

**MATHEMATICAL ANALYSIS 1**  
**HOMEWORK 10**

- (1) Prove the following proposition: (it is Proposition 9.4 in the lecture notes)

**Proposition.** *Any Maclaurin polynomial of an even function contains only even powers. Any Maclaurin polynomial of an odd function contains only odd powers.*

- (2) Let  $p_n(x) = a_0 + a_1x + \cdots + a_nx^n$  and  $q_m(x) = b_0 + b_1x + \cdots + b_mx^m$  be two polynomials of orders  $n$  and  $m$  respectively. Write the formula for  $p_n(x) \cdot q_m(x)$  (pay attention:  $n$  and  $m$  might be different). Explain your answer.

- (3) Suppose that two functions  $f, g$  can be written as

$$f(x) = p_n(x) + o(x^n) \quad g(x) = q_m(x) + o(x^m)$$

as  $x \rightarrow 0$ , where  $p_n, q_m$  are as in the previous question. Express the product  $f(x) \cdot g(x)$ . What is the order of the error? Explain your answer.

- (4) Write  $(Tf)_{n,x_0}(x)$  for the following  $f, n, x_0$ :

(a)  $f(x) = e^x, n = 4, x_0 = 2$

(d)  $f(x) = \sin x, n = 6, x_0 = \frac{\pi}{2}$

(b)  $f(x) = \ln x, n = 3, x_0 = 3$

(e)  $f(x) = \sqrt{2x+1}, n = 3, x_0 = 4$

(c)  $f(x) = 7 + x - 3x^2 + 5x^3, n = 2, x_0 = 1$

- (5) Write the Maclaurin expansions up to the indicated order with Peano's remainder:

(a)  $f(x) = x \cos 3x - 3 \sin x, n = 2$

(c)  $f(x) = \ln \frac{1+x}{1+3x}, n = 4$

(b)  $f(x) = e^{x^2} \sin 2x, n = 5$

(d)  $f(x) = \frac{x}{\sqrt[6]{1+x^2}} - \sin x, n = 5$

- (6) Prove that there exists a neighborhood of 0 on which the following inequality holds:

$$2 \cos(x + x^2) \leq 2 - x^2 - 2x^3.$$

- (7) (a) Write the Maclaurin polynomial of  $e^x$  of order  $n$  with Lagrange's remainder.  
(b) What is the minimal  $n$  we should take if we want to approximate the number  $e$  to within  $\frac{1}{1000000}$  (one millionth)? Justify your answer.

# HOMEWORK 10 SOLUTIONS

- (1) Prove the following proposition: (it is Proposition 9.4 in the lecture notes)

**Proposition.** *Any Maclaurin polynomial of an even function contains only even powers. Any Maclaurin polynomial of an odd function contains only odd powers.*

*Proof.* Let  $P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k$  be the  $n$ -th Maclaurin polynomial of  $f(x)$ . The proof relies on showing which derivatives  $f^{(k)}(0)$  must be zero.

**Case 1:  $f$  is an Even Function.** If  $f$  is an even function,  $f(-x) = f(x)$ .

We differentiate this identity repeatedly using the chain rule:

$$\begin{aligned} f'(-x)(-1) &= f'(x) & \Rightarrow f'(-x) &= -f'(x) \quad (f' \text{ is odd}) \\ f''(-x)(-1) &= -f'(-x)(-1) = f'(-x) & \Rightarrow f''(-x) &= f''(x) \quad (f'' \text{ is even}) \\ f'''(-x)(-1) &= f'''(x) & \Rightarrow f'''(-x) &= -f'''(x) \quad (f''' \text{ is odd}) \end{aligned}$$

By induction, all odd derivatives  $f^{(k)}$  are odd functions, and all even derivatives  $f^{(k)}$  are even functions.

Since an odd function must satisfy  $g(0) = -g(0)$ , we must have  $g(0) = 0$ . Therefore, for all odd  $k$ ,  $f^{(k)}(0) = 0$ . The Maclaurin coefficients  $\frac{f^{(k)}(0)}{k!}$  are zero for all odd  $k$ , leaving only terms corresponding to  $k = 0, 2, 4, \dots$  (even powers).

