Chapter 2

Functions

2.1 Definitions and examples

Definition

Let X and Y be two sets. A **function** f **from** X **to** Y is a rule that associates to any element $x \in X$ at most one element $y \in Y$. The subset of elements in X to which f associates an element in Y is called the domain of f and is denoted $\mathrm{dom}(f)$. We write

$$f : \mathrm{dom}(f) \subseteq X \to Y.$$

For $x \in \mathrm{dom}(f)$, the element $y \in Y$ associated to it by f is called the **image of** x **under** f and is denoted $y = f(x)$. We often write

$$f : x \mapsto f(x).$$

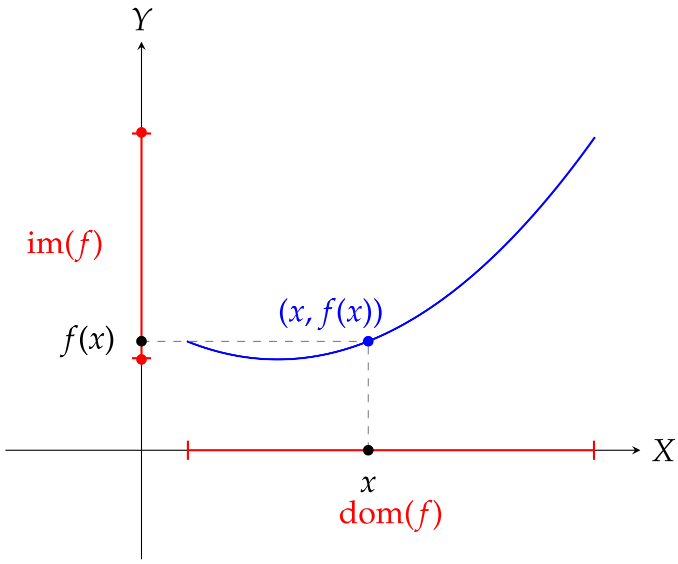
The subset of Y of all images of elements in X is called the **range of** f and is denoted:

$$\mathrm{im}(f) = \{y \in Y \mid \exists x \in \mathrm{dom}(f),\ y = f(x)\}.$$

If $Y = \mathbb{R}$ we say that the function f is **real-valued**. Finally, the **graph of** f is the following subset of the Cartesian product $X \times Y$:

$$\Gamma(f) = \{(x, f(x)) \in X \times Y \mid x \in \mathrm{dom}(f)\}.$$

Here is an example of how we can visualize these properties:



Example 2.1: Some notable examples for $f : \mathbb{R} \to \mathbb{R}$ include:

1. *Linear functions*: $f(x) = ax$ where $a \in \mathbb{R}, a \neq 0$. The graph is a straight line through the origin with slope a (the line cannot be vertical).

2. *Affine functions*: $f(x) = ax + b$ where $a, b \in \mathbb{R}$, $a \neq 0$. The graph is a straight line through the point $(0, b)$ with slope a (the line cannot be vertical).

3. *Quadratic functions*: $f(x) = ax^2 + bx + c$ where $a, b, c \in \mathbb{R}$, $a \neq 0$. The graph is a parabola.

4. *Square root*: $f(x) = \sqrt{x}$. This is the first function mentioned here whose domain is not $\mathbb{R}$: $\mathrm{dom}(\sqrt{\ }) = \{x \in \mathbb{R} \mid x \geq 0\}$.

5. *Absolute value*:
$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

6. *Sign function*:
$$f : \mathbb{R} \to \mathbb{Z}, \qquad f(x) = \mathrm{sign}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

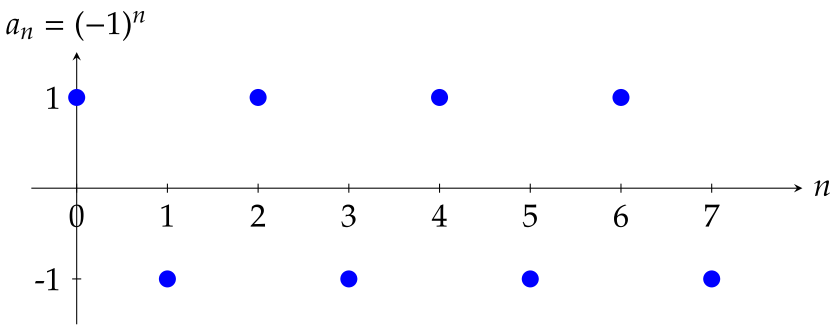
7. *Ceiling* ('rounding up'):
$$f : \mathbb{R} \to \mathbb{Z}, \qquad f(x) = \lceil x \rceil = \text{smallest } n \in \mathbb{Z} \text{ s.t. } n \geq x.$$

