MATHEMATICAL ANALYSIS 1
HOMEWORK 4

(1) Prove De Moivre's formula (*without* using the exponential form of complex numbers): for any complex number $z = r(\cos\theta + i\sin\theta)$, where $r > 0$ and $\theta \in \mathbb{R}$, and any $n \in \mathbb{N}$,
$$z^n = r^n(\cos(n\theta) + i\sin(n\theta)).$$
Hint: by induction.

(2) Let $z = 2 - 2i$ and $w = -1 + i\sqrt{3}$.
   (a) Write z and w in exponential form.
   (b) Compute $z \cdot w$ and express the result in both Cartesian and exponential forms.
   (c) Compute $\frac{z}{w}$ and express the result in both Cartesian and exponential forms.
   (d) Find the complex conjugate $\overline{z}$ and compute $z \cdot \overline{z}$.

(3) Let $z = 3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ and $w = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$.
   (a) Write z and w in Cartesian form.
   (b) Compute z^2 and w^3 using both De Moivre's formula and direct multiplication.
   (c) Verify that $|z \cdot w| = |z| \cdot |w|$ and $\arg(z \cdot w) = \arg(z) + \arg(w)$.

(4) Find all complex numbers z that satisfy:
   (a) $z^2 = -4$
   (b) $z^3 = 8i$
   (c) $z^4 = -16$

   Express your answers in both Cartesian and exponential forms.

(5) Let $z = 1 + i\sqrt{3}$.
   (a) Write z in exponential form.
   (b) Compute z^4 and express the result in both exponential and Cartesian forms.
   (c) Find all complex solutions to $w^3 = z$.

(6) Let $z = \frac{1+i}{1-i}$.
   (a) Simplify z to Cartesian form.
   (b) Write z in exponential form.
   (c) Compute z^{2023} (hint: use the exponential form).

(7) Let $\{a_n\}_{n\in\mathbb{N}}$ be a real-valued sequence. Prove that it can have at most one limit as $n \to \infty$ (hint: by contradiction).

(8) We proved that a monotone increasing sequence $\{a_n\}_{n\in\mathbb{N}}$ has a limit (which can be finite or infinite) and that it is equal to the supremum of the sequence (see the lecture notes for the precise statement). State and prove the analogous result for a monotone *decreasing* sequence.

(9) Let $\{a_n\}_{n\in\mathbb{N}}$ be a real-valued sequence. Suppose that the limit $\lim_{n\to\infty} |a_n|$ exists. Does $\lim_{n\to\infty} a_n$ exist? If so, prove it. If not, give a counterexample.

(10) Let $x_0 \in \mathbb{R}$.
   (a) Let $\varepsilon > 0$. Define the ε-neighborhood of x_0 (this is another name for the neighborhood of x_0 of size ε).
   (b) Prove that the intersection of two neighborhoods of x_0 is also a neighborhood of x_0.

(11) Consider the sequence $\{a_n\}_{n\in\mathbb{N}}$ defined by $a_n = \frac{n^2+(-1)^n n}{n^2+1}$.
   (a) Determine whether the sequence has a finite limit, infinite limit, or is indeterminate.
   (b) Prove your answer using the definition of limit.

(12) Let $f(x) = \frac{2x^2-3x+1}{x^2+4}$. Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. Prove your answers.

(1) *Proof.* We prove De Moivre's formula by induction on n.

Base case ($n = 1$): $z^1 = r^1(\cos(1\theta) + i\sin(1\theta))$, which is true by definition of z. (The base case $n = 0$ is also trivially true, but $n = 1$ is often preferred for multiplication-based induction.)

