MATHEMATICAL ANALYSIS 1
HOMEWORK 3

(1) Prove the following lemma:

Lemma. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be monotonically increasing on some $A \subseteq \mathbb{R}$. Then $f + g$ is also monotonically increasing on A. If either f or g are* strictly *increasing on A, then so is $f + g$. The same statements hold if we replace everywhere the word 'increasing' with the word 'decreasing'.*

(2) Prove the following lemma:

Lemma. *Let $f, g : \mathbb{R} \to \mathbb{R}$. Then:*

$$\begin{aligned} f, g \text{ are both monotone increasing} \quad &\Rightarrow \quad g \circ f \text{ is monotone increasing.} \\ f, g \text{ are both monotone decreasing} \quad &\Rightarrow \quad g \circ f \text{ is monotone increasing.} \\ f, g \text{ are monotone of different kinds} \quad &\Rightarrow \quad g \circ f \text{ is monotone decreasing.} \end{aligned}$$

(3) Show that if $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is monotone increasing on A, then $-f$ is monotone decreasing on A.

(4) Let t_c and s_c be the translation and scaling functions, respectively. Let r be the reflection function. Consider the function $f(x) = x^3$ restricted to the interval $[-1, 1]$. Sketch the following:
- (a) $f \circ t_c$ for $c = -1, 0, 1$
- (b) $t_c \circ f$ for $c = -1, 0, 1$
- (c) $f \circ s_c$ for $c = \frac{1}{2}, 2$
- (d) $s_c \circ f$ for $c = \frac{1}{2}, 2$
- (e) The difference $f \circ r - r \circ f$.

(5) Sketch the graphs of the following functions (choose a relevant part of the graph to sketch):
- (a) $\sin(x)$, $\sin(2x)$, $\sin(-x)$
- (b) $\log_{\frac{1}{5}} x$, $\log_5 x$, $\log_5(x^2)$
- (c) $\cos(x)$, $\cos(2x)$, $\cos(x - \frac{\pi}{2})$

(6) For the following pairs of functions f and g, compute $f \circ g$ and $g \circ f$, and determine their domains:
- (a) $f(x) = x^2$, $g(x) = x + 1$
- (b) $f(x) = \sqrt{x}$, $g(x) = x - 2$
- (c) $f(x) = \frac{1}{x}$, $g(x) = x^2 + 1$
- (d) $f(x) = \sin(x)$, $g(x) = 2x$

(7) For the following functions f and subsets $A \subseteq \mathrm{dom}(f)$, determine $\sup_A f$, $\inf_A f$, and whether the maximum and minimum are attained on A:
- (a) $f(x) = x^2$, $A = [0, 2]$
- (b) $f(x) = x^2$, $A = (0, 2)$
- (c) $f(x) = x^3 + x^2 + 2x + 1$, $A = [0, 2]$
- (d) $f(x) = \frac{1}{x}$, $A = (0, 1]$
- (e) $f(x) = \frac{1}{x}$, $A = [1, +\infty)$
- (f) $f(x) = e^x$, $A = [-1, 1]$

(8) Using trigonometric identities, simplify the following expressions:
- (a) $\frac{\sin(x)}{\cos(x)} + \frac{\cos(x)}{\sin(x)}$
- (b) $(\sin(x) + \cos(x))^2 + (\sin(x) - \cos(x))^2$
- (c) $\frac{1-\cos(2x)}{\sin(2x)}$

(9) Prove the following equalities:
- (a) $\tan(x) + \cot(x) = \frac{2}{\sin(2x)}$
- (b) $\cos(3x) = 4\cos^3(x) - 3\cos(x)$

Additional Practice Problems

$\star\star$ These problems are for you to practice $\star\star$
$\star\star$ They are not part of the homework assignment $\star\star$