8. *Floor* ('rounding down'):
$$f : \mathbb{R} \to \mathbb{Z}, \qquad f(x) = \lfloor x \rfloor = \text{greatest } n \in \mathbb{Z} \text{ s.t. } n \leq x.$$

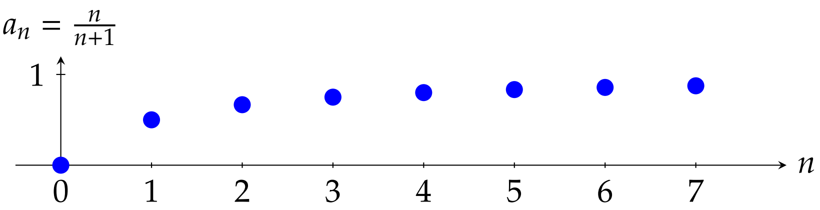
Sequences

A sequence of real numbers $a_0, a_1, a_2, \ldots$ can be viewed as a function $f : \mathrm{dom}(f) \subseteq \mathbb{N} \to \mathbb{R}$, where $f(n) = a_n$ for all $n \in \mathrm{dom}(f)$.

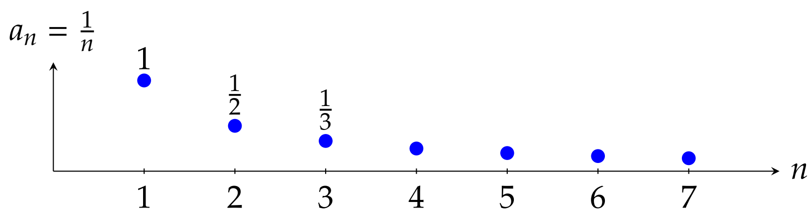
We start with the graph of the sequence $a_n = (-1)^n$, $n \in \mathbb{N}$. This sequence is simply given by $(-1)^n = \begin{cases} 1 & \text{when } n \text{ is even} \\ -1 & \text{when } n \text{ is odd} \end{cases}$ and looks as follows:



Here is the graph of $a_n = \frac{n}{n+1}$, $n \in \mathbb{N}$:



Here is the graph of $a_n = \frac{1}{n}$, with the smaller domain $n \in \mathbb{N}_+$:



2.2 Range and pre-image

Definitions

Let $f : X \to Y$ and let $A \subseteq X$. The **image of** A **under** f is the subset of Y

$$f(A) = \{f(x) \mid x \in A\} \subseteq \mathrm{im}(f) \subseteq Y.$$

Let $y \in Y$. The **pre-image of** y **under** f is the subset of X

$$f^{-1}(y) = \{x \in X \mid f(x) = y\} \subseteq \mathrm{dom}(f) \subseteq X.$$

Let $B \subseteq Y$. The **pre-image of** B **under** f is the subset of X

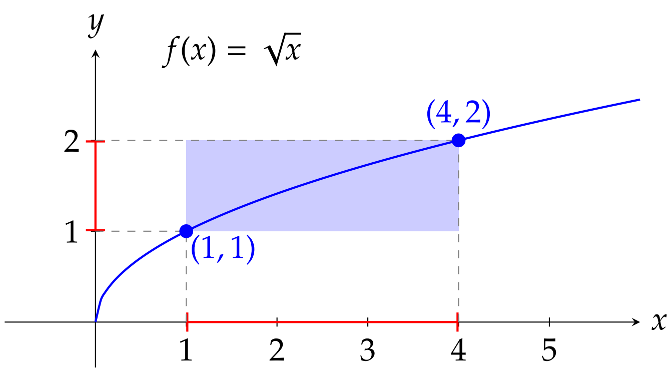
$$f^{-1}(B) = \{x \in X \mid f(x) \in B\} \subseteq \mathrm{dom}(f) \subseteq X.$$

