By definition of the supremum, immediately to the right of x_1 there are points where g_{x_0} is negative. By the continuity of g_{x_0}, it must hold that

$$g_{x_0}(x_1) = 0.$$

Claim: $g_{x_0}(x) = 0$ for all $x \in [x_0, x_1]$.

Proof of claim. **By contradiction**. If the claim is not true, then since g_{x_0} is non-negative, it must hold that $M = \max_{x \in [x_0,x_1]} g_{x_0}(x)$ is strictly positive: $M > 0$. By Weierstrass' Theorem, the maximum of a continuous function on a closed interval is attained, so that there exists $\overline{x} \in (x_0, x_1)$ such that

$$g_{x_0}(\overline{x}) = M.$$

By Fermat's Theorem (Theorem 8.9),

$$g'_{x_0}(\overline{x}) = 0$$

since $\overline{x}$ is an extremum.

Now recall that g_{x_0} is a convex function. Since $g_{x_0}(\overline{x}) = M$ and $g'_{x_0}(\overline{x}) = 0$, convexity implies that $g_{x_0}(x) \geq M$ on a neighborhood of $\overline{x}$. Since M is the maximum of g_{x_0}, it must hold that $g_{x_0}(x) = M$ on that neighborhood. How far to the right can this neighborhood extend? Define

$$Q = \{x > \overline{x} \;\;|\;\; g_{x_0}(y) = M,\ \forall y \in [\overline{x}, x]\}$$

and

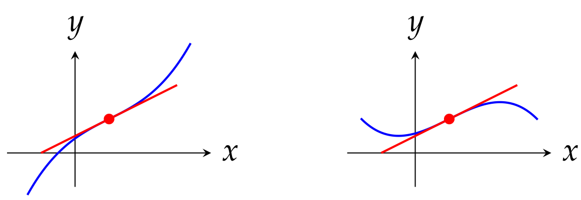
$$x_2 = \max Q$$

(the maximum is attained since g_{x_0} is a continuous function). Observe that $x_2 < x_1$ since $g_{x_0}(x_1) = 0$. So we have found a point $x_2 \in (x_0, x_1)$ where the function g_{x_0} attains the value M, and, moreover, g_{x_0} attains the value M in a left-neighborhood of x_2. Hence the left-derivative of g_{x_0} at x_2 must be 0. Since g_{x_0} is differentiable at x_2, it must hold that $g'_{x_0}(x_2) = 0$. But then due to convexity at x_2 it must hold that $g_{x_0} \geq 0$ in a neighborhood of x_2, and in particular in a right-neighborhood. But this contradicts the definition of x_2 as $\max Q$. Therefore the claim is proved. □

We can now conclude the proof of the theorem. The claim implies that $g_{x_0}(x) = 0$ for all $x \in [x_0, x_1]$. But then, as in the preceding argument, the left-derivative of g_{x_0} at x_1 is 0. Consequently, (by convexity at x_1) it must hold that $g_{x_0} \geq 0$ to the right of x_1, in contradiction to the definition of x_1 as $\sup P$ and the assumption that x_1 is an internal point of I. Therefore it must hold that $g_{x_0} \geq 0$ on I. □

Inflection point

An inflection point is a point x_0 where the graph of the function lies above (or at) the tangent line t_{x_0} on one side, and below (or at) on the other side.



Theorem 8.18: Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable on $I \subseteq \mathbb{R}$. Then

$$f \text{ is convex on } I \quad \Leftrightarrow \quad f' \text{ is increasing on } I,$$

and

$$f \text{ is strictly convex on } I \quad \Leftarrow \quad f' \text{ is strictly increasing on } I.$$

Proof. We skip the proof. □

Corollary 8.19: Let $f : \mathbb{R} \to \mathbb{R}$ be twice-differentiable on $I \subseteq \mathbb{R}$. Then

$$f \text{ is convex on } I \quad \Leftrightarrow \quad f''(x) \geq 0, \ \forall x \in I,$$

and

$$f \text{ is strictly convex on } I \quad \Leftarrow \quad f''(x) > 0, \ \forall x \in I.$$

