Therefore, for any $x_0, x \in I$

$$\frac{f(x) - f(x_0)}{x - x_0} \geq 0.$$

By Theorem 5.3 (local sign of limits) the limit as $x \to x_0$ is also non-negative:

$$f'(x_0) \geq 0.$$

This proves the assertion for all x_0 that are not on the boundary of I. If x_0 is on the boundary of I, the same argument can be repeated with one-sided limits.

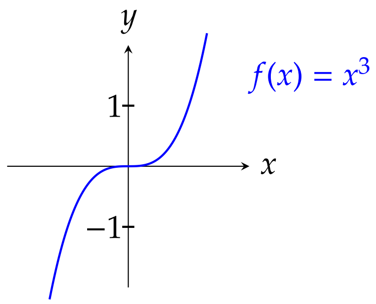
Now we prove (a)(⇒). Let $x_1, x_2 \in I$ with $x_1 < x_2$. By the second increment formula, there exists $\overline{x} \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1) = f'(\overline{x})(x_2 - x_1).$$

By assumption, $f'(\overline{x}) \geq 0$ and $x_2 - x_1 > 0$, so that $f(x_2) \geq f(x_1)$ which completes the proof.

The proof of (b) follows immediately, since in the above argument $f'(\overline{x}) > 0$, hence $f(x_2) > f(x_1)$. □

Observe that part (b) has a one-sided implication; the other implication is not true. For example, the function $f(x) = x^3$ is strictly increasing on $\mathbb{R}$, however its derivative function is not strictly positive (it vanishes at $x = 0$).



Corollary 8.16: Let $f : \mathbb{R} \to \mathbb{R}$ and suppose that f is differentiable on an interval I. Let $x_0 \in I$ be in the interior of I (not on the boundary). Then:

- If $f'(x) \geq 0$ to the left of x_0 and $f'(x) \leq 0$ to the right of x_0, then x_0 is a local maximum.
- If $f'(x) \leq 0$ to the left of x_0 and $f'(x) \geq 0$ to the right of x_0, then x_0 is a local minimum.

Proof. This simple proof is left as an exercise. □

Finding extrema and monotonicity intervals of a function

Using Theorem 8.15 and Corollary 8.16, we see that to find extrema and monotonicity intervals of a function, all we need to do is to know the sign and zeroes of its derivative.

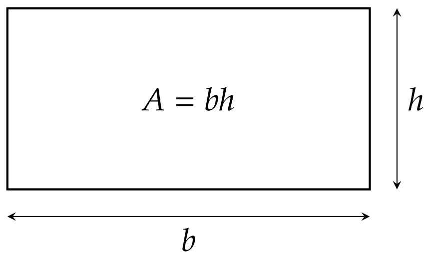
Example 8.19: Let $f(x) = xe^{2x}$. Then

$$f'(x) = e^{2x}(1 + 2x)$$

vanishes only when $1 + 2x = 0$, i.e. for $x = -\frac{1}{2}$. We see that $f'(x) > 0$ for $x > \frac{-}{1}2$ (f is strictly increasing on $[-\frac{1}{2}, +\infty)$) and $f'(x) < 0$ for $x < -\frac{1}{2}$ (f is strictly decreasing on $(-\infty, -\frac{1}{2}]$), so that $f(-\frac{1}{2})$ is a global minimum.

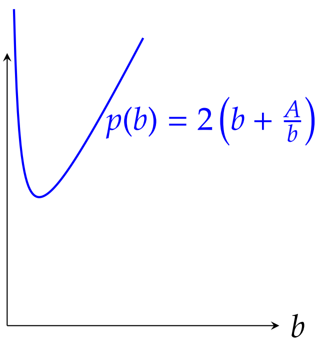
Example 8.20: Consider a rectangle with sides of length b and h, area $A = bh$ and perimeter

$$p = 2(b + h).$$



1. **Q: for fixed area A, what is the minimal perimeter?**

   We fix A, retain b as a variable, and express $h = \frac{A}{b}$. Then $p = 2(b + \frac{A}{b})$.