Notice that

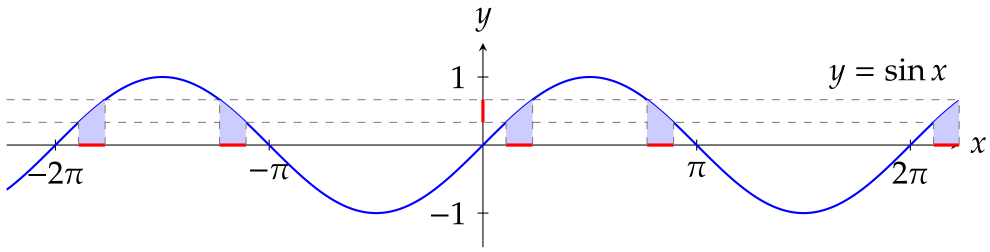
- $f(X) = \mathrm{im}(f)$.
- It is possible that $f^{-1}(y)$ or that $f^{-1}(B)$ are empty. For example, for $f(x) = x^2$, $f^{-1}(-5) = \emptyset$ and $f^{-1}([-4,-2]) = \emptyset$.

Example 2.2: Here are some examples of functions $\mathbb{R} \to \mathbb{R}$:

1. Let f be given by $f(x) = 2x$. Let $A = (a,b)$, where $a < b$. Then $f(A) = (2a, 2b)$. For any $y \in \mathbb{R}$, $f^{-1}(y) = \frac{y}{2}$.
2. Let f be given by $f(x) = 4$. Then for any non-empty $A \subseteq \mathbb{R}$, $f(A) = \{4\}$. Moreover, $f^{-1}(4) = \mathbb{R}$, while $f^{-1}(y) = \emptyset$ for any $y \neq 4$.
3. Let $f(x) = \mathrm{sign}(x)$. Then $f([0,1]) = \{0,1\}$, and $f^{-1}(-1) = \mathbb{R}_-$. Note that $f(0) = 0$, and $f(\{0\}) = \{0\}$.
4. Let $f(x) = \sqrt{x}$. Then $f((1,4)) = (1,2)$, $f^{-1}([1,2]) = [1,4]$, $f^{-1}(-1) = \emptyset$.



5. For $f(x) = \sin x$, we can see that $f^{-1}([\frac{1}{3}, \frac{2}{3}])$ is the union of infinitely many intervals.



We can now talk about the supremum, infimum, maximum and minimum of the image of various sets under a real-valued function f:

Supremum and infimum of a real-valued function

Let $f : X \to \mathbb{R}$ be a real-valued function. Let $A \subseteq \mathrm{dom}(f)$. The **supremum of** f **on** A is the supremum of the image of A under f:

$$\sup_A f = \sup_{x \in A} f(x) = \sup\{f(x) \mid x \in A\}.$$

Similarly, the **infimum of** f **on** A is the infimum of the image of A under f:

$$\inf_A f \inf_{x \in A} f(x) = \inf\{f(x) \mid x \in A\}.$$

As we have already seen, the supremum can be an element of $\mathbb{R} \cup \{+\infty\}$ and the infimum can be an element of $\{-\infty\} \cup \mathbb{R}$.

Boundedness of a real-valued function

If $\sup_{x \in A} f(x) < +\infty$ (i.e. it is a real number), we say that f is **bounded from above on** A. If $\inf_{x \in A} f(x) > -\infty$ (i.e. it is a real number), we say that f is **bounded from below on** A. If f is bounded from above and below on A, we say that it is **bounded on** A.

Maximum and minimum of a real-valued function

If $\sup_{x \in A} f(x) < +\infty$ and it belongs to $f(A)$ then it is the **maximum of** f **on** A. It is denoted

$$\max_A f \qquad \text{or} \qquad \max_{x \in A} f(x).$$

If $\inf_{x \in A} f(x) > -\infty$ and it belongs to $f(A)$ then it is the **minimum of** f **on** A. It is denoted

$$\min_A f \qquad \text{or} \qquad \min_{x \in A} f(x).$$

Since the minimum and the maximum of f on A belong to $f(A)$, there exist $x_m \in A$ and $x_M \in A$ such that

$$f(x_M) = \max_A f \qquad \text{and} \qquad f(x_m) = \min_A f.$$