which we did not prove (but we *nearly* proved, only a few minor manipulations were missing). Suppose the $x_0 \in \mathrm{dom}(f)$, then

$$\begin{aligned}
f'(x_0) &= \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} \\
&= \lim_{\Delta x\to 0} \frac{(x_0+\Delta x)^\alpha - x_0^\alpha}{\Delta x} \\
&= x_0^{\alpha-1} \lim_{\Delta x\to 0} \frac{(1+\frac{\Delta x}{x_0})^\alpha - 1}{\frac{\Delta x}{x_0}} \\
\left(\textit{substitute } y = \frac{\Delta x}{x_0}\right) \quad &= x_0^{\alpha-1} \lim_{y\to 0} \frac{(1+y)^\alpha - 1}{y} \\
&= \alpha x_0^{\alpha-1}.
\end{aligned}$$

This is true for all x_0 for which the expression $x_0^{\alpha-1}$ is well-defined. This means that the derivative function is $f'(x) = \alpha x^{\alpha-1}$ and its domain consists of all $x \in \mathrm{dom}(f)$ for which $x^{\alpha-1}$ is well-defined.

Example 8.5: The function

$$f(x) = \sqrt{x} = x^{\frac{1}{2}} \quad \text{has domain } \mathrm{dom}(f) = [0, +\infty)$$

while its derivative

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} \quad \text{has domain } \mathrm{dom}(f') = (0, +\infty).$$

Example 8.6: For the function $f(x) = \sin x$ we use the formula

$$\sin(\alpha) - \sin(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

as follows:

$$\begin{aligned}
f'(x_0) &= \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} \\
&= \lim_{x\to x_0} \frac{\sin x - \sin x_0}{x-x_0} \\
&= \lim_{x\to x_0} \frac{2\cos\left(\frac{x+x_0}{2}\right)\sin\left(\frac{x-x_0}{2}\right)}{x-x_0} \\
&= \underbrace{\lim_{x\to x_0} \cos\left(\frac{x+x_0}{2}\right)}_{=\cos x_0} \underbrace{\lim_{x\to x_0} \frac{\sin\left(\frac{x-x_0}{2}\right)}{\frac{x-x_0}{2}}}_{=1} \\
&= \cos x_0.
\end{aligned}$$

Since this is true for all $x_0 \in \mathbb{R}$, we find that the derivative function of $f(x)$ is $f'(x) = \cos x$.

Example 8.7: Similarly it can be shown that for $f(x) = \cos x$, $f'(x) = -\sin x$.

Example 8.8: Let $f(x) = a^x$. Then

$$\begin{aligned} f'(x_0) &= \lim_{\Delta x \to 0} \frac{a^{x_0+\Delta x} - a^{x_0}}{\Delta x} \\ &= a^{x_0} \underbrace{\lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}}_{=\ln a} \\ &= a^{x_0} \ln a \end{aligned}$$

Since this is true for all $x_0 \in \mathbb{R}$, we find that the derivative function of $f(x)$ is $f'(x) = a^x \ln a$. In particular,

$$(e^x)' = e^x.$$

This is the only function that is equal to its own derivative at every point, and that is one reason why the number e is so special.

Example 8.9: The area A of a disc of radius r is given by πr^2. Thinking of A as being a function of r, we can write

$$A(r) = \pi r^2.$$

Its derivative is

$$A'(r) = 2\pi r$$

which is exactly the circumference of the circle of radius r. That is, the rate of change of area of a disc of radius r is equal to the circumference of its boundary.

Example 8.10: If the location $x(t)$ at time t of a car driving along the x-axis is given by

$$x(t) = z + vt, \qquad z, v \in \mathbb{R},$$

(here the independent variable is *time* t, and x is the *dependent* variable) then the rate of change of the location is give by

$$x'(t) = v$$

which is the *speed* of the car.