**Case 2:  $f$  is an Odd Function.** If  $f$  is an odd function,  $f(-x) = -f(x)$ .

Differentiating this identity repeatedly:

$$\begin{aligned} f'(-x)(-1) &= -f'(x) & \Rightarrow f'(-x) &= f'(x) \quad (f' \text{ is even}) \\ f''(-x)(-1) &= f''(-x) & \Rightarrow f''(-x) &= -f''(x) \quad (f'' \text{ is odd}) \end{aligned}$$

By induction, all even derivatives  $f^{(k)}$  are odd functions, and all odd derivatives  $f^{(k)}$  are even functions.

Since an odd function must be zero at 0, for all even  $k$ ,  $f^{(k)}(0) = 0$ . (This includes  $f^{(0)}(0) = f(0)$ ). The Maclaurin coefficients  $\frac{f^{(k)}(0)}{k!}$  are zero for all even  $k$ , leaving only terms corresponding to  $k = 1, 3, 5, \dots$  (odd powers).  $\square$

- (2) **Product of two polynomials**

The product  $p_n(x) \cdot q_m(x)$  is obtained by distributing every term of  $p_n$  with every term of  $q_m$ :

$$p_n(x) \cdot q_m(x) = \left( \sum_{i=0}^n a_i x^i \right) \left( \sum_{j=0}^m b_j x^j \right) = \sum_{i=0}^n \sum_{j=0}^m a_i b_j x^{i+j}.$$

Collecting terms with the same power  $k = i + j$ , we get:

$$p_n(x) \cdot q_m(x) = \sum_{k=0}^{n+m} \left( \sum_{\substack{i+j=k \\ 0 \leq i \leq n \\ 0 \leq j \leq m}} a_i b_j \right) x^k = \sum_{k=0}^{n+m} \left( \sum_{j=0}^k a_{k-j} b_j \right) x^k$$

(where the last expression is valid with our understanding that a coefficient with a non-existent index is 0). Thus the product is a polynomial of degree at most  $n + m$  (exactly  $n + m$  if  $a_n \neq 0$  and  $b_m \neq 0$ ). The coefficient of  $x^k$  is the sum of all products  $a_i b_j$  where  $i + j = k$ , with  $i$  and  $j$  ranging over the appropriate indices.

- (3) **Product of asymptotic expansions**

We have  $f(x) = p_n(x) + o(x^n)$  and  $g(x) = q_m(x) + o(x^m)$  as  $x \rightarrow 0$ . Their product is:

$$f(x)g(x) = (p_n(x) + o(x^n))(q_m(x) + o(x^m)).$$

Expanding:

$$f(x)g(x) = p_n(x)q_m(x) + p_n(x)o(x^m) + q_m(x)o(x^n) + o(x^n)o(x^m).$$

We analyze each error term:

- $p_n(x) = O(1)$  as  $x \rightarrow 0$  (since it's a polynomial with constant term  $a_0$ ). Hence  $p_n(x)o(x^m) = o(x^m)$ .
- Similarly,  $q_m(x) = O(1)$ , so  $q_m(x)o(x^n) = o(x^n)$ .
- $o(x^n)o(x^m) = o(x^{n+m})$  (product of functions tending to 0 faster than  $x^n$  and  $x^m$  tends to 0 faster than  $x^{n+m}$ ).

Thus the error term is  $o(x^m) + o(x^n) + o(x^{n+m})$ . The dominant (slowest vanishing) error is  $o(x^{\min\{n,m\}})$  because  $o(x^n)$  and  $o(x^m)$  vanish at least as fast as  $x^{\min\{n,m\}}$  (possibly faster), while  $o(x^{n+m})$  vanishes even faster. More precisely,  $o(x^n) + o(x^m) = o(x^{\min\{n,m\}})$ . Hence:

$$\begin{aligned} f(x)g(x) &= p_n(x)q_m(x) + o(x^{\min\{n,m\}}) \\ &= \sum_{k=0}^{\min\{n,m\}} \left( \sum_{j=0}^k a_{k-j}b_j \right) x^k + o(x^{\min\{n,m\}}) \quad \text{as } x \rightarrow 0. \end{aligned}$$

[\*\* Notice the upper bound on the  $k$  sum !! \*\*] **Order of the error:** The error is  $o(x^{\min\{n,m\}})$ , i.e., it tends to zero faster than  $x^{\min\{n,m\}}$ .

