Example 11.6: Investigate integrals of the form (for $\alpha, \beta > 0$)

$$\int_2^{+\infty} \frac{1}{x^\alpha (\ln x)^\beta}\, \mathrm{d}x = ?$$

Case 1: $\alpha = 1$. In this case we can make the substitution $y = \ln x$ so that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}$ and we get

$$\int_2^{+\infty} \frac{1}{x(\ln x)^\beta}\, \mathrm{d}x = \int_{\ln 2}^{+\infty} \frac{1}{y^\beta}\, \mathrm{d}y$$

which converges if $\beta > 1$ and diverges if $\beta \leq 1$.

Case 2: $\alpha > 1$. Since $\ln x$ is strictly increasing, we have

$$\frac{1}{x^\alpha (\ln x)^\beta} \leq \frac{1}{x^\alpha (\ln 2)^\beta} \qquad \forall x \geq 2.$$

The Comparison Test implies that the integral will *converge* (regardless of β) since

$$\int_2^{+\infty} \frac{1}{x^\alpha (\ln 2)^\beta}\, \mathrm{d}x = \frac{1}{(\ln 2)^\beta} \int_2^{+\infty} \frac{1}{x^\alpha}\, \mathrm{d}x < +\infty.$$

Case 3: $\alpha < 1$. In this case we can write the integrand as

$$\frac{1}{x^\alpha (\ln x)^\beta} = \frac{1}{x} \frac{x^{1-\alpha}}{(\ln x)^\beta}$$

The function $\frac{x^{1-\alpha}}{(\ln x)^\beta}$ tends to $+\infty$ for any β (we've seen that x to any positive power grows faster than $\ln x$ to any positive power), so that there exists $M > 0$ such that

$$\frac{1}{x^\alpha (\ln x)^\beta} \geq \frac{M}{x}, \qquad \forall x \geq 2.$$

The Comparison Test implies that the integral *diverges* in this case.

Remark: The integral $\int_{-\infty}^{b} f(x)\, \mathrm{d}x$ is defined in an analogous way, and all the analysis follows analogously:

$$\int_{-\infty}^{b} f(x)\, \mathrm{d}x = \lim_{a \to -\infty} \int_a^b f(x)\, \mathrm{d}x.$$

11.1.2 Type II improper integrals

Improper integrals of type II occur when the integrand has a singularity (e.g., vertical asymptote) at a finite point of the integration domain.

Improper integral (type II)

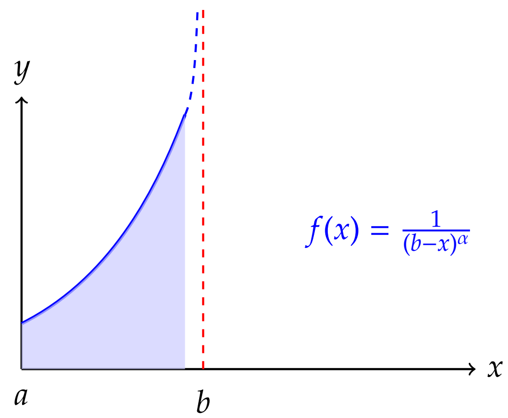
Let $f : \mathbb{R} \to \mathbb{R}$ be a function integrable on $[a, c]$ for any $c < b$, but with $\lim_{x\to b^-} |f(x)| = +\infty$ (or undefined). We define the **improper integral** (type II) of f on $[a, b)$ to be

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{t\to b^-} \int_a^t f(x)\,\mathrm{d}x.$$

Similarly, if f has a singularity at $x = a$, we define

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{t\to a^+} \int_t^b f(x)\,\mathrm{d}x.$$

They will *converge*, *diverge* or be *indeterminate* depending on the limit on the right hand side.



Example 11.7: Investigate the convergence of the integral

$$\int_0^1 \frac{1}{x^\alpha}\,\mathrm{d}x,$$

where $\alpha > 0$. The integrand has a singularity at $x = 0$ when $\alpha > 0$.

We analyze two main cases: $\alpha = 1$ and $\alpha \neq 1$.

