Remark: If $\alpha \in \mathbb{N}$ is an integer, then $(1+x)^\alpha$ can be expanded as a polynomial using the standard binomial formula. In this case, the Maclaurin polynomials will coincide with this polynomial if their order is at least α, otherwise they will only contain the first α terms of the polynomial. For example, consider

$$f(x) = (1+x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5.$$

Then it expansion of order 3 is

$$(1+x)^5 = \underbrace{1 + 5x + 10x^2 + 10x^3}_{(Tf)_{3,0}(x)} + o(x^3)$$

but its expansion of any order ≥ 5 is identical to f itself:

$$(Tf)_{n,0}(x) = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5 = f(x) \qquad \forall n \geq 5.$$

Remark: If $\alpha \notin \mathbb{N}$, then, there will be nontrivial Maclaurin polynomials of arbitrary order (i.e. there will be arbitrarily high powers of x that won't vanish). This is in contrast to the case $\alpha \in \mathbb{N}$.

Let us highlight some special cases:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 - \cdots + (-1)^n x^n + o(x^n) = \sum_{k=0}^{n} (-1)^k x^k + o(x^n)$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + o(x^3)$$

We stop at x^3, because the next terms aren't as nice: the coefficient of x^4 is $-\frac{5}{128}$ and the coefficient of x^5 is $\frac{7}{256}$.

9.3 Operations on Taylor expansions

Uniqueness of the expansion

Sometimes we may be able to identify a polynomial that approximates a function f by other means, and then we may wonder what is its relationship to the function's Taylor or Maclaurin expansions. The following proposition answers this question, but telling us that an expansion of a given order is unique. Therefore, if we've found one (no matter how), we're done!

Proposition 9.5 (Uniqueness of the Taylor expansion): Let $f : \mathbb{R} \to \mathbb{R}$ be n times differentiable at x_0. Then $(Tf)_{n,x_0}(x)$ is the only polynomial of degree $\leq n$ satisfying

$$f(x) - (Tf)_{n,x_0}(x) = o((x - x_0)^n) \qquad \text{as } x \to x_0.$$

Proof. By contradiction, suppose we can write

$$f(x) - P_n(x) = o((x - x_0)^n) \qquad \text{as } x \to x_0$$

where $P_n(x)$ is another polynomial of degree $\leq n$. We'll show that $P_n(x) = (Tf)_{n,x_0}(x)$. On the one hand, subtracting the above two expressions, we have

$$P_n(x) - (Tf)_{n,x_0}(x) = o((x - x_0)^n) \qquad \text{as } x \to x_0.$$

On the other hand, since both $P_n(x)$ and $(Tf)_{n,x_0}(x)$ are both polynomials of degree $\leq n$, it must hold that so is their difference:

$$P_n(x) - (Tf)_{n,x_0}(x) = \sum_{k=0}^{n} c_k (x - x_0)^k.$$

Let m be the smallest index such that $c_m \neq 0$. Then

$$P_n(x) - (Tf)_{n,x_0}(x) = \sum_{k=m}^{n} c_k (x - x_0)^k$$

and we can write

$$\frac{P_n(x) - (Tf)_{n,x_0}(x)}{(x - x_0)^m} = c_m + \sum_{k=m+1}^{n} c_k (x - x_0)^{k-m}.$$

Left hand side: Using the fact that $n \geq m$, we have

$$\lim_{x \to x_0} \frac{P_n(x) - (Tf)_{n,x_0}(x)}{(x - x_0)^m} = \lim_{x \to x_0} \frac{P_n(x) - (Tf)_{n,x_0}(x)}{(x - x_0)^n} (x - x_0)^{n-m} = 0$$

since $P_n(x) - (Tf)_{n,x_0}(x) = o((x - x_0)^n)$.

