Homework 8

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Exercise (3.14). *Show that $\oplus$ is well-defined on $\mathbb{Z}_n$.*

Solution. $2 \oplus 3 = 5$ and $12 \oplus 15 = 29 = 5 + 2(12) = 5$ So the representation of the number did not change the result, so this is well-defined.

Exercise (A.1). *Simply each of the following expressions modulo the specified number.*

- $(a+b)^3 \bmod 3$
- $103 - 28 + 5(6) \bmod 2$
- $3(4) \bmod 6$
- $1586749 \bmod 10$

Solution.

- $a+b)^3 \bmod 3$
  $n(a+b)^3 = 3$
- $103 - 28 + 30 = 75 + 30 = 105$
  $105 \bmod 2 = 1$
- $12 \bmod 6 = 0$
- $1586749 \bmod 10 = 9$

Exercise (3.16). *Looking at the Cayley tables for C_5 and $\mathbb{Z}_5$, do you notice any features of the rows and columns that could be generalized to all Cayley tables? Make a conjecture end prove it.*

Solution. I notice that lines of diagonal have the same number. This makes sense since the rows and columns of a Cayley table are in sequential order. With the operation of addition, when you move up one column, you also move right one row (making the diagonal), so you start with $a + b$ and move a up $(a-1)$ and move b right $(b+1)$. So $a + b = (a-1) + (b+1)$. These numbers are equivalent, so the numbers along a diagonal are equivalent. □

Exercise (3.18). *Consider a square in the plane. How many distinct symmetries does it have? Give each symmetry a concise, meaningful label. For each pair of symmetries, S and T, compose them in both orders. Record this in a Cayley table.*

Solution. There are 8 symmetries: $R_0, R_{90}, R_{180}, R_{270}, D_1, D_2, M_1, M_2,$ where R_n is a rotation of n degrees. D_1 and D_2 are both flips along a diagonal line, and M_1 and M_2 are flips are the midpoints (horizontal and vertical) of the sides.

Composing these symmetries in different orders will result in different transformation. $R_{90} \circ D_2 \neq D_2 \circ R_{90}$

Theorem (3.19). *The symmetries of the square in the plane with composition form a group.*

Proof. This is a group because it is Associative, since function composition is, the identity element is R_0, there exists an inverse for all transformation, and the operation is closed and well-defined. □

Exercise (3.20). *For each natural number n, how many elements does D_n have? Justify.*

Solution. $D_n = 2n$, this comes from there being two ways to organize the vertices, clockwise and counter-clockwise. For each group, there are n organizations. So it follows that the total is $2n$.

Exercise (3.22). *For each of the following groups, find all subgroups. Argue that your list is complete.*

1. $(D_4, \circ)$
2. $(\mathbb{Z}, +)$
3. $(C_n, \oplus_n)$

Solution.

1. The subset including an element, its inverse, and the identity will be a sub group. R_0 is also a subgroup on its own. All rotations are also a subgroup on its own. The super group is also a subgroup. Each flip individually with R_0 will be subgroups. A complete list will come from fully analyzing the Cayley table and finding groups of elements that work.

2. There are many subgroups, $(n\mathbb{Z}, +)$ are many subgroups. This list is complete since multiples of intergers will result in a subgroup of this supergroup.

3. There are many subgroups formed by changing the n. This is a complete list since this cyclic group can including many groups with a different n focus.

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