Inductive step: Assume the formula holds for some $n \in \mathbb{N}$. Then for $n + 1$:

$$\begin{aligned} z^{n+1} &= z^n \cdot z \\ &= r^n(\cos(n\theta) + i\sin(n\theta)) \cdot r(\cos\theta + i\sin\theta) \\ &= r^{n+1}[(\cos(n\theta)\cos\theta - \sin(n\theta)\sin\theta) + i(\sin(n\theta)\cos\theta + \cos(n\theta)\sin\theta)] \\ &= r^{n+1}[\cos((n+1)\theta) + i\sin((n+1)\theta)] \end{aligned}$$

where we used the trigonometric identities:

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$
$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

Thus the formula holds for $n + 1$, completing the induction. □

(2) (a) For $z = 2 - 2i$:

$$|z| = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}, \quad \arg(z) = -\frac{\pi}{4}$$

Exponential form: $z = 2\sqrt{2}e^{-i\pi/4}$

For $w = -1 + i\sqrt{3}$:

$$|w| = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2, \quad \arg(w) = \frac{2\pi}{3}$$

Exponential form: $w = 2e^{i2\pi/3}$

(b) $z \cdot w$: Exponential form: $z \cdot w = (2\sqrt{2}e^{-i\pi/4})(2e^{i2\pi/3}) = 4\sqrt{2}e^{i(2\pi/3-\pi/4)} = 4\sqrt{2}e^{i5\pi/12}$

Cartesian form: $4\sqrt{2}(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12})$

(c) $\frac{z}{w}$: Exponential form: $\frac{z}{w} = \frac{2\sqrt{2}e^{-i\pi/4}}{2e^{i2\pi/3}} = \sqrt{2}e^{-i(\pi/4+2\pi/3)} = \sqrt{2}e^{-i11\pi/12}$

Cartesian form: $\sqrt{2}(\cos(-\frac{11\pi}{12}) + i\sin(-\frac{11\pi}{12}))$

(d) $\overline{z} = 2 + 2i$, and $z \cdot \overline{z} = (2 - 2i)(2 + 2i) = 4 + 4 = 8 = |z|^2$

(3) (a) $z = 3(\frac{1}{2} + i\frac{\sqrt{3}}{2}) = \frac{3}{2} + i\frac{3\sqrt{3}}{2}$

$w = 2(\frac{\sqrt{3}}{2} + i\frac{1}{2}) = \sqrt{3} + i$

(b) Using De Moivre: $z^2 = 3^2(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}) = 9(-\frac{1}{2} + i\frac{\sqrt{3}}{2}) = -\frac{9}{2} + i\frac{9\sqrt{3}}{2}$

Direct: $z^2 = (\frac{3}{2} + i\frac{3\sqrt{3}}{2})^2 = \frac{9}{4} + 2\cdot\frac{3}{2}\cdot i\frac{3\sqrt{3}}{2} + (i\frac{3\sqrt{3}}{2})^2 = \frac{9}{4} + i\frac{9\sqrt{3}}{2} - \frac{27}{4} = -\frac{18}{4} + i\frac{9\sqrt{3}}{2} = -\frac{9}{2} + i\frac{9\sqrt{3}}{2}$

Using De Moivre: $w^3 = 2^3(\cos\frac{3\pi}{6} + i\sin\frac{3\pi}{6}) = 8(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) = 8(0 + i) = 8i$

Direct: $w^3 = (\sqrt{3} + i)^3 = 3\sqrt{3} + 9i - 3\sqrt{3} - i = 8i$

(c) $z \cdot w = (3 \cdot 2)(\cos(\frac{\pi}{3} + \frac{\pi}{6}) + i\sin(\frac{\pi}{3} + \frac{\pi}{6})) = 6(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) = 6i$

$|z \cdot w| = |6i| = 6$ $|z| \cdot |w| = 3 \cdot 2 = 6$ (Verification holds)

$\arg(z \cdot w) = \frac{\pi}{2}$ $\arg(z) + \arg(w) = \frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}$ (Verification holds)

(4) (a) $z^2 = -4$. In polar form: $-4 = 4(\cos\pi + i\sin\pi) = 4e^{i\pi}$.

