MATHEMATICAL ANALYSIS 1
HOMEWORK 12

(1) Recall the *Fundamental Theorem of Integral Calculus*:
$F_{x_0}(x) = \int_{x_0}^{x} f(y)dy \qquad \Rightarrow \qquad F'_{x_0}(x) = f(x).$
(a) Prove the following:
Corollary. Let $G(x)$ be any antiderivative of $f(x)$. Then $F_{x_0}(x) = G(x) - G(x_0)$.
(b) Prove the following:
Corollary. Let f be continuous on $[a, b]$ and let G be any antiderivative of f. Then

$$\int_a^b f(x)dx = G(b) - G(a). \tag{$\star$}$$

For the following problems we shall use the formula ($\star$):

(2) **Calculation of Definite Integrals (I).** Compute the following integrals.

(a) $\int_0^2 (3x^2 - 4x + 1)\, dx$.
(b) $\int_1^4 \left(\sqrt{x} + \frac{1}{x^2}\right)\, dx$.
(c) $\int_0^{\pi/4} \sec^2(x)\, dx$.
(d) $\int_0^{\ln 3} 2e^x\, dx$.
(e) $\int_1^e \frac{1}{x}\, dx$.
(f) $\int_0^1 \frac{1}{1+x^2}\, dx$.
(g) $\int_0^{\pi} \sin\left(\frac{x}{2}\right)\, dx$.
(h) $\int_0^1 (4x - e^{2x})\, dx$.

(3) **Calculation of Definite Integrals (II).** Compute the following integrals.

(a) $\int_0^1 \frac{x^3+x^2+1}{x+1}\, dx$.
(b) $\int_0^{\pi/4} \frac{1+\sin^2 x}{\cos^2 x}\, dx$.
(c) $\int_{1/e}^{e} \frac{x^2+x^3+x}{x^4}\, dx$.
(d) $\int_2^4 \frac{1}{\sqrt{x^2-1}}\, dx$.
(e) $\int_0^1 \frac{1}{\sqrt{4-x^2}}\, dx$.
(f) $\int_{-1}^{1} (e^{2x} - e^{-2x})\, dx$.
(g) $\int_{-\pi/4}^{\pi/4} (3\sin x + 2\tan x)\, dx$.
(h) $\int_0^1 \frac{1}{x^2+4}\, dx$.

(4) **Mean value of a function.**
(i) Compute the average $m(f; a, b)$ of $f(x)$ over the given interval $[a, b]$.
(ii) Write the equation for a point $z \in [a, b]$ such that $f(z) = m(f; a, b)$ (if there is such a z). If possible, write z explicitly.

(a) $f(x) = 2x + 1$ on $[0, 4]$.
(b) $f(x) = 3x^2 - 4x$ on $[1, 3]$.
(c) $f(x) = (x - 2)^2$ on $[0, 5]$.
(d) $f(x) = e^{2x}$ on $[0, \ln 3]$.
(e) $f(x) = \frac{1}{x}$ on $[1, e^2]$.
(f) $f(x) = \sin(x)$ on $[0, \pi]$.
(g) $f(x) = \sec^2(x)$ on $[0, \frac{\pi}{4}]$.
(h) $f(x) = \sqrt{x + 1}$ on $[3, 8]$.
(i) $f(x) = \frac{x^2-1}{x^4}$ on $[1, 2]$.
(j) $f(x) = x \cdot e^{-x^2}$ on $[0, 1]$.