(4) **Taylor polynomials at given points**

Recall  $(Tf)_{n,x_0}(x) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$ .

- (a)  $f(x) = e^x$ ,  $n = 4$ ,  $x_0 = 2$ . All derivatives are  $e^x$ , so  $f^{(k)}(2) = e^2$ . Thus

$$(Tf)_{4,2}(x) = e^2 \sum_{k=0}^4 \frac{(x-2)^k}{k!} = e^2 \left[ 1 + (x-2) + \frac{(x-2)^2}{2} + \frac{(x-2)^3}{6} + \frac{(x-2)^4}{24} \right].$$

- (b)  $f(x) = \ln x$ ,  $n = 3$ ,  $x_0 = 3$ . Compute derivatives:

$$\begin{aligned} f(x) &= \ln x, & f(3) &= \ln 3, \\ f'(x) &= \frac{1}{x}, & f'(3) &= \frac{1}{3}, \\ f''(x) &= -\frac{1}{x^2}, & f''(3) &= -\frac{1}{9}, \\ f'''(x) &= \frac{2}{x^3}, & f'''(3) &= \frac{2}{27}. \end{aligned}$$

Thus

$$(Tf)_{3,3}(x) = \ln 3 + \frac{1}{3}(x-3) - \frac{1}{9 \cdot 2}(x-3)^2 + \frac{2}{27 \cdot 6}(x-3)^3 = \ln 3 + \frac{x-3}{3} - \frac{(x-3)^2}{18} + \frac{(x-3)^3}{81}.$$

- (c)  $f(x) = 7 + x - 3x^2 + 5x^3$ ,  $n = 2$ ,  $x_0 = 1$ . Since  $f$  itself is a polynomial, its Taylor polynomial of order 2 at  $x_0 = 1$  is simply the quadratic part of  $f$  expanded in powers of  $(x-1)$ . Compute:

$$\begin{aligned} f(1) &= 7 + 1 - 3 + 5 = 10, & f'(1) &= 1 - 6 + 15 = 10, \\ f'(x) &= 1 - 6x + 15x^2, & f''(1) &= -6 + 30 = 24, \\ f''(x) &= -6 + 30x, \end{aligned}$$

Hence

$$(Tf)_{2,1}(x) = 10 + 10(x-1) + \frac{24}{2}(x-1)^2 = 10 + 10(x-1) + 12(x-1)^2.$$

One can check that this equals the original cubic polynomial when expanded, up to terms of order  $(x-1)^2$ .

(d)  $f(x) = \sin x$ ,  $n = 6$ ,  $x_0 = \frac{\pi}{2}$ . Compute derivatives:

$$\begin{array}{ll} f(x) = \sin x, & f(\pi/2) = 1, \\ f'(x) = \cos x, & f'(\pi/2) = 0, \\ f''(x) = -\sin x, & f''(\pi/2) = -1, \\ f'''(x) = -\cos x, & f'''(\pi/2) = 0, \\ f^{(4)}(x) = \sin x, & f^{(4)}(\pi/2) = 1, \\ f^{(5)}(x) = \cos x, & f^{(5)}(\pi/2) = 0, \\ f^{(6)}(x) = -\sin x, & f^{(6)}(\pi/2) = -1. \end{array}$$

