Example 9.4: Expand $h(x) = \sqrt{1+x} \cdot e^x$ to second order. To second order, these two functions have the expansions:

$$f(x) = \sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + o(x^2)$$

and

$$g(x) = e^x = 1 + x + \frac{x^2}{2} + o(x^2)$$

$$\begin{aligned}
f(x) \cdot g(x) &= \left(1 + \frac{x}{2} - \frac{x^2}{8} + o(x^2)\right)\left(1 + x + \frac{x^2}{2} + o(x^2)\right) \\
&= 1 \cdot \left(1 + x + \frac{x^2}{2}\right) + \frac{x}{2} \cdot \left(1 + x + \frac{x^2}{2}\right) - \frac{x^2}{8} \cdot \left(1 + x + \frac{x^2}{2}\right) + o(x^2) \\
&= 1 + \left(x + \frac{x}{2}\right) + \left(\frac{x^2}{2} + \frac{x^2}{2} - \frac{x^2}{8}\right) + \underbrace{\left(\frac{x^3}{4} - \frac{x^3}{8}\right) + \left(-\frac{x^4}{16}\right)}_{o(x^2)} + o(x^2) \\
&= 1 + \frac{3}{2}x + \frac{7}{8}x^2 + o(x^2) \qquad x \to 0.
\end{aligned}$$

Example 9.5: Let us give an example of a different flavor. Suppose we want to approximate the product $\pi \cdot e$. Here are the expressions for both number to 5 decimal places:

$$\pi = 3.14159 \pm 5 \cdot 10^{-6} \qquad e = 2.71828 \pm 5 \cdot 10^{-6}$$

Let's compare the product of these two approximation to the actual product of π and e:

$$\begin{aligned}
3.14159 \times 2.71828 &= 8.5397\mathbf{212652} \\
\pi \times e &= 8.5397\mathbf{342226}\cdots
\end{aligned}$$

We see that they already disagree starting from the 5th decimal place. This demonstrates that when multiplying approximations we need to exercise caution! *But please do note that the type of error here is different from our 'little o' errors. The example here has to do with rounding errors, so there's no exact comparison. However this example serves as a warning to be careful with approximations: just because we can get a longer sequence of numbers when multiplying two approximations, doesn't mean it's correct.*

9.4 Local behavior of a function via its Taylor expansion

In this section we consider a function $f : \mathbb{R} \to \mathbb{R}$ and suppose that near the point $x_0 \in \mathbb{R}$ it can be written as

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n + o((x - x_0)^n)$$

Order and principal part of infinitesimal functions

Suppose that in the expansion of f near x_0, all coefficients a_i, $i = 0, \ldots, m-1$, are 0, and $m > 0$ is the first one such that $a_m \neq 0$. Then, near x_0, f can be expressed as

$$f(x) = a_m(x - x_0)^m + a_{m+1}(x - x_0)^{m+1} + \cdots + a_n(x - x_0)^n + o((x - x_0)^n)$$

and since $m > 0$ it means that $\lim_{x \to x_0} f(x) = 0$, so that f is infinitesimal at x_0. The above expression for f can be written even more crudely as

$$f(x) = \underbrace{a_m(x - x_0)^m}_{p(x)} + o((x - x_0)^m)$$

which immediately reveals to us that f is of order m at x_0 with respect to $\varphi(x) = x - x_0$, and has principal part $p(x) = a_m(x - x_0)^m$.

Example 9.6: Consider the function

$$f(x) = \sin x - x \cos x - \frac{1}{3}x^3.$$

Let us study this function around $x_0 = 0$. The relevant Maclaurin polynomials are:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} - \cdots + (-1)^m \frac{x^{2m+1}}{(2m+1)!} + o(x^{2m+2})$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} - \cdots + (-1)^m \frac{x^{2m}}{(2m)!} + o(x^{2m+1})$$

so that

$$x \cos x = x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!} - \cdots + (-1)^m \frac{x^{2m+1}}{(2m)!} + o(x^{2m+2})$$

It follows that

$$\begin{aligned}
f(x) &= \sin x - x \cos x - \frac{1}{3}x^3 \\
&= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^6)\right) - \left(x - \frac{x^3}{2!} + \frac{x^5}{4!} + o(x^6)\right) - \frac{1}{3}x^3 \\
&= (1-1)x + \left(\frac{1}{2} - \frac{1}{6} - \frac{1}{3}\right)x^3 + \left(\frac{1}{5!} - \frac{1}{4!}\right)x^5 + o(x^6) \\
&= -\frac{1}{30}x^5 + o(x^6) \qquad \text{as } x \to 0.
\end{aligned}$$

So f is infinitesimal of order 5 at $x_0 = 0$ with respect to $\varphi(x) = x$, and its principal part is $p(x) = -\frac{1}{30}x^5$.