(5) Compute the following integral:

$$\int_{\frac{2}{41\pi}}^{\frac{2}{\pi}} \left(-\frac{1}{x^2}\cos\left(\frac{1}{x}\right)\right) dx.$$

(6) Compute the following integral:

$$\int_{-\pi}^{\pi} \frac{\sin x + x^3}{2 + \cos x + e^{x^2}} dx.$$

(7) (a) Compute the integral $\int_0^1 \frac{1}{(x+1)^2} dx$
(b) Use the previous result to prove that

$$\lim_{n\to\infty} n\left(\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(2n)^2}\right) = \frac{1}{2}$$

(1) **Fundamental Theorem of Integral Calculus**

(a) **Proof of Corollary: $F_{x_0}(x) = G(x) - G(x_0)$.**
We are given the function defined by the integral:

$$F_{x_0}(x) = \int_{x_0}^{x} f(y)\,dy.$$

By the Fundamental Theorem of Integral Calculus, the derivative of $F_{x_0}(x)$ is $f(x)$:

$$F'_{x_0}(x) = f(x).$$

Since $G(x)$ is defined as any antiderivative of $f(x)$, we have $G'(x) = f(x)$. Since $F'_{x_0}(x) = G'(x)$, we conclude that $F_{x_0}(x)$ and $G(x)$ must differ by a constant C:

$$F_{x_0}(x) = G(x) + C.$$

To find C, we evaluate $F_{x_0}(x)$ at $x = x_0$:

$$F_{x_0}(x_0) = \int_{x_0}^{x_0} f(y)\,dy = 0.$$

Substituting this into the equation above:

$$0 = G(x_0) + C \quad \Rightarrow \quad C = -G(x_0).$$

Therefore,

$$F_{x_0}(x) = G(x) - G(x_0).$$

(b) **Proof of Corollary: $\int_a^b f(x)\,dx = G(b) - G(a)$.**
Using the definition from the previous part, let $x_0 = a$. Then the integral is:

$$\int_a^b f(x)\,dx = F_a(b).$$

From the previous corollary, $F_a(x) = G(x) - G(a)$. Substituting $x = b$ into this result yields:

$$\int_a^b f(x)\,dx = F_a(b) = G(b) - G(a).$$

(2) **Calculation of Definite Integrals (I).**

(a) $\int_0^2 (3x^2 - 4x + 1)\,dx$.

$$\begin{aligned}\int_0^2 (3x^2 - 4x + 1)\,dx &= \left[x^3 - 2x^2 + x\right]_0^2 \\ &= (2^3 - 2(2^2) + 2) - (0^3 - 2(0^2) + 0) \\ &= (8 - 8 + 2) - 0 \\ &= \mathbf{2}.\end{aligned}$$

(b) $\int_1^4 \left(\sqrt{x} + \frac{1}{x^2}\right)\,dx$.

$$\begin{aligned}
\int_1^4 \left(x^{1/2} + x^{-2}\right)\,dx &= \left[\frac{x^{3/2}}{3/2} + \frac{x^{-1}}{-1}\right]_1^4 \\
&= \left[\frac{2}{3}x\sqrt{x} - \frac{1}{x}\right]_1^4 \\
&= \left(\frac{2}{3}(4\sqrt{4}) - \frac{1}{4}\right) - \left(\frac{2}{3}(1) - \frac{1}{1}\right) \\
&= \left(\frac{16}{3} - \frac{1}{4}\right) - \left(\frac{2}{3} - 1\right) \\
&= \left(\frac{16}{3} - \frac{2}{3}\right) + \left(1 - \frac{1}{4}\right) \\
&= \frac{14}{3} + \frac{3}{4} = \frac{56+9}{12} = \mathbf{\frac{65}{12}}.
\end{aligned}$$

(c) $\int_0^{\pi/4} \sec^2(x)\,dx$.

$$\begin{aligned}
\int_0^{\pi/4} \sec^2(x)\,dx &= [\tan x]_0^{\pi/4} \\
&= \tan\left(\frac{\pi}{4}\right) - \tan(0) \\
&= 1 - 0 = \mathbf{1}.
\end{aligned}$$

(d) $\int_0^{\ln 3} 2e^x\,dx$.

$$\begin{aligned}
\int_0^{\ln 3} 2e^x\,dx &= [2e^x]_0^{\ln 3} \\
&= 2e^{\ln 3} - 2e^0 \\
&= 2(3) - 2(1) \\
&= 6 - 2 = \mathbf{4}.
\end{aligned}$$