(1) Sketch the graphs of the following functions (choose a relevant part of the graph to sketch):
  (a) $\sin(x+\frac{\pi}{2})$, $\sin(x)+1$
  (b) $\log_{10}(\frac{2}{x})$, $\ln(-x)$, $\ln(x^{-1})$
  (c) $-\cos(x)$, $\cos(-x)$
  (d) $\tan(x)$, $\tan(2x)$, $\tan(x+\frac{\pi}{4})$, $-\tan(x)$
  (e) $\arcsin(x)$, $\arcsin(2x)$, $\arcsin(-x)$, $\arcsin(x)+\frac{\pi}{2}$
  (f) $\arctan(x)$, $\arctan(2x)$, $\arctan(-x)$, $-\arctan(x)$
(2) Sketch the graphs of the following power functions:
  (a) $x^{1/2}$, $x^{1/3}$, $x^{1/4}$
  (b) x^{-1}, x^{-2}, x^{-3}
  (c) $x^{2/3}$, $x^{3/2}$, $x^{-1/2}$
  (d) $(x-1)^{1/2}$, $(x+2)^{-1}$, $2x^3$
(3) For the following pairs of functions f and g, compute $f\circ g$ and $g\circ f$, and determine their domains:
  (a) $f(x)=e^x$, $g(x)=\ln(x)$
  (b) $f(x)=|x|$, $g(x)=x-3$
  (c) $f(x)=\sqrt{x}$, $g(x)=x^2$
  (d) $f(x)=\frac{1}{x-1}$, $g(x)=\frac{1}{x}$
  (e) $f(x)=x^{1/3}$, $g(x)=x^3$
  (f) $f(x)=x^2+1$, $g(x)=\sqrt{x-1}$
(4) For the following functions f and subsets $A\subseteq \mathrm{dom}(f)$, determine $\sup_A f$, $\inf_A f$, and whether the maximum and minimum are attained on A:
  (a) $f(x)=e^x$, $A=(-1,1)$
  (b) $f(x)=\sqrt{4-x^2}$, $A=(-2,2)$
  (c) $f(x)=\arctan(x)$, $A=\mathbb{R}$
  (d) $f(x)=\frac{1}{1-x}$, $A=[0,1)$
  (e) $f(x)=x(2-x)$, $A=(0,2)$
  (f) $f(x)=x^{1/3}$, $A=[-8,8]$
  (g) $f(x)=x^{-2}$, $A=[1,+\infty)$
(5) Determine whether the following functions are even, odd, or neither:
  (a) $f(x)=x^4+x^2$
  (b) $f(x)=x^5-x^3$
  (c) $f(x)=x^3+x^2$
  (d) $f(x)=\frac{x}{x^2+1}$
  (e) $f(x)=x^{-2}$
  (f) $f(x)=|x|+x$
(6) Using trigonometric identities, simplify the following expressions:
  (a) $\sin(x+y)\sin(x-y)$
  (b) $\cos^4(x)-\sin^4(x)$
  (c) $\frac{1+\tan^2(x)}{1+\cot^2(x)}$
(7) Prove the following equality:
  (a) $\frac{\sin(x)+\sin(3x)}{\cos(x)+\cos(3x)}=\tan(2x)$
(8) Simplify the following expressions involving powers:
  (a) $(x^3)^{1/2}\cdot x^{-1/2}$
  (b) $\frac{x^{2/3}\cdot x^{1/2}}{x^{-1/6}}$
  (c) $(x^{-2}y^3)^{-1/2}$
  (d) $\left(\frac{x^{1/3}}{x^{-2/3}}\right)^3$

(1) *Proof.* Let $x, y \in A$ with $x < y$. Since f and g are monotonically increasing:

$$f(x) \leq f(y)$$
$$g(x) \leq g(y)$$

Adding: $(f+g)(x) = f(x) + g(x) \leq f(y) + g(y) = (f+g)(y)$.
If either is strictly increasing, the corresponding inequality is strict, so $f + g$ is strictly increasing.
The decreasing case follows similarly. □

(2) *Proof.* Let $x < y$.
Case 1: Both increasing

$$f(x) \leq f(y)$$
$$g(f(x)) \leq g(f(y)) \quad \text{(since } g \text{ increasing)}$$

So $g \circ f$ is increasing.
Case 2: Both decreasing

$$f(x) \geq f(y)$$
$$g(f(x)) \leq g(f(y)) \quad \text{(since } g \text{ decreasing)}$$

So $g \circ f$ is increasing.
Case 3: Different types

$$f(x) \leq f(y) \quad \text{(if } f \text{ increasing)}$$
$$g(f(x)) \geq g(f(y)) \quad \text{(since } g \text{ decreasing)}$$

