7. Determine $\int \frac{1}{e^x + e^{-x}} \, dx$. Using the second approach, we have

$$g(x) = \frac{1}{e^x + e^{-x}}.$$

Choose $x(y) = \ln y$ to undo the exponents. Then we have

$$x(y) = \ln y \quad \text{so that} \quad \frac{dx}{dy} = \frac{1}{y} \quad \Rightarrow \quad dx = \frac{1}{y} \, dy$$

and we get

$$\int \frac{1}{e^x + e^{-x}} \, dx = \int \frac{1}{e^{\ln y} + e^{-\ln y}} \frac{1}{y} \, dy$$

$$= \int \frac{1}{y + \frac{1}{y}} \frac{1}{y} \, dy$$

$$= \int \frac{1}{1 + y^2} \, dy$$

$$= \arctan y + C = \arctan e^x + C.$$

Rational functions

It turns out that we can integrate rational functions such as

$$r(x) = \frac{p_n(x)}{q_m(x)}$$

The general theory is beyond the scope of this course, so we will stick to simple examples. However, we note that the key is the (deep) algebraic fact that every polynomial can be factorized into simple factors of the following forms:

$$(x + d)^s \quad \text{or} \quad (x^2 + bx + c)^t, \text{ with discriminant } \Delta = b^2 - 4c < 0$$

In particular, we can write

$$q_m(x) = (x + d_1)^{s_1}(x + d_2)^{s_2} \cdots (x^2 + b_1x + c_1)^{t_1}(x^2 + b_2x + c_2)^{t_2} \cdots$$

where $m = s_1 + s_2 + \cdots + 2(t_1 + t_2 + \cdots)$ is the degree of q_m, and hence the total number of roots (counting multiplicity). Consequently, the rational function $r(x)$ can be rewritten (roughly, there are a few missing terms) as

$$r(x) \approx \frac{A_1}{(x + d_1)^{s_1}} + \frac{A_2}{(x + d_2)^{s_2}} + \cdots + \frac{B_1x + C_1}{(x^2 + b_1x + c_1)^{t_1}} + \frac{B_2x + C_2}{(x^2 + b_2x + c_2)^{t_2}} + \cdots$$

where the constants A_i, B_i, C_i are determined by multiplying all the fractions on the right hand side and equating the resulting numerator to $p_n(x)$. The conclusion is that it is enough to know how to integrate rational functions of the forms

$$\frac{A_1}{(x + d_1)^{s_1}} \quad \text{and} \quad \frac{B_1x + C_1}{(x^2 + b_1x + c_1)^{t_1}}$$

Let us try to see how this might work.

Example 10.6: 1. The simplest case is easy:

$$\int \frac{1}{x+c}\,dx = \ln|x+c| + C.$$

2. The next case follows from what we know for power functions ($s > 1$):

$$\int \frac{1}{(x+c)^s}\,dx = \frac{1}{1-s}\frac{1}{(x+c)^{s-1}} + C.$$

3. Next we consider $\int g(x)\,dx$ where

$$g(x) = \frac{1}{x^2 + bx + c} \qquad \text{with discriminant } \Delta = b^2 - 4c < 0$$

so that the polynomial in the denominator is strictly positive (no real roots). We can write

$$x^2 + bx + c = x^2 + bx + \underbrace{c - \frac{1}{4}b^2}_{-\frac{1}{4}\Delta} + \frac{1}{4}b^2 = \underbrace{x^2 + bx + \frac{1}{4}b^2}_{(x+\frac{1}{2}b)^2} - \frac{1}{4}\Delta = (x + \frac{1}{2}b)^2 - \frac{1}{4}\Delta$$

$$= -\frac{1}{4}\Delta\left[1 + \left(\frac{x + \frac{1}{2}b}{\frac{1}{2}\sqrt{-\Delta}}\right)^2\right]$$

Make the substitution

$$y(x) = \frac{x + \frac{1}{2}b}{\frac{1}{2}\sqrt{-\Delta}}$$

whose derivative is

$$\frac{dy}{dx} = y'(x) = \frac{1}{\frac{1}{2}\sqrt{-\Delta}} \qquad \Rightarrow \qquad dx = \frac{1}{2}\sqrt{-\Delta}\,dy$$

