MATHEMATICAL ANALYSIS 1
HOMEWORK 13

(1) Compute the area under the graph of $f(x) = |\ln x|$ on the interval $[e^{-1}, e]$.

(2) Determine the following function:

$$F(x) = \int_{-1}^{x} (|y-1|+2)\,\mathrm{dy}.$$

(3) Compute the area between f and g:
   (a) $f(x) = |x|$ and $g(x) = \sqrt{1-x^2}$.
   (b) $f(x) = x^2 - 2x$ and $g(x) = -x^2 + x$.

(4) **Improper integrals I.** Compute the following improper integrals:

   (a) $\int_0^{+\infty} \frac{1}{x^2+3x+2}\,\mathrm{dx}$.
   (b) $\int_2^{+\infty} \frac{1}{x\sqrt{x-2}}\,\mathrm{dx}$.
   (c) $\int_0^{+\infty} \frac{x}{(x+1)^3}\,\mathrm{dx}$.
   (d) $\int_{-1}^{1} \frac{1}{\sqrt{x}(x-4)}\,\mathrm{dx}$.

(5) **Improper integrals II.** Discuss the convergence of the following improper integrals:

   (a) $\int_0^{+\infty} \frac{\sin x}{x\sqrt{x}}\,\mathrm{dx}$.
   (b) $\int_0^{+\infty} \frac{1}{\ln^2(2+e^x)}\,\mathrm{dx}$.
   (c) $\int_0^{+\infty} xe^{-x}\,\mathrm{dx}$.
   (d) $\int_0^{\pi} \frac{x-\frac{\pi}{2}}{\cos x\sqrt{\sin x}}\,\mathrm{dx}$.

(6) For which $n \in \mathbb{N}$ does the following integral converge

$$\int_2^{+\infty} \frac{x}{\sqrt{(x^2+3)^n}}\,\mathrm{dx} \quad ?$$

Compute the integral for the smallest n for which is converges.

(7) For which $\alpha \in \mathbb{R}$ does the integral

$$\int_2^3 \frac{x(\sin(x-2))^\alpha}{\sqrt{x^2-4}}\,\mathrm{dx}$$

converge? What is its value when $\alpha = 0$?

(8) **Geometric series.**
   (a) Let $r \in \mathbb{R}$ and let $\{r^k\}_{k=0}^{\infty} = (1, r, r^2, \dots)$ be a geometric sequence. Show that the partial sum $s_n = \sum_{k=0}^{n} r^k$ satisfies

$$s_n = \frac{1-r^{n+1}}{1-r} \qquad \text{for } r \neq 1.$$

   (b) Using a geometric series, write the number $2.3\overline{17} = 2.3171717\cdots$ as a fraction.

(9) **Positive-term series.** Study the convergence of the following series:

   (a) $\sum_{k=1}^{\infty} \frac{\ln k}{k^\alpha}$ for $\alpha \geq 0$.
   (b) $\sum_{k=1}^{\infty} \frac{1}{2^k-1}$
   (c) $\sum_{k=1}^{\infty} \sin\frac{1}{k}$
   (d) $\sum_{k=1}^{\infty} \frac{k+3}{\sqrt[3]{k^9+k^2}}$

(10) **Alternating sign series.** Study the convergence of the following series:

   (a) $\sum_{k=1}^{\infty} \sin\left(k\pi + \frac{1}{k}\right)$.
   (b) $\sum_{k=1}^{\infty} (-1)^{k+1}\frac{k^2}{k^3+1}$.
   (c) $\sum_{k=1}^{\infty} (-1)^k \ln\left(\frac{1}{k}+1\right)$.
   (d) $\sum_{k=1}^{\infty} (-1)^k \sqrt{\frac{k^3+3}{2k^3-5}}$.

(11) **Absolute convergence.** Do the following series converge absolutely?

(a) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}5^{k-1}}{(k+1)^2 4^{k+2}}$.

(b) $\sum_{k=1}^{\infty} \frac{\cos 3k}{k^3}$.