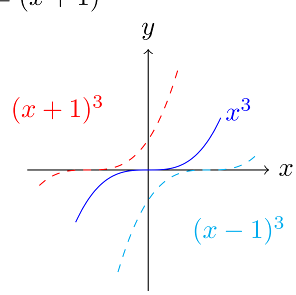
So $g \circ f$ is decreasing. □

(3) *Proof.* Let $x, y \in A$ with $x < y$. Since f is increasing:

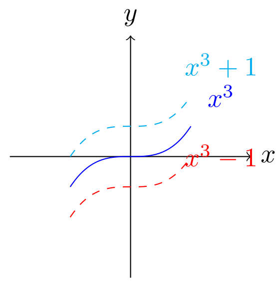
$$f(x) \leq f(y) \Rightarrow -f(x) \geq -f(y)$$

So $-f$ is decreasing. □

(4) (a) $f \circ t_c$ for $c = -1, 0, 1$:
- $c = -1$: $f \circ t_{-1}(x) = f(x-1) = (x-1)^3$
- $c = 0$: $f \circ t_0(x) = f(x) = x^3$
- $c = 1$: $f \circ t_1(x) = f(x+1) = (x+1)^3$

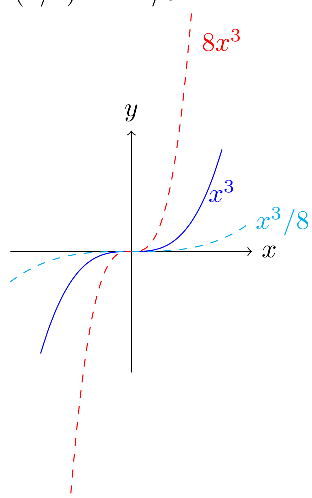


(b) $t_c \circ f$ for $c = -1, 0, 1$:
- $c = -1$: $t_{-1} \circ f(x) = f(x) - 1 = x^3 - 1$
- $c = 0$: $t_0 \circ f(x) = f(x) = x^3$
- $c = 1$: $t_1 \circ f(x) = f(x) + 1 = x^3 + 1$



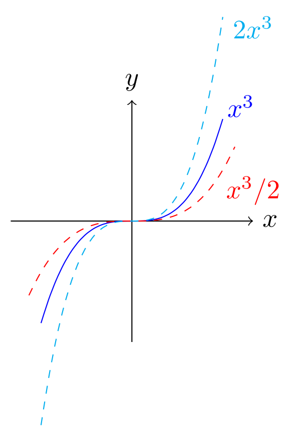
(c) $f \circ s_c$ for $c = \frac{1}{2}, 2$:

- $c = \frac{1}{2}$: $f \circ s_{1/2}(x) = f(2x) = (2x)^3 = 8x^3$
- $c = 2$: $f \circ s_2(x) = f(x/2) = (x/2)^3 = x^3/8$



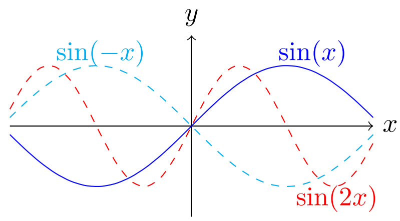
(d) $s_c \circ f$ for $c = \frac{1}{2}, 2$:

- $c = \frac{1}{2}$: $s_{1/2} \circ f(x) = \frac{1}{2}f(x) = \frac{1}{2}x^3 = x^3/2$
- $c = 2$: $s_2 \circ f(x) = 2f(x) = 2x^3$

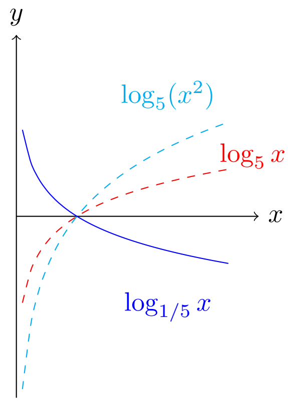


(e) $f \circ r - r \circ f = x^3 - (-x)^3 = x^3 + x^3 = 2x^3$

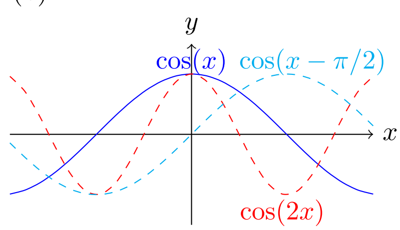
(5)



(a)



(b)



(c)