(e) $\int_1^e \frac{1}{x}\,dx$.

$$\begin{aligned}
\int_1^e \frac{1}{x}\,dx &= [\ln|x|]_1^e \\
&= \ln e - \ln 1 \\
&= 1 - 0 = \mathbf{1}.
\end{aligned}$$

(f) $\int_0^1 \frac{1}{1+x^2}\,dx$.

$$\begin{aligned}
\int_0^1 \frac{1}{1+x^2}\,dx &= [\arctan x]_0^1 \\
&= \arctan(1) - \arctan(0) \\
&= \frac{\pi}{4} - 0 = \mathbf{\frac{\pi}{4}}.
\end{aligned}$$

(g) $\int_0^{\pi} \sin\left(\frac{x}{2}\right)\,dx$.

$$\begin{aligned}
\int_0^{\pi} \sin\left(\frac{x}{2}\right)\,dx &= \left[-2\cos\left(\frac{x}{2}\right)\right]_0^{\pi} \\
&= -2\cos\left(\frac{\pi}{2}\right) - \left(-2\cos\left(\frac{0}{2}\right)\right) \\
&= -2(0) + 2\cos(0) \\
&= 0 + 2(1) = \mathbf{2}.
\end{aligned}$$

(h) $\int_0^1 (4x - e^{2x})\,dx$.

$$\begin{aligned}\int_0^1 (4x - e^{2x})\,dx &= \left[2x^2 - \frac{1}{2}e^{2x}\right]_0^1\\ &= \left(2(1)^2 - \frac{1}{2}e^2\right) - \left(2(0)^2 - \frac{1}{2}e^0\right)\\ &= \left(2 - \frac{e^2}{2}\right) - \left(0 - \frac{1}{2}\right)\\ &= 2 - \frac{e^2}{2} + \frac{1}{2} = \mathbf{\frac{5 - e^2}{2}}.\end{aligned}$$

(3) **Calculation of Definite Integrals (II).**

(a) $\int_0^1 \frac{x^3+x^2+1}{x+1}\,dx$. We simplify the integrand: $\frac{x^3+x^2+1}{x+1} = \frac{x^2(x+1)+1}{x+1} = x^2 + \frac{1}{x+1}$.

$$\begin{aligned}\int_0^1 \left(x^2 + \frac{1}{x+1}\right)dx &= \left[\frac{x^3}{3} + \ln|x+1|\right]_0^1\\ &= \left(\frac{1}{3} + \ln 2\right) - \left(\frac{0}{3} + \ln 1\right)\\ &= \mathbf{\frac{1}{3}} + \ln \mathbf{2}.\end{aligned}$$

(b) $\int_0^{\pi/4} \frac{1+\sin^2 x}{\cos^2 x}\,dx$. We simplify the integrand using $\frac{1}{\cos^2 x} = \sec^2 x$ and $\frac{\sin^2 x}{\cos^2 x} = \tan^2 x$, followed by the identity $\tan^2 x = \sec^2 x - 1$:

$$\begin{aligned}\frac{1+\sin^2 x}{\cos^2 x} &= \sec^2 x + \tan^2 x\\ &= \sec^2 x + (\sec^2 x - 1)\\ &= 2\sec^2 x - 1.\end{aligned}$$

$$\begin{aligned}\int_0^{\pi/4} (2\sec^2 x - 1)\,dx &= [2\tan x - x]_0^{\pi/4}\\ &= \left(2\tan\left(\frac{\pi}{4}\right) - \frac{\pi}{4}\right) - (2\tan(0) - 0)\\ &= 2(1) - \frac{\pi}{4} = \mathbf{2} - \frac{\pi}{\mathbf{4}}.\end{aligned}$$

(c) $\int_{1/e}^{e} \frac{x^2+x^3+x}{x^4}\,dx$. We simplify the integrand by term-by-term division: $\frac{x^2+x^3+x}{x^4} = x^{-2} + x^{-1} + x^{-3}$.

