# The model

We'll examine three different cases: One oscillator, two oscillators, and a general n-oscillator case. All cases are 1-dimensional. Meaning the movement is only in the x axis. When reading, make sure you first understand the simplest case of one oscillator, which is not trivial by itself. Only then should you read about two and more oscillators. You should apply the same idea when you begin writing your code - begin from the simplest case, and make sure it's working properly.

## One Oscillator

Let us consider a particle with mass $m$ connected to 2 walls with 2 springs, both with force constant $\kappa$ and rest length $l_0$, and no friction (See figure 1). We take the left wall to be $x = 0$, and assume the x-axis is pointing to the right.

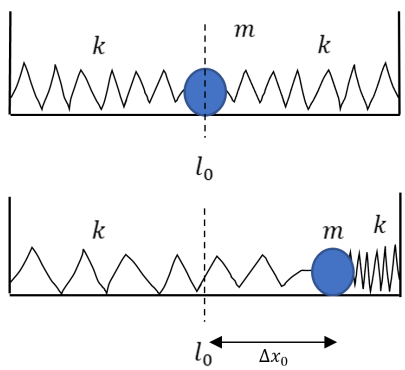


Figure 1: A single oscillator - mass connected to two walls with a two springs.

Assume both springs are not stretched/contracted, meaning the particle is positioned at rest at $x = l_0$, there will be no movement. In other words, $x = l_0$ is the stability point of the system, where nothing interesting is happening.
Now assume the particle is displaced just a bit to the right by some $\Delta x_0$ and is let go at rest ($v(0) = 0$). One spring will be stretched with $\Delta x_0$ and the other will be contracted with $\Delta x_0$, so a force from each spring will be applied - pulling and pushing the particle to the left. It will begin to accelerate to the left $a < 0$. As the particle moves closer to $x = l_0$ the springs are less stretched/contracted, the force applied gets smaller and so is the particle's acceleration. When it reaches $x = l_0$ the acceleration is $a = 0$ (But $v \neq 0$), and when the particle is crossing to left side $x < l_0$ the springs are again stretched/contracted, but with a total force to the right. So the particle begins to accelerate back to the right $a > 0$, slowing down it's velocity, until it's stops and turn back.
When it stops at $x = l_0 - \Delta x_0$, the opposite process will occur - moving to the right until it reaches $x = l_0 + \Delta x_0$ and stops. And thus a full period is done! This is one oscillation, which is repeated forever.

In a more rigorous manner, we can solve $x(t)$ using Hooke's law (at some general state/time of the system):

$$F = -2\kappa \Delta x = -2\kappa (x(t) - l_0) \tag{1}$$

and Newton's second law:

$$F = ma \tag{2}$$

to have a differential equation of $x(t)$ and $\ddot{x}(t)$:

$$\begin{aligned} ma &= -2\kappa (x - l_0) \\ \ddot{x} &= -\frac{2\kappa}{m}(x - l_0) \end{aligned} \tag{3}$$

Where we note $v = \dot{x} = \frac{dx}{dt}$ , $a = \ddot{x} = \frac{d^2x}{dt^2}$. If we consider a new coordinate $\tilde{x} \equiv x - l_0 \implies \ddot{\tilde{x}} = \ddot{x}$, and note

$$\omega \equiv \sqrt{\frac{2\kappa}{m}} \tag{4}$$

the equation becomes:

$$\ddot{\tilde{x}}(t) = -\omega^2 \tilde{x}(t) \tag{5}$$

Which is a second order differential equation, that we know the solution to. We won't go over the full solution of the equation, and you might have seen/will see it in mechanical physics course. The solution is:

$$\tilde{x}(t) = A\cos(\omega t) + B\sin(\omega t) \tag{6}$$

Where $A$ and $B$ are constant which depend on the initial state of the system (in short "initial conditions").