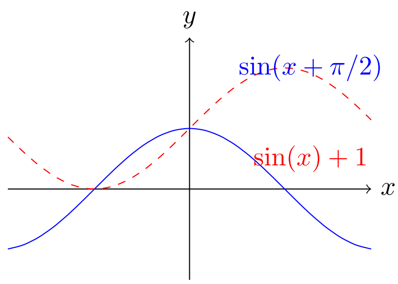
(6) (a) $f \circ g(x) = (x+1)^2$, dom $= \mathbb{R}$
$g \circ f(x) = x^2 + 1$, dom $= \mathbb{R}$
(b) $f \circ g(x) = \sqrt{x-2}$, dom $= [2, \infty)$
$g \circ f(x) = \sqrt{x} - 2$, dom $= [0, \infty)$
(c) $f \circ g(x) = \frac{1}{x^2+1}$, dom $= \mathbb{R}$
$g \circ f(x) = \frac{1}{x^2} + 1$, dom $= \mathbb{R} \setminus \{0\}$
(d) $f \circ g(x) = \sin(2x)$, dom $= \mathbb{R}$
$g \circ f(x) = 2\sin(x)$, dom $= \mathbb{R}$

(7) (a) $\sup = 4$, $\inf = 0$, max attained at $x = 2$, min at $x = 0$
(b) $\sup = 4$, $\inf = 0$, neither attained
(c) $\sup = 15$, $\inf = 1$, max at $x = 2$, min at $x = 0$
(d) $\sup = \infty$, $\inf = 1$, min attained at $x = 1$, no max
(e) $\sup = 1$, $\inf = 0$, max attained at $x = 1$, no min
(f) $\sup = e$, $\inf = e^{-1}$, max at $x = 1$, min at $x = -1$

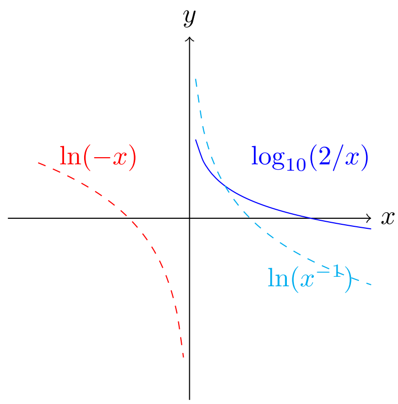
(8) (a) $\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \tan x + \cot x = \frac{\sin^2 x + \cos^2 x}{\sin x \cos x} = \frac{1}{\sin x \cos x} = \frac{2}{\sin 2x}$
(b) $(\sin x + \cos x)^2 + (\sin x - \cos x)^2 = (1 + 2\sin x \cos x) + (1 - 2\sin x \cos x) = 2$
(c) $\frac{1 - \cos 2x}{\sin 2x} = \frac{2\sin^2 x}{2\sin x \cos x} = \frac{\sin x}{\cos x} = \tan x$

(9) (a) $\tan x + \cot x = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \frac{\sin^2 x + \cos^2 x}{\sin x \cos x} = \frac{1}{\sin x \cos x} = \frac{2}{\sin 2x}$
(b) $\cos 3x = \cos(2x + x) = \cos 2x \cos x - \sin 2x \sin x$
$= (2\cos^2 x - 1)\cos x - 2\sin^2 x \cos x$
$= 2\cos^3 x - \cos x - 2(1 - \cos^2 x)\cos x$
$= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x = 4\cos^3 x - 3\cos x$

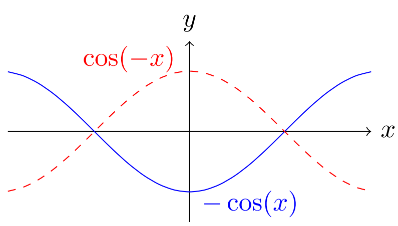
(1) (a)

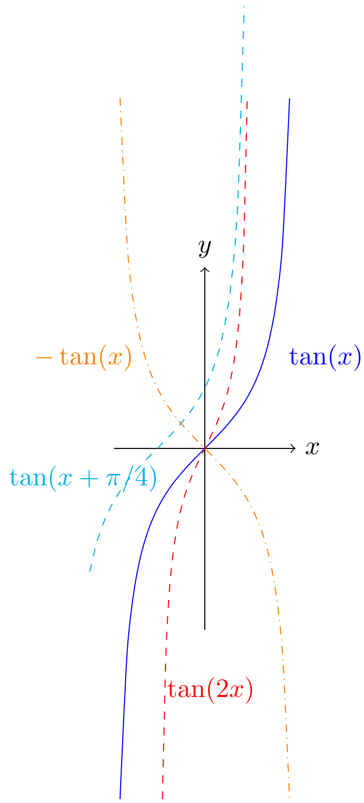


(b)