Case 1: $\alpha = 1$**.** In this case, the integrand is $f(x) = \frac{1}{x}$. We compute:

$$\int_t^1 \frac{1}{x}\,\mathrm{d}x = [\ln|x|]_t^1 = \ln|1| - \ln|t| = -\ln t$$

Now we take the limit as $t \to 0^+$:

$$\int_0^1 \frac{1}{x}\,\mathrm{d}x = \lim_{t\to 0^+}(-\ln t) = +\infty.$$

Therefore, the integral *diverges* for $\alpha = 1$.

Case 2: $\alpha \neq 1$**.** We have:

$$\int_t^1 \frac{1}{x^\alpha}\,\mathrm{d}x = \left[\frac{1}{1-\alpha}x^{1-\alpha}\right]_t^1 = \frac{1}{1-\alpha}\left(1 - t^{1-\alpha}\right)$$

Now we analyze the limit $\lim_{t\to 0^+} t^{1-\alpha}$:

- If $\mathbf{1} - \alpha > \mathbf{0}$ (i.e., $\alpha < \mathbf{1}$): Since $1-\alpha$ is positive, $\lim_{t\to 0^+} t^{1-\alpha} = 0$. The limit is:

  $$\int_0^1 \frac{1}{x^\alpha}\,\mathrm{d}x = \lim_{t\to 0^+} \frac{1}{1-\alpha}\left(1 - t^{1-\alpha}\right) = \frac{1}{1-\alpha}(1-0) = \frac{1}{1-\alpha}.$$

  The integral *converges* to $\frac{1}{1-\alpha}$ when $\alpha < \mathbf{1}$.

- If $\mathbf{1} - \alpha < \mathbf{0}$ (i.e., $\alpha > \mathbf{1}$): Since $1-\alpha$ is negative, we rewrite $t^{1-\alpha} = \frac{1}{t^{\alpha-1}}$, and $\lim_{t\to 0^+} \frac{1}{t^{\alpha-1}} = +\infty$. The limit is:

  $$\int_0^1 \frac{1}{x^\alpha}\,\mathrm{d}x = \lim_{t\to 0^+} \frac{1}{1-\alpha}\left(1 - t^{1-\alpha}\right) = \frac{1}{1-\alpha}(1-\infty) = +\infty.$$

  The integral *diverges* for $\alpha > \mathbf{1}$.

We conclude:

$$\int_0^1 \frac{1}{x^\alpha}\,\mathrm{d}x \qquad \begin{cases} \text{converges} & \text{if } 0 < \alpha < 1, \\ \text{diverges} & \text{if } \alpha \geq 1. \end{cases}$$

More generally, for an integral $\int_a^b \frac{1}{(b-x)^\alpha}\,\mathrm{d}x$ with singularity at b, the same convergence conditions hold.

Convergence tests

The comparison tests for type II improper integrals are analogous to those for type I.

Comparison Test

Let f, g be integrable on $[a, t]$ for any $t < b$, with $0 \leq f(x) \leq g(x)$ for all $x \in [a, b)$. Then

$$0 \leq \int_a^b f(x)\,\mathrm{d}x \leq \int_a^b g(x)\,\mathrm{d}x$$

Therefore

$$\int_a^b g(x)\,\mathrm{d}x < +\infty \text{ (converges)} \quad \Rightarrow \quad \int_a^b f(x)\,\mathrm{d}x < +\infty \text{ (converges)}$$

$$\int_a^b f(x)\,\mathrm{d}x = +\infty \text{ (diverges)} \quad \Rightarrow \quad \int_a^b g(x)\,\mathrm{d}x = +\infty \text{ (diverges)}$$

Example 11.8: Determine whether

$$\int_0^1 \frac{|\ln x|}{x^{1/3}}\,\mathrm{d}x$$

converges or diverges.