For example, if we move the mass and just let go, the initial velocity (at $t = 0$) will be zero:

$$\dot{x}(0) = \dot{\tilde{x}}(0) = 0 \tag{7}$$

and the initial position will be:

$$\tilde{x}(0) = x(0) - l_0 \equiv x_0 - l_0 \tag{8}$$

for some length $x_0$ that represents the initial displacement.
If we use these two, we can find $A$ by placing $t = 0$ and eq 8 in eq 6:

$$\tilde{x}(0) = x_0 - l_0 = A \tag{9}$$

and we can find $B$ by placing $t = 0$ and eq 7 in the derivative of eq 6:

$$\dot{\tilde{x}}(0) = 0 = -A\omega\sin(\omega \cdot 0) + B\omega\cos(\omega \cdot 0) = B\omega \tag{10}$$

$$\Downarrow$$

$$B = 0 \tag{11}$$

So overall we get:

$$\tilde{x}(t) = \tilde{x_0} \cos(\omega t) \tag{12}$$

where $\tilde{x_0} = x_0 - l_0$.

Later in the simulation we'll have a list/array of particles connected to each other, so we can think of the walls as two other non-moving particles (See figure 2).

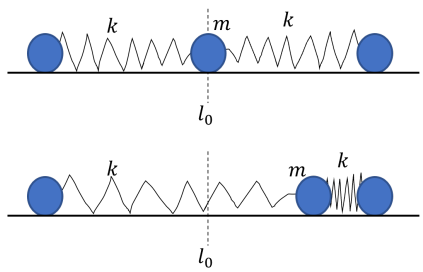


Figure 2: A single oscillator - mass connected to another two not moving masses with springs

In other words, for the single oscillator case, we already begin from a list/array $x_n = [x_0, x_1, x_2]$, where $x_0$ and $x_2$ are the walls, and simply will not move. The first and last coordinate in the array are the edges of the chain, and we'll have to consider them carefully later.

## Two Oscillators

Now, let us consider two particles both with mass $m$ connected to 2 walls with 3 springs, all of them with force constant $\kappa$ and rest length $l_0$, and no friction (See figure 3). We take the left wall to be $x = 0$.

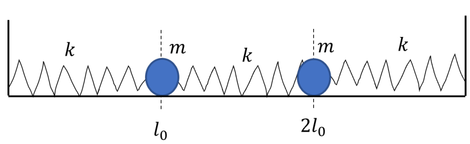


Figure 3: Two oscillators - two masses connected to one another and walls with springs

Like in the case of one oscillator, when the two particles are positioned at rest such that the springs are in their rest length, meaning $x_1 = l_0$ and $x_2 = 2l_0$, there will be no movement. But if we displace one of them with some small $\Delta x$, eventually both of them will begin to oscillate around their stability point. First, assume only the right mass is displaced at some small $\Delta x$ to the right (See figure 4).

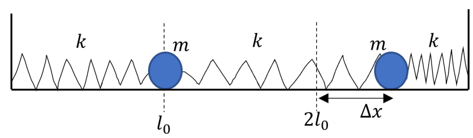


Figure 4: Two oscillators - displacement of only the right particle.

The most right spring will contract with $\Delta x$ and the middle spring will be stretched with $\Delta x$, and so the force on the right particle will be of the two springs:

$$\begin{aligned} F &= -2\kappa\Delta x \\ &= -2\kappa(x - 2l_0) \end{aligned} \tag{13}$$

What will be the force on the left particle? Although the most left spring is still at rest length, the middle spring is not! In other words, there will be a force applied to the left particle, although we only moved the right particle. The fact that the middle spring is affecting both of the particles (and vice versa) makes this problem more difficult to solve, though it is solvable!

