Università degli Studi di Roma "Tor Vergata"

Dipartimento di Matematica

Analysis 1 (Engineering Sciences) 2025-2026

Instructor: Prof. Jonathan Ben-Artzi

Final Examination — *Call 1 of 6*

27 January 2026

First Name (CAPITALS): ______________________________

Last Name (CAPITALS): ______________________________

Matricola: ______________________________

Grading Summary

Quest.	1	2	3	4	5	6	7	8	9	10	**Total**
Points	1	1	1	1	1	1	1	1	1	1	**10**
Score											

Quest.	11	12	13	14	15	**Total**
Points	3	3	3	3	3	**15**
Score						

FINAL GRADE	**/ 25**

Examination Rules:

- **Duration**: 2 hours and 30 minutes.
- **NO** cellphones, **NO** calculators, **NO** books, **NO** notes, and **NO** headphones.
- Write full solutions clearly within the provided spaces.
- Part B will only be graded if the student achieves a score of at least 9/10 in Part A.
- Any student caught copying or engaging in academic misconduct will face disciplinary action.
- Use only blue or black ink. Additional paper will be provided upon request.

Do not turn this sheet over until instructed to do so.

Part A

Exercise 1

As $x \to +\infty$, the function $f(x) = \ln\left(9 + \sin\frac{2}{x}\right) - 2\ln 3$ can be written in the form $f(x) = \frac{b}{x^\alpha} + o(x^{-\alpha})$. Find b and α. ____/ 1 p.

Exercise 2

Find the following limit. Explain your answer. ____/ 1 p.

$$\lim_{x\to e} \frac{\ln x - 1}{x - e}.$$

Exercise 3

Show that the equation $x^2 = 7$ has no rational solution.

______/ 1 p.

Exercise 4

Prove that for any $r > 0$ the equation $x^2 \sin\left(\frac{1}{x}\right) = r$ has a solution with $x > 0$.

______/ 1 p.

Exercise 5

Compute the limit $\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$. Justify your answer. ____/ 1 p.

Exercise 6

Let f be continuously differentiable on (a, b) and let $x_0 \in (a, b)$ satisfy $f(x_0) = 0$. Prove that there exists exactly one number L such that ____/ 1 p.

$$\lim_{x \to x_0} \frac{f(x) - L(x - x_0)}{x - x_0} = 0.$$

Find L.

Exercise 7

Evaluate $\lim_{x\to 0} \frac{1-\cos x}{x^2}$ using the Maclaurin expansion of cosine. ______/ 1 p.

Exercise 8

Consider the sequence $a_n = \arctan\left(\frac{5n+6}{n+1}\right)$, $n \in \mathbb{N}$. (i) Is this a monotone sequence? (ii) Find its infimum and supremum. (iii) Does $\lim_{n\to\infty} a_n$ exist? If so, what is it? ______/ 1 p.

Exercise 9

Evaluate the definite intergral $\int_0^1 \arctan x \text{ dx}$. ——/ 1 p.

Exercise 10

Use a trigonometric substitution to evaluate the definite integral $\int_0^3 \sqrt{9-x^2} \text{ dx}$. ——/ 1 p.

Part B

Exercise 11

Let $a, \ell_f, \ell_g \in \mathbb{R}$, let $f, g : \mathbb{R} \to \mathbb{R}$. Prove the following statement: ______/ 3 p.

If $\lim_{x\to a} f(x) = \ell_f$ and $\lim_{x\to a} g(x) = \ell_g$, then $\lim_{x\to a} \big(f(x) + g(x)\big) = \ell_f + \ell_g$.

Exercise 12

Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable. State and prove Rolle's Theorem for f. In your proof you may rely on Weierstrass' Theorem and Fermat's Theorem without proving them. ____/ 3 p.

Exercise 13

______/ 3 p.

Compute $f^{(6)}(0)$ where f is the function

$$f(x) = \sinh(x^2 + 2\sin^2 x).$$

You are reminded that $\sinh u = \frac{1}{2}(e^u - e^{-u})$. You may leave factorials without computing them.

Exercise 14

Using the expression for the derivatives of $\arctan x$, show that ______/ 3 p.

$$\arctan\frac{1}{x} = \frac{\pi}{2} - \arctan x, \qquad \forall x > 0.$$

Exercise 15

Suppose that two functions f, g can be written as $f(x) = p_n(x) + o(x^n)$ and $g(x) = q_m(x) + o(x^m)$ as $x \to 0$, where $p_n(x) = a_0 + a_1 x + \cdots a_n x^n$ and $q_m(x) = b_0 + b_1 x + \cdots b_m x^m$ are two polynomials of orders n and m respectively. The product $f(x) \cdot g(x)$ can be expressed as

____/ 3 p.

$$f(x) \cdot g(x) = \sum_{k=0}^{\ell} c_k x^k + o(x^\ell) \quad \text{as } x \to 0.$$

(i) Express the coefficients c_k in terms of a_i and b_j.
(ii) Identify the order ℓ in terms of n and m.