-30) in [71]). The Tschirnhaus transformation with exact values (λ, μ) transforms the equation not yet to the Brioschi form with just one parameter C but such that the constant term may be different. However, the last step is just a simple scaling and then one is in the Brioschi form (11.2). However, when we apply the transformation with the approximated values $(\lambda_{k,j}, \mu_{k,j})$ or with (λ_k, μ_k) we do not arrive at the Brioschi form. So, we *force* the coefficients of the fourth and second powers to vanish and replace the coefficients of the first power to match with the coefficients in the third power. Finally, after scaling the constant terms we have the Brioschi quintics $B_{k,j}$ and B_k, e.g.

$$B_{k,j}(Z) = Z^5 - 10C_{k,j}Z^3 + 45C_{k,j}^2Z - C_{k,j}^2 = 0. \tag{11.4}$$

Provided that the Newton iterations converge, that is, the initial values (u_0, λ_0) are generic, these quintics converge to the exact one.

Here we apply the generally convergent iteration by Doyle and McMullen [36]. They specify a rational function

$$T_C(Z) = z - 12\frac{g_C(Z)}{g'_C(Z)}$$

where g is a polynomial of degree 6 in the variable C and of degree 12 in Z. Starting from an initial guess w_o from an initial guess $w_{n+1} = T_C(T_C(w_n))$ to convergence and applying T_C still once, we obtain, after a finite rational computation with these two numbers, two roots of the Brioschi, say z_I and z_{II}. If applied to the approximative quintics and if the iteration is truncated after n steps, together with the corresponding postprocessing, we have obtained e.g. a pair $(z_{I,n,k,j}, z_{II,n,k,j})$.

What remains is to invert the Tschirnhaus transformations. Suppose z is a root of the exact Brioschi form (11.2). Then the corresponding root of the principal quintic is obtained immediately from (11.3)

$$ty = \frac{\lambda + \mu z}{(z^2/C) - 3}.$$

Naturally, we can only apply this using approximated values for the parameters. Finally, one needs to transform the (approximative roots) of the principal quintic to (approximative) roots for the original general quintic $p(X)$. This is done by a rational function $X = r(Y)$ where $r(Y)$ is of second order in Y and the coefficients are polynomials of the coefficients if the original $p(X)$ and u and v (p. 127, eq. (6.8-3) in [71]). Again, we would be using only approximative values u_j in place of the exact u. In any case, at the end we obtain a pair of approximations to the roots of the original quintic. If we put $n_1 = j, n_2 = k$ and $n_3 = n$, then this pair could be denoted by $(x_{I,n_3,n_2,n_1}, x_{II,n_3,n_2,n_1})$.

The link to the SCI. In the same way as with the quartic, we assume that the initial value $d_0 = (u_0, \lambda_0, w_0) \in \mathbb{C}^3$ is generic, so that all iterations converge for large enough values and since the transformations are continuous functions of the parameters in it, all necessary limits exist and match with each others. The functions Γ_{n_3,n_2,n_1} can then be identified in a natural manner:

$$\begin{aligned}
\Gamma_{n_3} &: \mathbb{P}_5 \times \mathbb{C}^3 \to \mathbb{C}^2, \text{ by } (p, d_0) \mapsto (x_{I,n_3}, x_{II,n_3}),\\
\Gamma_{n_3,n_2} &: \mathbb{P}_5 \times \mathbb{C}^3 \to \mathbb{C}^2 \text{ by } (p, d_0) \mapsto (x_{I,n_3,n_2}, x_{II,n_3,n_2}),\\
\Gamma_{n_3,n_2,n_1} &: \mathbb{P}_5 \times \mathbb{C}^3 \to \mathbb{C}^2 \text{ by } (p, d_0) \mapsto (x_{I,n_3,n_2,n_1}, x_{II,n_3,n_2,n_1}),
\end{aligned}$$

where $(x_{I,n_3,n_2}, x_{II,n_3,n_2})$ and (x_{I,n_3}, x_{II,n_3}) are the limits as $n_1 \to \infty$ and $n_2 \to \infty$ respectively. These limits exist for initial values in an open dense subset of $\mathbb{C}^3$. Hence, we let $\Omega = \mathbb{P}_5 \times \mathbb{C}^3$, and $\Xi, \mathcal{M}, \Lambda$ be as in case of the quartic. Then, by the construction above and Theorem 11.2 we have, again in a slightly weakened sense, that

$$\mathrm{SCI}(\Xi, \Omega)_{\mathrm{DM}} \in \{2, 3\}.$$

11.4. **Particular initial guesses and height one towers.** The special feature of the above mentioned Doyle-McMullen towers is that they address the question whether one can achieve converge to the roots of a polynomial p for (almost) arbitrary initial guesses. With slight change of perspective one might also ask the question how big the SCI gets if one applies purely iterative algorithms *after a suitable clever choice* of initial values. And indeed, the answer to that question is really satisfactory: For polynomials of arbitrary degree one can compute the whole set of roots (more precisely: approximate it in the sense of the Hausdorff distance) by a tower of height one which just consists of Newtons method.

The key tool for the choice of the initial values is the main theorem of [65]:

Theorem 11.3 (Hubbard, Schleicher and Sutherland [65])**.** *For every $d \geq 2$ there is a set S_d consisting of at most $1.11 d \log^2 d$ points in $\mathbb{C}$ with the property that for every polynomial p of degree d and every root z of p*

there is a point $s \in S_d$ such that the sequence of Newton iterates $\{s_n\}_{n\in\mathbb{N}} := \{N_p^n(s)\}_{n\in\mathbb{N}}$ converges to z. In particular, the proof is constructive, and these sets S_d can easily be computed.

A further important property of Newton's method is that, in the case of convergence, the speed is at least linear: If $z_n := N_p^n(s)$ tend to a root z of p then there exists a constant c such that $|z_n - z| \leq c/n$. Finally we have the following.

Proposition 11.4. *Let p be a polynomial of degree d, $\epsilon > 0$ and $z_n := N_p^n(s)$. If $|z_n - z_{n+1}| < \frac{\epsilon}{d}$ then there is a root z of p with $|z_n - z| < \epsilon$.*

Proof. We have $\left|\frac{p(z_n)}{p'(z_n)}\right| = |z_n - z_{n+1}| < \frac{\epsilon}{d}$, hence $|p(z_n)| < \frac{\epsilon|p'(z_n)|}{d}$. Decompose $p(x) = a\Pi_{i=1}^d(x - x_i)$, notice that $p'(x) = a\sum_{j=1}^d \Pi_{i=1,i\neq j}^d(x - x_i)$, choose j such that $|\Pi_{i=1,i\neq j}^d(z_n - x_i)|$ is maximal, and conclude that

$$|a\Pi_{i=1}^d(z_n - x_i)| = |p(z_n)| < \frac{\epsilon|p'(z_n)|}{d} \leq \epsilon|a\Pi_{i=1,i\neq j}^d(z_n - x_i)|,$$

thus $|z_n - x_j| < \epsilon$. Now $z = x_j$ is a root as asserted. □

Let p be a polynomial of degree d. For each $s \in S_d$ let s_n denote the nth Newton iterates of s, and define

$$\Gamma_n(p) := \left\{ s_n : s \in S_d, |s_n - s_{n+1}| < \frac{1}{\sqrt{n}} \right\}. \tag{11.5}$$

Then $(\Gamma_n(p))$ converges to the set $\mathcal{Z}(p)$ of all zeros of p in the Hausdorff metric. Indeed, let z be a zero of p. By Theorem 11.3 there is an initial value $s \in S_d$ such that $s_n = N_p^n(s)$ tend to z with at least linear speed, i.e.

$$|s_n - s_{n+1}| \leq |s_n - z| + |s_{n+1} - z| \leq \frac{2c}{n} < \frac{1}{\sqrt{n}}$$

for all large n, hence $s_n \in \Gamma_n(p)$ for all large n. Conversely, each $s_n \in \Gamma_n(p)$ has the property that its distance to the set $\mathcal{Z}(p)$ is less than $\epsilon = \frac{d}{\sqrt{n}}$ by Proposition 11.4.

Therefore we define $\Omega_d = \mathbb{P}_d$ to be the set of polynomials of degree d, $\mathcal{M}$ the set of finite subsets of $\mathbb{C}$ equipped with the Hausdorff metric, and $\Xi : \Omega_d \to \mathcal{M}$ be the mapping that sends $p \in \Omega_d$ to the set of its zeros. Further Λ_d shall consist of the evaluation functions that read the coefficients of the polynomial $p \in \Omega_d$, and the constant functions with the values $s \in S_d$. Note again that these values can be effectively constructed.

NOTE: ACTUALLY HAVE ERROR CONTROL

Theorem 11.5. *Consider $(\Xi, \Omega_d, \mathcal{M}, \Lambda_d)$ as above. Then the algorithms* (11.5) *define an arithmetic tower of height one for the computation of the roots of each input polynomial p with error control, thus this tower realises $\{\Xi, \Omega_d, \mathcal{M}, \Lambda_d\} \in \Sigma_1^A$. Moreover, this tower employs just Newton's Method, i.e. a purely iterative algorithm.*

12. Proof of Proposition 4.12

Here we prove Proposition 4.12. Note that (i) and (ii) immediately imply (iii). Hence we start with (i), generalise to (ii) and finish with (iv). Before we can present our main contribution in this section, we need to recall some basic definitions and results.