The roots are $z_k = \sqrt{4}e^{i\frac{\pi+2k\pi}{2}}$ for $k = 0, 1$.

- $k = 0$: $z_0 = 2e^{i\pi/2} = 2i$
- $k = 1$: $z_1 = 2e^{i3\pi/2} = -2i$

(b) $z^3 = 8i$. In polar form: $8i = 8(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) = 8e^{i\pi/2}$.

The roots are $z_k = \sqrt[3]{8}e^{i\frac{\pi/2+2k\pi}{3}}$ for $k = 0, 1, 2$.

- $k = 0$: $z_0 = 2e^{i\pi/6} = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) = \sqrt{3} + i$
- $k = 1$: $z_1 = 2e^{i5\pi/6} = 2(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}) = -\sqrt{3} + i$
- $k = 2$: $z_2 = 2e^{i3\pi/2} = 2(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}) = -2i$

(c) $z^4 = -16$. In polar form: $-16 = 16(\cos\pi + i\sin\pi) = 16e^{i\pi}$.

The roots are $z_k = \sqrt[4]{16}e^{i\frac{\pi+2k\pi}{4}}$ for $k = 0, 1, 2, 3$.

- $k = 0$: $z_0 = 2e^{i\pi/4} = 2(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) = \sqrt{2} + i\sqrt{2}$

- $k = 1$: $z_1 = 2e^{i3\pi/4} = 2(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}) = -\sqrt{2} + i\sqrt{2}$
- $k = 2$: $z_2 = 2e^{i5\pi/4} = 2(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}) = -\sqrt{2} - i\sqrt{2}$
- $k = 3$: $z_3 = 2e^{i7\pi/4} = 2(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}) = \sqrt{2} - i\sqrt{2}$

(5) (a) $|z| = \sqrt{1^2 + (\sqrt{3})^2} = 2$, $\arg(z) = \frac{\pi}{3}$. Exponential form: $z = 2e^{i\pi/3}$

(b) z^4: Exponential form: $z^4 = (2e^{i\pi/3})^4 = 16e^{i4\pi/3}$
Cartesian form: $z^4 = 16(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}) = 16(-\frac{1}{2} - i\frac{\sqrt{3}}{2}) = -8 - 8i\sqrt{3}$

(c) $w^3 = z = 2e^{i\pi/3}$. The roots are $w_k = \sqrt[3]{2}e^{i\frac{\pi/3+2k\pi}{3}}$ for $k = 0, 1, 2$.

- $k = 0$: $w_0 = \sqrt[3]{2}e^{i\pi/9}$
- $k = 1$: $w_1 = \sqrt[3]{2}e^{i(\pi/9+2\pi/3)} = \sqrt[3]{2}e^{i7\pi/9}$
- $k = 2$: $w_2 = \sqrt[3]{2}e^{i(\pi/9+4\pi/3)} = \sqrt[3]{2}e^{i13\pi/9}$

(6) (a) $z = \frac{1+i}{1-i} = \frac{(1+i)^2}{(1-i)(1+i)} = \frac{1+2i-1}{1+1} = \frac{2i}{2} = i$

(b) $|z| = 1$, $\arg(z) = \frac{\pi}{2}$. Exponential form: $z = e^{i\pi/2}$

(c) $z^{2023} = (e^{i\pi/2})^{2023} = e^{i(1010\pi+3\pi/2)} = e^{i3\pi/2} = -i$.

(7) *Proof.* By contradiction, assume that $\{a_n\}_{n\in\mathbb{N}}$ has two distinct limits ℓ_1 and ℓ_2 with $\ell_1 \neq \ell_2$. Let $\varepsilon = \frac{|\ell_1-\ell_2|}{2} > 0$.