   We compute the derivative function

   $$p'(b) = 2\left(1 - \frac{A}{b^2}\right)$$

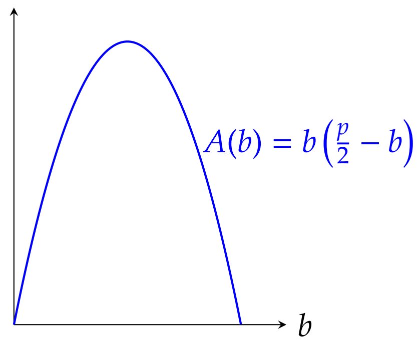
   which vanishes when $b = \sqrt{A}$. It is easy to check that $p'(b) < 0$ for $b < \sqrt{A}$ and $p'(b) > 0$ for $b > \sqrt{A}$. Hence $b = \sqrt{A}$ is the global minimum of this function, and

   $$p(\sqrt{A}) = 4\sqrt{A}.$$

   In this case $b = h$ and the rectangle is a square. This shows that **among all rectangles with a fixed area, the square minimizes the perimeter**.

2. **Q: for fixed perimeter p, what is the maximal area?**

   We fix p, retain b as a variable, and express $h = \frac{p}{2} - b$. Then $A = b(\frac{p}{2} - b)$.



We see that A vanishes for $b = 0$ and $b = \frac{p}{2}$. We compute

$$A'(b) = \frac{p}{2} - 2b$$

which vanishes when $b = \frac{p}{4}$. It is easy to check that $A'(b) > 0$ for $b < \frac{p}{4}$ and $A'(b) < 0$ for $b > \frac{p}{4}$. Hence $b = \frac{p}{4}$ is the global maximum of this function, and

$$A\left(\frac{p}{4}\right) = \frac{p^2}{16}.$$

In this case $b = h = \frac{p}{4}$ and the rectangle is a square. This shows that **among all rectangles with a fixed perimeter, the square maximizes the area**.

8.8 Higher-order derivatives

Second-order derivatives

Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. If the derivative function f' is differentiable at $x_0 \in \mathbb{R}$, we say that f **is twice differentiable at** x_0. We write

$$f''(x_0) = (f')'(x_0)$$

to express the **second derivative of** f **at** x_0. The derivative function of f' is denoted f'' and is called the **second derivative of** f. Some common symbols include

$$y''(x_0) \qquad \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(x_0) \qquad \frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x_0) \qquad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}(x_0) \qquad \frac{\mathrm{d}^2}{\mathrm{d}x^2} y(x_0) \qquad \mathrm{D}^2 f(x_0)$$

Higher-order derivatives

More generally, we can define derivatives of any integer order $k \in \mathbb{N}$ inductively. Since it is not convenient to write $f\overbrace{''''''}^{k\text{ times}}(x_0)$ we write $f^{(k)}$, defined inductively as

$$f^{(k)}(x_0) = (f^{(k-1)})'(x_0)$$

to express the k^{th} **derivative of** f **at** x_0. The k^{th} derivative function of f is denoted $f^{(k)}$ and is called the k^{th} **derivative of** f. Some common symbols include

$$y^{(k)}(x_0) \qquad \frac{\mathrm{d}^k f}{\mathrm{d}x^k}(x_0) \qquad \frac{\mathrm{d}^k}{\mathrm{d}x^k}f(x_0) \qquad \frac{\mathrm{d}^k y}{\mathrm{d}x^k}(x_0) \qquad \frac{\mathrm{d}^k}{\mathrm{d}x^k}y(x_0) \qquad \mathrm{D}^k f(x_0)$$

Example 8.21: The function $f(x) = x^n$, $n \in \mathbb{N}_+$ has the following derivatives, defined on $\mathbb{R}$:

$$\begin{aligned}
f'(x) &= nx^{n-1} \\
f''(x) &= n(n-1)x^{n-2} \\
&\vdots \\
f^{(k)}(x) &= n(n-1)\cdots(n-k+2)(n-k+1)x^{n-k} \\
&\vdots \\
f^{(n)}(x) &= n(n-1)\cdots 3\cdot 2\cdot 1x^{n-n} = n! \\
f^{(n+1)}(x) &= 0.
\end{aligned}$$

Example 8.22: Since $\frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x$, we have $\frac{\mathrm{d}^n}{\mathrm{d}x^n}e^x = e^x$ for all $n \in \mathbb{N}$.

Example 8.23: For sin and cos we have:

$$\begin{aligned}
\frac{\mathrm{d}^n}{\mathrm{d}x^n}\sin x &= \sin\left(x + n\frac{\pi}{2}\right) \\
\frac{\mathrm{d}^n}{\mathrm{d}x^n}\cos x &= \cos\left(x + n\frac{\pi}{2}\right)
\end{aligned}$$

Verify these formulas!!