MATHEMATICAL ANALYSIS 1
HOMEWORK 9

(1) Prove the following theorem: (it is Corollary 8.16 in the lecture notes)

Theorem. *Let $f : \mathbb{R} \to \mathbb{R}$ and suppose that f is differentiable on an interval I. Let $x_0 \in I$ be in the interior of I (not on the boundary). Then:*
- *If $f'(x) \geq 0$ to the left of x_0 and $f'(x) \leq 0$ to the right of x_0, then x_0 is a local maximum.*
- *If $f'(x) \leq 0$ to the left of x_0 and $f'(x) \geq 0$ to the right of x_0, then x_0 is a local minimum.*

(2) For each of the following functions, determine: (i) The domain. (ii) The critical points. (iii) The local maximum and minimum values. (iv) The inflection points. (v) The points where the function is not differentiable. (vi) All asymptotes. (vii) Sketch the function (or a relevant part of the function).

(a) $f(x) = x^3 - 6x^2 + 9x + 1$
(b) $f(x) = |x^2 - 4|$
(c) $f(x) = e^{\sqrt{x}}$
(d) $f(x) = \dfrac{1}{(x^2+4)^2}$
(e) $f(x) = \ln(\sin(x))$, $x \in (0, \pi)$
(f) $f(x) = x^{1/3}(x-2)^{1/3}$
(g) $f(x) = \cos(x^2)$
(h) $f(x) = \sqrt[3]{x^2 - 3x}$
(i) $f(x) = \ln(x + \sqrt{x^2+1})$
(j) $f(x) = \arctan\left(\dfrac{x}{x-1}\right)$

(3) Calculate the following limits. Identify the type of indeterminate form, and use De L'Hôpital's Theorem where appropriate.

(a) $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$
(b) $\lim_{x\to 0^+} x \ln x$
(c) $\lim_{x\to\infty} \frac{\ln(x)}{x^2}$
(d) $\lim_{x\to 0} \frac{\sin(x) - x}{x^3}$
(e) $\lim_{x\to\infty} \left(1 + \frac{3}{x}\right)^{2x}$
(f) $\lim_{x\to 0^+} x^x$
(g) $\lim_{x\to \frac{\pi}{2}^-} (\sec(x) - \tan(x))$
(h) $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{e^x - 1}\right)$

(4) Consider the function

$$f(x) = e^x(x^2 - 8|x-3| - 8)$$

Determine
(a) monotonicity intervals,
(b) local extrema and the range $\operatorname{im}(f)$,
(c) points where f is not continuous; points where f is not differentiable,
(d) and sketch a rough graph of f, highlighting the previous points.
(e) Does there exist a constant $\alpha \in \mathbb{R}$ such that the function

$$g(x) = f(x) - \alpha|x-3|$$

belongs to the functional space $\mathcal{C}^1(\mathbb{R})$?