$$\begin{aligned}\int_{1/e}^{e} \left(x^{-2} + x^{-1} + x^{-3}\right)dx &= \left[-x^{-1} + \ln|x| - \frac{1}{2}x^{-2}\right]_{1/e}^{e}\\ &= \left[-\frac{1}{x} + \ln|x| - \frac{1}{2x^2}\right]_{1/e}^{e}\\ &= \left(-\frac{1}{e} + \ln e - \frac{1}{2e^2}\right) - \left(-\frac{1}{1/e} + \ln(1/e) - \frac{1}{2(1/e)^2}\right)\\ &= \left(-\frac{1}{e} + 1 - \frac{1}{2e^2}\right) - \left(-e - 1 - \frac{e^2}{2}\right)\\ &= -\frac{1}{e} + 1 - \frac{1}{2e^2} + e + 1 + \frac{e^2}{2}\\ &= \mathbf{2 + e - \frac{1}{e} + \frac{e^2}{2} - \frac{1}{2e^2}}.\end{aligned}$$

(d) $\int_2^4 \frac{1}{\sqrt{x^2-1}}\,dx$. We have: $\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln|x+\sqrt{x^2-a^2}|$.

$$\begin{aligned}\int_2^4 \frac{1}{\sqrt{x^2-1}}\,dx &= \left[\ln|x+\sqrt{x^2-1}|\right]_2^4 \\ &= \ln|4+\sqrt{4^2-1}| - \ln|2+\sqrt{2^2-1}| \\ &= \ln(4+\sqrt{15}) - \ln(2+\sqrt{3}) \\ &= \ln\left(\frac{\mathbf{4}+\sqrt{\mathbf{15}}}{\mathbf{2}+\sqrt{\mathbf{3}}}\right).\end{aligned}$$

(e) $\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx$. We have: $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin(\frac{x}{a})$. Here $a=2$.

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx &= \left[\arcsin\left(\frac{x}{2}\right)\right]_0^1 \\ &= \arcsin\left(\frac{1}{2}\right) - \arcsin(0) \\ &= \frac{\pi}{6} - 0 = \frac{\pi}{\mathbf{6}}.\end{aligned}$$

(f) $\int_{-1}^1 (e^{2x}-e^{-2x})\,dx$. Let $f(x) = e^{2x}-e^{-2x}$. This function is odd because $f(-x) = e^{-2x} - e^{-(-2x)} = e^{-2x} - e^{2x} = -f(x)$. The integral of an odd function over a symmetric interval $[-a,a]$ is zero.

$$\int_{-1}^1 (e^{2x}-e^{-2x})\,dx = \mathbf{0}.$$

(Direct calculation: $\left[\frac{1}{2}e^{2x}+\frac{1}{2}e^{-2x}\right]_{-1}^1 = \left(\frac{e^2}{2}+\frac{e^{-2}}{2}\right) - \left(\frac{e^{-2}}{2}+\frac{e^2}{2}\right) = 0$).

(g) $\int_{-\pi/4}^{\pi/4} (3\sin x + 2\tan x)\,dx$. Let $f(x) = 3\sin x + 2\tan x$. Both $\sin x$ and $\tan x$ are odd functions, so $f(x)$ is an odd function. The integral of an odd function over a symmetric interval $[-\frac{\pi}{4}, \frac{\pi}{4}]$ is zero.

$$\int_{-\pi/4}^{\pi/4} (3\sin x + 2\tan x)\,dx = \mathbf{0}.$$

(h) $\int_0^1 \frac{1}{x^2+4}\,dx$. We have: $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan(\frac{x}{a})$. Here $a=2$.

$$\begin{aligned}\int_0^1 \frac{1}{x^2+4}\,dx &= \left[\frac{1}{2}\arctan\left(\frac{x}{2}\right)\right]_0^1 \\ &= \frac{1}{2}\arctan\left(\frac{1}{2}\right) - \frac{1}{2}\arctan(0) \\ &= \frac{\mathbf{1}}{\mathbf{2}}\arctan\left(\frac{\mathbf{1}}{\mathbf{2}}\right).\end{aligned}$$

(4) **Mean value of a function.** $\left(m(f;a,b) = \frac{1}{b-a}\int_a^b f(x)\,dx\right)$.