Now let us examine a more general case - we'll displace the left particle by some $\Delta x$ and the right particle by some $\Delta y$ (See figure 5). We want to solve $x(t)$, $y(t)$, for the left and right particles respectively, but we already know from the one-oscillator case, it probably will be easier to use coordinates relative to the stability points. We'll note $\tilde{x}(t) \equiv \Delta x = x(t) - l_0$ for the left and $\tilde{y}(t) \equiv \Delta y = y(t) - 2l_0$ for the right.

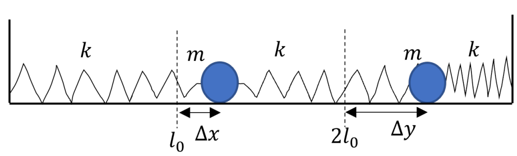


Figure 5: Two oscillators - displacement of both particles.

The most left spring will be stretched with $\tilde{x}$, and the most right spring contracted with $\tilde{y}$. The middle length will be changed both by $\tilde{y}$ and $\tilde{x}$. The forces on the left and right particles respectively will be:

$$
\begin{aligned}
F_x &= -\kappa\tilde{x} - \kappa(\tilde{x} - \tilde{y}) \\
F_y &= -\kappa\tilde{y} - \kappa(\tilde{y} - \tilde{x})
\end{aligned} \tag{14}
$$

And so from Newton's second law:

$$
\begin{aligned}
m\ddot{\tilde{x}} &= -\kappa\tilde{x} - \kappa(\tilde{x} - \tilde{y}) \\
m\ddot{\tilde{y}} &= -\kappa\tilde{y} - \kappa(\tilde{y} - \tilde{x})
\end{aligned} \tag{15}
$$

And now we've got two differential equations! Using $\tilde{x}$ and $\tilde{y}$ coordinates, these equations are very hard to solve. But, we can manipulate the equations and find new coordinates.
First, we'll add the equations:

$$
m(\ddot{\tilde{x}} + \ddot{\tilde{y}}) = -\kappa(\tilde{x} + \tilde{y}) \tag{16}
$$

Second, we'll subtract them:

$$
m(\ddot{\tilde{x}} - \ddot{\tilde{y}}) = -3\kappa(\tilde{x} - \tilde{y}) \tag{17}
$$

And consider the coordinates $X \equiv \tilde{x} + \tilde{y}$ and $Y \equiv \tilde{x} - \tilde{y}$ :

$$
\begin{aligned}
m\ddot{X} &= -\kappa X \\
m\ddot{Y} &= -3\kappa Y
\end{aligned} \tag{18}
$$

If we divide by $m$ both equations, and note $\omega_1^2 \equiv \frac{\kappa}{m}$ and $\omega_2^2 \equiv \frac{3\kappa}{m}$, we'll have the familiar harmonic oscillator equation (same as eq 5), with two frequencies!

$$
\begin{aligned}
\ddot{X} &= -{\omega_1}^2 X \\
\ddot{Y} &= -{\omega_2}^2 Y
\end{aligned} \tag{19}
$$

The solution is given as:

$$
\begin{aligned}
X(t) &= 2a\cos\left(\omega_1 t + \phi_1\right) \\
Y(t) &= 2b\cos\left(\omega_2 t + \phi_2\right)
\end{aligned} \tag{20}
$$

where $a$, $b$, $\phi_1$, $\phi_2$ can be determined by the initial state of the system. These coordinates represent the "normal modes"[1] of the system, and are a bit easier to study the system with than using the regular $\tilde{x}(t)$, $\tilde{y}(t)$ coordinate. We can go back to $\tilde{x}$, $\tilde{y}$ as linear combinations of $X$, $Y$:

[1] You'll study these cases further in waves mechanics course in second year.

$$
\begin{aligned}
\tilde{x}(t) &= \frac{1}{2}(X+Y) = a\cos(\omega_1 t + \phi_1) + b\cos(\omega_2 t + \phi_2) \\
\tilde{y}(t) &= \frac{1}{2}(X-Y) = a\cos(\omega_1 t - \phi_1) - b\cos(\omega_2 t + \phi_2)
\end{aligned}
\tag{21}
$$