Thus

$$\begin{aligned} (Tf)_{6, \frac{\pi}{2}}(x) &= 1 + 0 \cdot \left(x - \frac{\pi}{2}\right) + \frac{-1}{2!} \left(x - \frac{\pi}{2}\right)^2 + \frac{0}{3!} \left(x - \frac{\pi}{2}\right)^3 \\ &\quad + \frac{1}{4!} \left(x - \frac{\pi}{2}\right)^4 + \frac{0}{5!} \left(x - \frac{\pi}{2}\right)^5 + \frac{-1}{6!} \left(x - \frac{\pi}{2}\right)^6 \\ &= 1 - \frac{1}{2} \left(x - \frac{\pi}{2}\right)^2 + \frac{1}{24} \left(x - \frac{\pi}{2}\right)^4 - \frac{1}{720} \left(x - \frac{\pi}{2}\right)^6. \end{aligned}$$

(e)  $f(x) = \sqrt{2x+1} = (2x+1)^{1/2}$ ,  $n = 3$ ,  $x_0 = 4$ . Compute derivatives:

$$\begin{array}{ll} f(x) = (2x+1)^{1/2}, & f(4) = \sqrt{9} = 3, \\ f'(x) = \frac{1}{2}(2x+1)^{-1/2} \cdot 2 = (2x+1)^{-1/2}, & f'(4) = \frac{1}{3}, \\ f''(x) = -\frac{1}{2}(2x+1)^{-3/2} \cdot 2 = -(2x+1)^{-3/2}, & f''(4) = -\frac{1}{27}, \\ f'''(x) = \frac{3}{2}(2x+1)^{-5/2} \cdot 2 = 3(2x+1)^{-5/2}, & f'''(4) = \frac{3}{243} = \frac{1}{81}. \end{array}$$

Thus

$$(Tf)_{3,4}(x) = 3 + \frac{1}{3}(x-4) - \frac{1}{27 \cdot 2}(x-4)^2 + \frac{1}{81 \cdot 6}(x-4)^3 = 3 + \frac{x-4}{3} - \frac{(x-4)^2}{54} + \frac{(x-4)^3}{486}.$$

### (5) Maclaurin expansions with Peano remainder

Recall:  $f(x) = P_n(x) + o(x^n)$  as  $x \rightarrow 0$ , where  $P_n$  is the  $n$ th Maclaurin polynomial.

(a)  $f(x) = x \cos 3x - 3 \sin x$ ,  $n = 2$ . Use known expansions:

$$\begin{aligned} \cos 3x &= 1 - \frac{(3x)^2}{2!} + o(x^2) = 1 - \frac{9x^2}{2} + o(x^2), \\ x \cos 3x &= x \left(1 - \frac{9x^2}{2} + o(x^2)\right) = x - \frac{9}{2}x^3 + o(x^3) = x + o(x^2), \\ \sin x &= x - \frac{x^3}{6} + o(x^3) = x + o(x^2), \\ 3 \sin x &= 3x + o(x^2). \end{aligned}$$

Thus

$$f(x) = (x + o(x^2)) - (3x + o(x^2)) = -2x + o(x^2).$$

So the Maclaurin expansion up to order 2 is  $f(x) = -2x + o(x^2)$ . (The quadratic term is zero.)

(b)  $f(x) = e^{x^2} \sin 2x$ ,  $n = 5$ . We need expansions up to  $x^5$ :

$$\begin{aligned} e^{x^2} &= 1 + x^2 + \frac{x^4}{2!} + o(x^5) = 1 + x^2 + \frac{x^4}{2} + o(x^5), \\ \sin 2x &= 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + o(x^5) = 2x - \frac{8x^3}{6} + \frac{32x^5}{120} + o(x^5) = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 + o(x^5). \end{aligned}$$

Now multiply, keeping terms up to  $x^5$ :

$$\begin{aligned}
e^{x^2} \sin 2x &= \left(1 + x^2 + \frac{x^4}{2} + o(x^5)\right) \left(2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 + o(x^5)\right) \\
&= 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 \\
&\quad + x^2 \cdot 2x - x^2 \cdot \frac{4}{3}x^3 \quad (\text{only terms up to } x^5) \\
&\quad + \frac{x^4}{2} \cdot 2x \quad (\text{gives } x^5) \\
&\quad + o(x^5) \\
&= 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 + 2x^3 - \frac{4}{3}x^5 + x^5 + o(x^5) \\
&= 2x + \left(-\frac{4}{3} + 2\right)x^3 + \left(\frac{4}{15} - \frac{4}{3} + 1\right)x^5 + o(x^5) \\
&= 2x + \frac{2}{3}x^3 + \left(\frac{4}{15} - \frac{20}{15} + \frac{15}{15}\right)x^5 + o(x^5) \\
&= 2x + \frac{2}{3}x^3 - \frac{1}{15}x^5 + o(x^5).
\end{aligned}$$