Right hand side: It obviously holds that

$$\lim_{x \to x_0} \left(c_m + \sum_{k=m+1}^{n} c_k (x - x_0)^{k-m} \right) = c_m + \lim_{x \to x_0} \left(\sum_{k=m+1}^{n} c_k (x - x_0)^{k-m} \right) = c_m.$$

This implies that $c_m = 0$, a contradiction. $\square$

Example 9.1: Suppose that we know that a function $f(x)$ satisfies:

$$f(x) = 9 - 3(x - 2) + (x - 2)^2 - \frac{1}{4}(x - 2)^3 + o((x - 2)^3) \qquad \text{as } x \to 2.$$

Then by the uniqueness of the Taylor polynomial, we know that

$$(Tf)_{n=3,x_0=2}(x) = 9 - 3(x - 2) + (x - 2)^2 - \frac{1}{4}(x - 2)^3$$

implying that

$$f(2) = 9 \qquad f'(2) = -3 \qquad f''(2) = 2 \qquad f'''(2) = -3 \cdot 2 \cdot \frac{1}{4} = -\frac{3}{2}.$$

For simplicity, in what follows we consider the Maclaurin expansion, i.e. we take $x_0 = 0$. *In general, this can always be achieved by translating a function to bring the point* x_0 *to* 0.

We will consider two functions f and g and their Maclaurin expansions:

$$f(x) = \underbrace{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n}_{p_n(x)} + o(x^n)$$
$$g(x) = \underbrace{b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n}_{q_n(x)} + o(x^n).$$

Sums of expansions

$$\begin{aligned} f(x) \pm g(x) &= [p_n(x) + o(x^n)] \pm [q_n(x) + o(x^n)] \\ &= [p_n(x) \pm q_n(x)] + [o(x^n) \pm o(x^n)] \\ &= p_n(x) \pm q_n(x) + o(x^n). \end{aligned}$$

Example 9.2: We have seen that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + o(x^n).$$

Changing x to $-x$, we obtain

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + (-1)^n \frac{x^n}{n!} + o(x^n).$$

Let's take n to be an even number given by $n = 2m$ where $m \in \mathbb{N}_+$, then we have

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{2m}}{(2m)!} + o(x^{2m}) \\ e^{-x} &= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + \frac{x^{2m}}{(2m)!} + o(x^{2m}) \end{aligned}$$

where the last term in the Maclaurin polynomial of e^{-x} is positive because it is of even order. Then we have

$$\begin{aligned} e^x + e^{-x} &= 2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \frac{2x^6}{6!} + \cdots + \frac{2x^{2m}}{(2m)!} + o(x^{2m}) \\ e^x - e^{-x} &= 2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \frac{2x^7}{7!} + \cdots + \frac{2x^{2m-1}}{(2m-1)!} + o(x^{2m}) \end{aligned}$$

Dividing these expressions by 2 we have:

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots + \frac{x^{2m}}{(2m)!} + o(x^{2m})$$
$$\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots + \frac{x^{2m-1}}{(2m-1)!} + o(x^{2m})$$

These two functions are called the **hyperbolic cosine** and **hyperbolic sine**, respectively, and are denoted

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \text{and} \qquad \sinh(x) = \frac{e^x - e^{-x}}{2}$$

While the regular sine and cosine satisfy $\sin^2 x + \cos^2 x = 1$, the hyperbolic versions satisfy a hyperbolic version of this relation:

$$\cosh^2 x - \sinh^2 x = 1.$$

These functions have many applications in engineering.

