11.2 Numerical series

Improper integrals are a way of measuring the area of 'infinite' domains in the plane. Thinking of type I improper integrals, we can imagine replacing the function $f(x)$ with a step function that has the constant value $f(n)$ on any interval $[n, n+1)$, where $n \in \mathbb{N}$. Then,

$$\text{the integral} \qquad \int_N^{+\infty} f(x)\,\mathrm{d}x \qquad \text{is replaced by} \qquad \sum_{k=N}^{\infty} f(k).$$

This is called a numerical series. It can serve as an approximation for an improper integral, but it also is interesting irrespective. Thus, our goal in this section is to understand the meaning of an infinite sum of the form

$$a_0 + a_1 + a_2 + \cdots + a_n + \cdots = ?$$

Series and their partial sums

Given a sequence $\{a_n\}_{n\in\mathbb{N}}$ of real numbers, we form the **partial sums**

$$s_n = a_0 + a_1 + a_2 + \cdots + a_n = \sum_{k=0}^{n} a_k.$$

The **series** (or infinite sum) associated to $\{a_n\}_{n\in\mathbb{N}}$ is defined as the limit of the partial sums:

$$\sum_{k=0}^{\infty} a_k = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=0}^{n} a_k.$$

If the limit exists and is finite, we say the series *converges*; if the limit is infinite, the series *diverges*; if the limit does not exist, the series is *indeterminate*.

Geometric series

One of the most important examples is the geometric series.

Geometric series

For $r \in \mathbb{R}$, the geometric series is

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots .$$

Its partial sums are given by

$$s_n = 1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r} \quad \text{for } r \neq 1.$$

Taking the limit as $n \to \infty$, we obtain:

$$\sum_{n=0}^{\infty} r^n = \begin{cases} \frac{1}{1-r} & \text{if } |r| < 1, \\ \text{diverges} & \text{if } |r| \geq 1. \end{cases}$$

p-Series

For $p > 0$, a ***p*-series** is the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$. As we shall see below, it will converge for $p > 1$ and diverge for $0 < p \leq 1$. In the case $p = 1$, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is called the **harmonic series**.

Necessary condition

Necessary condition for convergence

If $\sum_{n=0}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Proof. Let $s_n = \sum_{k=0}^{n} a_k$ and $s = \lim_{n\to\infty} s_n$. Then

$$a_n = s_n - s_{n-1} \to s - s = 0 \quad \text{as } n \to \infty.$$

□

Remark: The converse is *false*! For example, the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges even though $\frac{1}{n} \to 0$. *(the fact that the harmonic series diverges will be a consequence of the integral test, which we will see below)*

Convergence tests for series with non-negative terms

For series with $a_n \geq 0$ for all n, the sequence of partial sums $\{s_n\}$ is non-decreasing, so it either converges to a finite limit or diverges to $+\infty$.

Comparison Test

Let $\{a_n\}$ and $\{b_n\}$ be sequences with $0 \leq a_n \leq b_n$ for all n. Then:

- If $\sum_{n=0}^{\infty} b_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.
- If $\sum_{n=0}^{\infty} a_n$ diverges, then $\sum_{n=0}^{\infty} b_n$ diverges.

Example 11.12: The p-series with $p = 2$, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges because $\frac{1}{n^2} \leq \frac{2}{n(n+1)} = 2\left(\frac{1}{n} - \frac{1}{n+1}\right)$ and

$$\sum_{n=1}^{N} 2\left(\frac{1}{n} - \frac{1}{n+1}\right) = 2\left(1 - \frac{1}{N+1}\right) \to 2.$$

Ratio Test (d'Alembert's test)

Let $\sum_{n=0}^{\infty} a_n$ be a series with $a_n > 0$ for all n, and let

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

Then:

- If $L < 1$, the series converges.
- If $L > 1$, the series diverges.
- If $L = 1$, the test is inconclusive.

Example 11.13: For $\sum_{n=1}^{\infty} \frac{n}{2^n}$, we have

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)/2^{n+1}}{n/2^n} = \frac{n+1}{2n} \to \frac{1}{2} < 1,$$

so the series converges by the Ratio Test.

Root Test (Cauchy's test)

Let $\sum_{n=0}^{\infty} a_n$ be a series with $a_n \geq 0$ for all n, and let

$$L = \lim_{n\to\infty} \sqrt[n]{a_n}.$$

Then:

- If $L < 1$, the series converges.
- If $L > 1$, the series diverges.
- If $L = 1$, the test is inconclusive.