(a) $f(x) = 2x+1 \quad$ on $[0,4]$.

$$\begin{aligned}m(f;0,4) = \frac{1}{4-0}\int_0^4 (2x+1)\,dx &= \frac{1}{4}\left[x^2+x\right]_0^4 \\ &= \frac{1}{4}\left[(16+4)-0\right] = \frac{20}{4} = \mathbf{5}.\end{aligned}$$

$$f(z) = 5 \Rightarrow 2z+1 = 5 \Rightarrow 2z = 4 \Rightarrow \mathbf{z} = \mathbf{2}.$$

(b) $f(x) = 3x^2 - 4x \quad \text{on } [1,3]$.

$$m(f;1,3) = \frac{1}{3-1}\int_1^3 (3x^2-4x)\,dx = \frac{1}{2}\left[x^3-2x^2\right]_1^3$$
$$= \frac{1}{2}\left[(27-18)-(1-2)\right] = \frac{1}{2}\left[9-(-1)\right] = \mathbf{5}.$$
$$f(z) = 5 \Rightarrow 3z^2-4z = 5 \Rightarrow 3z^2-4z-5 = 0.$$
$$z = \frac{-(-4)\pm\sqrt{(-4)^2-4(3)(-5)}}{2(3)} = \frac{4\pm\sqrt{16+60}}{6} = \frac{4\pm\sqrt{76}}{6}$$
$$z = \frac{4\pm 2\sqrt{19}}{6} = \frac{2\pm\sqrt{19}}{3}.$$

Since $\sqrt{19}\approx 4.36$, $z_1 = \frac{2+\sqrt{19}}{3}\approx 2.12 \in [1,3]$ and $z_2 = \frac{2-\sqrt{19}}{3}\approx -0.79 \notin [1,3]$. **The point is** $\mathbf{z} = \frac{\mathbf{2+\sqrt{19}}}{\mathbf{3}}$.

(c) $f(x) = (x-2)^2 \quad \text{on } [0,5]$.

$$m(f;0,5) = \frac{1}{5-0}\int_0^5 (x-2)^2\,dx = \frac{1}{5}\left[\frac{(x-2)^3}{3}\right]_0^5$$
$$= \frac{1}{15}\left[(5-2)^3-(0-2)^3\right] = \frac{1}{15}\left[3^3-(-2)^3\right]$$
$$= \frac{1}{15}\left[27+8\right] = \frac{35}{15} = \frac{\mathbf{7}}{\mathbf{3}}.$$
$$f(z) = \frac{7}{3} \Rightarrow (z-2)^2 = \frac{7}{3} \Rightarrow z-2 = \pm\sqrt{\frac{7}{3}}$$
$$\Rightarrow z = 2\pm\sqrt{\frac{7}{3}}.$$

Since $\sqrt{7/3}\approx 1.527$, $z_1 = 2+1.527 = 3.527 \in [0,5]$ and $z_2 = 2-1.527 = 0.473 \in [0,5]$. **The points are** $\mathbf{z} = \mathbf{2}\pm\sqrt{\frac{\mathbf{7}}{\mathbf{3}}}$.

(d) $f(x) = e^{2x} \quad \text{on } [0,\ln 3]$.

$$m(f;0,\ln 3) = \frac{1}{\ln 3}\int_0^{\ln 3} e^{2x}\,dx = \frac{1}{\ln 3}\left[\frac{1}{2}e^{2x}\right]_0^{\ln 3}$$
$$= \frac{1}{2\ln 3}\left[e^{2\ln 3}-e^0\right] = \frac{1}{2\ln 3}\left[e^{\ln 9}-1\right]$$
$$= \frac{9-1}{2\ln 3} = \frac{8}{2\ln 3} = \frac{\mathbf{4}}{\ln\mathbf{3}}.$$
$$f(z) = \frac{4}{\ln 3} \Rightarrow e^{2z} = \frac{4}{\ln 3} \Rightarrow 2z = \ln\left(\frac{4}{\ln 3}\right)$$
$$\Rightarrow \mathbf{z} = \frac{\mathbf{1}}{\mathbf{2}}\ln\left(\frac{\mathbf{4}}{\ln\mathbf{3}}\right).$$