Homework 13 Solutions

(1) The area is given by $\int_{e^{-1}}^{e} |\ln x|\,dx$. Since $\ln x \le 0$ on $[e^{-1},1]$ and $\ln x \ge 0$ on $[1,e]$, we split the integral:

$$\int_{e^{-1}}^{1} -\ln x\,dx + \int_{1}^{e} \ln x\,dx.$$

Using the antiderivative $\int \ln x\,dx = x\ln x - x$:

$$\begin{aligned}\text{Area} &= -[x\ln x - x]_{e^{-1}}^{1} + [x\ln x - x]_1^e\\ &= -[(0-1) - (e^{-1}(-1) - e^{-1})] + [(e-e) - (0-1)]\\ &= -[-1+2e^{-1}] + [1] = 2 - 2e^{-1}.\end{aligned}$$

(2) We evaluate $F(x) = \int_{-1}^{x}(|y-1|+2)\,dy$. If $x \le 1$, then $y \le 1$ and $|y-1| = 1-y$:

$$F(x) = \int_{-1}^{x}(1-y+2)\,dy = [3y - \frac{y^2}{2}]_{-1}^{x} = 3x - \frac{x^2}{2} - (-3-\frac{1}{2}) = -\frac{x^2}{2} + 3x + \frac{7}{2}.$$

If $x \ge 1$, we split the integral at $y=1$:

$$F(x) = F(1) + \int_{1}^{x}(y-1+2)\,dy = (-\frac{1}{2}+3+\frac{7}{2}) + [\frac{y^2}{2}+y]_1^x = 6 + (\frac{x^2}{2}+x) - (\frac{1}{2}+1) = \frac{x^2}{2}+x+\frac{9}{2}.$$

Thus, $F(x) = \begin{cases} -\frac{1}{2}x^2 + 3x + \frac{7}{2}, & x \le 1\\ \frac{1}{2}x^2 + x + \frac{9}{2}, & x > 1\end{cases}.$

(3) (a) Intersections: $|x| = \sqrt{1-x^2} \implies x^2 = 1-x^2 \implies x = \pm 1/\sqrt{2}$. Area is $2\int_0^{1/\sqrt{2}}(\sqrt{1-x^2} - x)\,dx$. Using $\int \sqrt{1-x^2}dx = \frac{1}{2}(x\sqrt{1-x^2} + \arcsin x)$:

$$\text{Area} = 2\left[\frac{1}{2}(x\sqrt{1-x^2} + \arcsin x) - \frac{x^2}{2}\right]_0^{1/\sqrt{2}} = 2\left(\frac{1}{4} + \frac{\pi}{8} - \frac{1}{4}\right) = \frac{\pi}{4}.$$

(b) Intersections: $x^2 - 2x = -x^2 + x \implies 2x^2 - 3x = 0 \implies x = 0, 3/2$. On $[0,3/2]$, $g(x) \ge f(x)$.

$$\text{Area} = \int_0^{3/2}(-2x^2+3x)\,dx = [-\frac{2}{3}x^3 + \frac{3}{2}x^2]_0^{3/2} = -\frac{2}{3}\frac{27}{8} + \frac{3}{2}\frac{9}{4} = -\frac{9}{4} + \frac{27}{8} = \frac{9}{8}.$$

(4) (a) $\int_0^\infty \frac{1}{(x+1)(x+2)}dx = \lim_{R\to\infty}[\ln|x+1| - \ln|x+2|]_0^R = \lim_{R\to\infty}\ln|\frac{R+1}{R+2}| - \ln(1/2) = 0 + \ln 2 = \ln 2.$

(b) Let $u = \sqrt{x-2}$, $du = \frac{1}{2\sqrt{x-2}}dx$. Then

$$\int_2^\infty \frac{1}{x\sqrt{x-2}}\,dx \;=\; \int_0^\infty \frac{1}{(u^2+2)\,u}\,2u\,du \;=\; \int_0^\infty \frac{2}{u^2+2}\,du \;=\; \left[\sqrt{2}\arctan\frac{u}{\sqrt{2}}\right]_0^\infty \;=\; \frac{\pi}{\sqrt{2}}.$$