8.2 Differentiation rules

Theorem 8.2 (Algebraic operations and linearity of the derivative)**:** Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable at $x_0 \in \mathbb{R}$. Then their sum, difference, product and ratio are all also differentiable at x_0 and satisfy:

$$\begin{aligned} (f \pm g)'(x_0) &= f'(x_0) \pm g'(x_0), \\ (fg)'(x_0) &= f'(x_0)g(x_0) + f(x_0)g'(x_0), \\ \left(\frac{f}{g}\right)'(x_0) &= \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{(g(x_0))^2}. \end{aligned}$$

Moreover, the formula for the sum and product implies that *differentiation is a linear operation*:

$$(\alpha f + \beta g)'(x_0) = \alpha f'(x_0) + \beta g'(x_0), \qquad \forall \alpha, \beta \in \mathbb{R}.$$

Proof. Let us prove the formula for the product of the two functions. The other proofs follow similar ideas.

$$\begin{aligned}
(fg)'(x_0) &= \lim_{x\to x_0} \frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0} \\
&= \lim_{x\to x_0} \frac{f(x)g(x) - f(x_0)g(x) + f(x_0)g(x) - f(x_0)g(x_0)}{x - x_0} \\
&= \lim_{x\to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0} g(x) + f(x_0) \frac{g(x) - g(x_0)}{x - x_0} \right] \\
&= \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} \lim_{x\to x_0} g(x) + f(x_0) \lim_{x\to x_0} \frac{g(x) - g(x_0)}{x - x_0} \\
&= f'(x_0)g(x_0) + f(x_0)g'(x_0).
\end{aligned}$$

□

The linearity of the derivative allows us to take a derivative of a **polynomial** since we already know the rule for a monomial $\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}$. Let $p(x) = \sum_{k=0}^{n} a_k x^k$, then

$$p'(x) = \sum_{k=0}^{n} k a_k x^{k-1} = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \cdots + 2a_2 x + a_1.$$

Example 8.11: Let $p(x) = 10x^6 - 3x^4 - 2x^3 + x^2 - 50$. Then

$$p'(x) = 60x^5 - 12x^3 - 6x^2 + 2x.$$

Example 8.12: We can also differentiate **rational functions**. Let

$$r(x) = \frac{5x^3 - 4x + 2}{3x - 10}.$$

Then

$$\begin{aligned}
r'(x) &= \frac{(15x^2 - 4)(3x - 10) - (5x^3 - 4x + 2) \cdot 3}{(3x - 10)^2} \\
&= \frac{45x^3 - 150x^2 - 12x + 40 - 15x^3 + 12x - 6}{9x^2 - 60x + 100} \\
&= \frac{30x^3 - 150x^2 + 34}{9x^2 - 60x + 100}.
\end{aligned}$$

Example 8.13: Let $f(x) = \tan x$. Then

$$\begin{aligned}
f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\sin x}{\cos x} \right) \\
&= \frac{\sin' x \cos x - \sin x \cos' x}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} = 1 + \tan^2 x.
\end{aligned}$$

Theorem 8.3 (Chain Rule): Let $f, g : \mathbb{R} \to \mathbb{R}$. Assume that f is differentiable at x_0 and that g is differentiable at $y_0 = f(x_0)$. Then $h = g \circ f$ is differentiable at x_0 and the derivative is given by:

$$h'(x_0) = g'(y_0)f'(x_0) = g'(f(x_0))f'(x_0).$$

Proof. Recall the definition of the derivative of g at y_0:

$$g'(y_0) = \lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0}$$

which can be rewritten as follows:

$$\begin{aligned} 0 &= \lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} - g'(y_0) \\ &= \lim_{y \to y_0} \frac{g(y) - g(y_0) - g'(y_0)(y - y_0)}{y - y_0}. \end{aligned}$$

Call the numerator in the above limit $\psi(y)$:

$$\psi(y) = g(y) - g(y_0) - g'(y_0)(y - y_0).$$

The function $\psi(y)$ is defined in a small neighborhood of y_0. Then we know that not only $\lim_{y \to y_0} \psi(y) = 0$, but also

$$\lim_{y \to y_0} \frac{\psi(y)}{y - y_0} = 0$$

(this means that $\psi(y) = o(y - y_0)$ as $y \to y_0$). That is, as $y \to y_0$, $\psi(y)$ tends to 0 faster than $y - y_0$. Denote:

$$\varphi(y) = \frac{\psi(y)}{y - y_0} \qquad \text{so that} \qquad \psi(y) = \varphi(y)(y - y_0).$$