Remark 12.1. Functions $R : \mathbb{Z}_+^n \to \{\mathit{true},\mathit{false}\}$ (or $\{\mathit{Yes}, \mathit{No}\}$ in our previous notation) are called n-ary relations. Note that such relations can also be regarded as the characteristic functions of the respective sets $\{(x_1, \dots, x_n) \in \mathbb{Z}_+^n : R(x_1, \dots, x_n)\}$.

Definition 12.2 (Arithmetical Hierarchy)**.** A relation R is in the *Arithmetical Hierarchy* if R is computable or if there exists a computable relation S such that R can be obtained from S by some finite sequence of complementation and/or projection operations. For this, the projection of $f : \mathbb{Z}_+^n \to \{\textit{true},\textit{false}\}$ along the jth coordinate is defined as the characteristic function of the set (which is a subset of $\mathbb{Z}_+^{n-1}$)

$$\{(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) : (\exists x_j) f(x_1, \dots, x_n)\}.$$

The following equivalent conditions describe the Arithmetical Hierarchy (see [82, IV.1] and [85, 14.1-4]). In particular, let R be an n-ary relation, then the following are equivalent:

(i) R is in the Arithmetical Hierarchy.

(ii) R can be represented as

$$(x_1, \dots, x_n) \mapsto (Q_1 y_1) \cdots (Q_m y_m) S(x_1, \dots, x_n, y_1, \dots, y_m)$$

where Q_i is either $(\forall)$ or $(\exists)$ for $i = 1, \dots, m$, and S is an $(n+m)$-ary computable relation.

(iii) R can be represented as

$$(x_1, \dots, x_n) \mapsto (Q_1 y_1) \cdots (Q_k y_k) T(x_1, \dots, x_n, y_1, \dots, y_k)$$

where (Q_i) is a list of alternated quantifiers ($(\forall)$ and $(\exists)$), and T is an $(n+k)$-ary computable relation (Prenex normal form, Kuratowsky, Tarski 1931).

(iv) R is definable in First-Order Arithmetic (Gödel, 1936).

Following [72] we define the classes of Σ_n, Π_n and Δ_n relations, proceeding by induction:

Definition 12.3 (Σ_m, Π_m, Δ_m)**.** Let $m \in \mathbb{Z}_+$. We then define the following.

(i) A relation is Σ_0 and Π_0 if it is computable.

(ii) A relation is Σ_{m+1} if it can be expressed in the form $(\exists y)S(x, y)$, where $S(x, y)$ is Π_m.

(iii) A relation R is Π_{m+1} if its complementary relation $\neg R$ is Σ_{m+1}.

(iv) $\Delta_m := \Sigma_m \cap \Pi_m$.

It is easily seen that a relation $R(x)$ is Σ_m iff it has a definition of the form

$$(x_1, \dots, x_n) \mapsto (\exists y_1)(\forall y_2) \cdots S(x_1, \dots, x_n, y_1, \dots, y_m),$$

where $S(x, y)$ is computable and there are m alternating quantifiers starting with $\exists$. An analogous observation holds for Π_m relations, with m alternating quantifiers starting with $\forall$. This hierarchy is the *Arithmetical Hierarchy*, or *Kleene-Mostowski Hierarchy* [82], and does not collapse. More precisely, we have the following [82, IV.1.13]:

Theorem 12.4 (Hierarchy theorem)**.** *For any $m \in \mathbb{N}$, we have the following:*

(i) $\Sigma_m \setminus \Pi_m \neq \emptyset$*, hence* $\Delta_m \subsetneq \Sigma_m$.

(ii) $\Pi_m \setminus \Sigma_m \neq \emptyset$*, hence* $\Delta_m \subsetneq \Pi_m$.

(iii) $\Sigma_m \cup \Pi_m \subsetneq \Delta_{m+1}$.

(iv) $R \in \bigcup_{n \in \mathbb{Z}_+} \Sigma_n$ *if and only if R is in the Arithmetical Hierarchy.*

It is the following result that builds the bridge between the SCI and the Arithmetical Hierarchy:

Theorem 12.5 (Shoenfield 1959, [82] (IV.1.19))**.** *For $m \in \mathbb{N}$ a function $f : \mathbb{Z}_+ \to \{\textit{true},\textit{false}\}$ is Δ_{m+1} if and only if there is a computable function $g : \mathbb{Z}_+^{m+1} \to \{\textit{true},\textit{false}\}$ such that*

$$f(y) = \lim_{x_1 \to \infty} \cdots \lim_{x_m \to \infty} g(y, x_1, \dots, x_m).$$

Using this, it is straightforward to prove the following proposition.

Proposition 12.6. *We have the following characterisation of the arithmetical hierarchy in terms of limiting recursion:*

(1) A set $A \subset \mathbb{N}$ lies in Δ_{m+1} if and only if there is a computable function $g : \mathbb{Z}_+^{m+1} \to \{0,1\}$ such that

$$\chi_A(y) = \lim_{x_1\to\infty} \cdots \lim_{x_m\to\infty} g(y, x_1, \ldots, x_m).$$

(2) A set $A \subset \mathbb{N}$ lies in Σ_m if and only if there is a computable function $g : \mathbb{Z}_+^{m+1} \to \{0,1\}$ such that

$$\chi_A(y) = \lim_{x_1\to\infty} \cdots \lim_{x_m\to\infty} g(y, x_1, \ldots, x_m),$$

with the additional property that all successive limits are alternating monotonic from above or below with the final limit from below.

(3) A set $A \subset \mathbb{N}$ lies in Π_m if and only if there is a computable function $g : \mathbb{Z}_+^{m+1} \to \{0,1\}$ such that

$$\chi_A(y) = \lim_{x_1\to\infty} \cdots \lim_{x_m\to\infty} g(y, x_1, \ldots, x_m),$$

with the additional property that all successive limits are alternating monotonic from above or below with the final limit from above.

(4) In parts (2) and (3) we can drop the alternating monotonic limits and only require the final limit is from below for Σ_m sets and from above for Π_m sets.

Proof. Part (1) is given in Theorem 12.5 and (3) follows from (2) by considering complements. We will prove Part (2) (and (3)) by induction.

If $m = 1$ then Σ_1 is the class of recursively enumerable sets and if $A \in \Sigma_1$ then there exists a recursive function $f \in \mathbb{N} \to \mathbb{N}$ listing all the elements of A at least once. Define the function $g(y, x_1) = 1$ if $y \in \{f(1), ..., f(x_1)\}$ and $g(y, x_1) = 0$ otherwise. Conversely suppose there is a computable function $g : \mathbb{Z}_+^2 \to \{0,1\}$ such that

$$\chi_A(y) = \lim_{x_1\to\infty} g(y, x_1),$$

with the additional property that

$$g(y, x_1) \le \chi_A(y), \quad \forall x_1.$$

Now set up a Turing machine which on input y successively computes $g(y, x_1)$ for $x_1 = 1, 2, ...$ and halts precisely when one of these is 1.

For $m > 1$, we assume that (2) and (3) hold for Σ_k, Π_k with $k < m$. If $A \in \Sigma_m$ then there exists $S(x, y) \in \Pi_{m-1}$ with

$$x \in A \text{ iff } \exists y S(x, y).$$

But by the inductive hypothesis there must exist a computable function $g_0 : \mathbb{Z}_+^m \to \{0,1\}$ such that

$$S(x, y) = \lim_{x_1\to\infty} \cdots \lim_{x_{m-1}\to\infty} g_0(x, y, x_1, \ldots, x_{m-1}),$$

with the required alternating monotonic successive limits. Define

$$g(x, x_1, ..., x_m) = \max_{1\le k\le x_1} g_0(x, k, x_2, \ldots, x_m).$$

Then we must have

$$\lim_{x_2\to\infty} \cdots \lim_{x_{m-1}\to\infty} g(x, x_1, ..., x_m) = \max_{1\le k\le x_1} S(x, k)$$

and it is clear that g satisfies the conditions of the Proposition. Conversely if χ_A has the characterisation described in Part (2) with the function $g(y, x_1, \ldots, x_m)$ then by the inductive hypothesis

$$S(x, y) = \lim_{x_2\to\infty} \cdots \lim_{x_m\to\infty} g(x, y \ldots, x_m)$$

lies in Π_{m-1}. But then we also have

$$x \in A \text{ iff } \exists y S(x, y)$$

and hence $A \in \Sigma_m$. Part (3) for m now also follows by taking complements and replacing g by $1 - g$.

For part (4), suppose that we have a computable function $g : \mathbb{Z}_+^{m+1} \to \{0,1\}$ such that

$$\chi_A(y) = \lim_{x_1\to\infty} \cdots \lim_{x_m\to\infty} g(y, x_1, \ldots, x_m),$$

with the final limit monotonic from below. Then

$$S(y,x_1) := \lim_{x_2\to\infty}\cdots\lim_{x_m\to\infty} g(y,x_1,\dots,x_m)$$

lies in Δ_m. Hence $S(y,x_1)$ lies in Σ_m and there exists a computable function $h:\mathbb{Z}_+^{m+2}\to\{0,1\}$ such that

$$S(y,x_1) = \lim_{x_2\to\infty}\cdots\lim_{x_{m+1}\to\infty} h(y,x_1,\dots,x_{m+1}),$$

but now successive limits are alternating from above and below with the final limit from below. Choose any recursive pairing function $\phi:\mathbb{N}^2\to\mathbb{N}$ and set

$$\tilde{g}(y,x_1,...,x_m) = \max_{1\leq k\leq x_1} h(y,\phi(x_1)_1,\phi(x_1)_2,x_2,...,x_m).$$

It is then clear that $\tilde{g}$ satisfies the requirements of Part (2) and hence $A\in\Sigma_m$. Again by considering compliments and $1-g$ we see that Part (4) also holds for Π_m sets. □

Given a subset $A\subset\mathbb{Z}_+$ with characteristic function χ_A being definable in First-Order Arithmetic, we are interested in the SCI of deciding whether a given number $x\in\mathbb{Z}_+$ belongs to A or not. In other words, we want to determine the value of the characteristic function of A at the point x. Thus, we want to consider Towers of Algorithms for χ_A where the functions/relations at the lowest level shall be computable, and we again ask for the minimal height. More precisely, we consider

- the primary set $\Omega := \mathbb{Z}_+$,
- the evaluation set $\Lambda = \{\lambda\}$ consisting of the function $\lambda:\mathbb{Z}_+\to\mathbb{C}$, $x\mapsto x$,
- the metric space $\mathcal{M} := (\{\mathit{true},\mathit{false}\}, d_{discr}) = (\{\mathit{Yes},\mathit{No}\}, d_{discr})$,

where d_{discr} denotes the discrete metric, and consider all functions $\Xi:\Omega\to\mathcal{M}$ in the Arithmetical Hierarchy. In honour of Kleene and Shoenfield we call a Tower of Algorithms that is computable a Kleene-Shoenfield tower.

