Example 7.1: The function

$$f(x) = \begin{cases} 1 & x \geq 10 \\ -1 & x < 10 \end{cases}$$

changes sign on the interval $[9, 11]$. However, there is no zero in $[9, 11]$ (i.e. there is no $x \in [9, 11]$ such that $f(x) = 0$. Why? The problem with this f is that it is not continuous on $[9, 11]$ (there's a jump discontinuity at $x = 10$).

Example 7.2: The function $f(x) = x^2$ is a continuous function on $\mathbb{R}$ that has a zero at $x = 0$, however it is always non-negative (i.e. it does not change sign). This shows that a continuous function can have a zero without changing sign. That is, changing sign is a *sufficient* condition for a continuous function to have a zero, but not a *necessary* condition.

Example 7.3: The function $f(x) = e^x + \sin x$ is continuous on $\mathbb{R}$. Let's look at the interval $[-\frac{\pi}{2}, 0]$. For $x = -\frac{\pi}{2}$ the function is negative: $f(-\frac{\pi}{2}) = e^{-\pi/2} + \sin(-\frac{\pi}{2}) = e^{-\pi/2} - 1 < e^0 - 1 = 1 - 1 = 0$ and for $x = 0$ the function is positive: $f(0) = e^0 - \sin 0 = 1 - 0 = 1 > 0$. Hence there exists $x_0 \in (-\frac{\pi}{2}, 0)$ such that $f(x_0) = 0$. Moreover, e^x is strictly increasing on $\mathbb{R}$, and $\sin x$ is strictly increasing on $[-\frac{\pi}{2}, 0]$, so that x_0 is the unique zero within this interval.

Corollary 7.2: Let $(\alpha, \beta) \subseteq \mathbb{R}$ be an interval (α may be $-\infty$ and β may be $+\infty$), and let $f : \mathbb{R} \to \mathbb{R}$ be defined on (a, b), satisfy

$$\lim_{x \to \alpha^+} f(x) = \ell_\alpha \in \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$$
$$\lim_{x \to \beta^-} f(x) = \ell_\beta \in \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$$

with ℓ_α and ℓ_β having opposite signs. Then f has a zero x_0 in (α, β): $f(x_0) = 0$. Moreover, if f is strictly monotone in (α, β) then x_0 is unique.

Proof. This proof is an exercise. □

Corollary 7.3: Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous on an interval $[a, b]$. If

$$f(a) < g(a) \quad \text{and} \quad f(b) > g(b)$$

or

$$f(a) > g(a) \quad \text{and} \quad f(b) < g(b)$$

then there exists a point $x_0 \in (a, b)$ satisfying

$$f(x_0) = g(x_0).$$

Moreover, if f and g are strictly monotone then x_0 is unique.

Proof. This proof is very simple: we consider the function

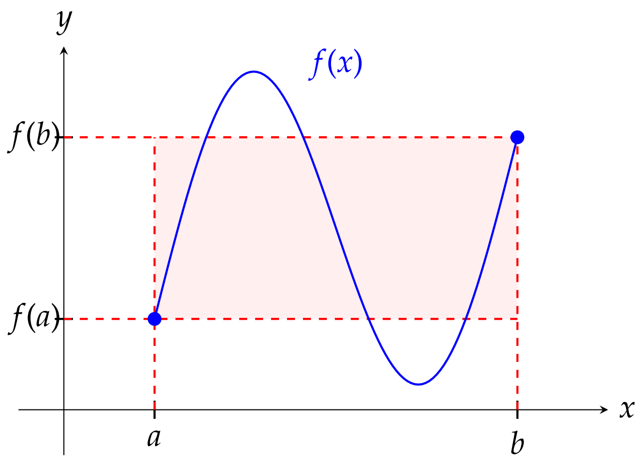
$$h(x) = f(x) - g(x).$$

Then h is continuous on $[a, b]$ (since it is the difference of two continuous functions on $[a, b]$). Furthermore, $h(a)$ and $h(b)$ have different sign (due to the assumptions on f and g). So h satisfies the conditions of Theorem 7.1, and there exists $x_0 \in (a, b)$ such that $h(x_0) = 0$. But this means (by definition of h) that $f(x_0) = g(x_0)$.