Proof. The corollary follows immediately from Theorem 8.18 by applying Theorem 8.15 to f'. □

Corollary 8.20: Let f be a twice-differentiable function in a neighborhood of x_0. Then:

$$x_0 \text{ is an inflection point} \quad \Rightarrow \quad f''(x_0) = 0,$$

and

$$f''(x_0) = 0 \text{ and } f'' \text{ changes sign at } x_0 \quad \Rightarrow \quad x_0 \text{ is an inflection point.}$$

Moreover,

$$f''(x_0) = 0 \text{ and } f'' \text{ doesn't change sign at } x_0 \quad \Rightarrow \quad x_0 \text{ isn't an inflection point.}$$

Proof. We still do not have all the tools to prove this. □

Remark: The preceding results can be stated for concave functions as well.

8.10 Qualitative study of a function

Qualitative study of a function f

STEP 1. Understand the domain of f and possible symmetries (such as the function being even or odd).

STEP 2. What happens at the end-points of the domain? Any asymptotes? Discontinuities?

STEP 3. Are there points where f is not differentiable?

STEP 4. Compute f' and understand its domain, where it vanishes and where it is positive/negative. This determines monotonicity intervals and (some) extrema of f.

STEP 5. Find inflection points and determine intervals of convexity/concavity using the tools we learned for f', or by differentiating again (if possible).

8.11 De l'Hôpital's Theorem

De l'Hôpital's Theorem, which we state below, is a very important theorem enabling us to evaluate some indeterminate forms of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. It also has an interesting history attached to it. It is named after Guillaume de l'Hôpital, a 17th century French mathematician, who appears to have been a decent mathematician, researching analysis and geometry. However, he was a contemporary of the Bernoulli brothers (Johann and Jacob) who were (and still are) considered to be exceptional mathematicians. De l'Hôpital, in addition to being a mathematician, also came from nobility. This meant that he had money. Enough money to offer Johann Bernoulli annual payments in return for his mathematical results. In short, the following theorem may be called "De l'Hôpital's Theorem", but it is, in fact, Johann Bernoulli's Theorem.

De l'Hôpital's Theorem

Let $f, g : \mathbb{R} \to \mathbb{R}$ and let $x_0 \in \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$. Suppose that f, g are differentiable in a neighborhood of x_0 (possibly excluding x_0 itself) and their limits as $x \to x_0$ exist and are equal:

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = L$$

where

$$L \text{ is either } 0 \text{ or } +\infty \text{ or } -\infty.$$

Then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

if the limit on the right hand side exists.

Proof. We prove only for the case $L = 0$, $x_0 \in \mathbb{R}$, and for the right-limit at x_0. In this case, even if f, g are not defined at x_0, we can define $f(x_0) = g(x_0) = L = 0$ to make the two functions continuous at x_0. Let $x > x_0$ be a nearby point to the right of x_0. Then since $f(x_0) = g(x_0) = 0$ we have

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)}.$$

By Cauchy's Theorem (Theorem 8.12) there exists a point $t \in (x_0, x)$, depending on x (so we write $t(x)$) such that

$$\frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(t(x))}{g'(t(x))}.$$

Now, letting $x \to x_0$, the point $t(x)$ will satisfy $\lim_{x \to x_0} t(x) = x_0$ by the Squeeze Theorem. Hence, using the Substitution Theorem (Theorem 5.12) we can replace the limit $x \to x_0$

with $t \to x_0$ and obtain:

$$\begin{aligned}
\lim_{x \to x_0} \frac{f(x)}{g(x)} &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{g(x) - g(x_0)} \\
&= \lim_{x \to x_0} \frac{f'(t(x))}{g'(t(x))} \\
&= \lim_{t \to x_0} \frac{f'(t)}{g'(t)}
\end{aligned}$$

and the proof (in this case) is complete. We skip the proofs of the other cases. □