Definition 12.7 (Kleene-Shoenfield tower)**.** A tower of algorithms given by a family $\{\Gamma_{n_k,\dots,n_1}:\Omega\to\mathcal{M}: n_k,\dots,n_1\in\mathbb{N}\}$ of functions at the lowest level is said to be a *Kleene-Shoenfield tower*, if the function

$$\mathbb{N}^k\times\Omega\to\mathcal{M},\quad (n_k,\dots,n_1,x)\mapsto\Gamma_{n_k,\dots,n_1}(x)$$

is computable. Given a computational problem $\{\Xi,\Omega,\mathcal{M},\Lambda\}$ as above, we will write $\mathrm{SCI}(\Xi,\Omega,\mathcal{M},\Lambda)_{\mathrm{KS}}$ to denote the SCI with respect to a Kleene-Shoenfield tower.

Given this set up, the following is an immediate consequence of Proposition 12.6 and shows that the Arithmetical Hierarchy is a particular case of the SCI hierarchy, proving part (i) of Proposition 4.12.

Proposition 12.8 (The SCI hierarchy vs the arithmetical hierarchy)**.** *For every $m\in\mathbb{N}$ we have*

$$\Xi\in\Delta_m \Leftrightarrow \{\Xi,\Omega\}\in\Delta_m^{\mathrm{KS}},$$
$$\Xi\in\Sigma_m \Leftrightarrow \{\Xi,\Omega\}\in\Sigma_m^{\mathrm{KS}},$$
$$\Xi\in\Pi_m \Leftrightarrow \{\Xi,\Omega\}\in\Pi_m^{\mathrm{KS}}.$$

This has an immediate corollary from Theorem 12.4 that shows how the SCI can become arbitrarily large. In particular, for any $k\in\mathbb{N}$ there exists a problem that has SCI equal to k.

Corollary 12.9 (The SCI can become arbitrarily large)**.** *For every $k\in\mathbb{N}$ there exists a problem function Ξ on Ω with $\mathrm{SCI}(\Xi,\Omega)_{\mathrm{KS}} = k$.*

12.1. **Generalisation to Arbitrary Decision Problems.** Next, we want to extend this to more general decision problems and more general types of towers of algorithms. The following can be viewed as a generalisation of the arithmetical hierarchy to a given Ω, a given evaluation set Λ and $\mathcal{M} = \{\textit{true}, \textit{false}\} = \{1, 0\}$. Inspired by the observations on the Arithmetical Hierarchy, we make the following two definitions:

Definition 12.10 (Alternating quantifier forms)**.** Given the general setup above we define the following:

(i) We say that $\Xi : \Omega \to \mathcal{M}$ permits a representation by an alternating quantifier form of length m if

$$\Xi = (Q_m n_m) \cdots (Q_1 n_1) \Gamma_{n_m,\ldots,n_1},$$

where (Q_i) is a list of alternating quantifiers $(\forall)$ and $(\exists)$, and all $\Gamma_{n_m,\ldots,n_1} : \Omega \to \mathcal{M}$ are general algorithms in the sense of Definition 4.3.

(ii) We say that $\{\Xi, \Omega\}$ is Σ_m if an alternating quantifier form of length m exists with Q_m being $(\exists)$, and that $\{\Xi, \Omega\}$ is Π_m if an alternating quantifier form of length m exists with Q_m being $(\forall)$.

(iii) We say that $\{\Xi, \Omega\}$ is Δ_m if $\{\Xi, \Omega\}$ is Σ_m and Π_m.

Definition 12.11 (Limit forms)**.** Given the general setup above we define the following with respect to a given type of tower of algorithms (arithmetical, radical general etc.):

(i) We say that $\{\Xi, \Omega\}$ is $\tilde{\Sigma}_m$ if there exists a height m tower solving the computational problem such that the final limit is monotonic from below. We say that $\{\Xi, \Omega\}$ is $\tilde{\Pi}_m$ if there exists a height m tower solving the computational problem such that the final limit is monotonic from above.

(ii) We say that $\{\Xi, \Omega\}$ is $\tilde{\Delta}_{m+1}$ if there exists a height m tower solving the computational problem.

The following theorem demonstrates how the SCI framework can be viewed in this special case as a generalization of the Arithmetical Hierarchy to arbitrary computational problems. In particular, one can define a hierarchy for any kind of tower. Here we do this for a general tower, and obviously, this can be done for any tower. We will call the hierarchy described below a *General Hierarchy*.

Theorem 12.12 (General Hierarchy)**.** *Following Definitions 4.10and 12.11, for any $m \geq 1$ we have that*

$$\tilde{\Sigma}_m = \Sigma_m, \quad \tilde{\Pi}_m = \Pi_m \textit{ and } \tilde{\Delta}_m = \Delta_m.$$

Proof of Theorem 12.12. **Step I:** We show that if $\mathrm{SCI}(\Xi, \Omega)_{\mathrm{G}} \leq m$ then Ξ is Δ_{m+1}. Let $p = \lim_i p_i$. Then

$$p = \textit{true} \quad \Leftrightarrow \quad \forall n \exists k (k \geq n \wedge p_k) \quad \Leftrightarrow \quad \exists n \forall k (k \leq n \vee p_k).$$

Further, let $\varphi : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, $k \mapsto (\varphi_1(k), \varphi_2(k))$ be a bijection which enumerates all pairs of natural numbers, and note that

$$\exists n \exists m (p_{n,m}) \Leftrightarrow \exists k (p(\varphi_1(k), \varphi_2(k))), \qquad \forall n \forall m (p_{n,m}) \Leftrightarrow \forall k (p(\varphi_1(k), \varphi_2(k))),$$

for any family $(p_{n,m})_{n,m \in \mathbb{N}} \subset \mathcal{M}$. Thus, every limit in a tower of height m can be converted alternately into an expression with two quantifiers ($\forall\exists$ or $\exists\forall$), and then $m-1$ doubles $\exists\exists$ or $\forall\forall$ can be replaced by single quantifiers. This easily gives the claim.

Step II: We show that if Ξ is Σ_m or Π_m then $\mathrm{SCI}(\Xi, \Omega)_{\mathrm{G}} \leq m$. In fact we show that $\Sigma_m \subset \tilde{\Sigma}_m$ and $\Pi_m \subset \tilde{\Pi}_m$. As a start let $(p_i) \subset \mathcal{M}$ be a sequence. Then

$$(\forall i (p_i)) = \textit{true} \quad \Leftrightarrow \quad \left(\lim_{n \to \infty} \bigwedge_{i=1}^{n} p_i \right) = \textit{true}, \qquad (\exists i (p_i)) = \textit{true} \quad \Leftrightarrow \quad \left(\lim_{n \to \infty} \bigvee_{i=1}^{n} p_i \right) = \textit{true}.$$

Furthermore, the conjunction (disjunction) of limits coincides with the limit of the elementwise conjunction (disjunction), hence

$$\forall n_m \exists n_{m-1} \cdots \forall n_1 \Gamma_{n_m,\cdots,n_1} = \lim_{k_m} \lim_{k_{m-1}} \cdots \lim_{k_1} \bigwedge_{i_m=1}^{k_m} \bigvee_{i_{m-1}=1}^{k_{m-1}} \cdots \bigwedge_{i_1=1}^{k_1} \Gamma_{i_m,i_{m-1},\cdots,i_1}$$

and similarly for any other possible alternating quantifier form. Since the $\Gamma_{n_m,\cdots,n_1}$ in the alternating quantifier form at the left hand side are General algorithms, the right hand side obviously yields a tower of algorithms of height m. Moreover, we obtain the required monotonic final limits.

Step III: We show that $\tilde{\Delta}_m = \Delta_m$. Let $m \in \mathbb{N}$ be the smallest number with Ξ being Δ_{m+1}. In the above steps we have already seen that $m \leq \mathrm{SCI}(\Xi, \Omega)_{\mathrm{G}} \leq m+1$, and we next prove the following: If

$$\Xi(y) = \exists i \forall j (g_0(i,j,y)) = \forall n \exists m (g_1(n,m,y))$$

then $\Xi(y) = \lim_{k\to\infty} g(k,y)$ with a function g being easily derivable from g_0, g_1. The following construction is adopted from [51, Proofs of Theorems 1 and 3]. Fix y and define a function $h_0 : \mathbb{N} \to \mathcal{M}$ recursively as follows:

$i(1) := 1$, $j(1) := 1$, $h_0(1) := g_0(i(1), j(1), y)$.
If $h_0(l) = \texttt{true}$
then: $i(l+1) := i(l)$, $j(l+1) := j(l) + 1$
else: $i(l+1) := i(l) + 1$, $j(l+1) := 1$.
$l := l + 1$.
$h_0(l) := g_0(i(l), j(l), y)$.