The only thing we want to take out of this result, is that there are two special situations with two calculable frequencies:

- The two particles move together, with

$$
\omega_1 \equiv \sqrt{\frac{\kappa}{m}} \tag{22}
$$

If both initial displacements are the same - $\tilde{x}(0) = \tilde{y}(0) = \Delta x_0$ (for some given initial displacement $\Delta x_0$), with no initial velocities $\dot{\tilde{x}}(0) = \dot{\tilde{y}}(0) = 0$. You can check and see that the parameters will be $a = \Delta x_0$, $b = 0$, $\phi_1 = \phi_2 = 0$:

$$
\tilde{x}(t) = \tilde{y}(t) = \Delta x_0 \cos(\omega_1 t) \tag{23}
$$

- The two particles move opposite to one another, with

$$
\omega_2 \equiv \sqrt{\frac{3\kappa}{m}} \tag{24}
$$

If initial displacements are opposite - $\tilde{x}(0) = \Delta x_0$, $\tilde{y}(0) = -\Delta x_0$, with no initial velocities $\dot{\tilde{x}}(0) = \dot{\tilde{y}}(0) = 0$. You can check and see that the parameters will be $a = 0$, $b = \Delta x_0$, $\phi_1 = \phi_2 = 0$:

$$
\begin{aligned}
\tilde{x}(t) &= \Delta x_0 \cos(\omega_2 t) \\
\tilde{y}(t) &= -\Delta x_0 \cos(\omega_2 t)
\end{aligned}
\tag{25}
$$

## N-Oscillators

In this model only close neighbors interactions are assumed. Meaning each particle know only of the one/two next to it. This model is useful in describing properties in some condensed matters like thermal and electric conductivity, radiation scattering and sound waves. We'll examine the last.

The idea is very similar to the previous two oscillators development. Examining the n-th oscillator somewhere in the chain (beside those in the ends), the force is:

$$
m\ddot{\tilde{x}}_n = -\kappa((\tilde{x}_n - \tilde{x}_{n-1}) + (\tilde{x}_n - \tilde{x}_{n+1})) \tag{26}
$$

Where $\kappa$ is the spring constant. So there are $N-2$ differential equations like eq. 26 (and 2 more equations of the ends - $x_0$ and $x_N$). The solution for each of these coordinates are a wave function:

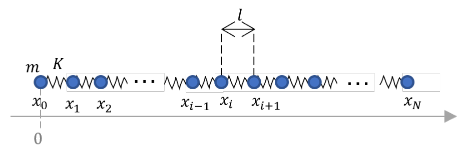


Figure 6: Two oscillators - two masses connected to one another and walls with springs

$$\tilde{x}_n(t) = A_0 \; cos(nkl_0 - \omega t) \tag{27}$$

where $l_0$ is the rest length and $n = 0, 1..N$ is the number of the $n - th$ oscillator. $k$ is the wave number and it's relation with the frequency is:

$$\omega(k) = 2\sqrt{\frac{\kappa}{m}}\left|\sin\left(\frac{kl_0}{2}\right)\right| \tag{28}$$

For small wave number ($k \ll \frac{\pi}{l_0}$) the relation is approximately linear:

$$\omega(k) \approx \sqrt{\frac{\kappa}{m}}kl_0 \tag{29}$$

The velocity of the wave propagation[2] is defined as:

$$C_s = \frac{\partial \omega}{\partial k} \tag{30}$$

(where $\frac{\partial}{\partial k}$ is a partial derivative by $k$) and gives the speed of sound in matter as:

$$\boxed{C_s = \sqrt{\frac{\kappa}{m}}l_0} \tag{31}$$

## Waves - brief introduction

We won't go deep into this subject as you'll study it properly in mechanics of waves course in your 2nd year. That being said, knowing the basics will help us test the simulation and better understand it.