Local behavior of a function

Since

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n + o((x - x_0)^n)$$

we can deduce that

$$\begin{aligned}
f(x_0) &= a_0 \\
f'(x_0) &= a_1 \\
f''(x_0) &= 2 \cdot a_2 \\
f'''(x_0) &= 3! \cdot a_3 \\
&\vdots \\
f^{(k)}(x_0) &= k! \cdot a_k \\
&\vdots
\end{aligned}$$

up to the nth derivative. In particular, knowing $f(x_0), f'(x_0), f''(x_0)$ means that we already know a lot about the function:

- $f(x_0)$ determines the sign of f near x_0 (if $f(x_0) \neq 0$)
- $f'(x_0)$ determines the monotonicity type of f near x_0 (if $f'(x_0) \neq 0$) or if x_0 is a critical point (if $f'(x_0) = 0$)
- $f''(x_0)$ determines convexity/concavity of f near x_0 (if $f''(x_0) \neq 0$)

Critical points

Theorem 9.6: Let f be differentiable $n \geq 2$ times at x_0 and suppose the for some $2 \leq m \leq n$

$$f'(x_0) = \cdots = f^{(m-1)}(x_0) = 0, \quad f^{(m)}(x_0) \neq 0$$

Then:

m even $\Rightarrow x_0$ is an extremum point (maximum if $f^{(m)}(x_0) < 0$ and minimum if $f^{(m)}(x_0) > 0$).

m odd $\Rightarrow x_0$ is an inflection point with horizontal tangent.

Proof. We skip the proof of this theorem. □

Example 9.7: Suppose that a function $f : \mathbb{R} \to \mathbb{R}$ can be written as

$$f(x) = 2 - 3(x-1)^4 + 2(x-1)^5 + o((x-1)^5), \qquad \text{as } x \to 1.$$

Then

$$f(1) = 2 \qquad f'(1) = f''(1) = f'''(1) = 0 \qquad f^{(4)}(1) = -3 \cdot 4! = -72 < 0$$

so that $x_0 = 1$ is a local maximum.

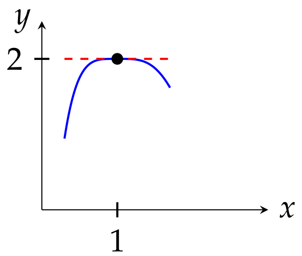


Figure 9.3: The Taylor polynomial $2 - 3(x-1)^4 + 2(x-1)^5$ near $x = 1$. It is extremely flat near $x = 1$.

Inflection points

In Corollary 8.20 we had the following statement which we did not prove since we lacked the tools:

Claim (Corollary 8.20): Let f be a twice-differentiable function in a neighborhood of x_0. Then:

$$x_0 \text{ is an inflection point} \qquad \Rightarrow \qquad f''(x_0) = 0,$$

and

$$f''(x_0) = 0 \text{ and } f'' \text{ changes sign at } x_0 \qquad \Rightarrow \qquad x_0 \text{ is an inflection point.}$$

Moreover,

$$f''(x_0) = 0 \text{ and } f'' \text{ doesn't change sign at } x_0 \qquad \Rightarrow \qquad x_0 \text{ isn't an inflection point.}$$

Proof. Let us only prove the first implication. Assume that x_0 is an inflection point. By definition, this means that f is above the tangent line at x_0 on one side of x_0, and below on the other side. First we recall the first- and second-order Taylor polynomials at x_0:

$$\begin{aligned}(Tf)_{1,x_0}(x) &= f(x_0) + f'(x_0)(x - x_0) \\ (Tf)_{2,x_0}(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 \\ &= (Tf)_{1,x_0}(x) + \frac{1}{2}f''(x_0)(x - x_0)^2\end{aligned}$$

and we keep in mind that $(Tf)_{1,x_0}(x)$ **is, in fact, the equation of the tangent** at x_0.