We observe that, if $\Xi(y) =$ *true* then $h_0(l)$ converges as $l \to \infty$ with limit *true*. Otherwise, the limit does not exist or is *false*. The same construction applies to $\neg(\forall n \exists m (g_1(n,m,y))) = \exists n \forall m \neg(g_1(n,m,y))$ and yields a function h_1 which converges to *true* if and only if $\Xi(y) =$ *false*. Clearly, exactly one of the functions h_0, h_1 converges to *true*. Now we derive the desired g from h_0 and h_1 as follows:

$\alpha(1) = 0$.
If $h_{\alpha(k)}(k) = \texttt{true}$
then: $\alpha(k+1) := \alpha(k)$
else: $\alpha(k+1) := 1 - \alpha(k)$.
$k := k + 1$.
If $\alpha(k) = 0$
then: $g(k,y) := \texttt{true}$
else: $g(k,y) := \texttt{false}$.

This provides $\Xi(y) = \lim_{k\to\infty} g(k,y)$.

Next, let g_0 and g_1 be of the form $g_s(i,j,y) = \lim_r f^s_{i,j,r}(y)$, $s \in \{0,1\}$. Fix y. Then for every pair (i,j) there is an $r(i,j)$ such that $f^s_{u,v,r}(y) = g_s(u,v,y)$ for all $u \leq i$, $v \leq j$, $s \in \{0,1\}$ and $r \geq r(i,j)$. Thus, g is also of the form $g(k,y) = \lim_r f_{k,r}(y)$ with $f_{k,r}$ being defined by the above procedure applied to the functions $(i,j,y) \mapsto f^s_{i,j,k}(y)$ instead of $g_s(i,j,y)$ ($s \in \{0,1\}$).

Now we are left with iterating this argument: If both functions g_s ($s \in \{0,1\}$) are of the form $g_s(i,j,y) = \lim_{k_{m-1}} \lim_{k_{m-2}} \cdots \lim_{k_1} f^s_{i,j,k_{m-1},\cdots,k_1}(y)$ with certain General algorithms $f^s_{i,j,k_{m-1},\cdots,k_1}$, then also g is of the form

$$g(k,y) = \lim_{k_{m-1}} \lim_{k_{m-2}} \cdots \lim_{k_1} f_{k,k_{m-1},\cdots,k_1}(y)$$

with $f_{k,k_{m-1},\cdots,k_1}$ being defined by the same procedure as before applied to the functions $(i,j,y) \mapsto f^s_{i,j,k_{m-1},\cdots,k_1}(y)$ instead of $g_s(i,j,y)$ ($s \in \{0,1\}$). The resulting functions $y \mapsto f_{k,k_{m-1},\cdots,k_1}(y)$ are General algorithms for every k, since their evaluation requires only finitely many evaluations of the General algorithms $f^s_{i,j,k_{m-1},\cdots,k_1}$.

Step IV: It remains to show that $\tilde{\Sigma}_m \subset \Sigma_m$ and $\tilde{\Pi}_m \subset \Pi_m$. Suppose that $\Xi \in \tilde{\Sigma}_m (\in \tilde{\Pi}_m)$ then by considering the first $m-1$ limits there exists a family $\Xi_{n_m} \in \tilde{\Delta}_m = \Delta_m$ (this is also trivially true if $m = 1$) such that

$$\Xi(y) = \lim_{n_m \to \infty} \Xi_{n_m}(y)$$

with the final limit monotonic from below (above). But then we must have $\Xi(y) = \exists n_m \Xi_{n_m}(y)$ ($\Xi(y) = \forall n_m \Xi_{n_m}(y)$). But $\Xi_{n_m} \in \Sigma_m (\in \Pi_m)$ and we can collapse the double quantifier $\exists\exists$ ($\forall\forall$) to a single $\exists$($\forall$). ☐

12.2. **Intersections and Sharpness Results.** In this subsection we prove part (iv) of Proposition 4.12. Let $(\mathcal{M}, d)$ be a metric space with the Attouch-Wets or Hausdorff topology induced by another metric space $(\mathcal{M}', d_{\mathcal{M}'})$. For the Attouch-Wets topology and any fixed $x_0 \in \mathcal{M}'$ we set

$$d_{AW}(C_1, C_2) = \sum_{n=1}^{\infty} 2^{-n} \min\{1, \sup_{d_{\mathcal{M}'}(x_0,x)\leq n} |\mathrm{dist}(x, C_1) - \mathrm{dist}(x, C_2)|\},$$

for $C_1, C_2 \in \mathrm{Cl}(\mathcal{M}')$, where $\mathrm{Cl}(\mathcal{M}')$ denotes the set of closed non-empty subsets of $\mathcal{M}'$. In the case that $\mathcal{M}' = \mathbb{C}$ with the usual metric we take $x_0 = 0$. Using the notation of §4, we have the following 'sandwich' lemma.

Lemma 12.13. *Suppose we have a sequence of sets $a_n \in \mathcal{M}$ with $a_n \subset_{\mathcal{M}'} B^{\mathcal{M}}_{2^{-n}}(C)$ and another sequence of sets $b_n \in \mathcal{M}$ with $B^{\mathcal{M}}_{2^{-n}}(b_n) \supset_{\mathcal{M}'} C$, where $C \in \mathcal{M}$. Then we have*

1. *$d_H(a_n, C) \leq d_H(a_n, b_n) + 2^{-n}$ if $(\mathcal{M}, d)$ is the Hausdorff topology on the collection of non-empty compact subsets of $(\mathcal{M}', d_{\mathcal{M}'})$.*
2. *$d_{AW}(a_n, C) \leq d_{AW}(a_n, b_n) + 2^{-n}$ if $(\mathcal{M}, d)$ is the Attouch-Wets topology on the collection of non-empty closed subsets of $(\mathcal{M}', d_{\mathcal{M}'})$.*

Proof. Suppose first that we are in case 1. If $x \in C$ then there exists S_x with $x \in S_x$ and $d_H(b_n, S_x) \leq 2^{-n}$. It follows that $\mathrm{dist}(x, a_n) \leq d_H(S_x, a_n) \leq 2^{-n} + d_H(a_n, b_n)$. If $x \in a_n$ then there exists some compact set S_x with $x \in S_x$ and $d_H(C, S_x) \leq 2^{-n}$. It follows that $\mathrm{dist}(x, C) \leq d_H(S_x, C) \leq 2^{-n}$.

For case 2, suppose that $d_{\mathcal{M}'}(x_0, x) \leq m$, then it is enough to show that

$$|\mathrm{dist}(x, a_n) - \mathrm{dist}(x, C)| \leq 2^m \big(d_{AW}(a_n, b_n) + 2^{-n}\big).$$

Note that we are given

$$|\mathrm{dist}(x, a_n) - \mathrm{dist}(x, b_n)| \leq 2^m d_{AW}(a_n, b_n). \tag{12.1}$$

Let $\delta > 0$ and choose $y \in a_n$ such that $d_{\mathcal{M}'}(x, y) \leq \mathrm{dist}(x, a_n) + \delta$. There exists some closed set S_y with $y \in S_y$ and $d_{AW}(S_y, C) \leq 2^{-n}$. Hence $|\mathrm{dist}(x, S_y) - \mathrm{dist}(x, C)| \leq 2^m 2^{-n}$. Using the fact that $\mathrm{dist}(x, S_y) \leq d_{\mathcal{M}'}(x, y)$, it follows that

$$\begin{aligned}\mathrm{dist}(x, C) - \mathrm{dist}(x, a_n) &\leq 2^m 2^{-n} + \delta + \mathrm{dist}(x, S_y) - d_{\mathcal{M}'}(x, y)\\ &\leq 2^m(2^{-n} + \delta 2^{-m})\end{aligned}$$

Now let $y \in C$ such that $\mathrm{dist}(x, C) \geq d_{\mathcal{M}'}(x, y) - \delta$. There exists some closed set S_y with $y \in S_y$ and $d_{AW}(S_y, b_n) \leq 2^{-n}$. Using this and (12.1),

$$\begin{aligned}\mathrm{dist}(x, C) - \mathrm{dist}(x, a_n) &\geq d_{\mathcal{M}'}(x, y) - \delta - \mathrm{dist}(x, a_n)\\ &\geq \mathrm{dist}(x, S_y) - \mathrm{dist}(x, b_n) - \delta - 2^m d_{AW}(a_n, b_n)\\ &\geq -2^m(\delta 2^{-m} + 2^{-n} + d_{AW}(a_n, b_n)).\end{aligned}$$

Since $\delta > 0$ was arbitrary, this completes the proof. ☐

Proposition 12.14. *Let $(\mathcal{M}, d)$ be either a metric space with the Attouch-Wets or Hausdorff topology induced by another metric space $(\mathcal{M}', d_{\mathcal{M}'})$ or a totally ordered metric space with order respecting metric. Suppose we have a computational problem*

$$\Xi : \Omega \to \mathcal{M},$$

with a corresponding convergent Σ^{α}_k tower $\Gamma^1_{n_k,\ldots,n_1}$ and a corresponding convergent Π^{α}_k tower $\Gamma^2_{n_k,\ldots,n_1}$ (either both arithmetic or both general). Suppose also that $1 \leq k \leq 3$ and that, in the case of arithmetic *towers, we can compute for every $A \in \Omega$ the distance $d(\Gamma^1_{n_k,\dots,n_1}(A), \Gamma^2_{n_k,\dots,n_1}(A))$ to arbitrary precision using finitely many arithmetic operations and comparisons. Then $\{\Xi, \Omega\} \in \Delta^\alpha_k$.*

Remark 12.15. This proposition essentially says that we can combine the two notions of error control Π_k and Σ_k to reduce the number of limits needed by one.