Thus  $f(x) = 2x + \frac{2}{3}x^3 - \frac{1}{15}x^5 + o(x^5)$ .

(c)  $f(x) = \ln \frac{1+x}{1+3x} = \ln(1+x) - \ln(1+3x)$ ,  $n = 4$ . Use  $\ln(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + o(u^4)$ :

$$\begin{aligned}
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + o(x^4), \\
\ln(1+3x) &= 3x - \frac{(3x)^2}{2} + \frac{(3x)^3}{3} - \frac{(3x)^4}{4} + o(x^4) \\
&= 3x - \frac{9x^2}{2} + 9x^3 - \frac{81x^4}{4} + o(x^4).
\end{aligned}$$

Subtract:

$$\begin{aligned}
f(x) &= (x - 3x) + \left(-\frac{1}{2} + \frac{9}{2}\right)x^2 + \left(\frac{1}{3} - 9\right)x^3 + \left(-\frac{1}{4} + \frac{81}{4}\right)x^4 + o(x^4) \\
&= -2x + 4x^2 + \left(\frac{1}{3} - \frac{27}{3}\right)x^3 + \left(\frac{80}{4}\right)x^4 + o(x^4) \\
&= -2x + 4x^2 - \frac{26}{3}x^3 + 20x^4 + o(x^4).
\end{aligned}$$

(d)  $f(x) = \frac{x}{\sqrt[6]{1+x^2}} - \sin x$ ,  $n = 5$ . Using the binomial expansion of  $(1+x^2)^{-1/6}$ , we have

$$\frac{x}{\sqrt[6]{1+x^2}} = x \left(1 - \frac{1}{6}x^2 + \frac{7}{72}x^4 + o(x^4)\right) = x - \frac{1}{6}x^3 + \frac{7}{72}x^5 + o(x^5).$$

Subtract  $\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)$ :

$$\begin{aligned}
f(x) &= \left(x - \frac{1}{6}x^3 + \frac{7}{72}x^5\right) - \left(x - \frac{x^3}{6} + \frac{x^5}{120}\right) + o(x^5) \\
&= \left(-\frac{1}{6} + \frac{1}{6}\right)x^3 + \left(\frac{7}{72} - \frac{1}{120}\right)x^5 + o(x^5) \\
&= 0 \cdot x^3 + \left(\frac{35}{360} - \frac{3}{360}\right)x^5 + o(x^5) \\
&= \frac{32}{360}x^5 + o(x^5) = \frac{4}{45}x^5 + o(x^5).
\end{aligned}$$

Thus  $f(x) = \frac{4}{45}x^5 + o(x^5)$ .

(6) **Inequality proof using Taylor expansion**

Consider the function  $h(x) = 2 \cos(x + x^2) - (2 - x^2 - 2x^3)$ . We want to show  $h(x) \leq 0$  for  $x$  near 0.

Compute the Maclaurin expansion of  $2 \cos(x + x^2)$  up to order 3. Let  $u = x + x^2$ . Then  $\cos u = 1 - \frac{u^2}{2} + \frac{u^4}{24} + \dots$ . Up to  $x^3$ :

$$u = x + x^2, \quad u^2 = x^2 + 2x^3 + x^4, \quad u^3 = O(x^3).$$

Thus

$$\begin{aligned} \cos(x + x^2) &= 1 - \frac{1}{2}(x^2 + 2x^3 + x^4) + o(x^3) \\ &= 1 - \frac{x^2}{2} - x^3 + o(x^3). \end{aligned}$$

Multiply by 2:

$$2 \cos(x + x^2) = 2 - x^2 - 2x^3 + o(x^3).$$

Therefore,

$$h(x) = (2 - x^2 - 2x^3 + o(x^3)) - (2 - x^2 - 2x^3) = o(x^3).$$

This means  $h(x) = \varepsilon(x)x^3$  where  $\varepsilon(x) \rightarrow 0$  as  $x \rightarrow 0$ . For  $x$  sufficiently small,  $|\varepsilon(x)| < 1$ , so  $|h(x)| < |x^3|$ .

