Chapter 9

Taylor expansions and applications

Our goal in this chapter is to approximate complicated functions using polynomials, which are easier to study. This is a powerful tool, with many applications, not least in engineering. Moreover, it is the first example of an approximation of a function using a well-known set of functions (polynomials in this case). Later on (not in this course), you will see examples of approximation using sines and cosines, as well as other sets of approximating functions.

9.1 Taylor formulas

Consider a function $f : \mathbb{R} \to \mathbb{R}$, and suppose that we can differentiate it at a given point x_0 as many times as we'd like. Then consider the following sequence of polynomials with increasing order, starting from 0th order:

0^{TH} **order polynomial.** Define

$$(Tf)_{0,x_0}(x) = f(x_0)$$

to simply be the constant function with value x_0 for all $x \in \mathbb{R}$. Then, obviously,

$$f(x_0) = (Tf)_{0,x_0}(x_0).$$

1^{ST} **order polynomial.** Define

$$(Tf)_{1,x_0}(x) = f(x_0) + f'(x_0)(x - x_0)$$

to be the affine function that passes through the point $(x_0, f(x_0))$ and has slope $f'(x_0)$. Then, since

$$(Tf)'_{1,x_0}(x) = f'(x_0), \quad \forall x \in \mathbb{R},$$

we have

$$\begin{aligned} f(x_0) &= (Tf)_{1,x_0}(x_0), \\ f'(x_0) &= (Tf)'_{1,x_0}(x_0). \end{aligned}$$

2^{ND} **order polynomial.** Define

$$(Tf)_{2,x_0}(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2$$

to be a parabola. Let's check its properties. Differentiating twice we have:

$$\begin{aligned}(Tf)'_{2,x_0}(x) &= f'(x_0) + f''(x_0)(x - x_0)\\ (Tf)''_{2,x_0}(x) &= f''(x_0)\end{aligned}$$

Thus, plugging $x = x_0$ into $(Tf)_{2,x_0}$ and its first two derivatives, we have

$$\begin{aligned}(Tf)_{2,x_0}(x_0) &= f(x_0)\\ (Tf)'_{2,x_0}(x_0) &= f'(x_0)\\ (Tf)''_{2,x_0}(x_0) &= f''(x_0).\end{aligned}$$

Hence $(Tf)_{2,x_0}(x)$ is a second-order polynomial (parabola) satisfying that: (a) it coincides with f at x_0, (b) its derivative coincides with the derivative of f at x_0, and (c) its *second* derivative coincides with the second derivative of f at x_0.

3^RD **order polynomial.** Defining

$$(Tf)_{3,x_0}(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + \frac{1}{3 \cdot 2}f'''(x_0)(x - x_0)^3$$

we have

$$\begin{aligned}(Tf)'_{3,x_0}(x) &= f'(x_0) + f''(x_0)(x - x_0) + \frac{1}{2}f'''(x_0)(x - x_0)^2\\ (Tf)''_{3,x_0}(x) &= f''(x_0) + f'''(x_0)(x - x_0)\\ (Tf)'''_{3,x_0}(x) &= f'''(x_0)\end{aligned}$$

so that

$$\begin{aligned}(Tf)_{3,x_0}(x_0) &= f(x_0)\\ (Tf)'_{3,x_0}(x_0) &= f'(x_0)\\ (Tf)''_{3,x_0}(x_0) &= f''(x_0)\\ (Tf)'''_{3,x_0}(x_0) &= f'''(x_0).\end{aligned}$$

n^{TH} **order polynomial.** Following the previous ideas, we can define

$$\begin{aligned}(Tf)_{n,x_0}(x) =& f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + \frac{1}{3 \cdot 2}f'''(x_0)(x - x_0)^3 + \cdots\\ &+ \frac{1}{n(n-1)(n-2)\cdots 3 \cdot 2}f^{(n)}(x_0)(x - x_0)^n\\ =& \sum_{k=0}^{n} \frac{1}{k!}f^{(k)}(x_0)(x - x_0)^k\end{aligned}$$

to find that

$$\begin{aligned}
(Tf)_{n,x_0}(x_0) &= f(x_0)\\
(Tf)'_{n,x_0}(x_0) &= f'(x_0)\\
(Tf)''_{n,x_0}(x_0) &= f''(x_0)\\
&\vdots\\
(Tf)^{(n)}_{n,x_0}(x_0) &= f^{(n)}(x_0).
\end{aligned}$$

Taylor and Maclaurin polynomials

Given a function $f : \mathbb{R} \to \mathbb{R}$ that is n-times differentiable at a point x_0, its **Taylor polynomial (expansion) of order (degree) n at x_0** is the polynomial

$$(Tf)_{n,x_0}(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k.$$

When the point $x_0 = 0$ is the origin, this polynomial is sometimes called the **Maclaurin polynomial (expansion)** of f.