Proof. **Step I:** For $k = 1$ and the case that $(\mathcal{M}, d)$ is either a metric space with the Attouch-Wets or Hausdorff topology, this is a trivial consequence of Lemma 12.13. Let $\epsilon > 0$ then simply choose $n_1 \in \mathbb{N}$ minimal such that $2^{-n_1} \leq \epsilon/2$ and

$$\tilde{d}(\Gamma^1_{n_1}(A), \Gamma^2_{n_1}(A)) \leq \epsilon/2,$$

where $\tilde{d}$ is an approximation of d to accuracy $1/n_1$. In the case that $(\mathcal{M}, d)$ is totally ordered with order respecting metric, this also works directly without $2^{-n_1} \leq \epsilon/2$ since

$$d(\Gamma^1_{n_1}(A), \Xi(A)) \leq d(\Gamma^1_{n_1}(A), \Gamma^2_{n_1}(A)),$$

and we can take n_1 large such that the right hand side is less than the given ϵ (recall we can compute the right hand side to arbitrary precision).

Step II: For larger k we use the same idea but we must be careful to ensure the first $k-1$ limits exist. For the rest of the proof, $\tilde{d}$ will denote an approximation of d to accuracy $1/n_1$ (which by assumption can always be computed). We first deal with the case $k = 2$. Let $\epsilon > 0$ and consider the intervals $J^1_\epsilon = [0, \epsilon]$ and $J^2_\epsilon = [2\epsilon, \infty)$. Define the quantity

$$\delta_{n_2,n_1}(A) = \tilde{d}(\Gamma^1_{n_2,n_1}(A), \Gamma^2_{n_2,n_1}(A)) + 2^{-n_2}.$$

It is clear that $\lim_{n_1\to\infty} \delta_{n_2,n_1}(A) = d(\Gamma^1_{n_2}(A), \Gamma^2_{n_2}(A)) + 2^{-n_2} =: \delta_{n_2}(A)$ and that $d(\Gamma^1_{n_2}(A), \Xi(A)) \leq \delta_{n_2}(A)$ (again appealing to Lemma 12.13 if we are in the case of the Attouch-Wets or Hausdorff topology). Given n_1, n_2, let $l(n_2, n_1) \leq n_1$ be maximal such that $\delta_{n_2,l}(A) \in J^1_\epsilon \cup J^2_\epsilon$. If no such l exists or $\delta_{n_2,l}(A) \in J^1_\epsilon$ then define $\mathrm{Osc}(\epsilon; n_1, n_2, A) = 1$ otherwise define $\mathrm{Osc}(\epsilon; n_1, n_2, A) = 0$. Since $\delta_{n_2,n_1}(A)$ cannot oscillate infinitely often between the two intervals J^1_ϵ and J^2_ϵ, it follows that

$$\mathrm{Osc}(\epsilon; n_2, A) := \lim_{n_1\to\infty} \mathrm{Osc}(\epsilon; n_1, n_2, A)$$

exists. Define $\Gamma^\epsilon_{n_1}(A)$ as follows. Choose $j \leq n_1$ minimal such that $\mathrm{Osc}(\epsilon; n_1, j, A) = 1$ if such a j exists, and define $\Gamma^\epsilon_{n_1}(A) = \Gamma_{j,n_1}(A)$. If no such j exists then define $\Gamma^\epsilon_{n_1}(A) = C_0$ where C_0 is any fixed member of $(\mathcal{M}, d)$. In particular, $\Gamma^\epsilon_{n_1}$ is a type α algorithm. Now for large n_2, we must have $\delta_{n_2}(A) < \epsilon$ and hence $\mathrm{Osc}(\epsilon; n_2, A) = 1$. It follows that $\Gamma^\epsilon(A) = \lim_{n_1\to\infty} \Gamma^\epsilon_{n_1}(A)$ exists and is equal to $\Gamma^1_N(A)$ where $N \in \mathbb{N}$ is minimal with $\mathrm{Osc}(\epsilon; N, A) = 1$. It follows that $d(\Gamma^\epsilon(A), \Xi(A)) \leq 2\epsilon$.

We will use the $\Gamma^\epsilon_{n_1}(A)$ to construct a height one tower. Observe first of all that by our assumptions we can compute $\tilde{d}(\Gamma^{\epsilon_1}_m(A), \Gamma^{\epsilon_2}_n(A))$ for $m, n \in \mathbb{N}$ and $\epsilon_1, \epsilon_2 > 0$. Given n_1, choose $j = j(n_1) \leq n_1$ maximal such that for all $1 \leq l \leq j$ we have

$$\tilde{d}(\Gamma^{2^{-j}}_{n_1}(A), \Gamma^{2^{-l}}_{n_1}(A)) \leq 4(2^{-j} + 2^{-l}). \tag{12.2}$$

If no such j exists then set $\Gamma_{n_1} = C_0$, otherwise set $\Gamma_{n_1}(A) = \Gamma^{2^{-j(n_1)}}_{n_1}(A)$. Again, this is easily seen to be a type α algorithm. Pick any $N \in \mathbb{N}$, then by the convergence of the $\Gamma^\epsilon_{n_1}(A)$ and $d(\Gamma^\epsilon(A), \Xi(A)) \leq 2\epsilon$, (12.2) must hold for $j = N$ and $1 \leq l \leq N$ if n_1 is large enough. Hence by definition of $j(n_1)$,

$$\limsup_{n_1\to\infty} d(\Gamma_{n_1}(A), \Xi(A)) \leq \limsup_{n_1\to\infty} d(\Gamma^{2^{-N}}_{n_1}(A), \Xi(A)) + 2^{3-N} \leq 2^{4-N}.$$

Since N was arbitrary we must have convergence to $\Xi(A)$.

Step III: We now deal with $k = 2$. The strategy will be similar to the $k = 1$ case but now we construct $\Gamma^\epsilon_{n_2,n_1}(A)$ such that $\Gamma^\epsilon_{n_2}(A) := \lim_{n_1\to\infty} \Gamma^\epsilon_{n_2,n_1}(A)$ exists and is 3ϵ close to $\Xi(A)$ for large n_2, but may not converge in $(\mathcal{M}, d)$. Using this, we will construct a height two α-type tower.

As in Step II, let $\epsilon > 0$ and consider the intervals $J_\epsilon^1 = [0, \epsilon]$ and $J_\epsilon^2 = [2\epsilon, \infty)$. Define the quantity

$$\delta_{n_3,n_2,n_1}(A) = \tilde{d}(\Gamma^1_{n_3,n_2,n_1}(A), \Gamma^2_{n_3,n_2,n_1}(A)) + 2^{-n_3}.$$

Again, we have $\lim_{n_2\to\infty}\lim_{n_1\to\infty}\delta_{n_3,n_2,n_1}(A) = d(\Gamma^1_{n_3}(A), \Gamma^2_{n_3}(A)) + 2^{-n_3} =: \delta_{n_3}(A)$ exists with $d(\Gamma^1_{n_3}(A), \Xi(A)) \leq \delta_{n_3}(A)$. Given n_1, n_2 and j, let $l(j, n_2, n_1) \leq n_1$ be maximal such that $\delta_{j,n_2,l}(A) \in J_\epsilon^1 \cup J_\epsilon^2$. If no such l exists or $\delta_{j,n_2,l}(A) \in J_\epsilon^1$ then define $\mathrm{Osc}(\epsilon; n_1, n_2, j, A) = 1$ otherwise define $\mathrm{Osc}(\epsilon; n_1, n_2, j, A) = 0$. Arguing as in Step I we have

$$\mathrm{Osc}(\epsilon; n_2, j, A) := \lim_{n_1\to\infty} \mathrm{Osc}(\epsilon; n_1, n_2, j, A)$$

exists. Now consider $\mathrm{Osc}(\epsilon; n_1, n_2, j, A)$ for $j \leq n_2$. If such a j exists with $\mathrm{Osc}(\epsilon; n_1, n_2, j, A) = 1$ then let $j(n_1, n_2)$ be the minimal such j and set $\Gamma^\epsilon_{n_2,n_1}(A) = \Gamma^1_{j(n_1,n_2),n_2,n_1}(A)$. Otherwise set $\Gamma^\epsilon_{n_2,n_1}(A) = C_0$, where again C_0 is some fixed member of $(\mathcal{M}, d)$. Since we only deal with finitely many $j \leq n_2$, it is clear that $\Gamma^\epsilon_{n_2,n_1}$ is a type α algorithm. Furthermore, we must have that $\Gamma^\epsilon_{n_2}(A) := \lim_{n_1\to\infty}\Gamma^\epsilon_{n_2,n_1}(A)$ exists and is defined as follows. Let $j(n_2) \leq n_2$ be minimal with $\mathrm{Osc}(\epsilon; n_2, j, A) = 1$ (if such a j exists). If such a j exists then $\Gamma^\epsilon_{n_2}(A) = \Gamma^1_{j(n_2),n_2}(A)$, otherwise $\Gamma^\epsilon_{n_2}(A) = C_0$.