(e) $f(x) = \frac{1}{x} \quad \text{on } [1,e^2]$.

$$m(f;1,e^2) = \frac{1}{e^2-1}\int_1^{e^2} \frac{1}{x}\,dx = \frac{1}{e^2-1}\left[\ln|x|\right]_1^{e^2}$$
$$= \frac{1}{e^2-1}\left[\ln(e^2)-\ln 1\right] = \frac{2-0}{e^2-1} = \frac{\mathbf{2}}{\mathbf{e^2-1}}.$$
$$f(z) = \frac{2}{e^2-1} \Rightarrow \frac{1}{z} = \frac{2}{e^2-1} \Rightarrow \mathbf{z} = \frac{\mathbf{e^2-1}}{\mathbf{2}}.$$

(f) $f(x) = \sin(x)$ on $[0, \pi]$.

$$\begin{aligned}
m(f; 0, \pi) &= \frac{1}{\pi - 0} \int_0^\pi \sin x \, dx = \frac{1}{\pi} \left[-\cos x \right]_0^\pi \\
&= \frac{1}{\pi} \left[-\cos(\pi) - (-\cos(0)) \right] = \frac{1}{\pi} \left[-(-1) + 1 \right] = \frac{\mathbf{2}}{\boldsymbol{\pi}}. \\
f(z) &= \frac{2}{\pi} \Rightarrow \sin(z) = \frac{2}{\pi}.
\end{aligned}$$

Since $2/\pi \approx 0.6366 < 1$, there are two solutions in $[0, \pi]$. $\mathbf{z_1} = \arcsin\left(\frac{\mathbf{2}}{\boldsymbol{\pi}}\right)$ and $\mathbf{z_2} = \pi - \arcsin\left(\frac{\mathbf{2}}{\boldsymbol{\pi}}\right)$.

(g) $f(x) = \sec^2(x)$ on $[0, \frac{\pi}{4}]$.

$$\begin{aligned}
m(f; 0, \pi/4) &= \frac{1}{\pi/4 - 0} \int_0^{\pi/4} \sec^2 x \, dx = \frac{4}{\pi} \left[\tan x \right]_0^{\pi/4} \\
&= \frac{4}{\pi} \left[\tan\left(\frac{\pi}{4}\right) - \tan(0) \right] = \frac{4}{\pi} [1 - 0] = \frac{\mathbf{4}}{\boldsymbol{\pi}}. \\
f(z) &= \frac{4}{\pi} \Rightarrow \sec^2(z) = \frac{4}{\pi} \Rightarrow \cos^2(z) = \frac{\pi}{4} \Rightarrow \cos(z) = \pm\sqrt{\frac{\pi}{4}}.
\end{aligned}$$

Since $z \in [0, \pi/4]$, $\cos z$ must be positive. $\mathbf{z} = \arccos\left(\frac{\sqrt{\boldsymbol{\pi}}}{\mathbf{2}}\right)$.

(h) $f(x) = \sqrt{x+1}$ on $[3, 8]$.

$$\begin{aligned}
m(f; 3, 8) &= \frac{1}{8 - 3} \int_3^8 (x+1)^{1/2} \, dx = \frac{1}{5} \left[\frac{2}{3} (x+1)^{3/2} \right]_3^8 \\
&= \frac{2}{15} \left[(8+1)^{3/2} - (3+1)^{3/2} \right] = \frac{2}{15} \left[9^{3/2} - 4^{3/2} \right] \\
&= \frac{2}{15} [27 - 8] = \frac{2(19)}{15} = \frac{\mathbf{38}}{\mathbf{15}}. \\
f(z) &= \frac{38}{15} \Rightarrow \sqrt{z+1} = \frac{38}{15} \Rightarrow z + 1 = \left(\frac{38}{15}\right)^2 \\
&\Rightarrow \mathbf{z} = \frac{\mathbf{1444}}{\mathbf{225}} - \mathbf{1} = \frac{\mathbf{1219}}{\mathbf{225}}.
\end{aligned}$$

($1219/225 \approx 5.418 \in [3, 8]$).

