

In the next sections we will give precise definitions, one due to Augustin-Louis Cauchy (18th-19th centuries, France) and one due to Bernhard Riemann (19th century, Germany). Cauchy's integral is simpler to understand, and lends itself better to computations (including on a computer), but Riemann's definition is the more 'correct' mathematical one. They lead to the same result.

10.4 Cauchy integral

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous on $[a, b]$ where $a < b$. Divide the interval $[a, b]$ into n equal subintervals of width

$$\Delta x = \frac{b - a}{n}$$

with the partition points x_k satisfying

$$x_0 = a, \quad x_1 = x_0 + \Delta x, \dots, \quad x_k = x_0 + k\Delta x, \dots, \quad x_n = x_0 + n\Delta x = b.$$

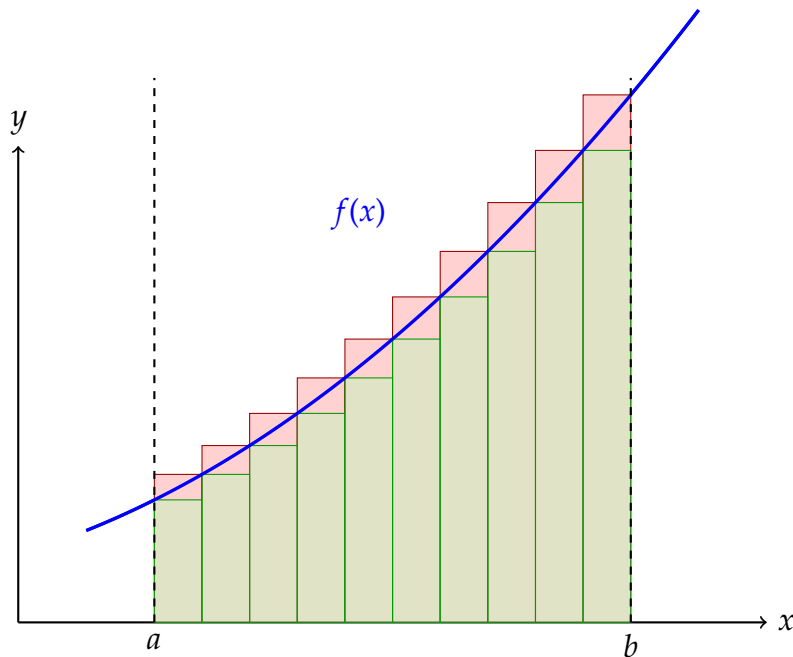
This is called an equal **partition** of $[a, b]$ into n subintervals, or an **n -partition**. Denote the n intervals by $I_k, k = 1, \dots, n$, so that the k th interval is

$$I_k = [x_{k-1}, x_k], \quad k = 1, \dots, n.$$

On each of these intervals we can define the minimum and the maximum

$$m_k = \min_{I_k} f(x) \quad \text{and} \quad M_k = \max_{I_k} f(x)$$

(their existence is guaranteed thanks to Weierstrass' Theorem).



Lower and upper sums

The area under the graph of f between a and b can be approximated from below by rectangles of height m_k over each interval I_k , and from above by rectangles of height M_k . We therefore define the **lower and upper sums**, respectively:

$$s_n = \sum_{k=1}^n m_k \Delta x \quad \text{and} \quad S_n = \sum_{k=1}^n M_k \Delta x$$

Since $\Delta x > 0$ and $m_k \leq M_k$, we have that $s_n \leq S_n$.

Definition 10.2: A partition of a partition is called a **refinement**.

Lemma 10.3: A refinement is in itself a partition of $[a, b]$. In particular, if an m -partition is refined by subdividing into ℓ further subintervals, the result is an $(m \cdot \ell)$ -partition.

Proof. An m -partition is given by

$$a = x_0 < x_1 < \cdots < x_m = b$$

where

$$x_k = x_0 + k \frac{b-a}{m}.$$

A refinement into ℓ sub-subintervals partitions the k th subinterval into

$$x_{k-1} = x_{k-1,0} < x_{k-1,1} < \cdots < x_{k-1,\ell} = x_k$$

where

$$x_{k-1,j} = x_{k-1} + j \frac{x_k - x_{k-1}}{\ell}.$$

But, using the fact that $x_k - x_{k-1} = \frac{b-a}{m}$ we have

$$x_{k-1,j} - x_{k-1,j-1} = \frac{x_k - x_{k-1}}{\ell} = \frac{\frac{b-a}{m}}{\ell} = \frac{b-a}{m \cdot \ell}.$$

This proves that the refinement is an $(m \cdot \ell)$ -partition with partition points

$$a = x_0 = x_{0,0} < x_{0,1} < \cdots < x_{0,\ell} = x_1 = x_{1,0} < x_{1,1} < \cdots < x_{m-1,\ell-1} < x_{m-1,\ell} = x_m = b.$$

□

Lemma 10.4: Under a refinement, the lower sum cannot decrease, and the upper sum cannot increase.