Since $\lim_{n\to\infty} a_n = \ell_1$, there exists N_1 such that for all $n > N_1$, $|a_n - \ell_1| < \varepsilon$.
Since $\lim_{n\to\infty} a_n = \ell_2$, there exists N_2 such that for all $n > N_2$, $|a_n - \ell_2| < \varepsilon$.
Let $N = \max\{N_1, N_2\}$. Then for $n > N$, by the triangle inequality:

$$|\ell_1 - \ell_2| = |(\ell_1 - a_n) + (a_n - \ell_2)| \leq |\ell_1 - a_n| + |a_n - \ell_2| < \varepsilon + \varepsilon = 2\varepsilon = |\ell_1 - \ell_2|$$

This gives the contradiction $|\ell_1-\ell_2| < |\ell_1-\ell_2|$. Therefore, the limit (if it exists) must be unique. □

(8)

Theorem. *A monotone decreasing sequence $\{a_n\}_{n\in\mathbb{N}}$ has a limit (finite or infinite) equal to the infimum of the sequence. Specifically:*

- *If $\{a_n\}_{n\in\mathbb{N}}$ is bounded below, then $\lim_{n\to\infty} a_n = \inf_{n\in\mathbb{N}}\{a_n\}$*
- *If $\{a_n\}_{n\in\mathbb{N}}$ is unbounded below, then $\lim_{n\to\infty} a_n = -\infty$*

Proof. Let $m = \inf_{n\in\mathbb{N}}\{a_n\}$.

Case 1: $m > -\infty$ (bounded below). For any $\varepsilon > 0$, $m + \varepsilon$ is not a lower bound. Therefore, by the definition of infimum, there exists N such that $a_N < m + \varepsilon$. Since the sequence is decreasing, for all $n > N$, we have $m \leq a_n \leq a_N < m + \varepsilon$. This implies $|a_n - m| < \varepsilon$ for all $n > N$, so by definition $\lim_{n\to\infty} a_n = m$.

Case 2: $m = -\infty$ (unbounded below). For any $M > 0$, since the sequence is unbounded below, there exists N such that $a_N < -M$. Since the sequence is decreasing, for all $n > N$, we have $a_n \leq a_N < -M$. Thus, by definition, $\lim_{n\to\infty} a_n = -\infty$. □

(9) No, the limit of a_n may not exist. Counterexample: Let $a_n = (-1)^n$. Then $|a_n| = 1$ for all n, so $\lim_{n\to\infty} |a_n| = 1$ exists. However, $\lim_{n\to\infty} a_n$ does not exist because the sequence oscillates between -1 and 1 and does not converge to a single value.

(10) (a) The ε-neighborhood of x_0 is the open interval centered at x_0 with radius ε:

$$N_\varepsilon(x_0) = \{x \in \mathbb{R} : |x - x_0| < \varepsilon\} = (x_0 - \varepsilon, x_0 + \varepsilon)$$

(b) Let $N_{\varepsilon_1}(x_0)$ and $N_{\varepsilon_2}(x_0)$ be two neighborhoods of x_0. Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. Since $\varepsilon > 0$, $N_\varepsilon(x_0)$ is a neighborhood of x_0. For any $x \in N_\varepsilon(x_0)$, we have $|x - x_0| < \varepsilon$. Since $\varepsilon \leq \varepsilon_1$ and $\varepsilon \leq \varepsilon_2$, we have $|x - x_0| < \varepsilon_1$ and $|x - x_0| < \varepsilon_2$. Thus $x \in N_{\varepsilon_1}(x_0)$ and $x \in N_{\varepsilon_2}(x_0)$, which means $x \in N_{\varepsilon_1}(x_0) \cap N_{\varepsilon_2}(x_0)$. Since $N_\varepsilon(x_0)$ is contained in the intersection, the intersection contains a neighborhood of x_0 and is therefore also a neighborhood of x_0.

(11) (a) The sequence has a **finite limit**: $\lim_{n\to\infty} a_n = 1$.