We will use Peano's estimate of the remainder for a second-order Taylor polynomial at x_0:

$$f(x) = (Tf)_{2,x_0}(x) + o((x - x_0)^2), \qquad x \to x_0.$$

The error can be written more explicitly:

$$f(x) = (Tf)_{2,x_0}(x) + \varphi(x)$$

where $\varphi(x) = o((x - x_0)^2)$ near x_0. That is, φ is a function that satisfies:

$$\lim_{x\to x_0} \frac{\varphi(x)}{(x - x_0)^2} = 0.$$

Define

$$\psi(x) = \frac{\varphi(x)}{(x - x_0)^2}.$$

Then

$$\varphi(x) = \psi(x)(x - x_0)^2 \qquad \text{and} \qquad \lim_{x\to x_0} \psi(x) = 0.$$

Now we can write a precise expression for f:

$$\begin{aligned}f(x) &= (Tf)_{2,x_0}(x) + \varphi(x) \\ &= (Tf)_{2,x_0}(x) + \psi(x)(x - x_0)^2 \\ &= (Tf)_{1,x_0}(x) + \frac{1}{2}f''(x_0)(x - x_0)^2 + \psi(x)(x - x_0)^2 \\ &= (Tf)_{1,x_0}(x) + \left[\frac{1}{2}f''(x_0) + \psi(x)\right](x - x_0)^2\end{aligned}$$

Moving $(Tf)_{1,x_0}(x)$ over to the left-hand-side will give us the difference between the function and the tangent:

$$f(x) - (Tf)_{1,x_0}(x) = \left[\frac{1}{2}f''(x_0) + \psi(x)\right](x - x_0)^2.$$

Recall our assumption: x_0 is an inflection point. By definition, this means that the right-hand-side should have different signs on either side of x_0. Since ψ is an infinitesimal function at x_0 (it is very small), if $f''(x_0) \neq 0$ then the term in the square brackets would have the same sign in a neighborhood of x_0. Since $(x - x_0)^2 \geq 0$, the entire right-hand-side would have the same sign in a neighborhood of x_0 (both sides of x_0). But since this is not the case, then necessarily $f''(x_0) = 0$. $\square$

Chapter 10

Intergral calculus

In this chapter we will discuss two fundamental problems for a given function f:

(1) finding another function F satisfying $F' = f$, and

(2) computing the area under its graph.

Through the *Fundamental Theorem of Integral Calculus* we will see that these two problems are, in fact, two aspects of the same problem.

10.1 Primitive functions and indefinite integrals

Primitive function

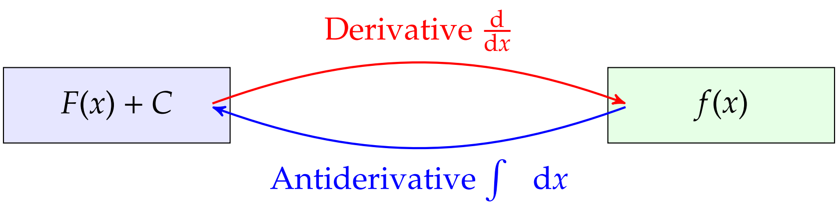
Given a function $f : \mathbb{R} \to \mathbb{R}$, defined on some interval $I \subseteq \mathbb{R}$, any function $F : \mathbb{R} \to \mathbb{R}$ satisfying

$$F'(x) = f(x), \qquad \forall x \in I,$$

is called a **primitive function (antiderivative)** of f. If $F'(x) = f(x)$ then also $\frac{\mathrm{d}}{\mathrm{d}x}(F(x) + C) = f(x)$ for any constant C, since the derivative of a constant is 0. Hence the antiderivative *isn't unique*. We therefore denote

$$\int f(x)\,\mathrm{d}x = F(x) + C$$

where the right hand side is an infinite set of functions.