Lemma 9.1: The derivative of the Taylor polynomial of f is the Taylor polynomial of the derivative f' of one lesser order:

$$(Tf)'_{n,x_0}(x) = (Tf')_{n-1,x_0}(x).$$

Proof. We compute

$$\begin{aligned}
(Tf)'_{n,x_0}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k\right)\\
&= \sum_{k=1}^{n} \frac{k}{k!} f^{(k)}(x_0)(x - x_0)^{k-1}\\
&= \sum_{k=1}^{n} \frac{1}{(k-1)!} f^{(k)}(x_0)(x - x_0)^{k-1}\\
(\textit{substitute } j = k-1) \qquad &= \sum_{j=0}^{n-1} \frac{1}{j!} f^{(j+1)}(x_0)(x - x_0)^{j}\\
&= \sum_{j=0}^{n-1} \frac{1}{j!} (f')^{(j)}(x_0)(x - x_0)^{j}\\
&= (Tf')_{n-1,x_0}(x)
\end{aligned}$$

and the proof is complete. □

We may take additional derivatives, to arrive at the formula, true for any $k = 0, \dots, n$:

$$(Tf)^{(k)}_{n,x_0}(x) = (Tf^{(k)})_{n-k,x_0}(x).$$

Therefore:

$$(Tf)^{(k)}_{n,x_0}(x_0) = (Tf^{(k)})_{n-k,x_0}(x_0) = f^{(k)}(x_0)$$

since the value of the Taylor polynomial of a function at x_0 equals the value of that function at x_0.

Theorem 9.2 (Peano's remainder)**:** For a function $f : \mathbb{R} \to \mathbb{R}$ that is n-times differentiable at a point x_0, the difference between f and its Taylor polynomial at a point x close to x_0 is of order $o((x - x_0)^n)$:

$$f(x) - (Tf)_{n,x_0}(x) = o((x - x_0)^n), \qquad x \to x_0.$$

This is called **Peano's approximation of the remainder**.

Proof. We must show that

$$\lim_{x \to x_0} \frac{f(x) - (Tf)_{n,x_0}(x)}{(x - x_0)^n} = 0.$$

This is an expression of the form $\frac{0}{0}$. Applying de l'Hôpital's Theorem we have

$$\begin{aligned} \lim_{x \to x_0} \frac{f(x) - (Tf)_{n,x_0}(x)}{(x - x_0)^n} &= \lim_{x \to x_0} \frac{f'(x) - (Tf)'_{n,x_0}(x)}{n(x - x_0)^{n-1}} \\ \textit{(using Lemma 9.1)} \quad &= \lim_{x \to x_0} \frac{f'(x) - (Tf')_{n-1,x_0}(x)}{n(x - x_0)^{n-1}} \end{aligned}$$

which is still of the form $\frac{0}{0}$ (if $n \geq 2$). Applying de l'Hôpital's Theorem and Lemma 9.1 $n - 1$ times we eventually arrive at

$$\begin{aligned} \lim_{x \to x_0} \frac{f(x) - (Tf)_{n,x_0}(x)}{(x - x_0)^n} &= \lim_{x \to x_0} \frac{f^{(n-1)}(x) - (Tf^{(n-1)})_{1,x_0}(x)}{n!(x - x_0)} \\ &= \frac{1}{n!} \lim_{x \to x_0} \frac{f^{(n-1)}(x) - f^{(n-1)}(x_0) - f^{(n)}(x_0)(x - x_0)}{x - x_0} \\ &= \frac{1}{n!} \lim_{x \to x_0} \left(\frac{f^{(n-1)}(x) - f^{(n-1)}(x_0)}{x - x_0} - f^{(n)}(x_0) \right) = 0 \end{aligned}$$

by the definition of the nth derivative at x_0. This justifies having applied de l'Hôpital's Theorem repeatedly, and proves the theorem. □

Theorem 9.3 (Lagrange's remainder)**:** Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is n-times differentiable at x_0 and $n + 1$ times differentiable in a neighborhood $I_r(x_0) \setminus \{x_0\}$. Then for $x \in I_r(x_0)$ the difference between f and its Taylor polynomial at x can be written as:

$$f(x) - (Tf)_{n,x_0}(x) = \frac{1}{(n + 1)!} f^{(n+1)}(\overline{x})(x - x_0)^{n+1},$$

where $\overline{x}$ is a point between x and x_0. This is called **Lagrange's formula for the remainder**.