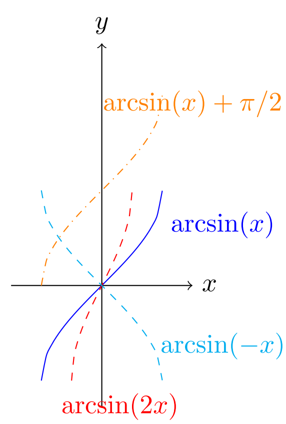
(c)





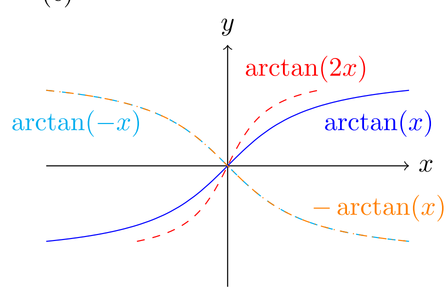
y
x
$-\tan(x)$
$\tan(x)$
$\tan(x+\pi/4)$
$\tan(2x)$

(d)



y
x
$\arcsin(x)+\pi/2$
$\arcsin(x)$
$\arcsin(-x)$
$\arcsin(2x)$

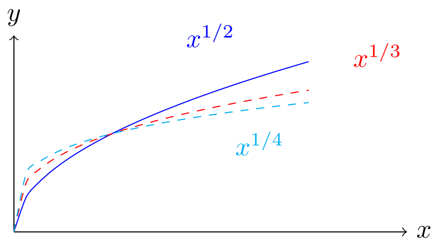
(e)



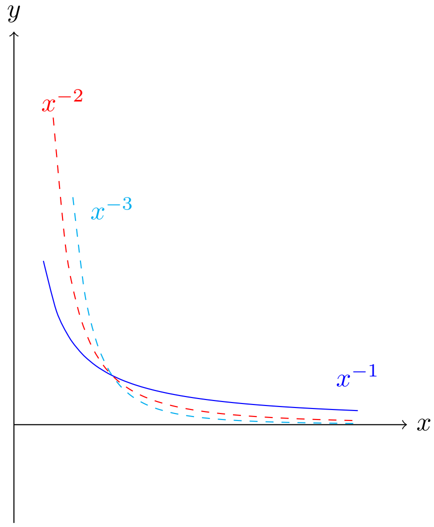
y
x
$\arctan(2x)$
$\arctan(-x)$
$\arctan(x)$
$-\arctan(x)$

(f)

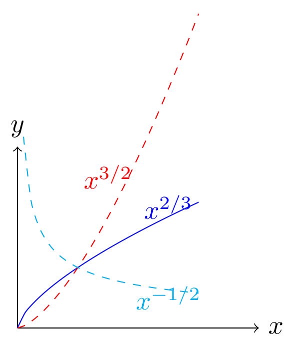
(2) (a)



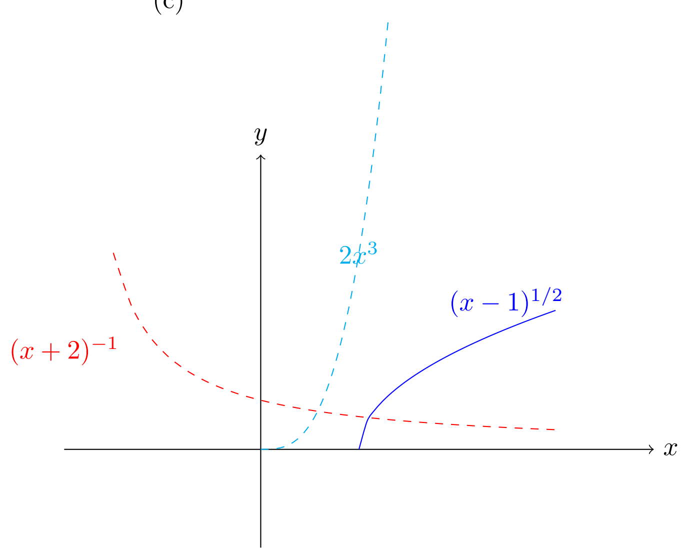
(b)



