

## Convergence tests

Now we give some criteria for the convergence/divergence of improper integrals of type I.

### Comparison Test

Fix  $a \in \mathbb{R}$ . Let  $f, g$  be integrable on  $[a, b]$  for any  $b > a$ . Assume that  $0 \leq f(x) \leq g(x)$  for all  $x \in [a, +\infty)$ . Then

$$0 \leq \int_a^{+\infty} f(x) \, dx \leq \int_a^{+\infty} g(x) \, dx$$

Therefore

$$\begin{aligned} \int_a^{+\infty} g(x) \, dx < +\infty \text{ (converges)} &\Rightarrow \int_a^{+\infty} f(x) \, dx < +\infty \text{ (converges)} \\ \int_a^{+\infty} f(x) \, dx = +\infty \text{ (diverges)} &\Rightarrow \int_a^{+\infty} g(x) \, dx = +\infty \text{ (diverges)} \end{aligned}$$

**Example 11.2:** We determine whether the integrals

$$\int_1^{+\infty} \frac{\arctan x}{x^2} \, dx \quad \text{and} \quad \int_1^{+\infty} \frac{\arctan x}{x} \, dx$$

converge or diverge. Recall that  $\arctan 1 = \frac{\pi}{4}$  and  $\arctan x$  is a strictly increasing function, with  $\lim_{x \rightarrow +\infty} \arctan x = \frac{\pi}{2}$ . Hence

$$\frac{\pi}{4} \leq \arctan x \leq \frac{\pi}{2}, \quad \forall x \in [1, +\infty).$$

It follows that for all  $x \in [1, +\infty)$ ,

$$\frac{\arctan x}{x^2} \leq \frac{\pi}{2} \frac{1}{x^2} \quad \text{and} \quad \frac{\pi}{4} \frac{1}{x} \leq \frac{\arctan x}{x}$$

We therefore have that

$$\int_1^{+\infty} \frac{\arctan x}{x^2} \, dx \leq \frac{\pi}{2} \int_1^{+\infty} \frac{1}{x^2} \, dx < +\infty \text{ (converges)}$$

and

$$\int_1^{+\infty} \frac{\arctan x}{x} \, dx \geq \frac{\pi}{4} \int_1^{+\infty} \frac{1}{x} \, dx = +\infty \text{ (diverges)}$$

### If $f = O(g)$

If  $f = O(g)$  are non-negative as  $x \rightarrow +\infty$ , then if the improper integral (type I) of  $g$  converges, then so does the integral of  $f$ .

**Example 11.3:** Since  $e^{-x^2} = o(x^{-2})$  as  $x \rightarrow +\infty$ , we deduce that  $\int_0^{+\infty} e^{-x^2} dx < +\infty$  (converges).

### Absolute Convergence Test

Fix  $a \in \mathbb{R}$ . Suppose that both  $f$  and  $|f|$  are integrable on any interval  $[a, b]$  for any  $b > a$ . Then if the improper integral (type I) of  $|f|$  converges, so does the integral of  $f$ , and

$$\left| \int_a^{+\infty} f(x) dx \right| \leq \int_a^{+\infty} |f(x)| dx.$$

**Example 11.4:** Consider the function  $f(x) = \frac{\cos x}{x^2}$ . Then  $|f(x)| \leq \frac{1}{x^2}$ . So  $|f(x)|$  is integrable on  $[1, +\infty)$  by the Comparison Test. It follows that  $f(x)$  is integrable by the Absolute Convergence Test:

$$\left| \int_1^{+\infty} \frac{\cos x}{x^2} dx \right| \leq \int_1^{+\infty} \left| \frac{\cos x}{x^2} \right| dx \leq \int_1^{+\infty} \frac{1}{x^2} dx = 1.$$

**Remark:** The converse is not necessarily true:

$$\int_a^{+\infty} f(x) dx < +\infty \quad \text{does not imply that} \quad \int_a^{+\infty} |f(x)| dx < +\infty.$$

### Asymptotic Comparison Test

Suppose that  $f \sim \frac{1}{x^\alpha}$  as  $x \rightarrow +\infty$ . Then:

$$\begin{aligned} \alpha > 1 &\Rightarrow \int_a^{+\infty} f(x) dx < +\infty \text{ (converges)} \\ \alpha \leq 1 &\Rightarrow \int_a^{+\infty} f(x) dx = +\infty \text{ (diverges)} \end{aligned}$$

**Example 11.5:** Investigate

$$\int_1^{+\infty} (\pi - 2 \arctan x) dx = ?$$

We know that  $\lim_{x \rightarrow +\infty} \arctan x = \frac{\pi}{2}$ , so that the integrand  $f(x) = \pi - 2 \arctan x$  tends to 0. Let's determine its order at  $+\infty$  using De l'Hôpital:

$$\lim_{x \rightarrow +\infty} \frac{\pi - 2 \arctan x}{\frac{1}{x}} = \lim_{x \rightarrow +\infty} \frac{2x^2}{1 + x^2} = 2.$$

Therefore  $f(x) \sim \frac{2}{x}$  as  $x \rightarrow +\infty$  and the integral diverges using the Asymptotic Comparison Test.