Mathematical Analysis Exercises

December 29, 2023

Basic Set Theory

Exercise 0.3.6

1. Prove $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

   (a) Prove $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$

   $$\begin{aligned}&\text{If } \ x \in A \cap (B \cup C) \implies x \in A \ \text{ and } \ (x \in B \ \text{ or } \ x \in C)\\ &\implies (x \in A \ \text{ and } \ x \in B) \ \text{ or } \ (x \in A \ \text{ and } \ x \in C)\\ &\therefore \ A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)\end{aligned}$$

   (b) Prove $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$

   $$\begin{aligned}&\text{If } \ x \in (A \cap B) \cup (A \cap C) \implies (x \in A \ \text{ and } \ x \in B) \ \text{ or } \ (x \in A \ \text{ and } \ x \in C)\\ &\implies x \in A \ \text{ or } \ (x \in B \ \text{ and } \ x \in C)\\ &\therefore \ (A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)\end{aligned}$$

   (c) Thus $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $\square$

2. Prove $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

   (a) Prove $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$

   $$\begin{aligned}&\text{If } \ x \in A \cup (B \cap C) \implies x \in A \ \text{ or } \ (x \in B \ \text{ and } \ x \in C)\\ &\implies (x \in A \ \text{ or } \ x \in B) \ \text{ and } \ (x \in A \ \text{ or } \ x \in C)\\ &\therefore \ A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)\end{aligned}$$

   (b) Prove $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$

   $$\begin{aligned}&\text{If } \ x \in (A \cup B) \cap (A \cup C) \implies (x \in A \ \text{ or } \ x \in B) \ \text{ and } \ (x \in A \ \text{ or } \ x \in C)\\ &\implies x \in A \ \text{ or } \ (x \in B \ \text{ and } \ x \in C)\\ &\therefore \ (A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)\end{aligned}$$

   (c) Thus $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $\square$

Exercise 0.3.11

Prove by induction that $n < 2^n$ for all $n \in \mathbb{N}$.

1. Step 1: $n = 1$

$$1 < 2^1$$
$$1 < 2$$

2. Step 2: Assume $k < 2^k$. Prove that $n < 2^n$ holds for $n = k + 1$

$$k + 1 < 2^{k+1}$$
$$k < 2^k \cdot 2^1 - 1$$

3. If $k < 2^k$ then $2 \cdot 2^k - 1 > k$ $\square$

Exercise 0.3.12

Show that for a finite set A of cardinality n, the cardinality of $\mathcal{P}(A)$ is 2^n.

1. Step 1: $n = 1$ $|A| = 1, |\mathcal{P}(A)| = 2^1 = 2$
2. Step 2: Assume $k = n + 1$. Prove that $|A| = k$ and $\{a\} \in A$, then $n = |A \backslash \{a\}|, |P(A \backslash \{a\})| = 2^n$