Near $x = 0$, $|\ln x| \to +\infty$ and $x^{-1/3} \to +\infty$, so the integrand is unbounded. To use the comparison test, observe that for any $\alpha > 0$, we have

$$\lim_{x\to 0^+} x^{\alpha}|\ln x| = 0,$$

which implies that there exists $\delta > 0$ such that for all $0 < x < \delta$,

$$|\ln x| \le x^{-\alpha}$$

(you can show this using De l'Hôpital, as in Example 8.28). Take $\alpha = 1/6$. Then for $0 < x < \delta$,

$$\frac{|\ln x|}{x^{1/3}} \le \frac{x^{-1/6}}{x^{1/3}} = x^{-1/2}.$$

Since

$$\int_0^1 x^{-1/2}\,\mathrm{d}x = 2$$

converges, we conclude by the comparison test that

$$\int_0^1 \frac{|\ln x|}{x^{1/3}}\,\mathrm{d}x$$

also converges.

Absolute Convergence Test

Suppose that both f and $|f|$ are integrable on $[a, t]$ for any $t < b$. Then if $\int_a^b |f(x)|\,\mathrm{d}x$ converges, so does $\int_a^b f(x)\,\mathrm{d}x$, and

$$\left|\int_a^b f(x)\,\mathrm{d}x\right| \le \int_a^b |f(x)|\,\mathrm{d}x.$$

Example 11.9: Consider $\int_0^1 \frac{\sin(1/x)}{\sqrt{x}}\,\mathrm{d}x$. The integrand oscillates wildly near $x = 0$, but

$$\left|\frac{\sin(1/x)}{\sqrt{x}}\right| \le \frac{1}{\sqrt{x}}.$$

Since $\int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x$ converges ($\alpha = \frac{1}{2} < 1$), the original integral converges absolutely.

Asymptotic Comparison Test

Suppose $f(x) \sim \frac{1}{(b-x)^\alpha}$ as $x \to b^-$ (or similarly near a). Then:

$$\alpha < 1 \quad \Rightarrow \quad \int_a^b f(x)\,\mathrm{d}x < +\infty \text{ (converges)}$$

$$\alpha \geq 1 \quad \Rightarrow \quad \int_a^b f(x)\,\mathrm{d}x = +\infty \text{ (diverges)}$$

Example 11.10: Investigate $\int_0^{\pi/2} \tan x\,\mathrm{d}x$.

The integrand has a singularity at $x = \pi/2$ since $\tan x \to +\infty$ as $x \to (\pi/2)^-$. We analyze the behavior near $\pi/2$:

$$\tan x = \frac{\sin x}{\cos x} \sim \frac{1}{\cos x} \quad \text{as } x \to \frac{\pi}{2}^-.$$

Using the substitution $t = \pi/2 - x$, we get $\cos x = \sin t \sim t$ as $t \to 0^+$. Thus $\tan x \sim \frac{1}{t}$ near $x = \pi/2$, i.e., $\alpha = 1$. Since $\int_0^{\pi/2} \frac{1}{(\pi/2-x)}\,\mathrm{d}x$ diverges, the original integral diverges.

Example 11.11: Investigate $\int_0^1 \frac{1}{\sqrt{x(1-x)}}\,\mathrm{d}x$.

This integrand has singularities at both endpoints $x = 0$ and $x = 1$. We split the integral at $x = 1/2$:

$$\int_0^1 \frac{1}{\sqrt{x(1-x)}}\,\mathrm{d}x = \int_0^{1/2} \frac{1}{\sqrt{x(1-x)}}\,\mathrm{d}x + \int_{1/2}^1 \frac{1}{\sqrt{x(1-x)}}\,\mathrm{d}x.$$

Near $x = 0$, $\frac{1}{\sqrt{x(1-x)}} \sim \frac{1}{\sqrt{x}}$ ($\alpha = 1/2 < 1$), so the first integral converges. Near $x = 1$, substitute $t = 1 - x$ to get $\frac{1}{\sqrt{x(1-x)}} \sim \frac{1}{\sqrt{t}}$ ($\alpha = 1/2 < 1$), so the second integral converges. Thus the original integral converges.

Remark: When dealing with type II improper integrals, always:

1. Identify all points where the integrand is unbounded (singularities).
2. Split the integral at each singularity.
3. Analyze convergence separately for each part.
4. The integral converges only if *all* parts converge.