(b) *Proof.* We want to prove $\lim_{n\to\infty} a_n = 1$. Let $\varepsilon > 0$ be given. We seek $N \in \mathbb{N}$ such that for all $n > N$, $|a_n - 1| < \varepsilon$.

First, simplify $|a_n - 1|$:

$$|a_n - 1| = \left| \frac{n^2 + (-1)^n n}{n^2+1} - 1 \right| = \left| \frac{n^2 + (-1)^n n - (n^2+1)}{n^2+1} \right| = \left| \frac{(-1)^n n - 1}{n^2+1} \right|$$

Using the triangle inequality on the numerator and the fact that $n^2 + 1 > 0$:

$$|a_n - 1| \leq \frac{|(-1)^n n| + |-1|}{n^2+1} = \frac{n+1}{n^2+1}$$

For $n \geq 1$, we can simplify the upper bound:

$$\frac{n+1}{n^2+1} \leq \frac{n+n}{n^2} = \frac{2n}{n^2} = \frac{2}{n}$$

We want $\frac{2}{n} < \varepsilon$, which means $n > \frac{2}{\varepsilon}$.
Choose $N = \lceil \frac{2}{\varepsilon} \rceil$. Then for all $n > N$:

$$|a_n - 1| \leq \frac{n+1}{n^2+1} \leq \frac{2}{n} < \frac{2}{N} \leq \varepsilon$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\lim_{n\to\infty} a_n = 1$. □

(12) Claim: We have $\lim_{x\to\infty} f(x) = 2$ and $\lim_{x\to-\infty} f(x) = 2$.

Proof. For $x \to \infty$: We want to find $A > 0$ such that for $x > A$, $|f(x) - 2| < \varepsilon$.

$$|f(x) - 2| = \left| \frac{2x^2 - 3x + 1}{x^2+4} - 2 \right| = \left| \frac{2x^2 - 3x + 1 - 2(x^2+4)}{x^2+4} \right| = \left| \frac{-3x - 7}{x^2+4} \right|$$

For $x > 1$, we have $3x + 7 > 0$, so $|-3x - 7| = 3x + 7$.

$$|f(x) - 2| = \frac{3x+7}{x^2+4} \leq \frac{3x+7}{x^2}$$

For $x > 1$, $7 < 7x$, so $3x + 7 < 3x + 7x = 10x$. Thus:

$$|f(x) - 2| < \frac{10x}{x^2} = \frac{10}{x}$$

We want $\frac{10}{x} < \varepsilon$, which means $x > \frac{10}{\varepsilon}$. Choose $A = \max\{1, \frac{10}{\varepsilon}\}$. Then for $x > A$, $|f(x) - 2| < \frac{10}{x} < \frac{10}{A} \leq \varepsilon$. Thus, $\lim_{x\to+\infty} f(x) = 2$.

For $x \to -\infty$: We want to find $A > 0$ such that for $x < -A$, $|f(x) - 2| < \varepsilon$. Let $x < -1$. Then $|x| = -x$. The expression is:

$$|f(x) - 2| = \left| \frac{-3x - 7}{x^2+4} \right| = \frac{|-3x - 7|}{x^2+4}$$

For x sufficiently large negative (e.g., $x < -3$, so $-3x > 9$), the numerator $-3x - 7$ is positive, and we can use the same bound as for $x \to +\infty$ by noting $x^2 = |x|^2$:

$$|f(x) - 2| = \frac{-3x - 7}{x^2+4} \leq \frac{-3x - 7}{x^2} < \frac{-3x}{x^2} = \frac{3}{|x|}$$

We want $\frac{3}{|x|} < \varepsilon$, so $|x| > \frac{3}{\varepsilon}$. Choose $A' = \max\{3, \frac{3}{\varepsilon}\}$. Then for $x < -A'$, $|f(x) - 2| < \frac{3}{|x|} < \frac{3}{A'} \leq \varepsilon$. Thus, $\lim_{x\to-\infty} f(x) = 2$. □