The strictly monotone case is also a consequence of Theorem 7.1 (can you think why if f and g are both strictly monotone, then h is strictly monotone? it is not immediately evident. Look at Lemma 2.2). □

7.2 Range of a continuous function defined on an interval

Theorem 7.4 (Intermediate Value Theorem)**:** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on the closed interval $[a, b]$, where $a < b$. Then f attains all values in the closed interval $[f(a), f(b)]$.



Proof. If $f(a) = f(b)$ the statement of the theorem is trivial: $[f(a), f(b)]$ is a single point, and it has both a and b in its pre-image. So we assume that $f(a) < f(b)$. The case $f(a) > f(b)$ follows similarly, and is left as an exercise.

We need to prove that for any $f(a) < y_0 < f(b)$ there exists $a < x_0 < b$ such that $f(x_0) = y_0$. Consider the constant function $g(x) = y_0$ for all $x \in [a, b]$. Then

$$f(a) < y_0 = g(a) \quad \text{and} \quad g(b) = y_0 < f(b).$$

Thus $f(a) < g(a)$ and $f(b) > g(b)$. By Corollary 7.3 there is a point $x_0 \in (a, b)$ where $f(x_0) = g(x_0)$. By the definition of g, we have that $f(x_0) = y_0$ and the proof is complete. □

Corollary 7.5: Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on an interval $I \subseteq \mathbb{R}$. Then $f(I)$ is also an interval, with endpoints give by $\inf_I f$ and $\sup_I f$.

Remark: The interval I can be open, closed, or half open half closed. Moreover, it may be infinite on one or both ends.

Proof. If $f(I)$ is a single point there's nothing to prove. So assume that there exist $y_1 \neq y_2$ both belonging to $f(I)$, and assume, without loss of generality, that $y_1 < y_2$. Then there exist $x_1 \neq x_2$, both elements of I, such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Assume that $x_1 < x_2$ (the other case will follow a similar proof). Then by the Intermediate Value Theorem (Theorem 7.4), f attains all values $y \in [y_1, y_2]$:

$$[y_1, y_2] \subseteq f([x_1, x_2]) \subseteq f(I).$$

Therefore we have shown that for any two points in $f(I)$, the closed interval that has these points as end-points is a subset of $f(I)$. This implies that $f(I)$ is itself an interval (possibly infinite). This fact is discussed in Lemma 7.6 below. By the definition of *infimum* and *supremum* of a set, it follows that the endpoints of $f(I)$ are $\inf_I f$ and $\sup_I f$ (they can be infinite). □

Lemma 7.6: Let $I \subseteq \mathbb{R}$ be a subset of $\mathbb{R}$ that satisfies the following condition: for any $x_1, x_2 \in I$ with $x_1 < x_2$, the entire closed interval $[x_1, x_2]$ is a subset of I. Then I is an interval, i.e. I could be an open, closed, or half-open-half-closed interval, and one or both of its endpoints can be infinite.

Proof. This is a simple proof (by contradiction) which we skip here. □

Example 7.4: Let $f(x) = \tan x$. Then $f((-\frac{\pi}{2}, \frac{\pi}{2})) = (-\infty, +\infty)$.

Example 7.5: Let $f(x) = \cos x$. Then $f((-\infty, +\infty)) = [-1, 1]$.

Example 7.6: Let $f(x) = \arctan x$. Then $f((-\infty, +\infty)) = (-\frac{\pi}{2}, \frac{\pi}{2})$.

Example 7.7: Let $f(x) = e^x$. Then $f([0, +\infty)) = [1, +\infty)$.