Now there exists $N \in \mathbb{N}$ such that $\delta_N(A) < \epsilon/2$ and hence $\delta_{N,n_2}(A) < \epsilon$ for large n_2. But this implies that $\mathrm{Osc}(\epsilon; n_2, N, A) = 1$. Hence for n_2 large we must have $j(n_2) \leq N$. If $\delta_l(A) > 2\epsilon$ then for large n_2 we must have $\delta_{l,n_2}(A) > 2\epsilon$ and hence $\mathrm{Osc}(\epsilon; n_2, l, A) = 0$. As n_2 increases, $j(n_2)$ may not converge. However, the above arguments show that for large n_2 it can take only finitely many values, say in the set $S = \{s_1, ..., s_m\}$, all of which must have $\delta_{s_i}(A) \leq 2\epsilon$. It follows that for large n_2 we must have

$$d(\Gamma^\epsilon_{n_2}(A), \Xi(A)) \leq 3\epsilon. \tag{12.3}$$

Now we get to work using these ‘towers’ (which don’t necessarily converge in the last limit) and the trick to avoid oscillations. Define

$$F(n_1, n_2, j, l, A) := \tilde{d}(\Gamma^{2^{-j}}_{n_2,n_1}(A), \Gamma^{2^{-l}}_{n_2,n_1}(A)),$$
$$F(n_2, j, l, A) := \lim_{n_1\to\infty} F(n_1, n_2, j, l, A) = d(\Gamma^{2^{-j}}_{n_2}(A), \Gamma^{2^{-l}}_{n_2}(A))$$

and the intervals $J^1_{j,l} = [0, 4(2^{-j}+2^{-l})]$, $J^2_{j,l} = [8(2^{-j}+2^{-l}), \infty)$. Given j, l, n_1 and n_2, let $i(j, l, n_2, n_1) \leq n_1$ be maximal such that $F(i, n_2, j, l, A) \in J^1_{j,l} \cup J^2_{j,l}$. If no such i exists or if it does and $F(i, n_2, j, l, A) \in J^1_{j,l}$ then define $\widehat{\mathrm{Osc}}(n_1, n_2, j, l, A) = 1$ otherwise define $\widehat{\mathrm{Osc}}(n_1, n_2, j, l, A) = 0$. Choose $j = j(n_1, n_2) \leq n_2$ maximal such that for all $1 \leq l \leq j$ we have $\widehat{\mathrm{Osc}}(n_1, n_2, j, l, A) = 1$. If no such j exists then set $\Gamma_{n_2,n_1} = C_0$, otherwise set $\Gamma_{n_2,n_1}(A) = \Gamma^{2^{-j(n_1,n_2)}}_{n_2,n_1}(A)$. Again, this is easily seen to be a is a type α algorithm.

Arguing as before, we have the existence of

$$\widehat{\mathrm{Osc}}(n_2, j, l, A) := \lim_{n_1\to\infty} \widehat{\mathrm{Osc}}(n_1, n_2, j, l, A).$$

Now define $h = h(n_2) \leq n_2$ maximal such that for all $1 \leq l \leq h$ we have $\widehat{\mathrm{Osc}}(n_2, h, l, A) = 1$. If no such h exists then we must have

$$\Gamma_{n_2}(A) := \lim_{n_1\to\infty} \Gamma_{n_2,n_1}(A) = C_0,$$

otherwise we must have

$$\Gamma_{n_2}(A) := \lim_{n_1\to\infty} \Gamma_{n_2,n_1}(A) = \Gamma^{2^{-h(n_2)}}_{n_2}(A).$$

By (12.3), for any fixed j, l we have $\widehat{\mathrm{Osc}}(n_2, j, l, A) = 1$ for large n_2 and hence $h(n_2)$ exists for large n_2 and diverges to ∞. Now let $N \in \mathbb{N}$ then it follows that

$$\begin{aligned}\limsup_{n_2\to\infty} d(\Gamma^{2^{-h(n_2)}}_{n_2}(A), \Xi(A)) &\leq \limsup_{n_2\to\infty} d(\Gamma^{2^{-N}}_{n_2}(A), \Xi(A)) + d(\Gamma^{2^{-h(n_2)}}_{n_2}(A), \Gamma^{2^{-N}}_{n_2}(A)) \\ &\leq 3\cdot 2^{-N} + \limsup_{n_2\to\infty} 8(2^{-h(n_2)} + 2^{-N}) \leq 11\cdot 2^{-N}.\end{aligned}$$

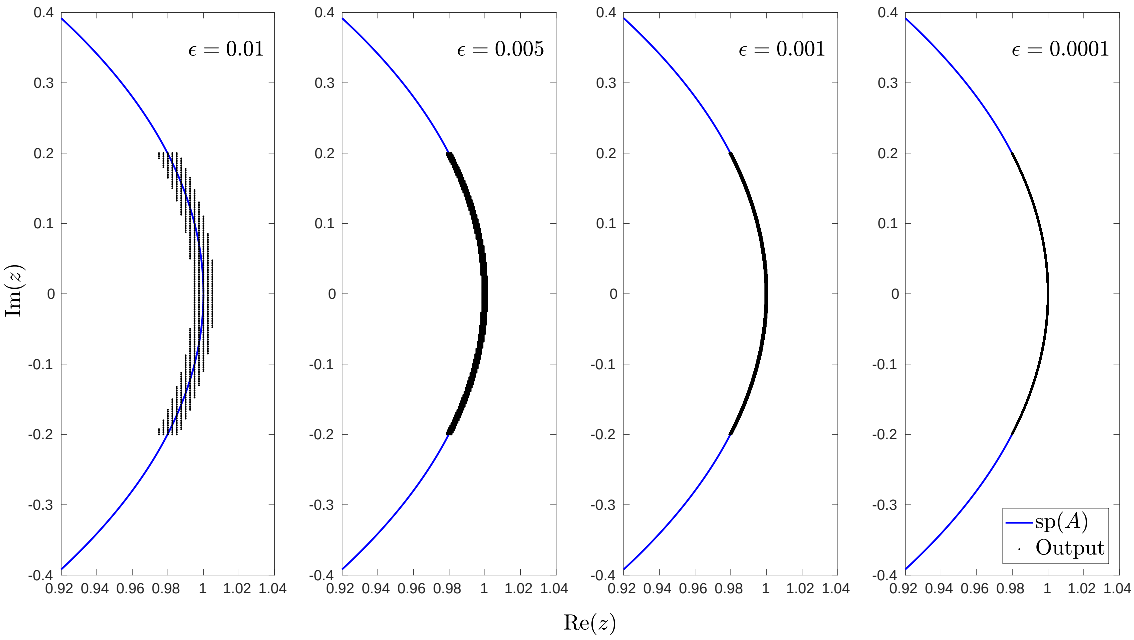

FIGURE 1. The figure shows a $\Gamma_n(A) \cap K$ (black) for a compact set $K \subset \mathbb{C}$ on top of a part of $\mathrm{sp}(A)$ (blue) for different increasing values of n corresponding to the chosen ϵ, where A is the shift operator on $l^2(\mathbb{Z})$.

Since N was arbitrary we must have convergence to $\Xi(A)$. □

Proof of Proposition 4.12 (iv). The first part follows directly from Proposition 12.14 and the following remark - no assumptions on being able to compute distances between output of algorithms is necessary. For the second part we deal with $X = \Sigma$ and the $X = \Pi$ follows from an identical argument. Suppose that $\Delta_k^G \not\ni \{\Xi, \Omega\} \in \Sigma_k^\alpha$. If $\{\Xi, \Omega\} \in \Pi_k^\alpha$, we would have $\{\Xi, \Omega\} \in \Sigma_k^\alpha \cap \Pi_k^\alpha \subset \Sigma_k^G \cap \Pi_k^G = \Delta_k^G$, a contradiction. □

13. NUMERICAL EXAMPLES

The purpose of this section is to demonstrate that the towers of algorithms developed to yield the sharp bounds on the SCI are indeed practical and yield implementable algorithms that are efficient. More detailed analysis of the practical use of these will appear elsewhere, and thus we will simply give a short demonstration of a small subset of the algorithms developed in this paper. Note that these algorithms are the first that are sharp with respect to the SCI.

13.1. **Toeplitz operators.** Toeplitz and Laurent operators are familiar test objects given that their spectra are very well understood [18, 19]. In this first example we are concerned with operators that are banded with known growth on their resolvents. In particular, the problem of computing the spectrum lies in Σ_1^A and has $\mathrm{SCI} = 1$. Since the problem does not lie in Π_1^G, we monitor the changes of $\Gamma_n(A)$ as $n \to \infty$. This is common practice in computations when error control is not available. In particular, we choose an $\epsilon > 0$ and $K \in \mathbb{N}$ and stop the iteration when

$$\max\{E_n(A), d(\Gamma_n(A), \Gamma_{n+k}(A))\} \leq \epsilon \text{ for all } k \leq K. \tag{13.1}$$

Here $E_n(A)$ refers to the error guarantee $\Gamma_n(A) \subset \mathrm{sp}(A) + B_{E_n(A)}(0)$ provided by the algorithm. To visualize the convergence we tested the tower of height one on the shift operator in Figure 1. Note that it is crucial to know the SCI of the problem so that one can apply the tower of algorithms with the correct height.

In particular, trying to solve this problem with a tower of height two would make the computation incredibly more complex. Compare for example with the experiment in §13.4.

13.2. **Graphene.** Graphene is a two-dimensional material with carbon atoms situated at the vertices of a honeycomb lattice whose unusual properties and potential applications led to a huge amount of attention in the condensed matter community, see e.g. [81]. In particular, magnetic properties of graphene have become an important research direction due to the experimental observation of the quantum Hall effect and Hofstadter's butterfly [30]. In this next example, we will demonstrate how our algorithms can compute the spectrum of a model of graphene.