Proof. Denote the remainder term $\varphi(x)$:

$$\begin{aligned}\varphi(x) &= f(x) - (Tf)_{n,x_0}(x)\\ &= f(x) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0)(x-x_0)^k\end{aligned}$$

and define

$$\psi(x) = (x-x_0)^{n+1}.$$

By our construction of the Taylor polynomial, it holds that for all $j = 0, \ldots, n$,

$$\begin{aligned}&\varphi^{(j)}(x_0) = 0,\\ &\psi^{(j)}(x_0) = 0 \qquad \text{and} \qquad \psi^{(j)}(x) \neq 0, \text{ if } x \neq x_0.\end{aligned}$$

Applying Cauchy's Theorem (Theorem 8.12) to φ and ψ on the interval between x and x_0, we have

$$\frac{\varphi(x)}{\psi(x)} = \frac{\varphi(x)-\varphi(x_0)}{\psi(x)-\psi(x_0)} = \frac{\varphi'(x_1)}{\psi'(x_1)}$$

where x_1 is some point between x and x_0. Repeating this argument with the derivatives (considering now the interval between x_0 and x_1), we have

$$\frac{\varphi'(x_1)}{\psi'(x_1)} = \frac{\varphi'(x_1)-\varphi'(x_0)}{\psi'(x_1)-\psi'(x_0)} = \frac{\varphi''(x_2)}{\psi''(x_2)}$$

where x_2 is some point between x_0 and x_1. Looking now between x_0 and x_2, we repeat the argument with the second derivatives to find

$$\frac{\varphi''(x_2)}{\psi''(x_2)} = \frac{\varphi''(x_2)-\varphi''(x_0)}{\psi''(x_2)-\psi''(x_0)} = \frac{\varphi'''(x_3)}{\psi'''(x_3)}$$

where x_3 is some point between x_0 and x_2. Repeating this argument $n+1$ times, we arrive at

$$\frac{\varphi(x)}{\psi(x)} = \frac{\varphi'(x_1)}{\psi'(x_1)} = \frac{\varphi''(x_2)}{\psi''(x_2)} = \cdots = \frac{\varphi^{(n+1)}(x_{n+1})}{\psi^{(n+1)}(x_{n+1})}$$

where x_{n+1} is a point lying between x_0 and x_n, x_n is a point lying between x_0 and x_{n-1}, and so on... However, observe that

$$\begin{aligned}\varphi^{(n+1)}(x) &= f^{(n+1)}(x)\\ \psi^{(n+1)}(x) &= (n+1)!\end{aligned}$$

So, by defining $\overline{x} = x_{n+1}$ the proof is complete since we have found that

$$\frac{f(x)-(Tf)_{n,x_0}(x)}{(x-x_0)^{n+1}} = \frac{\varphi(x)}{\psi(x)} = \frac{\varphi^{(n+1)}(\overline{x})}{\psi^{(n+1)}(\overline{x})} = \frac{f^{(n+1)}(\overline{x})}{(n+1)!}.$$

□

Remark: Recalling the definition of the Taylor polynomial, and using Lagrange's formula for the remainder, we have

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k + \frac{1}{(n+1)!} f^{(n+1)}(\overline{x})(x - x_0)^{n+1}.$$

So we see that the function f is nearly equal to the Taylor polynomial, and that the remainder term has a form that is very similar to the $n+1$ terms of the polynomial. The only (important) difference, is that the derivative term within the remainder is evaluated at a different point $\overline{x}$ which lies somewhere between x_0 and x (and we do not know where exactly).

9.2 Expanding the elementary functions

We can now see how the Taylor expansions of some of the most commonly used functions. Recall that a function f, the Taylor polynomial of order n at x_0 is given by

$$(Tf)_{n,x_0}(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k.$$

Hence, to construct this polynomial for a given function one needs to know the first n derivatives at a given point x_0.

The exponential function

For the function $f(x) = e^x$ we know that $f^{(k)}(x) = e^x$ for all $k \in \mathbb{N}$ and for all $x \in \mathbb{R}$. Plugging this into the expression for the Taylor polynomial, we find that

$$(Tf)_{n,x_0}(x) = \sum_{k=0}^{n} \frac{1}{k!} e^{x_0}(x - x_0)^k.$$

For simplicity, we choose $x_0 = 0$, so that $e^{x_0} = 1$ and we're left with the *Maclaurin expansion*

$$(Tf)_{n,0}(x) = \sum_{k=0}^{n} \frac{x^k}{k!}.$$