Now, for  $x > 0$  small,  $x^3 > 0$ , so  $h(x)$  could be positive or negative but is bounded in absolute value by  $x^3$ . However, we need to check the sign more precisely. Let's compute the next term to determine local behavior. Compute expansion to  $x^4$  to see the leading non-zero term of  $h(x)$  beyond the  $o(x^3)$ :

$$u^2 = x^2 + 2x^3 + x^4, \quad u^4 = x^4 + O(x^5).$$

Then

$$\cos u = 1 - \frac{u^2}{2} + \frac{u^4}{24} + o(x^4) = 1 - \frac{x^2}{2} - x^3 - \frac{x^4}{2} + \frac{x^4}{24} + o(x^4) = 1 - \frac{x^2}{2} - x^3 - \frac{11}{24}x^4 + o(x^4).$$

Thus

$$2 \cos(x + x^2) = 2 - x^2 - 2x^3 - \frac{11}{12}x^4 + o(x^4).$$

Then

$$h(x) = -\frac{11}{12}x^4 + o(x^4).$$

Since  $-\frac{11}{12}x^4$  is negative for  $x \neq 0$ , and  $o(x^4)$  is negligible compared to  $x^4$  as  $x \rightarrow 0$ , there exists  $\delta > 0$  such that for  $0 < |x| < \delta$ ,  $h(x) < 0$ . That is,

$$2 \cos(x + x^2) < 2 - x^2 - 2x^3.$$

For  $x = 0$ , equality holds. Hence, in a neighborhood of 0,  $2 \cos(x + x^2) \leq 2 - x^2 - 2x^3$ .

(7) (a) **Maclaurin polynomial of  $e^x$  with Lagrange remainder**

For  $f(x) = e^x$ , we have  $f^{(k)}(x) = e^x$  for all  $k$ . The Maclaurin polynomial of order  $n$  is

$$P_n(x) = \sum_{k=0}^n \frac{x^k}{k!}.$$

The Lagrange form of the remainder is: for any  $x$ , there exists  $\xi$  between 0 and  $x$  such that

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1} = \frac{e^\xi}{(n+1)!} x^{n+1}.$$

Thus

$$e^x = \sum_{k=0}^n \frac{x^k}{k!} + \frac{e^\xi}{(n+1)!} x^{n+1}.$$

(b) **Approximating  $e$  with error  $< 10^{-6}$**

We want to approximate  $e = e^1$  using the Maclaurin polynomial at  $x = 1$ . The remainder term is

$$R_n(1) = \frac{e^\xi}{(n+1)!}, \quad \text{where } 0 < \xi < 1.$$

Since  $e^\xi < e < 3$  (we could use a tighter bound, but  $e < 3$  suffices for finding  $n$ ), we have

$$|R_n(1)| < \frac{3}{(n+1)!}.$$

We require  $|R_n(1)| < 10^{-6}$ . So it suffices to find  $n$  such that

$$\frac{3}{(n+1)!} < 10^{-6} \quad \Leftrightarrow \quad (n+1)! > 3 \times 10^6.$$

Compute factorials:

$$9! = 362880,$$

$$10! = 3628800 > 3 \times 10^6.$$

Thus  $(n+1)! > 3 \times 10^6$  when  $n+1 \geq 10$ , i.e.,  $n \geq 9$ . Therefore, the minimal  $n$  is 9. Thus the Maclaurin polynomial  $\sum_{k=0}^9 \frac{1}{k!}$  approximates  $e$  with error less than  $10^{-6}$ .