(i) $f(x) = \frac{x^2 - 1}{x^4}$ on $[1, 2]$.

$$\begin{aligned}
\int_1^2 \left(x^{-2} - x^{-4} \right) dx &= \left[-x^{-1} + \frac{1}{3} x^{-3} \right]_1^2 = \left[-\frac{1}{x} + \frac{1}{3x^3} \right]_1^2 \\
&= \left(-\frac{1}{2} + \frac{1}{24} \right) - \left(-1 + \frac{1}{3} \right) = \left(-\frac{11}{24} \right) - \left(-\frac{2}{3} \right) \\
&= -\frac{11}{24} + \frac{16}{24} = \frac{5}{24}. \\
m(f; 1, 2) &= \frac{1}{2 - 1} \cdot \frac{5}{24} = \frac{\mathbf{5}}{\mathbf{24}}. \\
f(z) &= \frac{5}{24} \Rightarrow \frac{z^2 - 1}{z^4} = \frac{5}{24} \Rightarrow 24(z^2 - 1) = 5z^4.
\end{aligned}$$

Let $u = z^2$. $5u^2 - 24u + 24 = 0$. Using the quadratic formula: $u = \frac{24 \pm \sqrt{24^2 - 4(5)(24)}}{10} = \frac{24 \pm \sqrt{576 - 480}}{10} = \frac{24 \pm \sqrt{96}}{10} = \frac{12 \pm 2\sqrt{6}}{5}$. $u_1 \approx 3.379$ and $u_2 \approx 1.421$. Since $u = z^2$ and $z \in [1, 2]$, $u \in [1, 4]$. Both are possible. $\mathbf{z} = \sqrt{\frac{\mathbf{12 \pm 2\sqrt{6}}}{\mathbf{5}}}$. (Two positive roots).

(j) $f(x) = xe^{-x^2}$ on $[0,1]$. We use substitution $u = -x^2$, so $du = -2x\,dx$.

$$\int_0^1 xe^{-x^2}\,dx = -\frac{1}{2}\int_0^{-1} e^u\,du = -\frac{1}{2}\left[e^u\right]_0^{-1}$$
$$= -\frac{1}{2}(e^{-1} - e^0) = -\frac{1}{2}\left(\frac{1}{e} - 1\right) = \frac{e-1}{2e}.$$
$$m(f;0,1) = \frac{1}{1-0}\cdot\frac{e-1}{2e} = \frac{\mathbf{e-1}}{\mathbf{2e}}.$$
$$f(z) = \frac{e-1}{2e} \Rightarrow \mathbf{ze^{-z^2}} = \frac{\mathbf{e-1}}{\mathbf{2e}}.$$

(This is a transcendental equation; it cannot be solved explicitly for z).

(5) **Compute** $\int_{\frac{2}{41\pi}}^{\frac{2}{\pi}}\left(-\frac{1}{x^2}\cos\left(\frac{1}{x}\right)\right)dx$**.**

The integrand is the derivative of $f(x) = \sin\left(\frac{1}{x}\right)$: $\frac{d}{dx}\left[\sin\left(\frac{1}{x}\right)\right] = -\frac{1}{x^2}\cos\left(\frac{1}{x}\right)$.

