Homework 12

Jeremy Benedek

April 17, 2016

Exercise (3.79)**.** *Classify each of the following homomorphisms as a monomorphism, an epimorphism, an isomorphism, or none of these special types of homomorphisms.*

1. $\phi : \mathbb{Z}_{12} \to \mathbb{Z}_{24}$ *defined by* $\phi([a]_{12}) = [2a]_{24}$
2. $\phi : \mathbb{Z} \to \mathbb{Z}_9$ *defined by* $\phi(a) = [3a]_9$
3. $\phi : \mathbb{Z}_6 \to \mathbb{Z}_3$ *defined by* $\phi([a]_6) = [a]_3$
4. $\phi : \mathbb{Z}_n \to \mathbb{Z}_n$ *defined by* $\phi([a]_n) = [-a]_n$

Solution.

1. Monomorphism as the function is 1-1, so the domain maps to the image.
2. Homomorphism as the function is neither 1-1 or onto.
3. Epimorphism as the function if onto but not 1-1.
4. Isomorphic as the function is 1-1 and onto.

Theorem (3.83)**.** *For every natural number n, the two groups $(C_n, \oplus_n)$ and $(\mathbb{Z}_n, \oplus)$ are isomorphic*

Proof. Write your proof here. □

Theorem (3.84)**.** *Let $\phi : G \to H$ be an isomorphism. Then $\phi^{-1} : H \to G$ is an isomorphism.*

Proof. Since ϕ is isomorphic, and therefore onto, for every $h \in H$ there exists some $g \in G$ such tat $\phi^{-1}(h) = g$. Assume

Let $h_1, h_2 \in H$ and $g_1, g_2 \in G$. ϕ is injective since $\phi^{-1}(h_1) =$ □

Theorem (3.86)**.** *Let G be a group with an element g. Define $\phi_g : G \to G$ by $\phi_g(h) = ghg^{-1}$. Then $\phi_g : G \to G$ is an isomorphism called conjugation by g.*

Proof. Write your proof here. □

Theorem (3.87)**.** *Let $\phi : G \to H$ be a homomorphism. The ϕ is a monomorphism if and only if $Ker(\phi) = \{e_G\}$. In particular, ϕ is an isomorphism if and only if $Im(\phi) = H$ and $Ker(\phi) = \{e_G\}$.*

Proof. Write your proof here. □

Theorem (3.89)**.** *Let k and n be natural numbers. The map $\phi : \mathbb{Z}_n \to \mathbb{Z}_n$ defined by $\phi([a]_n) = [ka]_n$ is a homomorphism.*

Proof. ϕ respects the product as: $\phi([a]_n)+\phi([b]_n) = [ka]_n+[kb]_n = [ka+kb]_n = [k(a+b)]_n$ and $phi([a]_n) + \phi([b]_n) = \phi([a+b]_n) = [k(a+b)]_n$ ϕ is well defined as: Assume $[a]_n = [b]_n$ and $a = b + jn, j \in \mathbb{Z}$ $[ka]_n = [kb]_n$
k(b+jn)
$_n = [kb + kjn]_n = [kb]_n$ ϕ is a homomorphism. □

Exercise (3.90)**.** *Make and prove a conjecture that gives necessary and sufficient conditions on the natural numbers k and n to conclude that $\phi : \mathbb{Z}_n \to \mathbb{Z}_n$ defined by $\phi([a]_n) = [ka]_n$ is an isomorphism. Use this insight to show that there are several different isomorphisms $\phi : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$*

Solution. My conjecture is there must no common factors between k and n. Using this insight, there are many isomorphisms since you set $n = 12$, so k could equal $5, 7, 11$.

Exercise (3.91)**.** *1. Use conjugation to find two different isomorphisms from D_4 to D_4.*
2. Why does conjugation not give any interesting isomorphisms from $\mathbb{Z}_n$ to itself?

Solution. 1. If $g = R_0$ and $g^{-1} = R_0$ that would be an isomorphism. Also, if $g = R_{180}$ and $g^{-1} = R_{180}$ that would also be an isomorphism.
2. There are no interesting isomorphisms since the commutative property present, and by rearranging terms, you find $g * g^{-1} * h$ and $g * g^{-1}$ is the identity element, so there are no interesting isomorphisms since you $h * e = h$