(c) Let $u = x+1$: $\int_1^\infty \frac{u-1}{u^3}\,du = \int_1^\infty (u^{-2} - u^{-3})\,du = [-u^{-1} + \frac{1}{2}u^{-2}]_1^\infty = 0 - (-1+1/2) = 1/2.$

(d) *Domain note.* The expression $\sqrt{x}$ is only real for $x \ge 0$. The stated limits $[-1,1]$ include negative values where $\sqrt{x}$ is not real, so the only meaningful real improper integral here is over $(0,1]$. Interpreting the problem this way and setting $u = \sqrt{x}$ (so $x = u^2$, $dx = 2u\,du$), we get

$$\int_0^1 \frac{1}{\sqrt{x}(x-4)}\,dx \;=\; \int_0^1 \frac{2}{u^2-4}\,du \;=\; \left[\frac{1}{2}\ln\left|\frac{u-2}{u+2}\right|\right]_0^1 \;=\; \frac{1}{2}\ln\left(\frac{1}{3}\right) \;=\; -\frac{\ln 3}{2}.$$

(5) (a) Near 0, $\frac{\sin x}{x\sqrt{x}} \sim \frac{x}{x^{3/2}} = x^{-1/2}$, which is integrable at 0. As $x \to \infty$, $\left|\frac{\sin x}{x\sqrt{x}}\right| \le \frac{1}{x^{3/2}}$, which is integrable at ∞. Hence: **Converges**.

(b) As $x \to \infty$, $\ln(2+e^x) \sim \ln(e^x) = x$, so $\frac{1}{\ln^2(2+e^x)} \sim \frac{1}{x^2}$, which is integrable at ∞. No singularity on $(0,\infty)$. Hence: **Converges**.

(c) Use integration by parts with $u = x$, $dv = e^{-x}dx$. Then

$$\int_0^\infty xe^{-x}\,dx = \Big[-xe^{-x} - e^{-x}\Big]_0^\infty = 1.$$

Converges.

(d) Near $x = 0$, $\sin x \sim x$ and $\cos x \to 1$, so the integrand behaves like $\dfrac{-\pi/2}{1 \cdot \sqrt{x}}$, which is integrable at 0. At $x = \pi/2$, $\lim_{x\to\pi/2} \dfrac{x - \pi/2}{\cos x} = -1$, and $\sqrt{\sin x} \to 1$, so there is no divergence. Hence: **Converges**.

(6) For large x, $\dfrac{x}{\sqrt{(x^2+3)^n}} \sim \dfrac{x}{x^n} = x^{1-n}$. The integral $\int_2^\infty x^{1-n}\,dx$ converges iff $1-n < -1$, i.e. $n > 2$. The smallest such n is $n = 3$. For $n = 3$, set $u = x^2 + 3$, $du = 2x\,dx$:

$$\int_2^\infty \frac{x}{\sqrt{(x^2+3)^3}}\,dx = \int_7^\infty \frac{1}{2}u^{-3/2}\,du = \Big[-u^{-1/2}\Big]_7^\infty = \frac{1}{\sqrt{7}}.$$

(7) As $x \downarrow 2$, write $x = 2 + t$ with $t \downarrow 0$. Then $\sin(x-2) = \sin t \sim t$ and $\sqrt{x^2-4} = \sqrt{(2+t)^2-4} \sim \sqrt{4t}$, while $x \sim 2$. Thus the integrand behaves like

$$\frac{2\,t^\alpha}{2\sqrt{t}} = t^{\alpha - \frac{1}{2}}.$$

This is integrable near $t = 0$ iff $\alpha - \frac{1}{2} > -1$, i.e. $\alpha > -\frac{1}{2}$. No other singularities occur on $[2,3]$. Hence the integral converges iff $\alpha > -\frac{1}{2}$. For $\alpha = 0$,

$$\int_2^3 \frac{x}{\sqrt{x^2-4}}\,dx = \Big[\sqrt{x^2-4}\Big]_2^3 = \sqrt{5}.$$

(8) (a) $s_n = 1 + r + \cdots + r^n$. Then $rs_n = r + \cdots + r^{n+1}$. Subtract: $s_n(1-r) = 1 - r^{n+1}$, hence $s_n = \dfrac{1-r^{n+1}}{1-r}$ for $r \neq 1$.