11.2 Numerical series

Improper integrals are a way of measuring the area of 'infinite' domains in the plane. Thinking of type I improper integrals, we can imagine replacing the function $f(x)$ with a step function that has the constant value $f(n)$ on any interval $[n, n+1)$, where $n \in \mathbb{N}$. Then,

$$\text{the integral} \quad \int_N^{+\infty} f(x)\,\mathrm{d}x \quad \text{is replaced by} \quad \sum_{k=N}^{\infty} f(k).$$

This is called a numerical series. It can serve as an approximation for an improper integral, but it also is interesting irrespective. Thus, our goal in this section is to understand the meaning of an infinite sum of the form

$$a_0 + a_1 + a_2 + \cdots + a_n + \cdots = ?$$

Series and their partial sums

Given a sequence $\{a_n\}_{n\in\mathbb{N}}$ of real numbers, we form the **partial sums**

$$s_n = a_0 + a_1 + a_2 + \cdots + a_n = \sum_{k=0}^{n} a_k.$$

The **series** (or infinite sum) associated to $\{a_n\}_{n\in\mathbb{N}}$ is defined as the limit of the partial sums:

$$\sum_{k=0}^{\infty} a_k = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=0}^{n} a_k.$$

If the limit exists and is finite, we say the series *converges*; if the limit is infinite, the series *diverges*; if the limit does not exist, the series is *indeterminate*.

Geometric series

One of the most important examples is the geometric series.

Geometric series

For $r \in \mathbb{R}$, the geometric series is

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots .$$

Its partial sums are given by

$$s_n = 1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r} \quad \text{for } r \neq 1.$$

Taking the limit as $n \to \infty$, we obtain:

$$\sum_{n=0}^{\infty} r^n = \begin{cases} \frac{1}{1-r} & \text{if } |r| < 1, \\ \text{diverges} & \text{if } |r| \geq 1. \end{cases}$$

Necessary condition

Necessary condition for convergence

If $\sum_{n=0}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Proof. Let $s_n = \sum_{k=0}^{n} a_k$ and $s = \lim_{n\to\infty} s_n$. Then

$$a_n = s_n - s_{n-1} \to s - s = 0 \quad \text{as } n \to \infty.$$

□

Remark: The converse is *false*! For example, the **harmonic series** $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges even though $\frac{1}{n} \to 0$. *(the fact that the harmonic series diverges will be a consequence of the integral test, which we will see below)*

Convergence tests for series with non-negative terms

For series with $a_n \geq 0$ for all n, the sequence of partial sums $\{s_n\}$ is non-decreasing, so it either converges to a finite limit or diverges to $+\infty$.

Comparison Test

Let $\{a_n\}$ and $\{b_n\}$ be sequences with $0 \leq a_n \leq b_n$ for all n. Then:

- If $\sum_{n=0}^{\infty} b_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.
- If $\sum_{n=0}^{\infty} a_n$ diverges, then $\sum_{n=0}^{\infty} b_n$ diverges.

Example 11.12: The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges because $\frac{1}{n^2} \leq \frac{2}{n(n+1)} = 2\left(\frac{1}{n} - \frac{1}{n+1}\right)$ and

$$\sum_{n=1}^{N} 2\left(\frac{1}{n} - \frac{1}{n+1}\right) = 2\left(1 - \frac{1}{N+1}\right) \to 2.$$

Ratio Test (d'Alembert's test)

Let $\sum_{n=0}^{\infty} a_n$ be a series with $a_n > 0$ for all n, and let

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

Then:

- If $L < 1$, the series converges.
- If $L > 1$, the series diverges.
- If $L = 1$, the test is inconclusive.

Example 11.13: For $\sum_{n=1}^{\infty} \frac{n}{2^n}$, we have

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)/2^{n+1}}{n/2^n} = \frac{n+1}{2n} \to \frac{1}{2} < 1,$$

so the series converges by the Ratio Test.

Root Test (Cauchy's test)

Let $\sum_{n=0}^{\infty} a_n$ be a series with $a_n \geq 0$ for all n, and let

$$L = \lim_{n\to\infty} \sqrt[n]{a_n}.$$

Then:

- If $L < 1$, the series converges.
- If $L > 1$, the series diverges.
- If $L = 1$, the test is inconclusive.