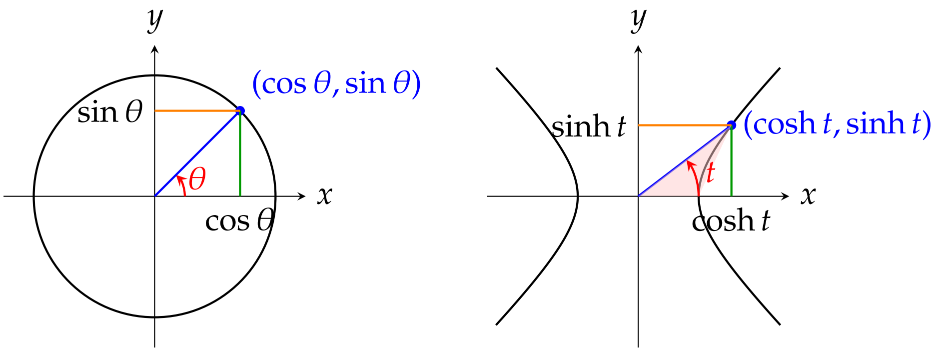

Figure 9.2: Geometric interpretation of trigonometric functions. *Left:* Circular functions $(\cos\theta, \sin\theta)$ on the unit circle $x^2 + y^2 = 1$, where θ is the angle. *Right:* Hyperbolic functions $(\cosh t, \sinh t)$ on the unit hyperbola $x^2 - y^2 = 1$, where t is twice the shaded area.

Example 9.3: Consider the function

$$h(x) = e^x - \sqrt{1 + 2x}$$

which vanishes as $x \to 0$. We want to understand its order at 0. We have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + o(x^n)$$
$$\sqrt{1 + 2x} = 1 + x - \frac{x^2}{2} + \frac{x^3}{2} + o(x^3)$$

Observe that if we stop at order 1 for both expansions there's complete cancellation and we get

$$h(x) = (1 + x + o(x)) - (1 + x + o(x)) = o(x) \quad \text{as } x \to 0.$$

However, taking a further order we have:

$$h(x) = \left(1 + x + \frac{x^2}{2!} + o(x^2)\right) - \left(1 + x - \frac{x^2}{2} + o(x^2)\right) = x^2 + o(x^2) \quad \text{as } x \to 0.$$

Taking another order:

$$\begin{aligned} h(x) &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + o(x^3)\right) - \left(1 + x - \frac{x^2}{2} + \frac{x^3}{2} + o(x^3)\right) \\ &= x^2 - \frac{x^3}{3} + o(x^3) \quad \text{as } x \to 0. \end{aligned}$$

Products of expansions

$$\begin{aligned} f(x) \cdot g(x) &= [p_n(x) + o(x^n)] \cdot [q_n(x) + o(x^n)] \\ &= p_n(x) \cdot q_n(x) + p_n(x)o(x^n) + q_n(x)o(x^n) + o(x^n)o(x^n) \\ &= p_n(x) \cdot q_n(x) + o(x^n) + o(x^n) + o(x^{2n}) \\ &= p_n(x) \cdot q_n(x) + o(x^n). \end{aligned}$$

Remark: A couple of remarks about the above computation. First, the fact that $p_n(x)o(x^n) = q_n(x)o(x^n) = o(x^n)$ follows from the fact that both functions p_n and q_n are bounded near $x = 0$, and therefore when they multiply something small of order $o(x^n)$ they do not affect its order. Second, we have $o(x^n)+o(x^n)+o(x^{2n}) = o(x^n)$ since the $o(x^{2n})$ is of higher order and therefore is negligible with respect to $o(x^n)$. Then, we can sum as many $o(x^n)$ as we want, and they will still result in an element that is of order $o(x^n)$.

Observe that the product of the two polynomials p_n and q_n will include monomials up to x^{2n} coming from the product of $a_n x^n$ with $b_n x^n$:

$$\begin{aligned} p_n(x) \cdot q_n(x) &= \left(\sum_{i=0}^{n} a_i x^i\right)\left(\sum_{j=0}^{n} b_j x^j\right) \\ &= \sum_{k=0}^{2n} \left(\sum_{m=0}^{k} a_m b_{k-m}\right) x^k \\ &= \sum_{k=0}^{n} \left(\sum_{m=0}^{k} a_m b_{k-m}\right) x^k + \underbrace{\sum_{k=n+1}^{2n} \left(\sum_{m=0}^{k} a_m b_{k-m}\right) x^k}_{o(x^n)} \end{aligned}$$

However, since our our order of accuracy is $o(x^n)$ there is no point in maintaining terms that are of that order. Hence we can write:

$$p_n(x) \cdot q_n(x) = \sum_{k=0}^{n} \left(\sum_{m=0}^{k} a_m b_{k-m}\right) x^k + o(x^n)$$