We therefore have

$$\int \frac{1}{x^2 + bx + c}\,dx = -\frac{1}{\frac{1}{4}\Delta}\int \frac{1}{1 + \left(\frac{x+\frac{1}{2}b}{\frac{1}{2}\sqrt{-\Delta}}\right)^2}\,dx$$

$$= -\frac{1}{\frac{1}{4}\Delta}\int \frac{1}{1 + y^2}\frac{1}{2}\sqrt{-\Delta}\,dy$$

$$= \frac{1}{\frac{1}{2}\sqrt{-\Delta}}\int \frac{1}{1 + y^2}\,dy$$

$$= \frac{1}{\frac{1}{2}\sqrt{-\Delta}}\arctan y + C$$

$$= \frac{1}{\frac{1}{2}\sqrt{-\Delta}}\arctan\left(\frac{x + \frac{1}{2}b}{\frac{1}{2}\sqrt{-\Delta}}\right) + C$$

4. Determine $\int \frac{4x-5}{x^2-2x+10}\, dx$. We write

$$\int \frac{4x-5}{x^2-2x+10}\, dx = 2\int \frac{2x-2}{x^2-2x+10}\, dx - \int \frac{1}{x^2-2x+10}\, dx$$

The first integral is of the form $\int \frac{\varphi'(x)}{\varphi(x)}\, dx = \ln|\varphi(x)| + C$, while for the second integral we see that

$$\Delta = (-2)^2 - 4 \cdot 10 = -36 < 0$$

so that, using our previous computation, we have

$$\int \frac{1}{x^2-2x+10}\, dx = \frac{1}{\frac{1}{2}\sqrt{36}}\arctan\left(\frac{x-1}{\frac{1}{2}\sqrt{36}}\right) = \frac{1}{3}\arctan\left(\frac{x-1}{3}\right)$$

and we therefore conclude that

$$\int \frac{4x-5}{x^2-2x+10}\, dx = 2\ln(x^2-2x+10) - \frac{1}{3}\arctan\left(\frac{x-1}{3}\right) + C.$$

5. Determine $\int \frac{2x^3+x^2-4x+7}{x^2+x-2}\, dx$. First we simplify the rational function:

$$\frac{2x^3 + x^2 - 4x + 7}{x^2 + x - 2} = \frac{2x(x^2 + x - 2) - (x^2 + x - 2) + x + 5}{x^2 + x - 2} = 2x - 1 + \frac{x+5}{x^2 + x - 2}$$

The next step is to see if we can factor the denominator in the last term. In this case we can indeed see that

$$x^2 + x - 2 = (x+2)(x-1)$$

so that we write

$$\begin{aligned}
\frac{x+5}{x^2+x-2} &= \frac{A_1}{x-1} + \frac{A_2}{x+2} \\
&= \frac{A_1(x+2) + A_2(x-1)}{(x-1)(x+2)} \\
&= \frac{(A_1 + A_2)x + (2A_1 - A_2)}{(x-1)(x+2)}
\end{aligned}$$

So we find that A_1 and A_2 solve the system of equations

$$A_1 + A_2 = 1 \quad \text{and} \quad 2A_1 - A_2 = 5$$

so that

$$A_1 = 2 \quad \text{and} \quad A_2 = -1.$$
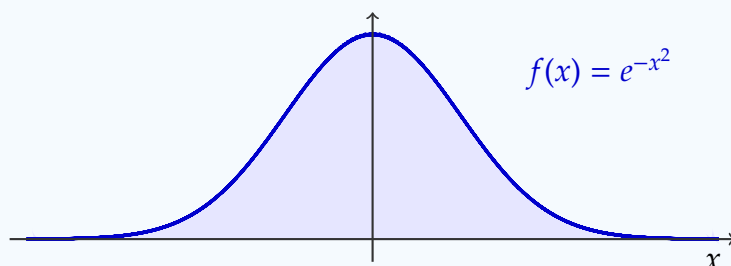
So we finally have:

$$\begin{aligned}
\int \frac{2x^3 + x^2 - 4x + 7}{x^2 + x - 2}\, dx &= \int (2x-1)\, dx + \int \frac{2}{x-1}\, dx - \int \frac{1}{x+2}\, dx \\
&= x^2 - x + 2\ln|x-1| - \ln|x+2| + C.
\end{aligned}$$

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Not all functions can be integrated

We have seen that integrating functions is not easy. It is important to understand that it is not even guaranteed to be possible. There exist functions that do not have an antiderivative that can be expressed using any of the functions we know. An example of this is the function

$$f(x) = e^{-x^2}$$

which is known as a **Gaussian** or a **bell curve**. This function is extremely important in probability theory, and yet we cannot express its antiderivatives using any of the functions that we know.

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10.3 Definite integrals

The *definite* integral, denoted

$$\int_a^b f(x)\,dx$$

is the measurement of the area under the graph of f between $x = a$ and $x = b$, where whenever the graph lies below the x-axis we count the corresponding area as a *negative* area.