In general, a wave is a propagating dynamic disturbance. Here are some examples:

- Waves in a string

[2] There are actually two velocities to a wave - "Group velocity", as defined in equation 30 and "Phase velocity", defined as $v_p = \frac{\omega}{k}$. We use the former.

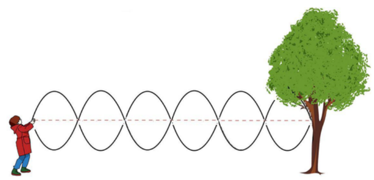

Figure 7: Illustration of a wave in a string/rope. The man is moving one end of the rope, creating a disturbance that is propagating through the rope until the tree, and then reflected back to the man.

- Water waves

Figure 8: A picture of a circular wave in water. The disturbance can be a rock thrown to a pond, making circular wave that is propagating radially.

- Sound waves

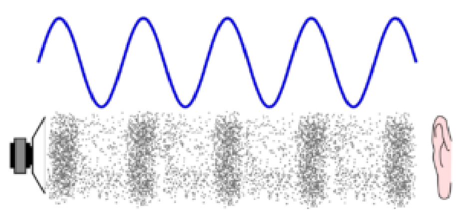

Figure 9: Illustration of a sound wave in air - changes the air's density with time. The speaker move the air molecules in one side, which hit the other molecules next to them and so on, until the last molecules next to our ear are moved.

- Electromagnetic waves

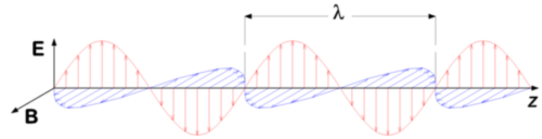


Figure 10: EM wave - the wave is propagating in the z axis direction, where the electric and magnetic fields oscillating perpendicular to propagation direction.

One way you can think of a wave, is like a moving function - say we have some 1-dim shape in space that we can represent with a function:

$$f(x) = A \sin(kx) \tag{32}$$

where $A$ and $k$ are some constants. You can see the function plotted in fig 11, where the $y$ axis can represent many things, like the height of a string at a giving moment, air's density function at a giving moment etc.

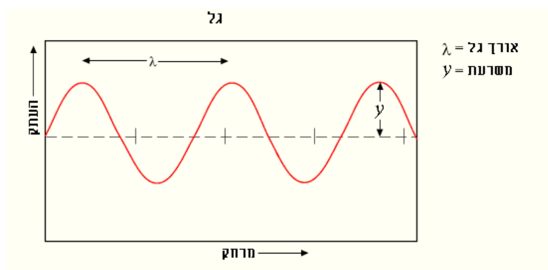


Figure 11: Illustration of simple sinusoidal wave, and it's properties.

Now say the shape is moving with time with some constant velocity $v$ (so instead of $f(x)$ we have $f(x,t)$). Without going into too much details, we can solve this case [3]:

$$f(x,t) = A \sin(kx - \omega t + \phi) \tag{33}$$

Seems familiar? The $A$, $\omega$ and $\phi$ parameters represent the same properties as of an oscillator - Amplitude, angular frequency and phase respectively.
In fact now we have analogous spatial parameters:

- Wave length $\lambda$ - spatial period. Analogous to period $T$ in time, but spatial. The length interval between two repeating points.
- Wave number $k$ - analogous to the frequency $\omega$, but spatial:

$$k = \frac{2\pi}{\lambda} \tag{34}$$

[3] Using a partial differential equation, called the wave equation, surprisingly.

But where is the velocity? Well, first you should know that the time frequency $\omega$ and the spatial frequency $k$ are not independent, and might have a non-trivial relation - some function $\omega(k)$. The velocity of the wave is a parameter which connects them. In other words, we have a function $\omega(k)$ from which we can calculate the wave velocity. The velocity of the wave's group propagation is defined as in eq 30.