Then

$$\lim_{y \to y_0} \varphi(y) = 0.$$

Returning to the definition of ψ, we have:

$$g(y) - g(y_0) - g'(y_0)(y - y_0) = \psi(y) = \varphi(y)(y - y_0).$$

Hence, for all y in a neighborhood of y_0:

$$g(y) - g(y_0) = g'(y_0)(y - y_0) + \varphi(y)(y - y_0).$$

Plugging in $y = f(x)$ and $y_0 = f(x_0)$, and dividing by $x - x_0$ this becomes

$$\frac{g(f(x)) - g(f(x_0))}{x - x_0} = g'(f(x_0))\frac{f(x) - f(x_0)}{x - x_0} + \varphi(f(x))\frac{f(x) - f(x_0)}{x - x_0}.$$

Recalling that $h = g \circ f$ we have

$$\frac{h(x) - h(x_0)}{x - x_0} = g'(f(x_0))\frac{f(x) - f(x_0)}{x - x_0} + \varphi(f(x))\frac{f(x) - f(x_0)}{x - x_0}.$$

Taking the limit $x \to x_0$, we get

$$\begin{aligned}\lim_{x\to x_0}\frac{h(x)-h(x_0)}{x-x_0} &= g'(f(x_0))\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} + \lim_{x\to x_0}\left[\varphi(f(x))\frac{f(x)-f(x_0)}{x-x_0}\right] \\ &= g'(f(x_0))\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} + \lim_{x\to x_0}\varphi(f(x))\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} \\ &= g'(f(x_0))f'(x_0) + 0\cdot f'(x_0) \\ &= g'(f(x_0))f'(x_0).\end{aligned}$$

Hence we find that the derivative of $h = g \circ f$ at x_0 exists and is equal to

$$h'(x_0) = (g \circ f)'(x_0) = g'(f(x_0))f'(x_0).$$

□

Chain rule: avoiding confusion!

It is easy to get confused with notation when dealing with the chain rule, so it is best to be cautious. If we write $h = g \circ f$ then the chain rule can be expressed as

$$\frac{\mathrm{d}h}{\mathrm{d}x}(x_0) = \frac{\mathrm{d}g}{\mathrm{d}y}(y_0)\frac{\mathrm{d}f}{\mathrm{d}x}(x_0)$$

where $y_0 = f(x_0)$. Alternatively, if we write $y = f(x)$ and $z = g(y)$ then we can write the chain rule also as:

$$\frac{\mathrm{d}z}{\mathrm{d}x}(x_0) = \frac{\mathrm{d}z}{\mathrm{d}y}(y_0)\frac{\mathrm{d}y}{\mathrm{d}x}(x_0)$$

where $y_0 = f(x_0)$. This latter expression is easier to remember, because one could imagine that the terms $\mathrm{d}y$ on the right hand side cancel out (though we are not allowed to actually do that!).

Example 8.14: Consider the function $z = h(x) = \sqrt{1-x^2}$ which is the composition of $y = f(x) = 1 - x^2$ with $z = g(y) = \sqrt{y}$. Recalling that

$$f'(x) = -2x \qquad \text{and} \qquad g'(y) = \frac{1}{2\sqrt{y}}$$

we have:

$$\frac{\mathrm{d}h}{\mathrm{d}x}(x) = \frac{\mathrm{d}g}{\mathrm{d}y}(y)\frac{\mathrm{d}f}{\mathrm{d}x}(x) = \frac{1}{2\sqrt{1-x^2}}(-2x) = -\frac{x}{\sqrt{1-x^2}}.$$

Theorem 8.4 (Derivative of the inverse function)**:** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and invertible in a neighborhood of $x_0 \in \mathbb{R}$. Moreover, suppose that f is differentiable at x_0 and that $f'(x_0) \neq 0$. Then the inverse function $f^{-1}(y)$ is differentiable at $y_0 = f(x_0)$ and the derivative there is given by

$$\frac{\mathrm{d}}{\mathrm{d}y}f^{-1}(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}.$$