(c)



(d)



(3) (a) $f \circ g(x) = e^{\ln x} = x$, dom $= (0, +\infty)$
$g \circ f(x) = \ln(e^x) = x$, dom $= \mathbb{R}$
(b) $f \circ g(x) = |x - 3|$, dom $= \mathbb{R}$
$g \circ f(x) = |x| - 3$, dom $= \mathbb{R}$
(c) $f \circ g(x) = \sqrt{x^2} = |x|$, dom $= \mathbb{R}$
$g \circ f(x) = (\sqrt{x})^2 = x$, dom $= [0, +\infty)$
(d) $f \circ g(x) = \frac{1}{\frac{1}{x}-1} = \frac{x}{1-x}$, dom $= \mathbb{R} \setminus \{0, 1\}$
$g \circ f(x) = \frac{1}{\frac{1}{x-1}} = x - 1$, dom $= \mathbb{R} \setminus \{1\}$
(e) $f \circ g(x) = (x^3)^{1/3} = x$, dom $= \mathbb{R}$
$g \circ f(x) = (x^{1/3})^3 = x$, dom $= \mathbb{R}$
(f) $f \circ g(x) = (\sqrt{x-1})^2 + 1 = x$, dom $= [1, +\infty)$
$g \circ f(x) = \sqrt{(x^2+1) - 1} = |x|$, dom $= \mathbb{R}$
(4) (a) $\sup = e$, $\inf = e^{-1}$, neither attained
(b) $\sup = 2$, $\inf = 0$, neither attained
(c) $\sup = \pi/2$, $\inf = -\pi/2$, neither attained
(d) $\sup = \infty$, $\inf = 1$, min attained at $x = 0$, no max
(e) $\sup = 1$, $\inf = 0$, neither attained
(f) $\sup = 2$, $\inf = -2$, both attained
(g) $\sup = 1$, $\inf = 0$, max attained at $x = 1$, no min
(5) (a) Even: $f(-x) = (-x)^4 + (-x)^2 = x^4 + x^2 = f(x)$
(b) Odd: $f(-x) = (-x)^5 - (-x)^3 = -x^5 + x^3 = -f(x)$
(c) Neither: $f(-x) = (-x)^3 + (-x)^2 = -x^3 + x^2 \neq \pm f(x)$
(d) Odd: $f(-x) = \frac{-x}{(-x)^2+1} = \frac{-x}{x^2+1} = -f(x)$
(e) Even: $f(-x) = (-x)^{-2} = x^{-2} = f(x)$
(f) Neither: $f(-x) = |-x| + (-x) = |x| - x \neq \pm f(x)$
(6) (a) $\sin(x+y)\sin(x-y) = \frac{1}{2}[\cos(2y) - \cos(2x)] = \sin^2 y - \sin^2 x$
(b) $\cos^4 x - \sin^4 x = (\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x) = \cos 2x$
(c) $\frac{1+\tan^2 x}{1+\cot^2 x} = \frac{\sec^2 x}{\csc^2 x} = \frac{\sin^2 x}{\cos^2 x} = \tan^2 x$
(7) *Proof.* Using sum-to-product identities:

$$\begin{aligned} \frac{\sin x + \sin 3x}{\cos x + \cos 3x} &= \frac{2\sin(2x)\cos x}{2\cos(2x)\cos x} \\ &= \frac{\sin 2x}{\cos 2x} = \tan 2x \end{aligned}$$

□

(8) (a) $(x^3)^{1/2} \cdot x^{-1/2} = x^{3/2} \cdot x^{-1/2} = x^1 = x$
(b) $\frac{x^{2/3} \cdot x^{1/2}}{x^{-1/6}} = x^{2/3+1/2-(-1/6)} = x^{2/3+1/2+1/6} = x^{4/3}$
(c) $(x^{-2}y^3)^{-1/2} = x^1 y^{-3/2} = \frac{x}{y^{3/2}}$
(d) $\left(\frac{x^{1/3}}{x^{-2/3}}\right)^3 = (x^{1/3-(-2/3)})^3 = (x^1)^3 = x^3$