Example 10.1: 1. Consider $f(x) = x$. We know that $\frac{\mathrm{d}}{\mathrm{d}x}(x^2) = 2x$ so that $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(x^2) = x$. It follows that

$$\int x\,\mathrm{d}x = \frac{1}{2}x^2 + C.$$

2. Consider $f(x) = x^2$. We know that $\frac{\mathrm{d}}{\mathrm{d}x}(x^3) = 3x^2$ so that $\frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}x}(x^3) = x^2$. It follows that
$$\int x^2 \,\mathrm{d}x = \frac{1}{3}x^3 + C.$$

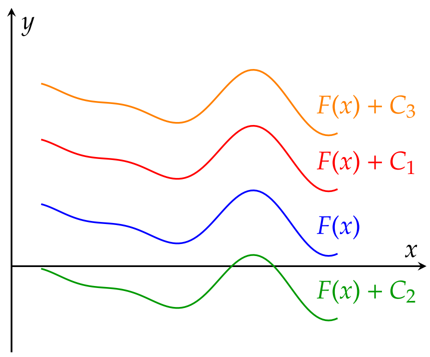
3. More generally, consider $f(x) = x^n$. We know that $\frac{\mathrm{d}}{\mathrm{d}x}(x^{n+1}) = (n+1)x^n$ so that $\frac{1}{n+1}\frac{\mathrm{d}}{\mathrm{d}x}(x^{n+1}) = x^n$. It follows that
$$\int x^n \,\mathrm{d}x = \frac{1}{n+1}x^{n+1} + C.$$

4. Consider $f(x) = e^x$. We know that $\frac{\mathrm{d}}{\mathrm{d}x}(e^x) = e^x$ so that
$$\int e^x \,\mathrm{d}x = e^x + C.$$

Following these ideas, we can list some of the fundamental antiderivatives:

$$\int x^\alpha \,\mathrm{d}x = \frac{x^{\alpha+1}}{\alpha+1} + C \quad (\alpha \neq -1)$$
$$\int \frac{1}{x} \,\mathrm{d}x = \log|x| + C \quad (\text{for } x > 0 \text{ or } x < 0)$$
$$\int \sin x \,\mathrm{d}x = -\cos x + C$$
$$\int \cos x \,\mathrm{d}x = \sin x + C$$
$$\int e^x \,\mathrm{d}x = e^x + C$$
$$\int \frac{1}{1+x^2} \,\mathrm{d}x = \arctan x + C$$
$$\int \frac{1}{\sqrt{1-x^2}} \,\mathrm{d}x = \arcsin x + C$$
$$\int \sinh x \,\mathrm{d}x = \cosh x + C$$
$$\int \cosh x \,\mathrm{d}x = \sinh x + C$$

It must be emphasized that there is no unique primitive function: if F is an antiderivative of f, then any vertical translation of F is also an antiderivative. There are *infinitely many antiderivatives*!



We can pinpoint which of these antiderivatives we are looking for by requiring that at some given x_0 the antiderivative has a particular value y_0: $F(x_0) = y_0$.

Example 10.2: 1. Identify the antiderivative $F(x)$ of $f(x) = \cos x$ that satisfies $F(2\pi) = 5$.

The antiderivative is $\int f(x)\,\mathrm{d}x = \sin x + C = F(x)$. If we require that $F(2\pi) = 5$ then we find that

$$\sin 2\pi + C = 5 \qquad \Rightarrow \qquad C = 5.$$

This has forced a choice of C.

2. Find the value at $x_1 = 3$ of the antiderivative of $f(x) = 6x^2 + 5x$ that vanishes at the point $x_0 = 1$.

First we identify the antiderivative:

$$\int f(x)\,\mathrm{d}x = \int \left(6x^2 + 5x\right)\,\mathrm{d}x = \frac{6x^3}{3} + \frac{5x^2}{2} + C = 2x^3 + \frac{5}{2}x^2 + C = F(x).$$

Plugging in the given condition we have:

$$0 = F(1) = 2 + \frac{5}{2} + C \qquad \Rightarrow \qquad C = -\frac{9}{2}.$$

Hence our antiderivative is:

$$F(x) = 2x^3 + \frac{5}{2}x^2 - \frac{9}{2}$$

and

$$F(3) = 2\cdot 3^3 + \frac{5}{2}\cdot 3^2 - \frac{9}{2} = 2\cdot 27 + \frac{5}{2}\cdot 9 - \frac{9}{2} = 72.$$

3. Find the antiderivative of $f(x) = \sin 3x$. We see that $\frac{\mathrm{d}}{\mathrm{d}x}(\cos 3x) = -3\sin 3x$, so that

$$F(x) = \int \sin 3x\,\mathrm{d}x = -\frac{1}{3}\cos 3x + C.$$