The following theorem tells us that the image of a closed interval under a continuous function is always a closed interval:

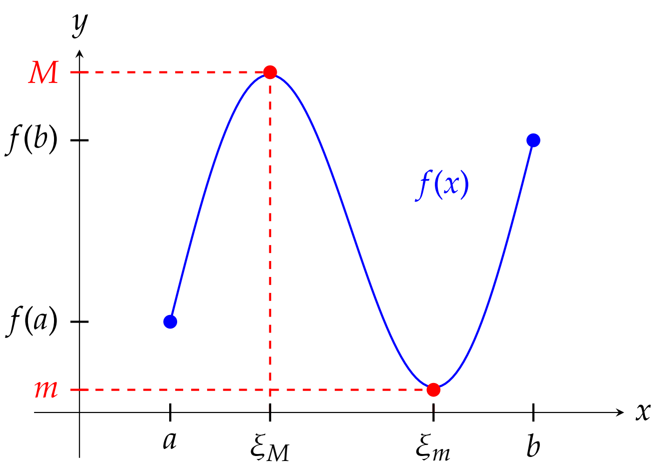
Weierstrass' Theorem

Let f be continuous on the interval $[a, b]$, where $a < b$ are real numbers. Then f is bounded on $[a, b]$ and it attains its minimum and maximum on $[a, b]$:

$$m = \min_{x \in [a,b]} f(x) \qquad \text{and} \qquad M = \max_{x \in [a,b]} f(x).$$

It follows that

$$f([a, b]) = [m, M].$$



Proof. **Step A.** The supremum. Define

$$M = \sup_{x \in [a,b]} f(x)$$

which can be a finite number or $+\infty$.

Case 1: M is finite. In this case, we know by definition of the supremum that for any $\varepsilon > 0$ there exists $x_\varepsilon \in [a, b]$ satisfying $M - \varepsilon < f(x_\varepsilon) \leq M$. Our goal is to construct a sequence $\{a_n\}_{n\in N_+}$, as follows: instead of ε take, for each $n \geq 1$, $\frac{1}{n}$, so that there exists $x_n \in [a, b]$ such that

$$M - \frac{1}{n} < f(x_n) \leq M.$$

By the Squeeze Theorem for sequences (Theorem 5.15(5)) if follows that

$$\lim_{n\to\infty} f(x_n) = M.$$

Case 2: M **is infinite.** In this case, again by the definition of the supremum, for each $n \in \mathbb{N}_+$ there exists $x_n \in [a, b]$ such that

$$f(x_n) > n.$$

By the Squeeze to $\pm\infty$ Theorem for sequences (Theorem 5.15(4)) we must have

$$\lim_{n\to\infty} f(x_n) = +\infty = M.$$

Step B. The sequence $\{x_n\}_{n\in\mathbb{N}_+}$. In both cases, we obtained a sequence $\{x_n\}_{n\in\mathbb{N}_+} \subset [a, b]$. This is a bounded sequence (it is contained in a bounded interval). By the Bolzano-Weierstrass Theorem (see below) it has a convergent subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$. Call its limit $\xi_M \in [a, b]$. Then we have:

$$\lim_{k\to\infty} x_{n_k} = \xi_M.$$

Moreover, since $f(x_n)$ converges, so does its subsequence $f(x_{n_k})$ and they share the same limit:

$$M = \lim_{n\to\infty} f(x_n) = \lim_{k\to\infty} f(x_{n_k}).$$

Using the continuity of f at the point ξ_M, we have

$$M = \lim_{k\to\infty} f(x_{n_k}) = f(\lim_{k\to\infty} x_{n_k}) = f(\xi_M).$$

But if $M = f(\xi_M)$ it must be a real number, so M cannot be $+\infty$. Furthermore, since M is attained at ξ_M, it belongs to the range f on $[a, b]$, hence

$$M = \max_{x\in[a,b]} f(x).$$

The proof for m follows the same ideas.

Finally, the fact that $f([a, b]) = [m, M]$ is an immediate consequence of Corollary 7.5. $\square$

The Bolzano-Weierstrass Theorem

Let $\{x_n\}_{n\in\mathbb{N}}$ be a bounded sequence (i.e., there exist real numbers $a < b$ such that $a < x_n < b$ for all $n \in \mathbb{N}$). Then $\{x_n\}_{n\in\mathbb{N}}$ has a convergent subsequence: there exists a sequence of indices $\{n_k\}_{k\in\mathbb{N}} \subseteq \mathbb{N}$ such that the subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ converges.