Example 11.14: For $\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n$, we have

$$\sqrt[n]{a_n} = \frac{n}{3n+1} \to \frac{1}{3} < 1,$$

so the series converges by the Root Test.

Integral Test

Let $f : [1, \infty) \to [0, +\infty)$ be a continuous, decreasing function with $f(n) = a_n$ for all $n \in \mathbb{N}$. Then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \quad \Leftrightarrow \quad \int_1^{\infty} f(x)\,\mathrm{d}x \text{ converges.}$$

Moreover, we have the error estimate:

$$\int_{N+1}^{\infty} f(x)\,\mathrm{d}x \le \sum_{n=N+1}^{\infty} a_n \le \int_N^{\infty} f(x)\,\mathrm{d}x.$$

Example 11.15: Consider the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$. Let $f(x) = \frac{1}{x^p}$, which is continuous and decreasing for $x \ge 1$. We know from improper integrals that

$$\int_1^{\infty} \frac{1}{x^p}\,\mathrm{d}x \text{ converges if and only if } p > 1.$$

Therefore, by the Integral Test:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges if and only if } p > 1.$$

Alternating series

A series is called **alternating** if its terms alternate in sign.

Leibniz's Alternating Series Test

Let $\{a_n\}$ be a sequence such that:

1. $a_n \ge 0$ for all n,
2. $\{a_n\}$ is decreasing: $a_{n+1} \le a_n$ for all n,
3. $\lim_{n\to\infty} a_n = 0$.

Then the alternating series $\sum_{n=0}^{\infty} (-1)^n a_n$ converges. Moreover, if $S = \sum_{n=0}^{\infty} (-1)^n a_n$ and $s_N = \sum_{n=0}^{N} (-1)^n a_n$, then the error satisfies:

$$|S - s_N| \le a_{N+1}.$$

Example 11.16: The alternating harmonic series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ converges by Leibniz's test, since $\frac{1}{n}$ is positive, decreasing, and tends to 0.

Note that the ordinary harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so this shows that convergence of an alternating series does not imply absolute convergence.

Absolute convergence

Absolute Convergence Test

If $\sum_{n=0}^{\infty} |a_n|$ converges, then $\sum_{n=0}^{\infty} a_n$ converges. In this case, we say the series **converges absolutely**.

Proof. Let $s_n = \sum_{k=0}^{n} a_k$ be the partial sums of $\sum_{n=0}^{\infty} a_n$, and let $t_n = \sum_{k=0}^{n} |a_k|$ be the partial sums of $\sum_{n=0}^{\infty} |a_n|$.

Since $\sum_{n=0}^{\infty} |a_n|$ converges, the sequence $\{t_n\}_{n\in\mathbb{N}}$ converges to some limit T. This means that for any $\varepsilon > 0$, there exists N such that for all $m > n \geq N$,

$$t_m - t_n = \sum_{k=n+1}^{m} |a_k| < \varepsilon.$$

Now, for the same $m > n \geq N$, using the triangle inequality:

$$|s_m - s_n| = \left| \sum_{k=n+1}^{m} a_k \right| \leq \sum_{k=n+1}^{m} |a_k| = t_m - t_n < \varepsilon.$$

Thus the partial sums $\{s_n\}$ satisfy that: for any $\varepsilon > 0$, there exists N such that $|s_m - s_n| < \varepsilon$ whenever $m > n \geq N$. In this course we haven't learned this precise condition, however this means $\{s_n\}_{n\in\mathbb{N}}$ is a convergent sequence (it is called a *Cauchy* sequence), so $\sum_{n=0}^{\infty} a_n$ converges. □

Example 11.17: The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges absolutely because

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges (p-series with $p = 2 > 1$).

Remark: Absolute convergence is stronger than conditional convergence:

- *Absolutely convergent* series can be rearranged without changing the sum.
- *Conditionally convergent* series (convergent but not absolutely convergent) can be rearranged to converge to any real number or even diverge (Riemann rearrangement theorem).

Example 11.18: The alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges conditionally but not absolutely, since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.