However, taking a further order we have:

$$h(x) = \left(1 + x + \frac{x^2}{2!} + o(x^2)\right) - \left(1 + x - \frac{x^2}{2} + o(x^2)\right) = x^2 + o(x^2) \quad \text{as } x \to 0.$$

Taking another order:

$$\begin{aligned} h(x) &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + o(x^3)\right) - \left(1 + x - \frac{x^2}{2} + \frac{x^3}{2} + o(x^3)\right) \\ &= x^2 - \frac{x^3}{3} + o(x^3) \quad \text{as } x \to 0. \end{aligned}$$

Products of expansions

$$\begin{aligned} f(x) \cdot g(x) &= [p_n(x) + o(x^n)] \cdot [q_n(x) + o(x^n)] \\ &= p_n(x) \cdot q_n(x) + p_n(x)o(x^n) + q_n(x)o(x^n) + o(x^n)o(x^n) \\ &= p_n(x) \cdot q_n(x) + o(x^n) + o(x^n) + o(x^{2n}) \\ &= p_n(x) \cdot q_n(x) + o(x^n). \end{aligned}$$

Remark: A couple of remarks about the above computation. First, the fact that $p_n(x)o(x^n) = q_n(x)o(x^n) = o(x^n)$ follows from the fact that both functions p_n and q_n are bounded near $x = 0$, and therefore when they multiply something small of order $o(x^n)$ they do not affect its order. Second, we have $o(x^n)+o(x^n)+o(x^{2n}) = o(x^n)$ since the $o(x^{2n})$ is of higher order and therefore is negligible with respect to $o(x^n)$. Then, we can sum as many $o(x^n)$ as we want, and they will still result in an element that is of order $o(x^n)$.

Observe that the product of the two polynomials p_n and q_n will include monomials up to x^{2n} coming from the product of $a_n x^n$ with $b_n x^n$:

$$\begin{aligned} p_n(x) \cdot q_n(x) &= \left(\sum_{i=0}^{n} a_i x^i\right)\left(\sum_{j=0}^{n} b_j x^j\right) \\ &= \sum_{k=0}^{2n}\left(\sum_{m=0}^{k} a_m b_{k-m}\right)x^k \\ &= \sum_{k=0}^{n}\left(\sum_{m=0}^{k} a_m b_{k-m}\right)x^k + \underbrace{\sum_{k=n+1}^{2n}\left(\sum_{m=0}^{k} a_m b_{k-m}\right)x^k}_{o(x^n)} \end{aligned}$$

However, since our our order of accuracy is $o(x^n)$ there is no point in maintaining terms that are of that order. Hence we can write:

$$p_n(x) \cdot q_n(x) = \sum_{k=0}^{n}\left(\sum_{m=0}^{k} a_m b_{k-m}\right)x^k + o(x^n)$$

Example 9.4: Expand $h(x) = \sqrt{1+x} \cdot e^x$ to second order. To second order, these two functions have the expansions:

$$f(x) = \sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + o(x^2)$$

and

$$g(x) = e^x = 1 + x + \frac{x^2}{2} + o(x^2)$$

$$\begin{aligned}
f(x) \cdot g(x) &= \left(1 + \frac{x}{2} - \frac{x^2}{8} + o(x^2)\right)\left(1 + x + \frac{x^2}{2} + o(x^2)\right) \\
&= 1 \cdot \left(1 + x + \frac{x^2}{2}\right) + \frac{x}{2} \cdot \left(1 + x + \frac{x^2}{2}\right) - \frac{x^2}{8} \cdot \left(1 + x + \frac{x^2}{2}\right) + o(x^2) \\
&= 1 + \left(x + \frac{x}{2}\right) + \left(\frac{x^2}{2} + \frac{x^2}{2} - \frac{x^2}{8}\right) + \underbrace{\left(\frac{x^3}{4} - \frac{x^3}{8}\right) + \left(-\frac{x^4}{16}\right)}_{o(x^2)} + o(x^2) \\
&= 1 + \frac{3}{2}x + \frac{7}{8}x^2 + o(x^2) \qquad x \to 0.
\end{aligned}$$

Example 9.5: Let us give an example of a different flavor. Suppose we want to approximate the product $\pi \cdot e$. Here are the expressions for both number to 5 decimal places:

$$\pi = 3.14159 \pm 5 \cdot 10^{-6} \qquad e = 2.71828 \pm 5 \cdot 10^{-6}$$

Let's compare the product of these two approximation to the actual product of π and e:

$$\begin{aligned}
3.14159 \times 2.71828 &= 8.5397\mathbf{212652} \\
\pi \times e &= 8.5397\mathbf{342226}\cdots
\end{aligned}$$

We see that they already disagree starting from the 5th decimal place. This demonstrates that when multiplying approximations we need to exercise caution! *But please do note that the type of error here is different from our 'little o' errors. The example here has to do with rounding errors, so there's no exact comparison. However this example serves as a warning to be careful with approximations: just because we can get a longer sequence of numbers when multiplying two approximations, doesn't mean it's correct.*

9.4 Local behavior of a function via its Taylor expansion

In this section we consider a function $f : \mathbb{R} \to \mathbb{R}$ and suppose that near the point $x_0 \in \mathbb{R}$ it can be written as

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n + o((x - x_0)^n)$$