Example 8.25: Compute

$$\lim_{x \to 0} \frac{e^{2x} - e^{-2x}}{\sin 5x}.$$

We see that we obtain a limit of the form $\frac{0}{0}$. Using De l'Hôpital:

$$\lim_{x \to 0} \frac{e^{2x} - e^{-2x}}{\sin 5x} = \lim_{x \to 0} \frac{\frac{\mathrm{d}}{\mathrm{d}x}(e^{2x} - e^{-2x})}{\frac{\mathrm{d}}{\mathrm{d}x}(\sin 5x)} = \frac{2}{5} \lim_{x \to 0} \frac{e^{2x} + e^{-2x}}{\cos 5x} = \frac{4}{5}.$$

Example 8.26: Compute

$$\lim_{x \to 0} \frac{1 + 3x - (1 + 2x)^{3/2}}{x \sin x}.$$

This has the indeterminate form $\frac{0}{0}$. Differentiating numerator and denominator, we consider the limit:

$$\lim_{x \to 0} \frac{3 - \frac{3}{2} \cdot 2 \cdot (1 + 2x)^{1/2}}{\sin x + x \cos x}.$$

This is still indeterminate of the form $\frac{0}{0}$. So we differentiate *again*:

$$\lim_{x \to 0} \frac{-\frac{3}{2} \cdot 2 \cdot \frac{1}{2} \cdot 2 \cdot (1 + 2x)^{-1/2}}{\cos x + \cos x - x \sin x} = -\frac{3}{2}.$$

Hence:

$$\lim_{x \to 0} \frac{1 + 3x - (1 + 2x)^{3/2}}{x \sin x} = -\frac{3}{2}.$$

Example 8.27: We can now prove that $x^n = o(e^x)$ as $x \to +\infty$. First we write

$$\lim_{x \to +\infty} \frac{x^n}{e^x}$$

which is of the indeterminate form $\frac{+\infty}{+\infty}$. Differentiating over and over again, we have:

$$\begin{aligned}
\lim_{x \to +\infty} \frac{x^n}{e^x} &= \lim_{x \to +\infty} \frac{n x^{n-1}}{e^x} \\
&= \lim_{x \to +\infty} \frac{n(n-1) x^{n-2}}{e^x} \\
&= \lim_{x \to +\infty} \frac{n(n-1)(n-2) x^{n-3}}{e^x} \\
&= \cdots \\
\textit{after } n \textit{ derivatives} &= \lim_{x \to +\infty} \frac{n!}{e^x} = 0.
\end{aligned}$$

Example 8.28: We can also show that $\ln x = o(x^\alpha)$ for any $\alpha > 0$ as $x \to +\infty$:

$$\lim_{x\to+\infty} \frac{\ln x}{x^\alpha} = \lim_{x\to+\infty} \frac{\frac{1}{x}}{\alpha x^{\alpha-1}} = \frac{1}{\alpha} \lim_{x\to+\infty} \frac{1}{x^\alpha} = 0$$

Example 8.29: The function $f(x) = e^x - 1 - \sin x$ is infinitesimal as $x \to 0$. What is its order with respect to $\varphi(x) = x$ and what is its principal part?

We consider the limit, for $\alpha > 0$ to be determined:

$$\lim_{x\to 0} \frac{f(x)}{x^\alpha} = \lim_{x\to 0} \frac{e^x - 1 - \sin x}{x^\alpha}$$

which is indeterminate of the form $\frac{0}{0}$. Differentiating we have

$$\lim_{x\to 0} \frac{e^x - \cos x}{\alpha x^{\alpha-1}}$$

which is still of the form $\frac{0}{0}$. So we differentiate again:

$$\lim_{x\to 0} \frac{e^x + \sin x}{\alpha(\alpha-1)x^{\alpha-2}}.$$

Now the numerator tends to 1 as $x \to 0$, and if we choose $\alpha = 2$ then the denominator is 2. So we have:

$$\lim_{x\to 0} \frac{f(x)}{x^2} = \frac{1}{2}$$

which means that $f(x)$ is infinitesimal of order 2 with respect to $\varphi(x) = x$ at $x = 0$, and the principal part is $p(x) = \frac{1}{2}x^2$.