Proof. Let the refinement be as in the previous proof. Denote

$$I_{k,j} = [x_{k-1,j-1}, x_{k-1,j}], \quad k = 1, \dots, m, \quad j = 1, \dots, \ell,$$

and observe that

$$I_{k,j} \subset I_k = [x_{k-1}, x_k].$$

Then, since the minimum over a larger set can only be smaller, we have

$$m_k = \min_{I_k} f(x) \leq \min_{I_{k,j}} f(x) = m_{k,j}, \quad k = 1, \dots, m, \quad j = 1, \dots, \ell.$$

It follows that

$$s_m = \sum_{k=1}^m m_k \frac{b-a}{m} \leq \sum_{k=1}^m \sum_{j=1}^{\ell} m_{k,j} \frac{b-a}{m \cdot \ell} = s_{m \cdot \ell}.$$

This proves that the lower sum cannot decrease. Similarly, the upper sum cannot increase. \square

Theorem 10.5: The sequences $\{s_n\}_{n \in \mathbb{N}_+}$ and $\{S_n\}_{n \in \mathbb{N}_+}$ are both convergent, and both have the same limit.

Definite integral

The limit of s_n and S_n is called the **definite integral of f on $[a, b]$** and is denoted

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} S_n.$$

Proof. **Step 1. Compare two different partitions.** We know that $s_n \leq S_n$, but we want to be able to compare any s_m to any S_ℓ . The first corresponds to a partition into m subintervals and the second corresponds to a partition into ℓ subintervals. These have nothing to do with one another, but we can refine them to create a common $m \cdot \ell$ partition. From Lemma 10.4, we have

$$s_m \leq s_{m \cdot \ell} \quad \text{and} \quad S_{m \cdot \ell} \leq S_\ell.$$

But since we know that for any n , $s_n \leq S_n$, we have

$$s_m \leq s_{m \cdot \ell} \leq S_{m \cdot \ell} \leq S_\ell$$

and we therefore have a comparison of s_m and S_ℓ for any m and ℓ . It follows that the sequence $\{s_n\}_{n \in \mathbb{N}_+}$ has an upper bound, and the sequence $\{S_n\}_{n \in \mathbb{N}_+}$ has a lower bound. So we can define their supremum and infimum, respectively, which are finite, and satisfy:

$$s = \sup_n s_n \quad \text{is less than or equal to} \quad S = \inf_n S_n$$

Step 2. Prove that $s = S$. Since f is continuous on the closed interval $[a, b]$, the Heine-Cantor Theorem (Theorem 7.11) implies that it is *uniformly continuous* there. That is, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|x_1 - x_2| < \delta$ implies $|f(x_1) - f(x_2)| < \varepsilon$.

Fix $\varepsilon > 0$. Let N_ε be big enough so that

$$\frac{b-a}{N_\varepsilon} < \delta.$$

Let $n > N_\varepsilon$. Then

$$\Delta x = \frac{b-a}{n} < \frac{b-a}{N_\varepsilon} < \delta.$$

Consider the k th subinterval, I_k . There are $\xi_k \in I_k$ and $\eta_k \in I_k$ such that

$$f(\xi_k) = m_k = \min_{I_k} f \quad \text{and} \quad f(\eta_k) = M_k = \max_{I_k} f.$$

Since both points belong to I_k , their distance is less than δ . Uniform continuity implies that

$$M_k - m_k = f(\eta_k) - f(\xi_k) < \varepsilon.$$

We therefore have that

$$\begin{aligned} S_n - s_n &= \sum_{k=1}^n M_k \Delta x - \sum_{k=1}^n m_k \Delta x \\ &= \sum_{k=1}^n (M_k - m_k) \Delta x \\ &< \varepsilon \sum_{k=1}^n \Delta x \\ &= \varepsilon(b - a). \end{aligned}$$

By definition of s and of S , we have

$$0 \leq S - s \leq S_n - s_n < \varepsilon(b - a).$$

Since ε was arbitrary, we find that $s = S$ (by the Squeeze Theorem, for instance).

Step 3. Prove that the limits exist. Since $S_n \geq s = S$, we also have that

$$0 \leq S - s_n \leq S_n - s_n < \varepsilon(b - a)$$

so that the limit of $\{s_n\}_{n \in \mathbb{N}_+}$ exists and

$$\lim_{n \rightarrow \infty} s_n = S.$$

Similarly, since $s_n \leq s = S$ we have

$$0 \leq S_n - S \leq S_n - s_n < \varepsilon(b - a)$$

so that the limit of $\{S_n\}_{n \in \mathbb{N}_+}$ exists and

$$\lim_{n \rightarrow \infty} S_n = S.$$

□

Piecewise continuous functions

Definition 10.6: A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is said to be **piecewise continuous on $[a, b]$** if it is continuous everywhere in $[a, b]$, except for finitely many points where it might have either a removable discontinuity or a jump discontinuity.