Homework 11

Jeremy Benedek

April 10, 2016e

Exercise (3.69)**.** *Confirm the function is a homomorphism*
1. $\phi : \mathbb{Z}_{12} \to \mathbb{Z}_{24}$ *defined by* $\phi([a]_{12}) = [2a]_{24}$
3. $\phi : \mathbb{Z}_6 \to \mathbb{Z}_3$ *defined by* $\phi([a]_6) = [a]_3$

Solution. 1. $\phi(g_1 *_G g_2) = \phi([g_1 + g)2]_{12}) = [2(g_1 + g_2)]_{24} = [2g_1 + 2g_2]_{24}$ and $\phi(g_1) *_H \phi(g_2) = [2g_1]_{24} + [2g_2]_{24} = [2g_1 + 2g_2]_{24}$ Additionally, this is well defined: $[a]_{12} = [b]_{12}[2a]_{24} = [2b]_{24}[2(b + k(12))]_{24} = [2b]_{24}[2b + 24k]_{24} = [2b]_{24}$ So this is a homomorphism.
3. $\phi(g_1 *_G g_2) = \phi([g_1 + g_2]_6 = [g_1 + g_2]_3$ $\phi(g_1) *_H \phi(g_2) = [g_1]_3 + [g_2]_3 = [g_1 + g_2]_3$ This is also well-defined, so is a homomorphism.

Exercise (3.69 Extra)**.** *Find the image of the homomorphism and the preimage of the identity.*

Solution. 1. $Im_\phi = \{[0]_{24}, [2]_{24}, ...[22]_{24}\} Preim_\phi([0]_{24}) = \{[0]_{12}\}$
3. $Im_\phi(\mathbb{Z}_6) = \{[0]_3, [1]_3, [2]_3\} Preim_\phi([0]_3) = \{[0]_6, [3]_6\}$

Theorem (3.70)**.** *Let H be a subgroup of group G. Then the inclusion of H into G, $i_{H \subset G} : H \to G$, is a homomorphism*

Proof. Let $i_{H \subset G}(h_1 *_H h_2) = h_3$ and $i_{H \subset G}(h_1) *_G i_{H \subset G}(h_2) = h_1 *_G h_2 = h_3$. Since $h_L \in H$, H will be closed. Therefore, this is a homomorphism. $\square$

Theorem (3.71)**.** *If $\phi : G \to H$ is a homomorphism, the $\phi(e_G) = e_H$.*

Proof. Chose g in G. $\phi(g) = \phi(g *_G e_G) = \phi(g) *_H \phi(e_G).\phi(g) * e_H = \phi(g) *_H \phi(e_G)$. By the cancellation law, $e_H = \phi(e_G)$, so $\phi(e_G) = e_H$ $\square$

Theorem (3.72)**.** *If $\phi : G \to H$ is a homomorphism and $g \in G$, then $\phi(g^{-1}) = [\phi(g)]^{-1}$.*

Proof. $\phi(g *_G g^{-1}) = \phi(g) *_H \phi(g^{-1} = e_H *_H \phi(g^{-1}$. So, $\phi(g) = e_H$. $\square$

Theorem (3.73)**.** *Let G and H be groups, let K be a subgroup of the group G, and let $\phi : G \to H$ be a homomorphism, the $\phi(K)$ is a subgroup of H.*

Proof. Since this is a homomorphism, it is closed. Since this was a group, the well-defined as associative properties are also found. By Theorem 3.72, the inverse element is also present. Additionally, by Theorem 3.71, the identity element is also found, so this is a group. Since all members of $\phi(K)$ will be found in H since the definition of ϕ, this is a subgroup of H. $\square$

Corollary (3.74)**.** *If ϕ is a homomorphism from the group G to the group H, then $Im(\phi)$ is a subgroup of H.*

Proof. By Thm. 3.73, this is a subgroup. Additionally, by Thm. 3.25, a group is a subgroup to itself. Since by the definition of ϕ, $Im(\phi)$ is the same group as H, so it is a subgroup. $\square$

Theorem (3.77)**.** *Let G and H be groups, let L be a subgroup of group H and let $\phi : G \to H$ be a homomorphism, the $\phi^{-1}(L)$, is a subgroup of G.*

Proof. $\phi^{-1}(L)$ is closed because ϕ is a homomorphism. The identity element is present because of Theorem 3.71, and has the inverse property because of Theorem 3.72. The associativity and well-defined properties will also be present since these were already groups. Therefore this is a group. By definition of ϕ, all elements in $\phi^{-1}(L)$ will be in G, therefore, we have a subgroup of G¿ $\square$

Corollary (3.78)**.** *Let G and H be groups. For any homomorphism $\phi : G \to H$, $Ker(\phi)$ is a subgroup of G.*

Proof. By Thm. 3.77, this is a subgroup. Additionally, by Thm. 3.24, the group containing only the identity element is a subgroup to the group that has that identity element. Since by the definition of kernel, $Ker(\phi)$ is a subgroup of G.

$\square$