Graphene in magnetic fields can be described by a one-dimensional Schrödinger operator

$$H := (-i\frac{d}{dx_e} - A_e)^2 + V_e$$

on an infinite hexagonal quantum graph where A_e is the projection of the magnetic vector potential onto edges e and V_e is a square-integrable potential that is the same for every edge [8]. In the case of constant magnetic fields it can be shown that the spectrum of the Schrödinger operator H is fully determined by an effective Hamiltonian known as the tight-binding operator. Let Φ be the total magnetic flux through one honeycomb, the tight-binding operator for the hexagonal lattice is

$$Q_\Lambda(\Phi) := \frac{1}{3}\begin{pmatrix} 0 & 1+\tau_0+\tau_1 \\ (1+\tau_0+\tau_1)^* & 0 \end{pmatrix} \in \mathcal{L}(\ell^2(\mathbb{Z}^2;\mathbb{C}^2)) \tag{13.2}$$

with translation operators $\tau_0, \tau_1 \in \mathcal{L}(l^2(\mathbb{Z}^2;\mathbb{C}))$ that for $\gamma \in \mathbb{Z}^2$ and $u \in l^2(\mathbb{Z}^2;\mathbb{C})$ are defined as

$$(\tau_0(u))_{\gamma_1,\gamma_2} := u_{\gamma_1-1,\gamma_2} \text{ and } (\tau_1(u))_{\gamma_1,\gamma_2} := e^{-i\Phi\gamma_1}u_{\gamma_1,\gamma_2-1}. \tag{13.3}$$

Using supersymmetry, one can express the spectrum of $Q_\Lambda(\Phi)$ in terms of the spectrum of a one-dimensional Jacobi operator $H_{\Phi,\theta} \in \mathcal{L}(\ell(\mathbb{Z}))$ with $c(\theta) = 1 + e^{-2\pi i\theta}$ and $v(\theta) = 2\cos(2\pi\theta)$ [8, Lemma 5.2] which is given by

$$(H_{\Phi,\theta}u)_m = c\left(\theta + m\frac{\Phi}{2\pi}\right)u_{m+1} + \overline{c\left(\theta + (m-1)\frac{\Phi}{2\pi}\right)}u_{m-1} + v\left(\theta + m\frac{\Phi}{2\pi}\right)u_m. \tag{13.4}$$

It can be shown [8, Theorem 3] that the spectrum of $Q_\Lambda(\Phi)$ is a finite union of intervals and purely absolutely continuous for $\frac{\Phi}{2\pi} = \frac{p}{q} \in \mathbb{Q}$, with the measure estimate $|\mathrm{sp}(Q_\Lambda(\Phi))| \leq \frac{C}{\sqrt{q}}$. However, if $\frac{\Phi}{2\pi} \in \mathbb{R}\setminus\mathbb{Q}$, then the spectrum is singular continuous and a zero measure Cantor set with Hausdorff dimension $\dim_H(\mathrm{sp}(Q_\Lambda(\Phi))) \leq 1/2$ for generic Φ.

In Figure 2 we display the computation of $\mathrm{sp}(Q_\Lambda(\Phi))$ for $\theta = 0$ and various $\Phi/(2\pi)$ with the stopping criteria (13.1) and error bound $\epsilon = 5 \times 10^{-3}$ (the spectrum of $H_{\Phi,\theta}$ was computed to an accuracy of order 10^{-5}). We have also shown the corresponding result when one attempts to compute $\mathrm{sp}(Q_\Lambda(\Phi))$ via $\mathrm{sp}(Q_\Lambda(H_{\Phi,\theta}))$ with finite section (square truncations of the matrix). Finite section produces heavy spectral pollution. Note that just because the SCI $= 1$ of a spectral problem (even in the self-adjoint case) does not mean that the finite section method converges.Figure 3 shows a finite portion of the corresponding spectrum of the full operator H.

13.3. **Schrödinger operator with quasi-periodic potential.** We now test the algorithm that computes spectra of Schrödinger acting on $\mathrm{W}^{2,2}(\mathbb{R})$ (i.e. a continuum model) with bounded potential. Recall that our algorithm uses only evaluations of the potential itself. This problem lies in Σ_1^A and has SCI $= 1$, hence we shall use the stopping criterion in (13.1). We chose the class of potentials

$$V_\lambda(x) = \cos(2\pi x) - \sin(\lambda 2^{3/2}x), \tag{13.5}$$

which are not periodic when $\lambda \in \mathbb{Q}$.

This type of potential is known as quasi-periodic and there is a vast literature, in both the mathematics and physics community, concerning the interesting spectral properties of these types of operators. This covers

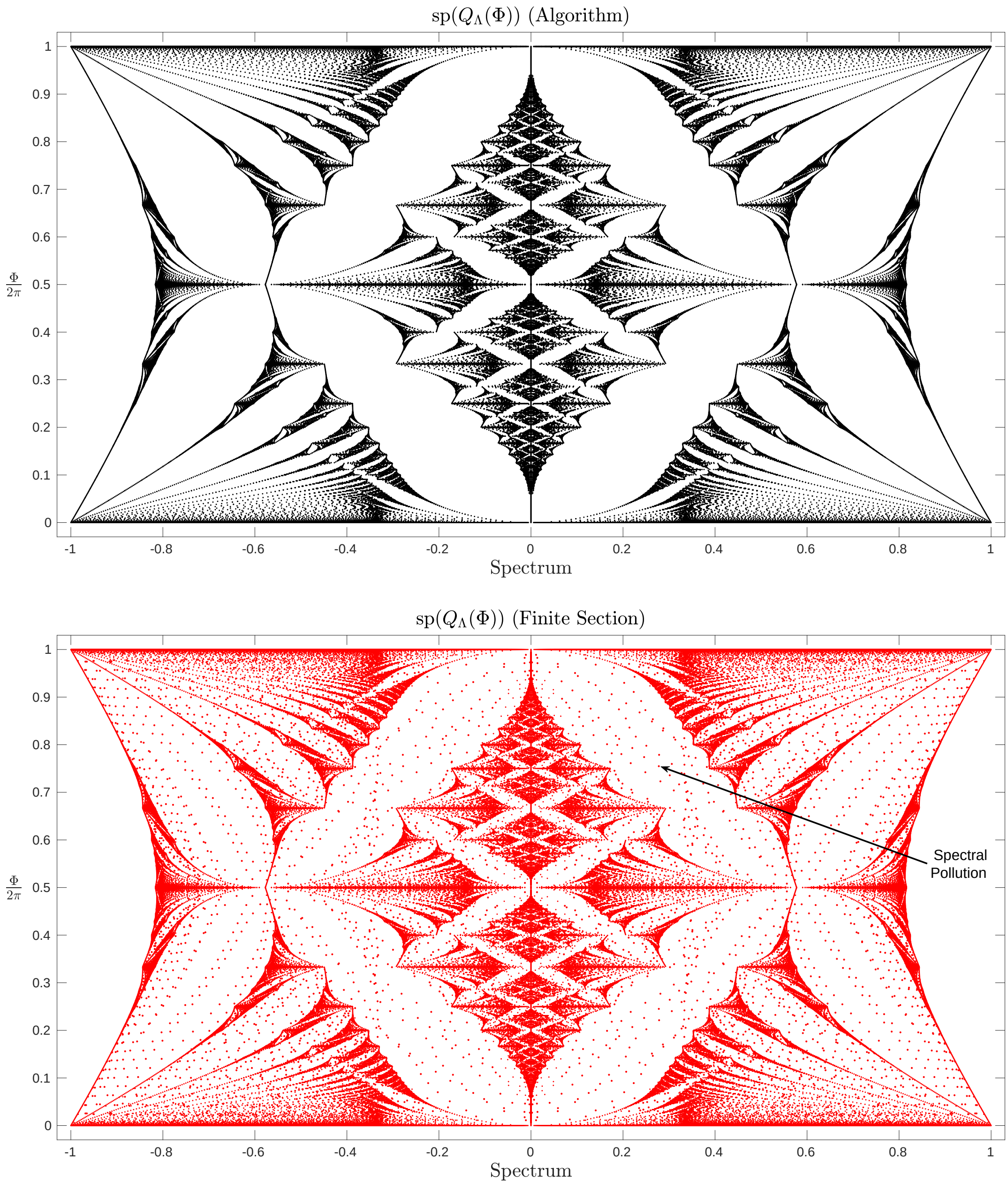

FIGURE 2. Computation of spectrum of $Q_\Lambda(\Phi)$ (model of graphene). Top: output of new algorithm, with guaranteed error bound of 10^{-5}. Bottom: output of finite section which does not converge and suffers from spectral pollution.

the discrete case[4] and, more recently, the continuum case [26, 37, 48, 97, 99]. Quasi-periodic operators arise naturally for crystals that are either inherently quasi-periodic or when the structure of the crystal is periodic and an external quasi-periodicity is enforced, e.g. by electromagnetic fields or impurities. In a wider sense, quasi-periodic operators are also studied to understand properties of random Schrödinger operators with ergodicity being the link between the two families of operators [25]. In Figure 4 we display the computation of a portion of $\Gamma_n(V_\lambda)$ (for $z \leq 10$) for various choice of λ and the error bound $\epsilon = 0.05$.

[4]The most famous is the almost Mathieu operator [6,66] which is an operator given considerable amount of attention in the literature, in particular, in connection with the Ten Martini Problem.