$$\int_{\frac{2}{41\pi}}^{\frac{2}{\pi}}\left(-\frac{1}{x^2}\cos\left(\frac{1}{x}\right)\right)dx = \left[\sin\left(\frac{1}{x}\right)\right]_{\frac{2}{41\pi}}^{\frac{2}{\pi}}$$
$$= \sin\left(\frac{1}{\frac{2}{\pi}}\right) - \sin\left(\frac{1}{\frac{2}{41\pi}}\right)$$
$$= \sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{41\pi}{2}\right).$$

Since $\sin(\frac{\pi}{2}) = 1$ and $\frac{41\pi}{2} = 20\pi + \frac{\pi}{2}$, we have:

$$\sin\left(\frac{41\pi}{2}\right) = \sin\left(20\pi + \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1.$$
$$\int_{\frac{2}{41\pi}}^{\frac{2}{\pi}}\left(-\frac{1}{x^2}\cos\left(\frac{1}{x}\right)\right)dx = 1 - 1 = \mathbf{0}.$$

(6) **Compute** $\int_{-\pi}^{\pi}\frac{\sin x + x^3}{2+\cos x + e^{x^2}}dx$**.**

Let the integrand be $f(x)$. We check the parity of the numerator and the denominator.

- **Numerator** $N(x) = \sin x + x^3$. Since $\sin(-x) = -\sin x$ and $(-x)^3 = -x^3$, $N(-x) = -\sin x - x^3 = -N(x)$. $N(x)$ is odd.
- **Denominator** $D(x) = 2 + \cos x + e^{x^2}$. Since $\cos(-x) = \cos x$ and $e^{(-x)^2} = e^{x^2}$, $D(-x) = D(x)$. $D(x)$ is even.

The entire integrand $f(x) = N(x)/D(x)$ is an $\frac{\text{odd}}{\text{even}}$ function, which means $f(x)$ is an odd function. Since the integral is over a symmetric interval $[-\pi,\pi]$, the value of the integral is zero.

$$\int_{-\pi}^{\pi}\frac{\sin x + x^3}{2+\cos x + e^{x^2}}dx = \mathbf{0}.$$

(7) **Integral and Limit of Sum.**

(a) **Compute the integral** $\int_0^1 \frac{1}{(x+1)^2}dx$**.**

$$\int_0^1 \frac{1}{(x+1)^2}dx = \int_0^1 (x+1)^{-2}dx$$
$$= \left[-(x+1)^{-1}\right]_0^1 = \left[-\frac{1}{x+1}\right]_0^1$$
$$= \left(-\frac{1}{1+1}\right) - \left(-\frac{1}{0+1}\right)$$
$$= -\frac{1}{2} + 1 = \frac{\mathbf{1}}{\mathbf{2}}.$$

(b) **Prove the limit of the sum is $\frac{1}{2}$.**

The sum can be rewritten by factoring n from the denominators:

$$S_n = n\left(\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(2n)^2}\right)$$

We rewrite the general term $\frac{1}{(n+k)^2}$ (where k runs from 1 to n) as:

$$\frac{1}{(n+k)^2} = \frac{1}{n^2(1+k/n)^2} = \frac{1}{n^2}\frac{1}{(1+k/n)^2}.$$

The sum is:

$$S_n = n\sum_{k=1}^{n}\frac{1}{n^2}\frac{1}{(1+k/n)^2} = \frac{1}{n}\sum_{k=1}^{n}\frac{1}{(1+k/n)^2}.$$

This is a Riemann sum for the function $f(x) = \frac{1}{(1+x)^2}$ over the interval $[0, 1]$, where the interval is partitioned into n subintervals of width $\Delta x = \frac{1}{n}$, and the sample points are $x_k = \frac{k}{n}$.

The limit of the Riemann sum is the definite integral:

$$\begin{aligned}\lim_{n\to\infty} S_n &= \lim_{n\to\infty}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\Delta x \\ &= \int_0^1 f(x)\,dx = \int_0^1 \frac{1}{(1+x)^2}\,dx.\end{aligned}$$

From part (a), we know the value of the integral is $\frac{1}{2}$.

$$\lim_{n\to\infty} n\left(\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(2n)^2}\right) = \mathbf{\frac{1}{2}}.$$