(b) $2.3\overline{17} = \dfrac{23}{10} + \dfrac{17}{10^3}\sum_{k=0}^\infty \left(\dfrac{1}{100}\right)^k = \dfrac{23}{10} + \dfrac{17}{1000} \cdot \dfrac{1}{1 - \frac{1}{100}} = \dfrac{23}{10} + \dfrac{17}{990} = \dfrac{2277+17}{990} = \dfrac{1147}{495}$.

(9) **Positive-term series.**

(a) $\displaystyle\sum_{k=1}^\infty \frac{\ln k}{k^\alpha}$.

Using the integral test or comparison with $\frac{1}{k^{\alpha-\varepsilon}}$ for small $\varepsilon > 0$: If $\alpha > 1$, then $\dfrac{\ln k}{k^\alpha} \le \dfrac{C}{k^{\alpha-\varepsilon}}$ with $\alpha - \varepsilon > 1$, so it **converges**. If $\alpha \le 1$, then $\dfrac{\ln k}{k^\alpha} \ge \dfrac{c}{k}$ for large k (since $\ln k$ grows and k^α is too small), hence it **diverges** by comparison with $\dfrac{1}{k}$.

(b) $\displaystyle\sum_{k=1}^\infty \frac{1}{2^k-1}$. For large k, $\frac{1}{2^k-1} \sim \frac{1}{2^k}$. By the **Comparison Test** with the geometric series $\sum(1/2)^k$, it **converges**.

(c) $\displaystyle\sum_{k=1}^\infty \sin\frac{1}{k}$. Since $\sin(1/k) \sim 1/k$, by the **Comparison Test** with $\sum \frac{1}{k}$, it **diverges**.

(d) $\displaystyle\sum_{k=1}^\infty \frac{k+3}{\sqrt[3]{k^9+k^2}}$. As $k \to \infty$, $\sqrt[3]{k^9+k^2} \sim k^3$, so the term $\sim \dfrac{k+3}{k^3} = \dfrac{1}{k^2} + \dfrac{3}{k^3}$. By the **Comparison Test** with $\sum \frac{1}{k^2}$, it **converges**.

(10) **Alternating sign series.**

(a) $\displaystyle\sum_{k=1}^\infty \sin\left(k\pi + \frac{1}{k}\right) = \sum_{k=1}^\infty (-1)^k \sin\left(\frac{1}{k}\right)$. Since $\sin(1/k) \searrow 0$, the series **converges** by the alternating series (Leibniz) test.

(b) $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k^2}{k^3+1}$. Here $a_k = \frac{k^2}{k^3+1} \searrow 0$. Hence it **converges** by Leibniz.

(c) $\displaystyle\sum_{k=1}^{\infty}(-1)^k \ln\left(1+\frac{1}{k}\right)$. Since $\ln\bigl(1+\frac{1}{k}\bigr) \searrow 0$, it **converges** by Leibniz.

(d) $\displaystyle\sum_{k=1}^{\infty}(-1)^k \sqrt{\frac{k^3+3}{2k^3-5}}$. The term tends to $\sqrt{\frac{1}{2}} \neq 0$, so the series **diverges** (term test).

(11) **Absolute convergence.**

(a) $\displaystyle\sum_{k=1}^{\infty}\left|\frac{(-1)^{k+1}5^{k-1}}{(k+1)^2 4^{k+2}}\right| = \sum_{k=1}^{\infty}\frac{5^{k-1}}{(k+1)^2 4^{k+2}}$. Using the ratio test on the absolute values,

$$\lim_{k\to\infty}\frac{a_{k+1}}{a_k} = \frac{5}{4} > 1,$$

so it **does not** converge absolutely (indeed, it diverges).

(b) $\displaystyle\sum_{k=1}^{\infty}\left|\frac{\cos(3k)}{k^3}\right| \leq \sum_{k=1}^{\infty}\frac{1}{k^3}$, which **converges**. Hence the series **converges absolutely**.