Example 11.14: For $\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n$, we have

$$\sqrt[n]{a_n} = \frac{n}{3n+1} \to \frac{1}{3} < 1,$$

so the series converges by the Root Test.

Integral Test

Let $f : [1, \infty) \to [0, +\infty)$ be a continuous, decreasing function with $f(n) = a_n$ for all $n \in \mathbb{N}$. Then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \quad \Leftrightarrow \quad \int_1^{\infty} f(x)\, \mathrm{d}x \text{ converges.}$$

Moreover, we have the error estimate:

$$\int_{N+1}^{\infty} f(x)\, \mathrm{d}x \le \sum_{n=N+1}^{\infty} a_n \le \int_N^{\infty} f(x)\, \mathrm{d}x.$$

Example 11.15: Consider the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$. Let $f(x) = \frac{1}{x^p}$, which is continuous and decreasing for $x \ge 1$. We know from improper integrals that

$$\int_1^{\infty} \frac{1}{x^p}\, \mathrm{d}x \text{ converges if and only if } p > 1.$$

Therefore, by the Integral Test:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges if and only if } p > 1.$$

Alternating series

A series is called **alternating** if its terms alternate in sign.

Leibniz's Alternating Series Test

Let $\{a_n\}$ be a sequence such that:

1. $a_n \ge 0$ for all n,
2. $\{a_n\}$ is decreasing: $a_{n+1} \le a_n$ for all n,
3. $\lim_{n\to\infty} a_n = 0$.

Then the alternating series $\sum_{n=0}^{\infty} (-1)^n a_n$ converges. Moreover, if $S = \sum_{n=0}^{\infty} (-1)^n a_n$ and $s_N = \sum_{n=0}^{N} (-1)^n a_n$, then the error satisfies:

$$|S - s_N| \le a_{N+1}.$$

Example 11.16: The alternating harmonic series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ converges by Leibniz's test, since $\frac{1}{n}$ is positive, decreasing, and tends to 0.

Note that the ordinary harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, so this shows that convergence of an alternating series does not imply absolute convergence.

Absolute convergence

Absolute Convergence Test

If $\sum_{n=0}^{\infty}|a_n|$ converges, then $\sum_{n=0}^{\infty}a_n$ converges. In this case, we say the series **converges absolutely** and we have

$$\left|\sum_{n=0}^{\infty}a_n\right| \le \sum_{n=0}^{\infty}|a_n|.$$

Proof. Let $s_n = \sum_{k=0}^{n}a_k$ be the partial sums of $\sum_{n=0}^{\infty}a_n$, and let $t_n = \sum_{k=0}^{n}|a_k|$ be the partial sums of $\sum_{n=0}^{\infty}|a_n|$.

Since $\sum_{n=0}^{\infty}|a_n|$ converges, the sequence $\{t_n\}_{n\in\mathbb{N}}$ converges to some limit T. This means that for any $\varepsilon > 0$, there exists N such that for all $m > n \ge N$,

$$t_m - t_n = \sum_{k=n+1}^{m}|a_k| < \varepsilon.$$

Now, for the same $m > n \ge N$, using the triangle inequality:

$$|s_m - s_n| = \left|\sum_{k=n+1}^{m}a_k\right| \le \sum_{k=n+1}^{m}|a_k| = t_m - t_n < \varepsilon.$$

Thus the partial sums $\{s_n\}$ satisfy that: for any $\varepsilon > 0$, there exists N such that $|s_m - s_n| < \varepsilon$ whenever $m > n \ge N$. In this course we haven't learned this precise condition, however this means $\{s_n\}_{n\in\mathbb{N}}$ is a convergent sequence (it is called a *Cauchy* sequence), so $\sum_{n=0}^{\infty}a_n$ converges. □

Example 11.17: The series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ converges absolutely because

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

converges (p-series with $p = 2 > 1$).

Remark: Absolute convergence is stronger than conditional convergence:

- *Absolutely convergent* series can be rearranged without changing the sum.
- *Conditionally convergent* series (convergent but not absolutely convergent) can be rearranged to converge to any real number or even diverge (Riemann rearrangement theorem).

Example 11.18: The alternating harmonic series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ converges conditionally but not absolutely, since $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.