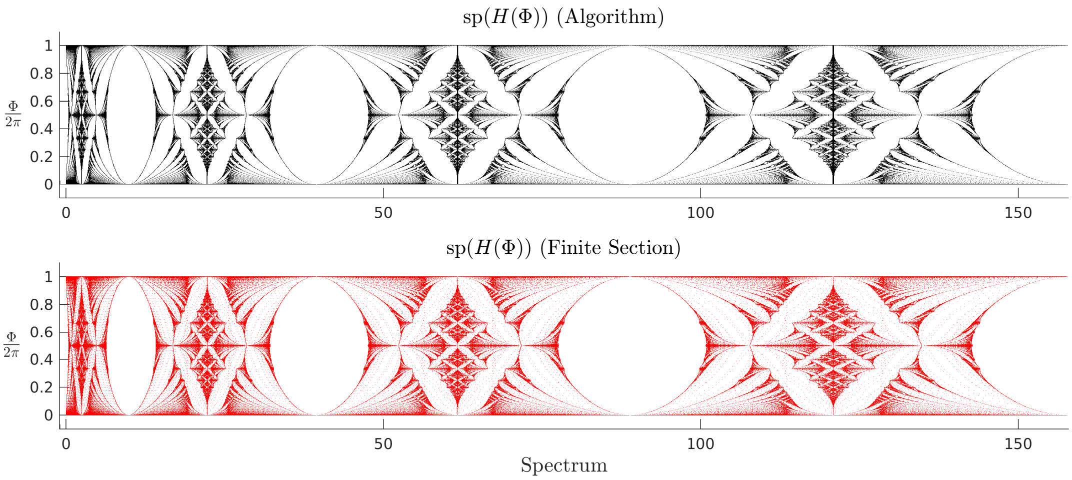

FIGURE 3. Portion of full spectrum of graphene computed using the results of Figure 2.

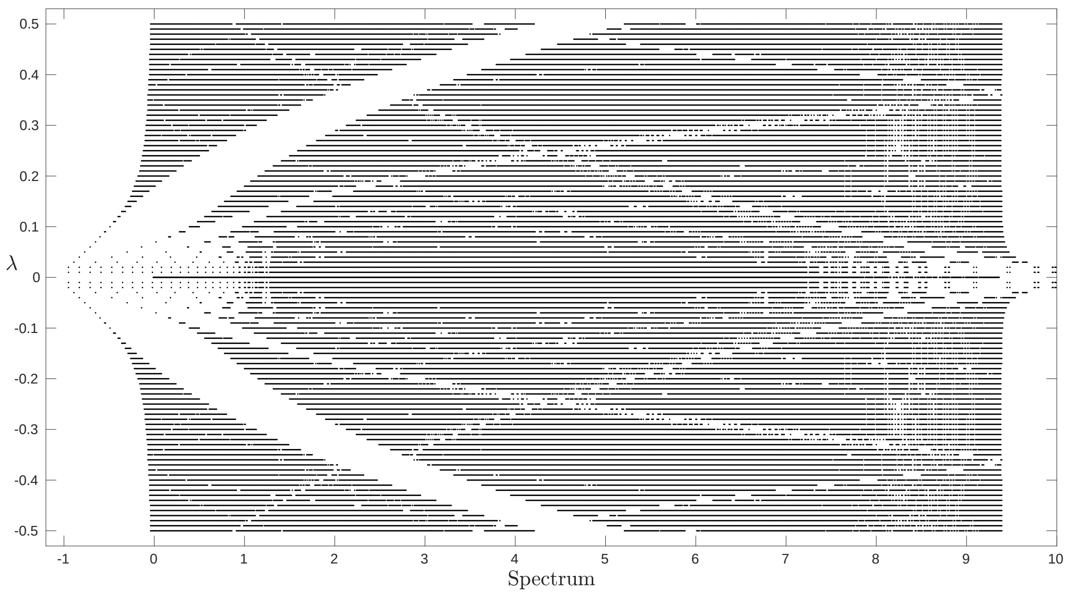

FIGURE 4. A portion of the computed spectrum of the one dimensional Schrödinger operator with potential V_λ and the error bound $\epsilon = 0.05$.

13.4. **The operator** $f(Q)$**.** If we consider the multiplication operator $(Qg)(x) = xg(x)$ on $L^2(\mathbb{R})$, then, for a bounded continuous function $f : \mathbb{R} \to \mathbb{R}$, the spectrum of $f(Q)$ is the range of the function f. In this example we use $f(x) = \frac{i(\exp(-2\pi i x)-1)}{2\pi x}$. To create an infinite matrix representation of $f(Q)$ we first consider the following Gabor basis for $L^2(\mathbb{R})$:

$$e^{2\pi i m x}\chi_{[0,1]}(x-n), \qquad m,n \in \mathbb{Z},$$

(where χ is the characteristic function) and then chose some enumeration of $\mathbb{Z} \times \mathbb{Z}$ into $\mathbb{N}$ to obtain a basis $\{\psi_j\}$ that is just indexed over $\mathbb{N}$. To get our basis we let $\varphi_j = \mathcal{F}\psi_j$, where $\mathcal{F}$ is the Fourier Transform. Finally we obtain the infinite matrix representation $A_{ij} = \langle f(Q)\varphi_j, \varphi_i\rangle$. Note that this becomes a full infinite matrix, however, we know the growth of the resolvent of the operator, thus, this is a problem in the class Σ_2^A with SCI $= 2$. As there are now two limits, our algorithm depends on two parameters, namely m

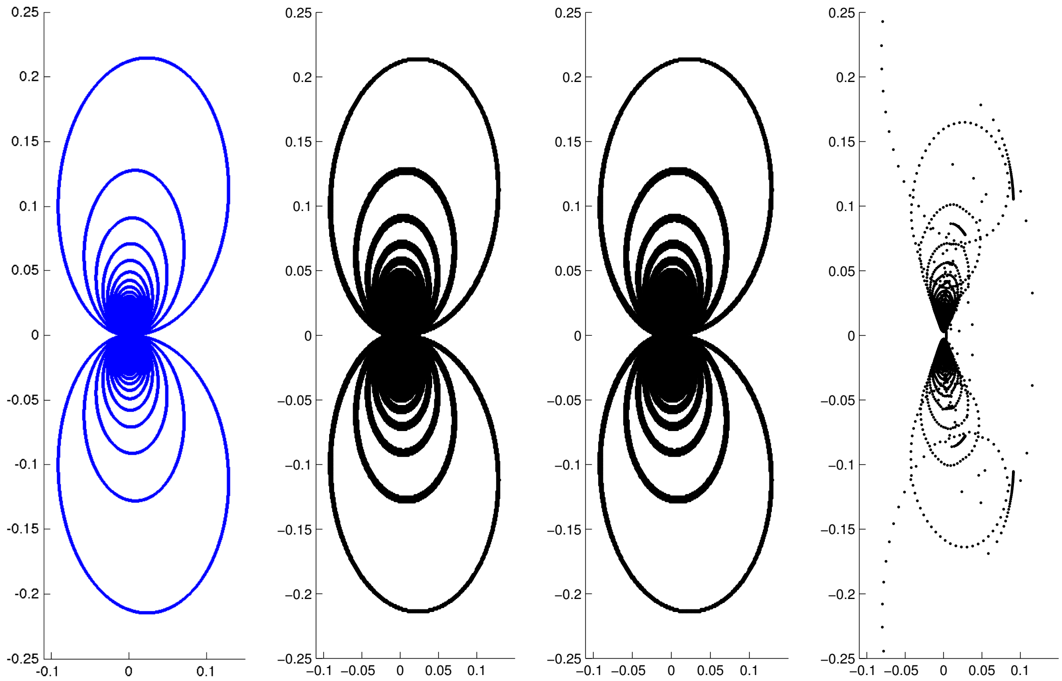

FIGURE 5. The left figure is a zoomed in part of $\mathrm{sp}(f(Q))$. The two following figures are $\Gamma_{n,m}(A)$ and $\Gamma_{n+p,m+s}(A)$ (restricted to the zoomed in part) to visualize the stopping criterion in (13.6). To get a better approximation a smaller ϵ must be chosen. A is the matrix representation of $f(Q)$. The right figure is the result of the finite section method trying to compute $\mathrm{sp}(f(Q))$.

and n, and we compute $\Gamma_{n,m}(A)$. This means that the stopping criterion from (13.1) becomes as follows. Choose $\epsilon > 0$ and $K \in \mathbb{N}$. Define, for any $n, l \in \mathbb{N}$,

$$
\begin{aligned}
\tilde{\Gamma}_n(A) &:= \Gamma_{n,m}(A), \quad m = \min\{p : d(\Gamma_{n,p}(A), \Gamma_{n,p+k}(A) \leq \epsilon \text{ for all } k \leq K\} \\
\tilde{\Gamma}(A) &:= \tilde{\Gamma}_l(A), \qquad l = \min\{p : d(\tilde{\Gamma}_p(A), \tilde{\Gamma}_{p+k}(A) \leq \epsilon \text{ for all } k \leq K\},
\end{aligned}
\tag{13.6}
$$

and let the output be $\tilde{\Gamma}(A)$. This stopping criterion is obviously a generalization of (13.1) and extends in an obvious way to several limits. Note however how incredibly more complex it gets by adding one more limit. In Figure 5 we have plotted $\Gamma_{n,m}(A)$ and $\Gamma_{n+p,m+s}(A)$ visualizing an output based on the two limit stopping criterion in (13.6). We also plotted the result of the finite section method. As we are computing within the class of problems with SCI $= 2$, there is of course no way that the finite section method could work.

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